THE ZARISKI AND NISNEVICH DESCENT THEOREMS

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This is a short expository note giving what I hope are careful proofs of the Zariski and Nisnevich descent theorems, appearing here as (1.5) and (2.3). For the Zariski topology the result is due to Brown-Gersten [BG], whereas the Nisnevich analog was proven by Morel-Voevodsky [MV]. The proofs that I give here are my 'translations' of the proofs from those sources.

As in [MV] all schemes are smooth and finite-type over a given Noetherian base, although the smoothness will not be used. In fact the results only depend on the schemes being Noetherian.

1. The Zariski Descent Theorem

Two basic pieces of terminology: (1) If p is a point on a scheme X, then the 'codimension of p' means the dimension of the local ring $\mathcal{O}_{X,p}$. (2) Given a map of simplicial sets $s\colon \partial\Delta^k \to K$, we'll say that s is 'null' if it can be extended over the simplex Δ^k. Although this latter terminology is not completely appropriate, it will work out well for us in the situations we need it.

Consider the following properties of a simplicial presheaf F, where n is a fixed integer with $n \geq -1$:

- (R_n) Given any scheme X, any point $p \in X$ of codimension n, and any map $s\colon \partial\Delta^k \to F(X)$, there exists an open neighborhood of p such that the restriction $s|_U\colon \partial\Delta^k \to F(U)$ is null.
- (SR_n) Given any scheme X, any subset $S \subseteq X$ consisting of points with codimension $\leq n$, and any map $s\colon \partial\Delta^k \to F(X)$, there exists an open neighborhood U of S such that $s|_U\colon \partial\Delta^k \to F(U)$ is null.

Condition (R_n) says that the Zariski stalks of F at any codimension n point are fibrant and contractible. Condition (SR_n) in some sense says the same for the 'stalks' around any codimension n subspace. Notice that (SR_n) implies conditions (R_0) through (R_n). The 'R' stands for 'refinement', and 'SR' is for 'strong refinement'. Observe that (R_{-1}) is vacuous, but that (SR_{-1}) it not—applying it in the case where $X = S = \emptyset$ gives the requirement that $F(\emptyset)$ be fibrant and contractible. We refer the reader to Appendix B for more reflections on the emtpyset.

The general theme in what follows is that if F satisfies these refinement conditions in low codimension AND it has the BG-property, then we can deduce them—at least to some extent—in higher codimension.

We start with two basic lemmas which will be used often. In fact they are the heart of the proof. The first of them is an exercise in point-set topology:

Lemma 1.1. *Suppose that $U \subseteq Y$ is an open inclusion of schemes, p is a point in $Y - U$, and W is an open subset of U. If W contains every point $q \in U$ which specializes to p (meaning that $p \in \overline{\{q\}}$, the closure of q in Y), then p has a neighborhood V in Y such that $W = U \cap V$.*

Proof. W is open in Y, and so it has the form $Y - C$ where C is closed. Any closed set is a finite union $\overline{\{y_1\}} \cup \cdots \overline{\{y_k\}}$ where the y_i are the generic points of the irreducible components of C. Let C' denote the union of those $\overline{\{y_i\}}$ which do not contain p, and let $V = Y - C'$. V is an open neighborhood of p, and clearly $W \subseteq V \cap U$. Showing the subset in the other direction is equivalent to showing $C - C' \subseteq (Y - U)$, so let $x \in C - C'$. This means there is a y_i with $x, p \in \overline{\{y_i\}}$. If we had $y_i \in U$ then our assumption on W would force y_i to be in W as well, which it is not. So y_i is not in U, and therefore x cannot be in U either. □

Unfortunately the statement of the second lemma is a little on the long side, but there seems to be no avoiding it:

Lemma 1.2. *Suppose that F is sectionwise-fibrant, and let $s\colon \partial\Delta^k \to F(X)$ be a map. Suppose that there exists an open subset U over which s is null, but also that there exists a point $p \in X\backslash U$ having a neighborhood on which s is null. Let n denote the codimension of p. If F satisfies both (SR_{n-1}) and the Zariski BG-property, then one can find an open set U' containing both U and p, with the property that on U' the map s is still null.*

Proof. Let V denote the neighborhood of p over which s is null. The BG-property says that the square

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \cap V) \end{array}$$

is homotopy cartesian. We choose a basepoint $* \in F(U \cup V)$ corresponding to a vertex of s, and we look at the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}F(U \cap V) \xrightarrow{\partial} \pi_k F(U \cup V) \xrightarrow{i} \pi_k F(U) \times \pi_k F(V) \longrightarrow \cdots \tag{1.1}$$

The class $[s] \in \pi_k F(U \cup V)$ becomes null under i, and so we can write $[s] = \partial[t]$ for some $t\colon \partial\Delta^{k+1} \to F(U \cap V)$.

