CAREER: Categorical representation theory of Hecke algebras

The primary goal of this proposal is to study the categorical representation theory (or *2-representation theory*) of the Iwahori-Hecke algebra $\mathbf{H}(W)$ attached to a Coxeter group W. Such 2-representations (and related structures) are ubiquitous in classical representation theory, including:

- the category $\mathcal{O}_0(\mathfrak{g})$ of representations of a complex semisimple lie algebra $\mathfrak{g}$, introduced by Bernstein, Gelfand, and Gelfand [9], or equivalently, the category $\mathrm{Perv}(Fl)$ of perverse sheaves on flag varieties [75],
- the category $\mathrm{Rep}(\mathfrak{g})$ of finite dimensional representations of $\mathfrak{g}$, via the geometric Satake equivalence [44, 67],
- the category $\mathrm{Rat}(G)$ of rational representations an algebraic group G (see [1, 78]),
- and most significantly, integral forms and/or quantum deformations of the above, which can be specialized to finite characteristic or to roots of unity.

Categorical representation theory is a worthy topic of study in its own right, but can be viewed as a fantastic new tool with which to approach these classical categories of interest.

In §1 we define 2-representations of Hecke algebras. One of the major questions in the field is the computation of local intersection forms, and we mention some related projects. In §2 we discuss a project, joint with Geordie Williamson and Daniel Juteau, to compute local intersection forms in the antispherical module of an affine Weyl group, with applications to rational representation theory and quantum groups at roots of unity. In §3 we describe a related project, partially joint with Benjamin Young, to describe Soergel bimodules for the complex reflection groups $G(m, m, n)$. This also describes the category which is dual, under the quantum geometric Satake equivalence, to quantum $\mathfrak{sl}_n$ at a root of unity. In §4 we introduce a program, joint with Matt Hogancamp, that tries to lift one of the foundational tools in linear algebra, diagonalization, to the categorical level.

1. Categorifying the Hecke algebra

The Iwahori-Hecke algebra $\mathbf{H}$ of a Coxeter group W is a $\mathbb{Z}[v, v^{-1}]$-deformation of the group algebra $\mathbb{Z}[W]$ of W. It has two well-known bases over $\mathbb{Z}[v, v^{-1}]$, the *standard basis* $\{H_w\}$ analogous to the usual basis of $\mathbb{Z}[W]$, and the *Kazhdan-Lusztig basis* or *canonical basis* $\{\underline{H}_w\}$ defined in [54]. The change of basis matrix can be computed algorithmically, though there is no known closed form.

Let W be a Weyl group, and let $\mathcal{O}_0$ denote the trivial block inside category $\mathcal{O}$ [9]. In 1979, Kazhdan and Lusztig [54] conjectured that the Grothendieck group $[\mathcal{O}_0]$ is naturally isomorphic to the regular representation of $\mathbf{H}$ at $v = 1$. Under this isomorphism, the change of basis matrix should encode multiplicities of simples inside Verma modules. Shortly thereafter, Beilinson-Bernstein [6] proved an equivalence of categories between $\mathcal{O}_0$ and $\mathrm{Perv} = \mathrm{Perv}(Fl)$ (with a similar approach due to Brylinski-Kashiwara [14]). Using the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [5], they were then able to prove that the sizes of the simple perverse sheaves agreed with the algebraic definition of the canonical basis, proving the Kazhdan-Lusztig conjecture.

There are still more bases of the Hecke algebra which arise from categorification: namely, the p-canonical basis for a prime p, or the q-canonical basis for a root of unity q (for the affine Weyl group). The change of basis matrices still encode important multiplicites in representation theory. Unforunately, there is currently no direct algorithm to compute these bases, without working in the categorification itself. Let us describe these categorifications, and the technology one uses to analyze them.

1.1. **Soergel bimodules.** Soergel's alternative approach [76] to proving the Kazhdan-Lusztig conjecture was to construct a functor $\mathbb{V}$ from $\mathcal{O}_0$ to R-modules, where $R = \mathrm{Sym}[\mathfrak{h}^*]$ is the polynomial ring attached to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This *Soergel functor* $\mathbb{V}$ is fully faithful on projectives; the images of projectives are known as *Soergel modules*, and form a full subcategory $\mathbb{S}\mathrm{Mod}$. More interesting than category $\mathcal{O}_0$ itself are the *projective functors* which act on it. These categorify the algebra $\mathbf{H}$ itself, acting on its regular representation. Under $\mathbb{V}$ these become R-bimodules, forming the monoidal category $\mathbb{S}\mathrm{Bim}$ of *Soergel bimodules*. Soergel reformulated the Kazhdan-Lusztig conjecture as *Soergel's conjecture*, stating that the indecomposable Soergel bimodules have the correct "size." (Soergel also constructed an analogous functor $\Gamma\colon \mathrm{Perv} \to \mathbb{S}\mathrm{Mod}$.)

Projective functors are generated by *wall-bouncing functors* θ_s, one for each simple reflection s of W, which categorify $\underline{H}_s$. The corresponding Soergel bimodules are denoted B_s. Their tensor products are called *Bott-Samelson bimodules*, and form a category $\mathbb{BS}\mathrm{Bim}$. There is one indecomposable Soergel bimodule B_w for each $w \in W$, which can be defined as the unique summand of the tensor product $BS(\underline{w}) = B_{s_1} B_{s_2} \cdots B_{s_d}$ for a reduced expression $\underline{w} = s_1 s_2 \cdots s_d$ of w, which is not a summand of a shorter tensor product. We call this a *top summand*. Thus the size of B_w depends on which lower summands of $BS(\underline{w})$ split off. Soergel conjectured that (in characteristic 0), one has $[B_w] = \underline{H}_w$, equivalent to the statement that every possible lower "submodule" is split.

One vast advantage of Soergel's approach is that $\mathbb{S}\mathrm{Bim}$ can be defined algebraically, without reference to $\mathcal{O}_0$ or to geometry [79]. This allows one to define $\mathbb{S}\mathrm{Bim}(W)$ for arbitrary Coxeter groups W, and to show that $[\mathbb{S}\mathrm{Bim}(W)] \cong \mathbf{H}(W)$. Soergel's conjecture is the appropriate analog of the Kazhdan-Lusztig conjecture in this context. Many familiar structures on $\mathcal{O}_0$ have Coxeter analogs as well. *Rouquier complexes* [72] are complexes of Soergel bimodules which provide an action of the braid group $\mathbf{Br}_W$ on the homotopy category $K(\mathbb{S}\mathrm{Bim})$ for any Coxeter group, analogous to the twisting and shuffling functors on $\mathcal{O}_0$. Note that, for non-crystallographic Coxeter groups, there is no known geometry or representation theory underlying the category $\mathbb{S}\mathrm{Bim}$. The existence of non-geometric categories having "geometric" properties is a great mystery (see [53] for an analogous situation in toric geometry).

Another advantage of Soergel's construction is that it can be defined over any ring $\Bbbk$ in which a "reflection representation" or *realization* of W is defined. So long as this realization is *reflection faithful*, one has $[\mathbb{S}\mathrm{Bim}] \cong \mathbf{H}$. This gives an integral form of $\mathcal{O}_0$ for any Weyl group, and thus finite characteristic analogs of $\mathcal{O}$. In finite characteristic, some submodules of $BS(\underline{w})$ will no longer split, and the size of B_w may change. Analogously, the decomposition theorem for perverse sheaves fails in characteristic p.

1.2. **Diagrammatics.** In the seminal paper of categorical representation theory, Chuang and Rouquier [16] demonstrate that one must understand the natural transformations between functors to truly appreciate the deeper structures at play. Khovanov-Lauda [56] and Rouquier [73] followed this lead and described the *quiver Hecke algebra*, the algebra of natural transformations in a 2-representation of a quantum group, by generators and relations.

