SHEAVES AND HOMOTOPY THEORY

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The purpose of this note is to describe the homotopy-theoretic version of sheaf theory developed in the work of Thomason [14] and Jardine [7, 8, 9]; a few enhancements are provided here and there, but the bulk of the material should be credited to them. Their work is the foundation from which Morel and Voevodsky build their homotopy theory for schemes [12], and it is our hope that this exposition will be useful to those striving to understand that material. Our motivating examples will center on these applications to algebraic geometry.

Some history: The machinery in question was invented by Thomason as the main tool in his proof of the Lichtenbaum-Quillen conjecture for Bott-periodic algebraic K-theory. He termed his constructions 'hypercohomology spectra', and a detailed examination of their basic properties can be found in the first section of [14]. Jardine later showed how these ideas can be elegantly rephrased in terms of model categories (cf. [8], [9]). In this setting the hypercohomology construction is just a certain fibrant replacement functor. His papers convincingly demonstrate how many questions concerning algebraic K-theory or étale homotopy theory can be most naturally understood using the model category language.

In this paper we set ourselves the specific task of developing some kind of homotopy theory for schemes. The hope is to demonstrate how Thomason's and Jardine's machinery can be built, step-by-step, so that it is precisely what is needed to solve the problems we encounter. The papers mentioned above all assume a familiarity with Grothendieck topologies and sheaf theory, and proceed to develop the homotopy-theoretic situation as a generalization of the classical case. In some sense the approach here will be the reverse of this: we will instead assume a general familiarity with homotopy theory, and show how the theory of sheaves fits in with perspectives already offered by the field.

1. Introduction

Our main question, then, is how might one associate a homotopy theory to something like the category of schemes? One may just as well ask how to associate a homotopy theory with the category of topological manifolds, or complex analytic spaces, or symplectic manifolds. The most obvious problem is that none of these categories are 'robust' enough. A homotopy theorist needs a category in which he can make essentially any construction imaginable, so he requires it to contain all limits and colimits. Categories like 'topological manifolds' simply don't have this property.

It turns out there is a way to fix this, using sheaf theory and Grothendieck topologies. These provide a method for enlarging a category in a sensible way, analagously to the way one enlarges the category of manifolds into that of all topological spaces. This will all be discussed in section 2, and for the purposes of this introduction we wish to ignore this point altogether and focus on a deeper question. Because even if we could enlarge the category of schemes to something more robust, why would we expect it to be a good place to do homotopy theory? After all, there are plenty of categories like $\mathcal{S}et$ which are perfectly complete and co-complete, but don't have any homotopy theory associated to them. Why would one expect schemes to be any different?

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To answer this question we look at a fundamental example, one which will motivate all of our later discussion. It provides one of the first pieces of evidence that schemes can be thought of as having a homotopy type associated to them. We then make some preliminary guesses about a scheme-theoretic notion of weak equivalence.

The algebraic K-groups of a scheme X are defined by producing a spectrum $\mathcal{K}(X)$ and setting $K^n(X) := \pi_{-n}\mathcal{K}(X)$ (be warned that K-theorists and geometers write $K_n(X)$ for our $K^{-n}(X)$—we will stick to homotopy-theoretic notation, however). $\mathcal{K}$ can be defined so that it actually gives a contravariant functor (*Schemes*) $\to$ (*Spectra*), and so the groups $K^n(X)$ do indeed start to look like a cohomology theory. For X a topological space, the analogue of $\mathcal{K}(X)$ turns out to be the mapping spectrum bu^X, where bu is the spectrum representing complex connective K-theory (so that bu is the connective cover of the spectrum BU). Thus, the functor $\mathcal{K}$ can be thought of as a substitute for the spectrum bu—it is sort of a device for storing all the same information that bu provides, but without an honest 'space' to house it in.

Most of the important results about algebraic K-theory arise from the study of the spectrum $\mathcal{K}(X)$, rather than that of the disembodied abelian groups $K^n(X)$. For example, if the scheme X is covered by two open sets U and V, one wants a Mayer-Vietoris sequence

$$\cdots \to K^n(X) \to K^n(U) \oplus K^n(V) \to K^n(U \cap V) \to K^{n+1}(X) \to \cdots$$

This follows formally once one proves the stronger result that

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}(U) \\ \downarrow & & \downarrow \\ \mathcal{K}(V) & \longrightarrow & \mathcal{K}(U \cap V) \end{array}$$

is a homotopy pullback diagram. More generally, it can be shown that for any open cover $\{U_\alpha\}$ of a scheme X one has

$$\mathcal{K}(X) \xrightarrow{\sim} \underleftarrow{\operatorname{holim}} \left[\prod_\alpha \mathcal{K}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{K}(U_{\alpha\beta}) \Rrightarrow \prod_{\alpha,\beta,\gamma} \mathcal{K}(U_{\alpha\beta\gamma}) \cdots \right]$$

(where we have omitted drawing the co-degeneracy maps, and have written $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$, etc.) For the moment we will refer to this as property $(*)$. In the case of a two-fold cover, it reduces to precisely the above homotopy-pullback statement. The fact that algebraic K-theory satisfies these properties is proven via Quillen's Localization Theorem, which is an analysis of the homotopy fibre of $f^* : \mathcal{K}(Y) \to \mathcal{K}(X)$ when f is a map $X \to Y$. The theorem depends heavily on particulars of the construction of $\mathcal{K}$.

When X is a topological space, it is of course also true that the assignment $X \mapsto bu^X$ has property $(*)$—i.e., that bu^X is the homotopy limit of the analagous cosimplicial spectrum

$$\prod_\alpha bu^{U_\alpha} \rightrightarrows \prod_{\alpha,\beta} bu^{U_{\alpha\beta}} \Rrightarrow \cdots$$

The proof of this topological fact uses *nothing* special about bu— it only depends on the more basic result that X can be recovered as the homotopy colimit of the U_α:

$$\underrightarrow{\operatorname{hocolim}} \left[\cdots \coprod U_{\alpha\beta\gamma} \Rrightarrow \coprod_{\alpha,\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha \right] \xrightarrow{\sim} X.$$

The analogy certainly makes one pause for thought. Is the 'real' reason that algebraic K-theory has property $(*)$ the fact that schemes have a hidden homotopy-type, one for which the analogue of statement (2) holds? Perhaps the question is meaningless. But let us *pretend* that schemes have a homotopy-type, and see what comes of it.

So if these mysterious homotopy-types exist, when should a map be a weak equivalence? Certainly the best answer to such a problem would be to give a brilliant, intrinsic definition depending only on geometric constructions. So far no one has accomplished this. Note that even in the case of simplicial sets it's difficult to give an 'intrinsic' definition of weak equivalence—in general one has to come up with the 'right' notions of cofibrant and fibrant, and build the corresponding cofibrant/fibrant-replacement functors. In our case we will not attempt this, but will instead try to get our hands on weak equivalences by a very indirect route.

Our first guess is motivated by the hypothesis that a contravariant functor $\mathcal{E} : (Schemes) \to (Spectra)$ satisfying property $(*)$, in analogy to the functor $\mathcal{K}$, must be something like a cohomology theory for schemes. A weak equivalence of schemes $X \xrightarrow{\sim} Y$ would be expected to have the property that $\mathcal{E}(Y) \to \mathcal{E}(X)$ is a weak equivalence of spectra for *every* such $\mathcal{E}$. So what if we were to *define* a map to be a weak equivalence precisely when it has this property?

First of all, in topology the requirement that $\mathcal{E}^Y \to \mathcal{E}^X$ be a weak equivalence for every spectrum $\mathcal{E}$ should characterize the map $X \to Y$ being a *stable* weak equivalence. So our proposed definition is already not quite right. However, we might try to get an unstable version by replacing $(Spectra)$ with $s\mathcal{S}et$. Functors $E : (Schemes) \to s\mathcal{S}et$ satisfying property $(*)$ might be thought of as the analogues of the functors $\mathcal{T}op \to s\mathcal{S}et$ given by $X \mapsto E^X$ for some space E. And then exactly as above, we could define a map of schemes $X \to Y$ to be a weak equivalence if it becomes a weak equivalence after applying any such E.

It turns out there is still something very wrong. Because the only information we are singling out is the property $(*)$, the belief that this notion of weak equivalence coincides with the 'correct' one is tantamount to believing that the requirements $\underrightarrow{\mathrm{hocolim}}\, U_\bullet \xrightarrow{\sim} X$ somehow 'generate' the homotopy theory—i.e., the belief that studying the homotopy-type of schemes can be reduced to studying these requirements. This turns out to be a ridiculous hope. As an example, let's work with spaces and consider the functor $E : \mathcal{T}op \to s\mathcal{S}et$ which sends a space X to the set of maps $\mathcal{T}op(X, S^1)$ (considered as a constant simplicial set). This functor does indeed have property $(*)$, yet $E(I) \to E(*)$ is not a weak-equivalence. This shows that our present definition, were we to apply it to the topological case, would not yield that $* \to I$ was a weak equivalence. We could just as easily have given this example for the category of schemes: for instance, the assignment $X \mapsto \mathcal{S}ch(X, \mathbb{A}^1)$ shows that $* \to \mathbb{A}^1$ would not be a weak equivalence.

Looking at this example in the context of our discussion, it's suddenly apparent that we've made a miscalculation. In fact, it's now obvious that functors $\mathcal{E} : (Schemes) \to (Spectra)$ satisfying property $(*)$ are *not* like cohomology theories after all—and that functors $F : (Schemes) \to s\mathcal{S}et$ having property $(*)$ are not like spaces. In particular, they don't have homotopy invariance!

Following Morel and Voevodsky, we will now try to fix this problem in the obvious way. Let us say that a map $X \to Y$ is a weak equivalence if $F(Y) \to F(X)$ is a weak equivalence for every functor $F : (Schemes) \to s\mathcal{S}et$ which satisfies property $(*)$ *and is homotopy invariant* in the sense that $F(Z) \xrightarrow{\sim} F(Z \times I)$ for every object Z. Here I will denote $\mathbb{A}^1$ when we are talking about schemes and the usual unit interval when we are talking about spaces; in either case, the map $Z \times I \to Z$ is meant to be the projection.

At this point the reader is probably feeling discouraged; after all, the definition seems a bit rigged. We have solved one difficulty by brute force, but there is no evidence that there aren't thousands of others. Again, believing that this gives the 'correct' definition of weak equivalence reduces to believing that the whole homotopy theory of schemes is captured in the two basic facts that $\underrightarrow{\mathrm{hocolim}}\, U_\bullet \to X$ and $X \times I \to X$ are weak equivalences. This doesn't sound particularly believable. Is it even true for topological spaces?

The surprising answer is yes. If we use this definition in the case of reasonable topological spaces (like CW-complexes) we *do* recover the usual notion of weak equivalence. This is essentially a result of Morel and Voevodsky, and it will be proven in the later sections of this paper. It says that the homotopy-theory of topological spaces can indeed be 'generated' by the two fundamental properties outlined above.

For schemes, on the other hand, the story is more complicated. This notion of weak equivalence does indeed turn out to be interesting, yet not as interesting as we would like it. The problem is that Zariski open covers don't tell the whole story about the homotopy-type of schemes. The first indications of this date back to Grothendieck's work on étale cohomology: looking at only Zariski covers makes the spectrum of a field look contractible, whereas in real life they behave more like $K(\pi,1)$'s (for example, they have nontrivial covering spaces). The story goes on, and in some ways is still unfinished—much of the recent mork on motivic cohomology can be seen as an attempt to figure out what the 'correct' substitutes for property $(*)$ should be. We will return to this issue in later sections, when we discuss Morel and Voevodsky's model category.

The idea that has been uncovered here, that of certain basic relations 'generating' a homotopy theory, turns out to be a key one. The purpose of this paper will essentially be to make this precise. Given a category $\mathcal{C}$ and a certain class of 'relations' like the ones encountered above (which required that the maps $\underrightarrow{\operatorname{hocolim}}\, U_\bullet \to X$ and $X \times I \to X$ be weak equivalences), we will show how to build a model category which represents something like the universal homotopy theory on $\mathcal{C}$ subject to these relations. When $\mathcal{C}$ is the category of topological manifolds, for instance, and the relations are the ones just mentioned, we will be able to write down a Quillen equivalence between the usual model category for spaces and the universal model category produced by our machinery. We will also give several examples of how one can prove theorems in a universal homotopy theory, even in those cases when it can't be identified with something familiar. It is in exactly this way that Morel and Voevodsky's model category for schemes turns out to be a useful setting in which to work.

1.1. **Contents.**

1. Introduction. A mystifying assortment of vagaries and half-truths, to which the reader has already been subjected.

2. Presheaves and Sheaves. A review of sheaves and Grothendieck topologies, but from a homotopy-theoretic perspective. Sheaves are developed as a way of 'co-completing' a category, and they appear via a close analogue of localization machinery. The sheafification functor is constructed using the small object argument.

3. Homotopy-theoretic sheaves. This is a generalization to the simplicial setting of the sheaf theory in Section 2. The analogues of sheaves appear as the fibrant objects in a certain model category. We explain how this 'Čech' model category can be thought of as the universal homotopy-theory built from a category $\mathcal{C}$, subject to certain relations. When applied to the category of manifolds (but with an additional class of relations imposed), the machinery yields a model category which is Quillen equivalent to that of topological spaces.

4. Points, the Godement construction, and Jardine's model category. This is a discussion of Jardine's model category and its relation with the Čech model category of Section 3. Jardine's weak equivalences are easier to identify, and homotopy classes of maps in his category are directly related to sheaf cohomology. We show that for the category of manifolds the Čech model category coincides with Jardine's.

5. The homotopy theory of schemes. The Morel-Voevodsky homotopy theory for schemes. This comes directly out of the machinery developed in previous sections. We discuss several applications, including the rigidity theorems of Suslin, Gillet-Thomason, and Gabber.

Miscellaneous. For mysterious notation or terminology the reader may consult the glossary at the end of the paper. This also contains our conventions about homotopy colimits and homotopy limits (which follow those of [6]). Finally, we have provided an appendix summarizing the basics of localization machinery.

It seems advisable to warn the reader that our approach here will not be the most general one possible. The discussion is deliberately centered around familiar geometric categories like schemes or topological manifolds, with the belief that it is easiest to understand the basic ideas in these settings. We will not, for instance, adopt the most general definition of a Grothendieck topology; nor will we work with an arbitrary topos when given the opportunity. We feel confident that the reader who is comfortable with these concepts will have no problem adapting our discussion to fit his language.

2. Presheaves and Sheaves

This section recalls the basic machinery of sheaves of sets on a Grothendieck site. Our approach will be somewhat non-traditional, however.

2.1. **Introduction.**

We begin with the observation that algebraic topologists, despite what we are usually led to believe, are really not very interested in the category of topological spaces. Of course there are certain subcategories, like the category of manifolds, in which they are *highly* interested—but the notion of 'topological space' is much too broad, admitting a host of pathological objects for which the machinery of algebraic topology is simply not intended.

On the other hand, categories such as topological manifolds end up being much too small for many purposes. The industrious homotopy-theorist finds himself wanting to glue manifolds together, quotient out by subspaces, divide out by group actions—in short, take various colimits—all of which have the disadvantage of perhaps producing something *which is no longer a manifold*. Gluing manifolds together might lead one to study some category of cell complexes, but of course these categories end up not being closed under colimits either. Thus, what the algebraic topologist desires is a setting in which he can study the category of manifolds (or finite complexes, if he is daring), have all small colimits at his disposal, and yet does not have to worry about the pathology that goes along with objects as diverse as topological spaces.

Now an interesting point is that there exist other settings in which one would like to do something similar. Instead of studying topological manifolds we might choose to study differentiable manifolds, or symplectic manifolds, or complex analytic spaces, or schemes. If there is to be any hope of applying the methods of homotopy theory to these objects, we again find ourselves needing a bigger category which contains them; it needs to be rich enough to permit us all the constructions we'd like, but also coarse enough so that the 'geometric' structure on our original spaces doesn't get lost. (For instance, embedding the category of symplectic manifolds into that of all topological spaces solves the first problem, but not the second in that it makes us lose sight of the symplectic structure).

The goal of this section will be to describe a general method for producing such 'enlarged' categories. The main idea is remarkably simple, and is based on the observation that the only thing an algebraic topologist ever really uses about spaces is that they can be built in some way from objects which he understands. In fact, it's precisely when the spaces *cease* to be built from such objects that the homotopy-theorist has to pull out mysterious adjectives like 'weak Hausdorff' and 'compactly-generated', in order to excommunicate the heretics.

What we plan to do, then, is to introduce a purely categorical construction which formalizes the notion of objects 'built from' the elements of a category $\mathcal{C}$. For example, there will be a category of creatures 'built from' symplectic manifolds, or creatures 'built from' schemes. The goal of this section is to make this precise—we must figure out what the phrase 'built from' should stand for.

The basic outline. Throughout the section we suppose given a category $\mathcal{C}$ which is somehow 'deficient' in colimits. Our goal will be to produce a category $\hat{\mathcal{C}}$ which is not deficient, which admits a map $\mathcal{C} \to \hat{\mathcal{C}}$, and which is as close as possible to $\mathcal{C}$ in the sense that it has an appropriate universal property. 'Deficient' will sometimes mean that certain colimits just don't exist in $\mathcal{C}$, but it may also mean that they *do* exist yet don't have some desired properties. The general procedure will be to add colimits to $\mathcal{C}$ in as 'free' a way as possible, and then to impose 'relations' on these colimits which reflect the geometric properties of our original objects. The former is accomplished via the theory of presheaves, the latter via the theory of sheaves.

Example 2.1.1.

(a) If $\mathcal{C}$ is the category of topological manifolds, the category $\hat{\mathcal{C}}$ will consist of something like 'formal gluings of manifolds', and will admit a 'realization functor' into $\mathcal{T}op$.

(b) Since manifolds are themselves just open subsets of Euclidean space which have been glued together, we might instead take $\mathcal{C}$ to be the category whose objects are such open sets, with morphisms the continuous maps between them. The category $\hat{\mathcal{C}}$ thus obtained will turn out to be equivalent to the category in (a).

(c) Continuing in this same vein and observing that every open subset of Euclidean space looks locally like an open *ball*, the initial category can be pared down even further, with objects just *one* open ball of each dimension. Again, we will get an equivalent category.

(d) Even better, we might also try to pare down the maps until there are just finitely many. For instance, if we substitute geometric simplices for the open balls, then we might take as morphisms just those generated by the face-inclusions and the elementary collapses. In this case our initial category $\mathcal{C}$ is just the simplicial category Δ, and our co-completed category will turn out to be the category of simpicial sets. This category is *not* equivalent to the ones in the above examples. However, when we figure out how to 'do homotopy theory' in these categories we will discover that the *homotopy theories* are equivalent.

(e) As a final example we mention the category of affine schemes over a field k, which by definition is the opposite category of $(k - algebras)$. The category of affine schemes *does* have small colimits, because the category of k-algebras has all limits. The difficulty is that these are somehow not the 'correct' colimits, at least as far as geometric considerations are concerned. For instance, consider the diagram

$$\begin{tikzcd} \mathbb{A}^1 - \{0\} \arrow[r, hook] \arrow[d] & \mathbb{A}^1 \\ \mathbb{A}^1 & \end{tikzcd}$$

where the horizontal map is the inclusion $z \mapsto z$ and the vertical map is the inversion $z \mapsto \frac{1}{z}$. The pushout in the category of affine k-schemes is the terminal object $\operatorname{Spec} k$, because the ring k is the intersection of $k[z]$ and $k[z^{-1}]$ inside $k[z, z^{-1}]$. However, geometry wants the pushout to be something like the projective line. One might say that what's 'wrong' with the colimits in the category of affine schemes is that the underlying topological space of a colimit is not the colimit of the underlying topological spaces. We will show how the category can be enlarged in a way that forces this property to hold.

