# Exercises - hints and remarks 3

SageMath online calculator `https://sagecell.sagemath.org/` with the relevant material summarized on
`https://doc.sagemath.org/html/en/thematic_tutorials/lie/weyl_character_ring.html`
Magma online calculator `http://magma.maths.usyd.edu.au/calc/`

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**Hints for Exercise 1**
For this exercise observe that all hom spaces are trivial and thus, all calculations are just shifting scalars around.

For the background on (strict and nonstrict monoidal) categories see for example Chapter 2 of `https://math.mit.edu/~etingof/egnobookfinal.pdf`, and a recollection on braided categories can be found in Chapter 8 of that book.

For a group $G$, one can define a cohomology theory $H^*(G, \mathbb{C}^*)$, called group cohomology. As usual these are constructed from a certain cochain complex and $H^i(G, \mathbb{C}^*) = Z^i(G, \mathbb{C}^*)/B^i(G, \mathbb{C}^*)$, so $i$ cocycles modulo $i$ coboundaries. All we need to know about group cohomology are the 3 cocycles which are functions $\omega\colon G \times G \times G \to \mathbb{C}^*$ satisfying

$$\omega(j,k,l)\omega(i,jk,l)\omega(i,j,k) = \omega(ij,k,l)\omega(i,j,kl),$$

$$\text{pictorially:}\quad \begin{tikzcd} & & \bullet & & \\ \bullet \arrow[urr, "\omega(ij,k,l)"] & & & & \bullet \arrow[ull, "\omega(i,j,k)"'] \\ & \bullet \arrow[ul, "\omega(i,j,kl)"] \arrow[rr, "\omega(j,k,l)"'] & & \bullet \arrow[ur, "\omega(i,jk,l)"'] & \end{tikzcd}.$$

These 3-cocycles give the obstruction set for twisting a monoidal structure on $\mathcal{V}\mathrm{ec}_G$. Moreover, $\mathcal{V}\mathrm{ec}_G^{\omega} \cong \mathcal{V}\mathrm{ec}_G^{\nu}$ if and only if $\omega$ and $\nu$ distinct in $H^3(G, \mathbb{C}^*)$. Monoidal categories of the form $\mathcal{V}\mathrm{ec}_G^{\omega}$ are nonstrict and skeletal, showing that MacLane's celebrated strictness theorem can not be proven by going to the skeleton.

The category $\mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}^{1}$ can be endowed with two braidings, the so-called standard braiding $\beta_{1,1}^{st} = 1$ and the super braiding $\beta_{1,1}^{su} = -1$. These are nonequivalent. For $\mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega}$ the 3-cocycle $\omega$ only allows one braiding up to equivalence.

**Hints for Exercise 2**
The Jordan decomposition over $\mathbb{C}$ (or rather $\overline{\mathbb{Q}}$ since the Jordan decomposition is unstable over inexact rings) and over $\overline{\mathbb{F}_3}$ can be done using SageMath as above by using:

```
matrix(QQbar,[[0,1,0],[0,0,1],[1,0,0]]).jordan_form(subdivide=False)

matrix(GF(3),[[0,1,0],[0,0,1],[1,0,0]]).jordan_form(subdivide=False)
```

Knowing this, you should be able to give a complete classification of indecomposables modules.

To guess the tensor product rule (and thus, the cell structure) use

```
M=matrix(GF(3),[[1,0],[1,1]]);
M.tensor_product(M).jordan_form()
```

```
M=matrix(GF(3),[[1,0,0],[1,1,0],[0,1,1]]);
M.tensor_product(M).jordan_form()
```

**Hints (or rather comments) for Exercise 3**

There is a Hopf algebra $T_n$ realizing the fiat monoidal category $\mathscr{S}$: the Taft algebra. It is defined by $T_n = \langle g, x | g^n = 1, x^n = 0, gx = \zeta xg \rangle$ where $\zeta$ is a complex primitive $n$th root of unity.

The Taft algebra is a notorious counterexample in Hopf algebra theory. For example, although $M \otimes N \cong N \otimes M$ holds, the category $\mathscr{S}$ is not braided. For $n = 2$ we also get an example of a fiat monoidal category with four indecomposable objects and infinitely many simples representations.

**Hints for Exercise 4**

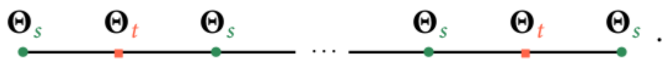
Let $U_k(X)$ be the Chebyshev polynomial defined by $U_0(X) = 1, U_1(X) = X$ and $U_{k+1}(X) = XU_k(X) - U_{k-1}(X)$ for $k > 1$. The defining relations of the $b_1$ and $b_2$ generators are the coefficients of these polynomials. That is, define $U_k(b_2, b_1)$ by replacing $X^k$ with an alternating string $\dots b_1 b_2 b_1$ of length $k$ (always having $b_1$ to the right), and define $U_{n-1}(b_2, b_1)$ similarly. Then $U_{n-1}(b_1, b_2) = 0 = U_{n-1}(b_2, b_1)$.

Thus, the graphs for which one gets a well-defined action must have their spectrum being a subset of the roots of the Chebyshev polynomial. The graphs satisfying this property are the ADE graphs.

If you are up for a challenge: you can construct the associated simple representations of the Soergel calculus. This is (up to some scaling) straightforward if you have worked with Soergel calculus before:

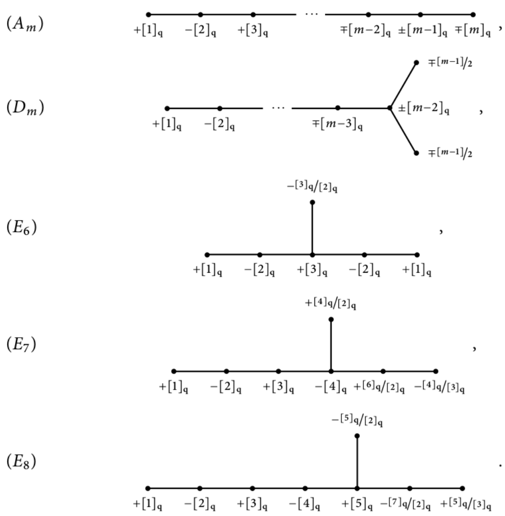
- You need an algebra whose category of projectives you would like to act on: take the zigzag algebra associated to the graph $\Gamma$.
- The projective endofunctors $\Theta$ you need to use for the generating KL basis elements are direct sums of projective endofunctors over the colored vertices:

  We sum over the graph of type $A_m$ as (in the case where $m$ is odd):

  Spinach = sum over tensoring with spinach projectives of the zigzag algebra and vise versa for tomato.
- All of the maps in Soergel land are then easy to guess.
- Warning: For Soergel calculus the scaling is often annoying, and this is the case here as well. The scaling has driven me insane.

  Anyway, the answer is not so bad in the end. You need to rescale everything using the entries of the Perron–Frobenius eigenvector of $\Gamma$, *e.g.* in ADE type:

  Here $[a]_q$ denote the usual quantum numbers evaluated at an $2n$th primitive complex root of unity. Scale the idempotents of the zigzag algebra using the values associated to the vertices, which are the aforementioned entries of the Perron–Frobenius eigenvector.
- Fingers crossed!