In the Hecke context, the analogous description was accomplished by the PI and collaborators ([32] with Khovanov for type A, [30] in dihedral type, [36] with Williamson in general type). The result is a monoidal category $\mathcal{D}(W)$ given by generators and relations, using the language of planar diagrammatics. A faithful functor $\mathcal{F}\colon \mathcal{D} \to \mathbb{BS}\mathrm{Bim}$ was constructed, which is an equivalence whenever Soergel bimodules are well-behaved (e.g. the realization is reflection faithful). This is a drastic computational improvement, as complicated operations involving polynomials can be described using simple planar graphs.

Definition 1.1. A *2-representation* of the Hecke algebra $\mathbf{H}(W)$ of a Coxeter group W is a monoidal action of $\mathcal{D}(W)$ on a (additive/abelian/triangulated) category.

However, Soergel bimodules are not always well-behaved: for example, there is no faithful representation of an affine Weyl group in positive characteristic (which is the setting of §2). Thankfully, the category $\mathcal{D}$ encodes only the "generic" morphisms between Bott-Samelson bimodules, not the extra morphisms which occur in degenerate cases. When $\mathcal{F}$ is not an equivalence, $\mathcal{D}$ is actually the correct object of study, in that $[\mathcal{D}] \cong \mathbf{H}$ always. When geometry is present, $\mathcal{D}$ is equivalent to the category of parity sheaves on the flag variety [52], which are a better-behaved analog of perverse sheaves in finite characteristic. (We abuse notation here, conflating $\mathcal{D}$ with its Karoubi envelope.)

When $\mathbb{k}$ has characteristic p, the basis of $\mathbf{H}$ given by the classes of indecomposables in $\mathcal{D}$ is known as the *p-canonical basis* ${}^p\underline{H}_w$. To compute it, one must compute *local intersection forms* in $\mathcal{D}$.

1.3. **Local intersection forms and clasps.** Let $\mathbb{k}$ be a field. For an indecomposable object X in a $\mathbb{k}$-linear, Krull-Schmidt category $\mathcal{A}$, $\mathrm{End}(X)$ is a local ring. Often one can guarantee that $\mathrm{End}(X)/\mathfrak{m} \cong \mathbb{k}$. In this case, for any object $B \in \mathcal{A}$, there is a *local intersection pairing*

$$\mathrm{Hom}(B, X) \times \mathrm{Hom}(X, B) \to \mathrm{End}(X)/\mathfrak{m} \cong \mathbb{k}$$

given by composition. The rank of this pairing is precisely the number of orthogonal copies of X in the direct sum decomposition of B. When $\mathcal{A}$ admits a duality functor such that $\mathrm{Hom}(B, X) \cong \mathrm{Hom}(X, B)$, the pairing becomes a *local intersection form* or *LIF* on $\mathrm{Hom}(B, X)$.

In the ideal situation, one can compute LIFs within $\mathcal{A}_{\mathbb{Z}}$, an integral form of $\mathcal{A}$, and can compute their invariant factors. This yields the rank of the LIF after specialization to any characteristic. Moreover, if $\mathrm{Hom}(B, X)$ admits a combinatorial basis, one can hope to compute the local intersection form on this basis. This will allow one to explicitly construct the orthogonal projection to each summand.

Dream 1. Compute local intersection forms in $\mathcal{D}(W)$, for all indecomposables X and all tensor products of indecomposables B.

This is an incredibly difficult problem. Let us look at a related problem (actually a special case).

Consider $\mathrm{Rep}_q(\mathfrak{sl}_n)$, the finite dimensional representations of the quantum group $U_q(\mathfrak{sl}_n)$ over $\mathbb{Q}(q)$, and its full monoidal subcategory $\mathrm{Fund}_q(\mathfrak{sl}_n)$ of tensor products of fundamental representations. Cautis, Kamnitzer, and Morrison [15] gave a diagrammatic description of Fund_q using *$\mathfrak{sl}_n$-webs* (c.f. the earlier works [68, 58]). A long-standing open problem of Kuperberg [58] is to compute the *clasps* in terms of $\mathfrak{sl}_n$-webs. Clasps are the projections from a tensor product in Fund_q to its top summand, the irreducible V_λ. For $\mathfrak{sl}_2$, clasps are called *Jones-Wenzl projectors* [51, 81]. Partial progress has been made for $\mathfrak{sl}_3$ by Kim [57], but beyond that nothing is known.

In forthcoming work [26], the PI has found a combinatorial cellular basis for $\mathfrak{sl}_n$-webs, akin to the basis produced by Fontaine [41] via the geometric Satake equivalence. It is still a basis for the $\mathbb{Z}[q, q^{-1}]$-integral form of Fund_q. Computing in this basis, we reached the following conjecture. Let ω_i be a fundamental weight, μ be a weight in V_{ω_i}, and λ be a dominant weight. If $\lambda + \mu$ is dominant, $V_{\lambda+\mu}$ appears as a summand in $V_\lambda \otimes V_{\omega_i}$ exactly once, and the LIF of $V_\lambda \otimes V_{\omega_i}$ at $V_{\lambda+\mu}$ is a (generically invertible) 1×1-matrix.

Conjecture 1. *Let $w \in S_n$ be a minimal length element which sends ω_i to μ, and let Φ_w denote the positive roots sent to negative roots by w^{-1}. Then the LIF of $V_\lambda \otimes V_{\omega_i}$ at $V_{\lambda+\mu}$ is equal to $\prod_{\alpha \in \Phi_w} \frac{[(\alpha,\lambda)+1]}{[(\alpha,\lambda)]}$. Here, (α, λ) is an integer, pairing a root with a weight, and $[n]$ denotes the n-th quantum number. Note that $\lambda + \mu$ is not dominant if and only if some $(\alpha, \lambda) = 0$.*

This conjecture allows one to develop a *double clasp recursive formula* for clasps, generalizing the formulas for Jones-Wenzl projectors in [81, 10]. The work in progress [26] of the PI will prove Conjecture 1 for $\mathfrak{sl}_n$ for $n = 3, 4$ by direct computation. We hope to generalize this proof to all n.

This is an important first step to analyzing what happens at a root of unity, when various quantum numbers will vanish. However, this "inductive step" is not sufficient to understand Fund_q at a root of unity, because it assumes that V_λ is already defined. The next step is to compute the LIF for more difficult tensor products.

The conjecture gives a combinatorial, root- and weight-theoretic description of the value of an intersection form. Life is made simpler by the fact that fundamental representations in type A are miniscule, but there is a reasonable chance that the conjecture can be broadened to deal with a wide variety of situations: arbitrary tensor products of irreducibles, tensor products in other types, etcetera. It gives hope that the more ambitious program of §2 is tractable.

1.4. **Decompositions of Soergel bimodules.** The Soergel conjecture is equivalent to the non-degeneracy of every LIF (after accounting for the grading). In a paper published in *Annals of Mathematics*, the PI and Williamson [38] proved the Soergel conjecture when the base ring $\Bbbk$ is $\mathbb{R}$. This gave the first purely algebraic proof of the Kazhdan-Lusztig conjecture. The main approach was to adapt De Cataldo and Migliorini's proof of the decomposition theorem [17, 18] to our algebraic setting. They proved the non-degeneracy of an analogous local intersection form using the Hodge-Riemann bilinear relations on a global intersection form. Similarly, one proves that indecomposable Soergel modules satisfy the Hodge-Riemann bilinear relations, and uses this to bootstrap a decomposition theorem for the semismall tensor product $B_w B_s$ (semismall means that it splits into indecomposables without grading shifts). This adaptation was subtle, requiring a careful argument using Rouquier complexes to bypass the lack of a Lefschetz hyperplane theorem.

We expect many other Hodge-theoretic properties that hold for perverse sheaves to hold more generally for Soergel bimodules. For example, Williamson [84] proved the local Hodge-Riemann bilinear relations for "stalks" of Soergel bimodules. For non-semismall Soergel bimodules one expects the *relative Hodge-Riemann bilinear relations* to hold (see [18]). We believe we are close to proving the following conjecture.

