RESEARCH STATEMENT

BEN ELIAS

1. Introduction

Categorification is the art of taking a well-beloved algebraic theory and adding an extra layer of categorical structure. For example, the homology groups $H_*(M)$ of a manifold M form a categorification of the Euler characteristic of M. As this example illustrates, categorifications may provide more insight and understanding into the original algebraic story, but this is not their primary use; they are intrinsically interesting theories in their own right.

My primary focus as a mathematician is categorical representation theory. Traditional representation theory is concerned with representations of algebraic groups and their quantum groups, or of Hecke algebras and braid groups, etcetera. One by one, these algebras and their representations have been realized as the Grothendieck groups of certain additive or abelian categories, their categorifications. Two goals of categorical representation theory that I find most interesting are:

- Finding the categorical analogs of the deep **structural** results in traditional representation theory: (generic) semisimplicity, tensor products and braiding, induction and restriction functors, double-commutant actions, etc;
- Studying these categories in their own right, on a **morphism-theoretic** level. The composition maps between morphism spaces are the data in the categorification which is most invisible in its Grothendieck group.

One of the key paradigms in this field is that a morphism-theoretic understanding is needed to successfully approach any structural results; this is keenly illustrated by the seminal work of Chuang and Rouquier on $\mathfrak{sl}_2$ categorification [4].

In this research statement I will describe a number of broad dreams, motivating goals, and specific projects I plan to pursue. They are focused mainly in the following general areas:

- Categorical Hecke theory, which studies categorifications of Hecke algebras, their representations, and related algebras. Particular subtopics include Hodge-theoretic properties, the calculation of certain idempotents, categorifying Hecke algebras with unequal parameters, and the existence of higher Bruhat orders.
- Geometric Satake, which gives an equivalence between perverse sheaves on an affine Grassmannian and representations of a Lie algebra. I have recently found an algebraic reformulation and proof of this equivalence, as well as a quantum version, both in type A. The applications to both representation theory and geometry have yet to be exploited.
- Categorification at roots of unity, a phenomenon made possible by the Hopfological algebra of Khovanov and Qi. This is a generalization of homological algebra, some of whose fundamental properties I plan to explore.
- Categorical braid group actions, which arise naturally in a host of geometric and algebraic circumstances. In the literature these categorical actions are "weak:" one demonstrates isomorphisms corresponding to the braid relations, but fails to check compatibility between these isomorphisms. I hope to "strictify" many of these actions.

Before describing these projects, I will provide a section giving some background and philosophy. The goal is to describe the motifs of algebraicization, combinatorial replacement, and idempotent hunting, and to reach one of the key mysteries in the field: what makes geometry special?

2. Background

Given an abelian category $\mathcal{A}$, such as one of our favorite categorifications, how does one get a handle on it? One key tool could be called **algebraicization**, finding an algebra A which governs the behavior of the category. To accomplish this, one could find a set $\mathbb{P} = \{P_i\}$ of projective objects which collectively generate the category, and set $A = \mathrm{End}(\oplus P_i)^{\mathrm{op}}$. The full subcategory $\mathcal{A}_{\mathbb{P}}$ (consisting of direct sums of objects in $\mathbb{P}$) contains enough data to recover $\mathcal{A}$ itself. Now, A is an algebroid (i.e. it is equipped with special idempotents $\mathbb{1}_i \in \mathrm{End}(P_i)$), whose principal modules $A\mathbb{1}_i$ form an additive category A-Prin equivalent to $\mathcal{A}_{\mathbb{P}}$. The Koszul-dual approach (which works in less generality) would be to find a collection of generating semisimple objects $\mathbb{L} = \{L_i\}$, and compute the (graded) algebroid $A = \mathrm{Ext}^*(\oplus L_i)$.

In order for this algebraic reformulation to be useful, the algebroid A should be accessible to algebraic tools. If one could present A by generators and relations, then one has an explicit description of A-Prin and therefore of $\mathcal{A}_{\mathbb{P}}$. In practice, this requires a clever choice of projective generators. Consider the example of the Hecke algebra $\mathbf{H}$ of a Weyl group, which has two traditional categorifications: perverse sheaves $\mathcal{P}$ on the corresponding flag variety, or representations of the corresponding Lie algebra within the BGG category $\mathcal{O}$ [30, 31]. A naive approach might choose $\mathbb{P}$ to be the set of indecomposable projective objects in $\mathcal{O}$, or on the Koszul-dual side, might choose $\mathbb{L}$ to be the simple objects in $\mathcal{P}$. However, indecomposable projectives in $\mathcal{P}$ and simple perverse sheaves in $\mathcal{P}$ are notoriously difficult to pin down, and the corresponding algebra A is generally unknown. A more clever approach due to Soergel [59, 60] suggests that one let $\mathbb{P}$ be the set of Bott-Samelson projectives in $\mathcal{O}$, or Bott-Samelson sheaves in $\mathcal{P}$, which are certain combinatorially-defined objects. Several of my papers [13, 12, 19] (joint in parts with Geordie Williamson or Mikhail Khovanov) comprise an explicit and combinatorial presentation of the morphism algebra between Bott-Samelson objects. This presentation is combinatorial in the sense that morphisms are depicted as planar graphs, with relations coming from graphical manipulations.

When one has a set $\mathbb{P}$ whose endomorphism algebroid A is combinatorially presentable, one can call A-Prin a **combinatorial replacement** for $\mathcal{A}$. Many of the key algebras appearing in categorical representation theory, such as the quiver Hecke algebras of Khovanov-Lauda and Rouquier [34, 36, 55], or the Webster algebras of [65], also come from combinatorial replacements of categories of perverse sheaves. In fact, these successes motivate a general paradigm for finding combinatorial replacements in geometry: one should find combinatorially-defined resolutions of singularities (of orbit closures).