2.2. **Presheaves.**

The process of formally adding colimits is taken care of by the presheaf functor. We now recall this concept.

Definition 2.2.1. *A presheaf on a category $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{op} \to \mathcal{S}et$; morphisms of presheaves are just natural transformations of functors. The category of presheaves on $\mathcal{C}$ will be denoted $Pre(\mathcal{C})$ (and sometimes $Pre_{\mathcal{C}}$, and sometimes just Pre, depending on typographical considerations).*

Remark 2.2.2.

(i) Any object $X \in \mathcal{C}$ determines a presheaf rX, defined by $rX(Z) = \mathcal{C}(Z, X)$. rX is called the presheaf *represented* by X, and such presheaves are termed *representable.* The assignment $X \mapsto rX$ induces a full embedding $\mathcal{C} \xrightarrow{r} Pre(\mathcal{C})$ called the 'Yoneda embedding' (the fact that it is a *full* embedding follows from the Yoneda Lemma).

(ii) More generally, a useful fact is that $Pre_{\mathcal{C}}(rX, F) = F(X)$ for any presheaf F (again by the Yoneda Lemma).

(iii) $Pre(\mathcal{C})$ has all limits and colimits, for these are just inherited from the limits and colimits in $\mathcal{S}et$. Specifically, if $\{F_\alpha\}$ is a diagram of presheaves then its colimit is the presheaf $X \mapsto \underrightarrow{\mathrm{colim}}_\alpha F_\alpha(X)$, and similarly for the limit. This is sometimes described by saying that limits and colimits of presheaves are computed 'objectwise'.

(iv) Our final point is that any presheaf F may be *canonically* written as a colimit of representables. Indeed, one can check that

$$F = \operatorname*{colim}_{rX \xrightarrow{\phi} F} (rX)_\phi,$$

where $(rX)_\phi$ denotes a copy of rX indexed by the label ϕ.

Warning 2.2.3. In light of remark (i), we will often identify the category $\mathcal{C}$ with its image in $Pre(\mathcal{C})$. So we will write X and rX interchangeably, and will be very cavalier about going back and forth between the two notations. The reader will most likely adapt to this very quickly.

The above remarks show that $Pre(\mathcal{C})$ is a co-complete category to which $\mathcal{C}$ maps. The following proposition says that it is the universal example of such a category.

Proposition 2.2.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories such that $\mathcal{D}$ is co-complete, and let $\gamma : \mathcal{C} \to \mathcal{D}$ be a functor. Then there exists a colimit-preserving functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$ which factors γ:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & Pre(\mathcal{C}) \arrow[d, dashed, "Re"] \\ & \mathcal{D}. \end{tikzcd}$$

Moreover, for any two such functors there is a unique isomorphism between them.

Note 2.2.5. With some care, one can interpret the assignment $\mathcal{C} \mapsto Pre(\mathcal{C})$ as the left-adjoint of a forgetful functor. One way to do this is to introduce the notion of a *functorially co-complete* category (or FCC-category, for short), by which we mean a co-complete category together with a functorial method for constructing colimits. For instance, $\mathcal{S}et$ can be given the structure of an FCC-category. Once that is done, one observes that $Pre(\mathcal{C})$ inherits such a structure from $\mathcal{S}et$, and then a simple modification of the above proposition shows that $\mathcal{C} \mapsto Pre(\mathcal{C})$ is the left-adjoint to the forgetful functor

$$((\text{Categories})) \xleftarrow{U} ((\text{FCC-categories})).$$

(The point is that a functorial choice of colimits forces the factorization of the proposition to be *unique*, and not just unique up to unique isomorphism). Thus, in a very rigorous sense $Pre(\mathcal{C})$ is the free FCC-category generated by $\mathcal{C}$.

Discussion 2.2.6. Before giving the proof of the above proposition (which is very easy) we try to provide a little motivation. What do presheaves have to do with co-completing a category?

Suppose that X and Y are objects in a category $\mathcal{C}$, and that we want to formally add an object to $\mathcal{C}$ which will be the coproduct of X and Y. That is, we want to produce a category $\hat{\mathcal{C}}$ for which $\mathrm{ob}\,\hat{\mathcal{C}} = \mathrm{ob}\,\mathcal{C} \cup \{\Omega\}$ (for some Ω), for which there is an embedding $\mathcal{C} \to \hat{\mathcal{C}}$, and such that Ω is the coproduct of X and Y in $\hat{\mathcal{C}}$. And let's also require $\hat{\mathcal{C}}$ to be the 'universal' such category, in the sense of the above proposition.

The desire that $\Omega = X \amalg Y$ tells us that we know how to map *out* of Ω—namely, for $Z \in \mathcal{C}$ we need to have $\hat{\mathcal{C}}(\Omega, Z) = \mathcal{C}(X, Z) \times \mathcal{C}(Y, Z)$. The harder question is how to map *into* Ω. What we know is that for any object $Z \in \mathcal{C}$ we must have a map $\mathcal{C}(Z, X) \amalg \mathcal{C}(Z, Y) \to \hat{\mathcal{C}}(Z, \Omega)$ (induced by the maps $X \to \Omega$ and $Y \to \Omega$). For a general coproduct this map will be neither injective nor surjective: for surjectivity, think about the case of S^0 mapping into $I \amalg I$ in the category of topological spaces; for injectivity, think about the same map in the category of *pointed* topological spaces.

Thus, we see that in general there will be maps into a coproduct which don't come from maps to X or Y, and that some maps into X and Y may become identified when we map to the coproduct. However the reader may convince himself that if he wants a 'universal' coproduct then he should have neither of these things, and therefore should insist that $\hat{\mathcal{C}}(Z,\Omega) = \mathcal{C}(Z,X) \amalg \mathcal{C}(Z,Y)$.

Based on this information, we can actually construct the category $\hat{\mathcal{C}}$. As an exercise, the reader might determine for himself what $\hat{\mathcal{C}}(\Omega,\Omega)$ must be.

Now the more general problem is: given some diagram shape I and a diagram $\mathcal{D} : I \to \mathcal{C}$, how can we formally add an object to $\mathcal{C}$ which will serve as its colimit? The solution is provided by exactly the same procedure—we require that $\hat{\mathcal{C}}(\Omega, Z) = \lim_{\alpha\in I} \mathcal{C}(D_\alpha, Z)$, and that $\hat{\mathcal{C}}(Z,\Omega) = \varinjlim_{\alpha\in I} \mathcal{C}(Z, D_\alpha)$.

Using this procedure, one may theoretically go about adding all colimits to a category. The difficulty that arises, however, is that very different-looking diagrams may out of necessity have the same colimit, and we wouldn't want to mistakenly add the same colimit twice. As a simple example, the diagrams

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & & X \arrow[dl, "g"'] \arrow[dr, "h"] & \\ W & & Y & & Z \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ W & Y & Z \end{tikzcd}$$

must necessarily have the same colimit in any category. The reader may well imagine that much more complicated examples occur. Thus, a certain amount of bookkeeping must take place in order to keep track of such 'equivalent' diagrams. A very elegant solution to all this is provided by the use of presheaves.

A presheaf F may be thought of as encoding a diagram, which can be extracted in the following way. If $X \in \mathcal{C}$, write down one copy of the object X corresponding to each element in $F(X)$. If $f : X \to Y$ is a map in $\mathcal{C}$, look at the induced map $f^* : F(Y) \to F(X)$. Then for each $s \in F(Y)$, find in your diagram the copies of X and Y corresponding to $f^*(s)$ and s, respectively, and draw in the map f from X to Y. Doing this for every f yields the desired diagram. A simple example may be helpful:

Let $\mathcal{C}$ be the category with two objects X and Y, whose non-identity arrows are two distinct maps f, $g : X \to Y$. In other words, $\mathcal{C}$ is the category

$$\begin{tikzcd} \bullet \arrow[loop left, "id"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet \arrow[loop right, "id"] \end{tikzcd}$$

Let F be the presheaf for which $F(X) = \{0,1\}$, $F(Y) = \{0,1,2\}$, and $F(f)$ and $F(g)$ are the following maps $\{0,1,2\} \to \{0,1\}$:

$$\begin{tikzcd} 0 \arrow[dr] & 1 \arrow[d] & 2 \arrow[dl] \\ & 0 & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[dr] & 1 \arrow[dr] & 2 \arrow[d] \\ & 0 & 1. \end{tikzcd}$$

The resulting diagram (omitting identity maps) is

$$\begin{tikzcd} & Y & \\ X \arrow[ur, bend left, "g"] \arrow[ur, "f"'] \arrow[r, "f"'] \arrow[dr, "f"'] & Y & X \arrow[l, "g"] \arrow[dl, "g"] \\ & Y. & \end{tikzcd}$$

Formally, the diagram we have just described is precisely the one given by the natural map $(\mathcal{C} \downarrow F) \to \mathcal{C}$, where the diagram shape $(\mathcal{C} \downarrow F)$ is the over-category of F via the embedding $\mathcal{C} \hookrightarrow Pre(\mathcal{C})$. One may now interpret part (iv) of Remark **??** as saying that after pushing this diagram into $Pre(\mathcal{C})$ the colimit is exactly the presheaf F that we started with. F can be thought of as the object which has been formally added to the category $\mathcal{C}$ in order to serve as this colimit.

Exercise 2.2.7. It is not true that every diagram in $\mathcal{C}$ comes from a presheaf in this way. It can be shown, however, that every diagram is 'equivalent' to one which comes from a presheaf. Figure out how to make sense of this statement. (*Hint: What diagram corresponds to the presheaf rX?*)

Exercise 2.2.8. In the discussion above it was argued that the formal colimit Ω of a diagram D should have the property that for $X \in \mathcal{C}$, $\hat{\mathcal{C}}(\Omega, X) = \lim_\alpha \mathcal{C}(D_\alpha, X)$ and that $\hat{\mathcal{C}}(X, \Omega) = \varinjlim_\alpha \mathcal{C}(X, D_\alpha)$. Verify that a presheaf F does indeed behave as the formal colimit of the diagram it represents.

This discussion has perhaps been more lengthy than is justified, considering how formal all this machinery actually is. The reader is encouraged to read through the following proof, to see how these concepts play out in real life.

Proof of Proposition 2.2.4. Since any presheaf F may be canonically written as a colimit of representables

$$F = \operatorname*{colim}_{rX \to F} rX,$$

then if we want Re to be colimit-preserving we are forced to define $Re(F) = \operatorname*{colim}_{rX \to F} \gamma(X)$. (Here we mean just to pick arbitrarily one object representing this colimit; however if F had the form rZ, then we will go out of our way to pick the object $\gamma(Z)$, and not just something isomorphic to it). The universal property of colimits allows us to extend Re to maps, thus giving a functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$— in essence, this is the functor that takes the instructions for building a colimit and actually *builds* it (Re is short for 'realization').

The hard part of the proof is to show that Re preserves colimits. The reader may wish to think about what this means in terms of diagrams and formal colimits; it's a bit difficult to put into words. As an example, if $\{I_\alpha\}$ is a diagram of sets with colimit I and if $d \in \mathcal{D}$, we must show that the coproduct $I \cdot d$ coincides with $\varinjlim_\alpha (I_\alpha \cdot d)$ (and this is by far the simplest of examples—the reader is encouraged to try and prove it by naive methods).

There turns out to be a trick for showing this, which is to realize that Re is actually a left-adjoint (and therefore must preserve colimits). We may define $S : \mathcal{D} \to Pre(\mathcal{C})$ to be the functor sending the object $d \in \mathcal{D}$ to the presheaf $X \to \mathcal{D}(\gamma(X), d)$. The reader may check that Re and S are indeed an adjoint pair.

As the final step, we merely remark that the uniqueness part of the proposition follows directly from the uniqueness property of colimits. □

Remark 2.2.9.

(a) When $\mathcal{C}$ is the simplicial category Δ, $Pre(\mathcal{C})$ is precisely the category $s\mathcal{S}et$. So simplicial sets are nothing other than the formal colimits built from diagrams of the basic geometric simplices.

(b) It has already been mentioned that Re stands for 'realization'; in turn, the S appearing in the above proof is a kind of 'singular functor'. The reader may confirm that in the case $\mathcal{C} = \Delta$ and $\mathcal{D} = \mathcal{T}op$ these functors coincide with the usual ones which assume those names.

(c) If $\mathcal{C} = \Delta$ and $\mathcal{D} = \mathcal{C}at$ (the category of small categories), let $\mathcal{C} \to \mathcal{D}$ be the map which regards the ordered set $\{0, 1, \cdots, n\}$ as a *category* (as may be done with any poset). In this case $Pre(\mathcal{C}) = s\mathcal{S}et$ and the singular functor $S : \mathcal{C}at \to s\mathcal{S}et$ may be identified with the functor sending a category to its nerve. (This nice example was pointed out to us by Tibor Beke.)

(d) If $F : \mathcal{C} \to \mathcal{D}$ is a functor between categories, the composite $\mathcal{C} \to \mathcal{D} \to Pre(\mathcal{D})$ is a map from $\mathcal{C}$ to a co-complete category. The proposition tells us that it extends to a pair of adjoint functors $F^* : Pre(\mathcal{C}) \rightleftarrows Pre(\mathcal{D}) : F_*$.

2.3. Sheaves.

The presheaf functor gives a way of embedding any category into one that is co-complete. But if we apply this to the category of manifolds, for instance, what happens is that we lose the underlying 'geometry' which made manifolds interesting in the first place. The point is that the process of formally adding all colimits also destroys whatever colimits we might have already had.

As an example, consider the category of manifolds and let M_1 and M_2 be two objects. These already have a coproduct in our original category, namely the disjoint union $M_1 \cup M_2$. But if we embed everything in the presheaf category then rM_1 and rM_2 (i.e. the 'new' copies of M_1 and M_2) now have a 'formal' coproduct $rM_1 \amalg rM_2$, and this is *not* the same as $r(M_1 \cup M_2)$. To get a sense of the difference, let's compute the set of maps from S^0 into both objects. Mapping S^0 into $M_1 \cup M_2$ is equivalent to just picking two points on $M_1 \cup M_2$: they may both be on M_1, both on M_2, or one on each. On the other hand, we may compute that

$$\begin{aligned} Pre(rS^0, rM_1 \amalg rM_2) &= Pre(rS^0, rM_1) \amalg Pre(rS^0, rM_2) \\ &= \mathcal{C}(S^0, M_1) \amalg \mathcal{C}(S^0, M_2). \end{aligned}$$

Thus, mapping S^0 into $rM_1 \amalg rM_2$ is equivalent to giving either two points on M_1 or two points on M_2. Note the difference!

It is clear that the original $M_1 \cup M_2$ is the 'right' coproduct—it is the coproduct which geometry gives us. Thus, the upshot is that in passing to the presheaf category we have exchanged our interesting coproduct $M_1 \cup M_2$ for a formal and uninteresting one $rM_1 \amalg rM_2$. This is only the most basic example; in general any colimit of manifolds in which geometry should play a role (for instance, gluing two manifolds along an open set) is replaced in the presheaf category by a colimit which has no interesting information in it. This is what we meant in saying that the presheaf functor loses the underlying geometry of our category. The goal of this section will be to replace the act of taking presheaves by something which preserves this geometry.

There turns out to be an elegant method for tackling this, developed by Gothendieck. The idea is that we give ourselves a collection of cones $\{D_\alpha \to X\}$ in $\mathcal{C}$ (where by 'cone' we mean a diagram with a terminal vertex) which we want to become colimit cones in our expanded category $\hat{\mathcal{C}}$. For instance, the above example said that when we expand the category of manifolds we would still like the following cone to be a colimit:

$$\begin{tikzcd} M_1 \arrow[dr] & & M_2 \arrow[dl] \\ & M_1 \dot{\cup} M_2. & \end{tikzcd}$$

So our goal becomes that of producing a co-complete category $\hat{\mathcal{C}}$ admitting a map $\mathcal{C} \to \hat{\mathcal{C}}$ which sends our cones to colimit cones, and which is universal with respect to these properties. Grothendieck showed that under certain hypotheses on the collection of cones one can indeed find such a category, and this is what he called the category of sheaves. Requiring that these distinguished cones become colimits may loosely be thought of as imposing 'relations'.

Example 2.3.1. Consider again the category of topological manifolds. If $\{U_\alpha\}$ is an open cover of a manifold M, then geometric considerations show that M can be built by gluing together all the U_α's along their intersections. In other words, the following is a coequalizer diagram

$$\coprod_{\beta,\gamma} U_\beta \cap U_\gamma \rightrightarrows \coprod_\alpha U_\alpha \longrightarrow M$$

(the two parallel arrows are induced by the inclusions of $U_\beta \cap U_\gamma$ into U_β and U_γ).

The collection of cones of the above form turns out to be sufficient for encoding the essential geometry in our category of manifolds. Grothendieck realized that by generalizing the notion of 'cover' one could utilize this basic method to produce a sufficient collection of cones in other categories (see Example 2.3.8). He was thereby led to the following

Definition 2.3.2. *A* ***Grothendieck topology*** *on a category* $\mathcal{C}$ *is an assignment* $\tau : \mathrm{ob}(\mathcal{C}) \to \mathcal{S}et$ *such that every element of* $\tau(X)$ *is a subset of* $\mathrm{ob}(\mathcal{C} \downarrow X)$. *(Thus, to each object* X *we associate a family of* ***covers*** $\{U_\alpha \to X\}$.*) We require the following properties:*

- *If* $f : Y \to X$ *is an isomorphism then* $\{Y \to X\}$ *is a cover of* X.
- *If* $\{U_\alpha \to X\}$ *is a cover of* X *and* $\{V_{\alpha\beta} \to U_\alpha\}$ *are covers of each* U_α, *then the collection of composites* $V_{\alpha\beta} \to X$ *is a cover of* X.
- *If* $f : Y \to X$ *and* $\{U_\alpha \to X\}$ *is a cover, then each* $Y \underset{X}{\times} U_\alpha$ *exists and* $\{Y \underset{X}{\times} U_\alpha \to Y\}$ *is a cover.*

A ***Grothendieck site*** *is a small category equipped with a Grothendieck topology.*

Remark 2.3.3. The reasoning behind the above definition will become clear in time (we hope). The reader may find it helpful to check that the usual notion of 'open cover' as in Example 2.3.1 does indeed provide a Grothendieck topology for topological manifolds (or, more generally, for topological spaces). For the moment, the main thing to keep in mind is that each cover $\{U_\alpha \to X\}$ in a Grothendieck topology gives rise to a cone

$$\coprod_{\beta,\gamma} U_\beta \underset{X}{\times} U_\gamma \rightrightarrows \coprod_{\alpha} U_\alpha \longrightarrow X,$$

and that these are the cones which we will want to become colimits. (For an explanation of this mysterious notation, see the glossary.)

