

ANOTHER BOOK ON K -THEORY

DANIEL DUGGER

CONTENTS

Preface	2
Part 1. Topological techniques and applications	3
1. The Atiyah-Hirzebruch spectral sequence, Part 1	3
2. The Atiyah-Hirzebruch spectral sequence, Part 2	13
3. Algebraic cycles on complex varieties	23
4. Operations on K -theory	35
5. The Hopf invariant one problem	47
6. Calculation of KO for stunted projective spaces	61
7. Solution to the vector field problem	79
8. The immersion problem for $\mathbb{R}P^n$	90
9. The sums-of-squares problem and beyond	103
Part 2. An introduction to index theory	116
10. K -theory and Fredholm operators, part I	116
11. K -theory and Fredholm operators, part 2	131
12. ???	142
13. Complex-oriented cohomology theories, again	142
14. K -theory with compact support	146
Part 3. Bott periodicity	148
15. Periodicity via Bott-Morse theory	150
16. Bott periodicity and Clifford algebras	171
17. The Clifford menagerie	181
18. Pinning down Bott periodicity	190
19. Pinning down Bott periodicity, Part 2	201
20. Graded Clifford modules and K -theory	216
21. Spectral proofs of Bott periodicity	237
22. Other proofs of Bott periodicity	248
Part 4. What is K-theory?	264
23. The second star to the right	264
References	279

PREFACE

Right now this is a collection of leftovers that didn't make it into the book "A geometric introduction to K -theory". Maybe one day it will be another book (hence the working title). This material concentrates more on the topological side of K -theory, whereas the former book was focused more on the links between geometry and topology.

At the moment there are many broken references created by peeling off the whole first part of the book. These will get fixed some day. Sorry!

Part 1. Topological techniques and applications

In the next few sections we will mostly ignore the “geometric” perspective on K -theory that we have developed so far. Instead we will concentrate on the topological aspects of K -theory, in particular its use *as a cohomology theory* (forgetting about the complex-orientation). We will develop the basic topological tools for computing K -groups, use them to carry out some important computations, and then apply these computations to solve (or at least obtain information about) certain types of geometric, algebraic, and topological problems.

1. THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE, PART 1

In this section we review the basic workings of spectral sequences and set up the Atiyah-Hirzebruch spectral sequence, which is the main tool for computing the groups $K^*(X)$ and $KO^*(X)$. In the next section we will discuss several examples and then delve into some of the deeper structure of the spectral sequence, for example the nature of the differentials and the relation to the Chern character.

Let $\mathcal{E}$ be a cohomology theory. Let X be a CW-complex with cellular filtration

$$\emptyset = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq X.$$

So each F_k/F_{k-1} is a wedge of k -spheres, and $X = \bigcup_k F_k$. Since one knows the cohomology groups $\mathcal{E}^*(F_k/F_{k-1})$, one can attempt to inductively determine the cohomology groups $\mathcal{E}^*(F_k)$ and thus to eventually determine $\mathcal{E}^*(X)$. A spectral sequence is a device for organizing all the information in such a calculation, and it has the surprising feature that one can sometimes determine $\mathcal{E}^*(X)$ without *explicitly* determining each of the steps $\mathcal{E}^*(F_k)$. It is somewhat magical that this can be done.

We will not try to teach the theory of spectral sequences from scratch here. For a thorough treatment the reader may refer to [McCleary], for example. We will assume some familiarity with this theory, but we will also give a brief though self-contained review.

1.1. Generalities. Each inclusion $F_{q-1} \hookrightarrow F_q$ yields a long exact sequence on cohomology, and these long exact sequences braid together to yield the following infinite diagram:

$$(1.2) \quad \begin{array}{ccccccc} & & \downarrow i & & \downarrow i & & \\ \cdots \longrightarrow & \boxed{\mathcal{E}^{p-1}(F_q)} & \xrightarrow{j} & \mathcal{E}^p(F_{q+1}, F_q) & \xrightarrow{k} & \mathcal{E}^p(F_{q+1}) & \xrightarrow{j} \mathcal{E}^{p+1}(F_{q+2}, F_{q+1}) \xrightarrow{k} \\ & \downarrow i & & \downarrow i & & \downarrow i & \\ \cdots \longrightarrow & \boxed{\mathcal{E}^{p-1}(F_{q-1})} & \xrightarrow{j} & \boxed{\mathcal{E}^p(F_q, F_{q-1})} & \xrightarrow{k} & \boxed{\mathcal{E}^p(F_q)} & \xrightarrow{j} \mathcal{E}^{p+1}(F_{q+1}, F_q) \xrightarrow{k} \\ & \downarrow i & & \downarrow i & & \downarrow i & \\ \cdots \longrightarrow & \mathcal{E}^{p-1}(F_{q-2}) & \xrightarrow{j} & \mathcal{E}^p(F_{q-1}, F_{q-2}) & \xrightarrow{k} & \boxed{\mathcal{E}^p(F_{q-1})} & \xrightarrow{j} \boxed{\mathcal{E}^{p+1}(F_q, F_{q-1})} \xrightarrow{k} \\ & \downarrow i & & \downarrow i & & \downarrow i & \end{array}$$

The terms in boxes constitute one long exact sequence: the one for the inclusion $F_{q-1} \hookrightarrow F_q$. Shifting these terms vertically yields an infinite family of long exact sequences, each linked to the next via two of their three terms. A spectral sequence is a bookkeeping device for managing diagram chases in this setting. One obtains a spectral sequence of the form

$$E_1^{p,q} = \mathcal{E}^p(F_q, F_{q-1}) \Rightarrow \mathcal{E}^p(X).$$

Let us explain how this works, and in the course of doing so we will also explain what it means. By the way, notice that in our particular setup the columns of the diagram are eventually zero as one proceeds downward, because $F_i = \emptyset$ when i is negative. Notice as well that if $X = F_n$ for some value of n (so the filtration is finite) then the columns stabilize when moving upwards.

Remark 1.3. There seems to be no standard name for diagrams like the one above, consisting of families of long exact sequences braided together. But they are the source of essentially every spectral sequence. Massey defined the more general notion of an *exact couple*, but personally I have found that level of generality more overwhelming than useful. Briefly, define $D = \oplus_{p,q} \mathcal{E}^p(F_q)$ and $E = \oplus_{p,q} \mathcal{E}^p(F_q, F_{q-1})$. So the D groups form two-thirds of the columns in the above diagram, and the E -groups are the remaining third. The exact couple consists of the maps

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

and the requirement of exactness at each vertex. While this is an elegant abstraction of our situation, the suppression of the gradings can get in the way of understanding what is happening.

We will refer to our braided long exact sequences as an “exact couple”, even though this is a slight abuse of terminology. “Bigraded exact couple” might be more appropriate.

We return to our development of the spectral sequence machinery:

(0). Here is the basic idea for how the spectral sequence operates. Consider an element $x \in \mathcal{E}^p(F_q, F_{q-1})$, and proceed as follows:

- (i) Let $x_0 = kx$.
- (ii) If $j(x_0) = 0$ then $x_0 = i(x_1)$ for some $x_1 \in \mathcal{E}^p(F_{q+1})$. We may then look at jx_1 .
- (iii) If $jx_1 = 0$ then $x_1 = i(x_2)$ for some $x_2 \in \mathcal{E}^p(F_{q+2})$. We may then look at jx_2 .
- (iv) Continuing in this way, we get a sequence of “obstructions” jx_u , $u = 0, 1, 2, \dots$. Each one only exists if the previous one vanishes. Note that at each stage the vanishing of jx_u can depend on the choice of x_u , as well as the choices made for the x_v at previous stages. In this sense the obstructions are not unique: different choices of lifts may lead to different obstructions. We will see, though, that the obstructions become unique after quotienting out by a subgroup that we can maintain some control over.
- (v) Two questions to consider: (A) For each n , can we make a consistent choice of liftings $x_1, \dots, x_n$? In other words, can we lift kx arbitrarily far up in the diagram? And (B) Can we make a consistent choice of liftings x_i for all $i \geq 1$?

That is to say, can we lift kx *coherently* all the way up the diagram? Questions (A) and (B) are not equivalent, because it could be that extending from x_n to x_{n+1} is not possible even though it would have been possible with a different choice of the lower x_i . If one constantly has to be changing the lower choices in order to make each extension, it could be that there is no coherent choice of all the extensions as once.

The vanishing of our obstructions says something about these questions. If the filtration $F_\bullet$ was finite then the questions are the same, and imply that we have produced an element of $\mathcal{E}^p(X)$. In situation (B) something like this also works for infinite filtrations, although the resulting element of $\mathcal{E}^p(X)$ is only uniquely determined in good cases. The spectral sequence is a device for keeping track of these obstructions and liftings, and what they ultimately produce.

We will now go through all of the machinery needed to define and work with the spectral sequence associated to our exact couple. **Important advice:** Write out your own version of the exact couple (1.2) with a few more terms added, and carefully follow along in the diagram while reading.

(1). For $1 \leq r \leq \infty$ write

$$Z_r^{p,q} = \{x \in \mathcal{E}^p(F_q, F_{q-1}) \mid kx \text{ may be lifted at least } r \text{ times under } i\}.$$

This is called the group of **r -cycles** in the spectral sequence. The phrasing is ambiguous when $r = \infty$, but we mean $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$. We also define $B_r^{p,q} \subseteq Z_r^{p,q}$ to be the subgroup generated by all “obstructions” that arise from at most $r-1$ layers lower down in the diagram. To be precise, $B_r^{p,q}$ is spanned by the sets

$$ji^{-s}k(\mathcal{E}^{p-1}(F_{q-s-1}, F_{q-s-2}))$$

for $0 \leq s \leq r-1$. It is best to immediately forget this precise description and just remember the idea.

Notice that everything in $B_r^{p,q}$ maps to zero under k , and hence is contained in every $Z_t^{p,q}$. That is, we have

$$0 = B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_2^{p,q} \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} = \mathcal{E}^p(F_q, F_{q-1}).$$

Note that $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$ and $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$.

(2). It is an easy exercise to prove that $x \in B_r^{p,q}$ if and only if x can be written as $x = j(y_1 + y_2 + \cdots + y_r)$ for some y_u 's such that $i^u(y_u) = 0$, for all u . As an immediate corollary, $B_\infty^{p,q}$ coincides with the image of j , or equivalently with the kernel of k (here we are using that $F_{-1} = \emptyset$).

Let $x_0 \in \mathcal{E}^p(F_q)$ and assume that we lifted x_0 a total of r times: that is, assume we have chosen elements $x_u \in \mathcal{E}^p(F_{q+u})$ for $1 \leq u \leq r$ such that $i(x_u) = x_{u-1}$. If $j(x_r) \in B_s$ for some $s \leq r-1$ then (using the result of the previous paragraph) there exists a chain of elements $x'_u \in \mathcal{E}^p(F_{q+u})$ for $1 \leq u \leq r$ such that $i(x'_u) = x'_{u-1}$, $x'_u = x_u$ for $u \leq r-s$, and $j(x'_r) = 0$. That is to say, we can alter our chain of x_u 's in the top s spots and end up with a chain that can be extended upwards one more level. Indeed, just define $x'_u = x_u - i^{r-u}(y_1 + \cdots + y_s)$ where $j(x_r) = j(y_1 + \cdots + y_s)$ and the y_i 's are as in the preceding paragraph.

(3). Define $E_r^{p,q} = Z_r^{p,q}/B_{r-1}^{p,q}$. The process “apply k , lift $r-1$ times, then apply j ” yields a well-defined map $d_r: E_r^{*,*} \rightarrow E_r^{*,*}$. This is our “obstruction” map, and it

is now well-defined precisely because we are quotienting out by the subgroup $B_{r-1}^{*,*}$. Note that d_r shifts the bigrading, so that we have $d_r: E_r^{p,q} \rightarrow E_r^{p+1,q+r}$.

The map d_r satisfies $d_r^2 = 0$ because $kj = 0$. A little work shows that $E_{r+1}^{*,*}$ is precisely the homology of $E_r^{*,*}$ with respect to d_r . The sequence of chain complexes $E_1, E_2, E_3, \dots$, each the homology of the previous one, is the **spectral sequence** associated to our exact couple.

(4). For fixed values of p and q we have “entering” and “exiting” differentials

$$E_r^{?,?} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{?,?}$$

for certain unimportant values of ‘?’’. Using (2) it is easy to check that

- (i) The exiting map d_r is zero if and only if $Z_{r-1}^{p,q} = Z_r^{p,q}$, and
- (ii) The entering map d_r is zero if and only if $B_{r-1}^{p,q} = B_r^{p,q}$.

So if in degree (p, q) all of the entering and exiting d_r vanish for $r \geq N$ then we have $Z_\infty^{p,q} = Z_{N-1}^{p,q}$, $B_\infty^{p,q} = B_{N-1}^{p,q}$, and $E_\infty^{p,q} = Z_{N-1}^{p,q}/B_{N-1}^{p,q} = E_N^{p,q}$.

(5). It remains to interpret the E_∞ -term. Notice that, strictly speaking, this is not one of the stages of our spectral sequence—it is not by definition obtained as the homology of a previous complex. Still, in many examples one finds that E_∞ agrees with some finite stage E_r , at least through a range of degrees (as in (4) above).

Write $\mathcal{E}^p(F_q)_\infty$ for the set of all $x \in \mathcal{E}^p(F_q)$ such that $i(x) = 0$ and x lifts arbitrarily far up in the diagram (and note that this is *not* the same as saying that x lifts to the inverse limit). The map k induces a surjection $Z_\infty^{p,q} \twoheadrightarrow \mathcal{E}^p(F_q)_\infty$, and $B_\infty^{p,q}$ is clearly the kernel since it equals the image of j . So we have an induced isomorphism

$$\Gamma: E_\infty^{p,q} \xrightarrow{\cong} \mathcal{E}^p(F_q)_\infty.$$

Consider the groups $\mathcal{E}^p(\mathcal{F}) = \lim_q \mathcal{E}^p(F_q)$. As for any inverse limit these come with a filtration where we define

$$\mathcal{E}^p(\mathcal{F})_{ZPq} = \{\alpha \in \mathcal{E}^p(\mathcal{F}) \mid \text{the image of } \alpha \text{ in } \mathcal{E}^p(F_{q-1}) \text{ is zero}\}.$$

The “ ZPq ” subscript is supposed to remind us “zero past F_q ”. We call this the “ ZP -filtration”:

$$\mathcal{E}^p(\mathcal{F}) = \mathcal{E}^p(\mathcal{F})_{ZP0} \supseteq \mathcal{E}^p(\mathcal{F})_{ZP1} \supseteq \mathcal{E}^p(\mathcal{F})_{ZP2} \supseteq \dots$$

There is an evident map $\mathcal{E}^p(\mathcal{F})_{ZPq} \rightarrow \mathcal{E}^p(F_q)_\infty$ which induces an injection

$$(1.4) \quad \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)} := \mathcal{E}^p(\mathcal{F})_{ZPq} / \mathcal{E}^p(\mathcal{F})_{ZP(q+1)} \hookrightarrow \mathcal{E}^p(F_q)_\infty = E_\infty^{p,q}.$$

Notice our notation for the associated graded of the ZP -filtration.

(6). We are ultimately trying to get information about $\mathcal{E}^*(X)$. Observe that we have a natural map $\mathcal{E}^p(X) \rightarrow \mathcal{E}^p(\mathcal{F})$. This is always surjective; while not obvious, it is a consequence of the fact that if Z is any topological space then $[X, Z] \rightarrow \lim_q [X_q, Z]$ is surjective, which in turn is a routine application of the homotopy extension property for cellular inclusions (this also appears in the surjectivity part of the Milnor exact sequence). Let

$$\mathcal{E}^p(X)_{ZPq} = \ker(\mathcal{E}^p(X) \rightarrow \mathcal{E}^p(F_{q-1}))$$

and note that we have a map of filtrations $\mathcal{E}^p(X)_{ZP\bullet} \rightarrow \mathcal{E}^p(\mathcal{F})_{ZP\bullet}$. The map on associated graded groups

$$\mathcal{E}^p(X)_{ZP(q/q+1)} \rightarrow \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)}$$

is readily seen to be an isomorphism, for all q .

(7). If we are in “good” cases then the map in (1.4) will actually be an isomorphism. The question is whether an element of $\mathcal{E}^p(F_q)$ that can be lifted arbitrarily high in the diagram can also be lifted into the inverse limit—note that this is not automatic! It might be possible that higher and higher liftings exist but not “coherently”; that is, to get a higher lifting one needs to change arbitrarily many elements lower down in the chain.

Fix a p and consider the following condition:

(SSC_p) : There exists an N such that for all q the differentials entering and exiting the group $E_r^{p,q}$ are zero for all $r \geq N$.

If this “Spectral Sequence Convergence Condition” holds then by (4) we know $Z_{N-1}^{p,q} = Z_r^{p,q}$ and $B_{N-1}^{p,q} = B_r^{p,q}$ for all $r \geq N$. Therefore $Z_\infty^{p,q} = Z_{N-1}^{p,q}$, $B_\infty^{p,q} = B_{N-1}^{p,q}$, and consequently $E_\infty^{p,q} = E_N^{p,q}$. These statements hold for all values of q . So the $E_\infty^{p,*}$ groups coincide with the stable values of the $E_r^{p,*}$ groups.

The convergence condition (SSC_p) gives us one more important consequence, namely that the map of (1.4) is an isomorphism:

$$(SSC_p) \Rightarrow \left[\mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)} \cong E_\infty^{p,q}, \text{ for all } q \right].$$

To prove this, let $x_0 \in \mathcal{E}^p(F_q)_\infty$. Then there exist elements $x_u \in \mathcal{E}^p(F_{q+u})$ for $1 \leq u \leq 2N$ such that $i(x_u) = x_{u-1}$ for each u (nothing special about $2N$ is being used here, it is just a convenient large number). The element $j(x_{2N})$ lies in B_{2N+1} , which we have seen equals B_{N-1} by (SSC_p) . Therefore by (2) we may modify the x_u 's in dimensions $x_{2N}, x_{2N-1}, \dots, x_{N+2}$ in such a way that the chain extends to an x_{2N+1} . It is important here that only the top $N-1$ elements are affected, for we can now continue by induction and build an element of $\mathcal{E}^p(\mathcal{F}) = \lim_s \mathcal{E}^p(F_s)$ that maps to x in $\mathcal{E}^p(F_q)$. This completes the proof.

To summarize, the statement that the spectral sequence “converges to $\mathcal{E}^p(X)$ ” is usually interpreted to mean that

- The groups $E_r^{p,*}$ stabilize, and equal $E_\infty^{p,*}$, at some value of r (usually one that is independent of the grading $*$); and,
- The maps in (1.4) are isomorphisms, for all values of q .

Under these conditions the stable values of the spectral sequence give the associated graded groups of the ZP -filtration on $\mathcal{E}^p(X)$. We have seen that (SSC_p) implies this kind of convergence.

(8). Sometimes it is convenient to have in mind a variation on the exact couple diagram (1.2). Fix an integer q and consider the following:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{\mathcal{E}^p(F_{q+2}, F_{q-1})} & \xrightarrow{k} & \mathcal{E}^p(F_{q+2}) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+3}, F_{q+2}) \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{\mathcal{E}^p(F_{q+1}, F_{q-1})} & \xrightarrow{k} & \mathcal{E}^p(F_{q+1}) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+2}, F_{q+1}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}^p(F_q, F_{q-1}) & \xrightarrow{k} & \mathcal{E}^p(F_q) & \xrightarrow{j} & \mathcal{E}^{p+1}(F_{q+1}, F_q) \end{array}$$

The groups in the boxes are “new”, in the sense that they are not part of the exact couple (and they are being drawn in places which previously were occupied by other groups from the exact couple). Each trio of groups $\mathcal{E}^p(F_{q+r}, F_{q-1}) \rightarrow \mathcal{E}^p(F_{q+r-1}, F_{q-1}) \xrightarrow{jk} \mathcal{E}^{p+1}(F_{q+r}, F_{q+r-1})$ is part of the long exact sequence for a triple, and so is exact in the middle spot. From this one can see via a diagram chase that a class $u \in \mathcal{E}^p(F_q, F_{q-1}) = E_1^{p,q}$ lies in Z_r if and only if u lifts to a class in $\mathcal{E}^p(F_{q+r}, F_{q-1})$ (for this argument one needs to combine the above diagram with (1.2) and do the chase in both at once). So the differentials in the spectral sequence can be viewed as a sequence of obstructions for lifting u to a class in $\mathcal{E}^p(F_N, F_{q-1})$, for larger and larger N .

(9). (Summary of the general workings of spectral sequences). We have produced a sequence of bigraded chain complexes $E_1^{*,*}, E_2^{*,*}, \dots$ such that each equals the homology of the previous one. We also have a “limiting” collection of bigraded groups $E_\infty^{*,*}$, and we have seen that under certain convergence conditions these E_∞ groups really are the “stable values” in the sequence of E_r ’s, and moreover they give the associated graded groups for the ZP -filtration of $\mathcal{E}^*(X)$.

(10). Everything that we have said so far works for any increasing filtration $F_\bullet$ of X that is empty in small enough degrees. Now we use the fact that a CW-filtration is very special. Notice that

$$E_1^{p,q} = \mathcal{E}^p(F_q, F_{q-1}) \cong \tilde{\mathcal{E}}^p(F_q/F_{q-1}) \cong \tilde{\mathcal{E}}^p\left(\bigvee_{\alpha} S^q\right) \cong \bigoplus_{\alpha} \tilde{\mathcal{E}}^p(S^q) \cong \bigoplus_{\alpha} \mathcal{E}^{p-q}(pt)$$

where the wedges and direct sums are over the set of q -cells in X . We can identify this group with the cellular cochain group $C^q(X; \mathcal{E}^{p-q}(pt))$. The differential $d_1: E_1^{p,q} \rightarrow E_1^{p+1,q+1}$ is a map $C^q(X; \mathcal{E}^{p-q}(pt)) \rightarrow C^{q+1}(X; \mathcal{E}^{p-q}(pt))$ and it is readily checked to coincide with the differential in the cellular cochain complex. We conclude that

$$E_2^{p,q} \cong H^q(X; \mathcal{E}^{p-q}(pt)).$$

Notice that the E_2 -term is a homotopy invariant of X , whereas the E_1 -term was not.

(11). The bigraded groups forming each term of the spectral sequence can be reindexed in whatever way seems convenient, and topologists use various conventions in different settings. For the Atiyah-Hirzebruch spectral sequence the standard convention is to choose the grading $E_1^{p,q} = \mathcal{E}^{p+q}(F_p, F_{p-1})$ so that we get

$$E_2^{p,q} = H^p(X; \mathcal{E}^q(pt)).$$

Under this grading we have that the differential d_r is a map

$$d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}.$$

The groups in the spectral sequence are drawn on a grid where p is the horizontal axis and q the vertical one, with $E_2^{p,q}$ drawn in the (p, q) -spot. Finally, in this grading the Γ -map relating the E_∞ -term to the associated graded of $\mathcal{E}^*(X)$ has the form

$$\Gamma: \mathcal{E}^{p+q}(X)_{ZP(p/p+1)} \hookrightarrow E_\infty^{p,q}.$$

In terms of the charts, the “total degree” lines are the diagonals where $p + q$ is constant. The (SSC_t) condition, translated into this new indexing, says that on the diagonal $p + q = t$ all entering and exiting differentials vanish past some finite stage

of the spectral sequence. When this condition holds we are guaranteed convergence for the groups along this diagonal.

When one uses the Atiyah-Hirzebruch spectral sequence it is easy (after a while) to remember the indexing scheme for the groups and differentials, and the fact that the diagonals $p + q = n$ give the associated graded of $\mathcal{E}^n$. The other thing to keep track of is whether the $E_\infty^{p,q}$ groups give the p th or q th associated graded piece of the filtration (those are the only two obvious choices). To recall which one, just remember that the filtration is associated to cellular dimension, which is what also indexes cohomology H^* . The formula $E_2^{p,q} = H^p(X; \mathcal{E}^q)$ is enough to identify the p variable as the one that is relevant here, so that $E_\infty^{p,q}$ is the p th graded piece.

Remark 1.5 (Warning about indexing.) For the rest of this book we will adopt the indexing conventions from (11) above, which are the standard ones for the Atiyah-Hirzebruch spectral sequence. This is **different** than the indexing we used in (0)–(10).

Remark 1.6 (Independence of cell structure). As we have defined things, the spectral sequence depends on the chosen CW-structure on X . However, this dependence actually goes away from the E_2 -term onward. Let X_1 and X_2 denote the same space but with two different CW-structures. The identity map $X_1 \rightarrow X_2$ is homotopic to a cellular map $f: X_1 \rightarrow X_2$, and f gives us a map of spectral sequences by naturality. Since f is a homotopy equivalence it induces an isomorphism on the E_2 -terms, and therefore on all the finite pages of the spectral sequence as well.

The ZP -filtration on X was defined in terms of the CW-structure, but we can define it in a different way that doesn't make use of that. We leave it as an exercise to check that

$$\mathcal{E}^p(X)_{ZPq} = \{\alpha \in \mathcal{E}^p(X) \mid u^*(\alpha) = 0 \text{ for any map } u: A \rightarrow X \text{ where } A \text{ is a CW-complex of dimension less than } q\}.$$

These remarks show us that the spectral sequence from E_2 -onwards may be regarded as a natural homotopy invariant of X . In particular, any map $g: X \rightarrow Y$ gives a map of spectral sequences in the opposite direction (by replacing g with a cellular map).

Exercise 1.7. Verify the above assertion about $\mathcal{E}^p(X)_{ZPq}$.

1.8. The Postnikov tower approach. Let $\mathcal{E}$ be a spectrum representing the cohomology theory $\mathcal{E}^*$ and let $P_n\mathcal{E}$ denote the n th Postnikov section for $\mathcal{E}$ (the spectrum obtained from $\mathcal{E}$ by attaching cells to kill off π_{n+1} and higher). There is a tower of fibrations

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1\mathcal{E} & \longrightarrow & P_0\mathcal{E} & \longrightarrow & P_{-1}\mathcal{E} \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \Sigma H(\mathcal{E}_1) & & H(\mathcal{E}_0) & & \Sigma^{-1}H(\mathcal{E}_{-1}) \end{array}$$

where $\mathcal{E}_i = \pi_i(\mathcal{E}) = \mathcal{E}^{-i}(pt)$ and HA denotes the Eilenberg-MacLane spectrum for the group A . If we apply function spectra $F(X, -)$ to all the spots in this diagram

we get a new tower of fibrations

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & F(X, P_1 \mathcal{E}) & \longrightarrow & F(X, P_0 \mathcal{E}) & \longrightarrow & F(X, P_{-1} \mathcal{E}) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & F(X, \Sigma H(\mathcal{E}_1)) & & F(X, H(\mathcal{E}_0)) & & F(X, \Sigma^{-1} H(\mathcal{E}_{-1}))
 \end{array}$$

Each fibration sequence $F(X, \Sigma^q H(\mathcal{E}_q)) \rightarrow F(X, P_q \mathcal{E}) \rightarrow F(X, P_{q-1} \mathcal{E})$ gives rise to a long exact sequence in homotopy groups, and these long exact sequences intertwine to form an exact couple. The associated spectral sequence has

$$\begin{aligned}
 E_1^{p,q} &= \pi_{-p-q} F(X, \Sigma^q H(\mathcal{E}_q)) = H^{p+2q}(X; \mathcal{E}^{-q}), \\
 d_r: E_r^{p,q} &\longrightarrow E_r^{p-r+1, q+r},
 \end{aligned}$$

and it is trying to converge to

$$\pi_{-p-q} F(X, \mathcal{E}) = \mathcal{E}^{p+q}(X).$$

It is not obvious, but with some trouble it can be seen that after re-indexing this is “the same” as the previously-constructed spectral sequence but with the E_1 -term of this one corresponding to the E_2 -term of the one constructed via CW-structures.

We won’t really need this Postnikov version of the spectral sequence for anything, but it often provides a useful perspective. For example, note that this version of the spectral sequence is manifestly functorial in X and a homotopy invariant.

1.9. Differentials. The d_2 -differential in the Atiyah-Hirzebruch spectral sequence is a map $H^p(X; \mathcal{E}^q) \rightarrow H^{p+2}(X; \mathcal{E}^{q-1})$. This is natural in X and it is also stable under the suspension isomorphism; so it is a stable cohomology operation. The d_3 -differential is in some sense a secondary cohomology operation, and so on for all the differentials. This is often a useful perspective. For example, we can now prove the following general fact:

Proposition 1.10. *Suppose that the coefficient groups $\mathcal{E}^*(pt)$ are rational vector spaces. Then for any space X the differentials in the Atiyah-Hirzebruch spectral sequence all vanish. If X is homotopy equivalent to a finite-dimensional CW-complex then there are (non-canonical) isomorphisms*

$$\mathcal{E}^n(X) \cong \bigoplus_{p+q=n} H^p(X; \mathcal{E}^q(pt))$$

for every $n \in \mathbb{Z}$.

Proof. The point is that the only stable cohomology operation of nonzero degree on $H^*(-; \mathbb{Q})$ is the zero operation. This immediately yields that all d_2 -differentials are zero. But then d_3 is a stable cohomology operation (not a secondary operation anymore) and therefore it also vanishes. Continue by induction.

Since $\mathcal{E}^n(X)$ is a rational vector space and has a finite filtration where the quotients are the $\mathbb{Q}$ -vector spaces $H^p(X; \mathcal{E}^q)$, there are no extension problems and one obtains the isomorphism from the statement. $\square$

Corollary 1.11. *Suppose the groups $\mathcal{E}^*(pt)$ are all torsion-free, and that X is a space with $H^*(X; \mathbb{Z})$ free in each degree. Then all the differentials in the Atiyah-Hirzebruch spectral sequence vanish.*

Proof. Consider the natural transformation of cohomology theories $\mathcal{E}^* \rightarrow \mathcal{E}_{\mathbb{Q}}^*$, which is an injection on coefficients. There is a resulting map between the two Atiyah-Hirzebruch spectral sequences, and by the universal coefficient theorem this is an injection on E_2 -terms (this is where we use that $H^*(X)$ is free). So if there were a nonzero d_2 -differential in the $\mathcal{E}$ -spectral sequence, there would also be one in the $\mathcal{E}_{\mathbb{Q}}$ -spectral sequence—which there are not. So all d_2 differentials vanish, and continuing inductively shows the same for d_r for all r . $\square$

Remark 1.12. In the Postnikov approach to the Atiyah-Hirzebruch spectral sequence one sees the d_2 -differentials very explicitly. The Postnikov tower has “ k -invariants” of the form

$$\Sigma^q H(\mathcal{E}_q) \rightarrow \Sigma^{q+2} H(\mathcal{E}_{q+1})$$

which desuspend to give $H(\mathcal{E}_q) \rightarrow \Sigma^2 H(\mathcal{E}_{q+1})$. These are quite visibly stable cohomology operations $H^*(-; \mathcal{E}^{-q}) \rightarrow H^{*+2}(-; \mathcal{E}^{-q-1})$. The connection between higher differentials and higher cohomology operations has a similar realization, but the details are too cumbersome to recount here.

It is worth observing that from the E_2 -term onward there are never any differentials emanating from the $p = 0$ line of the spectral sequence. For convenience assume X is connected. The map $X \rightarrow *$ induces a map of Atiyah-Hirzebruch spectral sequences that is an isomorphism at the E_2 -page on the $p = 0$ line, and so naturality immediately forces the vanishing of these differentials. Alternatively, choose a cell structure on X where $F_0 = \{*\}$. By the remarks in (8) above, such differentials would be the obstructions for a class in $\mathcal{E}^q(F_0, \emptyset)$ to lift to $\mathcal{E}^q(F_r, \emptyset)$; but such a lifting necessarily exists, because $F_0 \hookrightarrow F_r$ is split.

Note that when X is connected $\mathcal{E}^q(X)_{ZP1} = \tilde{\mathcal{E}}^q(X)$, and $\mathcal{E}^q(X)_{ZP(0/1)} = \mathcal{E}^q(X)/\tilde{\mathcal{E}}^q(X) = \mathcal{E}^q(pt)$, giving further confirmation that the E_{∞} -term coincides with the E_2 -term on the $p = 0$ line.

In the original exact couple (1.2) we could replace all the $\mathcal{E}^*$ groups with $\tilde{\mathcal{E}}^*$ groups and still have an exact couple, with the resulting spectral sequence having the form

$$E_2^{p,q} = \tilde{H}^p(X; \mathcal{E}^q) \Rightarrow \tilde{\mathcal{E}}^{p+q}(X).$$

This just amounts to removing the entire $p = 0$ line from the Atiyah-Hirzebruch spectral sequence. Sometimes it is convenient to consider this reduced version of the spectral sequence.

1.13. Multiplicativity. Suppose that $\mathcal{E}$ is a multiplicative cohomology theory. Then for spaces X and Y we have the external product

$$\mathcal{E}^r(X) \otimes \mathcal{E}^s(Y) \rightarrow \mathcal{E}^{r+s}(X \times Y),$$

and this is readily checked to induce associated pairings on the ZP -filtration:

$$\mathcal{E}^r(X)_{ZPa} \otimes \mathcal{E}^s(Y)_{ZPb} \rightarrow \mathcal{E}^{r+s}(X \times Y)_{ZP(a+b)}$$

and

$$(1.14) \quad \mathcal{E}^r(X)_{ZP(a/a+1)} \otimes \mathcal{E}^s(Y)_{ZP(b/b+1)} \rightarrow \mathcal{E}^{r+s}(X \times Y)_{ZP(a+b/a+b+1)}.$$

The pairings $\mathcal{E}^q(pt) \otimes \mathcal{E}^{q'}(pt) \rightarrow \mathcal{E}^{q+q'}(pt)$ also can be fed into the cup product machinery to give

$$(1.15) \quad H^p(X; \mathcal{E}^q(pt)) \otimes H^{p'}(Y; \mathcal{E}^{q'}(pt)) \longrightarrow H^{p+p'}(X \times Y; \mathcal{E}^{q+q'}(pt)).$$

Since the Atiyah-Hirzebruch spectral sequence starts with the groups $E_2(-) = H^*(-; \mathcal{E}^*)$ and then converges to the groups $\mathcal{E}^*(-)$, it is natural to ask if the pairings of (1.15) and (1.14) are connected via this convergence process. The machinery for making this connection is somewhat cumbersome to write out, although in practice not so cumbersome to use. Here are some things one can ask for:

- (i) For each r there is a product $E_r^{p,q}(X) \otimes E_r^{p',q'}(Y) \rightarrow E_r^{p+p',q+q'}(X \times Y)$;
- (ii) The differential d_r satisfies the Leibniz rule

$$d_r(a \cdot b) = d_r(a) \cdot b + (-1)^p a \cdot d_r b$$

for all $a \in E_r^{p,q}(X)$ and $b \in E_r^{p',q'}(Y)$, and therefore induces a product on $H_*(E_r) = E_{r+1}$;

- (iii) The product on the E_{r+1} -term equals the one induced by the product on the E_r -term, for all r ;
- (iv) There is a product on the E_∞ -term that agrees with the products on the E_r -terms whenever such a comparison is possible.
- (v) The maps $\mathcal{E}^p(-)_{ZP(q/q+1)} \rightarrow E_\infty^{q,p-q}(-)$ are compatible with the products, in the evident sense.

When (i)—(iii) hold one often says that this is a **pairing of spectral sequences** $E_*(X) \otimes E_*(Y) \rightarrow E_*(X \times Y)$, though this is not so useful without something like (iv) and (v) to go along with it. It is tricky to turn (iv) into a precise statement at the maximum level of generality, but in all of our applications the $E_\infty^{p,q}$ groups will agree with the $E_r^{p,q}$ -terms for all $r \gg 0$ and (iv) is to be interpreted in that light.

To give a decent treatment of pairings between Atiyah-Hirzebruch spectral sequences it is best to work at the level of spectra, and to work with a category of spectra where there is a well-behaved smash product. This introduces several layers of foundational technicalities that we do not wish to dwell on, so let us just say that these things can all be worked out. See [D1, D2] for one treatment. In this setting the right notion of “multiplicative cohomology theory” consists of a spectrum $\mathcal{E}$ together with a map $\mathcal{E} \wedge \mathcal{E} \rightarrow \mathcal{E}$ that is associative and unital. Both the complex and real K -theory spectra can be given this structure. One has the following general result:

Theorem 1.16. *Let $\mathcal{E}$ be a spectrum with a product $\mathcal{E} \wedge \mathcal{E} \rightarrow \mathcal{E}$. Then there is a pairing of Atiyah-Hirzebruch spectral sequences where the product on E_2 -terms $H^p(X; \mathcal{E}^q) \otimes H^{p'}(Y; \mathcal{E}^{q'}) \rightarrow H^{p+p'}(X \times Y; \mathcal{E}^{q+q'})$ is equal to $(-1)^{p'q}$ times the cup product, and where the product converges to the product on $\mathcal{E}$ -theory in the sense of (iv) and (v) above.*

We will not prove the above theorem here, as this would take us too far afield. For a proof, see [D2, Section 3]. What is more important is how to use the theorem; we will give some examples in the following section.