By studying these combinatorial replacement algebras, a number of generalizations have been constructed, yielding categorifications which have no explicit representation-theoretic or geometric source! Examples are Bott-Samelson bimodules for non-crystallographic Coxeter groups [60], or the odd categorifications of quantum groups studied by Ellis, Khovanov, and Lauda [21, 22]. For other categorifications there are geometric sources but insufficient geometric tools, such as Lauda's categorification of the quantum group [41] (dealing with non-proper geometry), or perverse sheaves in finite characteristic [27] (where key tools like the Decomposition Theorem fail to work). One also has the recent categorifications of quantum groups at roots of unity [37, 15], using geometrically-motivated algebras but equipped with differentials whose geometric underpinning is not yet understood.

One of the most fascinating aspects of this field is that **geometry is not that special:** some non-geometric categorifications appear to have the desirable properties one usually proves using geometric methods! One of the recent advances in this direction is a proof of the Soergel conjecture due to myself and Geordie Williamson [20]. Let us work over $\mathbb{R}$, and consider the category of Soergel bimodules, which are summands of Bott-Samelson bimodules. The classes of indecomposable Soergel bimodules give a basis of the Hecke algebra; the Soergel conjecture states that this is the Kazhdan-Lusztig basis. This was proven in Weyl group type by Soergel, being an easy consequence of the decomposition theorem in geometry [58]. In 2000 de Cataldo and Migliorini [6, 7] provided a new proof of the Decomposition Theorem using elementary Hodge theory, inductively proving that certain spaces have the Hodge-Riemann bilinear relations, and then deducing the existence of idempotent splittings from this. We adapted this proof to show that, for an arbitrary Coxeter group, indecomposable Soergel bimodules have the Hodge-Riemann bilinear relations, and used this to deduce the Soergel conjecture. Along the way, we needed to produce combinatorial versions of certain geometric constructions in this precise setting (like ample line bundles and hyperplane sections).

Why these non-geometric categorifications should have such reasonable behavior is a tantalizing mystery. Is there some algebraic principle which explains the good behavior of both the geometric and non-geometric categorifications (e.g. Hodge-Riemann bilinear relations)? Is there some unknown geometric theory behind these examples?

Finally, we should note that finding a combinatorial replacement of an abelian category $\mathcal{A}$ is not the end of the story. One hopes to understand $\mathcal{A}$-Proj, the full subcategory of projective objects, which is the idempotent closure of $\mathcal{A}_{\mathbb{P}}$. In order to make $\mathcal{A}$-Proj explicit, one must **compute all the indecomposable idempotents** in A-Prin. This is no easy task. For Bott-Samelson bimodules over $\mathbb{R}$, the Hodge-theoretic results mentioned above give a general proof of the existence of idempotents with certain properties, and provide an inductive algorithm. However, no general combinatorial description or recursive formula for these idempotents is known.

3. Categorical Hecke Theory

Let W be a Coxeter group, and let $\mathfrak{h}$ be a "suitable" reflection representation over a field $\Bbbk$. The following definitions are due to Soergel (see [59, 60]).

Definition 3.1. Let R denote the coordinate ring of $\mathfrak{h}$, graded so that deg $\mathfrak{h} = 2$. For any simple reflection s, let R^s denote the subring of s-invariants. Let B_s denote the R-bimodule $R \otimes_{R^s} R(1)$. Tensor products of various bimodules B_s are known as *Bott-Samelson bimodules*, and form a monoidal category $\mathbb{BS}$Bim, a full subcategory of R-bimodules. The idempotent closure of $\mathbb{BS}$Bim is known as the category of *Soergel bimodules* $\mathbb{S}$Bim. Taking the left R-modules obtained from Soergel bimodules by killing the image of R_+ (the positive degree part of R) on the right, one obtains a non-monoidal graded category $\overline{\mathbb{S}\mathrm{Bim}}$.

The key results of Soergel are summarized in the following theorem.

Theorem 3.2. *The Grothendieck ring* $[\mathbb{S}Bim]$ *is isomorphic to the Hecke algebra* $\mathbf{H}$ *of* W*. The action of* $\mathbb{S}Bim$ *on* $\overline{\mathbb{S}Bim}$ *categorifies the regular representation of* $\mathbf{H}$*. When* W *is a Weyl group and* $\mathfrak{h}$ *the usual reflection representation,* $\overline{\mathbb{S}Bim}$ *is equivalent (in a graded sense) to the category of semisimple perverse sheaves on the flag variety, and after forgetting the grading, it is equivalent to the category of projective objects in* $\mathcal{O}$.

Moreover, in the Weyl group case, one can study non-semisimple perverse sheaves or non-projective objects in $\mathcal{O}$ using complexes of Soergel bimodules. There are also geometric interpretations of $\mathbb{S}$Bim for crystallographic Coxeter groups, but for non-crystallographic Coxeter groups $\mathbb{S}$Bim is the only known categorification of $\mathbf{H}$.

Theorem 3.3. *(joint with Geordie Williamson [19]) There is an explicit diagrammatic presentation for* $\mathbb{BS}Bim$*, by generators and relations. In this presentation, morphisms are represented by planar graphs called* Soergel diagrams. *This diagrammatic presentation exists for more general representations* $\mathfrak{h}$ *over more general rings* $\Bbbk$ *(e.g. we allow* $\Bbbk = \mathbb{Z}$ *for crystallographic groups), and the Grothendieck ring is still isomorphic to* $\mathbf{H}$.

In special cases this presentation had appeared previously: in type A by myself and Khovanov [13], in dihedral type in my PhD thesis [12], and independently in right-angled type by Libedinsky [43]. Our presentation is a powerful new tool, and Williamson and I are in the midst of a fruitful collaboration in this field.