Conjecture 2. *Bott-Samelson bimodules, as well as arbitrary tensor products of indecomposable Soergel bimodules, satisfy the relative Hodge-Riemann bilinear relations.*

Project 1. Prove similar Hodge-theoretic results in analogous contexts: for singular Soergel bimodules [82], and for quiver Hecke algebras.

Abstract proofs of non-degeneracy of the LIF do not help to compute it, leaving the explicit form of the indecomposables unknown, even when $[B_w] = \underline{H}_w$. After all, the Kazhdan-Lusztig basis is poorly understood. There are only closed formulas for the Kazhdan-Lusztig basis in very special cases (e.g. dihedral groups, universal Coxeter groups [23]), where a complete understanding of the indecomposables in $\mathcal{D}$ has been accomplished by the PI (c.f. [30], [33] joint with Libedinsky). Although the community was initially optimistic, it is now accepted that the local intersection forms can be arbitrarily nasty, as was shown by Williamson in his disproof of the "Lusztig conjecture" [83]. Unfortunately, there is no known analog of Hodge theory to help outside of characteristic zero. At least the results of [38] imply that, for any given w, $[B_w]$ agrees with $\underline{H}_w$ as $p >> 0$.

Nonetheless, as papers like [83, 48, 24] illustrate, diagrammatics are excellent computational tool. Local intersection forms can now be computed by hand (compared to difficult geometric computations like [85, Appendix]). They can be programmed into a computer. Williamson has programmed a number of tools (not publically available) to compute with morphisms in $\mathcal{D}$, and to compute p-canonical bases.

(I would eagerly support the development of publically available programs for analyzing $\mathcal{D}$. This would require finding the right person for the job. Hopefully I find such a person and can put him or her on the next grant!)

2. Tilting modules, p- and q-canonical bases

2.1. **Rational representation theory.** We now discuss a major application of the p-canonical basis to classical representation theory.

Let G be a semisimple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. A *rational representation* is a map $G \to GL(V)$ as varieties over $\Bbbk$ (for a vector space V), and they form a category Rat. This category admits standard, costandard, and tilting modules, and a result of Donkin [19] states there is one indecomposable tilting module $T(\lambda)$ for each dominant weight.

Let Rat_0 denote the block of the trivial representation $T(0)$. There is an action of the affine Weyl group W_a on weights, where the finite Weyl group W_f acts by its usual "shifted" action, and a general reflection involves crossing *p-walls*. For simplicity we assume the trivial character is regular (care must be taken otherwise for small values of p). Then minimal coset representatives $w \in W_a/W_f$ parametrize the indecomposable tilting modules $T(w \cdot 0)$ in Rat_0, which generate an additive subcategory Tilt_0.

Dream 2. Compute the sizes (i.e. characters) of simple modules in Rat_0, or analogously, of indecomposable tilting modules in Tilt_0.

For any given $w \in W_a/W_f$, the size of $T(w \cdot 0)$ for $p >> 0$ is encoded by Kazhdan-Lusztig combinatorics. This (or its simple module analog) was conjectured by Lusztig in [62], and was proven by the combined work of many (see [2] for a survey). Which primes are sufficient within certain regions of W_a/W_f is the topic of a number of conjectures [62, 3], but the answer is difficult, as demonstrated by [83]. Outside of SL_2, almost nothing is known about the sizes of indecomposable tiltings.

Soergel's approach [1, 77, 78] is again to study wall-crossing functors acting on Rat_0. He proves that wall-crossing functors induce an action of the affine Hecke algebra $\mathbf{H}_a$ on $[\mathrm{Rat}_0]$, making $[\mathrm{Rat}_0]$ isomorphic to M^{asph}, the *antispherical module*. This is the representation induced from the sign representation of the finite Hecke algebra $\mathbf{H}_f$.

One can also categorify the antispherical module with a category $\mathcal{M}^{\mathrm{asph}}$, obtained as a *parabolic quotient* of $\mathcal{D}(W_a)$ by killing diagrams with any subdiagram from $\mathcal{D}(W_f)$ on the right. This parabolic quotient has one indecomposable object M_w for each minimal coset representative w in W_a/W_f. The following conjecture is due to Williamson and Riche [71], and is part of a larger program on their part.

Conjecture 3. *There is a (grading-forgetful) equivalence* $\mathcal{M}^{\mathrm{asph}} \to \mathrm{Tilt}_0$*, compatible with the action of* $\mathcal{D}(W_a)$ *and wall-crossing functors, sending* M_w *to* $T(w)$*.*

Williamson and Riche believe they can prove this result (in type A) by directly constructing an action of $\mathcal{D}_a$ on Tilt_0. Meanwhile, Loseu and the PI have an independent proof in progress (in type A), by constructing an action of $\hat{\mathfrak{sl}}_p$ on a parabolic quotient of singular Soergel bimodules, and using uniqueness results from quantum group categorification theory. By implication, the p-canonical basis of M^{asph} encodes the sizes of indecomposables in Tilt_0 (and gives a graded analog of the characters). Now we have algebraic tools to investigate this difficult problem.

Lusztig [61] recently extrapolated the expected behavior of the Steinberg tensor product theorem to produce a conjectural formula for the sizes of simple modules. This was modified by Lusztig-Williamson [64] to deal with tilting modules. It posits a **fractal** behavior: beginning with the Kazhdan-Lusztig basis of M^{asph}, denoted S^0_λ, one modifies the basis with an operation coming from p-walls to get S^1_λ, then again from p^2-walls to get S^2_λ, and so forth. The infinite limit S^∞_λ is expected to be the p-canonical basis.

Project 2. Prove the Lusztig-Williamson conjecture for SL_3, and eventually SL_n.

2.2. **The quantum deformation.** In Andersen-Jantzen-Soergel's study of Tilt_0, they also studied wall-crossing functors acting on tilting modules Tilt_q for the quantum group $U_q(\mathfrak{g})$ at a p-th root of unity, and found that these also categorify the antispherical module, and agree with Tilt_0 for $p >> 0$. Soergel [77] conjectured that there should be some graded, Soergel-bimodule-style lift of Tilt_q.

The PI [28] has introduced a q-deformation of the affine Cartan matrix in type A. This leads to a q-deformation of the reflection representation of W_a, and thus to q-deformations $\mathbb{S}\mathrm{Bim}_q$, $\mathcal{D}_q$, and $\mathcal{M}^{\mathrm{asph}}_q$, of the corresponding constructions for W_a. The following conjecture, the *quantum Riche-Williamson conjecture* [71], is a more precise version of the conjecture of Soergel just mentioned. It was recently proven for $\mathfrak{sl}_2$ by Andersen-Tubbenhauer [4].

Conjecture 4. *There is a grading-forgetful equivalence* $\mathcal{M}^{\mathrm{asph}}_q \to \mathrm{Tilt}_q$*, compatible with the action of wall-crossing functors.*

This motivates the computation of the *q-canonical basis* of the antispherical module, given by the indecomposable objects in $\mathcal{M}_q^{\text{asph}}$. One also expects the q-canonical basis to be given by the first step, S^1_λ, in the fractal process discussed above. This corresponds to a general philosophy, that quantum groups at a p-th root of unity are a first-order approximation to rational representation theory in characteristic p.

Where did this q-deformation arise? That a one-parameter deformation of the reflection representation exists was pointed out by Lusztig in [60] (one also exists in type C, and no other types), but an explanation for this particular parametrization is still a mystery. In addition to the conjecture above, it plays a key role in *quantum geometric Satake*. The geometric Satake equivalence [44, 67] was reformulated by the PI as a 2-equivalence of 2-categories between $\text{Rep}(\mathfrak{g})$ (keeping track of the central character), and a 2-category living inside singular Soergel bimodules [82] for the affine Weyl group $\mathcal{S}\mathbb{S}\text{Bim}_a$. In finite and affine type A, the PI and Williamson have a presentation of $\mathcal{S}\mathbb{S}\text{Bim}$ as a diagrammatic 2-category $\mathfrak{D}$ (still in progress, but see the appendix of [28]). The PI [28] used the diagrammatic descriptions ($\mathfrak{D}$ and $\mathfrak{sl}_n$-webs) of both sides to give an elementary proof of geometric Satake in type A, and to q-deform it to an equivalence between representations of $U_q(\mathfrak{sl}_n)$ and a sub-2-category of $\mathfrak{D}_q$. In particular, Conjecture 1 (about clasps) also describes intersection forms within $\mathfrak{D}_q$, $\mathcal{D}_q$, and $\mathcal{M}_q^{\text{asph}}$.