Our main goal in this section is the following result:

Proposition 2.3.4. *Let* $\mathcal{C}$ *be a Grothendieck site. Then there exists a co-complete category* $Shv(\mathcal{C})$ *and a functor* $\mathcal{C} \xrightarrow{r} Shv(\mathcal{C})$ *such that* r *takes the distinguished cones of* $\mathcal{C}$ *to colimit cones in* $Shv(\mathcal{C})$. *Moreover,* $Shv(\mathcal{C})$ *has the following universal property:*
If $\mathcal{D}$ *is a co-complete category and* $\gamma : \mathcal{C} \to \mathcal{D}$ *a map taking distinguished cones to colimits, then* γ *admits a colimit-preserving factorization*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & Shv(\mathcal{C}) \arrow[d, dotted] \\ & \mathcal{D}. \end{tikzcd}$$

Any two such factorizations admit a unique isomorphism between them.

We shall actually give an explicit construction of $Shv(\mathcal{C})$. The idea will be to first embed $\mathcal{C}$ into $Pre(\mathcal{C})$, and then to modify $Pre(\mathcal{C})$ to somehow force our distinguished cones to become colimits. To this end, let us introduce the following terminology: given a cone $\{A_\alpha \to X\}$ in a category $\mathcal{D}$, an object $Z \in \mathcal{D}$ *sees* $\{A_\alpha \to X\}$ *as a colimit* if $\mathcal{D}(X, Z) = \lim_\alpha \mathcal{D}(A_\alpha, Z)$. Note that $\{A_\alpha \to X\}$ is an actual colimit precisely when this equality holds for *every* object $Z \in \mathcal{D}$.

We can now say what a sheaf is:

Definition 2.3.5. *When* $\mathcal{C}$ *is a category with a Grothendieck topology, a* ***sheaf*** *on* $\mathcal{C}$ *is a presheaf* $F \in Pre(\mathcal{C})$ *which sees all the distinguished cones as colimits. This means that for every cover* $\{U_\alpha \to X\}$ *the following is an equalizer diagram:*

$$Pre(X, F) \longrightarrow \prod_\alpha Pre(U_\alpha, F) \rightrightarrows \prod_{\beta,\gamma} Pre(U_\beta \underset{X}{\times} U_\gamma, F).$$

$Shv(\mathcal{C})$ *is the full subcategory of* $Pre(\mathcal{C})$ *whose objects are the sheaves.*

Remark 2.3.6. The usual way to phrase the sheaf condition is to use the identification $Pre(rY, F) = F(Y)$, so that the diagram that is required to be an equalizer becomes

$$F(X) \longrightarrow \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} F(U_\beta \underset{X}{\times} U_\gamma).$$

We may also phrase the condition somewhat differently. If $\underrightarrow{\mathrm{colim}}\, U_\bullet$ denotes the coequalizer of

$$\coprod_{\beta,\gamma} U_\beta \underset{X}{\times} U_\gamma \rightrightarrows \coprod_{\alpha} U_\alpha$$

in the category of *presheaves* (which is co-complete), then there is an induced map $\underrightarrow{\mathrm{colim}}\, U_\bullet \to X$. The condition that F sees the distinguished cones as colimits is equivalent to saying that F sees the maps $\underrightarrow{\mathrm{colim}}\, U_\bullet \to X$ as isomorphisms, in the sense that $Pre(\underrightarrow{\mathrm{colim}}\, U_\bullet, F) = Pre(X, F)$. One could say that sheaves are the 'local objects' with respect to the collection of maps $\{\underrightarrow{\mathrm{colim}}\, U_\bullet \to X\}$.

Using this phrasing, we can interpret $Shv(\mathcal{C})$ as a kind of localization of the category $Pre(\mathcal{C})$. $Shv(\mathcal{C})$ is a co-complete category equipped with a colimit-preserving functor $Pre(\mathcal{C}) \to Shv(\mathcal{C})$, this map carries each $\underrightarrow{\mathrm{colim}}\, U_\bullet \to X$ to an isomorphism, and $Shv(\mathcal{C})$ is the universal object with respect to these properties. This statement is a combination of Propositions 2.3.4 and 2.3.7 (below). It says that $Shv(\mathcal{C})$ is a localization of $Pre(\mathcal{C})$ in the category of co-complete categories.

Notice that the notion of sheaf could just as well have been defined for any category with a distinguished collection of cones—it doesn't depend in any way on the special properties of a Grothendieck topology. However, these properties are used heavily in the following proposition, which is the key to the whole construction.

Proposition 2.3.7. *Let $\mathcal{C}$ be a Grothendieck site.*
(a) The inclusion of categories $Shv(\mathcal{C}) \hookrightarrow Pre(\mathcal{C})$ has a left adjoint L called ***sheafification*** *(or the associated sheaf functor).*
(b) The composite $\mathcal{C} \xrightarrow{r} Pre(\mathcal{C}) \xrightarrow{L} Shv(\mathcal{C})$ takes distinguished cones to colimit cones.
(c) The unit $Id \to L$ of the adjunction in (a) is such that $F \to LF$ is an isomorphism precisely when F is a sheaf.

The statements in (b) and (c) are formal consequences of the existence of the associated sheaf functor, so that the real content of this proposition is in (a). Assuming these results for the moment, we show how they imply Proposition 2.3.4.

Proof of 2.3.4. First observe that $Shv(\mathcal{C})$ is co-complete, because the left-adjoint L creates colimits. In other words, if $\{D_\alpha\}$ is a diagram in $Shv(\mathcal{C})$ then we regard it as a diagram in $Pre(\mathcal{C})$, take its colimit in that category, and apply L to it— this object will be the colimit in $Shv(\mathcal{C})$.

Now consider the composite $\mathcal{C} \xrightarrow{r} Pre(\mathcal{C}) \xrightarrow{L} Shv(\mathcal{C})$. By abuse of notation we will also call this r, and rX is called the sheaf *represented by* X. Since L takes distinguished cones to colimit cones, r clearly also has this property.

Finally, let $\mathcal{D}$ be a co-complete category and $\gamma : \mathcal{C} \to \mathcal{D}$ be a functor taking distinguished cones to colimits. We duplicate the proof of Proposition 2.2.4 to produce the required factorization. The reader may verify that, just as for presheaves, any sheaf F may be canonically written as a colimit of representable sheaves,

$$F = \operatorname*{colim}_{rX \xrightarrow{\phi} F} (rX)_\phi.$$

This depends heavily on the properties of L outlined in the above theorem.

Using this fact, we may again define a functor $Re : Shv(\mathcal{C}) \to \mathcal{D}$ by

$$F \mapsto \operatorname*{colim}_{rX \xrightarrow{\phi} F} \gamma(X).$$

To show that this preserves colimits, recall from Proposition 2.2.4 that there is a functor $S : \mathcal{D} \to Pre(\mathcal{C})$ where $S(d)$ is the presheaf $c \mapsto \mathcal{D}(\gamma(c), d)$. The assumption that γ takes distinguished cones to colimits says precisely that every $S(d)$ is in fact a *sheaf*, and one may easily check that S is right-adjoint to Re. Therefore Re, being a left-adjoint, is colimit-preserving. □

Example 2.3.8. (The étale topology).

Let $\mathcal{C}$ be the category of topological manifolds with the Grothendieck topology given by open coverings. If $E \to X$ is a covering space, one knows that in $\mathcal{T}op$ the diagram

$$E \underset{X}{\times} E \rightrightarrows E \longrightarrow X$$

is a coequalizer. Using the fact that covering spaces are local homeomorphisms, it can be shown that the corresponding diagram in $Shv(\mathcal{C})$ is also a coequalizer. This confirms that sheaves are behaving the same way spaces do, at least in this particular way.

Proceeding analagously, consider the category of schemes with the Grothendieck topology given by Zariski covers. The analogues of covering spaces are the finite étale maps, but they are not local isomorphisms. It can easily be checked that if $E \to X$ is étale then the diagram of representable (Zariski-) sheaves

$$E \underset{X}{\times} E \rightrightarrows E \longrightarrow X$$

is usually *not* a coequalizer. (The simplest example is the case where E and X are the spectra of fields). Grothendieck realized that by forcing these diagrams to be coequalizers he could produce a category whose objects behaved more like topological spaces, and in particular admitted a 'reasonable' cohomology theory. This will be discussed futher in section ???.

Of course, the way to force the diagrams to be coequalizers is to include them in the Grothendieck topology. The *étale topology* on the category of schemes is the smallest topology whose covers include the Zariski coverings $\{U_\alpha \to X\}$ and the finite étale maps $\{E \to X\}$.

Exercise 2.3.9. Recall that a map of topological spaces $E \xrightarrow{p} B$ is called a quotient map if it is surjective and has the property that U is open in B iff $p^{-1}(U)$ is open in E. Show that being a quotient map implies that $E \times_B E \rightrightarrows E \longrightarrow B$ is a coequalizer. (Since fibre bundles—and in particular, covering spaces—are quotient maps, this proves the first statement in the above example).

One might ask for a Grothendieck topology on spaces for which the covers are quotient maps $E \to B$. The difficulty is that the pullback of a quotient map is generally *not* another quotient map (example???). Show that one *can* get a Grothendieck topology by weakening the notion of cover slightly, by defining covers to be the singleton sets $\{E \to B\}$ where $E \to B$ is an open surjection.

The following example gives another possible way of 'fixing' the notion of quotient space in order to get a Grothendieck topology.

Example 2.3.10. (Voevodsky's h-topology)

A map of schemes $X \to Y$ is called a *topological epimorphism* if the underlying map of topological spaces is a quotient map. (Be warned that an epimorphism in the category of topological spaces is simply a surjective continuous map, so the above terminology is perhaps not completely appropriate). A *universal topological epimorphism* is a topological epimorphism $X \to Y$ with the property that for any $Z \to Y$ the pullback $Z \times_Y X \to Z$ is again a topolgical epimorphism. The **h-topology** on the category of schemes is the Grothendieck topology whose covers are the singleton sets $\{X \to Y\}$ where $X \to Y$ is a universal topological epimorphism.

2.4. **The sheafification functor.**

We end this section with a construction of the associated sheaf functor and a proof of Proposition 2.3.7. For the traditional approach, see [1] or [11]; our description will be more along homotopy-theoretic lines, using the small object argument. Only a sketch will be given, with many of the details left as an exercise.

Two bits of notation will be extremely helpful. If F is a presheaf, $s \in F(X)$, and $Y \longrightarrow X$ is a map in $\mathcal{C}$, we will write $s|_Y$ for the image of s under $F(X) \longrightarrow F(Y)$. And if $\{U_\alpha \to X\}$ is a cover, we will write $U_{\alpha\beta}$ for the pullback $U_\alpha \underset{X}{\times} U_\beta$, $U_{\alpha\beta\gamma}$ for $U_\alpha \underset{X}{\times} U_\beta \underset{X}{\times} U_\gamma$, etc. Using this notation, a presheaf F is a sheaf provided for every cover $\{U_\alpha \to X\}$ it is true that:

(1) two elements $s, t \in F(X)$ which have the same image in each $F(U_\alpha)$ are themselves equal;
(2) any collection of elements $s_\alpha \in F(U_\alpha)$ satisfying $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$ extend to an element $s \in F(X)$.

These properties say precisely that $F(X) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_{\alpha\beta})$ is an equalizer. For us, property (2) will be the more interesting one. If we write $\underrightarrow{\operatorname{colim}}\, U_\bullet$ for the coequalizer of

$$\coprod_{\beta,\gamma} U_\beta \underset{X}{\times} U_\gamma \rightrightarrows \coprod_\alpha U_\alpha,$$

in the category of *presheaves*, it may be checked that giving a collection of elements $s_\alpha \in F(U_\alpha)$ which agree on the $U_{\alpha\beta}$ is the same as giving a map $\underrightarrow{\operatorname{colim}}\, U_\bullet \to F$. Hence, property (2) is equivalent to the statement that any diagram

$$\begin{tikzcd} \underrightarrow{\operatorname{colim}}\, U_\bullet \arrow[r] \arrow[d] & F \\ X \arrow[ur, dotted] & \end{tikzcd}$$

admits a lifting as shown. Modifying a presheaf F to make it a sheaf will involve imposing property (1) together with this lifting property, and in the case of the latter we are on familiar ground—we use the small object argument.

Given a presheaf F, the idea will be to introduce two new presheafs $\mathcal{A}(F)$ and $\mathcal{B}(F)$ which come about by formally imposing properties (1) and (2), respectively. To this end, define an equivalence relation on each set $F(X)$ by saying that $s \sim t$ if there is a cover $\{U_\alpha \to X\}$ for which $s|_{U_\alpha} = t|_{U_\alpha}$.

Exercise 2.4.1. Verify that $\sim$ is an equivalence relation, and show that one may define a presheaf $\mathcal{A}(F)$ whose value at X is $F(X)/\sim$ (both facts depend heavily on the axioms for a Grothendieck topology). Observe that there is a natural map $F \to \mathcal{A}(F)$.

For the lifting property, let $\mathcal{B}(F)$ be the pushout

$$\begin{tikzcd} \coprod \underrightarrow{\operatorname{colim}}\, U_\bullet \arrow[r] \arrow[d] & F \arrow[d, dotted] \\ \coprod X \arrow[r, dotted] & \mathcal{B}(F), \end{tikzcd}$$

where the coproducts are taken over all diagrams

$$\begin{tikzcd} \underrightarrow{\operatorname{colim}}\, U_\bullet \arrow[r] \arrow[d] & F \\ X. & \end{tikzcd}$$

Now since the category $\mathcal{C}$ is small, we may choose a regular cardinal λ which is larger than the set of all maps in $\mathcal{C}$. This gives a bound on the size of our covers—the elements in a cover can't possibly form a set of cardinality greater than λ. We now let $\tilde{F}$ be the λ-transfinite colimit of

$$F \to \mathcal{A}(F) \to \mathcal{B}\mathcal{A}(F) \to \mathcal{A}\mathcal{B}\mathcal{A}(F) \to \mathcal{B}\mathcal{A}\mathcal{B}\mathcal{A}(F) \dots$$

(See [6] for background on the transfinite small object argument).

Exercise 2.4.2. Verify that $\tilde{F}$ satisfies properties (1) and (2).

$F \mapsto \tilde{F}$ will be our sheafification functor. Establishing that it has the desired properties is a routine

Exercise 2.4.3. If G is a sheaf, show that a presheaf map $F \to G$ admits unique extensions to both $\mathcal{A}(F)$ and $\mathcal{B}(F)$. Conclude that $F \mapsto \tilde{F}$ is a left-adjoint to the inclusion $Shv(\mathcal{C}) \hookrightarrow Pre(\mathcal{C})$. Use this to prove parts (b) and (c) of Proposition 2.3.7.

Remark 2.4.4. The reader may be wondering about the appearance of the construction $\mathcal{A}(F)$ in the above. Certainly $\mathcal{B}(F)$ is very familiar, being the usual construction attached to any small object argument; but $\mathcal{A}(F)$ at first seems mysterious. The answer, in very vague terms, is that we are really only seeing the 0-simplex level of a whole simplicial construction. This will be discussed in the next section, so for now let us just give an analogy. When dealing with a map of simplicial sets, one can force the map to be surjective on π_i by attaching i-cells to the domain (i.e., by imposing a lifting property), and can force the maps on π_i to be *injective* by attaching $(i+1)$-cells (which also amounts to imposing a lifting property). The important fact is that for injectivity one has to go up a dimension.

Looking back at our construction of $\tilde{F}$, $\mathcal{B}(F)$ is imposing a kind of *surjectivity*—it helps produce *some* lifting. On the other hand, $\mathcal{A}(F)$ is imposing a kind of injectivity—it helps ensure that the liftings are *unique*. Because there is no dimension to 'go up to' in this setting, we have to do something unfamiliar. Instead of attaching cells to impose relations in homotopy, identifications are made on the nose. In fact, the axioms of a Grothendieck topology are set up precisely so that we can get away with this—notice that the ability to build $\mathcal{A}(F)$ depended heavily on these axioms, and this was the only place they were used in the construction of $\tilde{F}$. (Also note, however, that an important property of the sheafification functor is that it preserves finite limits, and this relies on the axioms of a Grothendieck topology in an essential way).

In the next section we will investigate sheaves in the simplicial setting, and will find that sheafification functors can be produced in contexts more general than Grothendieck topologies. Grothendieck topologies will still have a role, but it will be much less central—they will ensure that there are no 'lower derived functors' of a sections-functor.

Exercise 2.4.5. Use of the full power of the small object argument to construct the sheafification functor was in some ways an overkill. Show that $\mathcal{AB}(F)$ coincides with the presheaf F^+ (notation as in [1]), and conclude that $\mathcal{ABAB}(F)$ is already a sheaf. In other words, the small object construction can in this case be terminated after just a few steps.

2.5. **Some miscellaneous remarks on sheaves.**

Remark 2.5.1. A Grothendieck topology is called *subcanonical* if every representable presheaf is a sheaf. This is equivalent to requiring that all the distinguished cones in $\mathcal{C}$ are already colimit cones in $\mathcal{C}$. In other words, a subcanonical topology is not introducing any *new* colimit-type relations when we pass from $\mathcal{C}$ to $Shv(\mathcal{C})$; instead, it is only requiring that certain colimit-relations from $\mathcal{C}$ be *preserved*.

The open covering topology for spaces, as well as the Zariski and étale topologies for schemes, are all subcanonical. Voevodsky's h-topology is not subcanonical.

Remark 2.5.2. This section addressed two basic problems: that of co-completing a category in a universal way, and that of imposing colimit-type relations on this universal construction. The entire discussion may be dualized to obtain a method for *completing* a category and imposing *limit-type* relations. For instance, $[Pre(\mathcal{C}^{op})]^{op}$ (perhaps better written $(\mathcal{S}et^{\mathcal{C}})^{op}$) may be interpreted as the category obtained from $\mathcal{C}$ by formally adding all limits. We leave it to the reader to re-formulate all of our other results in this setting. For the most part we will have no need of the theory in this paper, but the general perspective will appear briefly in our discussion of 'points' (section 4).

3. Homotopy-Theoretic Sheaves

3.1. **Introduction.** The last section's approach to sheaf theory is very reminiscent of the machinery homotopy theorists know as *localization*. Recall that we were interested in modifying the category $Pre(\mathcal{C})$ so that certain diagrams would become colimits, but in such a way that the resulting category would remain co-complete. As in Remark 2.3.6, the requirement that our diagrams become colimits can be rephrased in terms of certain maps becoming isomorphisms. Sheaves appeared as the objects which were 'local' with respect to these maps, and we showed how an arbitrary presheaf could be functorially replaced with a sheaf.

Of course this exactly parallels the process of localizing a model category, except that in homotopy theory we would have the phrase 'weak equivalence' in place of 'isomorphism'. Via this comparison, sheafification becomes the analogue of the localization functor. Sheafification is a left adjoint, and localization is a left adjoint up to homotopy. In this section we will make these connections precise by constructing a homotopy-theoretic notion of 'sheaf'.

In the last section we started with a category $\mathcal{C}$ and sought to co-complete it, with the proviso that certain distinguished cones were to become colimits. Here we will have the more daring goal of expanding $\mathcal{C}$ into a category where the objects have a homotopy-type, and will require not that certain diagrams become colimits, but that they become *homotopy* colimits. Just as sheaves solved the former problem, our homotopy-theoretic sheaves will be the solution to the latter.