Dream 1. Fully understand categorical Hecke theory and Soergel bimodules in an algebraic framework, using the presentation of Theorem 3.3.

3.1. Idempotents, Hodge theory, and p-canonical bases.

To pass from an understanding of the combinatorial replacement $\mathbb{BS}$Bim to an understanding of its idempotent closure $\mathbb{S}$Bim, one must compute the idempotents which project to each indecomposable object.

Goal 1. There is one indecomposable Soergel bimodule B_w for each $w \in W$, and it is a summand of the Bott-Samelson bimodule $BS(\underline{w})$ attached to any reduced expression $\underline{w}$ of w. Find the idempotent in $\mathrm{End}(BS(\underline{w}))$ which projects to B_w.

This goal seems extremely difficult, though at least the presentation of Theorem 3.3 allows for an explicit approach. Special cases are more accessible, and do hold interest. I have computed these idempotents directly for dihedral groups [12], and longest elements in type A [9], and both cases have produced interesting connections with other areas of representation theory. Recently, Libedinsky and I [14] have computed these idempotents for universal Coxeter groups.

Project 1. Extend these results in a number of ways: to longest elements in other Weyl groups (see Manin-Schechtmann theory below), to certain elements of affine Weyl groups (see Satake equivalence below), and using pattern-avoidance methods to other special elements.

There is also an algorithm for computing these idempotents (discussed in [20], though known before to experts, e.g. [42]), which amounts to the computation of certain so-called *intersection forms* valued in $\Bbbk$.

Proposition 3.4. *If the determinants of these intersection forms are all invertible, then the indecomposable Soergel bimodules descend in the Grothendieck group to the Kazhdan-Lusztig basis of* $\mathbf{H}$*. This implies the positivity of structure coefficients in* $\mathbf{H}$ *and of Kazhdan-Lusztig polynomials.*

Goal 2. Find a combinatorial understanding of these intersection forms, and compute their determinants. Find a combinatorial description of which primes divide a given determinant.

When $\Bbbk$ is a field of finite characteristic p, these intersection forms will sometimes be degenerate, and the image of the indecomposable Soergel bimodules in the Grothendieck group is known as the **p-canonical** or **p-orthodox basis** of $\mathbf{H}$ [68]. These bases play a significant role in modular representation theory [28], and can also provide information (but not solve) the geometric question of when the characteristic cycle is reducible [63]. Computing these bases is now possible with our tools: such a computation recently allowed Williamson to find counterexamples to the James conjecture and the so-called Lusztig conjecture [67]. Further general progress will depend on a more combinatorial approach.

Project 2. Find a combinatorial basis of morphism spaces in $\mathbb{BS}$Bim which is adapted to the computation of intersection forms.

This is expected to be one of the most difficult and far-reaching of the projects proposed here. Note that $\mathbb{BS}$Bim already has a combinatorial cellular basis, the *light leaves basis* of Libedinsky [42, 19], though it does not seem well-adapted to the problem.

As mentioned previously, Williamson and I have recently proven [20] that Soergel bimodules satisfy the Hodge-Riemann bilinear relations, when $\Bbbk = \mathbb{R}$. One implication is that intersection forms above are all non-degenerate. In this proof, we essentially adapted the arguments of de Cataldo and Migliorini for semismall maps in [6]. There are a number of other positivity conjectures which do not follow directly from our results, but require further investigation. We already have strategies in mind towards approaching these projects.

Project 3. There is a more general Hodge-theoretic result for non-semismall maps, the relative Hodge-Riemann bilinear relations proved in [7]. Show that all Bott-Samelson bimodules possess the relative Hodge-Riemann bilinear relations, as do all tensor products of Soergel bimodules. This would imply the unimodularity of structure coefficients in $\mathbf{H}$.

Project 4. Apply Hodge-theoretic techniques to the study of standard filtrations in Soergel bimodules. These standard filtrations correspond, via Soergel's functor and work of Kubel [39], to the Janzten filtration on $\mathcal{O}$. A Hodge-theoretic result would allow us to prove the Jantzen conjecture.

When $\Bbbk = \mathbb{C}$ and the reflection representation $\mathfrak{h}$ does not have a real form, it is possible that the intersection forms are degenerate [11]. The reflection representations $\mathfrak{h}_q$ defined in the chapter below on Satake are an example. These particular reflection representations do not appear to have been studied previously in any depth. I have no concrete projects to propose at the moment, however.

3.2. **Higher representation theory of H.** The algebra $\mathbf{H}$ with its canonical basis has an interesting cell theory, and in type A is actually a cellular algebra. This has significant connotations for its representation theory, and one hopes to lift these to the categorical level. The general "cell theory" of categorifications is investigated by Mazurchuck and Miemietz in a series of papers [52, 49, 50, 51]. They show among other things that minimal categorifications of cell modules exist and are canonically defined.

Project 5. Describe the minimal categorifications of cell modules explicitly, analogously to the way they have been described in categorical Lie theory (using cyclotomic quotients).

One step in this direction has been my categorification of induced trivial representations from parabolic sub-Hecke algebras in type A [9]. Note that, in type A, irreducible representations can be understood by intersecting induced sign and induced trivial representations. Moreover, the Jucys-Murphy elements play a key role in branching rules, allowing one to study the representations of all type A Hecke algebras as a connected family. Symmetric polynomials in Jucys-Murphy elements span the center of $\mathbf{H}$.

Goal 3. Categorify representation theory in type A.

Project 6. Generalize the result of [9] to sign representations, and categorify the intersection procedure, producing cell module categorifications.

Project 7. Study categorified branching rules, and categorify the Okounkov-Vershik approach to the symmetric group [53].

Project 8. Find complexes of Soergel bimodules which categorify Jucys-Murphy elements and their symmetric polynomials (or similar elements).