Project 3. Find a q-deformation of $\mathcal{D}$ in other types.

2.3. **Computations and conjectures.** In joint work with Williamson and Juteau, we aim to accomplish the following task.

Project 4. Compute the local intersection forms for M_y inside $M_x B_w$ in $\mathcal{M}_q^{\text{asph}}$ explicitly for $\mathfrak{sl}_3$, and compute their invariant factors. Perform the same computations at a root of unity (considerably more difficult).

We have done many computations both generically and at a root of unity, and believe the task is tractable. Once completed, the lessons learned can hopefully convert our results to general results for $\mathfrak{sl}_n$.

Conjecture 1 already indicates that various combinatorial patterns for intersection forms exist. It is also an illustration of the following conjecture.

Conjecture 5. *Intersection forms in $\mathcal{M}_q^{\text{asph}}$ admit a* periodicity principle*, where translating elements of W_a/W_f by dominant weights has a predictable effect on the LIF by shifting certain quantum numbers.*

There is a vague philosophical hope that the periodicity principle "explains" the fractal relation between the q and p-canonical bases, so that our results specialized to $q = 1$ will help prove the fractal character conjecture of Lusztig-Williamson. At a p-th root of unity, $[p] = 0$. If p divides n then $[p]$ divides $[n]$ exactly once. So if an intersection form vanishes at a p-th root of unity, the periodicity principle says it will vanish in the same way after translation, and this roughly underlies the transformation from S^0_λ to S^1_λ. Meanwhile, at $q = 1$, p can divide n more than once, leading to extra vanishing at p^2 walls, etcetera. This is just a rough heuristic, but we believe the periodicity principle will play a key role in an eventual proof.

Even with a periodicity principle, many LIF computations remain before the p-canonical basis is understood. A key tool to handle these will come from the proposal in the next chapter.

3. Soergel bimodules for $G(m, m, n)$

In the previous chapter, we have been careful to state our results in terms of the diagrammatic category $\mathcal{D}_q$, rather than the algebraic category $\mathbb{S}\text{Bim}_q$. The two categories are quite different. However, there is still a faithful functor $\mathcal{F} \colon \mathcal{D}_q \to \mathbb{S}\text{Bim}_q$, so one can compute the LIF in $\mathbb{S}\text{Bim}_q$ instead.

3.1. **A new Grothendieck group.** When q is a m-th root of unity, the q-deformed reflection representation of W_a factors through a finite quotient, the complex reflection group $W = G(m, m, n)$. (Here we work with $\mathfrak{sl}_n$, so that W_a has type $\tilde{A}_{n-1}$.) Soergel's categorification results no longer apply, and $[\mathbb{S}\mathrm{Bim}_q]$ is no longer isomorphic to $\mathbf{H}(W_a)$. What exactly $[\mathbb{S}\mathrm{Bim}_q]$ is, is currently unknown.

One major difference with the generic case is that the inclusion of the invariant subring $R^W \subset R$ is now a Frobenius extension. We have glossed over the importance of Frobenius extensions in Soergel theory, but the implication is that this enriches the (singular) Soergel category, allowing for new 1-morphisms and new 2-morphisms to be easily expressed as diagrams. Augmenting the approach outlined in [40], we hope to accomplish the following.

Project 5. Find a diagrammatic description for $\mathbb{S}\mathrm{Bim}_q$ and $\mathcal{S}\mathbb{S}\mathrm{Bim}_q$ at an m-th root of unity. Compute its Grothendieck group.

Consider the special case $n = 2$, corresponding to $\tilde{A}_1$. In this case, $W = G(m, m, 2)$ is itself a Coxeter group, the finite dihedral group. The PI studied dihedral groups extensively in [30], where a precise relationship is demonstrated between $\mathbb{S}\mathrm{Bim}(W_a)$ and $\mathbb{S}\mathrm{Bim}_q$. Namely, $\mathbb{S}\mathrm{Bim}_q$ is generated over $\mathbb{S}\mathrm{Bim}(W_a)$ by a new cyclic $2m$-valent vertex, requiring only the imposition of two new relations. Similarly, $\mathcal{S}\mathbb{S}\mathrm{Bim}_q$ is generated over $\mathcal{S}\mathbb{S}\mathrm{Bim}(W_a)$ be allowing a new Frobenius extension, and imposing one new non-Frobenius relation related to the Jones-Wenzl projector for $\mathfrak{sl}_2$.

Eventually we plan to tackle the general case, but we have focused first on the case $n = 3$. Preliminary calculations have demonstrated that the following conjecture is reasonable.

Conjecture 6. *In type $\tilde{A}_2$, the diagrammatic description of $\mathbb{S}\mathrm{Bim}_q$ is obtained from $\mathcal{D}_q$ by adding a new cyclic $3m$-valent vertex, and imposing a small number of relations. The diagrammatic description of $\mathcal{S}\mathbb{S}\mathrm{Bim}_q$ is obtained from $\mathfrak{D}_q$ by adding a new Frobenius extension, and imposing a small number of non-Frobenius relations (perhaps related to $\mathfrak{sl}_3$ clasps).*

To understand the Frobenius extension $R^W \subset R$, it is essential to understand the *quantum nilCoxeter algebra*. This is the algebra inside $\mathrm{End}(R)$ generated by the *divided difference operators* $\partial_s \colon f \mapsto \frac{f - sf}{\alpha_s}$ for each simple reflection s. In familiar settings the operators ∂_s satisfy the braid relations; here they satisfy them only up to scalar. However, the additional relations which give the kernel of the map $W_a \to W$ are not satisfied by ∂_s. Instead there are interesting linear combinations of compositions of divided difference operators which vanish, yielding a finite dimensional graded algebra. One requires there to be a unique "longest element" in this algebra (i.e. a one-dimensional top degree space), for which the corresponding operator is the Frobenius trace map $R \to R^W$.

Example 3.1. The complex reflection group $G(2, 2, 3)$ is isomorphic to S_4 and has size 24. Meanwhile, the quantum nilCoxeter algebra has size 36, with Poincare polynomial $1 + 3q + 6q^2 + 9q^3 + 10q^4 + 6q^5 + q^6$. It is also possible that $[\mathbb{S}\mathrm{Bim}_q]$ has size 36, though this is not confirmed.

Remark 3.2. Shoji and Rampetas [70] also have a notion of a nilCoxeter algebra for complex reflection groups $G(m, m, n)$, which acts on R. However, their algebra is always the same size as W. They work with the Broue-Malle-Rouquier presentation of W [12], rather than the presentation which arises as a quotient of the affine Weyl group, so they have different generators.

Working with Ben Young, also at University of Oregon, the PI is currently investigating the quantum nilCoxeter algebra, as a preliminary to any serious investigation of $\mathbb{S}\mathrm{Bim}_q$. We have been able to gather data using computerized calculations. Performing linear algebra with polynomial rings and their operators is a serious task, as checking the equality of two operators naively involves checking a large number of coefficients for each monomial in relatively high degree. Working over finite fields will drastically reduce the required computations, but leads to its own difficulties (such as the division in the definition of ∂_s). We used the opportunity to give a project to a talented undergraduate (Kevin Wilson), who wrote a package to deal with divided difference operators for polynomial rings over characteristic p. Using this, we have been able to compute the relations for $m < 29$.