Let S be the set of points in $U \cap V$ which specialize to p—these all have codimension less than n. Using (SR_{n-1}) there is an open subset $W \subseteq U \cap V$ containing S, over which t extends to Δ^{k+1}. But by our choice of S it follows from Lemma 1.1 that p has a neighborhood $V' \subseteq V$ with $V' \cap U = W$. Comparing the long exact sequence (1.1) with its analogue for the square

$$\begin{array}{ccc} F(U \cup V') & \longrightarrow & F(V') \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \cap V'), \end{array}$$

we find that $[t]$ becomes null in $F(U \cap V')$ and therefore $[s] = \partial[t]$ is null in $F(U \cup V')$. So we have produced an open set $U \cup V'$ which contains both U and p, and over which s still extends to Δ^k.

Technically we have only dealt with the case $k \geq 1$, since we assumed that $\partial\Delta^k$ was nonempty and had a basepoint. The reader may check that the same general proof still works for $k = 0$, however. □

Proposition 1.3. *Suppose F is sectionwise-fibrant and has the Zariski BG-property. If $n \geq 0$ and F satisfies (R_n) and (SR_{n-1}), then it also satisfies (SR_n).*

Proof. Condition (SR_{n-1}) implies (R_0)–(R_{n-1}), and so F actually satisfies (R_0) through (R_n). Let S be a set of points, all with codimension $\leq n$, and let $s\colon \partial\Delta^k \to F(X)$ be a map. If $S = \emptyset$ then we use (R_0) (applied to a generic point of X) to get an open subset U over which s extends to Δ^k, and we are done. So we may assume S is nonempty.

Consider all open subsets U of X having the property that the restriction of s to U is null, and also that $U \cap S \neq \emptyset$. There is some point $p \in S$, and by the appropriate (R_k) we can find a neighborhood of p over which s is null; so our collection of opens is nonempty. Let U denote a maximal element of this collection.

Suppose that U does not contain S, so that there is a point q in S which is not in U. By the appropriate (R_k) we know there is a neighborhood V of q over which s extends to Δ^k. Using (SR_{n-1}) and the Zariski BG-property, Lemma 1.2 gives an open subset which is strictly larger than U, but over which s still extends to Δ^k. This is a contradiction. □

Proposition 1.4 (Brown-Gersten)**.** *Suppose that F has the Zariski BG-property and the map $F \to *$ is a Zariski weak equivalence. Then $F(X)$ is contractible for every scheme X.*

Proof. Applying Ex^∞ to all of the sections of F gives a new simplicial presheaf with the same properties, so we can assume F is sectionwise-fibrant. The fact that $F \to *$ is a Zariski weak equivalence is equivalent to saying that F satisfies (R_n) for every $n \geq 0$. From (1.3) we then know that F satisfies (SR_n) for all $n \geq 0$.

Suppose that X is a scheme and $s\colon \partial\Delta^k \to F(X)$ is a map. We will show that s can be extended over Δ^k. By applying (R_0) to a generic point of X, we know there is some nonempty open subset of X over which s extends—choose U to be a maximal one (using that X is Noetherian). Suppose there is a point $p \in X - U$, and let n denote its codimension. By (R_n) this point has an open neighborhood V over which s is null. But then using Lemma 1.2 we deduce that s can be extended over Δ^k on some open subset strictly larger than U. This is a contradiction. □

Finally we have the Zariski descent theorem:

Proposition 1.5. *Let $F \to G$ be a map between simplicial presheaves with the Zariski BG-property, and suppose this map is a local weak equivalence. Then it is actually a sectionwise weak equivalence.*

Proof. We must show that $f_X\colon F(X) \to G(X)$ is a weak equivalence for every X. It will be enough to show that for every 0-simplex $x \in G(X)$, the homotopy fiber of f_X is contractible.

Define a simplicial presheaf Φ on the overcategory of X by setting $\Phi(Y) = \mathrm{hofib}(F(Y) \to G(Y))$, where the homotopy fiber is taken over the restriction of x (and this is where we need that Y lies over X). It's easy to see that the BG-property for F and G gets inherited by Φ, essentially because any two homotopy limits commute with each other. The fact that $F \to G$ was a local weak equivalence implies that $\Phi \to *$ is a local weak equivalence.