Of these projects the last is likely the most difficult, as working with complexes of Soergel bimodules is computationally intensive. However, it may turn out to be the most significant, if it provides insight into the categorical center of $\mathbb{S}$Bim.

Definition 3.5. The categorical center of $\mathbb{S}$Bim is the braided monoidal category of complexes $F^\bullet$ possessing natural isomorphisms $F^\bullet \otimes B \cong B \otimes F^\bullet$ for any $B \in \mathbb{S}\mathrm{Bim}$.

It is known that this categorical center is equivalent to Lusztig's category of character sheaves [45]. There are not many tools available to study character sheaves, so the ability to explicitly describe them as complexes and study them in a straightforward way would be very exciting.

Dream 2. Find a combinatorial description of the categorical center of $\mathbb{S}$Bim. More specifically, describe a combinatorial generator and its dg algebra of endomorphisms. Find idempotents projecting to indecomposable objects.

3.3. **Folding, unequal parameters, and other embeddings.** Let G be a cyclic group acting on the Coxeter graph of W, whose orbits consist of commuting reflections. The invariant subgroup W^G is another Coxeter group, generated by the product of the simple reflections in each orbit.

Theorem 3.6. *([17]) The category of G-equivariant Soergel bimodules for W will categorify (in a certain twisted sense) the Hecke algebra of W^G with unequal parameters.*

This process is called **folding**, and was conventionally understood using geometry (the cyclic group being generated by a non-conventional Frobenius map; see [44] for more details). However, we have a new algebraic proof of folding, which applies in the non-geometric setting as well, and is far easier to study. Using this twisted Grothendieck group we have also provided a categorification of the Lusztig-Vogan representation [18], giving a proof of the Lusztig-Vogan conjecture [46] on signed positivity. We expect analogous results for quiver Hecke algebras and quantum groups as well.

Project 9. Compute the twisted Grothendieck groups of equivariant Soergel bimodules in more generality (say, for non-cyclic groups). These will not be Hecke algebras with unequal parameters, but some new, unexplored algebras. Study the representation theory and cell theory of these algebras.

Project 10. Adapt the cellular categorification theory of Mazorchuk and Miemietz [52, 49, 50, 51] to study equivariant categories.

Goal 4. Many Hecke algebras with unequal parameters do not arise from folding. Find a framework in which these algebras can be categorified.

There are other embeddings of Coxeter groups similar to folding, in that a simple reflection is sent to the product of commuting simple reflections, except that these commuting reflections do not form the orbit of any group action. For example, H_3 embeds into D_6, and H_4 embeds into E_8. The relationship between their Hecke algebras with unequal parameters is not well understood.

Goal 5. Find a framework in which the Hecke algebra of H_4 can be categorified using some structure on the Soergel bimodules of E_8, and similarly for other such embeddings.

Dream 3. Use this to produce a geometric framework which categorifies Hecke algebras of non-crystallographic Coxeter groups. Explain the mystery of why non-crystallographic Soergel bimodules possess seemingly geometric properties.

There are a number of other embeddings of Coxeter groups (such as the excellent subgroups of Lusztig, appearing in the generalized Springer correspondence) which have not yet been studied categorically, but can be plausibly approached using variants on the Hecke category.

3.4. **Manin-Schechtmann theory.** Manin and Schechtmann define a notion of higher Bruhat orders [48], attached to the family of type A Coxeter groups. One of these higher Bruhat orders allows one to place a (partial) orientation on the graph of reduced expressions of a given element in S_n. The significance of this orientation suddenly comes to the fore when one examines $\mathbb{BS}$Bim. To any path in the reduced expression graph one can associate a morphism between Bott-Samelson bimodules.

Theorem 3.7. *([9]) For the Manin-Schechtmann orientation, any two oriented paths with the same start and finish yield equal morphisms in $\mathbb{BS}$Bim. When $w = w_0 \in S_n$ is the longest element, the oriented path from source to sink can be used to construct the idempotent projecting to B_{w_0}.*

Computations with Soergel bimodules have indicated that this behavior in $\mathbb{BS}$Bim extends to type B, but does not extend to type H_3.

Conjecture 1. *There is also a Manin-Schechtmann theory of higher Bruhat orders in type B, and possibly to type D. In fact, there may be a type-independent, geometric description of higher Bruhat orders for all Weyl groups. This would yield a canonical orientation on all reduced expression graphs, up to automorphisms of the Dynkin diagram.*

Project 11. Define these higher Bruhat orders in types B and D. Check whether oriented paths yield idempotents for longest elements.

Computation of the higher Bruhat order in type B was a project assigned to a high school student under the auspices of MIT's PRIMES-USA program. She managed to give a conjectural description of the type B analog of a major tool in Manin-Schechtmann: the packet order. From this, a proof and general description of higher Bruhat orders in type B should not be far off.

There is a definite geometric implication of the fact that certain paths yield equal morphisms in $\mathbb{BS}$Bim, and others do not. Thus the geometric description of the higher Bruhat orders in Conjecture 1 is not unreasonable.

3.5. **Singular Soergel bimodules.** If Soergel bimodules tell you about perverse sheaves on the flag variety of a Weyl group (resp. category $\mathcal{O}$), then singular Soergel bimodules $\mathcal{S}\mathbb{S}$Bim tell you about perverse sheaves on partial flag varieties (resp. parabolic or singular category $\mathcal{O}$). A generalization of Soergel bimodules, $\mathcal{S}\mathbb{S}$Bim is a 2-category with one object for each finite parabolic subgroup $W_I \subset W$, whose 1-morphisms consist of (R^{W_I}, R^{W_J})-bimodules; it can be defined for any Coxeter group. Once again, there is a combinatorial subcategory $\mathcal{S}\mathbb{BS}$Bim of singular Bott-Samelson bimodules, given by compositions of induction and restriction bimodules. These were all studied by Williamson [69].