Conjecture 7. *When $n = 3$, the quantum nilCoxeter algebra can be described as a quotient of the nilCoxeter algebra of W_a by two new explicit relations, in degrees $2m$ and $3m-1$. The Poincare polynomial obeys an explicit formula.*

It is too early to make any precise conjectures beyond $n = 3$.

3.2. **Connections to tilting modules and other applications.** There is a faithful functor from $\mathcal{D}_q$ to $\mathbb{S}\mathrm{Bim}_q$. Therefore, local intersection forms may be computed after applying the functor. Assuming that $[\mathbb{S}\mathrm{Bim}_q]$ is well-understood and an analog of Soergel's Hom Formula [79] is proven, it should be possible to use structural results in $\mathbb{S}\mathrm{Bim}_q$ to compute the ranks of intersection forms in $\mathcal{D}_q$. Williamson, Juteau, and the PI have tested this concept for $n = 2$ where $\mathbb{S}\mathrm{Bim}_q$ is well-understood (and local intersection forms are non-degenerate), and in this case there is enough information to compute the ranks of intersection forms easily. Outside of $\mathfrak{sl}_2$, it may not be as easy, but the technique should still reduce greatly the number of computations required; it is too early to make any precise conjectures.

Representation theorists have long been searching for a categorification of the Hecke algebras of complex reflection groups. It does not appear that $\mathbb{S}\mathrm{Bim}_q$ is the answer, though we do expect a "character map" from $[\mathbb{S}\mathrm{Bim}_q]$ to $\mathbf{H}(G(m,m,n))$, which could shed new light on that Hecke algebra. However, the PI would suggest that $[\mathbb{S}\mathrm{Bim}_q]$ is another Hecke-style algebra related to $G(m,m,n)$, with admirable properties, which may serve as a fruitful alternative.

Admittedly, this project is proposing the study of a brand new category which, outside of the functor from Tilt_q at a root of unity, has no known connections to other fields in mathematics, and must (for the time being) stand on its own merits. We do expect this category to have many of the same "geometric" properties that Soergel bimodules have. Proving this, and studying the category in general, will require the development of techniques which are sure to be more widely applicable in the search for some of the "missing" categorical constructions in mathematics.

4. Categorical diagonalization

4.1. **Chuang-Rouquier filtrations.** In the classical representation theory of $\mathfrak{sl}_2(\mathbb{C})$, irreducible representations $L(\lambda)$ are parametrized by highest weights $\lambda \in \mathbb{N}$. Representations are semisimple, so an arbitrary representation V splits canonically into isotypic components $V = \oplus_{\lambda \in \mathbb{N}} V_\lambda$. Isotypic components are determined by a multiplicity space.

Chuang and Rouquier [16] provide the analogous results for 2-representations of $\mathfrak{sl}_2$. The "irreducible" 2-representations $\mathcal{L}(\lambda)$ are parametrized by highest weights $\lambda \in \mathbb{N}$. An arbitary 2-representation $\mathcal{V}$ has a canonical **filtration** whose subquotients are isotypic categorifications $\mathcal{V}_\lambda$. This filtration is in order of highest weight, with the trivial isotypic component being a quotient. Isotypic categorifications are determined by a multiplicity category.

These are beautiful abstract results, but one hopes to make them more concrete. For example, Khovanov and Lauda [56] constructed a cyclotomic quotient of the quiver Hecke algebra, whose module category they conjectured (correctly, as proven by Lauda-Vazirani [59], Webster [80], and others) to be a realization of $\mathcal{L}(\lambda)$. This has been an incredibly fruitful object of study (e.g. [13, 66, 50]). In similar fashion, one might hope for an explicit construction of the Chuang-Rouquier filtration; perhaps a functor taking a module over the quiver Hecke algebra to a module for the appropriate cyclotomic quotient.

Here we propose a general theory to describe the Chuang-Rouquier filtration, coming from a different angle. One of the key tools in classical representation theory is the Casimir operator c, which generates the center of the enveloping algebra $U(\mathfrak{sl}_2)$. The splitting into isotypic components $V = \oplus V_\lambda$ is actually the eigenspace decomposition of V with respect to c. Note that the dominance order on weights (i.e. the order in the Chuang-Rouquier filtration) provides an order on the eigenvalues of c. Meanwhile, Beliakova-Khovanov-Lauda [7] constructed a complex of 1-morphisms C from Khovanov-Lauda's category $\mathcal{U}$, categorifying c. They proved that C is in the Drinfeld center of the homotopy category of $\mathcal{U}$, which is to say that for any 1-morphism $M \in \mathcal{U}$ there is a canonical homotopy equivalence $CM \to MC$ which is natural over 2-morphisms $M \to N$ (and satisfies other compatibility axioms).

Our main question is this: can the canonical filtration of $\mathcal{V}$ into isotypic components $\mathcal{V}_\lambda$ be described using an "eigenspace decomposition" of the operator C? This vague question will be made more precise soon. One also hopes to recover uniqueness theorems, and even possibly the explicit construction of $\mathcal{L}(\lambda)$ using cyclotomic quotients, by studying the diagonalization of C. Diagonalization has been a foundational tool in linear algebra and representation theory, and we expect a categorical version of diagonalization to be a foundational tool in 2-representation theory as well.

4.2. **Filtrations for Hecke algebras.** Analogously, one hopes to lift classical results about representations of $\mathbf{H}$ to the categorical level. For Hecke algebras, one classifies simple representations not with weights but with *cells*. The notion of a cell lifts easily to additive monoidal categories. One can place a preorder on the set of indecomposable objects, where $X < Y$ if there exist objects F and G such that Y is a summand of FXG. The equivalence classes are known as *two-sided cells*. Morphisms which factor through an object in a given cell or higher cells form a monoidal ideal, giving a cellular filtration on the category. On the Grothendieck group of $\mathcal{D}$, one recovers the usual cell theory for the Kazhdan-Lusztig basis (in characteristic 0). Thus, one can categorify cell modules as subquotients of the regular 2-representation $\mathcal{D}$.

Mazorchuk and Miemietz have a beautiful series of papers ([65] and its sequels) where they study 2-representation theory in this light, and prove a sweeping generalization of these results. Under certain assumptions, they can show that the "irreducible" 2-representations categorify the cell modules, that isotypics are (not-quite-exterior) products of a cell module with a multiplicity category, and that arbitrary representations have filtrations with isotypic subquotients. Many of their results are quite general, but their most powerful results rely on the cell theory being *regular*, meaning that within a two-sided cell each left cell and right cell intersect in exactly one element. For Coxeter groups, unfortunately, this is only a type A phenomenon.

The explicit-ification of irreducible representations on the Hecke algebra side is a wide open problem. Special cases are known due to work of the PI, such as for Temperley-Lieb quotients in type A [29], or generalized Temperley-Lieb quotients for dihedral groups [30]. The remainder of this chapter will focus on canonical filtrations, by categorically diagonalizing the full twist in its action on $\mathcal{D}$.

4.3. **Eigenmaps and eigencones.** Suppose that f is an endomorphism of a finite-dimensional vector space V, satisfying $\prod_{\lambda_i \in \mathbb{S}}(f - \lambda_i I) = 0$ for a finite set $\mathbb{S}$. In other words, f is *diagonalizable*, and we know its *spectrum* $\mathbb{S}$. Linear algebra has the machinery to extract a great deal out of this small amount of information. For example, for $\lambda_i \neq \lambda_j \in \mathbb{S}$ let $c_{i,j} = \frac{f-\lambda_i}{\lambda_j - \lambda_i}$, which acts as the identity on the λ_j eigenspace, while killing the λ_i-eigenspace. Note that $1 - c_{i,j} = c_{j,i}$. Then $p_j = \prod_{i \neq j} c_{i,j}$ is projection to the λ_j-eigenspace. Thus one has constructed "for free" a collection $\{p_j\}$ of orthogonal idempotents, which sum to the identity of V.