Remark 1.17. The signs in the above theorem cannot, in general, be neglected. See [D2, Section 2] for a complete discussion. However, notice that in the case of complex K -theory it is irrelevant because the groups $H^p(X; K^q)$ are only nonzero when q is even. This is a pleasant convenience. A similar convenience occurs for

KO -theory: whereas the coefficient groups do have some nonzero terms in odd degrees, since these terms are all $\mathbb{Z}/2$'s one can once again neglect the signs.

◦ Exercises ◦

Exercise 1.18.

- (a) Use the Atiyah-Hirzebruch spectral sequence to compute the groups $K^*(S)$ for every compact surface S .
- (b) Let T be the torus. Using only the Atiyah-Hirzebruch spectral sequence, compute the groups $KO^*(T)$ for $* \neq 0 \pmod{8}$ and prove that $\widetilde{KO}^0(T)$ is either $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ or $(\mathbb{Z}/2)^3$.
- (c) The torus is the cofiber of a map $S^1 \rightarrow S^1 \vee S^1$, and this map is null after one suspension. So $T \simeq S^1 \vee S^1 \vee S^2$ in the stable homotopy category. Use this to give a different computation of $KO^*(T)$ and prove that $\widetilde{KO}^0(T) \cong (\mathbb{Z}/2)^3$.
- (d) Attempt to use the Atiyah-Hirzebruch spectral sequence to compute $KO^*(\mathbb{R}P^2)$. This is mostly straightforward, except for the case of KO^0 where there is again some indeterminacy. Perhaps consult Section ??.
- (e) Using either the method of (b) or (c) (or a combination), compute the groups $KO^*(S)$ for every compact surface S .

Exercise 1.19. In the Atiyah-Hirzebruch spectral sequence for complex K -theory, prove that all d_3 differentials vanish on classes in $H^p(X)$ for $p \leq 2$. [Hint: Use the naturality of the spectral sequence to reduce to a universal example.]

Exercise 1.20. Let $p: E \rightarrow B$ be a fibration with fiber F , where B is a CW-complex. If $F_\bullet \subseteq B$ is the CW-filtration, then using the filtration $p^{-1}(F_\bullet) \subseteq E$ with the cohomology theory H leads to the Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(B; \underline{H}^q(F)) \Rightarrow H^{p+q}(E).$$

Here $\underline{H}^q(F)$ refers to a local coefficient system on B (i.e., a locally constant sheaf), but if $\pi_1(B) = 0$ then all local coefficient systems are trivializable. Try to work out the details here, consulting [McCleary] if you get stuck.

If $\mathcal{E}$ is a cohomology theory, use the same ideas to produce a spectral sequence of the form

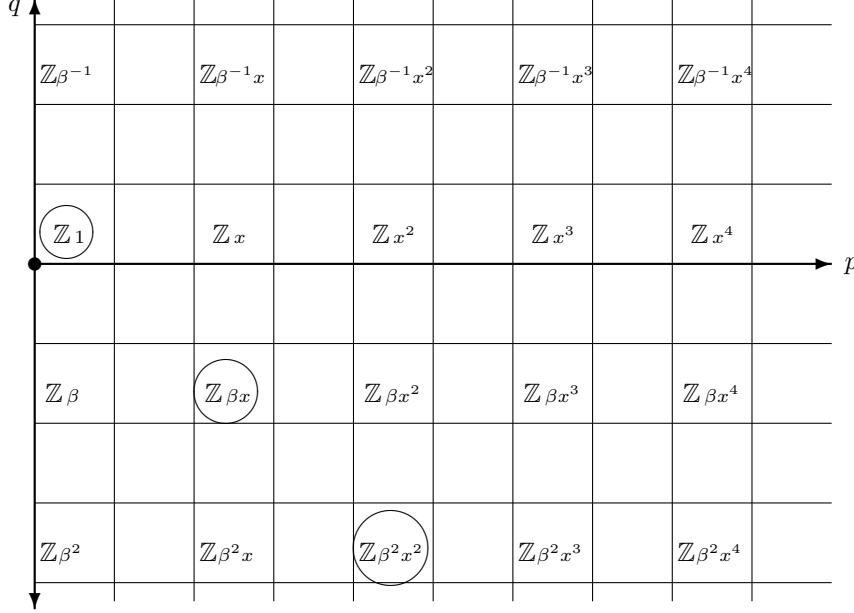
$$E_2^{p,q} = H^p(B; \underline{\mathcal{E}}^q(F)) \Rightarrow \mathcal{E}^{p+q}(E).$$

The only reasonable name for this is the Leray-Serre-Atiyah-Hirzebruch spectral sequence, though that is quite a mouthful.

2. THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE, PART 2

This section continues our tour of the Atiyah-Hirzebruch spectral sequence. We explore some examples and take a deeper look into the differentials. To keep things simple, we will focus entirely on complex K -theory. Further examples, for both K and KO , are in Section 6.

2.1. Some examples. Let us start by redoing the calculation $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$ where $X = 1 - L$, now using the Atiyah-Hirzebruch spectral sequence. The following is the E_2 -term:



Note that the E_2 -term vanishes to the right of the line $p = 2n$, since $H^*(\mathbb{C}P^n)$ vanishes in this range. The circled groups (and others along the same diagonal) are the ones that contribute to $K^0(\mathbb{C}P^n)$. Note that there is no room for any differentials, because the nonzero groups only occur when both p and q are even. So the spectral sequence immediately collapses, and $E_2 = E_\infty$. It follows that the filtration quotients for the ZP -filtration on $K^0(\mathbb{C}P^n)$ are as follows:

$$K^0(\mathbb{C}P^n) \xleftarrow{\mathbb{Z}} K^0(\mathbb{C}P^n)_{ZP_1} \xleftarrow{0} K^0(\mathbb{C}P^n)_{ZP_2} \xleftarrow{\mathbb{Z}} K^0(\mathbb{C}P^n)_{ZP_3} \xleftarrow{\cdots}$$

with the $\mathbb{Z}$'s appearing exactly $n + 1$ times. Since the quotients are free there are no extension problems and we conclude that $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$ as abelian groups.

The spectral sequence also gives information about the ring structure on $K^0(\mathbb{C}P^n)$. Note that $K^0(\mathbb{C}P^n)_{ZP_1} = \tilde{K}^0(\mathbb{C}P^n)$, and that $K^0(\mathbb{C}P^n)_{ZP_2} = K^0(\mathbb{C}P^n)_{ZP_1}$ by the previous paragraph. Consider the canonical map

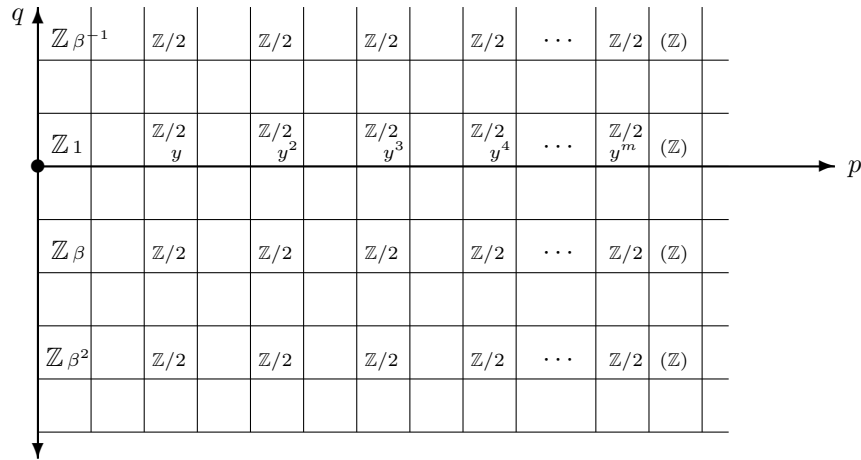
$$K^0(\mathbb{C}P^n)_{ZP(2/3)} \xrightarrow{\cong} E_\infty^{2,-2} = \mathbb{Z}\langle \beta x \rangle.$$

Let α denote a preimage for βx under this isomorphism. The multiplicativity of the spectral sequence tells us that α^k maps to $\beta^k x^k$ under the corresponding map $K^0(\mathbb{C}P^n)_{ZP(2k/2k+1)} \rightarrow E_\infty^{2k,-2k}$. In particular, α^k is nonzero for $k < n + 1$. We know $\alpha^{n+1} = 0$, either by Lemma ?? or by the fact that $\alpha^{n+1} \in K^0(\mathbb{C}P^n)_{ZP(2n+2)}$ and this filtration group is zero by the spectral sequence.

Now consider the map $\mathbb{Z}[\alpha]/(\alpha^{n+1}) \rightarrow K^0(\mathbb{C}P^n)$. This may be regarded as a map of filtered rings, where the domain is filtered by the powers of (α) and the target has the ZP -filtration. The spectral sequence tells us this is an isomorphism

$$K^0(\mathbb{C}P^n)_{ZP(2/3)} \rightarrow K^0(\mathbb{C}P^1)_{ZP(2/3)}$$

For our next example let us consider $K^0(\mathbb{RP}^n)$. The E_2 -term is very similar to the one before, with the difference that most $\mathbb{Z}$'s are changed to $\mathbb{Z}/2$'s:



The $\mathbb{Z}$'s in parentheses lie in degree $p = n$ when n is odd, and are not present when n is even. And $m = \lfloor \frac{n}{2} \rfloor$ (this is the largest power of y in the chart). As discussed in (1.9) above, there can be no differential emanating from the $p = 0$ column. So the only possible differentials allowed by the grading would occur when n is odd and would have a $\mathbb{Z}/2$ mapping into one of the $\mathbb{Z}$'s; but such a map must be zero. So all differentials vanish, and we again have $E_2 = E_\infty$. We conclude that the associated graded of the ZP -filtration on $\tilde{K}^0(\mathbb{R}P^n)$ consists of $\lfloor \frac{n}{2} \rfloor$ copies of $\mathbb{Z}/2$, and so $\tilde{K}^0(\mathbb{R}P^n)$ is an abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$. It remains to determine the group precisely. For this, use the map of spectral sequences induced by $j: \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. At the E_2 -term this consists of the maps $H^*(\mathbb{C}P^n) \xrightarrow{j^*} H^*(\mathbb{R}P^n)$, which in positive degrees are the surjections $\mathbb{Z} \rightarrow \mathbb{Z}/2$. Therefore we have surjections on E_∞ in the regions where the total degree is even. This surjection on E_∞ -terms shows that $\alpha = 1 - [j^*L]$ generates the filtration quotient $K^0(\mathbb{R}P^n)_{ZP(2/3)}$, since $1 - [L]$ generates the corresponding quotient in the $\mathbb{C}P^n$ case. Note that $\alpha^2, \dots, \alpha^{\lfloor \frac{n}{2} \rfloor}$ therefore generate the other filtration quotients, and so in particular are nonzero.

But we can compute

$$\alpha^2 = (1 - [j^*L])^2 = 1 - 2[j^*L] + [j^*L]^2 = 1 - 2[j^*L] + 1 = 2(1 - [j^*L]) = 2\alpha,$$

where in the third equality we have used that the square of any real line bundle is trivial (Corollary ??). It follows that $\alpha^i = 2^{i-1}\alpha$. Since $\alpha^{\lfloor \frac{n}{2} \rfloor} \neq 0$ this gives $2^{\lfloor \frac{n}{2} \rfloor - 1}\alpha \neq 0$. The only abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$ that admits such an element is $\mathbb{Z}/(2^{\lfloor \frac{n}{2} \rfloor})$, and so $\tilde{K}^0(\mathbb{R}P^n)$ is isomorphic to this cyclic group.

The spectral sequence also quickly shows that $K^1(\mathbb{R}P^n)$ is isomorphic to $\mathbb{Z}$ when n is odd, and 0 otherwise.

Many more calculations using these techniques can be found in Section 6 below.

2.2. More on differentials. We now turn to a deeper study of the differentials in the Atiyah-Hirzebruch spectral sequence for K -theory. Since $K^{odd}(pt) = 0$ it follows for degree reasons that all differentials d_{2r} vanish. So our first significant differential is d_3 , which is a stable cohomology operation $H^*(-; \mathbb{Z}) \rightarrow H^{*+3}(-; \mathbb{Z})$. It is an easy matter to compute *all* such stable operations, as they are parameterized by the group $H^3(H\mathbb{Z})$ of stable homotopy classes $H\mathbb{Z} \rightarrow \Sigma^3 H\mathbb{Z}$; this can be computed as the cohomology group $H^{n+3}(K(\mathbb{Z}, n))$ for $n \geq 3$. A routine calculation (say, with the Serre spectral sequence) shows that this group is $\mathbb{Z}/2$. The nonzero element is an operation α that is an integral lift of Sq^3 , in the sense that if $u \in H^*(X; \mathbb{Z})$ then

$$\overline{\alpha(u)} = \text{Sq}^3(\overline{u})$$

where $\overline{x}$ denotes the mod 2 reduction of a class x .

The above paragraph shows that our differential d_3 either equals zero (for all spaces X) or coincides with the operation α . The latter option is the correct one, and to see this it suffices to produce a single space X where d_3 is nonzero. For this we take the space X from Example ??, constructed as the cofiber of a map $\Sigma^3 \mathbb{R}P^2 \rightarrow S^3$ that gives a null-homotopy for 2η . This space has $H^3(X) = \mathbb{Z}$ and $H^6(X) = \mathbb{Z}/2$, so there is a potential d_3 in the spectral sequence. If this d_3 were zero then we would have $\tilde{K}^0(X) = \mathbb{Z}/2$, but we calculated in Example ?? (using the Chern character) that $\tilde{K}^0(X) = 0$. So d_3 is nonzero here. We have therefore proven:

Proposition 2.3. *The differential d_3 in the Atiyah-Hirzebruch spectral sequence is the unique nonzero cohomology operation $H^*(-; \mathbb{Z}) \rightarrow H^{*+3}(-; \mathbb{Z})$. It satisfies $2d_3(x) = 0$, for all x .*

Remark 2.4. We have now explained the motivation for the space X from Example ?? . It is literally the smallest space for which there is a nonzero α -operation in its cohomology.

The fact that $2d_3(x) = 0$ shows that d_3 must vanish on any class $x \in H^*(X)$ whose order is prime to 2. Alternatively, if we tensor the Atiyah-Hirzebruch spectral sequence with $\mathbb{Z}[\frac{1}{2}]$ then all d_3 differentials vanish. In that case d_5 is a cohomology operation $H^*(-; \mathbb{Z}[\frac{1}{2}]) \rightarrow H^{*+5}(-; \mathbb{Z}[\frac{1}{2}])$, and such things are classified by $H^5(H\mathbb{Z}; \mathbb{Z}[\frac{1}{2}])$. This group is readily calculated to be $\mathbb{Z}/3$ (see below). If one then also inverts 3 this will kill d_5 , but it turns out to also kill d_7 because $H^7(H\mathbb{Z}; \mathbb{Z}[\frac{1}{6}]) = 0$. The d_9 differential becomes a cohomology operation in $H^9(H\mathbb{Z}; \mathbb{Z}[\frac{1}{6}]) \cong \mathbb{Z}/5$, and so one can kill it by inverting 5. This process continues,

and shows that inverting all primes smaller than p kills all differentials below d_{2p-1} . Note that this gives another proof of Proposition 1.10 (but with more precise information), saying that after tensoring with $\mathbb{Q}$ all Atiyah-Hirzebruch differentials vanish.

The following result summarizes and expands the discussion in the last paragraph. The two parts are closely related and almost equivalent, but it is useful to have them both stated explicitly. The result and its proof are from [AH2, Section 7].

Proposition 2.5. *Fix a prime p .*

- (a) *Inverting $(p-1)!$ in the Atiyah-Hirzebruch spectral sequence results in $d_r = 0$ for $r < 2p-1$, together with $d_{2p-1}(u) = -\beta P^1(\bar{u})$ for all classes u , where $\bar{u}$ is reduction modulo p , P^1 is Steenrod's first reduced power operation (for the prime p), and β is the Bockstein for the sequence $0 \rightarrow \mathbb{Z}[\frac{1}{(p-1)!}] \xrightarrow{p} \mathbb{Z}[\frac{1}{(p-1)!}] \rightarrow \mathbb{Z}/p \rightarrow 0$.*
- (b) *Let $u \in H^*(X)$ be p^e -torsion, where p is a prime. Then in the Atiyah-Hirzebruch spectral sequence $d_i(u) = 0$ for $i < 2p-1$, and $d_{2p-1}(u) = -\beta P^1(\bar{u})$ where β is the Bockstein for $0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$ and P^1 is as in (a).*

Proof. Fixing a prime p , the following is known about $H^*(K(\mathbb{Z}, n))$ assuming $n > 2p-1$:

- (i) $H^i(K(\mathbb{Z}, n)) = 0$ for $0 < i < n$ and $H^n(K(\mathbb{Z}, n)) \cong \mathbb{Z}$;
- (ii) $H^i(K(\mathbb{Z}, n))$ is torsion, with all orders prime to p , for $n < i < n+2p-1$;
- (iii) $H^{n+2p-1}(K(\mathbb{Z}, n))$ is isomorphic to a direct sum $\mathbb{Z}/p \oplus A$ where A is a torsion group whose order only has prime factors smaller than p . The $\mathbb{Z}/p$ summand is generated by $\beta P^1(\bar{u})$, where $u \in H^n(K(\mathbb{Z}, n))$ is the fundamental class.

Alternatively, the above results imply that (and are almost equivalent to)

$$H^*\left(K(\mathbb{Z}, n); \mathbb{Z}[\tfrac{1}{(p-1)!}]\right) \cong \begin{cases} 0 & \text{if } i < n, \\ \mathbb{Z}[\tfrac{1}{(p-1)!}] & \text{if } i = n, \\ 0 & \text{if } n < i < n+2p-1, \\ \mathbb{Z}/p & \text{if } i = n+2p-1. \end{cases}$$

These facts are easy calculations with the Serre spectral sequence, using the methods of [MT, Chapter 9].

Consider the Atiyah-Hirzebruch spectral sequence for $K(\mathbb{Z}, n)$, and in particular the differentials on the fundamental class u . By naturality of the spectral sequence this serves as a universal example for what happens on all spaces. Since inverting $(p-1)!$ kills all of the cohomology of $K(\mathbb{Z}, n)$ in dimensions strictly between n and $n+2p-1$, this shows that it also kills the differentials $d_r(u)$ for $r < 2p-1$. By universality, inverting $(p-1)!$ kills these differentials for any space X .

Let $u \in H^*(X)$ be a p^e -torsion class. Then $d_r(u)$ is killed by p^e , but for $r < 2p-1$ it is also killed by a power of $(p-1)!$ by the preceding paragraph. Since these integers are relatively prime it follows that $d_r(u) = 0$ for $r < 2p-1$.

It remains to identify d_{2p-1} in both (a) and (b). We know by the calculation of $H^*(K(\mathbb{Z}, n))$ for $n \gg 0$ that after localization at $(p-1)!$ one must have $d_{2p-1}(u) = \lambda \cdot \beta P^1(\bar{u})$, for some $\lambda \in \mathbb{Z}/p$. We need to determine λ , and for this we can examine a single well-chosen example space. The sample space we choose is a generalization of the one from Example ???. Consider the projection $\pi: \mathbb{C}P^p \rightarrow \mathbb{C}P^p/\mathbb{C}P^{p-1} \cong$

S^{2p} . We claim that there is a stable map $f: S^{2p} \rightarrow \mathbb{C}P^p$ such that the composite $S^{2p} \rightarrow \mathbb{C}P^p \xrightarrow{\pi} S^{2p}$ has degree Mp for some integer M prime to p ; for the proof, see Lemma 2.7 below. The “stable map” phrase is to indicate that f might only exist after suspending some number of times, so really it is a map $f: \Sigma^r S^{2p} \rightarrow \Sigma^r \mathbb{C}P^p$. We can assume that r is even. Let X be the cofiber of f , and note that

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z} & \text{if } r+2 \leq i < r+2p \text{ and } i \text{ is even,} \\ \mathbb{Z}/(Mp) & \text{if } i = r+2p+1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the cofiber sequence

$$\Sigma^r \mathbb{C}P^p \xrightarrow{j} X \xrightarrow{p} S^{r+2p+1}$$

Let $x \in H^2(\mathbb{C}P^p)$ be a chosen generator, and let $u \in H^{r+2}(X)$ be a class that j^* maps onto the suspension $\sigma^r(x)$. Let $v \in H^{r+2p+1}(X)$ be the image under p^* of the canonical generator from $H^{r+2p+1}(S^{r+2p+1})$. Note that j^* is an isomorphism on $H^*(-; \mathbb{Z}/p)$ for $* \leq 2p+r$, and so we can write $\sigma^r \bar{x}^k$ for the generator of $H^{2k+r}(X; \mathbb{Z}/p)$ that maps to the element of the same name in $H^{2k+r}(\Sigma^r \mathbb{C}P^p; \mathbb{Z}/p)$. Observe that $\beta(\sigma^r \bar{x}^p) = Mv$ in $H^*(X)$.

In $H^*(\mathbb{C}P^p; \mathbb{Z}/p)$ we have $P^1(\bar{x}) = \bar{x}^p$, as this is how P^1 behaves on classes of degree 2. So in $H^*(\Sigma^r \mathbb{C}P^p; \mathbb{Z}/p)$ we have $P^1(\sigma^r \bar{x}) = \sigma^r(\bar{x}^p)$, by stability of the P^1 operation. It follows that $\beta P^1(\bar{u}) = Mv$.

In the Atiyah-Hirzebruch spectral sequence for X , after inverting $(p-1)!$ there is only one possible nonvanishing differential, namely $d_{2p-1}: \mathbb{Z}[\frac{1}{(p-1)!}] \rightarrow \mathbb{Z}/p$. We know by the definition of λ that

$$(2.6) \quad d_{2p-1}(u) = \lambda \cdot \beta P^1(\bar{u}) = \lambda Mv.$$

But we can also compute $d_{2p-1}(u)$ directly. The class u is represented by an element in $K^0(X_{r+2}, X_{r+1})$ (a cellular $(r+2)$ -cochain). As explained in (8) of Section 1 the differential can be computed by choosing a lift of $u \in K^0(X_{r+2}, X_{r+1})$ to $\xi \in K^0(X_{2p+r+1}, X_{r+1})$ and then applying the connecting homomorphism from the long exact sequence for the triple $(X_{2p+r+1}, X_{2p+r}, X_{r+1})$:

$$\begin{array}{ccc} \xi \in K^0(X_{2p+r+1}, X_{r+1}) & \xrightarrow{\delta} & K^1(X_{2p+r+1}, X_{2p+r}) \\ \downarrow & & \\ u \in K^0(X_{r+2}, X_{r+1}) & & \end{array}$$

Recall that $X_{r+1} = *$ and that u is represented by the class $\sigma^r(1 - [L]) \in \tilde{K}^0(\Sigma^r \mathbb{C}P^1)$. The element $1 - [L] \in \tilde{K}^0(\mathbb{C}P^1)$ lifts to the class with the same name in $\tilde{K}^0(\mathbb{C}P^p)$, and so we may take $\xi = \sigma^r(1 - [L]) \in K^0(\Sigma^r \mathbb{C}P^p, *)$. To compute $\delta(\xi)$ we can use the Chern character:

$$\begin{array}{ccc} K^0(X_{2p+r}, *) & \xrightarrow{\delta} & K^1(X_{2p+r+1}, X_{2p+r}) \\ \text{ch} \downarrow & & \downarrow \text{ch} \\ H^{ev}(X_{2p+r}, *; \mathbb{Q}) & \xrightarrow{\delta} & H^{odd}(X_{2p+r+1}, X_{2p+r}; \mathbb{Q}). \end{array}$$

We know from Lemma ?? that the right vertical map is an injection and that its image is the integral subgroup $H^{odd}(X_{2p+r+1}, X_{2p+r}; \mathbb{Z})$. So we compute

$$\text{ch}(\xi) = \text{ch}(\sigma^r(1 - [L])) = \sigma^r \text{ch}(1 - [L]) = \sigma^r(x - \frac{x^2}{2} + \frac{x^3}{6} - \dots),$$

recalling that $c_1(L) = -x$. Applying δ to this expresstion kills everything except the class in degree $2p + r$, which is $(-1)^{p+1}\sigma^r(x^p/p!)$. Let us again use v to denote the generator of $H^{2p+r+1}(X_{2p+r+1}, X_{2p+r})$, so that we have $\delta(x^p) = Mpv$. Therefore we get

$$\delta(\text{ch } \xi) = (-1)^{p+1} \cdot \frac{1}{p!} \cdot Mpv = (-1)^{p+1} \frac{M}{(p-1)!} \cdot v.$$

Note that commutativity of the above square implies that $\frac{M}{(p-1)!}$ must be an integer, and that

$$\delta(\xi) = (-1)^{p+1} \frac{M}{(p-1)!} \cdot v'$$

where v' denotes the preferred generator of $K^1(X_{2p+r+1}, X_{2p+r}) \cong \mathbb{Z}$.

Putting everything together, we have just proven that in the Atiyah-Hirzebruch spectral sequence for X with $(p-1)!$ inverted one has

$$d_{2p-1}(u) = \left[(-1)^{p+1} \frac{M}{(p-1)!} \right]_p \cdot v$$

where $[-]_p$ denotes the residue modulo p . Wilson's Theorem says that $(p-1)! \equiv -1 \pmod{p}$, and so

$$d_{2p-1}(u) = [(-1)^p M]_p \cdot v.$$

Comparing to (2.6) gives $\lambda = (-1)^p$. For $p > 2$ of course this sign is always -1 . For $p = 2$ the sign is $+1$, but for $p = 2$ the sign does not actually matter because the differential is 2-torsion (Proposition 2.3). So it is correct to just say $\lambda = -1$ for all p , and this finishes (a).

Part (b) can be deduced from (a) using the commutative square

$$\begin{array}{ccc} H^*(X) & \xrightarrow{d_{2p-1}} & H^{*+2p-1}(X) \\ \downarrow & & \downarrow \\ H^*(X; \mathbb{Z}_f) & \xrightarrow{d_{2p-1}} & H^{*+2p-1}(X; \mathbb{Z}_f) \end{array}$$

where we are writing $\mathbb{Z}_f = \mathbb{Z}[\frac{1}{(p-1)!}]$. Let $B \subseteq H^{*+2p-1}(X)$ be the subgroup of elements killed by a power of p , and note that B injects into $H^{*+2p-1}(X; \mathbb{Z}_f)$. If $p^e u = 0$ then both $d_{2p-1}u$ and $-\beta P^1(\bar{u})$ belong to B . The commutative square, together with part (a), show that the two classes map to the same element of $H^{*+2p-1}(X; \mathbb{Z}_f)$; hence, they are the same. $\square$

Lemma 2.7. *Fix a prime p . Then for some $r > 0$ there exists a map $S^{2p+r} \rightarrow \Sigma^r \mathbb{C}P^p$ such that the composite*

$$S^{2p+r} \rightarrow \Sigma^r \mathbb{C}P^p \xrightarrow{\Sigma^r \pi} \Sigma^r S^{2p}$$

has degree equal to Mp for some M relatively prime to p . Here $\pi: \mathbb{C}P^p \rightarrow S^{2p}$ is the map that collapses $\mathbb{C}P^{p-1}$ to a point.

Proof. This is a computation with stable homotopy groups. Consider the homology theory $X \mapsto E_*(X) = \pi_*^s(X) \otimes \mathbb{Z}_{(p)}$. When $X = S^0$ the groups $E_*(X)$ are the p -components of the stable homotopy groups of spheres, and it is known that $E_i(X) = 0$ for $0 < i < 2p - 3$ and $E_{2p-3}(X) \cong \mathbb{Z}/p$. An easy induction using the cofiber

sequences $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n \rightarrow S^{2n}$ shows that $E_{2p-1}(\mathbb{C}P^n) \cong \mathbb{Z}/p$ for all $1 \leq n \leq p-1$. The long exact sequence for $\mathbb{C}P^{p-1} \hookrightarrow \mathbb{C}P^p \rightarrow S^{2p}$ then gives

$$\cdots \rightarrow \pi_{2p}^s(\mathbb{C}P^{p-1}) \rightarrow \pi_{2p}^s(\mathbb{C}P^p) \rightarrow \mathbb{Z} \rightarrow \pi_{2p-1}^s(\mathbb{C}P^{p-1}) \rightarrow \cdots$$

The element $p \in \mathbb{Z}$ necessarily maps to zero in $\pi_{2p-1}^s(\mathbb{C}P^{p-1}) \otimes \mathbb{Z}_{(p)}$, since the latter group is $\mathbb{Z}/p$. This means that there exists $M \in \mathbb{Z}$ prime to p such that Mp maps to zero in $\pi_{2p-1}^s(\mathbb{C}P^{p-1})$. But then Mp is the image of an element in $\pi_{2p}^s(\mathbb{C}P^p)$, and this element is what we were looking for. $\square$

2.8. Differentials and the Chern character. In the proof of Proposition 2.5 there was a key step where we used the Chern character to help compute a differential in the Atiyah-Hirzebruch spectral sequence. We will next explain a generalization of this technique.

The E_1 -term of the Atiyah-Hirzebruch spectral sequence breaks up into chain complexes that look like

$$\cdots \rightarrow K^{-1}(F_{q-1}, F_{q-2}) \rightarrow K^0(F_q, F_{q-1}) \rightarrow K^1(F_{q+1}, F_q) \rightarrow \cdots$$

If q is odd this is the zero complex, and if q is even we have seen that it is isomorphic to the cellular chain complex for X with $\mathbb{Z}$ coefficients. The latter is via isomorphisms $K^0(F_q, F_{q-1}) \cong \tilde{K}^0(\vee_{\alpha} S^q) \cong \bigoplus_{\alpha} \tilde{K}^0(S^q) \cong \bigoplus_{\alpha} \mathbb{Z}$. Notice that we can also use the Chern character to obtain such an isomorphism, as we know $\text{ch}: K^0(F_q, F_{q-1}) \rightarrow H^*(F_q, F_{q-1}; \mathbb{Q})$ to be injective with image equal to $H^*(F_q, F_{q-1}; \mathbb{Z})$ (Lemma ??). Since the Chern character is natural it actually gives us an isomorphism of chain complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K^{-1}(F_{q-1}, F_{q-2}) & \longrightarrow & K^0(F_q, F_{q-1}) & \longrightarrow & K^1(F_{q+1}, F_q) \longrightarrow \cdots \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \cdots & \longrightarrow & H^*(F_{q-1}, F_{q-2}) & \longrightarrow & H^*(F_q, F_{q-1}) & \longrightarrow & H^*(F_{q+1}, F_q) \longrightarrow \cdots \end{array}$$

Taken on its own, this is just a matter of convenience: we already had a natural isomorphism between these complexes, so identifying it as the Chern character just gives it a nice name. But using the fact that the Chern character is defined more globally (i.e., on all pairs (Y, B)) allows us to push this a bit further and obtain a description of the Atiyah-Hirzebruch differentials. The following result is a combination of [AH2, Lemmas 1.2 and 7.3].

Proposition 2.9. *Let X be a CW-complex and let $u \in H^p(X)$, for $p \in \mathbb{Z}$. Let $r \geq 3$ be odd. Then in the Atiyah-Hirzebruch spectral sequence one has $d_i u = 0$ for all $2 \leq i < r$ if and only if there exist $\tilde{u} \in H^p(F_{p+r-1}, F_{p-1})$ and $\xi \in K^*(F_{p+r-1}, F_{p-1})$ such that*

- (i) $\tilde{u}$ is a lift for u under $H^p(F_{p+r-1}, F_{p-1}) \rightarrow H^p(F_{p+r-1}) \xleftarrow{\cong} H^p(X)$, and
- (ii) $\text{ch}(\xi) = \tilde{u} + \text{higher degree terms}$.

Moreover, if in the above situation α is a cellular cochain representing $\text{ch}(\xi)_{p+r-1}$ then its cellular coboundary $\delta\alpha$ is integral and represents the differential $d_r u$.

Proof. To symplify some typography we assume throughout the proof that p is even, although the odd case is identical (or else one could just replace X with its suspension).

Assume that $u \in H^p(X)$ satisfies $d_i u = 0$ for $2 \leq i < r$. Identifying u with an element in E_2 , this condition says that u can be represented by a class

$z \in E_1 = K^0(F_p, F_{p-1})$ with the property that $z \in Z_{r-1}$. The element associated to z by the isomorphism $K^0(F_p, F_{p-1}) \cong C_{cell}^p(X; \mathbb{Z})$ is a cellular p -cochain representative for u . By Lemma ?? we can realize this isomorphism via the Chern character $\text{ch}: K^0(F_p, F_{p-1}) \rightarrow H^p(F_p, F_{p-1}; \mathbb{Q})$, which is an injection with image $H^p(F_p, F_{p-1}; \mathbb{Z})$.

As remarked in (8) of Section 1.1, the condition $z \in Z_{r-1}$ is equivalent to saying that z lifts to a class $\tilde{z} \in K^0(F_{p+r-1}, F_{p-1})$. Now apply the Chern character to get the square

$$\begin{array}{ccc} K^0(F_{p+r-1}, F_{p-1}) & \longrightarrow & K^0(F_p, F_{p-1}) \\ \text{ch} \downarrow & & \downarrow \text{ch} \\ H^*(F_{p+r-1}, F_{p-1}; \mathbb{Q}) & \xrightarrow{j^*} & H^*(F_p, F_{p-1}; \mathbb{Q}), \end{array} \quad \begin{array}{ccc} \tilde{z} & \longrightarrow & z \\ \downarrow & & \downarrow \\ \text{ch}(\tilde{z}) & \longrightarrow & \text{ch}(z). \end{array}$$

The element $\text{ch}(z)$ is an (integral) cellular p -cochain that represents the class u . The groups $H^*(F_{p+r-1}, F_{p-1})$ are zero in degrees $* < p$, so $\text{ch}(\tilde{z})$ is of the form $\text{ch}(\tilde{z})_p + \text{higher order terms}$. The fact that $\text{ch}(z)$ is a cellular cochain representing u says that any lift of $\text{ch}(z)$ into $H^p(F_{p+i}, F_{p-1})$ has the same image in $H^p(F_{p+i})$ as u : consult the diagram

$$\begin{array}{ccccc} H^p(X) & \longrightarrow & H^p(F_{p+i}) & \longleftarrow & H^p(F_{p+i}, F_{p-1}) \\ & \searrow & \downarrow & & \downarrow \\ & & H^p(F_p) & \longleftarrow & H^p(F_p, F_{p-1}). \end{array}$$

In particular, $\text{ch}(\tilde{z})_p$ has the same image in $H^p(F_{p+r-1})$ as u . This completes the $(\Rightarrow)$ direction of the first statement in the proposition. The $(\Leftarrow)$ direction follows in the same way, as all the steps are reversible.

For the final statement of the proposition we continue to assume (for convenience) that p is even. Consider the diagram

$$\begin{array}{ccc} \xi \in K^0(F_{p+r-1}, F_{p-1}) & \xrightarrow{\delta} & K^1(F_{p+r}, F_{p+r-1}) \\ \text{ch} \downarrow & & \downarrow \text{ch} \\ H^*(F_{p+r-1}, F_{p-1}; \mathbb{Q}) & \xrightarrow{\delta} & H^{*+1}(F_{p+r}, F_{p+r-1}; \mathbb{Q}) \end{array}$$

where in both rows the map δ is the connecting homomorphism in the long exact sequence for the triple $(F_{p+r}, F_{p+r-1}, F_{p-1})$. Looking back on (8) of Section 1.1, the element $\delta(\xi)$ represents $d_r(u)$ in the E_r -term of the spectral sequence. But we know (again from Lemma ??) that the right vertical map is an injection whose image consists of the integral elements, and our isomorphism of the E_2 -term with $H^*(X; \mathbb{Z})$ identifies $\delta(\xi)$ with $\text{ch}(\delta(\xi))$. Commutativity of the square says this element is also $\delta((\text{ch } \xi)_{p+r-1})$ (and also verifies that this class is integral). $\square$

Proposition 2.9 does not surface as a useful tool as often as one might imagine. But we have seen that one can sometimes use algebraic geometry to produce convenient elements of $K^*(-, -)$, and that ch is fairly “computable” from a practical standpoint. Sometimes these stars line up and can be used to explain some Atiyah-Hirzebruch differentials. See Theorem 3.12 for one example.

