RESEARCH STATEMENT

BEN ELIAS

1. Introduction

My interests lie primarily in geometric representation theory, and more specifically in diagrammatic categorification. Geometric representation theory attempts to answer questions about representation theory by studying certain algebraic varieties and their categories of sheaves. Diagrammatics provide an efficient way of encoding the morphisms between sheaves and doing calculations with them, and have also been fruitful in their own right. I have been applying diagrammatic methods to the study of the Iwahori-Hecke algebra and its categorification in terms of Soergel bimodules, as well as the categorifications of quantum groups; I plan on continuing to research in these two areas. In addition, I would like to learn more about other areas in geometric representation theory, and see how understanding the morphisms between sheaves or the natural transformations between functors can help shed light on the theory. After giving a general overview of the field, I will discuss several specific projects.

In the most naive sense, a categorification of an algebra (or category) A is an additive monoidal category (or 2-category) $\mathcal{A}$ whose Grothendieck ring is A. Relations in A such as $xy = z + w$ will be replaced by isomorphisms $X \otimes Y \cong Z \oplus W$. What makes $\mathcal{A}$ a richer structure than A is that these isomorphisms themselves will have relations; that between objects we now have morphism spaces equipped with composition maps. 2-category). In their groundbreaking paper [CR], Chuang and Rouquier made the key observation that some categorifications are better than others, and those with the "correct" morphisms will have more *interesting* properties. Independently, Rouquier [Ro2] and Khovanov and Lauda (see [KL1, La, KL2]) proceeded to categorify quantum groups themselves, and effectively demonstrated that categorifying an algebra will give a precise notion of just what morphisms should exist within interesting categorifications of its modules. The question of what makes an "interesting" categorification in general (and just what structures one can hope for on the categorification) is very vague at the moment, but has led to some exciting mathematics.

Geometric representation theory gives an abundant source of interesting categorifications, including all those mentioned above. Instead of studying an algebra A directly, one produces a system of algebraic varieties with maps between them that "encodes" the algebra. Then, one can apply a plethora of tools to this setup. Using a basic tool like constructible functions or cohomology, one recovers the algebra A. Other tools, like the category of constructible sheaves or perverse sheaves, will yield categorifications of A. Typically, one can use the same setup to categorify the irreducible A-modules. One would expect geometric categorifications to have every property one could desire, since they are constructed so naturally. It is typically very easy to understand the *objects* in geometry, i.e. to show abstractly that there exists an isomorphism of sheaves $X \otimes Y \cong Z \oplus W$, or to classify irreducible sheaves. What is often difficult, especially when working with singular spaces, is to understand *morphisms* between sheaves, e.g. to classify $\mathrm{E}xt^*$ between perverse sheaves and how these extensions compose.

However, geometry has one additional benefit: in optimal setups (where the ambient varieties are smooth and the maps are proper, etc.) the categories which appear are *biadjoint cyclic 2-categories*. These are precisely the kinds of categories which can be studied with planar diagrams. A picture can represent a 2-morphism in an isotopy-invariant way, so that they can be drawn as decorated planar graphs. Relations between morphisms can often be *intuitively* expressed in terms of equalities of diagrams, and proofs thus reduced to simple graph theory. Planar diagrams are an efficient and comprehensible method of giving a 2-category by generators and relations; while it takes an expert to perform calculations with perverse sheaves, calculations with planar diagrams have even provided interesting research questions for undergraduates.

Planar diagrams also more than just tools for expressing morphisms in geometric categorifications. A number of diagrammatic categorifications are beginning to appear which are not linked to geometry. Recently, Khovanov [Kh1] gave a diagrammatic categorification that should conjecturally correspond to a setup in super-geometry ("varieties" associated to non-commutative super-algebras) for which the geometric methods are not yet in place. Khovanov and Lauda's diagrams for the full quantum group in [KL2] can only be derived from infinite-dimensional non-proper geometry, where again geometric techniques are insufficient. Discovering categorifications by generators and relations has the opposite advantages to the geometric approach. The morphisms are well-understood, and abstract properties of the categorification may be more easily proven. However, it is tricky even to know which objects are nonzero, and because one must take an idempotent completion, it is also difficult to classify indecomposables.

Finally, the diagrams themselves may be interesting in their own right. In the categorification of Hecke algebras described below, the planar diagrams appearing are actually 2D holograms of 3D singular spaces known as foams, and thus also encode interesting topological data.