Goal 6. Present $\mathcal{S}\mathbb{BS}$Bim by generators and relations in general type.

Project 12. Williamson and I have a conjectural presentation in type A. Prove that it works by connecting this presentation with the recent work of Cautis-Kamnitzer-Morrison [3] (see the Satake section below).

A presentation is found for dihedral groups in my thesis [12]. There are a number of reasons that presenting $\mathcal{S}\mathbb{BS}$Bim should be significant, as it plays a key role in several projects mentioned elsewhere, such as the higher representation theory of Hecke algebras above, and the Satake equivalence below. Also, the generating morphisms in $\mathbb{BS}$Bim are actually compositions of more elementary morphisms in $\mathcal{S}\mathbb{BS}$Bim, making certain relations more comprehensible.

3.6. **Knot theory.** Rouquier complexes are complexes of Soergel bimodules which satisfy the categorified braid relations [56]. Khovanov [33] showed that taking the Hochschild homology of Rouquier complexes in type A will yield a triply graded link homology theory. By passing to quotients of the Hecke category, one can understand $\mathfrak{sl}_n$ link homology as well [62, 47]. Though knot theory is not one of my main interests, I believe I can contribute something to the field.

Goal 7. As part of the proof of the Soergel conjecture in [20], Williamson and I showed that Rouquier complexes for positive lifts in the braid group satisfy a version of the Hodge-Riemann bilinear relations. Explore the implications of this for Khovanov's triply graded homology.

Goal 8. Provide a diagrammatic description of the Hochschild homologies of Soergel bimodules, in terms of Soergel diagrams on a cylinder.

In preliminary calculations of Hochschild cohomology, Koszul complexes associated to special non-regular sequences were a key tool [10]. The summands in any Koszul complex are indexed by vertices of a hypercube; for a special kind of non-regular sequence, the cohomology splits into terms for each vertex as well.

Project 13. Develop the theory of these special Koszul complexes, so that they can be applied to Hochschild homology computations in certain monoidal categories.

4. The Satake equivalence

Let G be a complex algebraic group with Langlands dual $G^\vee$, and let $\mathcal{O} = \mathbb{C}[[t]]$ and $\mathcal{K} = \mathbb{C}((t))$. The **geometric Satake equivalence** is an equivalence between two symmetric monoidal categories: on the geometric side, the $G(\mathcal{O})$-equivariant perverse sheaves on $G(\mathcal{K})/G(\mathcal{O})$, and on the representation theory side, the category $\mathrm{Rep}_{G^\vee}$. See [24] for more details. Rephrasing this result in another language, one arrives at a formulation independent of the choice of algebraic group for the Lie algebra $\mathfrak{g}$. This is explained in my forthcoming work [11].

Theorem 4.1. *([11]) There is an equivalence between the 2-category of* maximally singular Soergel bimodules *$m\mathcal{S}\mathbb{S}Bim$ for the affine Weyl group, and the 2-category of representations $Rep^{\Omega}_{\mathfrak{g}^\vee}$ of $\mathfrak{g}^\vee$ with the extra data of a central character in $\Omega = \pi_1(\mathfrak{g}^\vee)$. A simpler consequence is an equivalence between $Rep_{\mathfrak{g}^\vee}$ and a certain Hom category in $m\mathcal{S}\mathbb{S}Bim$.*

One might think that the Satake equivalence has an easy side (representation theory) and a difficult side (geometry), but this overstates the easiness of representation theory. The category $\mathrm{Rep}_{\mathfrak{g}^\vee}$ is semisimple, but its monoidal structure on morphisms is poorly understood. To algebraicize it, one should describe the endomorphism algebra $\mathrm{End}(\oplus P_{i_1} \otimes \cdots \otimes P_{i_d})$ of direct sums of tensor products of irreducibles, with both its usual and its monoidal product structures. This is a difficult open problem. In fact, I believe the geometric side is better adapted to this computation.

In type A one can find combinatorial replacements for both 2-categories in this equivalence. The full sub-2-category of *maximally singular Bott-Samelson bimodules* $m\mathcal{S}\mathbb{BS}\mathrm{Bim}$ inside $m\mathcal{S}\mathbb{S}\mathrm{Bim}$ has a presentation by generators and relations, arising from the conjectural presentation of $\mathcal{S}\mathbb{S}\mathrm{Bim}$ discussed in the previous chapter. The full sub-2-category $\mathrm{Fund}^{\Omega}_{\mathfrak{g}^\vee}$ consisting of tensor products of fundamental representations also has a presentation: for $\mathfrak{sl}_2$ it is the famous Temperley-Lieb algebra [61, 29]; in rank 2 one has Kuperberg's web algebras [40]; for $\mathfrak{sl}_n$ one has the $\mathfrak{sl}_n$-webs of Cautis-Kamnitzer-Morrison [3]. Using these presentations, one can give a straightforward proof of the following theorem, which yields a new proof of geometric Satake.

Theorem 4.2. *([11]) In type A there is an explicit equivalence of additive 2-categories between $m\mathcal{S}\mathbb{BS}Bim$ and $Fund^{\Omega}_{\mathfrak{g}^\vee}$.*

A surprising and unexplained fact is that this equivalence can be quantized! In type A, there is a q-deformed affine Cartan matrix, apparently new to the literature, yielding a non-traditional reflection representation $\mathfrak{h}_q$ of the affine Weyl group. When $q \in \mathbb{C} \setminus \mathbb{R}$, this representation has no real form. As discussed in Theorem 3.3, one can still define Soergel bimodules $\mathbb{S}\mathrm{Bim}_q$ for $\mathfrak{h}_q$, and its auxiliary 2-categories.