◦ Exercises ◦

Exercise 2.10. Mimic the computation of $K^*(\mathbb{R}P^n)$ to compute $KO^*(\mathbb{R}P^2)$ (the nontrivial part is $KO^0(\mathbb{R}P^2)$). Try to compute $KO^*(\mathbb{R}P^n)$ as well.

Remark 2.11. Note that we previously determined back in Section ?? that $\tilde{K}^0(\mathbb{R}P^2)$ was an abelian group of order 4. Comparing the “brute force” approach used there to the spectral sequence machinery really shows the power of the latter: the argument is basically the same, but the spectral sequence allows us to get at the conclusion much more quickly.

Exercise 2.12. Recall connective K -theory $ku^*(-)$ from Section ??, and recall there is a natural transformation $ku^*(-) \rightarrow K^*(-)$. Use the Atiyah-Hirzebruch to compute all of the groups $ku^k(\mathbb{R}P^4)$ for all k , and compare to $K^*(\mathbb{R}P^4)$. Generalize to $\mathbb{R}P^n$. [Hint: Use the comparison map between the ku - and K -spectral sequences as part of your argument.]

Exercise 2.13. In Proposition 2.3 we saw that $2d_3 = 0$ in the Atiyah-Hirzebruch spectral sequence for K . We also saw that after inverting 2 one has $3d_5 = 0$, because $d_5 = -\beta P^1$ and the composite 3β vanishes. It follows that any class $d_5(u)$ is annihilated by an integer of the form $2^N \cdot 3$ for some N . It is natural to ask whether we can find a single N that will work for all u , and what the smallest such N is. The following method gives an approach to this.

Let $H = H\mathbb{Z}$ denote the integral Eilenberg-MacLane spectrum. Let F be the homotopy fiber of the map $\alpha: H \rightarrow \Sigma^3 H$. Then F is the universal example of a spectrum having a vanishing d_3 -operation.

The Postnikov tower for the K -spectrum inherits a kind of periodicity from Bott periodicity. Applying Σ^{-2} everywhere is the same as shifting the tower down one spot. So analyzing the happenings around one “layer” actually tells us everything. The following diagram shows the fiber sequences $\Sigma^{2n}H \rightarrow P_{2n}K \rightarrow P_{2n-2}K$ together with the cofiber of the second map in that sequence, which is $\Sigma(\Sigma^{2n}H) = \Sigma^{2n+1}H$. The composites $\Sigma^{2n}H \rightarrow P_{2n}K \rightarrow \Sigma^{2n+3}H$ represent the d_3 differentials in the spectral sequence.

The composite from F to $\Sigma^3 H$ is null-homotopic by construction, so the map $F \rightarrow P_0 K$ lifts into $P_2 K$ as shown. The composite $F \rightarrow \Sigma^5 H$ represents the d_5 operation, in the sense that if $u \in H^p(X)$ is such that $d_3(u) = 0$ then the map $u: X \rightarrow \Sigma^p H$ lifts to $X \rightarrow \Sigma^p F$, and $d_5(u)$ is the composite $X \rightarrow \Sigma^p F \rightarrow \Sigma^{5+p} H$.

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 \Sigma^2 H & \longrightarrow & P_2 K & \longrightarrow & \Sigma^5 H \\
 & \nearrow \text{dotted} & \downarrow & & \\
 F & \longrightarrow & H & \longrightarrow & P_0 K \longrightarrow \Sigma^3 H \\
 & & \downarrow & & \\
 & & \vdots & &
 \end{array}$$

One way to get information about d_5 is therefore to compute the entire group $H^5(F)$. The long exact cohomology sequence associated to $F \rightarrow H \rightarrow \Sigma^3 F$ gives

$$\cdots \longleftarrow H^3(H) \longleftarrow H^5(F) \longleftarrow H^5(H) \longleftarrow H^2(H) \longleftarrow H^4(F) \longleftarrow \cdots$$

But $H^2 H = 0$, $H^3 H = \mathbb{Z}/2$, and $H^5 H = ??$ (these come from computing the cohomology of Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$ in the stable range). So $H^5 F$ is killed by ???, and this proves that ??? kills all d_5 operations.

The (rather open-ended) exercise is to fill in as many details as you can for this argument.

3. ALGEBRAIC CYCLES ON COMPLEX VARIETIES

Note: The material in this section requires the Atiyah-Hirzebruch spectral sequence from Section 1 below.

Let X be a smooth, projective algebraic variety over $\mathbb{C}$. As discussed in Section ?? every smooth subvariety Z of codimension q has a fundamental class $[Z] \in H^{2q}(X; \mathbb{Z})$. In fact it turns out that the smoothness of Z is not needed here: *every* subvariety $Z \hookrightarrow X$ of codimension q has such a fundamental class. This can be proven either by using resolution of singularities or by more naive methods—we will explain below.

Define $H_{\text{alg}}^{2q}(X; \mathbb{Z}) \subseteq H^{2q}(X; \mathbb{Z})$ to be the subgroup generated by the fundamental classes of all algebraic subvarieties. How large are these “algebraic” parts of the even cohomology groups? Are there examples of varieties X for which the algebraic part does not equal everything?

The answer to the latter question is provided by Hodge theory: yes, there do exist varieties X where not all of the even cohomology is algebraic. Hodge theory gives a decomposition of the cohomology groups $H^n(X; \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$, and the algebraic classes all lie in the $H^{i,i}$ pieces. The (even-dimensional) cohomology is entirely algebraic only when $H^{p,q} = 0$ for $p \neq q$ and $p+q$ even. But it is known that for an elliptic curve E one has $H^{1,0} = \mathbb{C} = H^{0,1}$, and the Künneth Theorem in this context says

$$H^{a,b}(X \times Y) = \bigoplus_{\substack{a=a_1+a_2 \\ b=b_1+b_2}} H^{a_1,b_1}(X) \otimes H^{a_2,b_2}(Y).$$

So for $E \times E$ one gets $H^{2,0}(E \times E) \cong \mathbb{C} \cong H^{0,2}(E \times E)$ and therefore not all of $H^2(E \times E)$ is algebraic.

The problem with Hodge theory is that it cannot see any torsion classes, as the coefficients of the cohomology groups need to be $\mathbb{C}$. Could it be true that torsion cohomology classes are always algebraic? A classical theorem of Lefschetz says that this holds for classes in H^2 . This is the so-called *Lefschetz (1,1)-theorem*, proven in modern times via sheaf cohomology (see [GH, Chapter 1, Section 2]). But for higher cohomology groups the answer is again no, and the first examples were given by Atiyah and Hirzebruch [AH2]. There are three basic components to their approach:

- (1) Every algebraic class must survive the Atiyah-Hirzebruch spectral sequence. The vanishing of the differentials therefore gives a sequence of obstructions for a given cohomology class to be algebraic.
- (2) The differentials in the spectral sequence can be analyzed in terms of cohomology operations. If p is a prime then on a p^e -torsion class the first nonzero differential is d_{2p-1} and coincides with the operation $u \mapsto -\beta P^1(\bar{u})$ where $\bar{u}$ is

reduction modulo p , P^1 is the first Steenrod p th power operation, and β is the mod p Bockstein. Note that this cohomology operation has degree $2p - 1$.

- (3) A clever construction of Serre's shows how to obtain smooth algebraic varieties whose cohomology contains that of BG through a range of dimensions, for G any finite group. Using this, one readily finds smooth varieties with even-dimensional, p -torsion cohomology classes for which βP^1 does not vanish; such a class cannot be algebraic.

The Atiyah-Hirzebruch proof is no longer the most efficient way to obtain conditions for torsion classes to be algebraic. Resolution of singularities shows that every algebraic class is actually a pushforward of the fundamental class of a manifold—i.e., every algebraic class lifts into complex cobordism. In the 1950s Thom had already obtained some necessary conditions for such a lifting to exist, in terms of Steenrod operations. Via this method K -theory is not needed at all, and moreover Thom's theory yields a stronger set of conditions: all odd degree mod p cohomology operation must vanish, for every prime p . Note that resolution of singularities was not proven by Hironaka until 1964 [Hironaka1], and so of course was not available at the time of [AH2].

Despite the modern shortcomings of Atiyah and Hirzebruch's method, we will spend this section describing it in detail. It sheds some light on the relationship between K -theory and singular cohomology, and also offers some interesting observations about algebraic varieties. We will also describe the approach via resolution of singularities, for comparison.

Remark 3.1. We should mention that [AH2] treats the case of *analytic* cycles in addition to algebraic cycles. The proofs are essentially the same, with one or two key differences. We will not cover the material on analytic cycles here.

Remark 3.2. The modern Hodge conjecture states that any class $\alpha \in H^{2n}(X; \mathbb{Q})$ whose image in $H^{2n}(X; \mathbb{C})$ lies in the Hodge group $H^{n,n}(X)$ is necessarily algebraic—that is, it lies in the subgroup $H^{2n}(X; \mathbb{Q})_{\text{alg}}$. When Hodge originally raised this question he did not explicitly specify rational coefficients. Since any torsion class in $H^{2n}(X; \mathbb{Z})$ would map to zero in $H^{2n}(X; \mathbb{C})$, and therefore lie inside $H^{n,n}(X)$, an integral version of the Hodge conjecture would imply that all torsion classes are algebraic. One of the main points of [AH2] was to demonstrate that this integral form of the Hodge conjecture does not hold in general.

How complicated does a smooth complex variety need to be in order to violate the integral Hodge conjecture? The smallest such varieties produced by the methods of [AH2] were of complex dimension 7. Arguments in [BCC] showed the existence of such varieties in dimension 3 (in fact sufficiently general hypersurfaces in $\mathbb{C}P^4$ of large degree), but without producing explicit examples. Examples of dimension 5 were produced in [SV], and then explicit examples of dimension 3 were obtained in [Totaro3] and later [BO]. This is just a sample of what is a very active area of research.

It turns out that the integral Hodge conjecture does hold for certain special classes of algebraic varieties, and this remains a topic of current interest. See [Totaro4] and [Voisin] as just two examples.

Up until Section 3.22 the material in this section closely follows [AH2].

3.3. Fundamental classes for subvarieties. If X is a smooth complex algebraic variety and $Y \subseteq X$ is a smooth subvariety of codimension q then we have seen

that complex orientability yields a fundamental class $[Y] \in H^{2q}(X)$ and a relative fundamental class $[Y]_{rel} \in H^{2q}(X, X - Y)$. This comes about by choosing a tubular neighborhood U of Y together with an orientation-preserving homeomorphism to the normal bundle, and using the isomorphisms

$$H^{2q}(X, X - Y) \cong H^{2q}(U, U - Y) \cong H^{2q}(N, N - 0).$$

Assuming Y is connected it follows that $H^{2q}(X, X - Y) \cong \mathbb{Z}$ (by Thom isomorphism), and $[Y]_{rel}$ is defined to be the image of the Thom class $u_N \in H^{2q}(N, N - 0)$. The fundamental class $[Y]$ is just the image of $[Y]_{rel}$ under $H^*(X, X - Y) \rightarrow H^*(X)$. Note that the argument shows the choice of U and orientation-preserving homeomorphism $U \cong N$ to be irrelevant: there are only two generators in $H^{2q}(X, X - Y)$, and any choices will lead to the same generator.

If Y is disconnected but all components have the same codimension then we define $[Y]$ to be $\sum_{i=1}^r [Y_i]$ where the Y_i are the connected components of Y . Note that in this case $H^{2q}(X, X - Y) \cong \mathbb{Z}^r$, with the generators corresponding to the images of the $[Y_i]_{rel}$ under $H^{2q}(X, X - Y_i) \rightarrow H^{2q}(X, X - Y)$.

We aim to show the existence of fundamental classes $[Y]$ even when the subvariety Y is not smooth. The approach we follow is to throw away the singular set and then observe that because this is in smaller dimension it didn't matter anyway.

In our level of generality Y might contain irreducible components of different dimensions. In this case we define $\text{codim}_X(Y)$ to be the *minimum* of the codimensions of the different components. We say that Y has **pure codimension** q if all of its irreducible components have codimension q .

Lemma 3.4. *Let X be a smooth algebraic variety and let $W \hookrightarrow Y \hookrightarrow X$ be closed subvarieties. Assume that Y has codimension $q \geq 1$ inside of X , and that W has codimension at least one inside of Y .*

- (a) $H^*(X) \rightarrow H^*(X - Y)$ is an isomorphism for $* \leq 2q - 2$ and a monomorphism for $* = 2q - 1$.
- (b) $H^*(X, X - Y) \rightarrow H^*(X - W, X - Y)$ is an isomorphism for $* \leq 2q$ and a monomorphism when $* = 2q + 1$.
- (c) If Y is irreducible then $H^{2q}(X, X - Y) \cong \mathbb{Z}$.

Proof. First note that part (a) is true when Y is smooth, using the long exact sequence for the pair $(X, X - Y)$, the isomorphism $H^*(X, X - Y) \cong H^*(N, N - 0)$, and the Thom isomorphism theorem. For the general case, we can filter Y by subvarieties

$$\emptyset \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_s = Y$$

where each Y_i is the singular set of Y_{i+1} (so that $Y_{i+1} - Y_i$ is smooth). Since Y_0 is smooth, we can assume by induction that $H^*(X) \rightarrow H^*(X - Y_i)$ is an isomorphism for $* \leq 2(\text{codim } Y_i) - 2$ and a monomorphism in the next degree. Now look at the composition

$$H^*(X) \rightarrow H^*(X - Y_i) \rightarrow H^*(X - Y_{i+1}) = H^*((X - Y_i) - (Y_{i+1} - Y_i)).$$

Since $Y_{i+1} - Y_i$ is smooth in $X - Y_i$, the second map is an isomorphism for $* \leq 2(\text{codim } Y_{i+1}) - 2$ and a monomorphism in the next degree. It follows that the same is true for the composite, and now the desired result follows by induction.

Part (b) follows from part (a) via the long exact sequences

$$\begin{array}{ccccccc}
 \cdots \rightarrow H^{*-1}(X - Y) & \longrightarrow & H^*(X, X - Y) & \longrightarrow & H^*(X) & \longrightarrow & H^*(X - Y) \rightarrow \cdots \\
 & & \downarrow = & & \downarrow j^* & & \downarrow = \\
 \cdots \rightarrow H^{*-1}(X - Y) & \rightarrow & H^*(X - W, X - Y) & \rightarrow & H^*(X - W) & \rightarrow & H^*(X - Y) \rightarrow \cdots
 \end{array}$$

By (a) the map labelled j^* is an isomorphism for $* \leq 2q$ and a monomorphism for $j = 2q + 1$, and so the result follows by the five lemma.

Finally, for (c) we let Z be the singular set of Y . Then $H^{2q}(X, X - Y) \cong H^{2q}(X - Z, (X - Z) - (Y - Z))$ by (b). But $Y - Z$ is a smooth closed subvariety of $X - Z$ of codimension q . A nonempty open subset of an irreducible is also irreducible (see [Hartshorne, Example I.1.1.3]). Therefore $Y - Z$ is irreducible, since Y is. By [Shafarevich2, Chapter 7, Section 2, Theorem 7.1], $Y - Z$ is connected in the complex topology. The remarks from the beginning of this section then show that $H^{2q}(X - Z, (X - Z) - (Y - Z))$ is isomorphic to $\mathbb{Z}$, with generator $[Y - Z]_{rel}$. $\square$

Remark 3.5 (Fundamental classes for singular subvarieties). Let $Y \hookrightarrow X$ be an algebraic subvariety of pure codimension q , and let Z be the singular set. Since $Y - Z$ is a smooth subvariety of $X - Z$ of codimension q , we have a fundamental class $[Y - Z] \in H^{2q}(X - Z)$. But by Lemma 3.4(a) the map $H^{2q}(X) \rightarrow H^{2q}(X - Z)$ is an isomorphism. We define $[Y] \in H^{2q}(X)$ to be the preimage of $[Y - Z]$ under this map. Additionally, note that we have the diagram

$$\begin{array}{ccc}
 H^{2q}(X - Z) & \longleftarrow & H^{2q}(X) \\
 \uparrow & & \uparrow \\
 H^{2q}(X - Z, X - Y) & \longleftarrow & H^{2q}(X, X - Y)
 \end{array}$$

and that both horizontal maps are isomorphisms by Lemma 3.4. We know the fundamental class $[Y - Z] \in H^{2q}(X - Z)$ is the image of a relative fundamental class $[Y - Z]_{rel} \in H^{2q}(X - Z, X - Y)$ (note that $(X - Z) - (Y - Z) = X - Y$), and so we define $[Y]_{rel} \in H^{2q}(X, X - Y)$ to be its preimage. Then $[Y]_{rel}$ maps to $[Y]$, as should be expected.

Define $H_{alg}^*(X) \subseteq H^*(X)$ to be the subgroup generated by the fundamental classes of all the algebraic subvarieties of X of pure codimension (this last hypothesis is mainly for convenience, as for cases of non-pure codimension one should define the fundamental class to be the sum of the fundamental classes of the irreducible components).

Sometimes it helps to be able to focus on classes $[Y]$ where Y is irreducible. To this end, the following is useful:

Proposition 3.6. *If $Y \hookrightarrow X$ is a pure codimension q subvariety with irreducible components $Y_1, \dots, Y_r$ then $[Y] = \sum_i [Y_i]$ in $H^{2q}(X)$. Therefore the subgroup $H_{alg}^*(X)$ is spanned by fundamental classes of irreducible subvarieties.*

Proof. Let $Z \subseteq Y$ be the singular set. This contains the points in each $Y_i \cap Y_j$, so $Y - Z$ is the disjoint union of the $Y_i - (Z \cap Y_i)$. So $[Y - Z] = \sum_i [Y_i - (Z \cap Y_i)]$ in $H^*(X - Z)$. Now consider the maps

$$H^{2q}(X) \longrightarrow H^{2q}(X - Z) \longleftarrow H^{2q}(X - (Z \cap Y_i)) \longleftarrow H^{2q}(X).$$

On the left, $[Y]$ is the unique class that maps to $[Y - Z] = \sum_i [Y_i - (Z \cap Y_i)]$. On the right, $[Y_i]$ is the unique class that maps to $[Y_i - (Z \cap Y_i)]$. It follows at once that $[Y] = \sum_i [Y_i]$ in $H^{2q}(X)$. $\square$

In the above results we saw the homological connectivity of the pairs $(X, X - Y)$. In fact this can easily be upgraded to a *homotopical* connectivity, as the following results show.

Proposition 3.7. *Let M be a smooth real manifold and let $Z \hookrightarrow M$ be a codimension k real submanifold. Then $M - Z \hookrightarrow M$ is $(k - 1)$ -connected.*

Proof. Let $U \subseteq M$ be a tubular neighborhood of N and consider the square

$$\begin{array}{ccc} U - Z & \xrightarrow{\quad} & M - Z \\ \downarrow & & \downarrow \\ U & \xrightarrow{\quad} & M. \end{array}$$

The map $U - Z \hookrightarrow M - Z$ is 0-connected by inspection. If N is the normal bundle of Z in M then the map $U - Z \hookrightarrow U$ is, up to homotopy, the bundle projection $N - 0 \rightarrow Z$, which is $(k - 1)$ -connected (the fibers are homotopy equivalent to S^{k-1}). The Blakers-Massey theorem implies that $\pi_i(U, U - Z) \rightarrow \pi_i(M, M - Z)$ is an isomorphism for $i \leq k - 2$ and an epimorphism for $i = k - 1$. But $\pi_i(U, U - Z) \cong \pi_i(N, N - 0) \cong \pi_{i-1}S^{k-1} = 0$ for $i \leq k - 1$, and therefore $M - Z \hookrightarrow M$ is $(k - 1)$ -connected. $\square$

Remark 3.8. The Blakers-Massey theorem is very homotopical, so it is worth keeping in mind the underlying geometric intuition behind the above result. If Z is a point and $\dim M = n$, then $M - Z$ is (up to homotopy) the complement of an n -disk in M and M is therefore obtained from $M - Z$ by attaching an n -cell. More generally, after removing a tubular neighborhood of Z one glues it back in by attaching a $(k + i)$ -cell for every i -cell of Z . So M is obtained from $M - Z$ by attaching cells of dimension k and higher.

Corollary 3.9. *Let X be a smooth complex algebraic variety and let $Y \hookrightarrow X$ be a subvariety of codimension q . Then $X - Y \hookrightarrow X$ is $(2q - 1)$ -connected. In particular, up to homotopy $(X, X - Y)$ may be replaced by a CW-pair (W, A) where W is obtained from A by attaching $(2q - 1)$ -cells and higher.*

Proof. If Y is smooth then this is a special case of Proposition 3.7. For the general case, filter Y by $\emptyset \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_s = Y$ where each Y_i is the singular set of Y_{i+1} . Consider the chain of inclusions

$$X - Y \subseteq X - Y_{s-1} \subseteq X - Y_{s-2} \subseteq \cdots \subseteq X - Y_0 \subseteq X.$$

Since $X - Y_i = (X - Y_{i-1}) - (Y_i - Y_{i-1})$ and $Y_i - Y_{i-1}$ is smooth, we have that $X - Y_i \hookrightarrow X - Y_{i-1}$ is $(2 \operatorname{codim}_X(Y_i) - 1)$ -connected. But $\operatorname{codim}_X(Y) \leq \operatorname{codim}_X(Y_i)$ for all i , and so every inclusion in our chain is $(2 \operatorname{codim}_X Y - 1)$ -connected. The result follows at once. $\square$

3.10. Vanishing of differentials on algebraic classes. Let X be a smooth complex algebraic variety, and let $Y \subseteq X$ be a subvariety. Let $\mathcal{F}_\bullet$ be a bounded resolution of $\mathcal{O}_Y$ by locally-free coherent $\mathcal{O}_X$ -modules (such a resolution exists by Remark ??), and write $F_\bullet$ for the associated chain complex of $\mathbb{C}$ -vector bundles on X . Then $[F_\bullet]$ defines a class in $K^0(X, X - Y)$ which we will denote $[Y]_{K, \text{rel}}$. Note

that when Y is smooth this agrees with the relative fundamental class provided by the complex orientation of K -theory, by Theorem ??.

Proposition 3.11. *Suppose that Y has pure codimension in X . Then $\text{ch}([Y]_{K,rel}) = [Y]_{H,rel} + \text{higher degree terms}$.*

Proof. We first prove this when Y is smooth. If N denotes the normal bundle for Y in X , then the result will follow once we know $\text{ch}(\mathcal{U}_N^K) = \mathcal{U}_N^H + \text{higher order terms}$, since both $[Y]_{K,rel}$ and $[Y]_{H,rel}$ are obtained from the Thom classes by applying natural maps. However, we have already seen in our discussion of Riemann-Roch that the complete formula is in fact

$$\text{ch}(\mathcal{U}_N^K) = \mathcal{U}_N^H \cdot \text{Td}(N)^{-1}$$

(see Proposition ??). Now just observe that $\text{Td}(N)^{-1} = 1 + \text{higher order terms}$.

Now let Y be arbitrary. Let Z be the singular locus, and let q denote the codimension of Y . Consider the diagram

$$\begin{array}{ccc} K^0(X, X - Y) & \xrightarrow{j^*} & K^0(X - Z, X - Y) \\ \text{ch} \downarrow & & \downarrow \text{ch} \\ H^*(X, X - Y) & \xrightarrow{j^*} & H^*(X - Z, X - Y). \end{array}$$

Both maps j^* are simply restriction to an open set, and so $j^*([Y]_{K,rel}) = [Y - Z]_{K,rel}$ and $j^*([Y]_{H,rel}) = [Y - Z]_{H,rel}$. By Lemma 3.4(b) the bottom map is an isomorphism for $* \leq 2q$. So the desired result for Y follows from the corresponding result for $Y - Z$, which has already been proven because $Y - Z$ is smooth in $X - Z$. $\square$

Theorem 3.12. *Let X be a smooth algebraic variety, and let $Y \subseteq X$ be a subvariety of pure codimension. Then $[Y]_H$ survives the Atiyah-Hirzebruch spectral sequence; that is, all differentials vanish on this class.*

Proof. This is now easy. We might as well assume that Y is irreducible, by Proposition 3.6. The class $[Y]_H$ is the image of $[Y]_{H,rel}$ under the natural map $H^*(X, X - Y) \rightarrow H^*(X)$. By naturality of the Atiyah-Hirzebruch spectral sequence, it suffices to show that all differentials on $[Y]_{H,rel}$ are zero. This follows at once from Proposition 3.11 and Proposition 2.9, in the latter taking $r = \infty$ and using the relative version for $(X, X - Y)$. Note that by Corollary 3.9 the pair $(X, X - Y)$ can be replaced up to homotopy by a CW-pair (W, A) where W is obtained from A by adding $2q$ -cells and higher, or equivalently by the space W/A which is a CW-complex having cells only in dimensions $2q$ and higher. So in this case the F_{2q-1} terms would all disappear from the statement of Proposition 2.9. $\square$

Corollary 3.13. *Let X be a smooth algebraic variety, and let p be a fixed prime. If a p^e -torsion class $u \in H^{ev}(X)$ is algebraic then $\beta P^1(\bar{u}) = 0$.*

Proof. Proposition 2.5(b) identifies the first possible differential on a p^e -torsion class in the Atiyah-Hirzebruch spectral sequence as $u \mapsto -\beta P^1(\bar{u})$. Using this, the result is immediate from Theorem 3.12. $\square$

3.14. Construction of varieties with non-algebraic cohomology classes. At this point our job is to construct a smooth, projective algebraic variety X that has a class $u \in H^{ev}(X)$ for which $\beta P^1 \bar{u} \neq 0$, for some prime p . Such a class cannot be algebraic by Corollary 3.13. It turns out that p can be any prime we like—that is, for any given p we can find an example of an X and a u . Moreover, u can be taken to lie in degree 4. The construction comes out of the following three results:

Theorem 3.15 (Serre). *Let G be a finite group and let $n \geq 1$. Then there exists a linear action of G on a projective space $\mathbb{CP}^N$ together with an n -dimensional, closed, smooth subvariety $X \hookrightarrow \mathbb{CP}^N$ which is a complete intersection, invariant under G , and has G acting freely. Under these conditions X/G is also a smooth projective variety.*

Corollary 3.16. *If X is a variety having the properties in Theorem 3.15 then there is an n -connected map $X/G \rightarrow \mathbb{CP}^\infty \times BG$. In particular, the homotopy $(n-1)$ -type of X/G is the same as that of $K(\mathbb{Z}, 2) \times BG$.*

Proposition 3.17. *Let p be a prime. Then there exists a finite group G and a class $u \in H^4(BG; \mathbb{Z})$ that is killed by p and is such that $\beta P^1(\bar{u}) \neq 0$. (In fact, one can take $G = \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$.)*

We postpone the proofs for one moment so that we can observe the immediate consequence:

Corollary 3.18 (Atiyah-Hirzebruch). *Fix a prime p . There exists a smooth, projective, complex algebraic variety X and a class $u \in H^{ev}(X)$ that is killed by p such that u is not algebraic.*

Proof. By Proposition 3.17 there exists a finite group G and a p -torsion class $u \in H^4(BG; \mathbb{Z})$ such that $\beta P^1(\bar{u}) \neq 0$. Note that P^1 has degree $2(p-1)$ and so $\beta P^1(\bar{u}) \in H^{2p+3}(BG; \mathbb{Z}/p)$.

By Corollary 3.16 there is a smooth, projective variety W of dimension $2p+3$ that admits a $(2p+3)$ -connected map $W \rightarrow K(\mathbb{Z}, 2) \times BG$. But then $H^*(BG; \mathbb{Z})$ injects into $H^*(W)$ up through dimension $2p+3$, and likewise for $\mathbb{Z}/p$ coefficients. So there is a class $w \in H^4(W)$ such that $\beta P^1(\bar{w}) \neq 0$, and by Corollary 3.13 this class cannot be algebraic. $\square$

Theorem 3.15 requires some algebraic geometry, but the proofs of both Corollary 3.16 and Proposition 3.17 are purely topological. We tackle these in reverse order:

Proof of Proposition 3.17. We start by considering $p = 2$ (the odd case turns out to be extremely similar). As a first attempt we might try to take $G = \mathbb{Z}/2$. Then $B\mathbb{Z}/2 = \mathbb{RP}^\infty$, $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$, and $H^*(\mathbb{RP}^\infty; \mathbb{Z}) = \mathbb{Z}/2[x^2]$. Note that we can regard the integral cohomology as being contained in the mod 2 cohomology. Unfortunately $\text{Sq}^3 = \beta \text{Sq}^2$ vanishes on all the integral classes, by an easy calculation. So this attempt doesn't work.

Next look at $B(\mathbb{Z}/2 \times \mathbb{Z}/2) = \mathbb{RP}^\infty \times \mathbb{RP}^\infty$. The mod 2 cohomology is $\mathbb{Z}/2[x, y]$, and because the integral cohomology is all 2-torsion it coincides with the subring of $H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}/2)$ consisting of all elements whose Bockstein vanishes. Such elements of course include all polynomials in x^2 and y^2 , but it also includes $\theta = x^2y + xy^2 = \beta(xy)$. In fact, the integral cohomology is spanned by the elements $x^{2i}y^{2j}$ and $x^{2i}y^{2j} \cdot \theta$, for $i, j \geq 0$. As before, the operation Sq^3 vanishes on all

elements of the first type. Another easy calculation shows that Sq^3 applied to an element of the second type is $x^{2i}y^{2j}\text{Sq}^3(\theta)$, and $\text{Sq}^3(\theta) = x^4y^2 + x^2y^4 \neq 0$. This gives us lots of classes that are not killed by Sq^3 , however they are all in odd dimensions. So this still doesn't solve our problem.

In the preceding paragraph, the reason things didn't work ultimately came down to the fact that $\beta(xy)$ had odd dimension. This gets fixed once we move to $B(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$. The mod 2 cohomology is $\mathbb{Z}/2[x, y, z]$, and again the integral cohomology is 2-torsion and so coincides with the elements in the mod 2 cohomology where the Bockstein vanishes. One such element is $\theta = \beta(xyz) = x^2yz + xy^2z + xyz^2$. It is easy to calculate that $\text{Sq}^3\theta = x^4y^2z + x^4yz^2 + \cdots \neq 0$. So finally we have an even-dimensional integral cohomology class where Sq^3 is nonzero.

The argument for odd primes works the same way. Recall that $H^*(B\mathbb{Z}/p; \mathbb{Z}/p) = \Lambda(u) \otimes \mathbb{F}_p[v]$, with $|u| = 1$, $|v| = 2$, $\beta(u) = v$ and $P^1(v) = v^p$. Take $G = \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$. The integral cohomology of BG is all p -torsion and so coincides with the subring of $H^*(BG; \mathbb{Z}/p)$ where the Bockstein vanishes. Look at $\theta = \beta(u_1u_2u_3) \in H^4(BG; \mathbb{Z})$. A simple calculation shows that $\beta P^1(\theta) \neq 0$. $\square$

Exercise 3.19. Verify all of the calculations outlined in the above proof.

Next we prove the corollary to Serre's theorem, again following [AH2]:

Proof of Corollary 3.16. Let $j: X \hookrightarrow \mathbb{C}P^N$ be the subvariety provided by Theorem 3.15(a). Since $X \hookrightarrow \mathbb{C}P^N$ is a complete intersection, the homotopical form of the Lefschetz Hyperplane Theorem (e.g. [Milnor6, Theorem 7.4]) yields that $X \rightarrow \mathbb{C}P^N$ is an isomorphism on π_i for $i < \dim X = n$ and an epimorphism on $\pi_{\dim X}$. In other words, the map $X \rightarrow \mathbb{C}P^N$ is n -connected.

Let $L \rightarrow \mathbb{C}P^N$ be the tautological line bundle $\mathbb{C}^{N+1} - 0 \rightarrow \mathbb{C}P^N$. Note that G acts on $\mathbb{C}^{N+1}$, and so G acts on the bundle L . Hence G acts on the pullback bundle $j^*L \rightarrow X$. Since the action of G on X is free we get a line bundle $(j^*L)/G \rightarrow X/G$ which pulls back to j^*L along the projection $X \rightarrow X/G$. Let the classifying map for this line bundle be $X/G \rightarrow \mathbb{C}P^\infty$. The diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{C}P^N \\ \downarrow & & \downarrow \\ X/G & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

necessarily commutes up to homotopy, as the two compositions classify the same bundle j^*L . Since $X \rightarrow \mathbb{C}P^N$ and $\mathbb{C}P^N \rightarrow \mathbb{C}P^\infty$ are both n -connected, so is the composite $X \rightarrow \mathbb{C}P^\infty$.

Consider the composite map $X \rightarrow X/G \rightarrow \mathbb{C}P^\infty$. This is a G -equivariant map, where the target is given the trivial G -action. Recall the homotopy coinvariants $X_{hG} = (X \times EG)/G$ (where we quotient by the diagonal G -action). The collapse map $X \rightarrow pt$ induces a fibration $X_{hG} \rightarrow BG$ with fiber X , and our equivariant map $X \rightarrow \mathbb{C}P^\infty$ gives a map of fibration sequences

$$\begin{array}{ccccc} X & \longrightarrow & X_{hG} & \twoheadrightarrow & BG \\ \downarrow & & \downarrow & & \parallel \\ \mathbb{C}P^\infty & \longrightarrow & (\mathbb{C}P^\infty)_{hG} & \twoheadrightarrow & BG. \end{array}$$

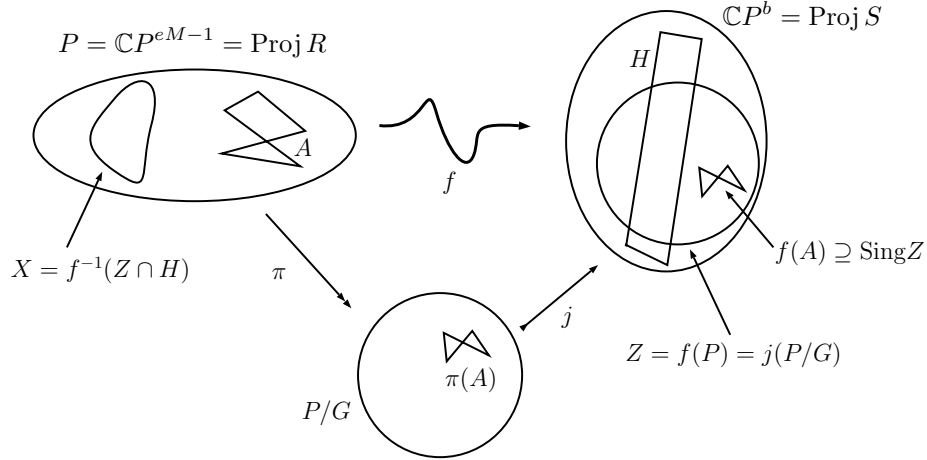
Using the long exact homotopy sequence and the five-lemma, the fact that $X \rightarrow \mathbb{C}P^\infty$ is n -connected implies the same for $X_{hG} \rightarrow \mathbb{C}P_{hG}^\infty$. But since the G -action on X is free one has $X_{hG} \simeq X/G$, and since the action on $\mathbb{C}P^\infty$ is trivial one has $(\mathbb{C}P^\infty)_{hG} \simeq \mathbb{C}P^\infty \times BG$. This completes the proof. $\square$

Finally we prove Serre's theorem, following [Serre1, Section 20]. Serre credits the method to Godeaux.

Proof of Theorem 3.15. Recall that we start with a finite group G and an $n \geq 1$. Let V be the regular representation of G and let $P = \mathbb{P}(V^M)$ where $M \geq 1$. Eventually we will assume that M is larger than a certain bound that depends on n , but for now let us keep that open. Let $e = \#G$, so that $\dim P = eM - 1$. The only reason for starting with the regular representation here is that the resulting G -action on P is generically free—that is, the set of points $A \subseteq P$ with nontrivial stabilizer is a proper closed subvariety. In fact, A is just the union of the projectivizations of the eigenspaces of all of the $g \in G - \{1_G\}$ acting on V^M . In a moment we will bound the dimension of A , but for now just regard A as a “small”.