The hope is that by explicitly describing categorifications using planar diagrams, one can pin down just what it is in the morphisms which makes those categorifications interesting. Working backwards, these structures may teach us something new about geometry.

2. Plans and Projects

2.1. **Categorical Hecke Theory.** For every Coxeter group there is an associated Iwahori-Hecke algebra $\mathcal{H}$, which deforms the group algebra of the Coxeter group. This algebra is fundamental in many areas of mathematics, from number theory to knot theory; it is also a crucial link between geometry and representation theory. In Lie type, the regular representation of $\mathcal{H}$ is isomorphic to the Grothendieck group of two different categories: perverse sheaves on the flag variety $\mathcal{P}$, and a category $\mathcal{O}$ associated the Lie algebra. In each case, $\mathcal{H}$ itself is categorified by a category of endofunctors. Therefore, structure coefficients in $\mathcal{H}$ encode data about multiplicities in both $\mathcal{P}$ and $\mathcal{O}$. In finite type and characteristic 0, the *Kazhdan-Lusztig basis* of $\mathcal{H}$ is the image of both the simples in $\mathcal{P}$ and the projectives in $\mathcal{O}$, yielding an equality of multiplicities known as the Kazhdan-Lusztig conjectures. There are analogous statements in other types and characteristics which are still unproven. Moreover, no simple algebraic proof of the Kazhdan-Lusztig conjectures is known.

Soergel [So1, So2] explained this link on the categorical level by finding a graded polynomial ring R and a functor from both $\mathcal{P}$ and $\mathcal{O}$ to R-modules. The image of the endofunctors associated to $\mathcal{H}$ forms a full subcategory of R-bimodules, known as Soergel bimodules $\mathcal{B}$, and it also categorifies $\mathcal{H}$. It is easier to understand morphisms between R-bimodules than in these other categories, but the morphisms can still become exceedingly complex. The Kazhdan-Lusztig conjecture now boils down to questions about indecomposables in $\mathcal{B}$ and their image in the Grothendieck group, which is the topic of the Soergel conjecture (see [So3]). The category $\mathcal{B}$ contains a subcategory $\mathcal{B}_{\mathrm{BS}}$ of Bott-Samelson bimodules, which come from the perverse sheaves associated to certain smooth resolutions of Schubert varieties. The category $\mathcal{B}_{\mathrm{BS}}$ is simpler still, and its idempotent completion is $\mathcal{B}$. Objects in $\mathcal{B}_{\mathrm{BS}}$ are given by sequences of indices from the Coxeter graph. In [EKh], M. Khovanov and I found a diagrammatic presentation for the category $\mathcal{B}_{\mathrm{BS}}$ in type A, defined over $\mathbb{Z}$. From here, the number of projects which can be tackled diagrammatically explodes.

- **Higher Representation Theory of Hecke algebras and cellular algebras** What is the general theory of categories upon which $\mathcal{B}$ acts? As we have seen, many useful such categories exist, and this information would tell us what extra structure is induced by the "correct" morphisms of $\mathcal{B}$. In [CR] Chuang and Rouquier give the answer for categorical $\mathfrak{sl}_2$ representations (generalized by Rouquier to the Kac-Moody case [Ro2]): while $\mathfrak{sl}_2$ representations are semisimple and split into isotypic components, their categorical analogs have filtrations by isotypic categories, along the lines of the cellular structure of finite quotients of quantum $\mathfrak{sl}_2$. This seems to be a very general idea, that the categorified representation theory of cellular algebras should be in some sense "cellular." The same should be true for the Hecke algebra. Some of the building blocks of this cellularity, the induced representations, have already been categorified diagrammatically (see below). These are ideas that I am currently pondering with G. Williamson.

On a "nice" cellular algebra like $\mathcal{H}$, the space of possible trace maps T is a free module with one component for each cell. In this case, cellular quotients can be cut out by the vanishing of certain traces. Similarly, categorifications of cellular quotients via quotients of $\mathcal{B}$ should induce the corresponding trace map via the dimension of Hom spaces. This is true for the categorification of the Temperley-Lieb algebra, given diagrammatically in [El2]. Moreover, the Hom spaces appearing there are somewhat natural, associated to the coordinate ring of the 1-skeleton of the Coxeter complex. It is an ill-formed question, but I would like to explore the general notion of cellular categorifications and their Hom spaces, and see what comes out. A more precise and possibly far-fetched question is whether there is a category $\mathcal{T}$ which categorifies the trace space T, and a categorification of $\mathcal{H}$ with morphisms enriched in $\mathcal{T}$ which induces the universal trace, and whether categorified cellularity could be seen in this light.