So we now apply Proposition 1.4 to Φ (where our Grothendieck site has changed to the overcategory of X) and conclude that $\Phi(X)$ is contractible. This does it. □

2. The Nisnevich Descent Theorem

The properties (R_n) and (SR_n) will mean the same thing they did in the last section—in particular, they still only refer to the Zariski topology, not the Nisnevich. But we will also make use of the Nisnevich analogs:

Given a scheme X and a point $p \in X$, a **Nisnevich neighborhood** of p consists of an étale map $V \to X$ and a point $q \in V$ such that q maps to p and the induced map on residue fields $\kappa(q) \to \kappa(p)$ is an isomorphism. For some reason these are usually called 'étale neighbohoods' (for example, cf. [Mi, p. 36]), but that terminology seems very ill-conceived to me—it comes from a time before the Nisnevich topology was around.

We can now consider the following property:

(R_n^{Nis}) Given any scheme X, any point $p \in X$ of codimension n, and any map $s\colon \partial\Delta^k \to F(X)$, there exists a Nisnevich neighborhood (V, q) of p such that s becomes null when pulled back to V.

The above condition implies that the Nisnevich stalks of F at any codimension n point are contractible. We could also introduce the properties (SR_n^{Nis}) and then follow the same general outline as for the Zariski topology, but instead we'll go a slightly different route. The result we're after is of course the following:

Proposition 2.1 (Morel-Voevodsky). *Suppose that F is a simplicial presheaf with the Nisnevich BG-property, such that $F \to *$ is a Nisnevich weak equivalence. Then $F \to *$ is actually a sectionwise weak equivalence, meaning that each $F(X)$ is contractible.*

We will prove this by first showing that $F \to *$ is a *Zariski* weak equivalence, and then appealing to the Zariski descent theorem from section 1.

Proof. As in the Zariski case, we can assume that F is sectionwise-fibrant by applying Ex^∞ to all the sections. Consider the following implications:

- $(SR_{n-1}) + (R_n) + (\mathrm{BG}_{Zar}) \Rightarrow (SR_n)$ (Lemma 1.3);
- $(SR_n) + (R_{n+1}^{Nis}) + (\mathrm{BG}_{Nis}) \Rightarrow (R_{n+1})$ (Lemma 2.2 below).

Since $F \to *$ is a Nisnevich weak equivalence we know (R_n^{Nis}) for all $n \geq 0$. We also know by the BG-property that $F(\emptyset)$ is contractible, which implies (SR_{-1}). Using inducton, the above implications now allow us to deduce (R_n) for every $n \geq 0$. In other words, $F \to *$ is a Zariski weak equivalence. By the Zariski Descent Theorem it then follows that each $F(X)$ is contractible. □

Lemma 2.2. *Suppose that F is a sectionwise-fibrant simplicial presheaf with the Nisnevich BG-property. If F satisfies (SR_n) and (R_{n+1}^{Nis}) then it also must satisfy (R_{n+1}).*

Proof. Let X be a scheme and $s\colon \partial\Delta^k \to F(X)$ be a map. If p is a point of X of codimension $n+1$, we must produce a Zariski neighborhood of p over which s is null. What we know is that there is a Nisnevich neighborhood (V, q) such that s is null when pulled back to V. By Corollary A.2 there is an open neighborhood

V' of q and an open neighborhood X' of p such that in $\pi\colon V' \to X'$ one has that $\pi^{-1}(\overline{\{p\}}) \to \overline{\{p\}}$ is an isomorphism. We replace V by V' and X by X'.

Consider the open subscheme $X - \overline{\{p\}} \hookrightarrow X$ and let S be the set of points in $X - \overline{\{p\}}$ which specialize to p. Using (SR_n) we know there is an open subset $U \subseteq X - \overline{\{p\}}$ containing S and such that s becomes null on U. By Lemma 1.1 there is an open neighborhood $\Omega \subseteq X$ of p such that $\Omega \cap (X - \overline{\{p\}}) = U$. Let $V' = V \times_X \Omega$. It is easy to see that $\{U, V' \to \Omega\}$ is now an elementary Nisnevich cover, and we have arranged things so that s becomes null on both U and V'.

So we have produced a Zariski neighborhood Ω of p and an elementary Nisnevich cover $\{U, V \to \Omega\}$ with the property that s becomes null on both U and V. Let π denote the map $V \to \Omega$, and write $\pi^{-1}(U)$ for $U \times_\Omega V$. The BG-property says that the square

$$\begin{array}{ccc} F(\Omega) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(\pi^{-1}(U)) \end{array}$$

is homotopy cartesian.