Theorem 4.3. *([11]) There is an explicit equivalence between $m\mathcal{S}\mathbb{BS}Bim_q$ and $Fund^{\Omega}_{U_q(\mathfrak{sl}_n)}$, the tensor products of fundamental representations for the quantum group $U_q(\mathfrak{sl}_n)$.*

Goal 9. Find a geometric explanation for the q-deformed affine Cartan matrix.

Note that Gaitsgory [23] also has a quantum geometric Satake equivalence, but the connection between these two constructions is entirely unknown.

To recover information about $\mathrm{Rep}_{\mathfrak{g}^\vee}$ from $\mathrm{Fund}_{\mathfrak{g}^\vee}$ one must compute the idempotents in Fund which project to the irreducible representations in Rep. For $\mathfrak{sl}_2$ one has the Jones-Wenzl projectors [26, 66], and one has the so-called "clasps" in rank 2 [38], but in $\mathfrak{sl}_n$ for $n \geq 4$ these idempotents are still unknown. However, Nicolas Libedinsky and I have done enough computations to convince ourselves this problem is tractable.

Project 14. Compute the $\mathfrak{sl}_n$ clasps. Via the Satake equivalence, this yields the idempotents projecting to indecomposable Soergel bimodules for special elements of the affine Weyl group $\widetilde{A}_{n-1}$.

One other obvious project is to explore what can be generalized to other types. Maximally singular Bott-Samelson bimodules should only correspond to tensor products of miniscule weights, though other combinatorial replacements may potentially be found. Neither side of the equivalence has any known presentation, and I expect the problem to be extremely difficult. The source of a possible q-deformation is also unknown.

Technically, the Satake equivalence sends Fund to the maps of degree 0 in $m\mathcal{S}\mathbb{BS}$Bim. However, the presentation for $m\mathcal{S}\mathbb{BS}$Bim makes it easy to understand morphisms of nonzero degree as well, which correspond (also see [24]) to interesting operations between representations of $\mathfrak{g}^\vee$ (and $U_q(\mathfrak{g}^\vee)$). One of the reasons I believe that $m\mathcal{S}\mathbb{BS}$Bim is easier than Fund is that the degree 0 maps are not elementary, being compositions of various elementary generators of nonzero degree.

5. Categorification at roots of unity

The word "categorification" was coined by Crane and Frenkel [5], who originally conjectured that one could categorify quantum groups at roots of unity. The (integral form of a) quantum group at generic q is a $\mathbb{Z}[q, q^{-1}]$-module, so it can be categorified by a graded additive category. This category was constructed by Khovanov and Lauda [35]. However, categorification at a root of unity requires a different kind of category, made possible by Khovanov's 2005 foray [32] into **hopfological algebra**.

In [32], a philosophy is presented whereby ordinary homological algebra is just a theory one can associate to the graded (super) Hopf algebra $\Bbbk[\partial]/\partial^2$. However, for any finite dimensional graded (normal or super) Hopf algebra one has a corresponding theory, yielding for any ring a pair of triangulated categories which are analogs of its homotopy and derived categories. For the partial (non-super) Hopf algebra $\Bbbk[\partial]/\partial^p$ over a field $\Bbbk$ of characteristic p, one obtains a theory where the Grothendieck group of the derived category is naturally a module over $\mathbb{Z}[\zeta_p]$, for ζ_p a primitive p-th root of unity. You Qi [54] has developed hopfological algebra to a great extent, though the theory is still, relatively speaking, in its infancy.

Khovanov and Qi [37] categorified the positive half U_q^+ of the quantum group at $q = \zeta_p$, by taking quiver Hecke algebras and endowing them with the structure of a *p-dg-algebra*. Qi and I [15] continued this story, categorifying all of $U_q(\mathfrak{sl}_2)$ at $q = \zeta_p$ (both the Lusztig and the small form), and developing some additional techniques to deal with p-dg-algebras.

Goal 10. Categorify $U_q(\mathfrak{g})$ at roots of unity for arbitrary $\mathfrak{g}$.

Dream 4. Study the structural properties of categorical p-$\mathfrak{g}$-actions. One may be able to produce results similar to those of Chuang-Rouquier [4] and Rouquier [55], but categorifying the properties of modular representation theory.

The techniques we developed for $\mathfrak{sl}_2$ relied on explicit idempotent decompositions, which are not known outside of $\mathfrak{sl}_2$. They also rely on the fact that indecomposable objects descend to a canonical basis. We believe that the reliance on explicit idempotent decompositions can be bypassed with additional technology, which would allow one to address the simply-laced case (thanks to results of Webster [64]).

Project 15. Category $U_q(\mathfrak{g})$ at roots of unity for simply-laced $\mathfrak{g}$.

The reliance of our proof on categorification of a canonical basis is more complicated, but I believe that additional structure governing decompositions into irreducibles, such as the technology of intersection forms and Lefschetz operators from [20], will also allow us to bypass most of the difficulty. This technology has not yet been developed for quantum group categorifications, but for Hecke categorifications it is in place. We have already computed the relevant p-dg-algebra structure on $\mathbb{BS}$Bim.

Project 16. Categorify **H** at a root of unity, by endowing Soergel diagrams with a p-dg-algebra structure.

On another note, one of the major features that homological algebra possesses and hopfological algebra lacks is t-structures. Instead, hopfological algebra tends to have collections of homological functors which collectively exhaust the category.

Dream 5. Find a general theory of homological functors in hopfological algebra, and analogs of spectral sequences which compare them.