More specifically: if $\mathcal{C}$ denotes a small category endowed with a Grothendieck topology our task will be to construct a model category $\mathcal{M}$ equipped with a functor $\theta\colon \mathcal{C} \to \mathcal{M}$. We will require the property that for each cover $\{U_\bullet \to X\}$ the natural map

$$\underrightarrow{\text{hocolim}} \Big[\cdots \textstyle\coprod_{\alpha\beta\gamma} \theta(U_{\alpha\beta\gamma}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \coprod_{\alpha,\beta} \theta(U_{\alpha\beta}) \substack{\longrightarrow\\[-1ex]\longrightarrow} \coprod_{\alpha} \theta(U_\alpha) \Big] \longrightarrow \theta(X). \tag{3.1}$$

is a weak equivalence, and the hope will be to produce the most general possible $\mathcal{M}$ for which this is true.

The basic idea of the construction is fairly simple. In order to transform $\mathcal{C}$ into a model category, we must first alter the category so that it is has all limits and colimits. There are two methods for doing this in a universal manner: we may add formal colimits by passing to $Pre(\mathcal{C})$ or we may add formal limits by passing to the category $(\mathcal{S}et^{\mathcal{C}})^{op} = [Pre(\mathcal{C}^{op})]^{op}$. Since we are interested in colimit-type properties, we choose the former route.

The next step is to note that any model category admits a *framing*, which is something like an action of $s\mathcal{S}et$ (cf. [6]). So we now take the category $Pre(\mathcal{C})$ and formally add objects $X_\bullet \otimes F$ for $X_\bullet \in s\mathcal{S}et$ and $F \in Pre(\mathcal{C})$. This is tantamount to looking at the category of *simplicial* presheaves $sPre(\mathcal{C})$. There is a natural choice of weak equivalences in this category, and at least two Quillen-equivalent model structures. We will think of these as representing something like a 'universal homotopy-theory' built on the category $\mathcal{C}$ (for precision, see ???? and ????).

Finally, we are left with the task of forcing the objects rX to be the homotopy colimits of the Čech diagrams built from their covers. This is accomplished by localizing $sPre(\mathcal{C})$ at an obvious set of maps. The model category thus obtained will be called the *Čech model category* on $sPre(\mathcal{C})$. We will end the section by discussing a few of its basic properties.

Why are we interested in imposing the property in (3.1)? The motivation of course comes from the fact that it holds for topological spaces, and that it appears as a key element in several applications. Perhaps it's worth stepping aside a moment to recall some of these, before we embark on the long journey through the machinery.

Example 3.1.1.
For an open covering $\{U_\alpha \to X\}$, let $\check{U}_\bullet$ denote the simplicial space which has $\coprod_{i_0,\dots,i_n} U_{i_0\cdots i_n}$ in dimension n, with the obvious face and degneracy maps.

(a) (Čech cohomology.) When $\{U_\alpha\}$ is an open covering of a space X, one may form an associated simplicial complex $\check{N}$ called the *Čech nerve* of $\{U_\alpha\}$. Recall that this complex is constructed by starting with a 0-simplex for every nonempty U_α, attaching a 1-simplex for each nonempty $U_{\alpha\beta}$, a 2-simplex for each nonempty $U_{\alpha\beta\gamma}$, etc. The following result may be found in discussions of Čech cohomology (we assume that X and all the $U_{i_0\cdots i_n}$ are cofibrant spaces):

If there is a number m such that the iterated intersections $U_{i_0\cdots i_k}$ are all contractible for $k \leq m$, then $H^k_{\text{simp}}(\check{N}) \cong H^k(X)$ for $k \leq m$.

Note that if X is triangulated then the open star covering associated to the mth barycentric subdivision has the above property. It follows (with some work) that if X is triangulable then there is an isomorphism

$$H^k(X) \cong \operatorname*{colim}_{U_\alpha \to X} H^k_{\text{simp}}(\check{N}_U)$$

where the colimit is taken over all covers $\{U_\alpha\}$, indexed by refinement. In other words, Čech cohomology agrees with singular cohomology for triangulable spaces.

Property $(*)$ can be used to give a very easy proof of the above result. In fact, we can do more: not only does the Čech nerve provide a good approximation to the cohomology of X, it actually provides a good approximation of the homotopy as well. We will produce a homotopy class of maps $X \to \check{N}$ which induces isomorphisms $\pi_k(X) \to \pi_k(\check{N})$ for $k < m$ and a surjection $\pi_m(X) \to \pi_m(\check{N})$. The above isomorphism on cohomology will then follow from the Whitehead theorems.

To accomplish this, let us regard $\check{N}$ as a simplicial *space* which is discrete in every dimension (so that the space in level k is just the set consisting of one point for every nonempty $U_{i_0\cdots i_k}$). Then there is an obvious map of simplicial spaces $\check{U}_\bullet \to \check{N}$ which collapses every nonempty $U_{i_0\cdots i_k}$ to a point. Since the $U_{i_0\cdots i_k}$ are contractible for $k \leq m$, this map is an objectwise weak equivalence on the m-skeletons $\operatorname{sk}_m \check{U}_\bullet \to \operatorname{sk}_m \check{N}$. All the simplicial spaces involved are Reedy cofibrant, so this also induces a weak equivalence on the realizations. In other words, we have a square

$$\begin{tikzcd}
{|\check{U}_\bullet|} \arrow[r] & {|\check{N}|} \\
{|\operatorname{sk}_m \check{U}_\bullet|} \arrow[u] \arrow[r, "\sim"] & {|\operatorname{sk}_m \check{N}|} \arrow[u]
\end{tikzcd}$$

Now one can show for any (Reedy cofibrant?) simplicial space $Z_\bullet$ that $|sk_m Z_\bullet| \to |Z_\bullet|$ induces isomorphisms on π_i for $i < m$ and a surjection on π_m, and so we conclude that $|\check{U}_\bullet| \to |\check{N}|$ also has this property. The fact that there is a weak equivalence $|\check{U}_\bullet| \to X$ now yields the desired result.

(b) (Vector bundles and classifying spaces.) Let X be a space and let E be a principle G-bundle over X (G a topological group). Choose an open cover $\{U_i \to X\}$ over which E is trivial. As usual, a choice of trivializations $h_i \colon E|_{U_i} \to G \times U_i$ yields transition functions $g_{ij} \colon U_{ij} \to G$ which encode the rule for changing from U_j-coordinates to U_i-coordinates. In detail, the g_{ij} are defined so that for $x \in U_{ij}$

$$(h_i \circ h_j^{-1})(g, x) = (g_{ij}(x) \cdot g, x).$$

The cocycle condition $g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x)$ implies that these fit into a map of simplicial spaces

$$\begin{tikzcd}
\coprod_i U_i \arrow[d] & \coprod_{ij} U_{ij} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \coprod_{ijk} U_{ijk} \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d] \\
* & G \arrow[l, shift left] \arrow[l, shift right] & G \times G \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2]
\end{tikzcd}$$

Here the lower object is the usual bar construction for G, and the maps $g_{i_1\cdots i_k}\colon U_{i_1\cdots i_k} \to G\times\cdots\times G$ are defined by

$$g_{i_1\cdots i_k}(x) = (g_{i_1i_2}(x), g_{i_2i_3}(x), \cdots, g_{i_{k-1}i_k}(x)).$$

Taking homotopy colimits of the rows in the above ladder yields a map $X \to BG$; this will of course be the map classifying the bundle E. These ideas can be pushed further to obtain the following:

Let $\underline{s\mathcal{T}op}(\check{U}_\bullet, BG)$ denote the function space between the simplicial spaces $\check{U}_\bullet$ and BG. Let $\mathcal{B}_{(X,G)}$ denote the category of principal G-bundles over X (where maps are isomorphisms), and let $\mathcal{B}_{(X,G,U_\bullet)}$ denote the subcategory consisting of G-bundles which are trivializable over each U_i.

(i) There is a natural weak equivalence $\underline{s\mathcal{T}op}(\check{U}_\bullet, BG) \xrightarrow{\sim} N(\mathcal{B}_{(X,G,U_\bullet)})$, where $N(\mathcal{C})$ denotes the nerve of the category $\mathcal{C}$.

(ii) It's possible to form a filtering system $\mathcal{J}$ consisting of open coverings of X under refinement, with the property that any open cover of X has a refinement in $\mathcal{J}$ (see Remark ???). It follows that

$$\operatorname*{colim}_{U\in\mathcal{J}}\Big[\underline{s\mathcal{T}op}(\check{U}_\bullet, BG)\Big] \xrightarrow{\sim} \operatorname*{colim}_{U\in\mathcal{J}}\Big[N(\mathcal{B}_{(X,G,U_\bullet)})\Big] \cong N(\mathcal{B}_{(X,G)}).$$

(iii) The weak equivalences $|\check{U}_\bullet| \to X$ tell us that each $\underline{s\mathcal{T}op}(\check{U}_\bullet, BG)$ is weakly equivalent to $\underline{\mathcal{T}op}(X, BG)$, and so we obtain the equivalence

$$\underline{\mathcal{T}op}(X, BG) \simeq N(\mathcal{B}_{(X,G)}).$$

Note that taking π_0 gives the usual bijection between $[X, BG]$ and isomorphism classes of G-bundles.

(c) (The bar construction.) If $E \to X$ is a covering space, we remarked in Example 2.3.8 that X is the coequalizer of $E\times_X E \rightrightarrows E$. It is in fact also true that X is the *homotopy* colimit of the simplicial space

$$E \leftleftarrows E\times_X E \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; E\times_X E\times_X E \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots$$

(at least, assuming X and E are cofibrant; the result is actually true for any fibration, assuming that E, $E\times_X E$, etc. are all cofibrant.)

As an application of this fact, consider the case where X is a $K(G,1)$ and E is its universal cover. Note that E is contractible, and that the iterated pullbacks $E\times_X\cdots\times_X E$ are homotopy discrete—i.e., they are weakly equivalent to their discrete set of path components. A little analysis reveals that $\pi_0(E\times_X E) = G$, $\pi_0(E\times_X E\times_X E) = G\times G$, etc. So we obtain a levelwise weak equivalence

$$\begin{array}{ccccccc}
E & \leftleftarrows & E\times_X E & \substack{\leftarrow\\\leftarrow\\\leftarrow} & E\times_X E\times_X E & \substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow} & \cdots\\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\
\pi_0(E) & \leftleftarrows & \pi_0(E\times_X E) & \substack{\leftarrow\\\leftarrow\\\leftarrow} & \pi_0(E\times_X E\times_X E) & \substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow} & \cdots\\
\| & & \| & & \| & & \\
* & \leftleftarrows & G & \substack{\leftarrow\\\leftarrow\\\leftarrow} & G\times G & \substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow} & \cdots
\end{array}$$

The fact that X is the homotopy colimit of the upper diagram now allows us to recover the usual bar construction for $K(G,1)$'s.

We now recall the basic machinery needed to construct our Čech model category.

3.2. Diagrams in a model category.

In section 2 it was revealed that the category $Pre(\mathcal{C}) = \mathcal{S}et^{\mathcal{C}^{op}}$ represented the universal co-complete category built from $\mathcal{C}$. The next goal is to take this further, by expanding $\mathcal{C}$ into a category where the objects actually have a homotopy type associated to them. We will see that the category of *simplicial* presheaves accomplishes this, and is in some ways the universal solution.

Recall that if $\mathcal{C}$ is a small category and $\mathcal{M}$ is a model category, then the category of diagrams $\mathcal{M}^{\mathcal{C}}$ can often be given a model structure—in fact, this can sometimes be done in more than one way. We will only be interested in the case when $\mathcal{M} = s\mathcal{S}et$, and will not state these results in their full generality (for which the reader is referred to [6].)

If D_1 and D_2 are diagrams in $s\mathcal{S}et$, let us say that a map $f\colon D_1 \to D_2$ is an **objectwise weak equivalence** (resp. **objectwise cofibration**, **objectwise fibration**) if $D_1(i) \xrightarrow{f} D_2(i)$ is a weak equivalence (resp. cofibration, fibration) for every object $i \in \mathcal{C}$. The following theorem catalogs results of [3] and [5]:

Theorem 3.2.1. *Let $\mathcal{C}$ be a small category.*

(a) (Bousfield-Kan) The category of diagrams $s\mathcal{S}et^{\mathcal{C}}$ has a model structure in which the weak equivalences and fibrations are objectwise. (The cofibrations are then the maps having the left-lifting-property with respect to arrows which are both objectwise fibrations and weak equivalences).

(b) (Heller) The category $s\mathcal{S}et^{\mathcal{C}}$ has a model structure in which the weak equivalences and cofibrations are objectwise (and the fibrations are maps with the appropriate right-lifting-property).

(c) These two model structures are Quillen-equivalent.

(d) Both model structures are compatible with the natural simplicial action on diagrams (see following remark), so that $s\mathcal{S}et^{\mathcal{C}}$ becomes a simplicial model category in these two different ways (cf. [6] *for the definition of simplicial model category).*

Remark 3.2.2. The natural simplicial action on $s\mathcal{S}et^{\mathcal{C}}$ referred to in the theorem is the one given as follows: for $K_\bullet \in s\mathcal{S}et$ and $D\colon \mathcal{C} \to s\mathcal{S}et$, $K_\bullet \otimes D$ and $D^{K_\bullet}$ are the diagrams defined by

$$(K_\bullet \otimes D)(c) = K_\bullet \times D(c) \qquad \text{and} \qquad (D^K)(c) = D(c)^K.$$

In other words, this is the *objectwise* simplicial action.

Since the objects of $\mathcal{S}et^{\mathcal{C}^{op}}$ were called 'presheaves', it will be natural for us think of the objects of $s\mathcal{S}et^{\mathcal{C}^{op}}$ as 'presheaves of simplicial sets'. These are really the same thing as simplicial objects in the category of presheaves, so they may also be referred to as 'simplicial presheaves'. We will indiscriminately go back and forth between these two ways of viewing the same objects. For instance, if F is a simplicial presheaf then we will write F_n for the presheaf which occupies the dimension n level of F, and we will also write $F(X)$ for the simplicial set which is the value of F on the object $X \in \mathcal{C}$. (So the n-simplices of $F(X)$ are the elements of $F_n(X)$.) The category of simplicial presheaves will usually be written $sPre(\mathcal{C})$ rather than $s\mathcal{S}et^{\mathcal{C}^{op}}$.

Definition 3.2.3. *$sPre(\mathcal{C})_{BK}$ and $sPre(\mathcal{C})_H$ denote the Bousfield-Kan and Heller model categories, respectively.*

Remark 3.2.4. We collect here some basic (but extremely useful) facts about simplicial presheaves. The reader may wish to compare these with the attributes of (non-simplicial) presheaves outlined in Remark 2.2.2.

(i) Any presheaf F determines a *simplicially constant presheaf* $sF_\bullet$. This is the simplicial object which has a copy of F in every dimension, and whose face and degeneracy operators are all identity maps. In particular, any representable presheaf rX yields a simplicially constant presheaf in this way, giving a map $\mathcal{C} \to sPre(\mathcal{C})$. We will usually just write rX instead of $s(rX)$.

(ii) Given a simplicial set $K_\bullet$, one may form the simplicial presheaf whose value at every X in $\mathcal{C}$ is $K_\bullet$. This will be called the *constant simplicial presheaf*, and will also be denoted $K_\bullet$ by abuse of notation. Context will always make clear whether we are talking about 'constant simplicial presheaves' or 'simplicially constant presheaves', so we will not worry too much about the precise order of the adjectives.

(iii) If F is a simplicial presheaf then $sPre(rX \times \Delta^n, F) = F_n(X)$ (isomorphism of sets). Hence, if we look at simplicial mapping spaces we find

$$\underline{sPre}(rX, F) = F(X)$$

(isomorphism of simplicial sets). This is a kind of 'simplicial Yoneda Lemma'.

(iv) Limits and colimits of simplicial presheaves are computed objectwise, just as they are for ordinary presheaves. A slightly more subtle fact is that *homotopy* limits and colimits may be computed objectwise—e.g., if $D\colon I \to sPre(\mathcal{C})$ is a diagram of simplicial presheaves then $\underrightarrow{\text{hocolim}}\, D$ is weakly equivalent to the simplicial presheaf given by $X \to \underrightarrow{\text{hocolim}}_\alpha D_\alpha(X)$. This is simply because in a simplicial model category $\underrightarrow{\text{hocolim}}\, D$ may be constructed in terms of coequalizers and simplicial actions, and in $sPre(\mathcal{C})$ these constructions are all just the objectwise versions of their parallels in $s\mathcal{S}et$.

(v) Any functorial construction in $s\mathcal{S}et$ admits an objectwise extension to $sPre$. For instance, if F is a group object in $sPre$ then each $F(X)$ is a simplicial group. Therefore $F(X)$ has a classifying space $BF(X)$ (obtained from $F(X)$ by applying the bar construction in each dimension and taking the diagonal). The assignment $X \mapsto BF(X)$ forms a simplicial presheaf which will naturally be called BF.

Other common examples: If G is a simplicial object in $sPre$ (i.e., a bisimplicial presheaf) then $\operatorname{diag} G \in sPre$ is obtained by taking the diagonal objectwise. If $F \in sPre$ then $\pi_0(F)$ is the presheaf (often regarded as a constant simplicial presheaf) obtained by applying π_0 levelwise.

(vi) An important fact about presheaves F was that they could always be written as a colimit of representables, namely

$$F \cong \operatorname*{colim}_{rX \xrightarrow{\phi} F} (rX)_\phi.$$

Any colimit may be expressed as a coequalizer in a standard way (cf. [10]), so that we could have written F as the coequalizer

$$\coprod_{rX \to rY \to F} (rX) \rightrightarrows \coprod_{rZ \to F} (rZ) \longrightarrow F.$$

This admits an elegant generalization to the simplicial setting:

Proposition 3.2.5 (Resolution by representables.)**.**

(a) Let F be a presheaf on $\mathcal{C}$, and let $Q_\bullet F$ be the simplicial presheaf whose nth level is given by

$$Q_n F = \coprod_{rX_1 \to rX_2 \to \cdots \to rX_n \to F} rX_1,$$

and whose face and degeneracy maps are the obvious candidates. $Q_\bullet F$ comes equipped with an augmentation $Q_0 F \to F$, which may be thought of as a map of simplicial presheaves $Q_\bullet F \to sF_\bullet$. This map is an objectwise weak equivalence.

(b) If $F_\bullet$ is an arbitrary simplicial presheaf, one gets a bisimplicial presheaf $Q_\bullet F_\bullet$ by applying the functor Q levelwise to $F_\bullet$. Let $(QF)_\bullet = \operatorname{diag}(Q_\bullet F_\bullet)$ (the simplicial presheaf obtained by applying the diagonal functor objectwise). Then once again there is a map $(QF)_\bullet \to F_\bullet$, and this is an objectwise weak equivalence.

Proof. In (a) we must show that for any $U \in \mathcal{C}$ the map $(Q_\bullet F)(U) \to F_\bullet(U)$ is a weak equivalence of simplicial sets. The key point is that $(Q_\bullet F)(U)$ admits an extra degeneracy (or contracting homotopy)—we leave the reader to produce this for himself, but see the following exercise if you have trouble.