We choose a basepoint $* \in F(\Omega)$ corresponding to a vertex of s, and we look at the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}F(\pi^{-1}(U)) \xrightarrow{\partial} \pi_k F(\Omega) \xrightarrow{i} \pi_k F(U) \times \pi_k F(V) \longrightarrow \cdots \tag{2.1}$$

The class $[s] \in \pi_k F(\Omega)$ becomes null under i, and so we can write $[s] = \partial[t]$ for some $t\colon \partial\Delta^{k+1} \to F(U \cap V)$.

Let S be the set of points in $\pi^{-1}(U)$ which specialize to q in V—these all have codimension less than $n+1$. Using (SR_n) there is an open subset $W \subseteq \pi^{-1}(U)$ containing S, over which t becomes null. But by our choice of S it follows from Lemma 1.1 that q has an open neighborhood $V' \subseteq V$ with $V' \cap \pi^{-1}(U) = W$. Let $\Omega' = U \cup \pi(V')$, which is an open subset of Ω. Comparing the long exact sequence (2.1) with its analogue for the square

$$\begin{array}{ccc} F(\Omega') & \longrightarrow & F(V') \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(W), \end{array}$$

we find that $[t]$ becomes null in $F(W)$ and therefore $[s] = \partial[t]$ is null in $F(\Omega')$. So we have produced an open set Ω' containing p, over which s becomes null. □

Now we have the full Nisnevich descent theorem:

Proposition 2.3 (Morel-Voevodsky). *Let $F \to G$ be a map between simplicial presheaves with the Nisnevich BG-property, and suppose this map is a local weak equivalence. Then it is actually a sectionwise weak equivalence.*

Proof. This exactly matches the proof for the Zariski case. □

Appendix A. Basic results about schemes

The statements in this section are elementary results in algebraic geometry. I have included the proofs because these things tend to confuse me, and because I haven't found great references.

Proposition A.1. *Suppose $f\colon X\to Y$ is a map locally of finite type, where X and Y are integral schemes. If the map induces an isomorphism of function fields $K(Y)\xrightarrow{\cong}K(X)$ then there exist open sets $U\subseteq X$ and $V\subseteq Y$ such that f induces an isomorphism $f\colon U\xrightarrow{\cong}V$.*

Proof. This is easy. We immediately reduce to the case where X and Y are affine, in which case we have a map of domains $R\to S$ (where S is finitely-generated as an R-algebra) which induces an isomorphism on quotient fields. It's easy to see that one gets the isomorphism after only inverting finitely many elements. $\square$

Corollary A.2. *Suppose that $f\colon X\to Y$ is locally of finite type. Let $x\in X$ and set $y=f(x)$. If f induces an isomorphism on residue fields $\kappa(y)\to\kappa(x)$ then there are neighborhoods $x\in U\subseteq X$ and $y\in V\subseteq Y$ such that*

(i) f restricts to a map $U\to V$ and,
(ii) this map induces an isomorphism $\overline{\{x\}}\to\overline{\{y\}}$, where these closed sets are given the reduced induced subscheme structure.

Moreover, we can arrange U and V to be irreducible.

Proof. Let $Z=\overline{\{x\}}$ and $W=\overline{\{y\}}$. First remove from X all of the irreducible components which do not contain Z, and remove from Y all the irreducible components not containing W. This gives a map $X'\to Y'$ having the same properties as $X\to Y$, but in which the domain and codomain are irreducible. So we can reduce to this case.

Z and W are integral schemes, and f restricts to a map $Z\to W$ which is an isomorphism on function fields. So by Proposition A.1 there are open sets $U_Z\subseteq Z$ and $U_W\subseteq W$ so that f induces an isomorphism $U_Z\to U_W$.

Let C_Z be the closed set $Z\backslash U_Z$ and let $C_W=W\backslash U_W$. Let $U=X-C_Z$ and $U_Y=Y-C_W$. It is easy to see that $U\cap Z=U_Z$ and $U_Y\cap W=U_W$. Finally, let $U_X=U\cap f^{-1}(U_Y)$.