The quotient P/G can be constructed as a projective algebraic variety (see ???), and its dimension will be the same as P . We can choose a closed embedding $j: P/G \hookrightarrow \mathbb{C}P^b$ for some $b > eM - 1$. Let $\pi: P \rightarrow P/G$ denote the quotient map, and let $f = j \circ \pi$. The map $P - A \rightarrow P/G - \pi(A)$ is a covering space (a finite étale map). Let R and S be the homogeneous coordinate rings of P and $\mathbb{C}P^b$.

FIGURE 3.19.1. Schematic diagram of the map $f: P \rightarrow \mathbb{C}P^b$



The reader might find Figure 3.19.1 helpful for navigating the next few steps of the proof. Let $H \subseteq \mathbb{C}P^b$ be a generic linear subspace of dimension equal to $b + n - (eM - 1)$, where now we assume $n < eM - 1$ so that $H \subsetneq \mathbb{C}P^b$. Then we can choose H so that it intersects Z transversely, and $Y = H \cap Z$ will be smooth of dimension n . Our setup now forces $X = f^{-1}(Y)$ to be stable under G . If $f(A)$ is small enough then we can also insist that H avoid it: this will be possible as long as $b + n - (eM - 1) + \dim A < b$, or equivalently $n + 1 + \dim A < eM$. So if we

have this bound then we can arrange for $Y \cap f(A) = \emptyset$, in which case $X \cap A = \emptyset$ and so the G -action on X is free.

Let us pause here and analyze the dimension condition. Let $g \in G - \{1_G\}$ and write $H = \langle g \rangle$. Let $G = \bigcup_i Hg_i$ be the right-coset decomposition. Then $V = \mathbb{C}\langle G \rangle = \bigoplus_i \mathbb{C}\langle Hg_i \rangle$ as H -sets. The eigenspaces of g acting on $G\langle H \rangle$ are all at most one-dimensional, so the eigenspaces in V have dimension equal to at most the number of right cosets, which is $\#G/\# \langle g \rangle$. In particular, this number is at most $\#G/2 = e/2$. The eigenspaces of g acting on V^M thus have dimension at most $eM/2$, and from this one derives that $\dim A \leq (eM/2) - 1$. So to guarantee that $n + 1 + \dim A < eM$ it is enough to insist that M was chosen to satisfy $\frac{2n}{e} < M$.

At this point we have produced a projective variety $X \subseteq P$ on which G acts freely. It is smooth because $X \rightarrow Y$ is a finite étale map (a covering space) and Y is smooth. It only remains to show that X is a complete intersection in P . For this, let $h_1, \dots, h_t \in S$ be linear forms such that $H = V(h_1, \dots, h_t)$, where here $t = (eM - 1) - n$. Then set-theoretically X is the vanishing locus of the functions $h_1f, \dots, h_tf \in R$. We have $\dim V(h_1f, \dots, h_tf) = \dim V(h_1, \dots, h_t) = \dim Z - i = \dim P - i$ for all i , so $h_1f, \dots, h_tf$ is a regular sequence in R (see Exercise ??). This shows that X is a set-theoretic complete intersection.

To show that X is actually a complete intersection it remains to show that $I = (h_1f, \dots, h_tf) \subseteq R$ is a radical ideal. In a commutative ring T , an ideal J is radical if and only if JT_m is radical for every maximal ideal $J \subseteq m$. The analogous result holds in the graded case, where one uses homogeneous maximal ideals and the degree zero part of the localization. So it suffices to show, for every homogeneous maximal ideal $I \subseteq m$, that $I\mathcal{O}_{P,m}$ is radical. Let $m' = f(m)$. The transversality of the intersection $H \cap Z$ implies that $h_1, \dots, h_t$ is part of a regular system of parameters for $\mathcal{O}_{Z,m'}$. That is to say, their images in the Zariski cotangent space $m'/(m')^2$ are independent. But $\mathcal{O}_{Z,m'} \rightarrow \mathcal{O}_{P,m}$ is étale and therefore induces an isomorphism on Zariski cotangent spaces. So $h_1f, \dots, h_tf$ are also part of a regular system of parameters for $\mathcal{O}_{P,m}$. This implies that $(h_1f, \dots, h_tf)$ is prime in $\mathcal{O}_{P,m}$, and in particular it is radical. $\square$

Remark 3.20. Since the above proof used some facts about quotient spaces X/G in algebraic geometry, let us give a few references. Let G be a finite group acting freely on a smooth complex projective variety X . The results of [SGA1, Exposé V, Sections 1-2] show that X/G exists as a scheme and that $X \rightarrow X/G$ is finite étale, which implies that X/G is smooth. Specifically, one readily checks that every finite set of points in X is contained in an affine open, and so X satisfies the hypotheses of [SGA1, Proposition V.1.8]. This guarantees the existence of X/G as a scheme. The projectivity is not explicitly stated in [SGA1], but is an easy exercise (???). Finally, [SGA1, Corollary V.2.3] shows that the freeness of the action implies that $X \rightarrow X/G$ is étale. Fogarty-Mumford??? Goresky-MacPherson??? See also Exercises ????

Exercise 3.21. Think through some details of the above proof in the special case $G = \mathbb{Z}/2$. Let $\mathbb{C}^{2M}$ have coordinates $x_1, \dots, x_M, y_1, \dots, y_M$, and let G act trivially on the x 's and by negation on the y 's. Consider the induced action on $P = \mathbb{P}(\mathbb{C}^{2M}) = \mathbb{C}P^{2M-1}$. The homogeneous coordinate ring is $R = \mathbb{C}[x_1, \dots, x_M, y_1, \dots, y_M]$, and the ring of invariants $S = R^G$ is the subring generated by the x_i 's and the $y_i y_j$'s (including $i = j$). Show that every homogeneous element of S having even degree is a polynomial in the elements $x_i x_j$ and $y_i y_j$. That

is, if $S\langle 2 \rangle \subseteq S$ is the $\mathbb{C}$ -linear span of all even-dimensional homogeneous elements then $S\langle 2 \rangle$ is generated (as a subring) by elements of degree 2.

Let $\alpha_1, \dots, \alpha_s$ denote the degree 2 monomials in the x_i , in some order. Similarly, let $\beta_1, \dots, \beta_s$ denote the degree 2 monomials in the y_j , and let $b = 2s - 1 = 2\binom{M+1}{2} - 1 = M^2 + M - 1$. Let $f: P \rightarrow \mathbb{C}P^b$ be the map

$$f([x_1 : \dots : x_M : y_1 : \dots : y_M]) = [\alpha_1 : \dots : \alpha_s : \beta_1 : \dots : \beta_s].$$

Check that f is G -equivariant, where the target has the trivial action, and that f induces an embedding of topological spaces $P/G \hookrightarrow \mathbb{C}P^b$. Its image Z is the closed subvariety whose homogeneous coordinate ring is $S\langle 2 \rangle$, regarded as a quotient of the polynomial algebra $\mathbb{C}[\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_s]$. The scheme Z is the categorical quotient of P by G : here we are using $\text{Proj } R^G = \text{Proj } S\langle 2 \rangle$ and Exercises ?? and ??.

Let $A \subseteq P$ be the set of fixed points for the $\mathbb{Z}/2$ -action. Check that A is the disjoint union $\mathbb{C}P^{M-1} \amalg \mathbb{C}P^{M-1}$, where the first piece corresponds to the vanishing of the x_i 's and the second piece to the vanishing of the y_j 's.

3.22. Thom's theory. Although this part of the story doesn't use K -theory, the obstructions to algebraicity obtained from resolution of singularities are so simple that they are worth discussing here.

Theorem 3.23. *Let X be a smooth algebraic variety, and let p be a fixed prime. If $u \in H^{ev}(X; \mathbb{Z})$ is algebraic then all odd-degree cohomology operations vanish on the mod p reduction $\bar{u} \in H^{ev}(X; \mathbb{Z}/p)$. In particular, all of the odd Steenrod squares vanish on the mod 2 reduction of u .*

The above theorem was probably folklore since the 1960s. It explicitly appears in the beautiful paper [Totaro1]. The key to this result is the cohomology theory called complex cobordism, denoted MU . This was first introduced and studied by Thom. The definitions are somewhat involved, but the corresponding homology groups $MU_*(X)$ are formed from classes of manifolds—with a certain kind of structure—mapping into X , with an equivalence relation coming from bordism of such things. See [Conner, Chapter 1] for a general introduction to bordism theories, and [Stong] and [Rudyak] for additional details. The spectrum MU can be constructed without going through the geometrical considerations, and the reader is taken through that in Exercise ?? below. Our discussion will sweep most of these details under the rug, as we only need a few carefully selected properties of MU .

First, MU is a complex-oriented cohomology theory. In fact it is the universal such one, in a certain sense. Consequently, there is a map of cohomology theories $MU^*(-) \rightarrow H^*(-)$ that is compatible with the complex orientations. See Exercise ?? below for details. So the map sends Thom classes to Thom classes (i.e. $\mathcal{U}_E^{MU} \mapsto \mathcal{U}_E^H$) and consequently if $Y \hookrightarrow X$ is a smooth, closed subvariety it sends $[Y]_{MU}$ to $[Y]_H$.

The next part of the argument is best explained using the language of spectra. The above map of cohomology theories comes from a map of spectra $MU \rightarrow H\mathbb{Z}$. If $Y \hookrightarrow X$ is a smooth subvariety of codimension q then $[Y]_{MU} \in MU^{2q}(X)$ is represented by a map $X \rightarrow \Sigma^{2q} MU$, and likewise $[Y]_H \in H^{2q}(X)$ is represented by a map $X \rightarrow \Sigma^{2q} H\mathbb{Z}$. In the homotopy category of spectra we have the commutative

diagram

$$\begin{array}{ccc} & \Sigma^{2q} MU & \\ [Y]_{MU} \nearrow & \downarrow & \\ X & \xrightarrow{[Y]_H} & \Sigma^{2q} H\mathbb{Z}. \end{array}$$

Let θ be a stable cohomology operation of degree r on $H^*(-; \mathbb{Z}/p)$. This is a map of spectra $H\mathbb{Z}/p \rightarrow \Sigma^r H\mathbb{Z}/p$. The application of this operation to the mod p reduction of $[Y]_H$ enhances our diagram:

$$\begin{array}{ccccccc} & \Sigma^{2q} MU & & & & & \\ [Y]_{MU} \nearrow & \downarrow & & \xrightarrow{f} & & & \\ X & \xrightarrow{[Y]_H} & \Sigma^{2q} H\mathbb{Z} & \longrightarrow & \Sigma^{2q} H\mathbb{Z}/p & \xrightarrow{\theta} & \Sigma^{2q+r} H\mathbb{Z}/p. \end{array}$$

The map labelled f is just the evident composite. Note that f is an element of $H^r(MU; \mathbb{Z}/p)$. The “miracle” is that we can easily compute this group. The spaces making up the spectrum MU are just Thom spaces of the universal bundles $\gamma_n \rightarrow BU(n)$, and their integral cohomology is known by the Thom isomorphism. We leave the details to the reader (see Exercise ??(e,f)), but the conclusion here is that $H^*(MU; \mathbb{Z})$ is free abelian and concentrated in even degrees. It follows that $H^*(MU; \mathbb{Z}/p)$ vanishes in all odd degrees. In particular, the map f is null when r is odd! Thus, we have proven Thom’s theorem:

Proposition 3.24 (Thom). *Let X be a smooth algebraic variety and let $Y \hookrightarrow X$ be a smooth subvariety. Fix a prime p . Then all odd-degree stable cohomology operations vanish on the mod p reduction of the class $[Y]_H \in H^*(X)$.*

Remark 3.25. It is interesting to note how easy the language of spectra makes the above argument. As a challenge, try to unwind the argument and rephrase it without using spectra—it is not so pleasant.

One can deduce Theorem 3.23 from Proposition 3.24 using resolution of singularities and a little work. We are not going to give complete details, but we give a rough sketch. Complete details (and much more) can be found in [Totaro2].

Sketch of proof of Theorem 3.23. By Proposition 3.6 it suffices to prove Theorem 3.23 when u is the fundamental class of an irreducible subvariety $Y \hookrightarrow X$, say of codimension q . Such elements generate all algebraic cohomology classes. By Hironaka there is a resolution of singularities $\tilde{Y} \rightarrow Y$ obtained by successively blowing up Y at closed subschemes. Even more, one can successively blow up X at the same subschemes to produce a commutative square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\quad} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\quad} & X \end{array}$$

where the horizontal maps are closed inclusions and the vertical maps are compositions of successive blow-ups (in particular, they are proper). We of course have the class $[\tilde{Y}]_{MU} \in MU^{2q}(\tilde{X})$, and with a little work one can construct a pushforward map $\pi_*: MU^{2q}(\tilde{X}) \rightarrow MU^{2q}(X)$ and likewise in H instead of MU . We claim that

$\pi_1([\tilde{Y}]_{MU})$ is a lift of the class $[Y]_H$. To see this, let $Z \subseteq Y$ be the singular set and let $\tilde{Z} \subseteq \tilde{Y}$ be its preimage in $\tilde{Y}$. Consider the diagram

$$\begin{array}{ccccccc} [\tilde{Y}]_{MU} & \in & MU^{2q}(\tilde{X}) & \longrightarrow & H^{2q}(\tilde{X}) & \xrightarrow[\cong]{j^*} & H^{2q}(\tilde{X} - \tilde{Z}) \\ \downarrow & & \downarrow \pi_1 & & \downarrow \pi_1 & & \downarrow \pi_1 \cong \\ \pi_1([\tilde{Y}]_{MU}) & \in & MU^{2q}(X) & \longrightarrow & H^{2q}(X) & \xrightarrow[\cong]{j^*} & H^{2q}(X - Z). \end{array}$$

The right-most vertical map is an isomorphism because $\tilde{X} - \tilde{Z} \rightarrow X - Z$ is a homeomorphism. The maps j^* are isomorphisms by Lemma 3.4. If we follow $[\tilde{Y}]_{MU}$ across the top and then down, it maps to $[\tilde{Y}]_H$ and then on to $[Y - Z]$ in the bottom right corner. Since $[Y] \in H^{2q}(X)$ also maps to $[Y - Z]$, it must be that $\pi_1([\tilde{Y}]_{MU})$ maps to $[Y]_H$, as desired. In other words, we have the diagram

$$\begin{array}{ccc} & \Sigma^{2q} MU & \\ \nearrow \pi_1([\tilde{Y}]_{MU}) & \downarrow & \\ X & \xrightarrow[\pi_1([Y]_H)]{} & \Sigma^{2q} H\mathbb{Z} \end{array}$$

and at this point everything proceeds the same as before. This completes our sketch of a proof for Theorem 3.23. $\square$

Note that even with this approach, as opposed to the K -theory approach of Atiyah-Hirzebruch, one still needs to give examples of algebraic varieties with nontrivial odd-degree operations on even-dimensional cohomology classes. So the hard work that went on in Section 3.14 is still necessary.

◦ Exercises ◦

4. OPERATIONS ON K -THEORY

Experience has shown that when studying a cohomology theory it is useful to look not just at the cohomology groups themselves but also the natural *operations* on the cohomology groups. In the case of singular cohomology this is the theory of Steenrod operations. In the present section we will construct some useful operations $K^0(X) \rightarrow K^0(X)$. We start with the λ -operations, which are easy to define but have the drawback that they are not group homomorphisms. Then we modify these to obtain the Adams operations ψ^k , which are more nicely behaved.

For most of this section X will be a paracompact Hausdorff space that is also homotopically compact, so that $K^0(X)$ is equal to the Grothendieck group of vector bundles. In Section 4.21 we will explain how to extend to the case of general X .

4.1. The lambda operations. Fix a topological space X . We start with the exterior power constructions $E \mapsto \Lambda^k E$ on vector bundles over X . These are, of course, not additive: $\Lambda^k(E \oplus F) \not\cong \Lambda^k E \oplus \Lambda^k F$. So it is not immediately clear how these constructions induce maps on K -groups. The key lies in the formula

$$(4.2) \quad \Lambda^k(E \oplus F) \cong \bigoplus_{i+j=k} \Lambda^i E \otimes \Lambda^j F.$$

Construct a formal power series

$$\lambda_t(E) = \sum_{i=0}^{\infty} [\Lambda^i E] t^i = 1 + [E]t + [\Lambda^2 E]t^2 + \cdots \in K^0(X)[[t]].$$

Because the zero coefficient is 1, this power series is a unit in $K^0(X)[[t]]$. So λ_t is a function into the group of units inside $K^0(X)[[t]]$:

$$\begin{array}{ccc} \text{Vect}(X) & \xrightarrow{\lambda_t} & \left(K^0(X)[[t]]\right)^* \\ \downarrow & \nearrow \text{dotted arrow} & \\ K^0(X) & & \end{array}$$

Formula (4.2) says that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$, and this implies the existence of the dotted-arrow group homomorphism in the above diagram. We will call this dotted arrow λ_t as well.

Finally, define $\lambda^k: K^0(X) \rightarrow K^0(X)$ by letting $\lambda^k(w)$ be the coefficient of t^k in $\lambda_t(w)$. Note that if E is a vector bundle over X then $\lambda^k([E]) = [\Lambda^k E]$. However, λ^k is not a group homomorphism; instead one has the formula

$$\lambda^k(u + v) = \sum_{i+j=k} \lambda^i(u) \lambda^j(v).$$

Example 4.3. To get a feeling for these operations let us compute $\lambda^k(-[E])$ for E a vector bundle over X . Note first that

$$\lambda_t(-[E]) = \frac{1}{\lambda_t([E])} = \frac{1}{1 + [E]t + [\Lambda^2 E]t^2 + \cdots}.$$

For R a commutative ring and $a = 1 + a_1 t + a_2 t^2 + \cdots \in R[[t]]$, one has

$$\frac{1}{a} = 1 + P_1 t + P_2 t^2 + \cdots$$

where the P_i 's are certain universal polynomials in the a_i 's with coefficients in $\mathbb{Z}$. Equating coefficients in the identity $1 = (1 + a_1 t + a_2 t^2 + \cdots)(1 + P_1 t + P_2 t^2 + \cdots)$ gives

$$P_k + P_{k-1}a_1 + P_{k-2}a_2 + \cdots + P_1 a_{k-1} + a_k = 0,$$

which allows one to inductively determine each P_k . One finds that

$$P_1 = -a_1, \quad P_2 = a_1^2 - a_2, \quad P_3 = -a_1^3 + 2a_1 a_2 - a_3$$

So we conclude that

$$\begin{aligned} \lambda^1(-[E]) &= -[E], \\ \lambda^2(-[E]) &= [E]^2 - [\Lambda^2 E], \\ \lambda^3(-[E]) &= -[E]^3 + 2[E][\Lambda^2 E] - [\Lambda^3 E] \end{aligned}$$

and so forth.

Exercise 4.4. Work out the formulas for $\lambda^k([E] - [F])$ for $1 \leq k \leq 3$, where E and F are vector bundles.

4.5. Symmetric power operations. One can repeat everything from the previous section using the symmetric product construction $E \mapsto \text{Sym}^k E$ in place of the exterior product $\Lambda^k E$. One obtains a group homomorphism

$$\text{sym}_t: K^0(X) \rightarrow (K^0(X)[[t]])^*$$

and defines $\text{sym}^k(w)$ to be the coefficient of t^k in $\text{sym}_t(w)$. It turns out, however, that these operations do not give anything ‘new’—they are related to the λ -operations by the formula

$$\text{sym}^k(w) = (-1)^k \lambda^k(-w).$$

To explain this we need a brief detour to discuss the multilinear algebra that feeds into the de Rham and Koszul complexes. Our treatment here is largely based on [FLS, Section 3], but one can find similar material in [Eisenbud, Section A2.6] and elsewhere.

Let V be a vector space over a field F . Write $\text{Sym}^*(V) = \bigoplus_k \text{Sym}^k(V)$ and $\Lambda^*(V) = \bigoplus_k \Lambda^k V$. These each have a familiar algebra structure, and we have canonical isomorphisms

$$\text{Sym}^*(V) \otimes \text{Sym}^*(W) \xrightarrow{\cong} \text{Sym}^*(V \oplus W), \quad \Lambda^*(V) \otimes \Lambda^*(W) \xrightarrow{\cong} \Lambda^*(V \oplus W).$$

(In the former case, include $\text{Sym}^*(V)$ and $\text{Sym}^*(W)$ into $\text{Sym}^*(V \oplus W)$ and then multiply there; likewise for the exterior algebra version). These isomorphisms allow us to equip both $\text{Sym}^*(V)$ and $\Lambda^*(V)$ with coproducts Δ_{sym} and Δ_{ext} making them into Hopf algebras. The coproducts are

$$\text{Sym}^*(V) \longrightarrow \text{Sym}^*(V \oplus V) \xleftarrow{\cong} \text{Sym}^*(V) \otimes \text{Sym}^*(V)$$

and

$$\Lambda^*(V) \longrightarrow \Lambda^*(V \oplus V) \xleftarrow{\cong} \Lambda^*(V) \otimes \Lambda^*(V)$$

where in each case the first map is the one induced by the diagonal $\Delta: V \rightarrow V \oplus V$. If $v_1, \dots, v_k \in V$ and $I \subseteq \{1, \dots, k\}$ then write v_I for the monomial $v_{i_1} \cdots v_{i_r}$ where $I = \{i_1, \dots, i_r\}$ and $i_1 < \cdots < i_r$. Then

$$\Delta_{\text{sym}}(v_1 \cdots v_k) = \sum_{A \dot{\cup} B = \{1, \dots, k\}} v_A \otimes v_B$$

where the $\dot{\cup}$ symbol denotes disjoint union, and likewise

$$\Delta_{\text{ext}}(v_1 \cdots v_k) = \sum_{A \dot{\cup} B = \{1, \dots, k\}} (-1)^{[A, B]} v_A \otimes v_B$$

where the sign $(-1)^{[A, B]}$ is the one that makes the equation $v_1 \cdots v_k = (-1)^{[A, B]} v_A v_B$ valid in $\Lambda^*(V)$.

Exercise 4.6. Derive the above formulas for Δ_{sym} and Δ_{ext} . Explain the connection to the process of multiplying out the formal product

$$(v_1 + \tilde{v}_1)(v_2 + \tilde{v}_2) \cdots (v_k + \tilde{v}_k)$$

and rewriting each of the 2^k terms as a product of v ’s times a product of $\tilde{v}$ ’s.

Write $e: \text{Sym}^k(V) \rightarrow \text{Sym}^{k-1}V \otimes \text{Sym}^1(V)$ and $e': \Lambda^k(V) \rightarrow \Lambda^1V \otimes \Lambda^{k-1}(V)$ for the projections of the coproduct onto the indicated factors. Finally, write d and κ for the composites

$$\begin{array}{c} \text{Sym}^k(V) \otimes \Lambda^i(V) \xrightarrow{e \otimes \text{id}} \text{Sym}^{k-1}(V) \otimes \text{Sym}^1(V) \otimes \Lambda^i(V) \\ \parallel \\ \text{Sym}^{k-1}(V) \otimes \Lambda^1(V) \otimes \Lambda^i(V) \xrightarrow{\text{id} \otimes \mu} \text{Sym}^{k-1}(V) \otimes \Lambda^{i+1}(V) \end{array}$$

and

$$\begin{array}{c} \text{Sym}^k(V) \otimes \Lambda^i(V) \xrightarrow{\text{id} \otimes e'} \text{Sym}^k(V) \otimes \Lambda^1(V) \otimes \Lambda^{i-1}(V) \\ \parallel \\ \text{Sym}^k(V) \otimes \text{Sym}^1(V) \otimes \Lambda^{i-1}(V) \xrightarrow{\mu \otimes \text{id}} \text{Sym}^{k+1}(V) \otimes \Lambda^{i-1}(V). \end{array}$$

The maps d and κ are called the **de Rham** and **Koszul** differentials, respectively; in a moment we will give concrete formulas for them and see that $d^2 = k^2 = 0$. Figure 4.6.1 shows the maps d , and the maps κ go in the opposite direction:

FIGURE 4.6.1. The strands of the deRham complex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ & \nearrow d & & \nearrow d & & \nearrow d & \\ \text{Sym}^3 V \otimes \Lambda^2 V & & \text{Sym}^2 V \otimes \Lambda^2 V & & \text{Sym}^1 V \otimes \Lambda^2 V & & \text{Sym}^0 V \otimes \Lambda^2 V \\ & \nearrow d & & \nearrow d & & \nearrow d & \\ \text{Sym}^3 V \otimes \Lambda^1 V & & \text{Sym}^2 V \otimes \Lambda^1 V & & \text{Sym}^1 V \otimes \Lambda^1 V & & \text{Sym}^0 V \otimes \Lambda^1 V \\ & \nearrow d & & \nearrow d & & \nearrow d & \\ \text{Sym}^3 V \otimes \Lambda^0 V & & \text{Sym}^2 V \otimes \Lambda^0 V & & \text{Sym}^1 V \otimes \Lambda^0 V & & \text{Sym}^0 V \otimes \Lambda^0 V \end{array}$$

Let $e_1, \dots, e_n$ be elements of V . It will be convenient to use the notation de_j for the element of $\Lambda^1(V)$ corresponding to e_j under the canonical isomorphism $\Lambda^1(V) \cong V$. Let $m = e_{i_1} \cdots e_{i_r} \in \text{Sym}^r(V)$ and $\omega = de_{j_1} \wedge \cdots \wedge de_{j_s} \in \Lambda^s(V)$. It is an exercise to verify that

$$e(m) = \sum_u (e_{i_1} \otimes \cdots \otimes \widehat{e_{i_u}} \otimes \cdots \otimes e_{i_r}) \otimes e_{i_u}$$

and

$$e'(\omega) = \sum_u (-1)^{u-1} de_{j_u} \otimes (de_{j_1} \wedge \cdots \wedge \widehat{de_{j_u}} \wedge \cdots \wedge de_{j_s}).$$

So

$$d(m \otimes \omega) = \sum_u (e_{i_1} \otimes \cdots \otimes \widehat{e_{i_u}} \otimes \cdots \otimes e_{i_r}) \otimes (de_{i_u} \wedge \omega)$$

and

$$\kappa(m \otimes \omega) = \sum_u (-1)^{u-1} (m \cdot e_{j_u}) \otimes (de_{j_1} \wedge \cdots \wedge \widehat{de_{j_u}} \wedge \cdots \wedge de_{j_s}).$$

From these descriptions one readily sees that d is the usual deRham differential and κ is the usual Koszul differential. Consequently, $d^2 = 0$ and $\kappa^2 = 0$. In the two-dimensional array (4.6.1), if we take direct sums inside of each row then we get

$$(4.7) \quad \text{Sym}^*(V) \otimes \Lambda^0(V) \xrightarrow{d} \text{Sym}^*(V) \otimes \Lambda^1(V) \xrightarrow{d} \cdots$$

and this is an algebraic version of the deRham complex. We also get

$$\cdots \longrightarrow \text{Sym}^*(V) \otimes \Lambda^2(V) \xrightarrow{\kappa} \text{Sym}^*(V) \otimes \Lambda^1(V) \xrightarrow{\kappa} \text{Sym}^*(V) \otimes \Lambda^0(V)$$

which is a Koszul complex.

Proposition 4.8.

- (a) $d\kappa + \kappa d: \text{Sym}^r(V) \otimes \Lambda^s(V) \rightarrow \text{Sym}^r(V) \otimes \Lambda^s(V)$ is multiplication by the total degree $r + s$.
- (b) In Figure 4.6.1 every diagonal deRham chain complex is exact in dimensions where the total degree is prime to the characteristic of F .
- (c) In Figure 4.6.1 every diagonal Koszul chain complex is exact, regardless of the characteristic of the ground field, except for the diagonal in total degree 0.

Proof. Part (a) is a computation that is tedious but not particularly hard. If $v_1, \dots, v_r, w_1, \dots, w_s \in V$ then $d\kappa(v_1 \cdots v_r \otimes w_1 \cdots w_s)$ has two types of terms: “pure” terms where a v_i is moved to the right of the tensor and then moved back, and “mixed” terms where a v_i is moved to the right of the tensor and then a w_j is moved to the left. There are exactly r pure terms, leading to a formula

$$d\kappa(\underline{v} \otimes \underline{w}) = r(\underline{v} \otimes \underline{w}) + (\text{mixed terms}).$$

Similarly, κd has s pure terms where a w_j is moved to the left and then moved back, and also a bunch of mixed terms, so that $\kappa d(\underline{v} \otimes \underline{w}) = s(\underline{v} \otimes \underline{w}) + (\text{mixed terms})$. One checks that the signs on the mixed terms in $d\kappa$ exactly cancel those in κd .

Part (b) follows from (a): if $z \in \text{Sym}^r(V) \otimes \Lambda^s(V)$ is a cycle for d then

$$(r + s)z = (d\kappa + \kappa d)(z) = d\kappa(z)$$

and so as long as $r + s$ is invertible in F one has that z is a boundary.

We do not actually need part (c) below, but we include it to complete the story. The maps d give a chain homotopy for the κ -complexes, much like in the proof of (b), but this gives exactness only for some spots in the complex. The proof of exactness at all spots is something we have already seen in a somewhat more general context, in Theorem ??(a). If we pick a basis $x_1, \dots, x_n$ of V then $\text{Sym}^*(V) = F[x_1, \dots, x_n]$ and the κ -complex is the Koszul complex $K(x_1, \dots, x_n)$. $\square$

Exercise 4.9. Suppose that $\text{char}(F) = p > 0$. For nonzero $x \in V$ verify that x^p and $x^{p-1} dx$ are cycles in the deRham complex and that they are not boundaries, so that there is indeed nontrivial homology in total degree p . (If $x_1, \dots, x_n$ is a basis for V then as an aside we mention that the cohomology of the algebraic deRham complex (4.7) in degree i turns out to be isomorphic to $F[x_1^p, \dots, x_n^p]$ tensored with $\Lambda^i V$. This is the content of the so-called *Cartier isomorphism*.)

Now we apply the above results to K -theory. Since the deRham and Koszul complexes were canonical constructions, we can apply them to vector bundles. We are in characteristic zero, so for each $k \geq 1$ the deRham complex gives us an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Sym}^k E \otimes \Lambda^0 E & \longrightarrow & \mathrm{Sym}^{k-1} E \otimes \Lambda^1 E & \longrightarrow & \cdots & \longrightarrow & \mathrm{Sym}^1 E \otimes \Lambda^{k-1} E \\ & & & & & & \downarrow \\ & & & & & & \mathrm{Sym}^0 E \otimes \Lambda^k E \longrightarrow 0. \end{array}$$

This shows that in $K^0(X)$ one has (for $k \geq 1$)

$$\sum_{a+b=k} (-1)^b [\mathrm{Sym}^a E] \cdot [\Lambda^b E] = 0.$$

Consequently, $\mathrm{sym}_t([E]) \cdot \lambda_{-t}([E]) = 1$. So

$$\mathrm{sym}_t([E]) = \frac{1}{\lambda_{-t}([E])} = \lambda_{-t}(-[E]) \quad \text{and} \quad \lambda_{-t}([E]) = \frac{1}{\mathrm{sym}_t([E])} = \mathrm{sym}_t(-[E]).$$

Any class $w \in K^0(X)$ has the form $w = [E] - [F]$ for some vector bundles E and F , and therefore

$$\begin{aligned} \mathrm{sym}_t(w) &= \mathrm{sym}_t([E]) \cdot \mathrm{sym}_t(-[F]) = \lambda_{-t}(-[E]) \cdot \lambda_{-t}([F]) = \lambda_{-t}([F] - [E]) \\ &= \lambda_{-t}(-w). \end{aligned}$$

So the sym^k and λ^k operations on K -theory are essentially the same: $\mathrm{sym}^k(w) = (-1)^k \lambda^k(-w)$.

This has been a long discussion with somewhat of a negative conclusion: the sym^k operations can be completely ignored in favor of the λ^k 's (or vice versa). We have learned some useful things along the way, however.

4.10. The Adams operations. The usefulness of the λ^k operations is limited by the fact that they are not group homomorphisms. There is a clever method, however, for combining the λ -operations in a way that does produce a collection of group homomorphisms. This is originally due to Frank Adams [Adams2]. Before describing this construction we take a brief detour to develop the algebraic combinatorics that we will need.

Recall that we have a map $\lambda_t: K^0(X) \rightarrow (K^0(X)[[t]])^*$ and that this is a group homomorphism:

$$\lambda_t(x + y) = \lambda_t(x) \cdot \lambda_t(y).$$

If we want additive maps $K^0(X) \rightarrow K^0(X)$ a natural idea is to apply logarithms to the above formula. To be precise, start with the formal power series

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

Since $\lambda_t(x)$ has constant term equal to 1, we can use the above series to make sense of $\log(\lambda_t(x))$ —but only provided that we add denominators into $K^0(X)$, say by tensoring with $\mathbb{Q}$. If we set $\mu_t(x) = \log(\lambda_t(x))$ then we would have

$$\mu_t(x + y) = \mu_t(x) + \mu_t(y).$$

The coefficients of powers of t in $\mu_t(x)$ then give additive operations, with the only difficulty being that they take values in $K^0(X) \otimes \mathbb{Q}$.

We can, however, eliminate the need for $\mathbb{Q}$ -coefficients by applying the operator $\frac{d}{dt}$. Precisely, define

$$\nu_t(x) = \frac{d}{dt} \left[\mu_t(x) \right] = \frac{\lambda'_t(x)}{\lambda_t(x)} = (1 - z + z^2 - z^3 + \cdots)|_{z=\lambda_t(x)-1} \cdot \lambda'_t(x).$$

The two factors in the right-most expression both lie in $K^0(X)[[t]]$, and so this eliminates the problem with denominators: $\nu_t(x) \in K^0(X)[[t]]$ yet we still have $\nu_t(x+y) = \nu_t(x) + \nu_t(y)$. Taking the coefficient of t^k in $\nu_t(x)$ thereby yields an additive operation $\nu^k: K^0(X) \rightarrow K^0(X)$, for each $k \geq 0$.

We could stop here, but there is one more modification that makes things a bit simpler later on. Suppose that L is a line bundle over X , and take $x = [L]$. Then $\lambda_t(x) = 1 + [L]t = 1 + xt$, hence

$$\nu_t(x) = \frac{x}{1+xt} = x(1 - xt + x^2t^2 - x^3t^3 + \cdots) = x - x^2t + x^3t^2 - \cdots.$$

The ν -operations simply give powers of x , together with certain signs: $\nu^k(x) = (-1)^k x^{k+1}$. It is easy to adopt a convention that makes these signs disappear, and we might as well do this; and while we are at it, let us shift the indexing on the ν 's so that the k th operation sends x to x^k , since that will be easier to remember.

Putting everything together, we have arrived at the following definition:

$$\psi_t(x) = -t \frac{d}{dt} \left[\log(\lambda_{-t}(x)) \right] = -t \cdot \frac{\lambda_{-t}(x)'}{\lambda_{-t}(x)},$$

and $\psi^k(x)$ is the coefficient of t^k in $\psi_t(x)$. The operations ψ^k are called **Adams operations**. We have proven that

- (1) Each ψ^k is a group homomorphism $K^0(X) \rightarrow K^0(X)$, natural in X ;
- (2) If $x = [L]$ for L a line bundle then $\psi^k(x) = x^k$.

We will see in Corollary 4.17 that conditions (1) and (2) actually completely characterize the Adams operations.

Remark 4.11. Be careful of writing $\lambda'_{-t}(x)$ rather than $\lambda_{-t}(x)'$ for the derivative. The former can be interpreted as $\lambda'_t(x)|_{t \rightarrow -t}$, which is different than $\frac{d}{dt}[\lambda_{-t}(x)]$.

If R is a commutative ring and $\alpha = \sum_{i \geq 1} \alpha_i t^i$ is a power series with no constant term, then there is an identity

$$(4.12) \quad \frac{\alpha'}{1+\alpha} = \sum_{k=0}^{\infty} (-1)^{k-1} S_k(\alpha_1, \dots, \alpha_k) t^{k-1}$$

where the S_k are certain universal polynomials called the *Newton polynomials*. See Appendix ?? for a discussion, and Proposition ?? for the identity. Applying this with $\alpha = \lambda_{-t}(x) - 1$ yields part (a) of the following result. Part (b) is a standard recursive relation for the Newton polynomials, again derived in Appendix ??.

Proposition 4.13.

- (a) $\psi^k = S_k(\lambda^1, \dots, \lambda^k)$, where S_k is the k th Newton polynomial;
- (b) $\psi^k = \lambda^1 \psi^{k-1} - \lambda^2 \psi^{k-2} + \cdots + (-1)^k \lambda^{k-1} \psi^1 + (-1)^{k+1} k \lambda^k$.

