Sloan Fellowship Research Statement

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A critical advance in representation theory was the observation that many representation categories of interest have Grothendieck groups which are themselves interesting representations of a different algebra. For example, the Bernstein-Gelfand-Gelfand category $\mathcal{O}_0$ of representations of a Lie algebra has a Grothendieck group which is acted on by the corresponding Hecke algebra. This representation is the subject of the famed Kazhdan-Lusztig conjectures. Similarly, representations of symmetric groups can be glued into a category whose Grothendieck group is acted on by an affine Lie algebra. In both examples, the action on the Grothendieck group is generated by the actions of functors on the categories themselves. Chuang and Rouquier studied the latter example and made the crucial claim that, by studying the natural transformations between these functors, one can prove strong structural results about the categories themselves. From this idea, 2-representation theory was born.

My research focuses on the 2-representation theory of the Iwahori-Hecke algebra $\mathbf{H}(W)$ attached to a Coxeter group W. There is a monoidal additive category $\mathcal{H}(W)$ known as the *Hecke category*, whose Grothendieck group is canonically isomorphic to $\mathbf{H}(W)$. In all known cases, every action of $\mathbf{H}(W)$ on the Grothendieck group of a category has lifted to an action of $\mathcal{H}(W)$ on the category itself, which is called a *2-representation* of the Hecke algebra. Such 2-representations are ubiquitous in classical representation theory.

My work to date has laid the groundwork for the study of the Hecke category. The starting point is a presentation by generators and relations of the morphisms in the Hecke category, using the language of planar diagrammatics. This presentation is nice enough to allow for efficient manual and computer-based computation. Together with Williamson, we proved the *Soergel conjecture*, which states that, in characteristic 0, the indecomposable objects in $\mathcal{H}(W)$ have the expected size, and certain *local intersection forms* are non-degenerate. This provided the first algebraic proof of the Kazhdan-Lusztig conjectures. A third major result is quantum geometric Satake. I constructed a q-deformation $\mathcal{H}_q$ of $\mathcal{H}(W)$ for an affine Weyl group in type A, using a new q-deformed Cartan matrix. I proved directly an equivalence of (part of) $\mathcal{H}_q$ with representations of a quantum group.

My research plan has three major components. This first, joint with Williamson and Juteau, is to compute local intersection forms in the antispherical module of $\mathcal{H}_q$, and its characteristic p analog. This would provide character formulas for rational representations of algebraic groups, where almost nothing is known. We plan to prove a "translation principle," which should imply the fractal behavior conjectured by Lusztig-Williamson in characteristic p.

In my study of $\mathcal{H}_q$ for affine $\mathfrak{sl}_2$, I showed that, when q is a root of unity, one can enhance the category by adding new objects and morphisms, to obtain $\mathcal{H}(D)$ for a finite dihedral group D. Moreover, $\mathcal{H}(D)$ has much nicer properties than $\mathcal{H}_q$. For $\mathfrak{sl}_n$, $n \geq 3$, the same enhancement yields a category $\mathcal{G}$ which is entirely new and mysterious, and should correspond to the complex reflection group $G(m, m, n)$. I plan to study $\mathcal{G}$, finding a presentation and proving it has nice properties. If true, this would imply results about $\mathcal{H}_q$, shedding light on the first project. Researchers have long sought a categorification of the Hecke algebra of complex reflection groups; while $\mathcal{G}$ does not meet their criteria, I now believe that $\mathcal{G}$ is the correct object of study.

A third project, joint with Hogancamp, will study categorical diagonalization. Diagonalization is a fundamental tool in linear algebra, and we have found the correct categorical analog. We hope to prove that certain objects in the homotopy category of $\mathcal{H}(W)$ are categorically diagonalizable. If so, we can use our tools to construct functors which project to "eigencategories." This would provide an explicit understanding of categorified cell theory for Hecke algebras. One main approach to the study of symmetric groups S_n, as described in Okounkov-Vershik, is to induce a representation from S_n to S_{n+1} and then project to an eigenspace of a Jucys-Murphy operator. We would be able to lift this approach to the categorical level.