- **Standard Modules** In a somewhat related note, Soergel bimodules also have filtrations by R-bimodules known as standard modules, as one expects from a cellular theory. These standard modules are connected to the virtual braid group [Ka]. I would like to investigate whether they can be given diagrammatically, in a similar fashion to Webster's diagrammatic version of standard modules for tensor products of quantum group representations [We].
- **Other types** I am currently writing up a diagrammatic presentation for $\mathcal{B}_{\mathrm{BS}}$ for any dihedral group. This takes care of all possible interactions between two indices. For general reasons, one hopes that diagrammatics for the general case will unfold once interactions between triples of indices are understood (3-index relations). One would also expect that the diagrammatics for any simply-laced type are already evident, but the proofs are currently lacking. It would be especially important to understand affine type, because that would provide a tool to studying some of the most important geometric categories.
- **Idempotents** To understand $\mathcal{B}$ we need to understand the idempotents in $\mathcal{B}_{\mathrm{BS}}$ which pick out each indecomposable in $\mathcal{B}$. This is a very difficult question, probably too difficult: finding the coefficients which appear in these idempotents would tell one about other characteristics, and finding the idempotents themselves would answer the Soergel conjecture. However, partial results are fruitful. In [El1] I found the idempotents for the longest element of a parabolic subgroup, and I believe I know all the idempotents for the dihedral group. These results motivate the following conjecture (generalizing an idea of Libedinsky [Li]), which I believe is reasonably provable:

Conjecture 1. *For each element $w \in S_n$, and for each reduced expression x of w, there is a natural idempotent inside End(B_x), the corresponding Bott-Samelson bimodule (more will be said about these idempotents below). These idempotents pick out a single Soergel bimodule MS_w. While MS_w is not necessarily indecomposable, the collection of all MS_w descends to a basis of $\mathcal{H}$ which can be characterized by certain properties on $\mathcal{H}$ in similar fashion to the Kazhdan-Lusztig basis. This basis is characteristic-independent, and hopefully interesting.*

- **Hecke representations** Using the idempotents above from [El1], we gave a diagrammatic categorification of representations of $\mathcal{H}$ induced from trivial representations of subalgebras. This is a reasonable start to categorifying one approach to the representation theory of $\mathcal{H}$: categorifying induced trivial and sign representations, and the morphism spaces between them, one should be able to categorify all irreducibles. Work on this is underway. This should presumably accord with the categorifications already appearing in geometry and in variants on category $\mathcal{O}$ (see [MaSt]), and could help give diagrammatics for those categories.
- **Hecke representations II** There is another interesting approach to the representation theory of $\mathcal{H}$, as found in Okounkov-Vershik [OV]. I am excited about the possibility of categorifying Jucys-Murphy elements of $\mathcal{H}$ and trying to mimic their approach on the categorical level.
- **Drinfeld Center** It should be possible to give an explicit presentation of the Drinfeld center of $\mathcal{B}$. Outside of type A, this center is strictly bigger than what one might expect (i.e. one object for each conjugacy class). While interesting in its own right, there may be extremely important connections to representation theory and character sheaves, as seen in Bezrukavnikov-Finkelberg-Ostrik [BFO] in finite type.
- **Higher Coxeter Theory** Manin and Schechtman [MaSc] gave a detailed study of an n-category $\mathcal{S}$ associated with the symmetric group S_n. In a generalization of the Bruhat order, they put the structure of a partially oriented graph on: the elements of the symmetric group, reduced expressions of each element, transformation paths of reduced expressions, transformation paths of those transformations, etcetera. The idempotents found in [El1] and in Conjecture 1 come from following paths in one of these graphs. What is astonishing is that the category $\mathcal{B}_{\mathrm{BS}}$ seems to be governed precisely by a truncation of this n-category. The objects in $\mathcal{B}$ come from 1-morphisms in $\mathcal{S}$, the generators from 2-morphisms, and the 3-index relations from 3-morphisms. There are many potential consequences of this observation.

As the Grothendieck group of a derived category, $\mathcal{B}$ is really part of an $(\infty, 1)$-category which controls the Hecke algebra. One might ask whether $\mathcal{B}$ can itself be *explicitly* categorified by a 3-category, and that categorified by a 4-category, and so on, providing nice finite models of this $(\infty, 1)$-category. The first step would be to replace the 3-index relations with 3-isomorphisms: the category $\mathcal{S}$ should tell you precisely what the relations are between those new 3-morphisms, and they should be determined by interactions of 4 indices.