Proof. Equation (4.12) gives that $\psi^k = -(-1)^{k-1} S_k(-\lambda_1, \lambda_2, -\lambda_3, \dots)$, but this equals $S_k(\lambda_1, \lambda_2, \dots)$ by Proposition ??(a). Part (b) is then directly from Proposition ??. $\square$

We record the first few Adams operations:

$$\psi^1 = \lambda^1, \quad \psi^2 = (\lambda^1)^2 - 2\lambda^2, \quad \psi^3 = (\lambda^1)^3 - 3\lambda^1\lambda^2 + 3\lambda^3.$$

For more of these, just see the list of Newton polynomials given in Table ?? of Appendix ??.

4.14. Properties of the Adams operations.

Proposition 4.15. *Fix $k, l \geq 1$ and $x, y \in K^0(X)$. Then*

- (a) $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$
- (b) $\psi^k(xy) = \psi^k(x)\psi^k(y)$
- (c) $\psi^k(\psi^l(x)) = \psi^{kl}(x)$
- (d) *If ℓ is prime then $\psi^\ell(x) \equiv x^\ell \pmod{\ell}$.*

Note that part (a) has already been justified, as the construction of the ψ^k 's was designed precisely so that this would hold. Part (d) is likewise easy, as this is a property of the Newton polynomials—see Proposition ??. But parts (b) and (c) are far from obvious, and in fact seem quite difficult at first. There are essentially two approaches. One can figure out what properties of the λ -operations correspond to these, and then try to prove those by working with the exterior product construction. The disadvantage here is that the identities are quite complicated and can quickly devolve into a combinatorial nightmare. The other approach, which we take here, uses a general ‘splitting principle’ to reduce the identities to a particularly simple case, where they can be verified directly.

Define a **line element** in $K^0(X)$ to be any element $[L]$ where $L \rightarrow X$ is a line bundle. The span of the line elements consist of the classes of the form $[L_1] + \cdots + [L_a] - [L'_1] - \cdots - [L'_b]$ where the L_i 's and L'_j 's are all line bundles.

Proof of Proposition 4.15(b,c). Note that if x and y are line elements then the desired results are obvious because $\psi^r([L]) = [L^r]$. More generally, the results follow easily if x and y are in the span of the line elements. The general result now follows from the splitting principle in Proposition 4.16 below. Specifically, choose a $p: X_1 \rightarrow X$ such that p^* is injective on $K^0(X)$ and $p^*(x)$ is in the span of line elements. Then choose a $q: X_2 \rightarrow X_1$ such that q^* is injective and $q^*(p^*(y))$ is in the span of line elements. The identities in (b) and (c) hold for $x' = (pq)^*(x)$ and $y' = (pq)^*(y)$, and so the injectivity of $(pq)^*$ shows they hold for x and y as well. $\square$

Proposition 4.16 (The Splitting Principle). *Let X be any space, and let $x \in K^0(X)$. Then there exists a space Y and a map $p: Y \rightarrow X$ such that $p^*: K^0(X) \rightarrow K^0(Y)$ is injective and $p^*(x)$ is in the span of line elements.*

Proof. Write $x = [E] - [F]$ for vector bundles E and F . Consider the map $\pi: \mathbb{P}(E) \rightarrow X$. Then $\pi^*E \cong E' \oplus L$ where L is a line bundle, and $\pi^*: K^0(X) \rightarrow K^0(\mathbb{P}(E))$ is injective (????). Iterating this procedure we obtain a map $f: Y \rightarrow X$ such that f^*E is a sum of line bundles and f^* is injective. Now use the same process to obtain a map $g: Y' \rightarrow Y$ such that g^* is injective and $g^*(f^*F)$ is a sum of line bundles. The composite $Y' \rightarrow X$ has the properties from the statement of the proposition. $\square$

Corollary 4.17 (Characterization of Adams operations). *Fix $k \geq 1$. Suppose that $\theta: K^0(-) \rightarrow K^0(-)$ is a natural group homomorphism such that $\theta([L]) = [L^k]$ for any line bundle $L \rightarrow X$. Then $\theta = \psi^k$.*

Proof. Fix a space X , and let $\alpha \in K^0(X)$. Then $\alpha = [E] - [F]$ for some vector bundles E and F on X . By the Splitting Principle there exists a map $p: Y \rightarrow X$ such that p^*E and p^*F are direct sums of line bundles, and such that $p^*: K^0(X) \rightarrow K^0(Y)$ is injective. Our assumption on θ implies at once that $\theta(p^*\alpha) = \psi^k(p^*\alpha)$, or equivalently $p^*(\theta\alpha) = p^*(\psi^k\alpha)$. Injectivity of p^* now gives $\theta(\alpha) = \psi^k(\alpha)$. $\square$

The fact that ψ^k is natural and preserves (internal) products immediately yields that it also preserves external products. Recall that if $x \in K^0(X)$ and $y \in K^0(Y)$ then the external product can be written as $x \times y = \pi_1^*(x) \cdot \pi_2^*(y) \in K^0(X \times Y)$. Clearly $\psi^k(x \times y) = \psi^k(x) \times \psi^k(y)$. Using this, we easily obtain the following:

Proposition 4.18. *For $k \geq 1$, ψ^k acts on $\tilde{K}^0(S^{2n})$ as multiplication by k^n .*

Proof. Let $\beta = 1 - [L]$ be the Bott element in $\tilde{K}^0(S^2)$, and recall from Proposition ?? that the internal square β^2 is zero. From this it follows readily that

$$\psi^k(\beta) = 1 - L^k = 1 - (1 - \beta)^k = 1 - (1 - k\beta) = k\beta.$$

But the external power $\beta^{(n)} = \beta \times \beta \times \cdots \times \beta$ generates $\tilde{K}^0(S^{2n})$, and

$$\psi^k(\beta^{(n)}) = (\psi^k\beta)^{(n)} = (k\beta)^{(n)} = k^n\beta^{(n)}.$$

$\square$

Recall the canonical filtration of $K^0(X)$ coming from the Atiyah-Hirzebruch spectral sequence, as discussed in Section 1 and especially in Remark 1.6. In particular, recall that $F^{2n-1}K^0(X) = F^{2n}K^0(X)$ for every n , as a consequence of $K^*(pt)$ being concentrated in even dimensions. The naturality of the Adams operations shows that they respect the filtration, and Proposition 4.18 shows that ψ^k acts as a scalar on the associated graded:

Proposition 4.19. *Let $k \geq 1$.*

- (a) *If $x \in F^{2n}K^0(X)$ then $\psi^k(x) = k^n x + \text{terms of higher filtration}$. That is, $\psi^k(x) - k^n x \in F^{2n+2}K^0(X)$.*
- (b) *If the induced filtration on $K^0(X)_{\mathbb{Q}}$ is finite (e.g., if X is a finite-dimensional CW-complex) then the operations ψ^k are diagonalizable on $K^0(X)_{\mathbb{Q}}$, with eigenvalues of the form k^r for $r \geq 0$. For $k \neq 1$ the decomposition*

$$K^0(X)_{\mathbb{Q}} = \bigoplus_{r \geq 0} \text{Eig}_{\psi^k}(k^r)$$

restricts to give

$$F^{2i}K^0(X)_{\mathbb{Q}} = \bigoplus_{r \geq i} \text{Eig}_{\psi^k}(k^r)$$

for each i , and this decomposition is independent of k : for all $k, l \neq 1$ and $r \geq 0$ one has $\text{Eig}_{\psi^k}(k^r) = \text{Eig}_{\psi^l}(l^r)$.

Proof. For part (a) it suffices to replace X by a weakly equivalent CW-complex, so that $F^{2n}K^0(X) = \ker[K^0(X) \rightarrow K^0(X^{2n-1})]$. If $\alpha \in F^{2n}K^0(X)$ then let α_1

denote its image in $K^0(X^{2n})$. The cofiber sequence $X^{2n-1} \hookrightarrow X^{2n} \rightarrow X^{2n}/X^{2n-1}$ induces a long exact sequence

$$\cdots \rightarrow \tilde{K}^0(X^{2n}/X^{2n-1}) \rightarrow K^0(X^{2n}) \rightarrow K^0(X^{2n-1}) \rightarrow \cdots$$

The element $\alpha_1 \in K^0(X^{2n})$ maps to zero, and so it is the image of a class $\alpha_2 \in \tilde{K}^0(X^{2n}/X^{2n-1})$. By Proposition 4.18 one knows $\psi^k(\alpha_2) = k^n \alpha_2$, and so $\psi^k \alpha_1 = k^n \alpha_1$. It follows that $\psi^k \alpha - k^n \alpha$ maps to zero in $K^0(X^{2n})$, and hence lies in $F^{2n+1}K^0(X^{2n}) = F^{2n+2}K^0(X^{2n})$.

Part (b) is an algebraic consequence of (a). For the details, first let $x \in \text{Eig}_{\psi^k}(k^r)$ be nonzero. Pick i largest so that $x \in F^{2i}K^0(X)_{\mathbb{Q}}$. Part (a) implies that $\psi^k x - k^i x \in F^{2i+2}K^0(X)_{\mathbb{Q}}$. But $\psi^k x - k^i x = k^r x - k^i x = (k^r - k^i)x$. If $r \neq i$ we divide by $k^r - k^i$ to get $x \in F^{2i+2}K^0(X)_{\mathbb{Q}}$, contradicting the maximality of i . So $r = i$ and therefore $\text{Eig}_{\psi^k}(k^r) \subseteq F^{2r}K^0(X)_{\mathbb{Q}}$, for all $r \geq 0$.

Now let $2n$ be the largest even integer such that $F^{2n}K^0(X)_{\mathbb{Q}} \neq 0$. It follows from (a) that ψ^k acts as multiplication by k^n on $F^{2n}K^0(X)_{\mathbb{Q}}$. Putting this together with the last paragraph, we have that $F^{2n}K^0(X)_{\mathbb{Q}} = \text{Eig}_{\psi^k}(k^n)$. The last paragraph also tells us that $\text{Eig}_{\psi^k}(k^i) = 0$ for $i > n$.

We now prove by reverse induction that for all $i \leq n$ the space $F^{2i}K^0(X)_{\mathbb{Q}}$ is the sum of the k^j -eigenspaces of ψ^k for $j \geq i$. The case $i = n$ was done in the previous paragraph, and we have also shown that the named eigenspaces are all contained in $F^{2i}K^0(X)_{\mathbb{Q}}$. So assume we know the result for a particular value of i , and let $\alpha \in F^{2i-2}K^0(X)_{\mathbb{Q}}$. We know $\psi^k \alpha = k^{i-1} \alpha + \gamma$, where γ is some element of $F^{2i}K^0(X)_{\mathbb{Q}}$. The induction hypothesis says that $\gamma = \sigma_i + \sigma_{i+1} + \cdots + \sigma_n$ where each σ_r is an eigenvector for ψ^k with eigenvalue k^r . A routine calculation now shows that $\alpha - \sum_{r \geq i} \frac{1}{k^r - k^{i-1}} \sigma_r$ is a k^{i-1} -eigenvector for ψ^k , and hence α belongs to the sum of eigenspaces for eigenvalues k^r , $r \geq i-1$. This completes the induction step.

Note that the above result for $i = 0$ shows that ψ^k is diagonalizable on $K^0(X)_{\mathbb{Q}}$.

It remains to prove that the eigenspaces $\text{Eig}_{\psi^k}(k^r)$ are independent of k . This is a small variant on the classical argument that commuting diagonalizable operators (in this case ψ^k and ψ^l) can be simultaneously diagonalized. The key fact, easy to prove, is that if A and B are commuting diagonalizable operators then any eigenspace $\text{Eig}_B(\lambda)$ is A -invariant and is equal to the direct sum over all μ of $\text{Eig}_B(\lambda) \cap \text{Eig}_A(\mu)$.

First note that we have seen that $\text{Eig}_{\psi^k}(k^n) = F^{2n}K^0(X)_{\mathbb{Q}} = \text{Eig}_{\psi^l}(l^n)$. We also have

$$\text{Eig}_{\psi^k}(k^{n-1}) \oplus \text{Eig}_{\psi^k}(k^n) = F^{2n-2}K^0(X)_{\mathbb{Q}} = \text{Eig}_{\psi^l}(l^{n-1}) \oplus \text{Eig}_{\psi^l}(l^n).$$

Putting the two together, $\text{Eig}_{\psi^k}(k^{n-1}) \cap \text{Eig}_{\psi^l}(l^n) = 0$. But since ψ^k and ψ^l commute, the algebraic fact mentioned above now immediately yields that $\text{Eig}_{\psi^k}(k^{n-1}) = \text{Eig}_{\psi^l}(l^{n-1})$. Repeat this argument inductively. $\square$

Remark 4.20. The Adams operations ψ^k are defined for $k \geq 1$. Someone once had the clever idea of taking the operation induced by $E \mapsto E^*$ and calling it ψ^{-1} . So $\psi^{-1}([E] - [F]) = [E^*] - [F^*]$. Then for $k \geq 1$ one checks that $\psi^{-1} \circ \psi^k = \psi^k \circ \psi^{-1}$, and so it is reasonable to define ψ^{-k} to be this common expression. With this definition parts (a)–(c) of Proposition 4.15 now hold for all $k, l \in \mathbb{Z} - \{0\}$.

4.21. Operations for non-compact spaces. Suppose given a natural operation $\theta: K^0(X) \rightarrow K^0(X)$ (just a function of sets) defined for all compact Hausdorff

spaces X . We will explain why this extends uniquely to a natural operation defined for all spaces X . This will allow us to extend both the λ^k and ψ^k operations, and all of the relations previously established will still hold. For example, the new operations ψ^6 and $\psi^2\psi^3$ will both extend the operation ψ^6 that was defined on compact spaces—because of Proposition 4.15—and therefore the uniqueness of extension shows that $\psi^6 = \psi^2\psi^3$ on non-compact spaces as well.

Recall that $K^0(X) = [X, \mathbb{Z} \times BU]$. By the Yoneda Lemma, natural operations on $K^0(X)$ are in bijective correspondence with elements of the set $[\mathbb{Z} \times BU, \mathbb{Z} \times BU]$. Such a homotopy class boils down to the following information: for each $k \in \mathbb{Z}$ one must give an integer $Y(k)$ and an element of $[\{k\} \times BU, \{Y(k)\} \times BU] = [BU, BU]$. Because the infinitely-many connected components add an extra layer of bookkeeping to the arguments, let us first analyze the situation where we don't have to worry about them. We will prove the following:

Proposition 4.22. *Let $\theta: \tilde{K}^0(X) \rightarrow \tilde{K}^0(X)$ be a natural transformation of functors defined on the class of pointed compact Hausdorff spaces. Then there is a unique natural transformation $\Theta: \tilde{K}^0 \rightarrow \tilde{K}^0$ (defined on all pointed spaces) that equals θ when the spaces are compact Hausdorff.*

Before giving the proof let us set up some of the necessary elements. Using the equivalence $BU_n \simeq \text{Gr}_n(\mathbb{C}^\infty)$ we can express BU_n as the directed homotopy colimit of the spaces $\text{Gr}_n(\mathbb{C}^k)$ as $k \rightarrow \infty$. So we have the Milnor exact sequence

$$0 \rightarrow \lim_k^1 \tilde{K}^{-1}(\text{Gr}_n(\mathbb{C}^k)) \rightarrow \tilde{K}^0(BU_n) \rightarrow \lim_k \tilde{K}^0(\text{Gr}_n(\mathbb{C}^k)) \rightarrow 0.$$

Since the spaces $\text{Gr}_n(\mathbb{C}^k)$ have cellular filtrations with only even-dimensional cells, the Atiyah-Hirzebruch spectral sequence immediately yields that $K^1(\text{Gr}_n(\mathbb{C}^k)) = 0$ for all k . So the $\lim^1$ term on the left vanishes, and we have the isomorphism

$$(4.23) \quad \tilde{K}^0(BU_n) \xrightarrow{\cong} \lim_k \tilde{K}^0(\text{Gr}_k(\mathbb{C}^n)).$$

Let $\eta_{n,k}: \text{Gr}_n(\mathbb{C}^k)$ denote the tautological n -plane bundle. The elements $[\eta_{n,k}] - \underline{n}$ map to each other and therefore lift to a class in $\tilde{K}^0(BU_n)$. Let us denote this class as $\tilde{\eta}_n$. If $E \rightarrow Y$ is a rank n bundle over a b-compact space Y , then there is a classifying map $f: Y \rightarrow BU_n$ such that $f^*(\eta_n) \cong E$. We have $f^*(\tilde{\eta}_n) = [E] - \underline{n} \in \tilde{K}^0(Y)$; this follows from the fact that f will factor through some $\text{Gr}_n(\mathbb{C}^k)$.

If $j: BU_{n-1} \hookrightarrow BU_n$ is the inclusion then $j^*(\tilde{\eta}_n) = \tilde{\eta}_{n-1}$. This boils down to the fact that under $\text{Gr}_{n-1}(\mathbb{C}^k) \rightarrow \text{Gr}_n(\mathbb{C} \oplus \mathbb{C}^k)$ the bundle $\eta_{n,k+1}$ pulls back to $\eta_{n-1,k} \oplus \underline{1}$, and therefore $[\eta_{n,k+1}] - \underline{n}$ pulls back to $[\eta_{n-1,k}] - \underline{n-1}$.

Now repeat the above analysis for the chain $BU_1 \hookrightarrow BU_2 \hookrightarrow BU_3 \hookrightarrow \dots \hookrightarrow BU$. We have another Milnor sequence

$$0 \rightarrow \lim_n^1 \tilde{K}^{-1}(BU_n) \rightarrow \tilde{K}^0(BU) \rightarrow \lim_n \tilde{K}^0(BU_n) \rightarrow 0.$$

The Atiyah-Hirzebruch spectral sequence again shows that all of the $K^1(BU_n) = 0$ for all n (one can also use the Milnor sequence here), and so we have the isomorphism

$$(4.24) \quad \tilde{K}^0(BU) \xrightarrow{\cong} \lim_n \tilde{K}^0(BU_n).$$

The elements $\tilde{\eta}_n$ fit together across the inverse limit, and therefore lift to a class $\tilde{\eta} \in \tilde{K}^0(BU)$.

Under the isomorphism $[Y, BU]_* = \tilde{K}^0(Y)$, the element $\tilde{\eta}_n \in \tilde{K}^0(BU_n)$ corresponds to the inclusion $BU_n \hookrightarrow BU$ and the element $\tilde{\eta}$ corresponds to the identity map $BU \rightarrow BU$.

Proof of Proposition 4.22. Since the Grassmannians $\text{Gr}_n(\mathbb{C}^k)$ are compact Hausdorff spaces we have the elements $\theta([\eta_{n,k}] - \underline{n}) \in \tilde{K}^0(\text{Gr}_n(\mathbb{C}^k))$, and as k increases these elements map to each other by naturality. So by (4.23) there is a unique element $E_n \in \tilde{K}^0(BU_n)$ that maps to each of the $\theta([\eta_{n,k}] - \underline{n})$.

By uniqueness the elements E_n map to each other as n varies, so by (4.24) there is a unique element $E \in \tilde{K}^0(BU)$ that lifts them. Note that $\tilde{K}^0(BU) = [BU, BU]$, and so we may regard E as a map $BU \rightarrow BU$ in the homotopy category of spaces.

For any space X , define $\Theta: [X, BU] \rightarrow [X, BU]$ as post-composition with E . This is a natural operation $\tilde{K}^0 \rightarrow \tilde{K}^0$.

Assume that X is compact Hausdorff and let $w \in \tilde{K}^0(X)$. We know we can write $w = [E] - \underline{\text{rank}} E$ for some bundle $E \rightarrow X$. Let $n = \text{rank } E$. Then there is a k and a map $f: X \rightarrow \text{Gr}_n(\mathbb{C}^k)$ such that $E \cong f^* \eta_{n,k}$. So we have the diagram

$$\begin{array}{ccccccc}
 & & & \theta([\eta_{n,k}] - \underline{n}) & & & \\
 & & & \curvearrowright & & & \\
 X & \xrightarrow{f} & \text{Gr}_n(\mathbb{C}^k) & \longrightarrow & BU_n & \longrightarrow & BU \xrightarrow{E} BU \\
 & \searrow & & & \nearrow & & \\
 & & & w & & &
 \end{array}$$

that commutes up to homotopy. So

$$Ew = f^* \left(\theta([\eta_{n,k}] - \underline{n}) \right) = \theta \left(f^*([\eta_{n,k}] - \underline{n}) \right) = \theta(w)$$

where in the last equality we have used that f is a map between compact Hausdorff spaces and θ is natural for such maps.

We leave the reader to consider the uniqueness of Θ , but it is basically transparent from the above argument. $\square$

Now let us return to the unreduced situation. Suppose that $\theta: K^0(X) \rightarrow K^0(X)$ is a natural transformation defined when X is compact Hausdorff. For each $k \in \mathbb{Z}$ let k_θ denote the image of k under the map $\theta: K^0(pt) \rightarrow K^0(pt)$. If $w \in K^0(X)$ is a class that has rank k at a point $x \in X$, meaning that the pullback of w along the inclusion $\{x\} \hookrightarrow X$ is equal to k , then by naturality the element $\theta(w)$ must have rank k_θ .

Let (X, x) be a pointed space. For each $k \in \mathbb{Z}$ let $K^0(X)_{\text{rank}=k} \subseteq K^0(X)$ denote the subset of elements which map to k under $K^0(X) \rightarrow K^0(x)$. Note that $+k: \tilde{K}^0(X) \rightarrow K^0(X)_{\text{rank}=k}$ is a bijection. Let θ_k be the operation defined by the following diagram:

$$\begin{array}{ccc}
 \tilde{K}^0(X) & \xrightarrow{\theta_k} & \tilde{K}^0(X) \\
 +k \downarrow \cong & & \cong \downarrow +k_\theta \\
 K^0(X)_{\text{rank}=k} & \xrightarrow{\theta} & K^0(X)_{\text{rank}=k_\theta}
 \end{array}$$

One can write $\theta_k(w) = \theta(w + k) - k_\theta$. It is evident that θ_k is natural for maps of pointed compact Hausdorff spaces.

By Proposition 4.22 there is a unique natural transformation $\Theta_k: \tilde{K}^0 \rightarrow \tilde{K}^0$ that extends θ_k . We can do this for each $k \in \mathbb{Z}$. Each Θ_k corresponds to a homotopy class of maps $BU \rightarrow BU$. Define $\Theta: \mathbb{Z} \times BU \rightarrow \mathbb{Z} \times BU$ to be the map that sends $\{k\} \times BU$ to $\{k_\theta\} \times BU$ via the map Θ_k . Of course Θ can also be regarded as a natural transformation $K^0 \rightarrow K^0$. It only remains to check that when X is compact Hausdorff the map $\Theta: K^0(X) \rightarrow K^0(X)$ coincides with the original θ . It suffices to do this when X is path-connected. In this case there is a well-defined notion of rank for elements of $K^0(X)$, since the maps $\{x\} \hookrightarrow X$ are homotopic for all choices of x . For $w \in K^0(X)$ observe that

$$\begin{aligned} \theta(w) &= \theta_{\text{rank } w}(w - (\text{rank } w)) - (\text{rank } w)_\theta \\ &= \Theta_{\text{rank } w}(w - (\text{rank } w)) - (\text{rank } w)_\theta \\ &= \Theta(w). \end{aligned}$$

The last equality is simply the definition of Θ .

5. THE HOPF INVARIANT ONE PROBLEM

When $n > 1$ the Hopf invariant assigns an integer to every map $f: S^{2n-1} \rightarrow S^n$, giving a group homomorphism $H: \pi_{2n-1}(S^n) \rightarrow \mathbb{Z}$. Elementary arguments show that 2 is always in the image, and the natural question is then whether 1 is also in the image. This is the Hopf invariant one problem—determine all values of n for which H is surjective (or said differently, all values of n for which there exists a map of Hopf invariant one).

It was known classically that H is surjective when $n \in \{2, 4, 8\}$, because the classical Hopf maps all have Hopf invariant equal to one. The question for other dimensions was first settled by Adams in [Adams1], who proved that no other Hopf invariant one maps exist. Adams's proof is not simple, even by modern standards, being based on secondary cohomology operations associated to the Steenrod squares. Several years after Adams gave his original proof, Adams and Atiyah [AA] used K -theory to give a much simpler solution to the Hopf invariant one problem. Their ‘postcard proof’ takes less than a page, in dramatic contrast to Adams's original method. This was seen—rightly so—as a huge demonstration of the power of K -theory.

Our goal in this section will be to present the Adams-Atiyah proof, although we will not quite do this in their style. Specifically, when Adams and Atiyah wrote their paper they clearly had an agenda: to write down the proof in as small a space as possible. If the goal is to accentuate how much the use of K -theory simplifies the solution, this makes perfect sense. But at the same time, writing the proof in this way results in a certain air of mystery: the proof involves a strange manipulation with the Adams operations ψ^2 and ψ^3 that comes out of nowhere—it seems like a magic trick.

In our presentation below we try to put this (ψ^2, ψ^3) trick into its proper context: it is part of a calculation of a certain Ext^1 group. The full calculation of this group is not hard, and quite interesting for other reasons—e.g., it connects deeply to the study of the J -homomorphism. Our presentation doesn't fit on a postcard, but by the time we are done we will have a good understanding of several neat and important things. Hopefully it won't seem like magic.

5.1. Brief review of the problem. Let $f: S^{2n-1} \rightarrow S^n$ where $n > 1$ and consider the mapping cone Cf . One readily computes that

$$H^i(Cf) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0, n, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Fix an orientation on the two spheres, and let a and b be corresponding generators for $H^n(Cf)$ and $H^{2n}(Cf)$. Then $b^2 = h \cdot a$ for a unique integer $h \in \mathbb{Z}$, and this integer is called the **Hopf invariant** of f : we write $h = H(f)$.

Note that if n is odd then $b^2 = -b^2$ and so $h = 0$. Therefore the Hopf invariant is only interesting when n is even, which we implicitly assume from now on.

Remark 5.2. We follow [Hatcher1, Proposition 4B.1] to see that $H: \pi_{2n-1}(S^n) \rightarrow \mathbb{Z}$ is a group homomorphism. Given $f, g: S^{2n-1} \rightarrow S^n$ consider the diagram of mapping cones

$$\begin{array}{ccccc} S^{2n-1} & \xrightarrow{f+g} & S^n & \longrightarrow & C_{f+g} \\ \downarrow & & \parallel & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f \vee g} & S^n & \longrightarrow & X \end{array}$$

where the left vertical map is the equatorial collapse and $X = C_{f \vee g}$. Note that there are inclusions $Cf \hookrightarrow X$ and $Cg \hookrightarrow X$. The cohomology group $H^{2n}(X)$ has two generators a_1 and a_2 , and naturality applied to those inclusions shows that $b^2 = H(f)a_1 + H(g)a_2$. But under the map $C_{f+g} \rightarrow X$ both a_1 and a_2 are sent to our usual generator a , and from this one gets that $H(f+g) = H(f) + H(g)$.

Remark 5.3. It is easy to see that 2 (and therefore any even integer) is always in the image of H . We again follow [Hatcher1] here and let X be the pushout

$$\begin{array}{ccc} S^n \vee S^n & \xrightarrow{\quad} & S^n \times S^n \\ \nabla \downarrow & & \downarrow \alpha \\ S^n & \xrightarrow{\quad} & X. \end{array}$$

One readily checks that the cohomology of X consists of two copies of $\mathbb{Z}$, in degrees n and $2n$. So X is the mapping cone of a certain map $f: S^{2n-1} \rightarrow S^n$, the attaching map of the top cell. If $x \in H^n(S^n)$ is a fixed generator, then there is a generator $b \in H^n(X)$ that maps to $x \otimes 1 + 1 \otimes x$ under α^* . It follows that b^2 maps to

$$(x \otimes 1 + 1 \otimes x)^2 = 2(x \otimes x)$$

(recall that n is even) and therefore b^2 is twice a generator of $H^{2n}(X)$. One concludes that the Hopf invariant of f is ± 2 , depending on one's sign choices.

The problem arises of determining the precise image of $H: \pi_{2n-1}(S^n) \rightarrow \mathbb{Z}$, when n is even. By Remarks 5.2 and 5.3 the image is a subgroup that contains $2\mathbb{Z}$, so there are only two possibilities: either the image equals $2\mathbb{Z}$, or else it equals all of $\mathbb{Z}$. The latter happens if and only if there exists an element in $\pi_{2n-1}(S^n)$ having Hopf invariant equal to one. Thus, this is the ‘‘Hopf invariant one’’ problem.

Exercise 5.4. In our above discussion we needed to assume $n > 1$, otherwise the cohomology of Cf is not as claimed. In the case $n = 1$ one can do something similar, though, using $H^*(Cf; \mathbb{Z}/2)$. If $\deg(f)$ is even then $H^*(Cf; \mathbb{Z}/2)$ equals $\mathbb{Z}/2$ in degrees 1 and 2, and one can follow the above procedure to define a mod

2 Hopf invariant. Check that this defines a group homomorphism $2\pi_1(S^1) \rightarrow \mathbb{Z}/2$ (the domain is the set of maps $S^1 \rightarrow S^1$ of even degree), and that the degree two map has Hopf invariant one. For this reason the case $n = 1$ is often included in the problem, even though the details here are slightly different.

The following several paragraphs involve the Steenrod squares. The results will not be needed later in this section, but they constitute an interesting part of the overall story.

As soon as one is versed in the Steenrod squares it is easy to obtain a necessary condition for the existence of a Hopf invariant one map $f: S^{2n-1} \rightarrow S^n$. In the mod 2 cohomology of Cf we have $Sq^n(b) = b^2 = h \cdot a$. So if f has odd Hopf invariant then $Sq^n(b) = a$, and the mod 2 cohomology of Cf looks like this:

$$\begin{array}{c} a \circ \\ \left. \vphantom{\begin{array}{c} a \circ \\ b \circ \end{array}} \right\} Sq^n \\ b \circ \end{array}$$

This picture just says that the cohomology has generators a and b together with a Sq^n connecting b to a . As an immediate consequence we obtain Adem's theorem:

Proposition 5.5 (Adem). *If $f: S^{2n-1} \rightarrow S^n$ has Hopf invariant one then n is a power of 2.*

Proof. The above picture represents a module over the Steenrod algebra only if Sq^n is indecomposable. But by our knowledge of the Steenrod algebra, the indecomposables all have degrees equal to a power of 2 (they are represented by the elements Sq^{2^i}). $\square$

The reader might have noticed that there actually seem to be *two* problems here, that are interrelated. There is the Hopf invariant one problem, and there is the question of whether there exists a map $S^{k+n-1} \rightarrow S^k$ whose cofiber has a nonzero Sq^n in mod 2 cohomology. The first problem is inherently *unstable* in nature because it deals with the cup product, whereas the second problem is clearly stable. It is useful to note that the two problems are actually equivalent:

Proposition 5.6. *Fix $n \geq 1$. The following two statements are equivalent:*

- (a) *There exists a map $S^{2n-1} \rightarrow S^n$ of Hopf invariant one;*
- (b) *There exists a $k \geq 0$ and a map $S^{k+n-1} \rightarrow S^k$ whose mapping cone has a nonzero Sq^n operation.*

Proof. In the discussion preceding Adem's theorem we saw that (a) implies (b) by taking $k = n$. Conversely, if (b) holds for a certain map g then by suspending if necessary we can assume $k \geq n$. The Freudenthal Suspension Theorem guarantees that $\pi_{2n-1}(S^n) \rightarrow \pi_{k+n-1}(S^k)$ is surjective, so choose map a map $f: S^{2n-1} \rightarrow S^n$ that is a preimage of g . The spaces Cf and Cg are homotopy equivalent after appropriate suspensions, so the mod 2 cohomology of Cf has a nonzero Sq^n . It immediately follows that f has odd Hopf invariant, and consequently there exists a map of Hopf invariant one. $\square$

Adem's theorem is really an analysis of the stable problem, and it may be rephrased as follows. If $f: S^{n+k-1} \rightarrow S^k$ then there is a cofiber sequence

$S^{n+k-1} \rightarrow S^k \rightarrow Cf$, and if $n \geq 2$ the long exact sequence on mod 2 cohomology breaks up into a family of short exact sequences

$$0 \leftarrow \tilde{H}^*(S^k; \mathbb{Z}/2) \leftarrow \tilde{H}^*(Cf; \mathbb{Z}/2) \leftarrow \tilde{H}^*(S^{n+k}; \mathbb{Z}/2) \leftarrow 0.$$

These are maps of modules over the Steenrod algebra $\mathcal{A}$, and both the left and right terms are isomorphic to the trivial $\mathcal{A}$ -module $\mathbb{F}_2$ (graded to lie in the appropriate dimension). So the above short exact sequence represents an element of $\text{Ext}_{\mathcal{A}}^1(\mathbb{F}_2, \mathbb{F}_2)$. Standard homological algebra identifies this Ext^1 with the module of indecomposables I/I^2 , where I is the augmentation ideal of $\mathcal{A}$. Adem's Theorem works because we know this module of indecomposables precisely, and therefore can identify the Ext^1 groups precisely.

5.7. An Ext calculation. Most of this section will be spent in pursuit of a purely algebraic question, somewhat related to what we just saw. Let $(\mathbb{N}, \cdot)$ be the monoid of natural numbers under multiplication, and let $\mathcal{B} = \mathbb{Z}[\mathbb{N}]$ be the corresponding monoid ring. Write ψ^k for the element of $\mathcal{B}$ corresponding to $k \in \mathbb{N}$. Then $\mathcal{B}$ is simply the polynomial ring

$$\mathcal{B} = \mathbb{Z}[\psi^2, \psi^3, \psi^5, \dots],$$

with one generator corresponding to each prime number. We think of $\mathcal{B}$ as the ring of formal Adams operations, and note that $K^0(X)$ is naturally a $\mathcal{B}$ -module for any space X (Section 4).

Let $\mathbb{Z}(r)$ denote the following module over $\mathcal{B}$: as an abelian group it is a copy of $\mathbb{Z}$, with chosen generator g , and the $\mathcal{B}$ -module structure is $\psi^k \cdot g = k^r g$. Our goal will be to compute the groups

$$\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$$

for all values of r and s .

Before exploring this algebraic problem let us quickly indicate the application to topology. Let $f: S^{n+k} \rightarrow S^n$ be a map of spheres, and write Cf for the mapping cone. The Puppe sequence looks like

$$S^{n+k} \rightarrow S^n \rightarrow Cf \rightarrow S^{n+k+1} \rightarrow S^{n+1} \rightarrow \dots$$

and applying $\tilde{K}^0(-)$ to this yields

$$\tilde{K}^0(S^{n+k}) \leftarrow \tilde{K}^0(S^n) \leftarrow \tilde{K}^0(Cf) \leftarrow \tilde{K}^0(S^{n+k+1}) \leftarrow \tilde{K}^0(S^{n+1}) \leftarrow \dots$$

Any $\tilde{K}^0(-)$ group is naturally a $\mathcal{B}$ -module, via the Adams operations; and all the maps in the above sequence are maps of $\mathcal{B}$ -modules. Under the hypotheses that n is even and k is odd, the groups on the two ends vanish and we get a short exact sequence

$$0 \leftarrow \tilde{K}^0(S^n) \leftarrow \tilde{K}^0(Cf) \leftarrow \tilde{K}^0(S^{n+k+1}) \leftarrow 0.$$

Proposition 4.18 says that as a $\mathcal{B}$ -module $\tilde{K}^0(S^{2r})$ is isomorphic to $\mathbb{Z}(r)$, and hence the above sequence yields an element

$$A(f) \in \text{Ext}_{\mathcal{B}}^1\left(\mathbb{Z}\left(\frac{n}{2}\right), \mathbb{Z}\left(\frac{n+k+1}{2}\right)\right).$$

That is to say, we have obtained a topological invariant of f taking values in this Ext group. So we are motivated to determine what this group actually is.

Now we begin our computation. Let X be a $\mathcal{B}$ -module that sits in a short exact sequence

$$(5.8) \quad 0 \rightarrow \mathbb{Z}(s) \rightarrow X \rightarrow \mathbb{Z}(r) \rightarrow 0.$$

Write a for a chosen generator of $\mathbb{Z}(s)$ (as well as its image in X) and $\tilde{b}$ for a chosen generator of $\mathbb{Z}(r)$. Write b for a preimage of $\tilde{b}$ in X . Then we have

$$\psi^k b = k^r b + P_k a$$

for a unique $P_k \in \mathbb{Z}$. The $\mathcal{B}$ -module structure on X is completely determined by the ∞ -tuple of integers $\mathbf{P} = (P_2, P_3, P_5, \dots)$.