(b) follows immediately from (a) in a standard way: if we let $F_{ij} = F_j$ (so that we regard F as a bisimplicial gadget whose columns are constant) then there is a map of bisimplicial presheaves $Q_\bullet F_\bullet \to F_{\bullet\bullet}$. Upon evaluating at U this becomes a map of bisimplicial sets, which by (a) is a column-wise weak equivalence. By a standard result for simplicial sets, the map also induces a weak equivalence of the diagonals. This is what we wanted. □

Exercise 3.2.6. The above resolution of a presheaf may be obtained as the resolution associated to a certain cotriple. For any object $A \in \mathcal{C}$ there are a pair of adjoint functors

$$L_A : \mathcal{S}et \rightleftarrows Pre(\mathcal{C}) : R_A$$

where $R_A(F) = F(A) = Pre(rA, F)$ and $L_A(S) = \coprod_S rA$ (i.e., the coproduct of several copies of rA, one for each element of S). Taken together as A varies, the pairs (L_A, R_A) assemble in the obvious way into a single adjunction

$$L : \mathcal{S}et^C \rightleftarrows Pre(\mathcal{C}) : R.$$

The composite LR is a cotriple, and the corresponding cotriple-resolution of a presheaf F is exactly the above resolution by representables. This can all be done just as well in the simplicial setting. The exercise it to fill in the details of all this. (Note that, as with any construction of this sort, if we apply R levelwise to the resolution then we pick up an extra degeneracy.)

Remark 3.2.7. Recall that for a presheaf F we interpreted the statement

$$F \cong \operatorname*{colim}_{rX \xrightarrow{\phi} F} (rX)_\phi$$

as saying that F was the 'formal colimit' of the diagram of representables $D\colon \mathcal{C} \downarrow F \to Pre(\mathcal{C})$. By regarding each presheaf rX as a simplicial presheaf, we may also think of this as a diagram in $sPre(\mathcal{C})$. The homotopy colimit of this diagram may easily be identified with the object $Q_\bullet F$ appearing in the above proposition. In other words, the proposition says that F also serves as the 'formal *homotopy* colimit' of this diagram of representables. The reason the colimit and the homotopy colimit are weakly equivalent in this case can be traced to the fact that the diagram in question is extremely 'thick'.

The two model structures we've introduced enjoy the following properties:
In the Bousfield-Kan category

- Every representable presheaf (constant in the simplicial direction) is cofibrant;
- Being fibrant is the same as being *objectwise* fibrant;
- Any object which is constant in the simplicial direction is fibrant (as a consequence of the previous statement).

In the Heller model category

- Every object is cofibrant (in particular, this holds for any representable);
- Being fibrant implies being objectwise fibrant, but is stronger. (There are additional diagramatic conditions involving maps being fibrations, etc.)
- Any object which is constant in the simplicial direction is fibrant.

Consider the problem of mapping a representable rX into a simplicial presheaf F. (Of the kinds of situations we need to understand, this will be the most common by far). Let's work in the Bousfield-Kan category, just so we can say something specific. Then rX is cofibrant, so we only need to choose a fibrant replacement $\tilde{F}$ for F. The function space of maps $\underline{sPre}(rX, \tilde{F})$ may be identified with $\tilde{F}(X)$, but this is weakly equivalent to $F(X)$ because $F \to \tilde{F}$ is an objectwise weak equivalence. In other words: *when we're mapping from a representable rX to F, we can compute* *the function space even when F is not fibrant. In fact, the function space has the homotopy type of $F(X)$.* This also holds in the Heller category, by the same analysis.

As the Bousfield-Kan and Heller model categories are Quillen equivalent, they represent the same underlying 'homotopy theory'. We will tend to think of this as the 'universal homotopy theory' built from the category $\mathcal{C}$. Although it's not completely clear how to make this precise, we offer the following proposition and corollary as results in this direction.

Proposition 3.2.8. *Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be a simplicial model category with a functor $\mathcal{C} \to \mathcal{M}$. Then there is a Quillen pair $L : sPre(\mathcal{C})_{BK} \rightleftarrows \mathcal{M} : R$ which sits in a (non-commuting) triangle*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & sPre(\mathcal{C})_{BK} \arrow[d, bend right, "L"'] \\ & \mathcal{M}. \arrow[u, bend right, "R"'] \end{tikzcd}$$

L and R are simplicial functors, and there is a natural transformation from $L \circ r$ to γ which is a weak equivalence on objects.

Proof. Let Q be a cofibrant-replacement functor in $\mathcal{M}$, and set $\tilde{\gamma} = Q \circ \gamma$. Since $\mathcal{M}$ is co-complete, $\tilde{\gamma}$ induces an adjoint pair

$$\begin{tikzcd} Pre(\mathcal{C}) \arrow[r, bend left, "Re"] & \mathcal{M}. \arrow[l, bend left, "S"] \end{tikzcd}$$

Adjoint pairs always extend to the categories of simplicial objects, so there is also an adjunction

$$\begin{tikzcd} sPre(\mathcal{C}) \arrow[r, bend left, "Re"] & s\mathcal{M}. \arrow[l, bend left, "S"] \end{tikzcd}$$

Now there exists a pair of adjoint functors $|-| : s\mathcal{M} \rightleftarrows \mathcal{M} : Sing$ which are unfortunately also called 'realization' and 'singular' functors, although they are a little different from the ones just considered. The functor $|-|$ takes an object $X_\bullet \in s\mathcal{M}$ and builds its realization via the usual formula using the simplicial structure on $\mathcal{M}$, and $Sing$ maps an object $Y \in \mathcal{M}$ to the simplicial object $Y^{\Delta^\bullet}$.

Consider the composite of the pairs

$$\begin{tikzcd} sPre(\mathcal{C}) \arrow[r, bend left, "Re"] & s\mathcal{M} \arrow[l, bend left, "S"] \arrow[r, shift left, "|-|"] & \mathcal{M}, \arrow[l, shift left, "Sing"] \end{tikzcd}$$

and call the composite functors L and R. It's easy to see that R is the functor which maps an object $X \in \mathcal{M}$ to the simplicial presheaf $c \mapsto \underline{\mathcal{M}}(\tilde{\gamma}(c), X)$. To see that L and R form a Quillen pair, it is enough to show that R preserves fibrations and trivial fibrations. But if $X \to Y$ is a fibration in $\mathcal{M}$, then the map on function spaces $\underline{\mathcal{M}}(\tilde{\gamma}(c), X) \to \underline{\mathcal{M}}(\tilde{\gamma}(c), Y)$ is also a fibration, as a consequence of SM7 and the fact that $\tilde{\gamma}(c)$ is cofibrant. This says that $RX \to RY$ is an objectwise fibration, and therefore a fibration in $sPre(\mathcal{C})_{BK}$. The same argument shows that R preserves trivial fibrations.

The functors L and R are simplicial because they've been constructed as the composites of simplicial functors (here $s\mathcal{M}$ is given the *categorical* simplicial structure, not the levelwise simplicial structure induced by that on $\mathcal{M}$).

Finally, we have to provide a natural transformation $L \circ r \to \gamma$ which is a weak equivalence on objects. But by definition $L(rX) \cong |\tilde{\gamma}(X)_\bullet|$, where $\tilde{\gamma}(X)_\bullet$ is the constant object in $s\mathcal{M}$ which has

$\tilde{\gamma}(X)$ in every dimension. Of course there is a natural weak equivalence $|\tilde{\gamma}(X)_\bullet| \xrightarrow{\sim} \tilde{\gamma}(X)$, and we can compose this with the map $\tilde{\gamma}(X) \xrightarrow{\sim} \gamma(X)$ to get the desired natural transformation. □

Any diagram $D\colon I \to \mathcal{C}$ may be regarded as a diagram in $sPre(\mathcal{C})$ by looking at the composite $D\colon I \to \mathcal{C} \to sPre(\mathcal{C})$. Let HC_D denote the homotopy colimit of this diagram. (Since our definition of homotopy colimit uses only the simplicial structure of $sPre(\mathcal{C})$, it doesn't matter whether we are using the Bousfield-Kan or the Heller model category). It is tempting to think of HC_D as the 'formal homotopy colimit' of the diagram D. This is justified in part by the following result:

Corollary 3.2.9.
Let D and E be two diagrams in $\mathcal{C}$ (possibly of different shapes). Then $HC_D \simeq HC_E$ in $sPre(\mathcal{C})$ if and only if for every simplicial model category $\mathcal{M}$ and every functor $\mathcal{C} \xrightarrow{\gamma} \mathcal{M}$, $\mathbf{L}\underrightarrow{hocolim}(\gamma \circ D) \cong \mathbf{L}\underrightarrow{hocolim}(\gamma \circ E)$ in $Ho(\mathcal{M})$.

Proof. The 'if' part follows trivially, by taking $\mathcal{M}$ to be the category $sPre(\mathcal{C})$ itself. For the 'only if' part, first note that H_D and H_E are cofibrant in $sPre(\mathcal{C})_{BK}$. This follows because they are each defined as a homotopy colimit of representables, and representables are cofibrant. Since the functor $L : sPre(\mathcal{C})_{BK} \to \mathcal{M}$ is part of a Quillen pair, L takes weak equivalences between cofibrant objects to weak equivalences: hence, $L(H_D) \simeq L(H_E)$ in $\mathcal{M}$. The fact that L is a simplicial functor implies that it commutes with homotopy colimits for diagrams of cofibrant objects: there is a natural weak equivalence $\underrightarrow{\mathrm{hocolim}}(L \circ D) \xrightarrow{\sim} L(H_D)$. So $L \circ D$ and $L \circ E$ have weakly equivalent homotopy colimits in $\mathcal{M}$. Finally, the fact that $L(rX)$ is a cofibrant approximation to $\gamma(X)$ tells us that $\underrightarrow{\mathrm{hocolim}}(L \circ D)$ is isomorphic to $\mathbf{L}\underrightarrow{\mathrm{hocolim}}\, D$ in $Ho(\mathcal{M})$. This completes the proof. □

Any simplicial presheaf may to some extent be regarded as a 'formal' homotopy colimit of a diagram in $\mathcal{C}$. The above corollary then gives an intriguing characterization of the objectwise weak equivalences: two simplicial presheaves are weakly equivalent when their underlying diagrams have the same homotopy colimit in any homotopy theory encompassing $\mathcal{C}$. Let's consider this property for a moment. The condition would hold, for instance, if one diagram were homotopy-cofinal in the other. In general, the condition is capturing a kind of 'formal equivalence': two diagrams are related in this way if their homotopy colimits can be shown to be equivalent for 'formal' reasons, reasons not depending on any properties specific of the ambient homotopy theory. Since simplicial presheaves can to some extent be regarded as formal homotopy colimits, the above corollary is saying that the objectwise weak equivalences are capturing this notion of 'formal' equivalence. These are intriguing statements, although admittedly vague; it would be nice to have a clearer explanation of all this.

Remark 3.2.10. We have seen that $sPre(\mathcal{C})$ is a certain kind of 'universal homotopy theory' built from the category $\mathcal{C}$. Because of its universal nature this model category has many special properties one couldn't expect in a general model category. (In the same way, a category of presheaves has special properties which don't hold in arbitrary co-complete categories: for example, products distribute over direct sums in presheaf categories.)

The point is that the category of simplicial presheaves is very closely tied to $s\mathcal{S}et$, and as a result inherits certain properties which are very special. We mention two as examples:

- *Homotopy colimits commute with products.* If D is a diagram of simplicial presheaves and $F \in sPre(\mathcal{C})$, then $\underrightarrow{\mathrm{hocolim}}_\alpha(F \times D_\alpha) \simeq F \times \underrightarrow{\mathrm{hocolim}}_\alpha D_\alpha$.
- *Weak equivalences are preserved by filtered colimits.* Let I be a filtered indexing category, let $D_1, D_2\colon I \to sPre(\mathcal{C})$ be two diagrams, and let $D_1 \to D_2$ a natural transformation such that $D_1(\alpha) \to D_2(\alpha)$ is a weak equivalence for every $\alpha \in I$. Then $\underrightarrow{\mathrm{colim}}\, D_1 \to \underrightarrow{\mathrm{colim}}\, D_2$ is a weak equivalence.

These statements are true precisely because they are true in $s\mathcal{S}et$, and because all the constructions in $sPre(\mathcal{C})$ take place objectwise.

3.3. The Čech model category structure.

Let $\mathcal{C}$ be a category with Grothendieck topology. As above, for each cover $\{U_\alpha \to X\}$ we let $\check{U}_\bullet$ be the simplicial presheaf which in dimension n is $\coprod_{i_0,\dots,i_n} r(U_{i_0\dots i_n})$. There is a natural map $\check{U}_\bullet \to rX$, where the presheaf rX is regarded as a simplicial presheaf in the usual manner.

Definition 3.3.1. *The* ***Čech model category*** *associated to $\mathcal{C}$ is the model category obtained by starting with the Heller structure on $sPre(\mathcal{C})$ and localizing at the set of all maps $\check{U}_\bullet \to X$. The* ***Bousfield-Kan Čech model category*** *is obtained by starting with the BK model structure on $sPre(\mathcal{C})$ and localizing at the same set of maps. We will denote these model categories by $sPre(\mathcal{C})_{\check{C}ech}$ and $sPre(\mathcal{C})_{BK,\check{C}ech}$.*

These two Čech model categories are Quillen equivalent. The universal property of localization implies that they represent something like the universal homotopy theory built from $\mathcal{C}$ with the property that the maps $\underrightarrow{\mathrm{hocolim}}\, U_\bullet \to X$ are weak equivalences.

The weak equivalences in the Čech model category will be called *Čech weak equivalences*, and likewise for the cofibrations and fibrations. Observe that:

(i) Every objectwise weak equivalence is a Čech weak equivalence.
(ii) The Čech cofibrations are precisely the monomorphisms (which are the Heller cofibrations). In particular, every object is cofibrant.
(iii) The Čech fibrant objects are precisely the Heller fibrant objects F having the property that for every cover $\{U_\alpha \to X\}$ the maps

$$F(X) \xrightarrow{\sim} \underleftarrow{\mathrm{holim}} \left[\prod F(U_\alpha) \rightrightarrows \prod F(U_{\alpha\beta}) \cdots \right]$$

are weak equivalences.

Based on this last remark, we will regard the fibrant objects in the Čech model structure as 'homotopy-theoretic sheaves'. This is justified in part by the following result, which says that sheafification is a fibrant replacement functor for *constant* simplicial presheaves.

Proposition 3.3.2. *Let $\mathcal{C}$ be a Grothendieck site, and let F be a presheaf on $\mathcal{C}$.*

(a) The constant simplicial presheaf $F_\bullet$ is Čech-fibrant if and only if F is a sheaf.
(b) If $\tilde{F}$ denotes the sheafification of F, the map of constant simplicial presheaves $F_\bullet \to \tilde{F}_\bullet$ is a Čech weak equivalence.

Remark 3.3.3. A consequence of this proposition is that the map $Shv \to Ho(sPre)$ which sends a sheaf F to the associated constant simplicial presheaf is a full embedding. In other words, the category of sheaves has 'homotopy-theoretic meaning'.

Exercise 3.3.4. Let $\mathcal{M}an$ denote the category of manifolds, with Grothendieck topology given by open covers. Proposition 3.2.8 gives a Quillen pair $sPre(\mathcal{M}an)_{BK} \rightleftarrows \mathcal{T}op$, and we know that the realization functor takes the maps $\check{U}_\bullet \to X$ to weak equivalences. The Quillen pair therefore extends to the Čech model category: $sPre(\mathcal{M}an)_{BK,\text{Čech}} \rightleftarrows \mathcal{T}op$. Use the above proposition to show that the map $* \to \mathbb{R}$ is not a Čech weak equivalence, and conclude that the above adjoint pair is not a Quillen equivalence.
(*Hint: This was essentially done in the introduction to this paper. Be warned that the above proposition concerns the Heller Čech model category, whereas the exercise is about the BK structure.*)

Proof of (a). Since a constant simplicial presheaf is Heller-fibrant, $F_\bullet$ is Čech-fibrant if and only if

$$F(X) \xrightarrow{\sim} \underleftarrow{\mathrm{holim}} \left[\prod F(U_\alpha) \rightrightarrows \prod F(U_{\alpha\beta}) \cdots \right]$$

for every X and every cover $\{U_\alpha \to X\}$. But all the simplicial sets $F(W)$ are *constant*, so this homotopy limit is just an ordinary limit. It is also true that a limit of a co-simplicial diagram of sets is the same as the equalizer of the first two co-face maps, and so we find that $F_\bullet$ is Čech-fibrant iff

$$F(X) \longrightarrow \prod F(U_\alpha) \rightrightarrows \prod F(U_{\alpha\beta})$$

is an equalizer for every X and $U_\bullet$. This is the requirement that F be a sheaf. □

The proof of (b) is much more difficult. Note that the result is far from obvious, as there is no reason to expect that the fibrant replacement of a constant object will still be constant. We will see that it is the axioms of a Grothendieck topology that force this to be true.

The trouble one encounters in proving (b) is that it's often hard to recognize weak equivalences in a localized model category, other than in cases where a map has been built explicitly from the set of maps you're localizing. In the present setting, determining a fibrant replacement for an object is almost doomed to be a wrestling match with the small object argument. The idea will be to use the axioms of a Grothendieck topology in order to simplify the construction somewhat. We warn the reader that the proof is fairly involved, although interesting in a weird sort of way. It can be safely skipped without any great loss in understanding.

The first step is the following important lemma, which says that the simplicial presheaf $\check{U}_\bullet$ has no higher homotopy. Just as in Section 2, we will write $\varinjlim U_\bullet$ for the coequalizer of $\coprod U_{\alpha\beta} \rightrightarrows \coprod U_\gamma$ in the category of presheaves—in our setting it will be regarded as a constant *simplicial* presheaf, however. Note that this coequalizer could also go under the name $\pi_0(\check{U}_\bullet)$.

Lemma 3.3.5. *The map of simplicial presheaves $\check{U}_\bullet \to \varinjlim U_\bullet$ is an objectwise weak equivalence, given any cover $\{U_\alpha \to X\}$.*

Proof. Since the map in question may be identified with $\check{U}_\bullet \to \pi_0(\check{U}_\bullet)$, what must be shown is that for each $Y \in \mathcal{C}$ the simplicial set $\check{U}_\bullet(Y)$ is homotopy discrete. This will follow if $\check{U}_\bullet(Y)$ can be shown to have the extension property for the maps $\partial\Delta^n \to \Delta^n$ $(n \geq 2)$ and the maps $\Lambda^{2,k} \to \partial\Delta^2$ (which will also show that $\check{U}_\bullet(Y)$ is fibrant). Establishing these properties is an easy calculation, which we leave to the reader. □

Proof of (b). Given a presheaf F, recall the constructions $\mathcal{A}F$ and $\mathcal{B}F$ from Section 2. $\mathcal{B}F$ is obtained as a pushout

$$\begin{tikzcd}
\coprod \varinjlim U_\bullet \arrow[r] \arrow[d] & F \arrow[d, dotted] \\
\coprod X \arrow[r, dotted] & \mathcal{B}F,
\end{tikzcd}$$

where the coproducts run over all $X \in \mathcal{C}$, all covers $\{U_\alpha \to X\}$, and all maps $\varinjlim U_\bullet \to F$. We may regard all these presheaves as constant simplicial presheaves, in which case the maps $\varinjlim U_\bullet \to X$ are Čech trivial cofibrations by the above lemma. Since $F \to \mathcal{B}F$ is obtained as a co-base extension of these maps, it is also a Čech trivial cofibration. If we can only show $F \to \mathcal{A}F$ is a Čech weak equivalence then we will be done, since the sheafification functor is just the composite $\mathcal{ABAB}$. The idea will be to find a particularly efficient way of producing a chain of Čech trivial cofibrations

$$F = L_0 F \overset{\sim}{\rightarrowtail} L_1 F \overset{\sim}{\rightarrowtail} L_2 F \overset{\sim}{\rightarrowtail} \cdots \qquad \Rightarrow L_\infty F,$$

so that $L_\infty F$ comes with a *formal* weak equivalence $L_\infty F \to \mathcal{A}F$.

