Project Summary – Ben Elias

Ben Elias intends to study geometric representation theory and categorification. Categorification takes useful algebras and replaces them with categories which possess a richer structure. Ben is specifically interested in describing the morphism spaces in these categories explicitly. Geometric representation theory provides a plethora of interesting categories to study, and has much to say about the general properties of objects in these categories, but does not have many techniques which allow one to understand the morphisms easily. Ben plans to use diagrammatic algebra to unravel these morphisms, and explore what this understanding might yield for the general theory.

The Hecke algebra of the symmetric group has a categorification by Soergel bimodules (see [So1, So2, So3]), which relates the category of perverse sheaves on the flag variety to category $\mathcal{O}$ for $\mathfrak{sl}_n$. A diagrammatic presentation of this category was given by Ben Elias and Mikhail Khovanov in [EKh]. Ben has already used these diagrammatics to give explicit descriptions of various natural categories and isomorphisms appearing in geometric representation theory (see [El1, El2, EKr]). He plans on continuing to use this description to study Kazhdan-Lusztig theory and cellular theory.

The quantum group of a Kac-Moody lie algebra is another algebra which has been successfully categorified and diagrammaticized (see [KL1, KL2, Ro2]). Ben plans on continuing to study this category, looking at crystal structures (see [LV, We, KP]) and at connections to Soergel theory (see [MSV]).

Numerous other geometric setups exist that Ben intends to learn about while at MIT. These include character sheaves, coherent perverse sheaves, Cherednik algebras and W-algebras. He would like to study natural transformations between functors in these setups as well.

Intellectual Merit

- The Kazhdan-Lusztig conjecture and its analogs have been fundamental motivations in geometric representation theory. With this explicit description of the Soergel category, one has a chance to find a purely algebraic proof of the Kazhdan-Lusztig conjecture, for which there are currently only geometric proofs.
- Finding idempotents in the Soergel category explicitly and looking at the coefficients which appear, one can study geometry in specific finite characteristics, where the decomposition theorem fails and geometric techniques are not as well developed (see [BBD, JMW]).
- Categorified representations of $\mathfrak{sl}_2$ are ubiquitous, in geometry and modular representation theory among other places, and Chuang and Rouquier used the morphisms in these categorifications to find interesting derived equivalences, proving the Broué conjecture (see [CR]). Categorified representations of the Hecke algebra and the symmetric group are equally ubiquitous, and a similar structural theory could potentially prove results in many fields.
- Affine Hecke algebras appear in the geometric Satake correspondence and in the geometric Langlands program. Pending the ability to extend this calculus to arbitrary Hecke algebras, it could open up a wealth of applications to these exciting fields. Other Hecke algebras, as well as higher categorifications thereof, appear in topological quantum field theory (see [BFN] for instance).

Broader Impacts

- Diagrammatic algebras are very easy to work with compared to sophisticated geometric techniques. They provide accessible problems even to undergraduates, as demonstrated by a series of REUs at Columbia. Developing this theory could make numerous developments in geometric representation theory more accessible to a broad class of mathematicians.
- Soergel bimodules are also useful for studying knot theory, as shown by Khovanov [Kh2]. The work of Ben Elias and Dan Krasner [EKr] as well as current work with Pedro Vaz could help make the categorification of the HOMFLY-PT invariant incredibly explicit. This could lead to more effective computational tools, and a reasonable study of functoriality in knot homology theories.
- Categorification has deep connections with physics and topological quantum field theory. Diagrammatics may provide a new way to study results in those fields.

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