Does any choice of $\mathbf{P}$ correspond to a $\mathcal{B}$ -module? To be a $\mathcal{B}$ -module one must have $\psi^k \psi^l = \psi^l \psi^k$ on $\mathcal{B}$. But we can compute

$$\begin{aligned} \psi^k(\psi^l b) &= \psi^k(l^r b + P_l a) = l^r \cdot \psi^k(b) + P_l \cdot \psi^k a \\ &= l^r \cdot (k^r b + P_k a) + P_l k^s a \end{aligned}$$

and likewise

$$\begin{aligned} \psi^l(\psi^k b) &= \psi^l(k^r b + P_k a) = k^r \cdot \psi^l(b) + P_k \cdot \psi^l a \\ &= k^r \cdot (l^r b + P_l a) + P_k l^s a. \end{aligned}$$

Equating these expressions we find that $(l^r - l^s)P_k a = (k^r - k^s)P_l a$. Since a is infinite order in X it must be that

$$(l^r - l^s)P_k = (k^r - k^s)P_l,$$

and this holds for every two primes k and l . If $r = s$ this gives no condition and it is indeed true that any choice of $\mathbf{P}$ corresponds to a module X . But in the case $r \neq s$ we can write

$$\frac{P_l}{P_k} = \frac{l^r - l^s}{k^r - k^s}.$$

So once we fix a prime k , all other P_l 's are determined by P_k . For convenience we take k to be the smallest prime, and obtain

$$P_l = P_2 \cdot \left(\frac{l^r - l^s}{2^r - 2^s} \right)$$

for every prime l . This shows that the module X depends on the single parameter P_2 ; however, it is still not true that all possible integral choices for P_2 correspond to $\mathcal{B}$ -modules. Indeed, we will only get a $\mathcal{B}$ -module if the above formula for P_l yields an *integer* for every choice of l . To this end define

$$Z_{r,s} = \left\{ P \in \mathbb{Z} \mid P \cdot \left(\frac{l^r - l^s}{2^r - 2^s} \right) \in \mathbb{Z}, \text{ for all primes } l \right\}.$$

For $P \in Z_{r,s}$ let X_P denote the corresponding $\mathcal{B}$ -module for which $P_2 = P$.

Note that $Z_{r,s} \subseteq \mathbb{Z}$ is an ideal, and nonzero because it contains $2^r - 2^s$. In a moment we will compute this ideal in some examples. For now simply note that we have a map (in fact a surjection) $Z_{r,s} \rightarrow \text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$ sending P to the extension (5.8) in which $X = X_P$. It is an exercise to check that this is indeed a map of abelian groups.

We next need to understand when X_P and X_Q are isomorphic as elements of $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$. This is when there is a map of $\mathcal{B}$ -modules $X_P \rightarrow X_Q$ yielding a

commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}(s) & \longrightarrow & X_P & \longrightarrow & \mathbb{Z}(r) \longrightarrow 0 \\ & & \parallel & & \downarrow f & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}(s) & \longrightarrow & X_Q & \longrightarrow & \mathbb{Z}(r) \longrightarrow 0 \end{array}$$

Such an f must satisfy $f(a) = a$, and $f(b) = b + Ja$ for some $J \in \mathbb{Z}$ (note that the symbols a and b are being used to simultaneously represent elements of X_P and X_Q). The condition that f be a map of $\mathcal{B}$ -modules is that $\psi^k(f(b)) = f(\psi^k b)$ for all primes k . For $k = 2$ the left-hand-side is

$$\psi^2(b + Ja) = 2^r b + Qa + J \cdot 2^s a$$

and the right-hand-side is

$$f(\psi^2 b) = f(2^r b + Pa) = 2^r(b + Ja) + Pa.$$

We obtain the condition

$$Q - P = (2^r - 2^s)J.$$

The reader may check as an exercise that the condition for the other ψ^k 's follows as a consequence of this one.

The conclusion is that X_P and X_Q are isomorphic as elements of $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$ precisely when $P - Q$ is a multiple of $2^r - 2^s$. The map $Z_{r,s} \rightarrow \text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$ therefore descends to an isomorphism

$$Z_{r,s}/(2^r - 2^s) \xrightarrow{\cong} \text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s)).$$

Finally, it remains to determine the group $Z_{r,s}/(2^r - 2^s)$. This is a cyclic group (since $Z_{r,s} \cong \mathbb{Z}$), and we need to find its order. To this end, note that the condition $P(\frac{l^r - l^s}{2^r - 2^s}) \in \mathbb{Z}$ is equivalent to

$$\frac{2^r - 2^s}{\gcd(2^r - 2^s, l^r - l^s)} \mid P.$$

This is true for all l if and only if P is a multiple of

$$\text{lcm}\left\{ \frac{2^r - 2^s}{\gcd(2^r - 2^s, l^r - l^s)} \mid l \text{ prime} \right\} = \frac{2^r - 2^s}{\gcd(\{l^r - l^s \mid l \text{ prime}\})}.$$

So define

$$N_{r,s} = \gcd(2^r - 2^s, 3^r - 3^s, 5^r - 5^s, 7^r - 7^s, \dots).$$

Then we have just determined that $Z_{r,s} = ((2^r - 2^s)/N_{r,s})$, and hence

$$\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s)) \cong (\frac{2^r - 2^s}{N_{r,s}})/(2^r - 2^s) \cong \mathbb{Z}/N_{r,s}.$$

For ease of future reference we restate this as a proposition:

Proposition 5.9. *For $r, s \geq 0$ distinct, $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s)) \cong \mathbb{Z}/N_{r,s}$ where $N_{r,s} = \gcd\{l^r - l^s \mid l \text{ is prime}\}$.*

Exercise 5.10. Prove that $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(r)) \cong \mathbb{Z}^\infty$.

We will explore the numbers $N_{r,s}$ in a moment, but we have already done enough to be able to solve the Hopf invariant one problem. So let us pause and tackle that first.

5.11. Solution to Hopf invariant one. We are ready to give the Adams-Atiyah [AA] solution to the Hopf invariant one problem:

Theorem 5.12. *If $f: S^{2n-1} \rightarrow S^n$ has Hopf invariant one then $n \in \{1, 2, 4, 8\}$.*

Proof. We assume $n > 1$ and prove that $n \in \{2, 4, 8\}$. We of course know that n is even, since otherwise the Hopf invariant is necessarily zero. Write $n = 2r$, and let X be the mapping cone of f . We have an exact sequence of $\mathcal{B}$ -modules

$$0 \leftarrow \tilde{K}^0(S^n) \leftarrow \tilde{K}^0(X) \leftarrow \tilde{K}^0(S^{2n}) \leftarrow 0$$

which has the form

$$0 \leftarrow \mathbb{Z}(r) \leftarrow \tilde{K}^0(X) \leftarrow \mathbb{Z}(2r) \leftarrow 0.$$

Let $a \in \tilde{K}^0(X)$ be the image of a chosen generator for $\tilde{K}^0(S^{2n})$ and let $b \in \tilde{K}^0(X)$ be an element that maps to a chosen generator of $\tilde{K}^0(S^n)$. Then b^2 maps to 0 in $\tilde{K}^0(S^n)$, so we have $b^2 = h \cdot a$ for a unique $h \in \mathbb{Z}$. The Atiyah-Hirzebruch spectral sequence for X immediately shows that (perhaps up to sign) h is the Hopf invariant of the map f .

The key to the argument is the equivalence $\psi^2(b) \equiv b^2 \pmod{2}$ (Proposition 4.15(d)). Using our assumption that h is odd, this gives $\psi^2(b) \equiv a \pmod{2}$. However, recall our classification of extensions in $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(2r))$. Such extensions are determined by an integer $P_2 \in \mathbb{Z}$ satisfying

$$(5.13) \quad P_2 \cdot \left(\frac{l^{2r} - l^r}{2^{2r} - 2^r} \right) \in \mathbb{Z}$$

for all primes l , where P_2 is defined by the equation $\psi^2 b = 2^r b + P_2 a$. Our assumption about the Hopf invariant of f now gives that P_2 is odd. But equation (5.13) says that

$$P_2 \cdot \frac{l^r}{2^r} \cdot \frac{l^r - 1}{2^r - 1} \in \mathbb{Z},$$

and if P_2 is odd and l is odd then this implies that $2^r | l^r - 1$.

Let us pause here and summarize. From the topology we have extracted a number-theoretic condition: if $n = 2r$ and $S^{2n-1} \rightarrow S^n$ has Hopf invariant one, then $2^r | l^r - 1$ for all odd primes l .

This number-theoretic condition is very restrictive, and it turns out just looking at $l = 3$ is enough to give us what we want. The lemma below shows that r lies in $\{1, 2, 4\}$, implying that our original n belongs to $\{2, 4, 8\}$ as desired. $\square$

Lemma 5.14. *If $2^r | 3^r - 1$ then $r \in \{0, 1, 2, 4\}$.*

Proof. Let $\nu(n)$ be the 2-adic valuation of an integer n : that is, $n = 2^{\nu(n)} \cdot (\text{odd})$. Here is a table showing the numbers $\nu(3^r - 1)$ for small values of r :

TABLE 5.15.

r	1	2	3	4	5	6	7	8	9	10	11	12
$\nu(3^r - 1)$	1	3	1	4	1	3	1	5	1	3	1	4

The reader will reach the natural guess that $\nu(3^r - 1) = 1$ when r is odd, and this is easy to prove by working modulo 4. In $\mathbb{Z}/4$ we have $3 = -1$, and so $3^r = (-1)^r = -1$ when r is odd. Thus $3^r - 1 = 2$ in $\mathbb{Z}/4$, which confirms that $\nu(3^r - 1) < 2$.

When r is even the reader will note from the table that $\nu(3^r - 1)$ seems to grow quite slowly as a function of r . Again, this is easy enough to prove as soon as one has the idea to do so. If $r = 2u$ then

$$3^r - 1 = 3^{2u} - 1 = (3^u - 1)(3^u + 1).$$

Modulo 8 the powers of 3 are just 1 and 3, so the possible values for $3^u + 1$ are only 2 and 4. In particular, 8 does not divide $3^u + 1$: that is, $\nu(3^u + 1) < 3$ for all values of u . We therefore have $\nu(3^r - 1) \leq \nu(3^u - 1) + 2$. If u is odd we stop here, otherwise we again divide by 2 and apply the same formula; a simple induction along these lines yields

$$\nu(3^r - 1) \leq 1 + 2\nu(r).$$

The bound $1 + 2\nu(r)$ is generally substantially smaller than r . An easy exercise verifies that $r \leq 1 + 2\nu(r)$ only when $r \in \{1, 2, 4\}$. So to summarize, we have shown that if $r \notin \{1, 2, 4\}$ then $1 + 2\nu(r) < r$, hence $\nu(3^r - 1) < r$, and so $2^r \nmid 3^r - 1$. $\square$

Exercise 5.16. Complete the above proof by showing that if $r \leq 1 + 2\nu(r)$ then $r \in \{1, 2, 4\}$. [Suggestion: There are several ways to piece such a proof together, don't worry about finding the best method.]

Exercise 5.17. We took an ad hoc approach to the above number theory problem, but there are tools known to the experts that could be applied here. The so-called “Lifting-the-Exponent Lemma”—a broad catchall for several related results—says the following:

- Suppose p is an odd prime, $p \nmid x$, $p \nmid y$, and $p \mid x - y$. Then $\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n)$.
- Suppose x and y are odd and n is even. Then $\nu_2(x^n - y^n) = \nu_2(x - y) + \nu_2(x + y) + \nu_2(n) - 1$.

Try to prove these results (or go learn about them) and think about how the second applies to our discussion above.

5.18. Completion of the Ext calculation. At this point we have finished with the solution to the Hopf invariant one problem. But there is another interesting problem that is still on the table, namely the exact computation of the groups

$$\mathrm{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s)) \cong \mathbb{Z}/N_{r,s}.$$

We need to determine the numbers $N_{r,s} = \gcd(2^r - 2^s, 3^r - 3^s, 5^r - 5^s, \dots)$. Note that $N_{r,s} = N_{s,r}$, so we can concentrate on the case $r \geq s$.

Of course this calculation is intriguing from a purely algebraic perspective—when an answer comes down to finding one specific number, it would be difficult not to take the extra step and determine just what that number is. But the answer is also interesting for topological reasons. We have seen that if $f: S^{n+k-1} \rightarrow S^n$ where n and k are both even, then we get an extension $A(f) \in \mathrm{Ext}_{\mathcal{B}}^1(\mathbb{Z}(\frac{n}{2}), \mathbb{Z}(\frac{n+k}{2}))$. A little work (see Exercise 5.26) shows that this actually gives a group homomorphism

$$A: \pi_{n+k-1}(S^n) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(\mathbb{Z}(\frac{n}{2}), \mathbb{Z}(\frac{n+k}{2})) \cong \mathbb{Z}/N_{\frac{n}{2}, \frac{n+k}{2}}.$$

It is important to determine how large the target group is, and how close A is to being an isomorphism. This was investigated by Adams [Adams3].

We begin our investigation of the numbers $N_{r,s}$ by looking at $N_{r,r-1}$. Since $l^r - l^{r-1} = l^{r-1}(l-1)$ this is

$$N_{r,r-1} = \gcd(2^{r-1}, 3^{r-1} \cdot 2, 5^{r-1} \cdot 4, 7^{r-1} \cdot 6, \dots).$$

The 2^{r-1} in the first entry tells us that the gcd will be a power of 2, and the $3^{r-1} \cdot 2$ tells us that it will be at most 2^1 . A moment's thought reveals that the gcd is precisely 2^1 , as long as $r \geq 2$. When $r = 1$ the gcd is just 1:

$$N_{r,r-1} = \begin{cases} 1 & \text{if } r = 1, \\ 2 & \text{if } r \geq 2. \end{cases}$$

Next consider the numbers $N_{r,r-2}$, requiring us to look at $l^r - l^{r-2} = l^{r-2}(l^2 - 1)$. We have

$$\begin{aligned} N_{r,r-2} &= \gcd(2^{r-2}(2^2 - 1), 3^{r-2}(3^2 - 1), 5^{r-2}(5^2 - 1), 7^{r-2}(7^2 - 1), \dots) \\ &= \gcd(2^{r-2} \cdot 3, 3^{r-2} \cdot 8, 5^{r-2} \cdot 24, 7^{r-2} \cdot 48, \dots). \end{aligned}$$

From the first entry we see that the gcd will only have twos and threes in its factorization, with at most one 3. Later entries show that the gcd has at most 3 twos, and a brief inspection leads to the guess that the gcd is 24 as long as $r \geq 5$. To prove this we need to verify that $24 \mid l^2 - 1$ for primes $l > 3$. This is easy, though. Consider the numbers $l-1$, l , and $l+1$. At least one is a multiple of 3, and our hypotheses on l say that it isn't l . So 3 divides $(l-1)(l+1) = l^2 - 1$. Likewise, both $l-1$ and $l+1$ are even and at least one must be a multiple of 4: so $8 \mid l^2 - 1$ as well. The reader will now find it easy to check the following numbers:

$$N_{r,r-2} = \begin{cases} 1 & \text{if } r = 2, \\ 6 & \text{if } r = 3, \\ 12 & \text{if } r = 4, \\ 24 & \text{if } r \geq 5. \end{cases}$$

Remark 5.19. The two cases we have analyzed so far yield an evident conjecture: that $N_{r,r-t}$ is independent of r for $r \gg 0$. We will see below that this is indeed the case.

Let us work out two more cases before discussing the general pattern.

$$\begin{aligned} N_{r,r-3} &= \gcd(2^{r-3}(2^3 - 1), 3^{r-3}(3^3 - 1), 5^{r-3}(5^3 - 1), \dots) \\ &= \gcd(2^{r-3} \cdot 7, 3^{r-3} \cdot 26, 5^{r-3} \cdot 124, \dots). \end{aligned}$$

A very quick investigation shows that

$$N_{r,r-3} = \begin{cases} 1 & \text{if } r = 3, \\ 2 & \text{if } r \geq 4. \end{cases}$$

Moving to $N_{r,r-4}$ we have

$$\begin{aligned} N_{r,r-4} &= \gcd(2^{r-4}(2^4 - 1), 3^{r-4}(3^4 - 1), 5^{r-4}(5^4 - 1), \dots) \\ &= \gcd(2^{r-4} \cdot 15, 3^{r-4} \cdot 80, 5^{r-4} \cdot 624, 7^{r-4} \cdot 2400, \dots). \end{aligned}$$

The numbers are getting larger now, and it is harder to see the patterns. The relevant fact is that $l^4 - 1$ is a multiple of $2^4 \cdot 3 \cdot 5$ for all primes $l > 5$; and for $l = 2$ it is a multiple of $3 \cdot 5$, for $l = 3$ it is a multiple of $2^4 \cdot 5$, and for $l = 5$ it is a multiple of $2^4 \cdot 3$. We leave it as an exercise for the reader to prove this, using

the factorization $l^4 - 1 = (l - 1)(l + 1)(l^2 + 1)$ and some easy number theory. The conclusion is that

$$N_{r,r-4} = \begin{cases} 1 & \text{if } r = 4, \\ 30 & \text{if } r = 5, \\ 60 & \text{if } r = 6, \\ 120 & \text{if } r = 7, \\ 240 & \text{if } r \geq 8. \end{cases}$$

By now it should be clear what the general pattern is, if not the specifics. To understand $N_{r,r-t}$ we consider the numbers

$$2^t - 1, 3^t - 1, 5^t - 1, 7^t - 1, \dots$$

Excluding some finite set of primes at the beginning, there will be an “interesting” gcd to this set of numbers. When r is large the bad primes at the beginning become irrelevant to the computation, and so here $N_{r,r-t}$ is equal to the aforementioned “interesting” gcd. We encourage the reader to do some investigation on their own at this point. The “large r ” values of $N_{r,r-t}$ are listed in the following table, together with their prime factorizations:

t	1	2	3	4	5	6	7	8	9	10	11	12
$N_{r,r-t}$	2	24	2	240	2	504	2	480	2	264	2	65520
factors	2	$2^3 \cdot 3$	2	$2^4 \cdot 5$	2	$2^3 3^2 \cdot 7$	2	$2^5 3 \cdot 5$	2	$2^3 3 \cdot 11$	2	$2^4 3^2 5 \cdot 7 \cdot 13$

If you have been around the stable homotopy groups of spheres you will see some familiar numbers in this table, which might make you sit up and take notice. For example: $\pi_3^s \cong \mathbb{Z}/24$, $\pi_7^s \cong \mathbb{Z}/240$, $\pi_{11}^s \cong \mathbb{Z}/504$, and $\pi_{15}^s \cong \mathbb{Z}/960$ (note that the last one does not quite match). It is remarkable to have these numbers coming up in a purely algebraic computation! It turns out that what we are seeing here is the so-called “image of J ”, or more precisely the image of the *complex* J -homomorphism. See Exercise 5.28 below for a small taste of this subject.

Fix $t \in \mathbb{Z}_+$. It turns out that there is a simple formula for the “stable” values of $N_{r,r-t}$, as a function of t . These stable values are also closely connected to the denominators of Bernoulli numbers. We close this section by explaining this.

Our examples have led to the hypothesis that the sequence of numbers

$$2^C(2^t - 1), 3^C(3^t - 1), 5^C(5^t - 1), 7^C(7^t - 1), \dots$$

has a greatest common divisor that is independent of C when $C \gg 0$. Our aim is to prove this, and to investigate this gcd. To this end, let $m_C(t)$ be this gcd:

$$m_C(t) = \gcd\{l^C(l^t - 1) \mid l \text{ is prime}\}.$$

Notice the relation to our Ext-calculation is that when $r \geq s$ we have $N_{r,s} = m_s(r - s)$.

Also define

$$m'_C(t) = \gcd\{k^C(k^t - 1) \mid k \in \mathbb{Z}_+\}.$$

Clearly $m'_C(t)$ divides $m_C(t)$, but in fact the two are equal:

Lemma 5.20. *For all t and C , $m'_C(t) = m_C(t)$.*

Proof. It will suffice to show that $m_C(t)$ divides $m'_C(t)$, or equivalently that every prime-power factor of the former is also a factor of the latter. So let p be a prime and suppose $p^e | m_C(t)$. Then $p^e | p^C(p^t - 1)$, so $e \leq C$. For any l such that $(l, p) = 1$ we have $p^e | l^t - 1$, so $l^t = 1$ in $\mathbb{Z}/p^e$.

Now let $k \in \mathbb{Z}$ with $k \geq 2$. If $p | k$ then $p^e | k^C(k^t - 1)$ since $e \leq C$. If $p \nmid k$ then write $k = l_1 l_2 \dots l_r$ where each l_i is a prime different from p . We know that $l_i^t = 1$ in $\mathbb{Z}/p^e$ for each i , and so $k^t = l_1^t l_2^t \dots l_r^t = 1$ in $\mathbb{Z}/p^e$ as well. That is, $p^e | k^t - 1$. We have therefore shown that $p^e | m'_C(t)$, which is what we wanted. $\square$

The next proposition proves that $m_C(t)$ stabilizes for $C \gg 0$, and it also determines an explicit formula for the stable value in terms of the prime factorization of t . Let $\nu_p(t)$ denote the exponent of the prime p in the prime factorization of t .

Proposition 5.21. *Let L be the supremum of all exponents in the prime factorization of t . Then $m_C(t)$ is independent of C for $C \geq L + 2$. If we call this stable value $m(t)$ then*

(a) $m(t) = 2$ when t is odd;

(b) When t is even $m(t) = 2^{2+\nu_2(t)} \cdot \prod_{p \text{ odd}, (p-1)|t} p^{1+\nu_p(t)}$.

(c) More generally,

$$m_C(t) = 2^{\min\{2+\nu_2(t), C\}} \cdot \prod_{p \text{ odd}, (p-1)|t} p^{\min\{1+\nu_p(t), C\}}.$$

Remark 5.22. The notation $m(t)$ comes from Adams [Adams3]. Note that the proposition completes our Ext-calculations, via the formula $N_{r,s} = m_s(r - s)$. But see also Exercise 5.27 for more about the Ext groups.

Before proving the proposition let us look at a couple of examples. To compute $m(50)$ we write $50 = 2 \cdot 5^2$. Next we make a list of all odd primes p such that $p - 1$ divides 50; these are 3 and 11. So

$$m(50) = 2^3 \cdot 3 \cdot 11 = 264.$$

For a harder example let us compute $m(12)$. We write $12 = 2^2 \cdot 3$, and our list of odd p such that $p - 1$ divides 12 is 3, 5, 7, and 13. So

$$m(12) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 65520.$$

Further, we have $m(12) = m_4(12)$ and

$$m_3(12) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$$

$$m_2(12) = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$$

$$m_1(12) = 2^1 \cdot 3^1 \cdot 5 \cdot 7 \cdot 13$$

$$m_0(12) = 1.$$

Here are several more values of m for the reader's curiosity (the numbers are, of course, better understood in terms of their prime factorizations):

t	2	4	6	8	10	12	14	16	18
$m(t)$	24	240	504	480	264	65520	24	16320	28728

To prove Proposition 5.21 we need a lemma from algebra. Most basic algebra courses prove that the group of units in $\mathbb{Z}/p$ is a cyclic group, necessarily isomorphic to $\mathbb{Z}/(p-1)$. One can also completely describe the group of units inside the ring $\mathbb{Z}/p^e$, for any e . Recall that this group simply consists of all residue classes of integers k such that $(p, k) = 1$. Here is the result:

Lemma 5.23. *Fix a prime p and consider the group of units $(\mathbb{Z}/p^e)^*$ inside the ring $\mathbb{Z}/p^e$.*

- (a) *If p is odd then $(\mathbb{Z}/p^e)^* \cong \mathbb{Z}/((p-1)p^{e-1}) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p^{e-1})$.*
- (b) *If $e \geq 2$ then $(\mathbb{Z}/2^e)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/(2^{e-2})$. Here the $\mathbb{Z}/2$ is the subgroup $\{1, -1\}$ and the $\mathbb{Z}/(2^{e-2})$ is the subgroup of all numbers congruent to 1 mod 4.*
- (c) $(\mathbb{Z}/2)^* = \{1\}$.

Proof. We first recall the proof that $(\mathbb{Z}/p)^*$ is cyclic. If a finite abelian group is noncyclic, then it contains a subgroup isomorphic to $\mathbb{Z}/k \times \mathbb{Z}/k$, for some prime k (this follows readily from the structure theorem for finite abelian groups). But if this were true for $(\mathbb{Z}/p)^*$ then the field $\mathbb{Z}/p$ would have k^2 solutions to the polynomial $x^k - 1$, and this is a contradiction.

Assume that p is odd. Reduction modulo p gives a surjective map $(\mathbb{Z}/p^e)^* \rightarrow (\mathbb{Z}/p)^*$. Let K denote the kernel. Note that $(\mathbb{Z}/p^e)^*$ coincides with the set

$$\{1, 2, \dots, p^e - 1\} - \{p, 2p, 3p, \dots, (p^{e-1} - 1)p\}$$

and so has order $p^e - p^{e-1}$. Thus, $|K| = p^{e-1}$. Since $|K|$ is prime to the order of $(\mathbb{Z}/p)^*$, our sequence must be split: $(\mathbb{Z}/p^e)^* \cong K \times (\mathbb{Z}/p)^*$. It remains to show that K is cyclic, and for this it suffices to verify that K has exactly $p-1$ elements of order p . Let $a \in K - \{1\}$, and let the base p representation of a be

$$a = 1 + a_f p^f + a_{f+1} p^{f+1} + \dots + a_{e-1} p^{e-1},$$

for $0 \leq a_i < p$ and $a_f \neq 0$ (note that $a_0 = 1$ by the definition of K). Write $b = a_f + a_{f+1}p + a_{f+2}p^2 + \dots$, so that

$$a^p = (1 + p^f b)^p = 1 + p \cdot p^f b + \binom{p}{2} p^{2f} b^2 + \dots$$

The terms after $p^{1+f}b$ all contain at least $f+2$ factors of p ; so modulo p^{f+2} one has $a^p \equiv 1 + p^{f+1}b \equiv 1 + a_f p^{f+1}$. So we can have $a^p = 1$ in $\mathbb{Z}/p^e$ only if $f = e-1$. Thus, we have shown that the elements of K that are p th roots of unity are precisely the elements $1 + ap^{e-1}$, for $0 \leq a < p$. In particular, we have only $p-1$ of these (excluding the identity element). This completes the proof.

The proof for $p = 2$ is similar. Of course $(\mathbb{Z}/4)^* \cong \mathbb{Z}/2$. For $e \geq 3$ consider the sequence $0 \rightarrow K \rightarrow (\mathbb{Z}/2^e)^* \rightarrow (\mathbb{Z}/4)^* \rightarrow 0$, where the right map is reduction modulo 4. This reduction map is split-surjective, with the splitting sending the generator of $(\mathbb{Z}/4)^*$ to -1 . The proof that K is cyclic proceeds exactly as in the odd primary case. $\square$

Proof of Proposition 5.21. Let p be an odd prime. Then one has

$$\begin{aligned} p^e \mid m_C(t) &\iff p^e \mid p^C(p^t - 1) \text{ and } p^e \mid l^C(l^t - 1) \text{ for all primes } l \neq p \\ &\iff e \leq C \text{ and } p^e \mid l^t - 1 \text{ for all primes } l \neq p \\ &\iff e \leq C \text{ and } p^e \mid k^t - 1 \text{ for all } k \in \mathbb{Z}_+ \text{ such that } p \nmid k \\ &\iff e \leq C \text{ and all units in } \mathbb{Z}/p^e \text{ are } t\text{th roots of 1} \\ &\iff e \leq C \text{ and } (p-1)p^{e-1} \mid t. \end{aligned}$$

The third equivalence is by Lemma 5.20, and in the last equivalence we have used that $(\mathbb{Z}/p^e)^* \cong \mathbb{Z}/((p-1)p^{e-1})$. This last line shows why C is redundant when it is large enough: the condition $p^{e-1}|t$ already forces $e \leq \nu_p(t) + 1$, and so $e \leq C$ is redundant if $C \geq \nu_p(t) + 1$.

Assuming now that C is large enough so that we are in the stable case, the above equivalences show that $p|m(t)$ only when $p-1|t$; and also that in this case $\nu_p(m(t)) = 1 + \nu_p(t)$.

The analysis of $p = 2$ is very similar. One finds

$$2^e | m_C(t) \iff e \leq C \text{ and all units in } \mathbb{Z}/2^e \text{ are } t\text{th roots of 1.}$$

When $e = 1$ the latter condition is just $e \leq C$. When $e > 1$ the latter condition is equivalent to $e \leq C$ and $2^{e-2}|t$, using Lemma 5.23(b). This readily yields the desired result. $\square$

Exercise 5.24. Verify the following characterization of $m(t)$:

$$m(t) = \prod_{p \text{ prime}} (\text{maximum } p^e \text{ such that all units in } \mathbb{Z}/p^e \text{ are } t\text{th roots of 1}).$$

Our final task is to make the connection between the numbers $m(t)$ and the Bernoulli numbers. For a review of the Bernoulli numbers and their basic properties, see Appendix A. The result we are after is the following:

Theorem 5.25. *When t is even, $m(t)$ is the denominator of the fraction $\frac{B_t}{2t}$ when expressed in lowest terms.*

The following table demonstrates this result in the first few cases:

t	2	4	6	8	10	12
$ B_t $	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$
$\frac{ B_t }{2t}$	$\frac{1}{24}$	$\frac{1}{240}$	$\frac{1}{504}$	$\frac{1}{480}$	$\frac{1}{264}$	$\frac{691}{65520}$
$m(t)$	24	240	504	480	264	65520

It should be remarked, perhaps, that this connection between $m(t)$ and Bernoulli numbers is worth more in effect than it is in practical value. The explicit formula for $m(t)$ from Proposition 5.21 is much more useful than its description as the denominator of $B_t/(2t)$. Moreover, the denominators of Bernoulli numbers are very simple: the denominator of B_n (when n is even) is simply the product of all primes p such that $p-1$ divides n (Theorem ??(b)). So in the end the result of Theorem 5.25 is neither deep nor particularly useful. Still, it provides a nice-sounding connection between topology and number theory.

Proof of Proposition 5.25. By the theorem of von Staudt and Clausen (Theorem ??) we know that when t is even the denominator of B_t (in lowest terms) is the product of all primes p such that $p-1$ divides t . Note that one such prime is $p = 2$, so the denominator is even (in fact, congruent to 2 modulo 4) and the numerator is odd. Note also that these are the same primes appearing in the factorization of $m(t)$, by Proposition 5.21(b).

Consider now $\alpha = B_t/(2t)$. Since the numerator of B_t is odd, the number of twos in the denominator of α is $1 + 1 + \nu_2(t)$. This is the same as the number of twos in the prime factorization of $m(t)$.

For every prime p such that $p-1|t$ we have one p appearing in the denominator of B_t and $\nu_p(t)$ of them appearing in t , so the total number of p 's in the denominator of α is $1 + \nu_p(t)$. At this point we have thereby shown that $m(t)|\text{den}(\alpha)$, and (from the last paragraph) that the two numbers have the same 2-adic valuation.

It remains to check that if p is an odd prime and $p^e | \text{den}(\alpha)$ then $p^e | m(t)$. This is the nontrivial part of the proof. Note that p^e will also be in the denominator of $2\alpha = B_t/t$. By Proposition ?? we know that

$$\frac{k^t(k^t-1)B_t}{t} \in \mathbb{Z}$$

for all $k \in \mathbb{Z}$. Consequently we have $p^e | k^t(k^t-1)$ for all $k \in \mathbb{Z}$. But this exactly says that p^e divides the greatest common divisor of those numbers, which is $m_t(t)$. We know $m_t(t) = m(t)$, though, from Proposition 5.21. This completes the proof. $\square$

◦ Exercises ◦

Exercise 5.26. Let $f, g: S^{n+k-1} \rightarrow S^n$ where n and k are even. Examine the ladder of Puppe sequences

$$\begin{array}{ccccccc}
 S^{n+k-1} & \xrightarrow{f+g} & S^n & \longrightarrow & C_{f+g} & \longrightarrow & S^{n+k} \\
 \downarrow p & & \downarrow \text{id} & & \downarrow & \boxed{2} & \downarrow p \\
 S^{n+k-1} \vee S^{n+k-1} & \xrightarrow{\nabla(f \vee g)} & S^n & \longrightarrow & C_{\nabla(f \vee g)} & \longrightarrow & S^{n+k} \vee S^{n+k} \\
 \uparrow \text{id} & & \uparrow \nabla & \boxed{1} & \uparrow & & \uparrow \text{id} \\
 S^{n+k-1} \vee S^{n+k-1} & \xrightarrow{f \vee g} & S^n \vee S^n & \longrightarrow & C_{f \vee g} & \longrightarrow & S^{n+k} \vee S^{n+k}
 \end{array}$$

where p is the pinch map. The squares labelled 1 and 2 are homotopy pushouts. Prove that after applying $\tilde{K}^0$ square 1 is a pullback square and 2 is a pushout square. Using this, prove that $A(f+g) = A(f) + A(g)$ where $A: \pi_{n+k-1}(S^n) \rightarrow \text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(\frac{n}{2}), \mathbb{Z}(\frac{n+k}{2}))$ is the map defined in Section 5.18.

For the next two exercises remember the isomorphisms

$$\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s)) \cong \frac{\{P \in \mathbb{Z} \mid \frac{l^r - l^s}{2^r - 2^s} P \in \mathbb{Z} \text{ for all primes } l\}}{(2^r - 2^s)} \cong \mathbb{Z}/N_{r,s} = \mathbb{Z}/(m_s(r-s))$$

where in the first isomorphism P corresponds to the extension module X_P .

Exercise 5.27. Let us return to our original motivation, which was to understand the groups $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$. As part of our computation we discovered two surprises: the groups are symmetric in r and s , and when $r \geq s$ there is a “stability” phenomenon related to s being large compared to $r - s$. In this exercise we will identify some algebraic structure behind these phenomena.

- (a) If M is a $\mathcal{B}$ -module then $M^* = \text{Hom}_{\mathcal{A}\mathcal{B}}(M, \mathbb{Z})$ becomes a $\mathcal{B}$ -module by precomposition: if $f: M \rightarrow \mathbb{Z}$ then define $\psi^k.f = f \circ \psi^k$. Note that $\mathbb{Z}(r)^* \cong \mathbb{Z}(r)$, and that if $0 \rightarrow \mathbb{Z}(s) \rightarrow M \rightarrow \mathbb{Z}(r) \rightarrow 0$ is an exact sequence then so is $0 \leftarrow \mathbb{Z}(s)^* \leftarrow M^* \leftarrow \mathbb{Z}(r)^* \leftarrow 0$. Verify that this gives an isomorphism between $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s))$ and $\text{Ext}_{\mathcal{B}}^1(\mathbb{Z}(s), \mathbb{Z}(r))$.

- (b) If M and N are $\mathcal{B}$ -modules verify that the abelian group $M \otimes_{\mathbb{Z}} N$ becomes a $\mathcal{B}$ -module via the formula $\psi^k(m \otimes n) = (\psi^k m) \otimes (\psi^k n)$. Check that $\mathbb{Z}(r)$ is a flat $\mathcal{B}$ -module, so that we get maps

$$\mathrm{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r), \mathbb{Z}(s)) \xrightarrow{\otimes \mathbb{Z}(1)} \mathrm{Ext}_{\mathcal{B}}^1(\mathbb{Z}(r+1), \mathbb{Z}(s+1)) \xrightarrow{\otimes \mathbb{Z}(1)} \dots$$

Verify that the maps in this sequence eventually become isomorphisms, with the stable value isomorphic to $\mathbb{Z}/(m(r-s))$. [Warning: The proofs that I know all make use of the number-theoretic computations we did in this section. I am not aware of an entirely conceptual proof.]