Our goal is to construct the analogous machinery when one has a functor $\mathcal{F}$ acting on a category $\mathcal{V}$. For simplicity of language, let us work with a graded monoidal additive category $\mathcal{A}$. A (bounded) complex $\mathcal{F}$ of objects in $\mathcal{A}$ can be viewed as a functor on the homotopy category $\mathcal{V} = K(\mathcal{A})$ via the monoidal product. For simplicity, we shall assume that all eigenvalues are *monomial*. That is, they have the form $\lambda_i = (-1)^{k_i} v^{n_i}$, so that they can be categorified by an *eigenshift* $\mathbb{1}\langle k_i \rangle (n_i)$, where $\langle k_i \rangle$ is a homological shift and (n_i) is a grading shift. If $M \in K(\mathcal{A})$ categorifies an eigenvector, one expects that $\mathcal{F}M \cong M\langle k_i\rangle(n_i)$. As usual in category theory, isomorphism is not a property, but a structure.

Definition 4.1. Let $\alpha\colon \mathbb{1}\langle k_i\rangle(n_i) \to \mathcal{F}$ be a morphism in $K(\mathcal{A})$, which we call a potential *(forward) eigenmap*. A nonzero complex $M \in K(\mathcal{A})$ is an *eigencomplex* for α, and α is actually an eigenmap, if αM is an isomorphism $M\langle k_i\rangle(n_i) \to \mathcal{F}M$. Let Λ_α denote the cone of α, the *eigencone*. The full subcategory of $K(\mathcal{A})$ consisting of eigencomplexes for α is known as the *eigencategory* $\mathcal{A}_\alpha$ of α.

The eigencone categorifies $(f - \lambda_i I)$, and M is an eigencomplex for α if and only if $\Lambda_\alpha M \cong 0$ in $K(\mathcal{A})$.

Definition 4.2. We say that $\mathcal{F}$ is *(forward) categorically diagonalizable* with *spectrum* $\mathbb{S} = \{\alpha_i\}$ if $\bigotimes \Lambda_{\alpha_i} = 0$.

These definitions, together with the following theorem, are to be found in forthcoming work [31] of the PI with Matt Hogancamp. There are some technical assumptions we ignore.

Theorem 4.3. *Place a preorder on the set of eigenmaps of $\mathcal{F}$, where $i < j$ if $n_i < n_j$, or $n_i = n_j$ and $k_i < k_j$. Let α_i and α_j be two eigenmaps with $i < j$. Then one can explicitly construct (infinite) complexes $C_{i,j}$ and $C_{j,i}$ lifting $c_{i,j}$ and $c_{j,i}$, with a natural map $C_{i,j} \to C_{j,i}$. There is an isomorphism*

$$\mathbb{1} \cong \mathrm{Cone}(C_{i,j} \to C_{j,i}).$$

Suppose that $\mathcal{F}$ is categorically diagonalizable with finite spectrum $\mathbb{S}$. For simplicity, assume that the preorder on eigenmaps is a total order. Then $P_j = \bigotimes_{i \neq j} C_{i,j}$ is well-defined, and there is a canonical filtration $Q^\bullet$ (an iterated mapping cone) with $0 = Q^0 \to Q^1 \to \ldots \to Q^d \cong \mathbb{1}$, such that

$$P_i \cong \mathrm{Cone}(Q^i \to Q^{i+1}).$$

One has $P_i P_j = 0$, so that tensoring this filtration with P_i gives a canonical isomorphism $P_i P_i \cong P_i$. Applying this filtration to $K(\mathcal{A})$, one obtains a canonical filtration of $K(\mathcal{A})$ by eigencategories.

This theorem lifts the linear algebra, giving one "for free" a filtration by orthogonal idempotent functors, once one knows that a complex is categorically diagonalizable. What is more significant, these constructions are straightforward enough that, in practice, once one has computed the eigenmaps, one can actually compute the complexes P_i. In the next section, we apply this definition fruitfully to the full twist in the homotopy category of Soergel bimodules.

Remark 4.4. We also have a notion for when two commuting functors are simultaneously categorically diagonalizable. Dealing with multiple functors simultaneously can help eliminate some of the awkwardness in the above theorem (such as when the preorder is not a total order).

Remark 4.5. In order for the Grothendieck group $[K(\mathcal{A})]$ to be well-behaved, one must choose a boundedness condition, involving both the homological and grading degrees. If done correctly, one can categorify elements in a completion of $\mathbb{Z}[v, v^{-1}]$, like $\mathbb{Z}((v))$, using infinite complexes. One reason we restrict to monomial eigenvalues is to have a nice expansion of $\frac{1}{\lambda_i - \lambda_j}$ in this completion. This is also the reason we restrict to $i < j$ when constructing $C_{i,j}$ and $C_{j,i}$. Our chosen boundedness conditions force upon us a preorder on eigenvalues; one expects a preorder anyway, given that eigenspace decompositions only lift to filtrations.

Theorem 4.3 and the applications below justify that our definition is along the "correct" path to categorifying the notion of diagonalization. While monomial eigenvalues are sufficient for 2-representations of $\mathbf{H}$, they are not sufficient for the Casimir element c. Some of the technology mentioned above will work for more general eigenvalues and eigenmaps, but some results are still limited in scope. Our hope is to expand this abstract technology to its appropriate level of generality, with the belief that it should be widely applicable in situations where diagonalizable operators act with discrete spectra.

Conjecture 8. *There is an analogous construction to Theorem 4.3 which deals with* compound eigenmaps*, where the eigencones are iterated cones. Applying this construction to the Casimir operator C will produce the filtration constructed by Chuang and Rouquier.*

Conjecture 9. *Consider Khovanov-Lauda's category $\mathcal{U}$ associated to $\mathfrak{sl}_n$. One can construct central complexes C_i which lift the central generators of $U(\mathfrak{sl}_n)$. They are simultaneously categorically diagonalizable, giving filtrations of arbitrary categorifications by isotypic components, ordered by the dominance order on weights.*

Remark 4.6. Similar notions underlie the geometric Langlands equivalence [42]. One side of this equivalence are Hecke eigensheaves on Bun_G: sheaves which are sent by a certain functor to a "multiple" of themselves. The PI has a very limited understanding of this topic, unfortunately.

4.4. **The full twists.** Let W be a finite Coxeter group with longest element w_0. Inside the braid group Br_W, let ht, the *half twist*, denote the (unique) positive lift of w_0. Let ft, the *full twist*, satisfy $ft = ht^2$; it is in the center of the braid group. In particular, if $W' \subset W$ is a parabolic subgroup, then ft_W and $ft_{W'}$ commute. We also use ht and ft to denote the images of these elements in the Hecke algebra $\mathbf{H}$ of W.

As noted previously, one can categorify the Hecke algebra using the category of Soergel bimodules, or its diagrammatic analog $\mathcal{D}$. Rouquier [72] has constructed complexes of Soergel bimodules, one for each braid word, which satisfy the braid relations up to canonical homotopy equivalence. Thus, there are canonical objects HT and FT in the homotopy category $K(\mathcal{D})$ which correspond to ht and ft.

Let r denote Lusztig's $\mathbf{a}$-function from two-sided cells to $\mathbb{N}$. For a two-sided cell λ, let $c(\lambda) = r(w_0\lambda)$, and let $x(\lambda) = c(\lambda) - r(\lambda)$.

Conjecture 10. *For any finite Coxeter group W, the categorical full twist FT_W is categorically diagonalizable. The spectrum $\mathbb{S}$ is in bijection with the 2-sided cells of $\mathbf{H}_W$. The eigenmap α_λ has eigenshift $\mathbb{1}\langle 2c(\lambda)\rangle(2x(\lambda))$.*

The PI and Hogancamp are currently writing [31] where we prove this conjecture for finite dihedral groups. We have also confirmed it in type A_n for $n \le 5$. Note that this conjecture is entirely computational: to prove it, one need only provide an eigenmap for each 2-sided cell, and check that the tensor product of the cones is zero. Various refinements of Conjecture 10 below will make it clear how the eigenmap should be defined, and why the tensor product of eigencones is zero.