Given a cover $\{U_\bullet \to X\}$, let $J^n(U_\bullet)$ be the pushout

$$\begin{tikzcd} \Delta^n \otimes \operatorname{\underrightarrow{colim}} U_\bullet \arrow[r] \arrow[d] & \partial\Delta^n \otimes X \arrow[d, dotted] \\ \partial\Delta^n \otimes \operatorname{\underrightarrow{colim}} U_\bullet \arrow[r, dotted] & J^n(U_\bullet). \end{tikzcd}$$

There is a natural map $J^n(U_\bullet) \to \Delta^n \otimes X$, and this map is necessarily a Čech trivial cofibration by SM7.

Now set $L_0F = F$ and let $L_{n+1}F$ be obtained from L_nF as the pushout

$$\begin{tikzcd} \coprod J^{n+1}(U_\bullet) \arrow[r] \arrow[d, tail, "\sim"'] & L_nF \arrow[d, dotted, tail, "\sim"] \\ \coprod \Delta^{n+1} \otimes X \arrow[r, dotted] & L_{n+1}F. \end{tikzcd}$$

Here the coproducts run over all $X \in \mathcal{C}$, all covers $U_\bullet \to X$, and all maps $J^{n+1}(U_\bullet) \to L_nF$. To get a feeling for these constructions, note that giving a map $J^1(U_\bullet) \to F$ is the same as giving $s, t \in F(X)$ with the property that $s|_{U_\alpha} = t|_{U_\alpha}$, for all α. Forming the pushout

$$(\Delta^1 \otimes X) \coprod\nolimits_{J^1(U_\bullet)} F$$

is tantamount to formally adding a 1-simplex to $F(X)$ which will equalize s and t in π_0. In particular, the reader should verify that $\pi_0 L_1F(X) \cong \mathcal{A}F(X)$ for all X.

The L_iF give us a chain of Čech trivial cofibrations as desired, and we let $L_\infty F$ be the colimit. The goal will be to show that for each $Y \in \mathcal{C}$ the simplicial set $L_\infty F(Y)$ is fibrant, homotopy discrete, and $\pi_0 L_\infty F(Y) = \mathcal{A}F(Y)$. This says that the natural map of simplicial presheaves $L_\infty F \to \pi_0(L_\infty F)$ is an objectwise weak equivalence, and that $\pi_0(L_\infty F) \cong \mathcal{A}F$.

The argument will proceed by establishing that the above chain has the following properties: Given $Y \in \mathcal{C}$,

(i) The map of simplicial sets $L_nF(Y) \to L_{n+1}F(Y)$ is an isomorphism in simplicial degrees less than or equal to n;
(ii) The simplicial set $L_nF(Y)$ is degenerate in degrees greater than n;
(iii) Given any n-simplex σ in $L_nF(Y)$, there is an open cover $\{U_\bullet \to Y\}$ so that $\sigma|_{U_\alpha}$ is in the image of the map $L_{n-1}F(Y) \to L_nF(Y)$;
(iv) Given any n-simplex σ in $L_nF(Y)$, there is an open cover $\{U_\bullet \to Y\}$ so that $\sigma|_{U_\alpha}$ is in the image of the map $F(Y) \to L_nF(Y)$;
(v) For $n \geq 2$, any map $\partial\Delta^n \to L_{n-1}F(Y)$ extends to a map $\Delta^n \to L_nF(Y)$:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & L_{n-1}F(Y) \arrow[d] \\ \Delta^n \arrow[r, dotted] & L_nF(Y); \end{tikzcd}$$

(vi) Any map $\Lambda^{2,k} \to L_1F(Y)$ extends as follows:

$$\begin{tikzcd} \Lambda^{2,k} \arrow[r] \arrow[d] & L_1F(Y) \arrow[d] \\ \partial\Delta^2 \arrow[r, dotted] & L_2F(Y). \end{tikzcd}$$

Granting these statements for the moment, let us show that they imply the desired result. To show that $L_\infty F(Y)$ is fibrant and homotopy discrete, it is enough to verify that it has the extension property with respect to the maps $\partial\Delta^n \to \Delta^n$ (for $n \geq 2$) and the maps $\Lambda^{2,k} \to \Delta^2$. This is an easy consequence of parts (i), (v), and (vi). For instance, part (i) gives that any map $\partial\Delta^n \to L_\infty F(Y)$ lifts to $L_{n-1}F(Y)$; part (v) then says that the map extends to $\Delta^n \to L_n F(Y)$, which can be pushed forward to a map $\Delta^n \to L_\infty F(Y)$.

Now part (i) tells us that $\pi_0 L_\infty F(Y) = \pi_0 L_1 F(Y)$, and we have already remarked that this is precisely $\mathcal{A}F(Y)$. This finishes the proof, granting the statements outlined above.

The task is then to prove the above statements. Part (i) follows from the fact that $J^n(U_\bullet)(Y) \to \big(\Delta^n \otimes rX\big)(Y)$ is an isomorphism in simplicial degrees less than or equal to n, and part (ii) follows from an induction, using that $\big(\Delta^n \otimes X\big)(Y)$ is degenerate in degrees greater than n.

The remaining parts are harder.

Parts (v) and (vi) are consequences of (iv): A map $\partial\Delta^n \to L_{n-1}F(Y)$ is a finite collection of $(n-1)$-simplices $\sigma_0, \dots, \sigma_n$ in $L_{n-1}F(Y)$ which 'fit together' correctly. Part (iv) says that for each σ_i we may find some cover over which the simplex is pushed-forward from F, and since there are only finitely many σ_i we may choose a common refinement $\{U_\alpha\}$ of all these covers. In other words, $\{U_\alpha \to Y\}$ has been chosen so there are $(n-1)$-simplices $\sigma_i^\alpha \in F(U_\alpha)$ $(i = 0, \dots, n)$ with the property that $\sigma_i^\alpha \mapsto \sigma_i|_{U_\alpha}$ under the map $F(U_\alpha) \to L_{n-1}F(U_\alpha)$. What is not immediately clear is that these simplices 'fit together' to give a map $\partial\Delta^n \to F(U_\alpha)$. But since the σ_i's fit together in $L_{n-1}F(U_\alpha)$ and the map $F \to L_{n-1}F$ is a *cofibration* (hence an objectwise monomorphism), the σ_i^α's must also fit together in $F(U_\alpha)$. Finally, note that we must have $\sigma_i^\alpha|_{U_{\alpha\beta}} = \sigma_i^\beta|_{U_{\alpha\beta}}$— this again follows using the fact that $F \to L_{n-1}F$ is an objectwise monomorphism.

To paraphrase this discussion, we started with a map $\partial\Delta^n \otimes Y \to L_{n-1}F$ and have produced a cover $\{U_\alpha \to Y\}$ and a map $\partial\Delta^n \otimes \varinjlim U_\bullet \to F$ which fits in a square as follows:

$$\begin{tikzcd}
\partial\Delta^n \otimes \varinjlim U_\bullet \arrow[r] \arrow[d] & F \arrow[d] \\
\partial\Delta^n \otimes Y \arrow[r] & L_{n-1}F.
\end{tikzcd}$$

Now we come to the main point, which is that F was a *constant* simplicial presheaf—so all the face and degeneracy maps are identities. This means that the only way the $\sigma_i^\alpha \in F(U_\alpha)$ can 'fit together' is if they are all equal, and so there will be a canonical extension of our map $\partial\Delta^n \to F(U_\alpha)$ to $\Delta^n \to F(U_\alpha)$. Such an extension exists for each α, and since everything was canonical they all still patch together correctly on the $U_{\alpha\beta}$. So we obtain a lifting

$$\begin{tikzcd}
\partial\Delta^n \otimes \varinjlim U_\bullet \arrow[r] \arrow[d] & F \\
\Delta^n \otimes \varinjlim U_\bullet, \arrow[ur, dotted] &
\end{tikzcd}$$

and the dotted map may be pushed forward from F into $L_{n-1}F$. So we have produced a commutative square

$$\begin{tikzcd}
\partial\Delta^n \otimes \varinjlim U_\bullet \arrow[r] \arrow[d] & \partial\Delta^n \otimes Y \arrow[d, "\sigma"] \\
\Delta^n \otimes \varinjlim U_\bullet \arrow[r] & L_{n-1}F,
\end{tikzcd}$$

and this induces a map

$$\Big(\Delta^n \otimes \varinjlim U_\bullet\Big) \amalg_{\partial\Delta^n \otimes \varinjlim U_\bullet} \Big(\partial\Delta^n \otimes Y\Big) \longrightarrow L_{n-1}F.$$

But by construction of the L_k such a map induces an extension $\Delta^n \otimes Y \to L_n F$; in other words, we have extended the original map $\partial\Delta^n \to L_{n-1}F(Y)$ to a map $\Delta^n \otimes Y \to L_n F$. This is what we wanted.

The proof that (iv)⇒(vi) is similar (but much easier); we leave it for the reader.

We are therefore reduced to showing (iii) and (iv). Part (iv) follows from (iii) by induction, so we just have to show (iii).

Let σ be an n-simplex in $L_n F(Y)$. From the definition of L_n, there is an object $X \in \mathcal{C}$, a cover $\{V_\alpha \to X\}$, and a map $J^n(V_\bullet) \to L_{n-1}$ so that σ is an n-simplex in

$$(\Delta^n \otimes rX(Y)) \coprod_{J^n(V_\bullet)(Y)} L_{n-1}(Y).$$

If σ is represented by an n-simplex of $L_{n-1}(Y)$ then we are done. Otherwise, σ is represented by a pair $(\tilde{\sigma}, f)$ where $\tilde{\sigma}$ is an n-simplex of Δ^n and f is a map $Y \to X$ (i.e., f is an element of $rX(Y)$.)

Let $U_\alpha = Y \times_X V_\alpha$. Then $\{U_\alpha \to Y\}$ is a cover, by the axioms for a Grothendieck topology. Note that the map $J^n(V_\bullet) \to L_{n-1}$ specifies n-simplices $\tau_\alpha \in L_{n-1}(V_\alpha)$ whose boundaries extend to a section over X. Pulling back via the map $f\colon U_\alpha \to V_\alpha$ produces n-simplices $f^*\tau_\alpha \in L_{n-1}(U_\alpha)$. It is merely a matter of chasing through the definitions to see that $f^*(\tau_\alpha)$ maps to $\sigma|_{U_\alpha}$ under the map $L_{n-1}F \to L_n F$. □

Remark 3.3.6. We speculate that the above result can be generalized to the following:

If F is a simplicial presheaf with the property that each $F(X)$ has no homotopy above dimension n, then the Čech-fibrant replacements of F also have this property.

To see the implications of this statement, consider the 'sections over X' functor $\Gamma_X : sPre(\mathcal{C})_{\text{Čech}} \to s\mathcal{S}et$ defined by $\Gamma_X(F) = F(X)$. If $F \to G$ is a Čech weak equivalence between Čech-fibrant objects, then formal properties of localizations tell us that $F \to G$ is necessarily an *objectwise* weak equivalence: in particular, $F(X) \xrightarrow{\sim} G(X)$. So Γ_X preserves weak equivalences between fibrant objects, and therefore has a total right derived functor

$$Ho(sPre(\mathcal{C})_{\text{Čech}}) \xrightarrow{\mathbf{R}\Gamma_X} Ho(s\mathcal{S}et).$$

Note that $\mathbf{R}\Gamma_X(F)$ has the same homotopy type as the simplicial mapping space $\underline{sPre}(X, \tilde{F})$, where $\tilde{F}$ is a Čech-fibrant replacement for F.

Now if F is a sheaf of abelian groups, we may form the simplicial presheaf $K(F,n)$ given by $X \mapsto K(F(X), n)$. We will see in the next section that $\pi_i\mathbf{R}\Gamma_X(K(F,n))$ is related to the sheaf cohomology group $H^{n-i}(X, F)$. (This should be contrasted to the statement that for topological spaces X and abelian groups A, $\pi_i(K(A,n)^X) \cong H^{n-i}(X; A)$.) Since each $K(F,n)(X)$ has no homotopy in dimensions greater than n, the result speculated above would imply that $\mathbf{R}\Gamma_X(F)$ also has this property; clearly this is related to the fact that there are no negative sheaf cohomology groups, or that the sections functor has no 'lower' derived functors. Weakening the axioms of a Grothendieck topology tends to make statements like these cease to hold.

3.4. **Application: The universal homotopy theory built from manifolds.**

In this section we will start with a category of manifolds and build the 'universal homotopy theory' subject to the following relations:

- For every manifold M and every open covering $\{U_\alpha \to M\}$, the natural map

$$\underrightarrow{\text{hocolim}}\left[\cdots \textstyle\coprod_{\alpha\beta\gamma} U_{\alpha\beta\gamma} \Rrightarrow \coprod_{\alpha,\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha\right] \longrightarrow X.$$

  is a weak equivalence;
- For every manifold M, the projection $M \times \mathbb{R} \to M$ is a weak equivalence.

The construction of this category is an easy application of the machinery we've developed so far. Our goal will be to show that the model category thus obtained is Quillen equivalent to the usual model category of topological spaces. Most of this section is just an elaboration of the material in [12].

Let $\mathcal{M}an$ denote the full subcategory of $\mathcal{T}op$ consisting of submanifolds of $\mathbb{R}^\infty$. (The point is that this gives us a *small* category which nevertheless contains any manifold we're likely to be interested in.) We will regard $\mathcal{M}an$ as a Grothendieck site in which the covers are just the open coverings. Let I denote the real number line $\mathbb{R}$, which will serve as an analogue of the unit interval (which unfortunately is not in our category).

Definition 3.4.1. *The **I-local Čech model category structure** on $sPre(\mathcal{M}an)$ is the localization of $sPre(\mathcal{M}an)_{\check{C}ech}$ at the set of maps $\{rX \times I \to rX\}_{X \in \mathcal{M}an}$. The category will be denoted $sPre(\mathcal{M}an)_I$.*

When dealing with this category we will speak of the 'I-local weak equivalences', 'I-local fibrations', etc. At times we will also consider the localization of $sPre(\mathcal{M}an)_{BK,\text{Čech}}$ with respect to the same set of maps as above. This is a Quillen equivalent model category, which we will denote $sPre(\mathcal{M}an)_{BK,I}$.

The reader may wonder why we didn't localize at *all* maps $F \times I \to F$, rather than just the maps where F is representable. The reason is that the collection $\{F \times I \to F : F \in sPre\}$ is much too big to be a set, and our machinery only lets us localize at sets. Since $\mathcal{M}an$ was small, restricting F to be representable fits this requirement. But it turns out that having $F \times I \xrightarrow{\sim} F$ for F representable actually implies that we have it for everything:

Lemma 3.4.2. *For all $F \in sPre(\mathcal{M}an)$, the map $F \times I \to F$ is an I-local equivalence.*

Proof. We know from Remark 3.2.4 that F may be written as a homotopy colimit of a diagram of representables. We also know from Remark 3.2.10 that the functor $I \times -$ will commute with homotopy colimits. The result follows immediately from these observations, using the fact that all objects are cofibrant. $\square$

The reader is encouraged to think of $sPre(\mathcal{M}an)$ as a category of 'spaces'. At this point we have given quite a bit of justification for this. The rest of the section will be spent showing that the I-local Čech structure on $sPre(\mathcal{M}an)$ is Quillen-equivalent to the usual model category of topological spaces. There are at least two ways of approaching this, by comparing our category either to $s\mathcal{S}et$ or to $\mathcal{T}op$. Both approaches revolve around the same basic idea, which we take a moment to explain.

If M is a particularly nice manifold (e.g. one admitting a locally finite triangulation), one can find an open covering $\{U_\bullet\}$ in which all the intersections $U_{i_1 \cdots i_k}$ are either empty or contractible. By construction, there is a Čech weak equivalence between rM and the homotopy colimit of the diagram

$$\cdots \coprod rU_{\alpha\beta} \rightrightarrows \coprod rU_\alpha.$$

But the covering has been chosen so that there is an I-local equivalence between each nonempty $rU_{i_1 \cdots i_k}$ and a point, so that rM is weakly equivalent to a homotopy colimit of points. Now any simplicial presheaf F may be expressed as a homotopy colimit of representables, and if all our manifolds are nice enough then we can express each representable as a homotopy colimit of points. In this way, any object in our category may be 'unravelled' into something which is really just a simplicial set. (Of course we will also have to show something to the effect that if an object is unravelled in two different ways, the resulting simplicial sets are weakly equivalent.) The only catch involves handling the fact that our manifolds may not actually have a 'nice enough' cover—this will require some technical machinery.

Consider the one-point category $*$, together with the map $* \to \mathcal{M}an$ which picks out the one-point manifold. As in Remark 2.2.9(d), this map induces a pair of adjoint functors $\mathcal{S}et = Pre(*) \rightleftarrows$

$Pre(\mathcal{M}an)$, and these extend to an adjunction on the simplicial level $s\mathcal{S}et \rightleftarrows sPre(\mathcal{M}an)$. It may easily be checked that the left-adjoint $L : s\mathcal{S}et \to sPre(\mathcal{M}an)$ sends a simplicial set $S_\bullet$ to the constant simplicial presheaf $cS_\bullet$. The right-adjoint $R : sPre(\mathcal{M}an) \to s\mathcal{S}et$ is the map sending a simplicial presheaf $F_\bullet$ to its value $F(*)$ on the one-point manifold. Note that the composite RL is naturally isomorphic to the identity functor.

Theorem 3.4.3. *The functors L and R above induce a Quillen-equivalence between the model categories $s\mathcal{S}et$ and $sPre(\mathcal{M}an)_{BK,I}$.*

We need the following lemmas:

Lemma 3.4.4 (Rigidity)**.** *If $F, G \in sPre(\mathcal{M}an)$ are I-fibrant, a map $F \to G$ is an I-weak equivalence if and only if $F(*) \to G(*)$ is a weak equivalence.*

Lemma 3.4.5. *Let K be a simplicial set and let $\widetilde{(cK)} \in sPre$ be an I-fibrant replacement for cK. Then $cK(*) \to \widetilde{(cK)}(*)$ is a weak equivalence.*

Proof of Theorem. We can show that L and R form a Quillen-pair by verifying that R preserves fibrations and trivial fibrations. Because $R(F)$ may be identified with $sPre(*, F)$, this is a trivial consequence of SM7.

The next thing to show is that for every $K \in s\mathcal{S}et$ (necessarily cofibrant) and every fibrant $F \in sPre_I$, a map $cK \to F$ is an I-local equivalence if and only if its adjoint $K \to F(*)$ is a weak equivalence of simplicial sets.