While the 1- and 2-index relations were complicated, one expects all higher-index maps to be isomorphisms, and thus the structure comes directly from $\mathcal{S}$. So we have:

Conjecture 2. *The* 2*-category* $\mathcal{B}_{\mathrm{BS}}$ *can be improved to a* 3*-category, and so on up to an* n*-category. At the* k*-th step, the new morphisms are the* k*-morphisms in* $\mathcal{S}$*, and the relations come from the* $k+1$*-morphisms. This can be described explicitly, maybe with higher-dimensional diagrams.*

Studying higher Coxeter theory in other types could give the missing 3-index relations for the general case, and might lead to similar results. Also, I would like to see what implications higher Coxeter theory via Soergel bimodules has for the study of flag varieties themselves.

- **Singular Soergel Bimodules** The entire story described above is actually part of a slightly richer story. The Hecke algebroid is an "idempotented" version of the Hecke algebra, and it is categorified by the 2-category of Singular Soergel bimodules, as shown by Williamson [Wi]. This 2-category acts on 2-categories associated to both geometry and category $\mathcal{O}$, so that it encodes even more interesting data analogously to $\mathcal{B}$. We are currently in the process of giving a diagrammatic presentation for this 2-category in type A, after which many of the above projects apply equally to this richer structure. In addition, this 2-category is more "local" than $\mathcal{B}$, meaning that the generators and relations are simpler and more generic, and thus perhaps more intuitive.
- **HOMFLY-PT Knot Homology** The Hecke algebra plays a pivotal role in knot homology theories. The surjection from the braid group Br to $\mathcal{H}$ lifts to a map from Br to isomorphism classes in the homotopy category of $\mathcal{B}$. Rouquier [Ro1] showed that this map is "strict" in the sense that braid words are sent to complexes, and braid relations to chain maps. Using diagrammatics, D. Krasner and I [EKr] gave these chain maps explicitly, and checked that they satisfied the "movie moves". Now the map $Br \to \mathcal{H}$ is truly categorified by a functor from the category of braid cobordisms to the homotopy category of $\mathcal{B}$.

  Since Khovanov [Kh2] proved that the Hochschild homology of Rouquier's complexes gives a categorification of the HOMFLY-PT polynomial, we see that this knot homology theory is functorial over braid cobordisms. Having an explicit description of these complexes and chain maps should make many calculations in knot theory much simpler. Pedro Vaz and I have found diagrammatics for the Hochschild homology of Soergel bimodules, and are working on a purely diagrammatic version of Rasmussen's spectral sequence [Ra], using the functor from Soergel bimodules to foams found by Vaz and Mackaay [Va, MV]. We hope these diagrammatics help knot theorists to calculate and understand HOMFLY-PT homology and its functoriality.

2.2. **Quantum Group Categorification.** Quantum groups are another class of algebras that have been successfully presented using planar diagrammatics, thanks to Khovanov and Lauda. Irreducible representations were categorified by Lauda and Vazirani in [LV], and tensor products thereof by Webster [We]. The field has become a hotbed of activity, to which I have not yet contributed but of which I have kept abreast. I plan on working on the following projects:

- **Schur-Weyl duality** Mackaay, Stosic and Vaz [MSV] have given a categorification of the Schur algebroid, a quotient of the quantum group, by using a quotient of Khovanov and Lauda's category. They also give a functor from Soergel bimodules to this quotient. Since the Hecke algebroid and the Schur algebroid are canonically isomorphic thanks to Schur-Weyl duality, we expect to be able to give an equivalence of categories from Singular Soergel bimodules to the Schur quotient category. This is joint with Williamson and Mackaay.
- **Tensor product crystals** It would be nice to prove the conjecture of Webster in [We], that his categorification of tensor products gives the correct crystal structure under induction-cosocle functors. It seems like the recent work of Kang and Park [KP] should generalize to a proof in this context. This is joint with E. Park. I would also like to examine induction-cosocle functors in the context of $\mathcal{B}$.

2.3. **Other Topics.** There are vast swaths of geometric representation theory which I have not studied, and where diagrammatic methods or a careful study of natural transformations could potentially be very useful. The same can be said about the theory of category $\mathcal{O}$. I plan on expanding my horizons, and seeing what connections can be made.

References

[BFO] R. Bezrukavnikov, M. Finkelberg and V. Ostrik, Character D-modules via Drinfeld center of Harish-Chandra bimodules, preprint 2009, arXiv:0902.1493v2.

[CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, *Annals of Math.* **167** (2008), 245-298, math.RT/0407205v2.

[El1] B. Elias, A diagrammatic category for generalized Bott-Samelson bimodules and a diagrammatic categorification of induced trivial modules for Hecke algebras, preprint 2010, arXiv:1009.2120.

[El2] B. Elias, A Diagrammatic Temperley-Lieb categorification, preprint 2010, arXiv:1003.3416. To appear in International Journal of Mathematics and Mathematical Sciences, special issue on Categorification in Representation Theory.

[EKh] B. Elias and M. Khovanov, Diagrammatics for Soergel categories, preprint 2009, arXiv:0902.4700.

[EKr] B. Elias and D. Krasner, Rouquier complexes are functorial over braid cobordisms, *Homology, Homotopy and Applications* **12** (2010), No. 2, pp.109-146, arXiv:0906.4761.

[KP] S. Kang and E. Park, Irreducible Modules over Khovanov-Lauda-Rouquier Algebras of type A_n and Semistandard Tableaux, preprint 2010, arXiv:1005.1373.

[Ka] L. H. Kauffman, Virtual knot theory, *European J. Combin.*, **20** (1999), Vol. 7, pp. 663690.

[Kh1] M. Khovanov, How to categorify one-half of quantum gl(1|2), preprint 2010, arXiv:1007.3517.

[Kh2] M. Khovanov, Triply-graded link homology and Hochschild homology of Soergel bimodules, *Int. Journal of Math.* **18** no. 8 (2007) 869-885. arXiv:math/0510265.

[KL1] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* **13** (2009), 309–347. arXiv:0803.4121.

[KL2] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups III, *Quantum Topology* **1** (2010), Issue 1, pp. 1-92. arXiv:0807.3250.

[La] A. Lauda, A categorication of quantum sl(2), preprint 2008, arXiv:0803.3652. To appear in *Adv. Math.*

[LV] A. Lauda and M. Vazirani, Crystals from categorified quantum groups, preprint 2009, math.RT/0909.1810.

[Li] N. Libedinsky, New bases of some Hecke algebras via Soergel bimodules, preprint 2009, arXiv:0907.0031.

[MV] M. Mackaay, P. Vaz, The diagrammatic Soergel category and sl(N) foams for N>3, preprint 2009, arXiv:0911.2485.

[MSV] M. Mackaay, M. Stosic and P. Vaz, A diagrammatic categorification of the q-Schur algebra, preprint 2010, arXiv:1008.1348.

[MaSc] Y. Manin and V. Schechtman, Arrangements of Hyperplanes, Higher Braid Groups and Higher Bruhat Orders, *Advanced Studies in Pure Math.* **17** (1989), 289–308.

[MaSt] V. Mazorchuk and C. Stroppel, Categorification of (induced) cell modules and the rough structure of generalized Verma modules, *Adv. Math.* **219** (2008), No 4, 1363-1426.

[OV] A. Okounkov and A. Vershik, A New Approach to the Representation Thoery of the Symmetric Groups 2, *Zapiski Seminarod POMI* (In Russian) **307**, (2004). arXiv:math/0503040.

[Ra] J. Rasmussen, Some differentials on Khovanov-Rozansky homology, preprint 2006, arXiv:math.GT/0607544.

[Ro1] R. Rouquier, Categorification of the braid groups, preprint 2004, arXiv:math.RT/0409593.

[Ro2] R. Rouquier, 2-Kac Moody algebras, preprint 2008, arXiv:math.RT/0812.5023.

[So1] W. Soergel, Category $\mathcal{O}$, perverse sheaves and modules over the coinvariants for the Weyl group, *J. Amer. Math. Soc.* **3** (1990), no. 2, 421–445.

[So2] W. Soergel, The combinatorics of Harish-Chandra bimodules, *J. Reine Angew. Math.* **429** (1992), 49–74.

[So3] W. Soergel, Kazhdan-Lusztig polynomials and indecomposable bimodules over polynomial rings, *J. Inst. Math. Jussieu* **6** (2007), no. 3, 501–525. English translation available on Soergel's webpage.

[Va] P. Vaz. The diagrammatic Soergel category and sl(2) and sl(3) foams, preprint 2009, math.QA/0909.3495.

[We] B. Webster, Knot invariants and higher representation theory, preprint 2010, math.GT/1001.2020

[Wi] G. Williamson, Singular Soergel Bimodules, PhD Thesis 2008. Can be found on Williamson's webpage.