- (c) Let n and k be even and consider the diagram

$$\begin{array}{ccc} \pi_{n+k-1}(S^n) & \xrightarrow{\Sigma^2} & \pi_{n+k+1}(S^{n+1}) \xrightarrow{\Sigma^2} \dots \\ A \downarrow & & A \downarrow \\ \mathrm{Ext}_{\mathcal{B}}^1\left(\mathbb{Z}\left(\frac{n}{2}\right), \mathbb{Z}\left(\frac{n+k}{2}\right)\right) & \xrightarrow{\otimes \mathbb{Z}(1)} & \mathrm{Ext}_{\mathcal{B}}^1\left(\mathbb{Z}\left(\frac{n+2}{2}\right), \mathbb{Z}\left(\frac{n+k+2}{2}\right)\right) \xrightarrow{\otimes \mathbb{Z}(1)} \dots \end{array}$$

Prove that the squares commute, so that one gets an induced map of colimits

$$\pi_{k-1}(S) \rightarrow \mathbb{Z}/(m(\frac{k}{2})).$$

Exercise 5.28. For n and k even we have the map

$$A: \pi_{n+k-1}(S^n) \rightarrow \mathrm{Ext}_{\mathcal{B}}^1(\mathbb{Z}(\frac{n}{2}), \mathbb{Z}(\frac{n+k}{2})) \cong \mathbb{Z}/N_{\frac{n}{2}, \frac{n+k}{2}} \cong \mathbb{Z}/(m(\frac{k}{2}))$$

where the last isomorphism holds for n large enough. So after stabilization this is a map $\pi_{k-1}(S) \rightarrow \mathbb{Z}/(m(\frac{k}{2}))$. The first few of these are

$$\pi_1(S) \rightarrow \mathbb{Z}/2, \quad \pi_3(S) \rightarrow \mathbb{Z}/24, \quad \pi_5(S) \rightarrow \mathbb{Z}/2, \quad \pi_7(S) \rightarrow \mathbb{Z}/240.$$

A natural question is to ask how close these are to being isomorphisms. For example, are they surjective? Such questions (any many more) were investigated by Adams in the series of papers [Adams3]. A full account would take us far afield, but let us at least give a little taste by exploring the first two maps.

- (a) Start with the Hopf map $\eta: S^3 \rightarrow S^2$, whose cofiber is $\mathbb{C}P^2$. Use our knowledge of $K^0(\mathbb{C}P^2)$ to calculate $A(\eta)$ and show that it is the generator of the target group $\mathbb{Z}/2$.
- (b) Next consider the quaternionic Hopf map $\nu: S^7 \rightarrow S^4$ whose cofiber is $\mathbb{H}P^2$. Calculate $A(\nu)$ and show that it generates the target group $\mathbb{Z}/24$. [Hint: Let $Q \rightarrow \mathbb{H}P^2$ denote the tautological quaternionic line bundle, which as a complex bundle has rank 2. If $u = [Q] - 2$ then the Atiyah-Hirzebruch spectral sequence yields that $K^0(\mathbb{H}P^2) = \mathbb{Z}[u]/(u^3)$. The key step is to compute the two coefficients in $\psi^2(Q-2) = ?(Q-2) + ??(Q-2)^2$ (the first is easy). Use that $\psi^2 = (\lambda^1)^2 - 2\lambda^2$ together with the fact that all complex line bundles on $\mathbb{H}P^2$ are trivial.]

6. CALCULATION OF KO FOR STUNTED PROJECTIVE SPACES

The goal in this section is to determine $KO^0(\mathbb{R}P^n)$ and $KO^0(\mathbb{R}P^n/\mathbb{R}P^a)$ for all values of n and a , together with the Adams operations on these groups. These computations are the key to solving the vector fields on spheres problem (introduced in Section ??), which we do in the next section. As intermediate steps we also

compute $K^*(\mathbb{R}P^n)$ and $K^*(\mathbb{R}P^n/\mathbb{R}P^a)$. The original written source for this material is Adams [Adams2], though he acknowledges unpublished work of Atiyah-Todd and Bott-Shapiro for portions of the calculation. We follow Adams's approach very closely.

Some words of warning about this material are in order. The complete calculation of KO for stunted projective spaces is fairly involved. Several things end up going on at once, so that there is much to keep track of. And calculations are just never very fun to read in the first place. We have attempted to structure our presentation to try to help with this, but of course that only gets us so far. After some preliminary material we give a section which has just the statements of the results, with a minimal amount of discussion (and no proofs). The intent is to give the reader the general picture, and also a convenient reference section. All of the proofs are then given in a subsequent section.

Some readers might want to skip the proofs the first time through, and this is not a problem. Later applications in the text only need the results, not details from the proofs. However, I **highly recommend** that algebraic topologists go through the proofs carefully at an early stage in their career. I cannot stress this enough. Going through the proofs will teach you something important about this subject that I do not have words for, and it will open up doors for you down the road. Trust me that this is an important thing to do.

This section will heavily use the complexification and restriction maps $c: KO(X) \rightarrow K(X)$ and $r_{\mathbb{R}}: K(X) \rightarrow KO(X)$. See ???, and also Theorem ?? for the behavior when X is a sphere. We will also use the Adams operations ψ^k defined for $k \in \mathbb{Z}$ (see Remark 4.20). Recall that $\psi^{-1}(E) = E^*$ for bundles E . Finally, we will throughout this section abbreviate KO^0 to KO and K^0 to K whenever convenient.

6.1. Initial material. The results in this section will make heavy use of the interplay between $\mathbb{R}P^n$ and $\mathbb{C}P^n$ in the homotopy category of spaces. We begin by reviewing the basics of what we will need.

Let η be the tautological complex line bundle on $\mathbb{C}P^n$, and let L be the tautological real line bundle on $\mathbb{R}P^n$. Let $j: \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ be the inclusion.

Lemma 6.2. *The complexification of L is the pullback of η : that is, $cL \cong j^*\eta$.*

Proof. Complex line bundles on a space X are classified by homotopy classes in $[X, \mathbb{C}P^\infty]$. The real line bundle L is classified by the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$, so the complexification of L is classified by the composition $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$. The complex line bundle η is classified by $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$, so $j^*\eta$ is classified by the composition $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$. The result is immediate. $\square$

Note that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so complex line bundles on X are classified by $[X, \mathbb{C}P^\infty] = H^2(X)$. If $n > 1$ then $H^2(\mathbb{R}P^n) = \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles: the trivial bundle and the nontrivial bundle. The bundle $cL = j^*\eta$ is nontrivial, since its classification map $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$ represents the generator of $H^2(\mathbb{R}P^n)$.

Remark 6.3. The powers η^k are all distinct line bundles on $\mathbb{C}P^n$ (e.g., the first Chern classes are $c_1(\eta^k) = kc_1(\eta) = kx$ where x is the generator of $H^2(\mathbb{C}P^n)$). The

situation upon pulling back to $\mathbb{R}P^n$ is very different, however. We have

$$[j^*\eta]^2 = (cL)^2 = c(L^2) = c(1_{\mathbb{R}}) = 1_{\mathbb{C}}.$$

So the even powers of $j^*\eta$ are all trivial, and the odd powers are just $j^*\eta$. We will see that this accounts for the main difference between $K^0(\mathbb{C}P^n)$ and $K^0(\mathbb{R}P^n)$.

In addition to the inclusion $j: \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ there is another interesting map from real to complex projective space. Every real line in $\mathbb{C}^{n+1}$ determines a complex line by taking the $\mathbb{C}$ -linear span, and therefore we get a map $\mathbb{P}_{\mathbb{R}}(\mathbb{C}^{n+1}) \rightarrow \mathbb{P}_{\mathbb{C}}(\mathbb{C}^{n+1}) = \mathbb{C}P^n$. It is easy to see that this is a fiber bundle with fiber S^1 (the space of real lines in $\mathbb{C}$). Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ shows that the domain is homeomorphic to $\mathbb{R}P^{2n+1}$, giving us a fiber bundle

$$S^1 \longrightarrow \mathbb{R}P^{2n+1} \xrightarrow{q} \mathbb{C}P^n.$$

In terms of homogeneous coordinates, q sends the point $[x_0 : x_1 : \cdots : x_{2n} : x_{2n+1}]$ to $[x_0 + ix_1 : x_2 + ix_3 : \cdots : x_{2n} + ix_{2n+1}]$.

Lemma 6.4. *The diagram*

$$\begin{array}{ccc} \mathbb{R}P^n & \xrightarrow{i} & \mathbb{R}P^{2n+1} \\ & \searrow j & \downarrow q \\ & & \mathbb{C}P^n \end{array}$$

commutes up to homotopy, where i is the standard inclusion. Consequently, q^ sends $x \in H^2(\mathbb{C}P^n)$ to the nonzero element of $H^2(\mathbb{R}P^{2n+1})$. (The latter statement can also be proven via the Serre spectral sequence for q).*

Proof. The diagram commutes on the nose if i is replaced by the inclusion sending $[x_0 : x_1 : \cdots : x_n]$ to $[x_0 : 0 : x_1 : 0 : \cdots : x_n : 0]$. But all linear inclusions from one projective space to another are homotopic. $\square$

Corollary 6.5. *There is an isomorphism of bundles $q^*\eta \cong cL$.*

Proof. As we have remarked before, there are only two isomorphism classes of complex bundles on $\mathbb{R}P^{2n+1}$. Since $j^*\eta = i^*(q^*\eta)$ is not trivial, the bundle $q^*\eta$ cannot be trivial. So the only possibility is $q^*\eta \cong cL$. $\square$

Note that for $a \leq n$ one has the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^{2a+1} & \xrightarrow{i} & \mathbb{R}P^{2n+1} \\ q \downarrow & & \downarrow q \\ \mathbb{C}P^a & \xrightarrow{i} & \mathbb{C}P^n, \end{array}$$

and therefore q induces a map $\mathbb{R}P^{2n+1}/\mathbb{R}P^{2a+1} \rightarrow \mathbb{C}P^n/\mathbb{C}P^a$. We will also use q to denote this induced map on quotients, as well as various small modifications.

Note that these induced maps fit together to give a big commutative diagram

$$\begin{array}{ccc}
 (6.6) \quad S^{2a+2} \cong \mathbb{R}P^{2a+2}/\mathbb{R}P^{2a+1} & & \\
 \downarrow & & \downarrow \\
 \mathbb{R}P^{2a+3}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^{a+1}/\mathbb{C}P^a \cong S^{2a+2} \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \mathbb{R}P^{2n+1}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^n/\mathbb{C}P^a \\
 \downarrow & & \downarrow \\
 \mathbb{R}P^{2n+2}/\mathbb{R}P^{2a+1} & & \\
 \downarrow & & \downarrow \\
 \mathbb{R}P^{2n+3}/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^{n+1}/\mathbb{C}P^a \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \mathbb{R}P^\infty/\mathbb{R}P^{2a+1} & \xrightarrow{q} & \mathbb{C}P^\infty/\mathbb{C}P^a.
 \end{array}$$

We will tend to use ‘ q ’ as a name for any composition of maps from this diagram that involves one horizontal step.

Notice that at the very top of the diagram we have a map from S^{2a+2} to itself. The next result identifies this map:

Lemma 6.7. *For $n \geq 1$ the composite*

$$S^{2n} \cong \mathbb{R}P^{2n}/\mathbb{R}P^{2n-1} \hookrightarrow \mathbb{R}P^{2n+1}/\mathbb{R}P^{2n-1} \xrightarrow{q} \mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}$$

is a homeomorphism.

Proof. The space $\mathbb{R}P^{2n}/\mathbb{R}P^{2n-1}$ consists of the basepoint and the affine space $\mathbb{R}^{2n}$ made up of points $[x_0 : x_1 : \cdots : x_{2n-2} : x_{2n-1} : 1]$. Likewise, the space $\mathbb{C}P^n/\mathbb{C}P^{n-1}$ consists of the basepoint and the affine space $\mathbb{C}^n$ made up of the points $[z_0 : z_1 : \cdots : z_{n-1} : 1]$. One readily uses the formula for q to see that it gives a continuous bijective correspondence, and is therefore a homeomorphism (the spaces involved being compact and Hausdorff). $\square$

6.8. The main results. Here we state the main theorems about the K -theory of real and complex projective spaces. The proofs will be deferred until the next section.

Theorem 6.9 (Complex K -theory of complex projective spaces and stunted projective spaces). *Let η be the tautological line bundle on $\mathbb{C}P^n$, and write $\mu = [\eta] - 1 \in \tilde{K}^0(\mathbb{C}P^n)$.*

(a) $K^0(\mathbb{C}P^n) = \mathbb{Z}[\mu]/(\mu^{n+1})$ and $K^1(\mathbb{C}P^n) = 0$.

(b) *The Adams operations on $K^0(\mathbb{C}P^n)$ are given by*

$$\psi^k(\mu^s) = [(1 + \mu)^k - 1]^s = k^s \mu^s + s \binom{k}{2} k^{s-1} \mu^{s+1} + (\text{higher order terms})$$

for $k \geq 1$.

(c) The sequence $0 \rightarrow \tilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^a) \rightarrow \tilde{K}^0(\mathbb{C}P^n) \rightarrow \tilde{K}^0(\mathbb{C}P^a) \rightarrow 0$ is exact, and identifies $\tilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^a)$ as the free abelian group $\mathbb{Z}\langle\mu^{a+1}, \mu^{a+2}, \dots, \mu^n\rangle \subseteq K^0(\mathbb{C}P^n)$. The ring structure and action of the Adams operations are determined by the corresponding structures on $K^0(\mathbb{C}P^n)$. Also, $K^1(\mathbb{C}P^n/\mathbb{C}P^a) = 0$.

Remark 6.10. Following Adams [Adams2] we write $\mu^{(i)}$ for the element of $K^0(\mathbb{C}P^n/\mathbb{C}P^a)$ that maps to μ^i in $K^0(\mathbb{C}P^n)$. The extra parentheses in the exponent remind us that this class is not a true i th power in the ring $K^0(\mathbb{C}P^n/\mathbb{C}P^a)$.

The isomorphism $L^2 \cong \underline{1}$ complexifies to the isomorphism $(cL)^2 \cong \underline{1}$. If we set $\nu = [j^*\eta] - 1 = [cL] - 1$ then

$$\nu^2 = [(cL)^2] - 2[cL] + 1 = 1 - 2[cL] + 1 = 2 - 2[cL] = -2\nu.$$

This is the key to understanding $K^*(\mathbb{R}P^n)$:

Theorem 6.11 (Complex K -theory of real projective spaces). *Let ν be the element $[j^*\eta] - 1 \in \tilde{K}^0(\mathbb{R}P^n)$, where as usual j is the standard inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$.*

(a) $K^0(\mathbb{R}P^n) = \mathbb{Z}[\nu]/(\nu^{\lfloor \frac{n}{2} \rfloor + 1}, \nu^2 + 2\nu)$ as rings and $\tilde{K}^0(\mathbb{R}P^n)$ is the ideal generated by ν . As abelian groups $\tilde{K}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^{\lfloor \frac{n}{2} \rfloor})$ with generator ν .

(b) $K^1(\mathbb{R}P^n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$

(c) The Adams operations on $K^0(\mathbb{R}P^n)$ are given by

$$\psi^k(\nu^e) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \nu^e & \text{if } k \text{ is odd} \end{cases}$$

for all $k \in \mathbb{Z}$.

Remark 6.12. Let E be a complex-oriented cohomology theory, with associated formal group F . The **2-series** of F is defined to be the power series $x +_F x = F(x, x)$. The isomorphism $\underline{1} \cong (j^*\eta)^2$ implies that

$$0 = c_1^E((j^*\eta)^2) = c_1^E(j^*\eta) +_F c_1^E(j^*\eta).$$

So the map $E^*(\mathbb{C}P^\infty) \rightarrow E^*(\mathbb{R}P^\infty)$ descends to give a map $E^*[[x]]/(x +_F x) \rightarrow E^*(\mathbb{R}P^\infty)$ where $x = c_1^E(\eta)$. In many cases this map is an isomorphism.

For $E = K$ one has $F(x, y) = x + y - xy$, $c_1^K(\eta) = 1 - [j^*\eta] = -\nu$, and so the 2-series relation is $0 = F(-\nu, -\nu) = -2\nu - \nu^2$.

Recall that we have calculated $\tilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^{a-1})$ to be the free abelian group $\mathbb{Z}\langle\mu^{(a)}, \mu^{(a+1)}, \dots, \mu^{(n)}\rangle$. For $k \leq 2n+1$ we may pull back these classes along the map $q: \mathbb{R}P^k/\mathbb{R}P^{2a-1} \rightarrow \mathbb{C}P^n/\mathbb{C}P^{a-1}$ to get elements of $\tilde{K}^0(\mathbb{R}P^k/\mathbb{R}P^{2a-1})$. We again follow Adams [Adams2] and set

$$\bar{\nu}^{(t)} = q^*(\mu^{(t)}) \in \tilde{K}^0(\mathbb{R}P^k/\mathbb{R}P^{2a-1})$$

for $a \leq t \leq n$. Note that these elements correspond nicely as k and n vary, due to the commutative diagram (6.6). We may as well take $n \mapsto \infty$ so that we have classes $\bar{\nu}^{(t)}$ for all $t \geq a$.

We claim that upon pulling back along the projection $\pi: \mathbb{R}P^k \rightarrow \mathbb{R}P^k/\mathbb{R}P^{2a-1}$ we have $\pi^*(\bar{\nu}^{(t)}) = \nu^t$; this explains our choice of notation. To prove this claim

we can deal with the cases $k = 2u$ and $k = 2u + 1$ simultaneously. Consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{R}P^{2u}/\mathbb{R}P^{2a-1} & \xrightarrow{\quad} & \mathbb{R}P^{2u+1}/\mathbb{R}P^{2a-1} & \xrightarrow{q} & \mathbb{C}P^u/\mathbb{C}P^{a-1} \\ \pi \uparrow & & \pi \uparrow & & \pi_{\mathbb{C}} \uparrow \\ \mathbb{R}P^{2u} & \xrightarrow{\quad} & \mathbb{R}P^{2u+1} & \xrightarrow{q} & \mathbb{C}P^u. \end{array}$$

We have $\pi^*(q^*(\mu^{(t)})) = q^*(\pi_{\mathbb{C}}^*(\mu^{(t)})) = q^*(\mu^t) = (q^*\mu)^t = \nu^t$, where the last equality is by Corollary 6.5 and Lemma 6.2.

Observe that $\tilde{K}^0(\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1}) = \tilde{K}^0(S^{2a}) \cong \mathbb{Z}$ and the class $\bar{\nu}^{(a)}$ is a generator. This follows because $\mu^{(a)}$ generates $\tilde{K}^0(\mathbb{C}P^a/\mathbb{C}P^{a-1}) \cong \mathbb{Z}$ and $\bar{\nu}^{(a)}$ is the pullback of $\mu^{(a)}$ along the map $\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1} \rightarrow \mathbb{C}P^a/\mathbb{C}P^{a-1}$, which by Lemma 6.7 is a homotopy equivalence.

For $n \geq 2a$ the inclusion $i: \mathbb{R}P^{2a}/\mathbb{R}P^{2a-1} \hookrightarrow \mathbb{R}P^n/\mathbb{R}P^{2a-1}$ induces a map

$$\mathbb{Z} = \tilde{K}^0(S^{2a}) = \tilde{K}^0(\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1}) \xleftarrow{i^*} \tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2a-1}).$$

This map is surjective, because the class $\bar{\nu}^{(a)}$ in the domain maps to the class $\bar{\nu}^{(a)}$ in the target, and the latter is a generator. In particular, not only do we know that the above map i^* is surjective but the class $\bar{\nu}^{(a)}$ in the domain gives a choice of splitting. This will be the main use of these $\bar{\nu}$ classes.

Theorem 6.13 (Complex K -theory of real stunted projective spaces).

Suppose $n > a > 0$.

(a) If $a = 2t$ then the sequence

$$0 \rightarrow \tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \rightarrow \tilde{K}^0(\mathbb{R}P^n) \rightarrow \tilde{K}^0(\mathbb{R}P^a) \rightarrow 0$$

is exact. It identifies $\tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ with the subgroup of $\tilde{K}^0(\mathbb{R}P^n)$ generated by $\nu^{t+1} = (-2)^t \nu$. As a group, $\tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z}/(2^g)$ where $g = \lfloor \frac{n-a}{2} \rfloor$. The ring structure and Adams operations are determined by the structures in $K^0(\mathbb{R}P^n)$. In particular, ψ^k acts as zero when k is even and as the identity when k is odd. We write $\nu^{(i)}$ for the element of $\tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ that maps to ν^i in $\tilde{K}^0(\mathbb{R}P^n)$, so that $\nu^{(t+1)}$ is a generator.

(b) Let $a = 2t - 1$. The cofiber sequence $\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1} \rightarrow \mathbb{R}P^n/\mathbb{R}P^{2t-1} \rightarrow \mathbb{R}P^n/\mathbb{R}P^{2t}$ induces a sequence

$$0 \leftarrow \tilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) \leftarrow \tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \leftarrow \tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t}) \leftarrow 0$$

that is short exact. Consequently, $\tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \cong \mathbb{Z} \oplus \mathbb{Z}/(2^f)$ where $f = \lfloor \frac{n}{2} \rfloor - t$; the former summand is generated by $\bar{\nu}^{(t)}$ and the latter summand is generated by $\nu^{(t+1)}$.

(c) The action of ψ^k on $\tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1})$ is given by

$$\psi^k(\nu^{(t+1)}) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \nu^{(t+1)} & \text{if } k \text{ is odd,} \end{cases}$$

and

$$\psi^k(\bar{\nu}^{(t)}) = k^t \bar{\nu}^{(t)} + \begin{cases} \frac{1}{2} k^t \nu^{(t+1)} & \text{if } k \text{ is even} \\ \frac{1}{2} (k^t - 1) \nu^{(t+1)} & \text{if } k \text{ is odd} \end{cases}$$

for all $k \in \mathbb{Z}$.

Most of the content to the above theorem is represented in the following convoluted but useful diagram:

$$\begin{array}{ccccccc}
 \mathbb{Z}/(2^{\lfloor \frac{n}{2} \rfloor}) = \tilde{K}(\mathbb{R}P^n) & & & & \tilde{K}(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) = \mathbb{Z} & & \\
 \uparrow & \swarrow & & \searrow & & & \\
 \cdots \leftarrow \tilde{K}(\mathbb{R}P^n/\mathbb{R}P^{2t-2}) & \leftarrow & \tilde{K}(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) & \leftarrow & \tilde{K}(\mathbb{R}P^n/\mathbb{R}P^{2t}) & \leftarrow & \cdots \\
 \parallel & & \parallel & & \parallel & & \\
 \mathbb{Z}/(2^{B+1}) & \leftarrow & \mathbb{Z} \oplus \mathbb{Z}/(2^B) & \leftarrow & \mathbb{Z}/(2^B) & & B = \lfloor \frac{n}{2} \rfloor - t \\
 \\
 \nu^{(t)} & \leftarrow & \bar{\nu}^{(t)}, & \nu^{(t+1)} & \leftarrow & \nu^{(t+1)} & \\
 & & \swarrow & & & & \\
 \nu^{(t+1)} = -2\nu^{(t)} & & & & & &
 \end{array}$$

The indicated maps are injections/surjections, and our chosen generators of the groups are written in the bottom two lines. The generators $\nu^{(t+1)}$ and $\bar{\nu}^{(t)}$ map to the elements ν^{t+1} and ν^t in $\tilde{K}(\mathbb{R}P^n)$, and $\bar{\nu}^{(t)}$ maps to a generator of $\tilde{K}(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) = \tilde{K}(S^{2t})$. The action of the Adams operations on $\nu^{(t+1)}$ is completely determined by what happens in $\tilde{K}(\mathbb{R}P^n)$. Likewise, the action on $\bar{\nu}^{(t)}$ is completely determined by using the surjection onto $\tilde{K}(S^{2t})$ together with the map into $\tilde{K}(\mathbb{R}P^n)$. These are instructive exercises; but if necessary see the proofs in Section 6.27 for details.

Remark 6.14. The action of ψ^k on the element $\bar{\nu}^{(t)}$ is of crucial importance to the solution of the vector fields on spheres problem. See Section 7.

We now move from the realm of K -theory to KO -theory. Recall that $L \rightarrow \mathbb{R}P^n$ always denotes the tautological line bundle.

Theorem 6.15 (Real K -theory of real projective spaces).

$\widetilde{KO}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^f)$ where $f = \#\{s \mid 0 < s \leq n, s \equiv 0, 1, 2, \text{ or } 4 \pmod{8}\}$. The group is generated by $\lambda = [L] - 1$, which satisfies $\lambda^2 = -2\lambda$ and $\lambda^{f+1} = 0$. The Adams operations are given by

$$\psi^k(\lambda) = \begin{cases} 0 & k \text{ even,} \\ \lambda & k \text{ odd.} \end{cases}$$

Definition 6.16. Because this number comes up so often, let us define

$$\varphi(n) = \#\{s \mid 0 < s \leq n, s \equiv 0, 1, 2, \text{ or } 4 \pmod{8}\}.$$

The following chart shows the groups $\widetilde{KO}^0(\mathbb{R}P^n)$ and $\tilde{K}^0(\mathbb{R}P^n)$ as functions of n . To save space we write $\mathbb{Z}_n$ instead of $\mathbb{Z}/n$; but all the groups are cyclic, and so really one only needs to keep track of the order.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
KO	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{128}$	$\mathbb{Z}_{128}$	$\mathbb{Z}_{128}$
K	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{128}$

Observe that the $\widetilde{KO}^0(\mathbb{R}P^n)$ groups follow the by-now-familiar 8-fold pattern from Bott periodicity and Clifford algebras: starting in multiples of 8 the orders of the

groups jump according to the pattern “jump-jump-nothing-jump-nothing-nothing-nothing-jump” (the first couple are not on the chart because the associated projective spaces are exceptions in some way). In particular, every eight steps a total of four jumps have occurred, resulting in the orders being multiplied by 16. This is the quasi-periodicity of the first line. The second line has the simpler quasi-periodicity of length 2, where every two steps the order of the group gets doubled. Note that the groups on the two lines coincide in dimensions congruent to 6, 7, and 8 modulo 8; in other dimensions there is a difference of a factor of 2.

For reference purposes we also give Table 6.16 below showing the numbers $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$. Even though this is really the same information as in the previous table, it is very handy to have around. The table provides some useful information about the comparison between $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$, which we record in a proposition:

TABLE 6.16. Comparison of $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\varphi(n)$	0	1	2	2	3	3	3	3	4	5	6	6	7	7	7	7	8
$\lfloor \frac{n}{2} \rfloor$	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8

Proposition 6.17. *For every $n \geq 2$, the number $\varphi(n)$ equals either $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$. The former occurs precisely when n is congruent to 6, 7, or 8 modulo 8.*

Recall the complexification map $c: \widetilde{KO}^0(\mathbb{R}P^n) \rightarrow \widetilde{K}^0(\mathbb{R}P^n)$. Both groups are cyclic, and by Lemma 6.2 the map sends the generator $\lambda = L - 1$ of the domain to the generator $\nu = j^*\eta - 1$ of the target. Hence, c is surjective. Our observations about the orders now proves part (a) of the following result. Part (b) follows at once from $c(\lambda) = \nu$ and the fact that $r_{\mathbb{R}}c = 2$.

Theorem 6.18. *Let $n \geq 2$.*

- (a) *The complexification map $c: \widetilde{KO}^0(\mathbb{R}P^n) \rightarrow \widetilde{K}^0(\mathbb{R}P^n)$ is always surjective. It is an isomorphism if n is congruent to 6, 7, or 8 modulo 8, and it has kernel $\mathbb{Z}/2$ otherwise.*
- (b) *The map $r_{\mathbb{R}}: \widetilde{K}^0(\mathbb{R}P^n) \rightarrow \widetilde{KO}^0(\mathbb{R}P^n)$ sends ν to 2λ .*

Finally, we turn our attention to KO -theory of the spaces $\mathbb{R}P^n/\mathbb{R}P^a$. It is almost true that the Atiyah-Hirzebruch spectral sequence for $\mathbb{R}P^n/\mathbb{R}P^a$ is a truncation of the one for $\mathbb{R}P^n$. The mod 2 cohomology groups $H^*(\mathbb{R}P^n/\mathbb{R}P^a; \mathbb{Z}/2)$ are indeed a truncation of $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$, and the integral cohomology groups are a similar truncation when a is even. But when a is odd there is a $\mathbb{Z}$ in $H^{a+1}(\mathbb{R}P^n/\mathbb{R}P^a)$ that does not appear in $H^{a+1}(\mathbb{R}P^n)$. This new $\mathbb{Z}$ will contribute to $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ only if it shows up along the main diagonal in the Atiyah-Hirzebruch spectral sequence, which will happen precisely when $a + 1$ is a multiple of 4. This explains the two cases in the following result:

Theorem 6.19 (Real K -theory of real stunted projective spaces; part 1).

- (a) *Suppose $a \not\equiv -1 \pmod{4}$. Then the map $\pi^*: \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \rightarrow \widetilde{KO}^0(\mathbb{R}P^n)$ is an injection whose image is the subgroup generated by $\lambda^{\varphi(a)+1}$. So $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z}/(2^g)$ where $g = \varphi(n) - \varphi(a) = \#\{s \mid a < s \leq n, s \equiv$*

$0, 1, 2$, or $4 \bmod 8$. Let $\lambda^{(\varphi(a)+1)}$ be the preimage for $\lambda^{\varphi(a)+1}$ under π^* , which generates the group. Then

$$\psi^k(\lambda^{(u)}) = \begin{cases} 0 & k \text{ even,} \\ \lambda^{(u)} & k \text{ odd} \end{cases}$$

for all $k \in \mathbb{Z}$.

(b) Assume that $a \equiv -1 \bmod 4$. The sequence

$$S^{a+1} = \mathbb{R}P^{a+1}/\mathbb{R}P^a \rightarrow \mathbb{R}P^n/\mathbb{R}P^a \rightarrow \mathbb{R}P^n/\mathbb{R}P^{a+1}$$

induces a split-exact sequence in KO -groups:

$$0 \leftarrow \widetilde{KO}^0(S^{a+1}) \leftarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \leftarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{a+1}) \leftarrow 0.$$

Consequently,

$$\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z} \oplus \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{a+1}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^h$$

where $h = \varphi(n) - \varphi(a+1) = \#\{s \mid a+1 < s \leq n, s \equiv 0, 1, 2, \text{ or } 4 \bmod 8\}$.

Notice that the above result does not give the action of the Adams operations in part (b). To do this we need to choose a specific generator for the $\mathbb{Z}$ summand, and this requires some explanation. It turns out (and this is not obvious) that the generator can always be chosen so that it maps to $\lambda^{\varphi(a+1)}$ in $\widetilde{KO}^0(\mathbb{R}P^n)$. This property is all that we will really need, but it is not so easy to prove; in fact there are always *two* such generators, and proving the desired existence seems to be best accomplished by having a method for systematically choosing a preferred generator out of the two possibilities. This is what we do next; in the chain

$$\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \leftarrow \widetilde{KO}(\mathbb{R}P^{4t+1}/\mathbb{R}P^{4t-1}) \leftarrow \cdots \leftarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \leftarrow \cdots$$

we wish to choose elements $\bar{\lambda}^{(\varphi(4t))}$ in each group with the property that they all map onto each other, they all map to a generator of the left-most group (which is $\mathbb{Z}$), and upon pulling back along the projection $\mathbb{R}P^n \rightarrow \mathbb{R}P^n/\mathbb{R}P^{4t-1}$ the element $\bar{\lambda}^{(\varphi(4t))}$ maps to $\lambda^{\varphi(4t)}$ (for any choice of n).

These elements will be produced by starting with the elements $\bar{\nu}^{(2t)}$ that we have already constructed, living in the bottom groups of the following diagram:

$$\begin{array}{ccccc} \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) & \longleftarrow \cdots \longleftarrow & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & & \\ \downarrow c \quad \nearrow r_{\mathbb{R}} & & \downarrow c \quad \nearrow r_{\mathbb{R}} & \searrow & \\ \widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) & \longleftarrow \cdots \longleftarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & & \widetilde{KO}(\mathbb{R}P^n) \\ & & \searrow & \searrow & \downarrow c \quad \nearrow r_{\mathbb{R}} \\ & & & & \widetilde{K}(\mathbb{R}P^n) \end{array}$$

First note that it suffices to construct the $\bar{\lambda}^{(\varphi(4t))}$ classes for n sufficiently large, as we can then construct the classes for smaller n by naturality. In particular, we may assume that n is congruent to 6 (or 7 or 8) modulo 8. This forces the right-most vertical c map to be an isomorphism, by Theorem 6.18(a).

The rest of the argument breaks into two cases, depending on whether t is even or odd. When t is even, $\varphi(4t) = 2t$ (see Table 6.16) and the vertical maps c in the above

diagram are all isomorphisms; this will be proven carefully in Theorem 6.25(b) below, but for now we just accept it. Define

$$(6.20) \quad \overline{\lambda}^{(\varphi(4t))} = c^{-1}(\bar{\nu}^{(2t)}).$$

The desired properties of $\overline{\lambda}^{(\varphi(4t))}$ are immediate by the naturality of c and the known properties of $\bar{\nu}^{(2t)}$.

When t is odd one has $\varphi(4t) = 2t + 1$. The vertical maps c are no longer isomorphisms (except the rightmost one), but we can use the map $r_{\mathbb{R}}$ instead. The idea for this comes from the fact that when t is odd the map $c: \widetilde{KO}(S^{4t}) \rightarrow \widetilde{K}(S^{4t})$ is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, and $r_{\mathbb{R}}$ sends a generator to a generator (Theorem ??). So $r_{\mathbb{R}}(\bar{\nu}^{(2t)})$ will give us an element of $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ that maps to a generator in $\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$.

However, note that $r_{\mathbb{R}}(\bar{\nu}^{2t})$ maps to $-\lambda^{(2t+1)}$ in $\widetilde{KO}(\mathbb{R}P^n)$. This follows at once from a simple calculation:

$$r_{\mathbb{R}}(\nu^{2t}) = r_{\mathbb{R}}((-2)^{2t-1} \cdot \nu) = (-2)^{2t-1} \cdot r_{\mathbb{R}}(\nu) = (-2)^{2t-1} \cdot 2\lambda = -(-2)^{2t} \cdot \lambda = -\lambda^{2t+1}.$$

The extra minus sign leads us to make the definition

$$(6.21) \quad \overline{\lambda}^{(\varphi(4t))} = -r_{\mathbb{R}}(\bar{\nu}^{(2t)})$$

in the case when t is odd.

We have now constructed the desired generators $\overline{\lambda}^{(\varphi(4t))}$. The group $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ is generated by the two elements $\lambda^{\varphi(4t)+1}$ (which is torsion) and $\overline{\lambda}^{(\varphi(4t))}$ (which is non-torsion). These elements map to the corresponding powers of λ in $\widetilde{KO}^0(\mathbb{R}P^n)$. Observe that $\varphi(4t) = \varphi(4t-1) + 1$. Combining this knowledge with Theorem 6.19(a) (and other previous work) gives the following uniform result:

Proposition 6.22.

- (a) For any $a \leq n$, the image of $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \rightarrow \widetilde{KO}^0(\mathbb{R}P^n)$ is the subgroup generated by $\lambda^{\varphi(a)+1}$.
- (b) For any $n \leq N$ the map $\widetilde{KO}^0(\mathbb{R}P^N/\mathbb{R}P^a) \rightarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ is surjective.

Exercise 6.23. Prove Proposition 6.22.

We next use our generators to describe the action of the Adams operations:

Theorem 6.24 (Real K -theory of real, stunted projective spaces; part 2). *Let $t \geq 1$ and let $f = \varphi(4t)$. The Adams operations on $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ are given by the formulas*

$$\begin{aligned} \psi^k(\lambda^{(f+1)}) &= \begin{cases} 0 & k \text{ even,} \\ \lambda^{(f+1)} & k \text{ odd;} \end{cases} \\ \psi^k(\overline{\lambda}^{(f)}) &= k^{2t} \overline{\lambda}^{(f)} + \begin{cases} \frac{1}{2} k^{2t} \lambda^{(f+1)} & k \text{ even,} \\ \frac{1}{2} (k^{2t} - 1) \lambda^{(f+1)} & k \text{ odd.} \end{cases} \end{aligned}$$

Just as we saw for $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^a)$, much of the information about KO -theory of stunted projective spaces is represented in the following useful diagram. The

above formulas for the Adams operations are obtained easily by chasing information around the diagram.

$$\begin{array}{ccccccc}
 \mathbb{Z}/(2^{\varphi(n)}) = \widetilde{K}(\mathbb{R}P^n) & & & & \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) = \mathbb{Z} & & \\
 \uparrow & \swarrow & & \searrow & \uparrow & & \\
 \leftarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-2}) \leftarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \leftarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t}) \leftarrow & & & & & & \\
 \parallel & & \parallel & & \parallel & & \\
 \mathbb{Z}/(2^{B+1}) \leftarrow \mathbb{Z} \oplus \mathbb{Z}/(2^B) \leftarrow \mathbb{Z}/(2^B) & & & & B=\varphi(n)-\varphi(4t) & & \\
 & & & & & & \\
 \lambda^{(f)} \leftarrow \bar{\lambda}^{(f)}, & \lambda^{(f+1)} \leftarrow \lambda^{(f+1)} & f=\varphi(4t) & & & & \\
 & \swarrow & & & & & \\
 \lambda^{(f+1)} = -2\lambda^{(f)} & & & & & &
 \end{array}$$

Due to lack of space the diagram does not show the groups $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-3})$, but these are similar to the $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-2})$ and $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t})$ cases in that these groups all inject into $\widetilde{KO}(\mathbb{R}P^n)$ and have the “expected” image.