If $cK \to F$ is an I-weak equivalence then F is an I-fibrant replacement for cK. Lemma 3.4.5 then says that $cK(*) \to F(*)$ is a weak equivalence. Since $cK(*) = K$, this settles the left-to-right direction.

To show the converse, it is enough to check that $c[F(*)] \to F$ is an I-weak equivalence when F is fibrant. Let G be the I-fibrant replacement of $c[F(*)]$, so that we have a diagram

$$\begin{tikzcd} c[F(*)] \arrow[r] \arrow[d, tail, "j"', "\sim"] & F \\ G. & \end{tikzcd}$$

Since F is fibrant, there is a lifting $G \xrightarrow{p} F$. Now Lemma 3.4.5 tells us that $F(*) \xrightarrow{j} G(*)$ is a weak equivalence. By construction, $G(*) \xrightarrow{p} F(*)$ is a right inverse for this map, so it is also a weak equivalence. Then $G \to F$ is a map between fibrant objects which is a weak equivalence over the basepoint, so by Lemma 3.4.4 it is an I-local equivalence. This completes the proof. $\square$

Proving the lemmas is more difficult, and will require some heavy technical machinery. Note that in Lemma 3.4.5 we actually have to get our hands on the fibrant replacement for something, and we've already seen that this can be quite hard. The machinery we're about to develop will give us a way of detecting Čech weak equivalences, and this will motivate the development of Jardine's model category in Section 4.

The proof of Lemma 3.4.4 also requires this machinery, but for a somewhat different reason. We'll explain this by giving an

Attempt at a proof of 3.4.4. If $F \to G$ is a weak equivalence between fibrant objects, then by SM7 it follows that $\underline{sPre}(*, F) \to \underline{sPre}(*, G)$ is a weak equivalence in $s\mathcal{S}et$. But these mapping spaces may be identified with $F(*)$ and $G(*)$, so this proves the left-to-right direction.

Conversely, now assume that $F(*) \to G(*)$ is a weak equivalence.

Step 1: $F(V) \to G(V)$ is a weak equivalence for every contractible $V \in \mathcal{M}an$.

Because F is homotopy invariant it follows that for contractible V we have $F(*) \xrightarrow{\sim} F(V)$. The same is true of G, so there is a square

$$\begin{tikzcd} F(*) \arrow[r, "\sim"] \arrow[d] & F(V) \arrow[d] \\ G(*) \arrow[r, "\sim"] & G(V). \end{tikzcd}$$

By assumption $F(*) \to G(*)$ is a weak equivalence, so $F(V) \to G(V)$ must be as well.

Step 2: $F(M) \to G(M)$ *is a weak equivalence for every* $M \in \mathcal{M}an$.

If M is nice enough, we can choose a cover $\{U_\alpha \to M\}$ such that each $U_{\alpha_1 \ldots \alpha_n}$ is either empty or contractible. Because F and G are fibrant (and therefore Čech -fibrant, in particular) we know that $F(M)$ and $G(M)$ may be recovered up to homotopy from the values of F and G on the U_α. In other words, we have the following diagram:

$$\begin{tikzcd} F(M) \arrow[r, "\sim"] \arrow[d] & \underleftarrow{\operatorname{holim}} \Big[\prod_\alpha F(U_\alpha) \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & \prod_{\alpha,\beta} F(U_{\alpha\beta}) \arrow[d] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \prod_{\alpha,\beta,\gamma} F(U_{\alpha\beta\gamma}) \cdots \Big] \arrow[d] \\ G(M) \arrow[r, "\sim"] & \underleftarrow{\operatorname{holim}} \Big[\prod_\alpha G(U_\alpha) \arrow[r, shift left] \arrow[r, shift right] & \prod_{\alpha,\beta} G(U_{\alpha\beta}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \prod_{\alpha,\beta,\gamma} G(U_{\alpha\beta\gamma}) \cdots \Big] \end{tikzcd}$$

But since all the $U_{\alpha_1 \ldots \alpha_n}$ were contractible (or empty), this tells us that the vertical maps inside the above homotopy limit are all weak equivalences. Since each $F(X)$ and $G(X)$ is fibrant (because F and G are BK fibrant), it follows that the weak equivalence passes to the homotopy limit: in other words, $F(M) \xrightarrow{\sim} G(M)$. As this holds for all $M \in \mathcal{M}an$, we have $F \xrightarrow{\sim} G$.

If M was not 'nice enough' then we have a problem. We could try taking a colimit as the covers of M get smaller and smaller, but this requires commuting the $\underrightarrow{\operatorname{colim}}$ and the $\underleftarrow{\operatorname{holim}}$. We will see that there is a way to do this... ☐

Detecting Čech weak equivalences.

Write B^n_k for the n-ball of radius $\frac{1}{k}$ centered about the origin in $\mathbb{R}^n$. If F is a simplicial presheaf, let

$$p_n(F) = \operatorname*{colim}_{k \to \infty} F(B^n_k).$$

$p_n(F)$ is called the **stalk** of F in dimension n. Note that $p_0(F) = F(*)$.

Definition 3.4.6. *A map of simplicial presheaves $F \to G$ will be called a* ***stalkwise weak equivalence*** *if the induced maps $p_n(F) \to p_n(G)$ are weak equivalences for all $n \in \mathbb{N}$.*

Remark 3.4.7. Note that every objectwise weak equivalence is a stalkwise weak equivalence. This follows because stalks are computed as filtered colimits, and filtered colimits preserve weak equivalences in $s\mathcal{S}et$.

Our goal is the following important result:

Proposition 3.4.8. *The classes of Čech weak equivalences and stalkwise weak equivalences are identical.*

This is not an easy proposition, and the proof will be postponed until section 4 when we have more machinery available. For now we will be content to prove containment in one direction, at least suggesting that the result is plausible. Then we will use the proposition to complete the proofs of Lemmas 3.4.4 and 3.4.5.

Lemma 3.4.9.

(a) Stalkwise weak equivalences satisfy the two-out-of-three property.
(b) The pushout of a stalkwise weak equivalence is a stalkwise weak equivalence.
(c) For a cover $\{U_\alpha \to X\}$, the maps $\check{U}_\bullet \to X$ are stalkwise weak equivalences.
(d) Every Čech weak equivalence is a stalkwise weak equivalence.

Proof. This is not difficult, and we will only give a sketch. Parts (a) and (b) are immediate, and part (d) follows from the previous parts and a knowledge of localization machinery. To prove (c) we recall that the map $\check{U}_\bullet \to X$ factors as $\check{U}_\bullet \to \underrightarrow{\operatorname{colim}}\, U_\bullet \to X$, and the former map is an objectwise (hence stalkwise) weak equivalence. It therefore suffices to show that the maps $\underrightarrow{\operatorname{colim}}\, U_\bullet \to X$ are stalkwise weak equivalences. They are actually stalkwise isomorphisms—we leave this to the reader. $\square$

Completion of the proof of 3.4.4. In our 'attempt at a proof' above, we reduced the problem to showing the following: if $F \to G$ is a map between I-fibrant objects such that $F(*) \to G(*)$ is a weak equivalence, then $F \to G$ is an I-local equivalence. This follows from looking at the stalks:

Since F is homotopy invariant, the maps $B^n_k \to *$ induce weak equivalences $F(*) \to F(B^n_k)$. Taking the colimit as $k \to \infty$ shows that the natural map $F(*) \to p_n(F)$ is a weak equivalence. The same argument applies to G, so that we obtain a square

$$\begin{tikzcd}
F(*) \arrow[r, "\sim"] \arrow[d, "\sim"] & G(*) \arrow[d, "\sim"] \\
p_n(F) \arrow[r] & p_n(G).
\end{tikzcd}$$

It follows that $p_n(F) \to p_n(G)$ is a weak equivalence, for all n. The above proposition then says that $F \to G$ is a Čech weak equivalence (and hence an I-local equivalence). $\square$

Proof of Lemma 3.4.5. Let $|K|$ denote the topological realization of the simplicial set K. Let Q be a cofibrant replacment functor for spaces, and let G be the simplicial presheaf given by $M \mapsto \mathcal{T}op(QM, |K|)$. There are natural maps $K \to \mathcal{T}op(*, |K|) \to \mathcal{T}op(QM, |K|)$, so that we get a map of simplicial presheaves $cK \to G$. It's easy to see that this map is a stalkwise weak equivalence, and therefore a Čech equivalence. We claim that G is in fact an I-fibrant model for cK; note this would prove the lemma because over the basepoint the map is $K \to G(*) \simeq \mathcal{T}op(*, |K|)$, and this is a weak equivalence.

It's easy to check that G is fibrant in the Bousfield-Kan category, so we just need to check that G is homotopy invariant and has the expected behavior on covers. But these are just standard facts about the homotopy theory of topological spaces. $\square$

Remark 3.4.10. This completes the proof that the functors $s\mathcal{S}et \rightleftarrows sPre_{BK,I}$ are a Quillen equivalence. We could also have used the I-local Heller structure and obtained the same result; in fact, there is a chain of Quillen functors

$$\begin{tikzcd}
s\mathcal{S}et \arrow[r, shift left, "L"] & sPre(\mathcal{M}an)_{BK,I} \arrow[l, shift left, "R"] \arrow[r, shift left, "Id"] & sPre(\mathcal{M}an)_{H,I} \arrow[l, shift left, "Id"]
\end{tikzcd}$$

and each is a Quillen equivalence. The same therefore holds for the composite.

Remark 3.4.11. The reader may wish to compare Lemma 3.4.4 with the rigidity theorems of Gabber and Gillet-Thomason, and especially with their generalization in section 4 of [13]. We will return to this point in Section 5.

We end this section by comparing $sPre(Man)$ directly with the category of topological spaces. It turns out that all of the hard work has been done already.

Of course there is an obvious functor $\mathcal{M}an \to Top$, therefore Proposition 3.2.8 gives us a Quillen pair $Re : sPre(\mathcal{M}an)_{BK} \rightleftarrows \mathcal{T}op : S$. The functor Re takes the maps $\check{U}_\bullet \to X$ and $X \times I \to X$ to weak equivalences in $\mathcal{T}op$, therefore our Quillen functors extend to the localization:

$$\begin{tikzcd} sPre(\mathcal{M}an)_{BK,I} \arrow[r, bend left, "Re"] & \mathcal{T}op. \arrow[l, bend left, "S"] \end{tikzcd}$$

In fact these functors form a composable pair with the Quillen functors from $s\mathcal{S}et$ which we have already considered, *and the composites are just the usual realization/singular functors between* $s\mathcal{S}et$ *and* $\mathcal{T}op$:

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "L"] & sPre(\mathcal{M}an)_{BK,I} \arrow[l, bend left, "R"] \arrow[r, bend left, "Re"] & \mathcal{T}op. \arrow[l, bend left, "S"] \end{tikzcd}$$

Exercise 3.4.12. Show that $Re : sPre(\mathcal{M}an)_{BK,I} \rightleftarrows \mathcal{T}op : S$ is a Quillen equivalence.
(*Hint: If* $F \in sPre$*, show that there is a simplicial set* K *and a weak equivalence* $LK \xrightarrow{\sim} F$*. Then make use of the Quillen equivalences you already know.*)

Exercise 3.4.13. Use the Quillen equivalence $sPre(\mathcal{M}an) \rightleftarrows \mathcal{T}op$ to show that a map of cofibrant manifolds $M \to N$ is a weak equivalence in $\mathcal{T}op$ if and only if $F(N) \to F(M)$ is a weak equivalence for every functor $F : (\mathcal{M}an)^{op} \to s\mathcal{S}et$ with the properties that

(i) $F(X) \xrightarrow{\sim} \underleftarrow{\text{holim}} \left[\prod F(U_\alpha) \rightrightarrows \prod F(U_{\alpha\beta}) \cdots \right]$ for every open cover $\{U_\alpha \to X\}$, and
(ii) $F(X) \xrightarrow{\sim} F(X \times I)$ for every manifold X.

(This justifies some remarks made in the introduction.)

Remark 3.4.14. Compactly-generated

Exercise 3.4.15. Let S^1_s denote the constant simplicial presheaf whose value is $\Delta^1/\partial\Delta^1$, and let S^1_t denote the simplicial presheaf represented by the manifold S^1. (These are called the 'simplicial' and 'topological' circles in $sPre(\mathcal{M}an)$, respectively). The above Quillen equivalences show that S^1_s and S^1_t are weakly equivalent in $sPre(\mathcal{M}an)_I$. Find an explicit chain of weak equivalences exhbiting this fact.

3.5. **Application: the I-local homotopy theory of schemes.**

Let k be a field and let Sch/k denote the category of schemes which are finite type over $\operatorname{Spec} k$. Sm/k will denote the full subcategory of Sch/k consisting of smooth schemes. The **Zariski topology** on either Sch/k or Sm/k is the Grothendieck topology whose covers are the Zariski open coverings. Write I for the scheme $\mathbb{A}^1$.

Definition 3.5.1. *The* ***I-local Čech model category structure*** *on* $sPre(Sm/k)$ *is the localization of* $sPre(Sm/k)_{\check{C}ech}$ *at the set of maps* $\{rX \times I \to rX\}_{X \in Sm/k}$*. The category will be denoted* $sPre(Sm/k)_I$*, and the associated pointed model category by* $sPre(Sm/k)_{I*}$*. We will write* Ho_I *and* Ho_{I*} *for the corresponding homotopy categories.*

We postpone a detailed investigation of these categories until Section 4, after we've introduced a more manageable notion of weak equivalence. But for the moment we wish to point out some simple applications.

For X a scheme, let $\operatorname{Vect}(X)$ denote the category of algebraic vector bundles over X. This is an exact category, so we may feed it into Quillen's Q-construction and obtain a pointed simplicial set $\mathcal{K}(X) := Q(\operatorname{Vect}(X))$. The assignment $X \mapsto \mathcal{K}(X)$ is not quite functorial in X, because $X \to$

Vect(X) is only a 'pseudo-functor': given $X \to Y \to Z$, the pullback functors

$$\begin{tikzcd} \mathrm{Vect}(Z) \arrow[rr] \arrow[dr] & & \mathrm{Vect}(Y) \arrow[dl] \\ & \mathrm{Vect}(X) & \end{tikzcd}$$

only commute up to natural isomorphism, not on the nose. We will not go into details here, but it's possible to get around this problem and to actually produce a functor $\mathcal{K} : (Sm/k)^{op} \to s\mathcal{S}et_*$ which models algebraic K-theory. We will regard this as an element of $sPre(Sm/k)_*$.

Let S_s^n denote the constant simplicial presheaf whose value is $\Delta^n/\partial\Delta^n$—this is called the 'simplicial sphere' in $sPre$. The following result tells us that algebraic K-theory is representable in our I-local model category.

Proposition 3.5.2. *For any smooth scheme X,*

$$\mathcal{H}o_{I*}(S_s^n \wedge X_+, \mathcal{K}) = K^n(X)$$

where $K^n(X)$ is the Quillen algebraic K-theory of X (usually written $K_{-n}(X)$ by K-theorists).

Lemma 3.5.3. *The Heller-fibrant replacement for $\mathcal{K}$ is also the I-fibrant replacement of $\mathcal{K}$.*

Proof. Let $\tilde{\mathcal{K}}$ be the Heller-fibrant replacement of $\mathcal{K}$. To check that this is I-fibrant we need to verify two things:

(a) For any smooth X and any cover $\{U_\alpha \to X\}$,

$$\tilde{\mathcal{K}}(X) \xrightarrow{\sim} \underleftarrow{\mathrm{holim}} \left[\prod_\alpha \tilde{\mathcal{K}}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \tilde{\mathcal{K}}(U_{\alpha\beta}) \Rrightarrow \prod_{\alpha,\beta,\gamma} \tilde{\mathcal{K}}(U_{\alpha\beta\gamma}) \cdots \right]$$

(b) For any smooth X, $\tilde{\mathcal{K}}(X) \to \tilde{\mathcal{K}}(X \times I)$ is a weak equivalence.

Since $\mathcal{K}(X) \to \tilde{\mathcal{K}}(X)$ is a weak equivalence for all X (because $\mathcal{K} \to \tilde{\mathcal{K}}$ is a *Heller* weak equivalence), it's sufficient to check corresponding properties for $\mathcal{K}$. But then these are just standard facts about Quillen K-theory. □

Proof of proposition. By general nonsense it follows that

$$Ho_{I*}(S_s^n \wedge X_+, \mathcal{K}) = \pi_n\left\{\underline{sPre}(X, \tilde{\mathcal{K}})\right\} = \pi_n\tilde{\mathcal{K}}(X)$$

where $\tilde{\mathcal{K}}$ is an I-fibrant model for $\mathcal{K}$. But the above lemma tells us that $\mathcal{K}(X) \xrightarrow{\sim} \tilde{\mathcal{K}}(X)$, and so $\pi_n\tilde{\mathcal{K}}(X) \cong \pi_n\mathcal{K}(X)$. The latter is by definition the group $K^n(X)$. □

Remark 3.5.4.

(a) Note that the map $\mathcal{K}(X) \to \mathcal{K}(X \times I)$ is generally *not* a weak equivalence for singular schemes X. This is why we had to use the Grothendieck site Sm/k instead of Sch/k if we wanted algebraic K-theory to be representable. It turns out that many constructions from algebraic geometry are not homotopy invariant for singular schemes, and this is a common source of trouble. Voevodsky's work on the h-topology is one attempt at dealing with this.

Exercise 3.5.5. Note that any scheme Z (smooth or not!) gives rise to an element $\tilde{r}Z$ of $sPre(Sm/k)$: $\tilde{r}Z$ is the presheaf defined by $X \mapsto \mathcal{S}ch(X, Z)$. When Z is smooth this is the same object we've always been calling rZ (because $Sm/k \hookrightarrow Sch/k$ is a full subcategory). Define

$$K^n_{new}(Z) := \mathcal{H}o_{I*}(S_s^n \wedge \tilde{r}Z_+, \mathcal{K}).$$

Find a scheme Z for which $K^*_{new}(Z) \ncong K^*(Z)$. How do the two groups differ?

Let us now consider the case of schemes over $\mathbb{C}$. There is a natural map $Sch/\mathbb{C} \to Top$ which associates a scheme X with the space of its $\mathbb{C}$-valued points $X(\mathbb{C})$. It's easy to see that this map preserves the 'relations' we've imposed, so that we get Quillen functors

$$sPre(Sm/\mathbb{C})_I \rightleftarrows \mathcal{T}op.$$

Just as for manifolds, we may also compare $sPre(Sm/\mathbb{C})$ with $s\mathcal{S}et$ via the functor which maps a simplicial set K to the constant simplicial presheaf cK. We therefore again have a composable pair of Quillen functors

$$s\mathcal{S}et \rightleftarrows sPre(Sm/\mathbb{C}) \rightleftarrows \mathcal{T}op$$

and the composites are the usual realization and singular functors. For manifolds the above Quillen functors were all equivalences, but that is not the case here:

Exercise 3.5.6. Let S^1_s denote the constant simplicial presheaf whose value is $\Delta^1/\partial\Delta^1$, and let S^1_t denote the presheaf represented by $\mathbb{A}^1 - \{0\}$. These are called the 'simplicial' and 'topological' circles, respectively. Show that these two circles are not weakly equivalent in $sPre(Sm/\mathbb{C})$, but that their images in $\mathcal{T}op$ *are* weakly equivalent. Conclude that the above adjoint pairs cannot be Quillen equivalences.
(*Hint: To show that S^1_s and S^1_t are not weakly equivalent in $sPre$, it is enough to show that they have different algebraic K-groups (why?). Verify that $K^0(S^1_s) = \mathbb{Z} \oplus \mathbb{C}^*$, whereas $K^0(S^1_t) = \mathbb{Z}$.*)

Exercise 3.5.7. Over a general field k, let $C_1 = \operatorname{Spec} k[x,y]/(xy(1-x-y))$ and let $C_2 = \operatorname{Spec} k[x,y]/(xy(1-x)(1-y))$. Even though these are singular schemes, they give rise to presheaves $\tilde{r}C_1$ and $\tilde{r}C_2$ as in Exercise 3.5.5. Like S^1_s and S^1_t, these are analogues of the circle. Show that S^1_s, $\tilde{r}C_1$, and $\tilde{r}C_2$ are all weakly equivalent in $sPre(Sm/k)$. Formulate a general principle along these lines.