One may now check that U_X is a neighborhood of x and U_Y is a neighborhood of y, f maps U_X into U_Y, and f restricts to an isomorphism $\overline{\{x\}}\to\overline{\{y\}}$. $\square$

Corollary A.3. *Suppose that $f\colon X\to Y$ is étale, and again let $x\in X$ and $y\in Y$ be such that $f(x)=y$ and $\kappa(y)\to\kappa(x)$ is an isomorphism. Then there are open neighborhoods $x\in U\subseteq X$ and $y\in V\subseteq Y$ such that f maps U into V and in the restriction $f\colon U\to V$ we have that $f^{-1}(\overline{\{y\}})\to\overline{\{y\}}$ is an isomorphism. Again, we may even arrange U and V to be irreducible.*

Proof. Let $\{q_1,\dots,q_k\}$ be the pre-images of y other than x. Let X' be the open subscheme $X-(\overline{\{q_1\}}\cup\cdots\cup\overline{\{q_n\}})$. By replacing X by X' we may assume that $f^{-1}(y)$ consists only of x.

From the previous corollary there are open neighborhoods U of x and V of y such that U maps to V and $f\colon U\to V$ induces an isomorphism $\overline{\{x\}}\to\overline{\{y\}}$. We may arrange U and V to be irreducible. Working within these open sets, we claim we must have $f^{-1}(\overline{\{y\}})=\overline{\{x\}}$: If $z\in f^{-1}(\overline{\{y\}})$ then by the Going Down theorem for flat maps (see [E, Lemma 10.11]) there is a w in U which specializes to z and is such that $f(w)=y$. But we have arranged things so that x is the only point mapping to y, so it must be that $w=x$. Therefore $z\in\overline{\{x\}}$, and we are done. $\square$

Proposition A.4. *Suppose that $\{\pi_i\colon V_i\to X\}$ is a Nisnevich cover of a scheme X. Then there is a refinement $\{W_j\to X\}$ in which $W_0\to X$ is an open immersion.*

Proof. Let η denote a generic point of an irreducible component of X. There is a V_j and a point $\xi \in V_j$ so that $\pi(\xi) = \eta$ and $\kappa(\eta) \to \kappa(\xi)$ is an isomorphism. By Corollary A.2 there are neighborhoods U of ξ and W of η so that π maps U to W and in $U \to W$ we have $\overline{\{\xi\}} \to \overline{\{\eta\}}$ an isomorphism (where these closed subschemes have the reduced induced structure). We can assume U and W are irreducible, in which case $\overline{\{\xi\}} = U$ and $\overline{\{\eta\}} = W$ *as topological spaces*.

If W is reduced then so is U (because $U \to W$ is étale), which means that we have scheme-theoretic equalities $\overline{\{\eta\}} = W$ and $\overline{\{\xi\}} = U$. So $U \to W$ is an isomorphism.

If W is not reduced then we consider the closed immersion $\overline{\{\eta\}} \hookrightarrow W$ and the induced functor $\mathrm{Et}/W \to \mathrm{Et}/\overline{\{\eta\}}$ obtained by pulling back. By [Mi, Theorem 3.23] this map is an equivalence of categories. The maps $U \to W$ and $\mathrm{id}\colon W \to W$ becomes isomorphic under this functor, and so we conclude that $U \to W$ is isomorphic to $W \to W$ as schemes over W. In other words, $U \to W$ is again an isomorphism.

In either case we have shown that $V_i \to X$ is split over W, and so $\{W, V_i \to X\}$ is a Nisnevich cover of X which refines $\{V_i \to X\}$. □

Appendix B. How to handle the emptyset

This section concerns a very slight technical detail. If $\emptyset$ denotes the empty scheme, then it yields a representable presheaf $r\emptyset$. Unfortunately this is *not* the initial object in the category of presheaves: that is the presheaf $c\emptyset$ given by $c\emptyset(X) = \emptyset$ for all X. The only difference between $r\emptyset$ and $c\emptyset$ is their value on $X = \emptyset$, as $r\emptyset(\emptyset) = *$.

What must happen is that in our Grothendieck topology on schemes (or smooth schemes, or whatever) we should make the convention that the 'empty cover' is a covering family for $\emptyset$. When we then pass to our model category on simplicial presheaves, we have forced the map $c\emptyset \to r\emptyset$ to be a weak equivalence—equivalently, we have guaranteed that a fibrant simplicial presheaf will have the property that $F(\emptyset)$ is contractible (because it will be weakly equivalent to the homotopy function complex $\mathrm{Map}(c\emptyset, F) \cong *$).

The BG-property is supposed to encode the descent conditions for certain naive covers, and among these we must include the empty cover of $\emptyset$. So simplicial presheaves having the BG-property must be such that $F(\emptyset)$ is contractible. Without this condition results such as the Zariski and Nisnevich descent theorems are not quite true.

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