Let us discuss the main application. Let $W = S_n$ be the symmetric group, and for $i < n$ let $S_i \subset S_n$ permute the first i letters. One famous approach to the representation theory of S_n and its Hekce algebra $\mathbf{H}$ is to study the tower $S_0 \subset S_1 \subset \ldots \subset S_n$, and how representations restrict along this tower. This approach has been popularized by the work of Okounkov-Vershik [69], though it can be found in earlier work of Cherednik, and many concepts go back further. The centralizer of $\mathbf{H}(S_{i-1})$ inside $\mathbf{H}(S_i)$ is generated over the center of $\mathbf{H}(S_i)$ by $ft_i = ft_{S_i}$.

For a standard tableau T of shape λ, let $c(i)$ denote the column number of the i-th box (starting from 0), let $r(i)$ denote its row number, and let $x(i) = c(i) - r(i)$ denote its content. Let $c(\lambda)$, $r(\lambda)$ and $x(\lambda)$ denote the sum over all boxes of the relevant statistic (this agrees with the notation above). The full twist ft_n acts on the irreducible V_λ by the eigenvalue $(-1)^{2c(\lambda)}v^{2x(\lambda)}$. The family $\{ft_i\}$ has an eigenbasis $\{e_T\}$ on V_λ, parametrized by tableaux of shape λ, where ft_i acts according to the partition made by the first i boxes. In [69] they give an proof, independent of the constructions in representation theory, that the spectrum of the commuting family $\{ft_i\}$ is given by the set of standard tableaux with n boxes.

Everything discussed above should lift categorically. That is:

Conjecture 1 (Refinement 1 of Conjecture 10)**.** *The full twist FT_{S_n} is categorically diagonalizable, with spectrum $\mathbb{S}$ in bijection with partitions of n. The family of full twists FT_i, $1 \le i \le n$ is simultaneously diagonalizable, with joint spectrum parametrized by tableaux of size n.*

Remark 4.7. The full twist ft_n can not distinguish between all irreducibles. For example, the partitions $\lambda = (3,1,1,1)$ and $\mu = (2,2,2)$ have $x(\lambda) = x(\mu)$ and $c(\lambda) = c(\mu)$. Nonetheless, these correspond to distinct eigenmaps, eigencones, etc for FT_n. The preorder on eigenvalues is not a total order, but simultaneous diagonalization allows one to still apply Theorem 4.3.

One can also examine how FT behaves under the cellular filtration on $\mathcal{D}$. It is known that the cellular subquotient for the two-sided cell λ, with a monoidal structure truncated in grading degree $-r(\lambda)$, is a rigid monoidal categorification of the J*-ring* or a piece of the *asymptotic Hecke algebra*. The monoidal identity of this category is given by the direct sum $\oplus B_d$ ranging over the *distinguished involutions* d of λ (in type A, all involutions are distinguished).

Conjecture 2 (Refinement 2 of Conjecture 10)**.** *Fix a finite Coxeter group with two-sided cell λ. In homological degree $2c(\lambda)$, the full twist FT consists only of indecomposables B_w for w in a higher cell than λ, and B_d for the distinguished involutions d in λ appearing in homological degree $(2x(\lambda) - r(\lambda))$. The eigenmap α_λ is the zero map to higher cells, and descends to the structure map of the monoidal identity for the asymptotic Hecke category in cell λ. Consequently, for any y in cell λ, $\Lambda_\alpha B_y$ is a complex constructed out of B_w for w in a higher cell.*

This refinement pins down the eigenmap precisely, and could eventually lead to a proof. However, it does not make the eigenmap entirely explicit, nor does it directly prove that the proposed map is a chain map. The final statement implies a quick proof that $\bigotimes \Lambda_\lambda \cdot \mathbb{1} = 0$, and thus that FT is categorically diagonalizable.

We have various proposed methods to tackling the conjecture, at least in type A. One is direct computation, which though time-consuming is straightforward and computerizable. It also has the potential for the highest payout, if the patterns found in type A can be extended to other types. Other methods involve further technology, such as Rouquier complexes for singular Soergel bimodules and cabled braids, braid group actions on the asymptotic Hecke category, or a deep understanding of the endomorphism ring of the projection functors P_λ to aid the inductive step. A favorite approach of the PI is to imitate Okounkov-Vershik's classification of the spectrum on a higher categorical level. It remains to be seen which method will pan out.

4.5. **Further applications.** The full twist, its eigencones, and the eigenmaps themselves, are all objects or morphisms in the Drinfeld center $\mathcal{Z}$ of $K(\mathcal{D})$. Projection functors and other complexes built from eigencones live in the completed Drinfeld center, involving infinite complexes. By work of Bezrukavnikov-Finkelberg-Ostrik [11], it is expected that $\mathcal{Z}$ is equivalent to a homotopy category of Lusztig's character sheaves [63]. As there have been no direct, algebraic approaches to character sheaves to date, we believe that this could crack open a wide new field, giving a concrete construction of many character sheaves as complexes of Soergel bimodules. This also justifies our main philosophy: the eigencones are critical new objects of study.

Remark 4.8. In separate work, Williamson and the PI have also examined the Drinfeld center of $\mathcal{D}$ in affine type A, and can construct complexes of Soergel bimodules which realize Gaitsgory's central sheaves [43]. This plays a role in the tilting story above.

Hogancamp has related work on constructing the projection functor P_λ attached to a one-row partition [49], or a one-column partition (forthcoming with Abel). Contained within this work is a further study of the endomorphism rings $\mathrm{End}(P_\lambda)$ of these projectors. (This is a commutative ring, by the Eckmann-Hilton argument.) This has led Hogancamp to the following conjecture.

Conjecture 11. *The endomorphism rings of P_λ for any partition, or P_T for any tableau, are generated by eigenmaps corresponding to partitions dominated by λ or T. This describes* $\mathrm{End}(P_\lambda)$ *as a quotient of a bigraded polynomial ring.*

The action of Rouquier complexes in type A is woven up in the HOMFLYPT homology theory of links due to Khovanov [55]. In particular, a combination of Gorsky, Oblomkov, Rasmussen, and Shende [46, 47] have a series of elegant conjectures (some still unpublished), eventually relating HOMFLYPT homology to the geometry of the flag Hilbert scheme on $\mathbb{C}^2$. The fixed points under a torus action are in bijection with tableaux with n boxes, and various constructions with Soergel bimodules should correspond to sheaves on the flag Hilbert scheme. Their work should connect the algebraic geometry of P_λ (i.e. of the commutative ring $\mathrm{End}(P_\lambda)$) with affine subschemes of the flag Hilbert scheme, where one expects an action of a polynomial ring.

The algebraic geometry of cell categorifications is interesting in another way. Soergel bimodules, as modules over R, can be thought of as living on the affine variety $\mathfrak{h}$. Cell quotients live on interesting reducible subvarieties, which the PI can compute in type A. For example, in [29] it was shown that the Temperley-Lieb quotient of $\mathbf{H}(S_n)$ is categorified by a category defined over the union of all *Weyl lines*, the lines cut out by transverse intersections of root hyperplanes in $\mathfrak{h}$. An algebro-geometric approach to cell quotients, similar to Soergel's approach to Soergel bimodules in [79], might be interesting (c.f work of Gobet [45]), and might be related to the flag Hilbert scheme as well.

For Hecke algebras with unequal parameters, the cell theory is far more interesting. The quasi-split case can be categorified (by work of Lusztig) using "folding", i.e. equivariant Soergel bimodules, with many explicit computations done by the PI [25]. Rouquier complexes have not yet been studied, but the question is extremely interesting.

Finally, a wild comment. Using the diagonalization of the full twist ft_n, one can construct the functor of *i-induction* from S_{n-1} to S_n, which takes a partition μ of $n-1$ and adds a box with content i, when such a thing is possible. These functors for all $i \in \mathbb{Z}$ give an action of $\mathfrak{sl}_\infty$ on the Grothendieck group of all $\mathbf{H}(S_n)$-representations for $n \in \mathbb{N}$. This, or its finite characteristic version, was the 2-representation which inspired the original paper of Chuang and Rouquier [16] on $\mathfrak{sl}_2$ categorification. Now it is possible to define *categorical i-induction*, which perhaps could give rise to a 3-representation of $\mathfrak{sl}_2$.