4. Points, the Godement construction, and Jardine's model category.

4.1. **Introduction.** Although the Čech model category arose in a very natural way, we've seen by now that it's not so easy to work with. This is a general problem about localizations of model categories—recognizing when a map is or is not a weak equivalence can be nearly impossible. The goal of this section will be to replace the Čech model structure with one which is similar in spirit, but where the weak equivalences are quite manageable. This is Jardine's model category of simplicial presheaves, and it will be the main object of study in subsequent sections.

It will turn out that Jardine's category is a further localization of the Čech category. So every Čech equivalence is a weak equivalence in Jardine's category, and every fibrant object in Jardine's category is Čech-fibrant. The difference between the two categories is very related to the difference between Čech cohomology and sheaf cohomology. (In fact sheaf cohomology can be interpreted as a certain set of maps in Jardine's homotopy category.) It's useful in applications to have both model categories around, and to be able to go back and forth between them; while Jardine's weak equivalences are more manageable, it's easier to understand what fibrancy means in the Čech category.

Our new notion of weak equivalence will generalize the concept of 'stalkwise weak equivalence' that was introduced in Section 3 for presheaves on $\mathcal{M}an$. The appropriate machinery was all worked out by Grothendieck & Co. for sheaves on a site, and is by now classical. We take a little time to recall some of this:

Definition 4.1.1. *Let $\mathcal{C}$ be Grothendieck site.*

*(a) A **point** of $\mathcal{C}$ is a pair of adjoint functors $p^* : Shv(\mathcal{C}) \rightleftarrows \mathcal{S}et : p_*$ for which the left adjoint p^* preserves finite limits.*

*(b) $\mathcal{C}$ is said to have **enough points** if there is a set of points $\{p_i\}_{i\in I}$ (I some indexing set) with the property that a map of sheaves $F \to G$ is an isomorphism iff for every $i \in I$, the maps of sets $p_i^*(F) \to p_i^*(G)$ is an isomorphism.*

We will usually refer to a point by its left adjoint (by abuse), for instance in the phrase 'Let p^* be a point of $\mathcal{C}$.'

Example 4.1.2. Let M be a manifold in $\mathcal{M}an$, and let $x \in M$ be a point (in the usual sense of the word). Define $p_x^*\colon Shv(\mathcal{M}an) \to \mathcal{S}et$ by the formula

$$p_x^*(F) = \underset{x\in U^{\text{open}}\subseteq M}{\operatorname{colim}} F(U).$$

It can be shown that p_x^* has a right adjoint $(p_x)_*$, and that p_x^* preserves finite limits. In the case where x is the origin of $\mathbb{R}^n$, p_x^* is what we were writing as p_n^* in Section 3.

It's not hard to see that the point p_x depends (up to isomorphism) only on the local dimension of M near x. So we've really produced only countably many different points, one for each possible dimension of a manifold. We may as well use the notation of Section 3 and write these as p_n^*. It's not hard to show that these form a set of 'enough' points.

Discussion 4.1.3. The above definition is extremely simple and compact, but also extremely obtuse. What does it mean? We can use the results of Section 2 to unravel this a little.

Recall from Proposition 2.3.4 that giving an adjoint pair $p^* : Shv(\mathcal{C}) \rightleftarrows \mathcal{S}et : p_*$ is equivalent to just giving a functor $F_p\colon \mathcal{C} \to \mathcal{S}et$ which takes distinguished cones to colimits. The discussion in section 2 showed how a contravariant functor F from $\mathcal{C}$ into $\mathcal{S}et$ could be thought of as encoding a certain diagram in $\mathcal{C}$, for which F would serve as the formal colimit. The dual theory (see Remark 2.5.2) allows us to interpret the *covariant* functor F_p as also encoding a diagram, for which it serves as the formal inverse limit. Stated rigorously, we look at the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{\mathcal{S}et}$, regard F_p as an element of the target category, and consider the overcategory $(\mathcal{C} \downarrow F_p)$. The natural map $P\colon (\mathcal{C} \downarrow F_p) \to \mathcal{C}$ is the diagram we're after.

The following statements are not hard, but somewhat lengthy to prove; we leave them for the industrious reader.

(i) $p^*\colon Shv(\mathcal{C}) \to \mathcal{S}et$ preserves finite limits if and only if the composite $Pre(\mathcal{C}) \to Shv(\mathcal{C}) \to \mathcal{S}et$ (also called p^*, by abuse) has the same property.
(ii) $p^*\colon Pre(\mathcal{C}) \to \mathcal{S}et$ preserves finite limits iff the diagram P if left-filtering.
(iii) F_p takes distinguished cones to colimits iff the diagram P has the following properties:
  (a) for any $X \in \mathcal{C}$ and any cover $\{U_\alpha \to X\}$, if X appears at some spot in the diagram then some U_α appears in the diagram and maps to this X (via the map $U_\alpha \to X$ which came with the cover);
  (b) if U_α and U_β appear in the diagram and map to a common X (also in the diagram), then there is a sequence $U_{i_1}, \dots, U_{i_n}$ such that $U_{i_1} = U_\alpha$, $U_{i_n} = U_\beta$, each U_i appears in the diagram mapping to the common X, and each $U_{i_j} \times_X U_{i_{j+1}}$ appears in the diagram mapping to U_{i_j} and $U_{i_{j+1}}$.
  (The reader is encouraged to sketch out these properties using pictures).
(iv) If G is a sheaf then $p^*(G) \cong \operatorname{colim}_{X\in P} G(X)$.

To get a feeling for the above conditions, consider a category $\mathcal{C}$ of topological spaces with Grothendieck topology given by open covers. If X is a space and $p \in X$ is a point (in the traditional sense), we may form the inverse limit system consisting of all open subsets of X containing p. It is easy to check that this diagram is left-filtering and has the other properties sketched above. It therefore gives rise to a point $p^*\colon Shv(\mathcal{C}) \to \mathcal{S}et$ defined by

$$p^*(F) = \operatorname*{colim}_{p\in U\subseteq X} F(U).$$

The reader may check by brute force that p^* preserves finite limits and admits a right adjoint.

Points can be naively thought of in the following way. Note that all the information about a sheaf G is contained in the hom-sets $Shv(X, G)$ where X ranges over all spaces. (This is a tautology, because $Shv(X, G) = G(X)$ and G is the functor $X \mapsto G(X)$.) But much of this information is redundant: for instance, if $\{U_1, U_2\}$ is an open covering of S^1, we know from the sheaf property that $Shv(S^1, G)$ is the equalizer of

$$Shv(U_1, G) \times Shv(U_2, G) \rightrightarrows Shv(U_1 \cap U_2, G).$$

This says that the information about G contained in $Shv(S^1, G)$ is in some sense unnecessary, because it can be recovered from information already contained in the sets $Shv(U_i, G)$. Intuitively, a point p^* of a Grothendieck site represents information about sheaves which can't be 'further reduced' by passing to covers (this accounts to some extent for the properties of the inverse limit system P sketched above). It is not always true that this kind of 'primitive' information encodes everything about a sheaf—rather, that this should hold is precisely the condition that the site have 'enough' points.

?????????

Theorem 4.1.4 (Jardine)**.** *Let $\mathcal{C}$ be a Grothendieck site with enough points. Then there is a model category structure on $sPre(\mathcal{C})$ in which*
(i) weak equivalences are pointwise weak equivalences;
(ii) cofibrations are monomorphisms;
(iii) fibrations are maps with the appropriate right lifting property.

Remark 4.1.5. Once we know that the above model category structure exists, it is a trivial consequence that it must be precisely the localization of $sPre(\mathcal{C})_H$ with respect to the pointwise weak equivalences.

We next show that Jardine's category is also a localization of our Čech model category. This follows directly from the

Lemma 4.1.6. *If $\{U_\alpha \to X\}$ is a cover then $\check{U}_\bullet \to rX$ is a point-wise weak equivalence of simplicial presheaves. Every Čech weak equivalence is therefore a pointwise weak equivalence.*

Proof. The latter statement follows from the former. To prove the former, the first step is to note that p^* commutes with coproducts and finite limits, so that we have a levelwise isomorphism

$$\begin{tikzcd} p^*(\coprod U_\alpha) \arrow[d, "\wr"] & p^*(\coprod U_{\alpha\beta}) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\wr"] & \cdots \arrow[l] \arrow[l, shift left] \arrow[l, shift right] \\ \coprod p^*(U_\alpha) & \coprod p^*(U_\alpha) \times_{p^*(X)} p^*(U_\beta) \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l] \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

If we set $E = \coprod p^*(U_\alpha)$ and $B = p^*(X)$, the fact that fibred products distribute over coproducts in $\mathcal{S}et$ may be used to identify the lower simplicial set with

$$E \leftleftarrows E \times_B E \Lleftarrow E \times_B E \times_B E \cdots$$

What we are reduced to showing is that the map from this object to the constant simplicial set on B is a weak equivalence. The simplicial set built from the $E \times_B E$'s is easily seen to be fibrant and homotopy discrete, so the only thing left to check is that

$$E \times_B E \rightrightarrows E \longrightarrow B$$

is a coequalizer diagram. But this diagram is the one obtained by applying p^* to

$$\coprod U_{\alpha\beta} \rightrightarrows \coprod U_\alpha \longrightarrow X,$$

so the result follows from the fact that p^* takes distinguished cones to colimits. $\square$

4.2. **The Godement construction.** The Godement construction was originally introduced in the context of sheaves of abelian groups over a topological space. Godement showed how to construct a canonical flasque resolution of such a sheaf, the importance being that this can be used to compute sheaf cohomology. In our present setting the importance is that the Godement construction provides a canonical fibrant replacement for simplicial presheaves (under some slight restrictions). We shall see that this fibrant replacement functor has several 'good' properties.

Let $\mathcal{C}$ be a Grothendieck site with enough points, so that there is a set of point $\{p_i \; : \; i \in I\}$ which detect isomorphisms. Each of these points is a pair of adjoint functors, and we can assemble them together into a single adjunction

$$p^* : Pre(\mathcal{C}) \rightleftarrows \mathcal{S}et^I : p_*.$$

As usual, we will also write p^* and p_* for the extensions of these functors to the simplicial categories: $sPre(\mathcal{C}) \rightleftarrows s\mathcal{S}et^I$.

Let $G = p_* \circ p^*$. Then G is a triple, so we use it to form cosimplicial resolutions. These are the *Godement resolutions.* For F a simplicial presheaf, let

$$\mathcal{G}(F) := \underleftarrow{\operatorname{holim}} \left[GF \rightrightarrows G^2F \cdots \right].$$

Note that there is a canonical map $F \to \mathcal{G}F$.

Lemma 4.2.1. *Both GF and $\mathcal{G}F$ are fibrant in Jardine's category.*

Proof. Since the homotopy limit of a diagram of fibrant object will always be fibrant, the claim about $\mathcal{G}F$ follows from that about GF. To show that GF is fibrant, ???? $\square$

We must next show that $F \to \mathcal{G}F$ is a stalkwise weak equivalence. This is only known to hold under certain assumptions.

Definition 4.2.2. *Let $\mathcal{C}$ be a Grothendieck site. Suppose that for each $X \in \mathcal{C}$ there is an $N \in \mathbb{N}$ such that ?????*

Proposition 4.2.3.

????????????

5. Appendix: Localizing a model category

When $\mathcal{M}$ is a model category and S a set of maps between cofibrant objects, we will be concerned with the problem of producing a new model structure on $\mathcal{M}$ in which the maps S are weak equivalences, and which is in some sense as close to our original model structure as possible. This problem has been well-studied, and the new model structure is called a *localization* of the old one. A theorem of Hirschhorn (Theorem 5.0.7 below) says that when $\mathcal{M}$ is a 'sufficiently nice' model category one can localize at any set of maps. 'Sufficiently nice' entails being cofibrantly generated together with having certain other finiteness properties; the exact notion is that of a *cellular* model category. We will not recall the definition here, but happily refer the reader to [6]. Suffice it to say that all the model categories we encounter in this note are cellular.

Note 5.0.4. To simplify matters we will from now on assume that all model categories are simplicial. This is not strictly necessary, but it allows us to avoid a certain amount of machinery required for dealing with the general case. The inquisitive reader is again referred to [6].

Definition 5.0.5. *Let $\mathcal{M}$ be a simplicial model category.*

(1) If $X \xrightarrow{f} Y$ is a map between cofibrant objects, a fibrant object $Z \in \mathcal{M}$ is called **f-*local*** *if the induced map $\underline{\mathcal{M}}(Y,Z) \to \underline{\mathcal{M}}(X,Z)$ is a weak equivalence. We will often say that Z 'sees f as a weak equivalence'.*

(2) If S is a set of maps between cofibrant objects, a fibrant object Z is called **S-*local*** *precisely when it is f-local for every $f \in S$.*

(3) If S is as above, an **S-*local equivalence*** *is a map which is seen as a weak equivalence by every S-local object. If $X \to Y$ is a cofibrant approximation to our map, the formal requirement is that $\underline{\mathcal{M}}(Y,Z) \to \underline{\mathcal{M}}(X,Z)$ be a weak equivalence for every S-local Z.*

Note 5.0.6. The requirement that our maps have cofibrant domain and target can be easily eliminated by a slight re-working of the definition. The added generality is more of a nuisance than a gain, however. In a model category, every map may be replaced up to weak equivalence by a map between cofibrant objects.

Theorem 5.0.7. *(Hirschhorn) Let $\mathcal{M}$ be a cellular, simplicial model category and S a set of maps between cofibrant objects. Then there exists a new model structure on $\mathcal{M}$ in which*

(i) the weak equivalences are the S-local equivalences;

(ii) the cofibrations are precisely the original cofibrations of $\mathcal{M}$;

(iii) the fibrations are the maps having the right-lifting-property with respect to cofibrations which are also S-local equivalences.

In addition, the fibrant objects of $\mathcal{M}$ are precisely the S-local objects, and this new model structure is again cellular and simplicial.

The model category whose existence is guaranteed by the above theorem is called that ***S*-localization** of $\mathcal{M}$. The underlying category is the same as that of $\mathcal{M}$, but there are more trivial cofibrations (and hence fewer fibrations). We will sometimes use $S^{-1}\mathcal{M}$ to denote the S-localization.

Note that the identity maps yield a Quillen pair $\mathcal{M} \rightleftarrows S^{-1}\mathcal{M}$, where the left Quillen functor is the map $Id\colon \mathcal{M} \to S^{-1}\mathcal{M}$. A fact which is important in the context of this paper is that the localization $S^{-1}\mathcal{M}$ is characterized by the following universal property:

Theorem 5.0.8 (Hirschhorn)**.** *Let $\mathcal{M}$ and S be as in the above theorem. Let $\mathcal{N}$ be a model category and $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ be a Quillen pair such that L takes the elements of S to weak equivalences in $\mathcal{N}$. Then the pair (L,R) extends to a Quillen pair $\tilde{L} : S^{-1}\mathcal{M} \rightleftarrows \mathcal{N} : \tilde{R}$, and this extension is unique up to unique isomorphism.*

6. Glossary

$C(a,b)$ maps from a to b in the category C.

co-complete	containing all small colimits.
cone	a diagram D whose indexing category has a terminal object.
$\underline{\mathcal{M}}(a,b)$	the mapping space between a and b in the simplicial model category $\mathcal{M}$.
$\bigsqcup_\alpha U_\alpha$	the diagram consisting of the objects U_α with only identity maps. (The reason for using $\bigsqcup$ instead of $\coprod$ is that we often have a functor $\mathcal{C} \to \mathcal{D}$ and want to push-forward a diagram in $\mathcal{C}$ to a diagram in $\mathcal{D}$. The problem is that the coproducts in $\mathcal{C}$ and $\mathcal{D}$ may not be respected by our functor; the notation $\bigsqcup U_\alpha$ circumvents unnecessary confusion.)
$\Lambda^{n,k}$	the k-horns in $\Delta[n]$.
$\check{U}_\bullet$	given a cover $\{U_\alpha \to X\}$, this is the simplicial object with $\coprod U_{i_0\dots i_n}$ in dimension n. This is sometimes thought of as a diagram, and sometimes (by regarding each $U_{i_1\dots i_n}$ as a representable presheaf) as a simplicial presheaf.
$\underrightarrow{\mathrm{hocolim}}\, D$	if $D : I \to \mathcal{M}$ is a diagram in a simplicial model category $\mathcal{M}$, $\underrightarrow{\mathrm{hocolim}}\, D$ is the coequalizer of the maps $$\coprod_{i\to j} D_i \otimes B(j\downarrow I)^{op} \rightrightarrows \coprod_{i\in I} D_i \otimes B(i \downarrow I)^{op}.$$ See section 19 of [6]. Note that $\underrightarrow{\mathrm{hocolim}}$ is not homotopy invariant when defined this way: if $D \to E$ is a map of diagrams which is an objectwise weak equivalence, it is not necessarily true that $\underrightarrow{\mathrm{hocolim}}\, D \to \underrightarrow{\mathrm{hocolim}}\, E$ is a weak equivalence. This statement does hold if the objects in D and E are all cofibrant.
$\mathbf{L}\underrightarrow{\mathrm{hocolim}}$	if $\mathcal{M}$ is a simplicial model category and I is an indexing category, this is the map $Ho(\mathcal{M}^I) \to Ho(\mathcal{M})$ which is the left total derived functor of $\underrightarrow{\mathrm{hocolim}}$. (Note that even though $\mathcal{M}^I$ may not have a model category structure, one can still ask about the localization of M^I with respect to the objectwise weak equivalences. This localization exists by results of [4].) $\mathbf{L}\underrightarrow{\mathrm{hocolim}}$ can be defined by choosing a cofibrant-replacement functor Q in $\mathcal{M}$ and setting $\mathbf{L}\underrightarrow{\mathrm{hocolim}}\, D = \underrightarrow{\mathrm{hocolim}}\,(Q \circ D)$.
$\mathcal{C} \rightleftarrows \mathcal{D}$	an adjoint pair of functors, with $\mathcal{C} \to \mathcal{D}$ the left adjoint.
$sPre(\mathcal{C})_{BK}$	the category of simplicial presheaves with the Bousfield-Kan model structure.
$sPre(\mathcal{C})_H$	the category of simplicial presheaves with the Heller model structure.

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