6. Categorical braid group actions

A weak action of a group G on a category $\mathcal{C}$ is a homomorphism $G \to [\mathrm{Aut}(\mathcal{C})]$ to the group of isomorphism classes of autofunctors of $\mathcal{C}$. One has a functor F_g for each $g \in G$, with $F_g \circ F_h \cong F_{gh}$. A strict action, on the other hand, is a 2-functor from Ω_G, the 2-groupoid of G, to the 2-category $\mathrm{Aut}(\mathcal{C})$ itself. That is, one needs to provide an explicit isomorphism $F_g \circ F_h \to F_{gh}$, satisfying some natural compatibility requirements. There are many important geometric examples of weak actions of braid groups $\mathbf{Br}_W$ and affine braid groups $\mathbf{Br}_W^{\mathrm{aff}}$ on categories of sheaves (such as [1]), and I would like to know if they can be made strict.

One issue is that these weak actions are not usually defined by giving a functor F_g for each braid g. Instead, one provides functors for the generators (simple reflections in a braid group, and the translations in $\mathbf{Br}^{\mathrm{aff}}$), and checks that the functors corresponding to either side of the braid relation are isomorphic. Williamson and I have a result [16] which greatly simplifies what additional data one needs to provide in order to upgrade such a weak action of $\mathbf{Br}$ into a strict action (vastly strengthening a similar result of Deligne [8]).

Project 17. Simplify the strictification data for $\mathbf{Br}^{\mathrm{aff}}$ in its lattice presentation.

Project 18. Use these results to prove the strictness of many geometric braid actions.

Our result in [16] was based on the topology of certain cell complexes relaxed to Coxeter complexes and Salvetti complexes [57]. More precisely, we give a diagrammatic interpretation of π_2 of this complex. The $K(\pi, 1)$ conjecture states that these complexes are classifying spaces, so that π_2 should be trivial. This is why we call the following project the $K(\pi, 1)$ conjecturette.

Project 19. Prove diagrammatically that π_2 of the Salvetti complex is trivial.

One fact, proven by Rouquier [56] using entirely different techniques, is that the action of Rouquier complexes on R-bimodules is strict. Joseph Grant and I have found a way to bootstrap this result into the following theorem.

Theorem 6.1. *(joint with Grant) All braid group actions by spherical twists are strict.*

Project 20. Spherical twists yield braid group actions which factor, in the Grothendieck group, through a very restrictive quotient of the Hecke algebra. Use similar bootstrapping techniques to strictify any braid group action which factors, in the Grothendieck group, through the Hecke algebra.

Grant has also explored in detail the actions of positive lifts of longest elements of parabolic subgroups in the spherical case [25], and this may also be generalized.

There are numerous braid group actions which do not factor through the Hecke algebra. For instance, the braid group acts on tensor products of quantum group representations, and this can be lifted to a categorical braid group action. The most accessible way to study these braid group actions in type A is using categorical Howe duality [2], using the Rickard complexes of Chuang-Rouquier.

Project 21. Prove that all such categorical braid group actions are strict.

References

[1] Roman Bezrukavnikov and Simon Riche. Affine braid group actions on derived categories of Springer resolutions. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(4):535–599 (2013), 2012.
[2] Sabin Cautis, Joel Kamnitzer, and Anthony Licata. Categorical geometric skew Howe duality. *Invent. Math.*, 180(1):111–159, 2010.
[3] Sabin Cautis, Joel Kamnitzer, and Scott Morrison. Webs and quantum skew Howe duality. Preprint. arXiv:1210.6437.
[4] Joseph Chuang and Raphaël Rouquier. Derived equivalences for symmetric groups and sl_2-categorification. *Ann. of Math. (2)*, 167(1):245–298, 2008.
[5] Louis Crane and Igor B. Frenkel. Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases. *J. Math. Phys.*, 35(10):5136–5154, 1994. Topology and physics.
[6] Mark Andrea A. de Cataldo and Luca Migliorini. The hard Lefschetz theorem and the topology of semismall maps. *Ann. Sci. École Norm. Sup. (4)*, 35(5):759–772, 2002.
[7] Mark Andrea A. de Cataldo and Luca Migliorini. The Hodge theory of algebraic maps. *Ann. Sci. École Norm. Sup. (4)*, 38(5):693–750, 2005.
[8] Pierre Deligne. Les immeubles des groupes de tresses généralisés. *Invent. Math.*, 17:273–302, 1972.
[9] Ben Elias. A diagrammatic category for generalized Bott-Samelson bimodules and a diagrammatic categorification of induced trivial modules for Hecke algebras. Preprint. arXiv:1009.2120.
[10] Ben Elias. Koszul complexes for quasi-regular sequences. In preparation.
[11] Ben Elias. Quantum algebraic geometric Satake. In preparation.
[12] Ben Elias. The two-color Soergel calculus. Preprint. arXiv:1308.6611.
[13] Ben Elias and Mikhail Khovanov. Diagrammatics for Soergel categories. *Int. J. Math. Math. Sci.*, pages Art. ID 978635, 58, 2010.
[14] Ben Elias and Nicolas Libedinsky. Soergel bimodules for universal Coxeter groups. In preparation.
[15] Ben Elias and You Qi. An approach to categorification of some small quantum groups II. Preprint, arXiv:1302.5478.
[16] Ben Elias and Geordie Williamson. Diagrammatics for Coxeter groups and their braid groups. Preprint. arXiv:0902.4700.
[17] Ben Elias and Geordie Williamson. Equivariant Soergel bimodules and folding. In preparation.
[18] Ben Elias and Geordie Williamson. Equivariant Soergel bimodules and Lusztig-Vogan polynomials. In preparation.
[19] Ben Elias and Geordie Williamson. Generators and relations for Soergel bimodules. In preparation.
[20] Ben Elias and Geordie Williamson. The Hodge theory of Soergel bimodules. Preprint. arXiv:1212.0791.
[21] Alexander Ellis, Mikhail Khovanov, and Aaron Lauda. The odd nilHecke algebra and its diagrammatics. Preprint, arXiv:1111.1320.
[22] Alexander Ellis and Aaron Lauda. An odd categorification of quantum sl(2). Preprint, arXiv:1307.7816.
[23] Dennis Gaitsgory. Twisted Whittaker model and factorizable sheaves. Preprint, arXiv:0705.4571.
[24] Victor Ginzburg. Perverse sheaves on a loop group and Langlands' duality. Preprint, arXiv:9511007.
[25] Joseph Grant. Lifts of longest elements to braid groups acting on derived categories. Preprint, arXiv:1207.2758.
[26] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983.
[27] Daniel Juteau, Carl Mautner, and Geordie Williamson. Parity sheaves. Preprint, arXiv:0906.2994.
[28] Daniel Juteau, Carl Mautner, and Geordie Williamson. Perverse sheaves and modular representation theory. In *Geometric methods in representation theory II*, volume 25 of *Séminaires et Congrès*, pages 313–350. Soc. Math. France, Paris, 2010.
[29] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.
[30] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. volume 53, pages 165–184. 1979.
[31] David Kazhdan and George Lusztig. Schubert varieties and Poincaré duality. In *Geometry of the Laplace operator (Univ. Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.
[32] Mikhail Khovanov. Hopfological algebra and categorification at a root of unity: the first steps. Preprint, arXiv:0509.5083.