5. Prior support

From September 2011 through August 2014 the PI was supported by an NSF Postdoctoral Research Fellowship, DMS-1103862, to the total of $135,000.

5.1. **Intellectual Merit.** Nine papers [38, 34, 40, 30, 36, 33, 37, 28, 39, 35] are publically distributed on `http://www.arxiv.org` from work done during this time, of which five [38, 30, 33, 37, 39] have already been published or accepted for publication (the others have been more recently submitted). This includes the paper on the Hodge theory of Soergel bimodules [38] joint with Williamson, which proved the long-outstanding Soergel conjecture, and has been published in *Annals of Mathematics*. It also includes the diagrammatic description of $\mathcal{D}$ in all types [36], which was one of the major projects outlined in this prior research proposal. Some of the other papers cover topics like: diagrammatics for Frobenius extensions, diagrammatics for standard modules and the 2-groupoid of W, a complete description of idempotents for universal Coxeter groups, and quantum geometric Satake.

Also begun during this time is a collaboration with You Qi, enriching quiver Hecke algebras and related structures with a p-DG algebra structure in order to categorify quantum groups at roots of unity. This project is an ongoing effort as well.

In addition, many works in progress were begun during the postdoctoral fellowship: a study of folding [25], of Gaitsgory's central sheaves, of knot homology [27], generators and relations for $\mathcal{S}\mathbb{S}$Bim, and a variety of other projects.

5.2. **Broader impacts.** While at MIT, I was the program coordinator for the inaugural year of PRIMES-USA. This extension of their PRIMES program sought to attract the best high school math students in the nation, to work remotely for an entire year with MIT faculty and graduate students. While advertising for this program, I gave several unrelated lectures for high schoolers, including a miniseries at Canada/USA Mathcamp, and a colloquium at SPUR.

On three occasions, I provided research problems and served as a faculty mentor for high school students and undergraduates in the PRIMES-USA, PRIMES, and SPUR programs. One of these projects recently appeared on the arXiv [74].

Since our proof of the Soergel conjecture, Williamson and I have been asked (separately, or together) to run several workshops aimed to bring young graduate students up to speed on the new theory and techniques. We gave a week-long, 20 lecture program at the QGM in Aarhus; I gave a 3 week-long program at the IMSc in Chennai; I gave a week-long program at the University of Oregon (and we will run a summer school at MSRI in 2017). The workshop at IMSc provided one Indian student with her thesis topic (hopefully). The program at Oregon has led several graduate students into this research field, such as Makimoto Shotaro, with whom I hope to collaborate soon.

6. Broader impacts of the proposed work

6.1. **Mentoring activities.** I consider the brand of research that I do to be especially well suited for elementary research projects, from the high school to the early graduate level. Diagrammatic algebra is an attempt to take difficult theories which require a great deal of background (e.g. perverse sheaves on flag varieties) and express them in simple, computable fashion. In addition, the field becomes accessible to computer-based analysis, which a talented programmer can take advantage of. Although I have no graduate students at the moment (having only just finished my first year at Oregon), I have many projects waiting for a graduate student or talented undergraduate, and look forward to having students soon. I also intend to offer my services to PRIMES-USA if they have students from the West coast seeking a local mentor.

6.2. **Graduate-level activities.** As mentioned above, I have ran several workshops with the stated goal of bringing graduate students, unfamiliar with Soergel bimodules and categorification in general, to the point where they understand the modern techniques. The most recent such workshop was at the University of Oregon, under the auspices of Nick Proudfoot's CAREER grant. I had attended Proudfoot's workshops as a student in 2011-2013 before running the 2014 edition, and I have found these workshops to be the most rewarding workshops/conferences I have ever attended. I modeled my other workshops after this example. I am not alone: many students who attended for a topic closer to their field of research have returned yearly for topics they had scarcely considered before.

Proudfoot's workshop series has been a demonstrably strong program, and I propose to inherit it. Proudfoot has given me the opportunity to take over his mantle as his CAREER grant ends, and if mine is funded, I will do so. This year I am helping to organize his last workshop (led by June Huh, on positivity in tropical geometry) as part of the segue.

More precisely, I plan to run a yearly week-long workshop at the University of Oregon during the beautiful late summer of the Pacific Northwest (August or September). Each workshop will be focused on a single relatively-recent paper or series of papers, in the fields of representation theory, algebraic geometry, or related combinatorics, and will be led by an invited speaker. Emphasis will be placed on choosing a speaker known for their pedagogical skill. Topics with many connections to other fields (especially those favored by the Oregon community) will be preferred, so that in learning one result well, students will still receive the broad picture and be able to apply the ideas elsewhere.

The workshop will be aimed at graduate students and postdocs with a broad background (assuming standard graduate courses only), with the intent of bringing them deep into a very specific area which, for most, is not their area of expertise. There will be daily or twice-daily exercise sessions, Q&A sessions, and a morning "triage" where the speaker goes over the complicated aspects from the previous day's adventure, all of which are fantastic tools to create student engagement. Keeping students on track for five days is no simple task, and many other workshops fail in this regard, either by beginning at too specialized a level, or lacking the necessary focus and infrastructure. Proper exercise design is extremely important: the goal is to have numerous shorter computations which lead to the acquisition of key skills, and to use them soon thereafter.

Past iterations have gone through several lecturing formats, such as students lecturing just past their knowledge (being forced to learn something new in advance) similar to the MIT Talbot workshop, or primarily talks by the invited expert (and students already in the field) similar to the QGM workshops. The former requires much more preparation for the speaker, working with students for months in advance to give them the big picture, and has been less successful, so I will default to the latter. However, the specifics will be flexible, left up to myself and the speaker.

For students entering an area there are often very few resources, often an uncollated list of references at best. In addition to giving a boost to the students in attendance, these workshops have created (and hopefully, will create) a valuable resource for everyone trying to enter the field, as lecture notes, exercises, and sometimes videos are publically available. I still answer emails from students who watched the Aarhus videos online. Having a series of lectures, with a list of corresponding references, gives students a plan of attack, and an idea of what they should focus on as they read the literature, and exercises to test this understanding. In coordination with the speaker, I plan to make the workshop webpage a true resource for students (and to place more emphasis on this point than has been done previously).

I also hope to support three Oregon graduate students during the summer months, with the intent that they organize a weekly seminar on the topic of the workshop. This way the Oregon community can get the most out of the workshop. They would be encouraged to write up solutions to the exercises, for inclusion on the workshop webpage.

Here are several potential topics, some of which dovetail nicely with the research proposal above.

- The recent conjectures of Gorsky-Oblomkov-Rasmussen-Shende [47] on the stable knot homology of torus knots, and their relation to the geometry of flag Hilbert schemes, would be an ideal choice. Rasmussen is an excellent speaker. This should also interest the many topologists and the algebraic geometers at University of Oregon (e.g. Robert Lipshitz and Nick Proudfoot, to name a few).
- Alternatively, one could focus on Cherednik algebras and double affine Hecke algebras, which also play a role in this story, and in geometric representation theory at large. Various experts (David Jordan, Pavel Etingof) are known to be good speakers. Jordan's recent work [8] should also make this important for other topologists in the department, like Dev Sinha.
- One of the original applications of categorification was to the modular representation theory of finite groups, with the proof of Broué's conjecture for symmetric groups [16]. Now many finite group theorists are working with Deligne-Lusztig varieties to compute decomposition matrices and prove similar results. The recent work of Dudas [20, 21, 22] is quite interesting, and Dudas is a fantastic speaker.

I believe that the demand for focused, introductory workshops continues to outpace the supply. There are few better ways for a graduate student to spend their summer weeks.

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