[33] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.*, 18(8):869–885, 2007.
[34] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory*, 13:309–347, 2009.
[35] Mikhail Khovanov and Aaron D. Lauda. A categorification of quantum sl(n). *Quantum Topol.*, 1(1):1–92, 2010.
[36] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups II. *Trans. Amer. Math. Soc.*, 363(5):2685–2700, 2011.
[37] Mikhail Khovanov and You Qi. An approach to categorification of some small quantum groups. Preprint, arXiv:1208.0616.
[38] Dongseok Kim. *Graphical calculus on representations of quantum Lie algebras*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–University of California, Davis.
[39] Johannes Kubel. From Jantzen to Andersen filtration via tilting equivalence. Preprint, arXiv:1011.1794.
[40] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996.
[41] Aaron D. Lauda. A categorification of quantum sl(2). *Adv. Math.*, 225(6):3327–3424, 2010.
[42] Nicolas Libedinsky. Light leaves and Lusztig's conjecture. Preprint, arXiv:1304.1448.
[43] Nicolas Libedinsky. Presentation of right-angled Soergel categories by generators and relations. *J. Pure Appl. Algebra*, 214(12):2265–2278, 2010.
[44] G. Lusztig. *Hecke algebras with unequal parameters*, volume 18 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2003.
[45] George Lusztig. Character sheaves. I. *Adv. in Math.*, 56(3):193–237, 1985.
[46] George Lusztig and David A. Vogan, Jr. Hecke algebras and involutions in Weyl groups. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 7(3):323–354, 2012.
[47] Marco Mackaay and Pedro Vaz. The diagrammatic Soergel category and sl(N)-foams, for $N \geq 4$. *Int. J. Math. Math. Sci.*, pages Art. ID 468968, 20, 2010.
[48] Yu. I. Manin and V. V. Schechtman. Arrangements of hyperplanes, higher braid groups and higher Bruhat orders. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 289–308. Academic Press, Boston, MA, 1989.
[49] Volodymyr Mazorchuk and Vanessa Miemietz. Additive versus abelian 2-representations of fiat 2-categories. Preprint, arXiv:1112.4949.
[50] Volodymyr Mazorchuk and Vanessa Miemietz. Endomorphisms of cell 2-representations. Preprint, arXiv:1207.6236.
[51] Volodymyr Mazorchuk and Vanessa Miemietz. Morita theory for finitary 2-categories. Preprint, arXiv:1304.4698.
[52] Volodymyr Mazorchuk and Vanessa Miemietz. Cell 2-representations of finitary 2-categories. *Compos. Math.*, 147(5):1519–1545, 2011.
[53] Andrei Okounkov and Anatoly Vershik. A new approach to representation theory of symmetric groups. *Selecta Math. (N.S.)*, 2(4):581–605, 1996.
[54] You Qi. Hopfological algebra. Preprint, arXiv:1205.1814.
[55] Raphael Rouquier. 2-Kac-Moody algebras. Preprint, arXiv:0812.5023.
[56] Raphael Rouquier. Categorification of the braid groups. Preprint, arXiv:0409593.
[57] M. Salvetti. Topology of the complement of real hyperplanes in $\mathbf{C}^N$. *Invent. Math.*, 88(3):603–618, 1987.
[58] Wolfgang Soergel. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3(2):421–445, 1990.
[59] Wolfgang Soergel. The combinatorics of Harish-Chandra bimodules. *J. Reine Angew. Math.*, 429:49–74, 1992.
[60] Wolfgang Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.
[61] H. N. V. Temperley and E. H. Lieb. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. *Proc. Roy. Soc. London Ser. A*, 322(1549):251–280, 1971.
[62] Pedro Vaz. The diagrammatic Soergel category and sl(2) and sl(3) foams. *Int. J. Math. Math. Sci.*, pages Art. ID 612360, 23, 2010.

[63] Kari Vilonen and Geordie Williamson. Characteristic cycles and decomposition numbers. Preprint, arXiv:1208.1198.
[64] Ben Webster. Canonical bases and higher representation theory. Preprint, arXiv:1209.0051.
[65] Ben Webster. Knot invariants and higher representation theory. Preprint, arXiv:1309.3796.
[66] Hans Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.
[67] Geordie Williamson. Schubert calculus and torsion. Preprint, arXiv:1309.5055.
[68] Geordie Williamson. Some examples of parity sheaves. preprint, Oberwolfach reports. 5 pages.
[69] Geordie Williamson. Singular Soergel bimodules. *Int. Math. Res. Not. IMRN*, (20):4555–4632, 2011.