Because we have needed this already in the process of defining the classes $\bar{\lambda}^{(\varphi(4t))}$, we also include some more information on the complexification map for stunted projective spaces. We want to investigate $c: \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \rightarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$, but results are awkward to state in this generality: one runs into a multitude of cases depending on the congruence classes of n and a modulo 8. We start with the observation that it is essentially enough to solve the problem for n large enough. If $N \geq n$ then by Proposition 6.22 we have the diagram

$$\begin{array}{ccc}
 \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \leftarrow & \widetilde{KO}^0(\mathbb{R}P^N/\mathbb{R}P^a) \\
 \downarrow c & & \downarrow c \\
 \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \leftarrow & \widetilde{K}^0(\mathbb{R}P^N/\mathbb{R}P^a).
 \end{array}$$

So if we know the right vertical map then we can also figure out the left vertical map, using the horizontal surjections.

Notice that by choosing N so that it is congruent to 6 (or 7 or 8) modulo 8, we can get ourselves in the situation where $c: \widetilde{KO}^0(\mathbb{R}P^N) \rightarrow \widetilde{K}^0(\mathbb{R}P^N)$ is an isomorphism (see Theorem 6.18)—clearly this will simplify some matters in our analysis. This explains why we focus on this special case in the following result.

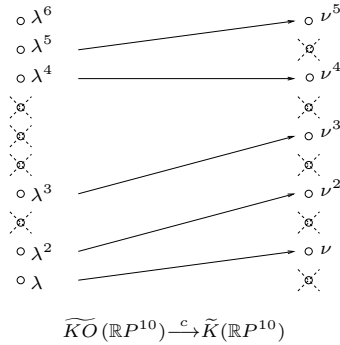
Here is a notational simplification that is very useful. At this point we have specified particular generators for the groups $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$, for all values of n and a . These are the elements $\lambda^{(i)}$ and $\bar{\lambda}^{(j)}$ for certain values of i and j that depend on a . To actually name i and j precisely requires separating various cases for a , and it is convenient to not always have to do this. We will write λ° and $\bar{\lambda}^\circ$ as abbreviations for our generators, but where we have not bothered to write the exact number in the exponent (it is uniquely specified, so we can be incautious about this). We also write ν° and $\bar{\nu}^\circ$ for our generators in $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$.

Theorem 6.25. Consider the map $c: \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \rightarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ and the map $r_{\mathbb{R}}$ going in the opposite direction. Assume that n is congruent to 6, 7, or 8 modulo 8.

- (a) Suppose a is even, so that both the groups are torsion.
 - (i) If $a \equiv 6, 8 \pmod{8}$ then c is an isomorphism, $c(\lambda^\circ) = \nu^\circ$, and $r_{\mathbb{R}}(\nu^\circ) = 2\lambda^\circ$.
 - (ii) If $a \equiv 2, 4 \pmod{8}$ then c is an injection with cokernel $\mathbb{Z}/2$. One has $c(\lambda^\circ) = -2\nu^\circ$ and $r_{\mathbb{R}}(\nu^\circ) = -\lambda^\circ$.
- (b) Suppose that $a = 4t - 1$. Here both the domain and target of c have copies of $\mathbb{Z}$ inside them.
 - (i) If $a \equiv 7 \pmod{8}$ (i.e., t is even) then c is an isomorphism.
 - (ii) If $a \equiv 3 \pmod{8}$ (i.e., t is odd) then c is a monomorphism, and the cokernel is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
- (c) Suppose that $a = 4t + 1$. In this case c maps its domain isomorphically onto the torsion subgroup of the target of c . One has $c(\lambda^\circ) = \nu^\circ$ and $r_{\mathbb{R}}(\nu^\circ) = 2\lambda^\circ$.

6.26. An extended example. Let us demonstrate much of what we have learned by looking at a specific example. The Atiyah-Hirzebruch spectral sequence for computing $\widetilde{KO}(\mathbb{R}P^{10})$ gives one $\mathbb{Z}/2$ for every dimension from 1 through 10 that is congruent to 0, 1, 2, or 4 modulo 8. These are the dimensions 1, 2, 4, 8, 9, and 10—so we have six $\mathbb{Z}/2$'s, and $\widetilde{KO}(\mathbb{R}P^{10}) \cong \mathbb{Z}/(2^6)$. In comparison, $\widetilde{K}(\mathbb{R}P^{10})$ is just $\mathbb{Z}/(2^5)$ (as $5 = \frac{10}{2}$).

There is a visual way of representing this information that is useful, especially when it comes to the stunted projective spaces. Draw a cell diagram for $\mathbb{R}P^{10}$, leaving out the 0-cell. For $\widetilde{KO}(\mathbb{R}P^{10})$ discard all cells except the ones in dimensions congruent to 0, 1, 2, or 4 modulo 8; then label the remaining cells with ascending powers of λ . For $\widetilde{K}(\mathbb{R}P^{10})$ discard all the odd-dimensional cells and label the remaining ones with ascending powers of ν . Always remembering that $\lambda^2 = -2\lambda$ (and $\nu^2 = -2\nu$), the cells now represent the associated graded of $\widetilde{KO}(\mathbb{R}P^{10})$ (or $\widetilde{K}(\mathbb{R}P^{10})$) with respect to the 2-adic filtration. The picture below also shows the complexification map $c: \widetilde{KO}(\mathbb{R}P^{10}) \rightarrow \widetilde{K}(\mathbb{R}P^{10})$. Recall that this is a ring map and sends λ to ν :



We see in this case that $c: \widetilde{KO}(\mathbb{R}P^{10}) \rightarrow \widetilde{K}(\mathbb{R}P^{10})$ is surjective with kernel $\mathbb{Z}/2$ (generated by $\lambda^6 = -32\lambda$). One has $r_{\mathbb{R}}(\nu) = r_{\mathbb{R}}(c(\lambda)) = 2\lambda$, and more generally

$$r_{\mathbb{R}}(\nu^k) = r_{\mathbb{R}}((-2)^{k-1}\nu) = (-2)^{k-1}r_{\mathbb{R}}(\nu) = (-2)^{k-1} \cdot 2\lambda = -(-2)^k\lambda = -\lambda^{k+1}.$$

Next let us consider the K -groups of $\mathbb{R}P^{10}/\mathbb{R}P^4$, referring to the diagram

$$\begin{array}{ccc} \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) & \xrightarrow{\quad} & \widetilde{KO}(\mathbb{R}P^{10}) \\ \downarrow c & & \downarrow c \\ \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) & \xrightarrow{\quad} & \widetilde{K}(\mathbb{R}P^{10}). \end{array}$$

In relation to our cell-diagrams above, the K -groups of $\mathbb{R}P^{10}/\mathbb{R}P^4$ are obtained by throwing away the bottom four cells. We obtain the picture

$$\begin{array}{ccc} \circ \lambda^{(6)} & \nearrow & \circ \nu^{(5)} \\ \circ \lambda^{(5)} & \xrightarrow{\quad} & \circ \nu^{(4)} \\ \circ \lambda^{(4)} & \xrightarrow{\quad} & \circ \nu^{(4)} \\ \circ & & \circ \\ \circ & & \circ \\ \circ & & \circ \\ \circ & & \circ \end{array}$$

$$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4)$$

This picture tells us that $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \cong \mathbb{Z}/(2^3)$, generated by $\lambda^{(4)}$, and also $\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) \cong \mathbb{Z}/(2^3)$ with generator $\nu^{(3)}$. These each embed into the respective K -group of $\mathbb{R}P^{10}$. The complexification map therefore sends $\lambda^{(4)}$ to $\nu^{(4)}$ and $\lambda^{(5)}$ to $\nu^{(5)}$, and we find that this map has both kernel and cokernel isomorphic to $\mathbb{Z}/2$.

The situation is a little different if we consider the K -groups of $\mathbb{R}P^{10}/\mathbb{R}P^3$. Here the bottom cell of $\mathbb{R}P^{10}/\mathbb{R}P^3$ gives rise to a $\mathbb{Z}$ in singular cohomology, and a corresponding $\mathbb{Z}$ in the K -groups. The picture becomes as follows:

$$\begin{array}{ccc} \circ \lambda^{(6)} & \nearrow & \circ \nu^{(5)} \\ \circ \lambda^{(5)} & \xrightarrow{\quad} & \circ \nu^{(4)} \\ \circ \lambda^{(4)} & \xrightarrow{\quad} & \circ \nu^{(4)} \\ \circ & & \circ \\ \circ & & \circ \\ \circ & & \circ \\ \bullet \bar{\lambda}^{(3)} & \xrightarrow{-2} & \bullet \bar{\nu}^{(2)} \end{array}$$

$$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3)$$

Here the black dots represent copies of $\mathbb{Z}$, so that $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ with the two summands generated by $\bar{\lambda}^{(3)}$ and $\lambda^{(4)}$, respectively. Likewise, $\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ with the two summands generated by $\bar{\nu}^{(2)}$ and $\nu^{(3)}$. The various maps

$$\begin{aligned} \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) &\longrightarrow \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \longrightarrow \widetilde{KO}(\mathbb{R}P^{10}), \text{ and} \\ \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) &\longrightarrow \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3) \longrightarrow \widetilde{K}(\mathbb{R}P^{10}) \end{aligned}$$

are the evident ones suggested by the diagrams. The only subtlety lies in determining the complexification map c . Of course $c(\lambda^{(i)}) = \nu^{(i)}$ for $i = 4, 5$, as this is forced by that happens on the subgroup $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \subseteq \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3)$. To compute $c(\bar{\lambda}^{(3)})$ we must remember from (6.21) that $\bar{\lambda}^{(3)}$ is defined by the equation

$$\bar{\lambda}^{(3)} = -r_{\mathbb{R}}(\bar{\nu}^{(2)})$$

(note that $3 = 4t - 1$ where $t = 1$, and so we are in the case where t is odd; in the case where t is even the definition of the $\bar{\lambda}$ classes is different). So we obtain

$$c(\bar{\lambda}^{(3)}) = -c(r_{\mathbb{R}}(\nu^{(2)})) = -(1 + \psi^{-1})(\bar{\nu}^{(2)}) = -[\bar{\nu}^{(2)} + \bar{\nu}^{(2)}] = -2\bar{\nu}^{(2)}.$$

Here we have used Theorem 6.13(c) for evaluating $\psi^{-1}(\bar{\nu}^{(2)})$. Note that the pull-back map induced by $\mathbb{R}P^{10} \rightarrow \mathbb{R}P^{10}/\mathbb{R}P^3$ sends $\bar{\lambda}^{(3)}$ to λ^3 and sends $-2\bar{\nu}^{(2)}$ to $-2\nu^2 = \nu^3$; hence the above formula is consistent with our previous computation of $c: \widetilde{KO}(\mathbb{R}P^{10}) \rightarrow \widetilde{K}(\mathbb{R}P^{10})$.

6.27. The proofs. We now give proofs for all of the results previously stated in this section.

Proof of Theorem 6.9. This is straightforward, and left to the reader. $\square$

Proof of Theorem 6.11. There is no room for differentials in the Atiyah-Hirzebruch spectral sequence for $K^*(\mathbb{R}P^n)$, so it collapses at E_2 . Part (b) follows immediately. It is also a direct consequence that $\widetilde{K}^0(\mathbb{R}P^n)$ is an abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$. It remains to solve the extension problems to determine precisely which group it is.

Observe that $L^2 = 1$, hence $(cL)^2 = c(L^2) = c(1) = 1$. So

$$\nu^2 = (cL - 1)^2 = (cL)^2 - 2(cL) + 1 = 2(1 - c(L)) = -2\nu.$$

Note that an immediate consequence is $\nu^t = (-2)^{t-1}\nu$.

Let $F^i = \ker(\widetilde{K}^0(\mathbb{R}P^n) \rightarrow \widetilde{K}^0(\mathbb{R}P^{i-1}))$. So

$$\widetilde{K}^0(\mathbb{R}P^n) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots \supseteq F^{n+1} = 0.$$

The quotients F^i/F^{i+1} are the groups in the E_∞ term of the spectral sequence, and so are

$$F^i/F^{i+1} = \begin{cases} \mathbb{Z}/2 & 0 < i \leq 2\lfloor \frac{n}{2} \rfloor \text{ and } i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

So $F^0 = F^1 = F^2$ and $F^2/F^3 \cong \mathbb{Z}/2$. The element ν generates F^2/F^3 : we know this by naturality of the spectral sequence, applied to the map $j: \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. The element μ generates F^2/F^3 for $\widetilde{K}^0(\mathbb{C}P^n)$, and $H^2(\mathbb{C}P^n) \rightarrow H^2(\mathbb{R}P^n)$ is the projection $\mathbb{Z} \rightarrow \mathbb{Z}/2$: the image of a generator is another generator. So $j^*\mu$ generates F^2/F^3 for $\widetilde{K}^0(\mathbb{R}P^n)$, and of course $\nu = j^*\mu$.

The multiplicativity of the spectral sequence then gives us that ν^2 generates F^4/F^5 , and in general ν^j generates F^{2j}/F^{2j+1} for $j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. In particular, ν^j is not equal to zero for j in this range. But $\nu^j = (-2)^{j-1}\nu$, so $2^{\lfloor \frac{n}{2} \rfloor - 1}\nu \neq 0$. This proves that the only possibility for $\widetilde{K}^0(\mathbb{R}P^n)$ is $\mathbb{Z}/2^{\lfloor \frac{n}{2} \rfloor}$, and that ν is a generator.

Note that we then have $0 = (-2)^{\lfloor \frac{n}{2} \rfloor}\nu = \nu^{\lfloor \frac{n}{2} \rfloor + 1}$.

For part (c) we just observe that $\psi^k(cL) = (cL)^k = c(L^k)$ and this equals 1 if k is even, and cL if k is odd. $\square$

Proof of Theorem 6.13. Parts (a) and (b) are straightforward. In each case one writes down the evident long exact sequence and quickly sees that the given sequence is short exact (use that $K^1(\mathbb{R}P^n/\mathbb{R}P^{2t})$ is torsion-free, by the Atiyah-Hirzebruch spectral sequence). The only slight subtlety is seeing in the case $a = 2t - 1$ that $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \rightarrow \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1})$ is surjective, but this was explained when we constructed the element $\bar{\nu}^{(t)}$ (which maps to a generator in the target group).

For part (c), the action of ψ^k on $\nu^{(t+1)}$ is determined by the corresponding action in $\tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t})$; so there is nothing to prove here. The action on $\bar{\nu}^{(t)}$ is more interesting. We can, of course, write

$$(6.28) \quad \psi^k(\bar{\nu}^{(t)}) = A\bar{\nu}^{(t)} + B\nu^{(t+1)}$$

where A is a unique integer and B is unique modulo 2^f . Applying the map $i^*: \tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \rightarrow \tilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1})$ kills $\nu^{(t+1)}$ and sends $\bar{\nu}^{(t)}$ to a generator g , so this equation becomes $\psi^k(g) = Ag$. But we already know that ψ^k acts on such a generator by k^t , so $A = k^t$.

Next we apply the map $\pi^*: \tilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t}) \rightarrow \tilde{K}^0(\mathbb{R}P^n)$ to equation (6.28). The map π^* sends $\bar{\nu}^{(t)}$ to ν^t and $\nu^{(t+1)}$ to ν^{t+1} , so using $A = k^t$ we obtain

$$\psi^k(\nu^t) = k^t\nu^t + B\nu^{t+1}$$

in $\tilde{K}^0(\mathbb{R}P^n)$. Now use that $\nu^{t+1} = -2\nu^t$ together with the action of ψ^k on $\tilde{K}^0(\mathbb{R}P^n)$ as given by Theorem 6.11(c). We get

$$(k^t - 2B)\nu^t = \psi^k(\nu^t) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \nu^t & \text{if } k \text{ is odd.} \end{cases}$$

The group $\tilde{K}^0(\mathbb{R}P^n)$ is $\mathbb{Z}/(2^g)$ with generator ν , and $\nu^t = (-2)^{t-1}\nu$. So the additive order of ν^t is 2^{g-t+1} , or equivalently 2^{f+1} . In the case that k is even it follows that $k^t - 2B$ is a multiple of 2^{f+1} , so that $\frac{k^t}{2} \equiv B \pmod{2^f}$ (recall that B is only well-defined modulo 2^f in the first place).

In the remaining case where k is odd we get $k^t - 2B \equiv 1$ modulo 2^{f+1} . So $\frac{k^t-1}{2} \equiv B$ modulo 2^f , which is what we wanted. $\square$

Proof of Theorem 6.15. In the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(\mathbb{R}P^n)$, the diagonal of the E_2 -term that is relevant to $\widetilde{KO}^0(\mathbb{R}P^n)$ consists of $\varphi(n)$ copies of $\mathbb{Z}/2$. The first concern is to determine if there are any differentials causing some of these copies to disappear by E_∞ , and the second concern is the problem of extensions.

Observe that the complexification map $c: \widetilde{KO}^0(\mathbb{R}P^n) \rightarrow \tilde{K}^0(\mathbb{R}P^n)$ is surjective, because ν generates the target and $\nu = c(L-1)$. So it follows from Theorem 6.11 that at least $\lfloor \frac{n}{2} \rfloor$ among our $\varphi(n)$ copies of $\mathbb{Z}/2$ must survive the spectral sequence.

The trick now is to not consider one n at a time, but rather to consider them all at once. When n is congruent to 6, 7, or 8 modulo 8 then we know $\varphi(n) = \lfloor \frac{n}{2} \rfloor$, and so here it must be that all the $\mathbb{Z}/2$'s along the main diagonal survive. That is, all differentials entering or exiting the main diagonal are zero. But then by naturality of the spectral sequence this is true for all n . We conclude that the order of $\widetilde{KO}^0(\mathbb{R}P^n)$ is $2^{\varphi(n)}$, no matter what n is.

When n is congruent to 6, 7, or 8 modulo 8 we now know that the orders of $\widetilde{KO}^0(\mathbb{R}P^n)$ and $\tilde{K}^0(\mathbb{R}P^n)$ are the same. Since the complexification map is surjective, it is therefore an isomorphism. So $\widetilde{KO}^0(\mathbb{R}P^n)$ is cyclic. Since $c(\lambda) = \nu$ it follows that λ is a generator. In particular, λ is a generator for the quotient F^1/F^2 .

But then by naturality of the spectral sequence (and with it, naturality of the filtration F^i) it follows that λ generates F^1/F^2 for *every* value of n . Since $L^2 = 1$ we of course have $\lambda^2 = -2\lambda$. At this point the argument follows the one in the proof of Theorem 6.11 to show that $\widetilde{KO}^0(\mathbb{R}P^n)$ is cyclic, for all values of n .

The computation of the Adams operations again follows from $\psi^k(L) = L^k$, which equals 1 if k is even and L if k is odd. Note that this even works for $k < 0$, since $L \cong L^*$. $\square$

Proof of Theorem 6.18. This was given just prior to the statement of the theorem. $\square$

Proof of Theorem 6.19. For part (a) one examines the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}^*(\mathbb{R}P^n/\mathbb{R}P^a)$. Note that the quotient $\mathbb{R}P^n \rightarrow \mathbb{R}P^n/\mathbb{R}P^a$ induces a map of spectral sequences in the other direction. The diagonal groups in the E_2 -term for $\widetilde{KO}^*(\mathbb{R}P^n/\mathbb{R}P^a)$ are a truncation of the diagonal groups appearing in the E_2 -term for $\widetilde{KO}^*(\mathbb{R}P^n)$. Since there are no entering or exiting differentials (along the diagonal) in the latter case, naturality of the spectral sequence guarantees there are no entering or exiting differentials for $\mathbb{R}P^n/\mathbb{R}P^a$. Passing to E_∞ -terms now, we see that the associated graded groups for $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a)$ are a truncation of the associated graded groups for $\widetilde{KO}(\mathbb{R}P^n)$. Examining the map of filtered groups

$$\begin{array}{ccccccc} \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a) & \longleftarrow & F_1 & \longleftarrow & F_2 & \longleftarrow & \cdots \\ \downarrow \pi^* & & \downarrow & & \downarrow & & \\ \widetilde{KO}(\mathbb{R}P^n) & \longleftarrow & F'_1 & \longleftarrow & F'_2 & \longleftarrow & \cdots \end{array}$$

we now find that $F_k \rightarrow F'_k$ is an isomorphism for $k \geq a+1$ and $F_k/F_{k+1} = 0$ for $k \leq a$. It follows that $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a) \rightarrow \widetilde{KO}(\mathbb{R}P^n)$ is an injection, with image equal to F'_{a+1} . In our analysis of $KO(\mathbb{R}P^n)$ we have already seen from the Atiyah-Hirzebruch spectral sequence that $F'_{a+1} \subseteq \widetilde{KO}(\mathbb{R}P^n)$ is the subgroup generated by λ^{a+1} . Everything else in part (a) is then immediate.

For (b) we only need to prove that $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \rightarrow \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$ is surjective, since the latter group is isomorphic to $\widetilde{KO}^0(S^{4t}) \cong \mathbb{Z}$. Everything else in part (b) is routine. To do this, consider the following diagram:

$$\begin{array}{ccccccc} & & & \mathbb{Z}/2 & & & \\ & & & \parallel & & & \\ \cdots \longleftarrow & \widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \longleftarrow & \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2}) & \xleftarrow{j_1^*} & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2}) & \longleftarrow \cdots \\ & \uparrow & & \uparrow i^* & & \uparrow & \\ \cdots \longleftarrow & \widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \longleftarrow & \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) & \xleftarrow{j_2^*} & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \longleftarrow \cdots \\ & & & \parallel & & & \\ & & & \mathbb{Z} & & & \end{array}$$

The two indicated vertical maps are surjections because they sit inside long exact sequences where the third term is $\widetilde{KO}^0(\mathbb{R}P^{4t-1}/\mathbb{R}P^{4t-2}) = \widetilde{KO}^0(S^{4t-1}) = 0$. The indicated group is $\mathbb{Z}/2$ by part (a) of the theorem, which also yields the diagram

$$\begin{array}{ccc} \widetilde{KO}^0(\mathbb{R}P^{4t}) & \longleftarrow & \widetilde{KO}^0(\mathbb{R}P^n) \\ \uparrow & & \uparrow \\ \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2}) & \xleftarrow{j_1^*} & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2}). \end{array}$$

The top horizontal map is surjective because both groups are generated by λ . The group $\widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2})$ is the subgroup of $\widetilde{KO}^0(\mathbb{R}P^{4t})$ generated by $\lambda^{1+\varphi(4t-2)}$, and $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2})$ is the similarly-described subgroup of $\widetilde{KO}^0(\mathbb{R}P^n)$. It follows at once that j_1^* is surjective.

Returning to the earlier diagram, the image of j_2^* is an ideal (r) inside of $\mathbb{Z}$. The fact that $i^*j_2^*$ is surjective (readily observed from the diagram) proves that r must be odd. But the quotient $\mathbb{Z}/r$ will inject into $\widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t})$, by the long exact sequence. The Atiyah-Hirzebruch spectral sequence shows that this latter group has no odd torsion, because it has a filtration where the quotients are only $\mathbb{Z}$'s and $\mathbb{Z}/2$'s. So the conclusion is that $r = 1$, hence j_2^* is surjective. $\square$

Proof of Theorem 6.24. The evaluation of $\psi^k(\lambda^{(f+1)})$ is immediate using naturality and Theorem 6.15. For the evaluation of $\psi^k(\bar{\lambda}^{(f)})$ one can repeat the proof of Theorem 6.13(c) almost verbatim. Alternatively, one can use the result of Theorem 6.13(c) together with the complexification map $c: \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \rightarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$, which is a monomorphism for n congruent to 6, 7, or 8 modulo 8; the result for other values of n can then be deduced by naturality. $\square$

Proof of Theorem 6.25. For part (a) we consider the two short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \longrightarrow & \widetilde{KO}^0(\mathbb{R}P^n) & \longrightarrow & \widetilde{KO}^0(\mathbb{R}P^a) \longrightarrow 0 \\ & & \downarrow c & & \downarrow c \cong & & \downarrow c \\ 0 & \longrightarrow & \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \longrightarrow & \widetilde{K}^0(\mathbb{R}P^n) & \longrightarrow & \widetilde{K}^0(\mathbb{R}P^a) \longrightarrow 0. \end{array}$$

If a is congruent to 6 or 8 modulo 8 then the right vertical map is an isomorphism, which means the left vertical map is as well. The desired results are immediate.

If a is congruent to 2 or 4 modulo 8 then the right vertical map is a surjection with kernel $\mathbb{Z}/2$. It follows from the zig-zag lemma that the left vertical map is an injection with cokernel $\mathbb{Z}/2$. The generator for the domain is $\lambda^{(\varphi(a)+1)}$, and we are in the case where $\varphi(a) = \lfloor \frac{a}{2} \rfloor + 1$. So c maps this generator to $\nu^{(\lfloor \frac{a}{2} \rfloor + 2)} = -2\nu^{(\lfloor \frac{a}{2} \rfloor + 1)}$. The statement $r(\nu^\circ) = -\lambda^\circ$ then follows using that $rc = 2$.

For (b) we look at the short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \longrightarrow & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \longrightarrow & \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \longrightarrow 0 \\ & & \downarrow c & & \downarrow c & & \downarrow c \\ 0 & \longrightarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \longrightarrow 0. \end{array}$$

When t is even the right vertical map is an isomorphism by Theorem ??, and the left vertical map is an isomorphism by (a). So the middle vertical map is also an isomorphism.

When t is odd the right vertical map is an injection with cokernel $\mathbb{Z}/2$, by Theorem ??. The left vertical map is an injection with cokernel $\mathbb{Z}/2$ by part (a). So by the zig-zag lemma the middle vertical map is also an injection, and its cokernel is either $(\mathbb{Z}/2)^2$ or $\mathbb{Z}/4$. The element $\bar{\nu}^{(2t)}$ maps to a generator for the right bottom group $\widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$. If we verify that $2\bar{\nu}^{(2t)} = 0$ in the cokernel of c then we

will have proven that this cokernel is $(\mathbb{Z}/2)^2$, not $\mathbb{Z}/4$. But note that

$$cr_{\mathbb{R}}(\bar{\nu}^{(2t)}) = (1 + \psi^{-1})(\bar{\nu}^{(2t)}) = 2\bar{\nu}^{(2t)}$$

where in the last equality we have used the formula for $\psi^{-1}(\bar{\nu}^{(2t)})$ from Theorem 6.13(c). So indeed, $2\bar{\nu}^{(2t)}$ vanishes in the cokernel of c .

For (c) we consider the following:

$$\begin{array}{ccccccc} 0 \rightarrow \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+2}) & \xrightarrow{p^*} & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \rightarrow & \widetilde{KO}(\mathbb{R}P^{4t+2}/\mathbb{R}P^{4t+1}) & \rightarrow & 0 \\ & \downarrow c & \downarrow c & & \downarrow c & & \\ 0 \rightarrow \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+2}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^{4t+2}/\mathbb{R}P^{4t+1}) & \longrightarrow & 0. \end{array}$$

We have not yet discussed exactness of the top row. On the left end this follows because $\widetilde{KO}^{-1}(S^{4t+2}) = 0$. By our computations, the cokernel of p^* is a group of order $2^{\varphi(4t+2)-\varphi(4t+1)}$; but by inspection this number is equal to 1 when t is odd and 2 when t is even. As this is the same as the order of the group $\widetilde{KO}(S^{4t+2})$, this justifies exactness on the right.

If t is odd then the left vertical map is an isomorphism by (a), and the horizontal map p^* is an isomorphism; the desired claims follow at once. When t is even we must argue more carefully. Note that the image of $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+2})$ inside $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is precisely the torsion subgroup; let us call this image T . Since the group $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is torsion, its image under c is also torsion; so this image is a subgroup of T . Moreover, $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is generated by $\lambda^{(\varphi(4t+1)+1)}$, and one readily computes that $\varphi(4t+1) = 2t+1$. Consider the square

$$\begin{array}{ccc} \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \xrightarrow{\quad} & \widetilde{KO}(\mathbb{R}P^n) \\ \downarrow c & & \downarrow c \cong \\ \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n). \end{array}$$

The left vertical map clearly must be injective. Compute that $\pi^*(c(\lambda^{(2t+2)})) = c(\lambda^{2t+2}) = \nu^{2t+2}$. There is only one element of the torsion subgroup of $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ that pulls back to ν^{2t+2} , namely $\nu^{(2t+2)}$. It follows that $c(\lambda^{(2t+2)}) = \nu^{(2t+2)}$. But $\nu^{(2t+2)}$ generates T , so c maps $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ isomorphically onto T . $\square$

◦ Exercises ◦

Exercise 6.29. Compute the action of ψ^{-1} on $K^0(\mathbb{C}P^n)$ as follows. Make sense of the relation

$$\eta^* = \frac{1}{\eta} = \frac{1}{1-(1-\eta)} = 1 + (1-\eta) + (1-\eta)^2 + \cdots + (1-\eta)^n$$

and then use this to compute $\psi^{-1}(\mu^k)$ for $1 \leq k \leq n$. Prove that ψ^{-1} acts on $\widetilde{K}^0(S^{2n})$ as multiplication by $(-1)^n$.

Exercise 6.30. The cohomology theory $H\mathbb{Z}/2$ is complex-oriented (???). Compare $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$ to $H^*(pt; \mathbb{Z}/2)[[x]]/(x +_F x)$. Are they equal? See Remark ???. How do things work out for $H\mathbb{Z}$ instead?

Exercise 6.31. Show that $\tilde{K}^0(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong (\mathbb{Z}/2)^3$ and $\tilde{K}^1(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \mathbb{Z}/2$. Here are three things that might be helpful:

- (1) The Atiyah-Hirzebruch spectral sequence;
- (2) Whenever X and Y are pointed spaces one has the cofiber sequence $X \vee Y \hookrightarrow X \times Y \rightarrow X \wedge Y$, and this is split after one suspension;
- (3) Smashing the cofiber sequence $S^1 \xrightarrow{2} S^1 \rightarrow \mathbb{R}P^2$ with $\mathbb{R}P^2$ gives the cofiber sequence $\Sigma \mathbb{R}P^2 \xrightarrow{2} \Sigma \mathbb{R}P^2 \rightarrow \mathbb{R}P^2 \wedge \mathbb{R}P^2$.

Use a combination of these to solve the problem. Along the way, show that $\tilde{K}^0(\mathbb{R}P^2 \wedge \mathbb{R}P^2) \cong \mathbb{Z}/2$.

Bonus question: Identify three generators for $\tilde{K}^0(\mathbb{R}P^2 \times \mathbb{R}P^2)$ in terms of classes of virtual bundles.

Exercise 6.32. Extend the above exercise to show $\tilde{K}^0(\mathbb{R}P^2 \times \mathbb{R}P^4) \cong (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4$ and $\tilde{K}^1(\mathbb{R}P^2 \times \mathbb{R}P^4) \cong \mathbb{Z}/2$. In addition, identify the precise pattern of differentials in the Atiyah-Hirzebruch spectral sequence. [For the latter, one approach is to prove that d_3 on classes of degree two must always vanish.]

7. SOLUTION TO THE VECTOR FIELD PROBLEM

In this section we conclude our story of the vector field problem, following the original paper by Adams [Adams2]. Let us first recall the Hurwitz-Radon function $\rho(n)$: if $n = 2^{4b+a} \cdot (\text{odd})$ then $\rho(n) = 2^a + 8b - 1$. We have seen in Theorem ?? that one can construct $\rho(n)$ independent vector fields on S^{n-1} . The vector field problem will be settled once we prove the following:

Theorem 7.1 (Adams). *There do not exist $\rho(n) + 1$ independent vector fields on S^{n-1} .*

Remark 7.2. There will inevitably come a time when the reader wishes to remember the formula for $\rho(n)$ but cannot immediately look it up. The key facts about the formula are:

- (i) $\rho(n)$ only depends on the power of 2 in the prime factorization of n ;
- (ii) For $a \leq 3$ one has $\rho(2^a) = 2^a - 1$;
- (iii) $\rho(16n) = \rho(n) + 8$.

These facts of course uniquely determine $\rho(n)$. Personally, I find the exact form of (iii) hard to remember when I haven't been working with this stuff for a while. What *is* able to stick in my head is that there are zero vector fields on S^0 , one on S^1 , three on S^3 , seven on S^7 —and then I have to remember that there are only eight on S^{15} . The jump from zero on S^0 to eight on S^{15} is the quasi-periodicity. From this it is easy to recover the formula $\rho(16n) = \rho(n) + 8$, and onward to the general formula for ρ .

The proof of Theorem 7.1 is quite involved—it requires a surprising amount of algebraic topology. KO -theory is usually regarded as the key tool in the proof, but one also needs Steenrod operations, James periodicity, and Atiyah duality in the stable homotopy category. This adds up to a sizable amount of material. We will take a modular approach to things; we start by giving an outline of the proof, and then we will fill in the details one by one.

7.3. Outline of the proof.

Step 1: We have the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv):

- (i) There exist $k - 1$ vector fields on S^{n-1} .
- (ii) There exist $k - 1$ vector fields on S^{un-1} for every $u \geq 1$.
- (iii) The map $\pi_1 : V_k(\mathbb{R}^{un}) \rightarrow S^{un-1}$ has a section, for every $u \geq 1$.
- (iv) The map $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1} \rightarrow S^{un-1}$ (projection onto the top cell) has a section in the homotopy category, for every u such that $un + 2 > 2k$. That is to say, $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1}$ splits off the top cell in the stable homotopy category.

We have seen these implications back in Section ??, but let us briefly recall why they hold. For (i) $\Rightarrow$ (ii) it is a direct construction: given $k-1$ orthogonal vector fields made from vectors with n coordinates, one can repeat those patterns in successive groups of coordinates to make $k-1$ orthogonal vector fields in un coordinates, for any u . The step (ii) $\Rightarrow$ (iii) is a triviality, essentially just a restatement of the problem. Then for (iii) $\Rightarrow$ (iv) it is because for $a + 2 > 2b$ the space $V_b(\mathbb{R}^a)$ has a cell structure where the a -skeleton is $\mathbb{R}P^{a-1}/\mathbb{R}P^{a-b-1}$ (Proposition ??).

Step 2: Steenrod operations allow one to prove that if $a + 1 = 2^r \cdot (\text{odd})$ then $\mathbb{R}P^a/\mathbb{R}P^{a-b}$ does not split off the top cell for $b > 2^r$.

The proof will be described in detail below, but here is a short summary. The hypothesis says that $a = 2^r(2t + 1) - 1 = 2^{r+1}t + 2^r - 1$, and this guarantees that there is a Sq^{2^r} operation in $H^*(\mathbb{R}P^\infty)$ connecting the class in degree $2^{r+1}t - 1$ to the class in degree a . If $b > 2^r$ then that Sq^{2^r} operation is still present in $\mathbb{R}P^a/\mathbb{R}P^{a-b}$, and this obstructs the splitting off of the top cell.

Step 3: Putting steps 1 and 2 together, we have that if $n = 2^m \cdot (\text{odd})$ then there do not exist 2^m vector fields on S^{n-1} .

For $m \leq 3$ this solves the vector field problem, because in this case $\rho(m) = 2^m - 1$.

Step 4: There are periodicities to the spaces $\mathbb{R}P^a/\mathbb{R}P^{a-b}$. If $L - 1$ has finite order r_b in $\widehat{KO}(\mathbb{R}P^{b-1})$, then

$$\mathbb{R}P^a/\mathbb{R}P^{a-b} \simeq \Sigma^{-sr_b} \left(\mathbb{R}P^{a+sr_b}/\mathbb{R}P^{a+sr_b-b} \right)$$

for every $s \geq 1$, where the homotopy equivalence is in the stable homotopy category. This is called *James periodicity*. The proof was given in Proposition ??.

Step 5: Now things get a bit more sophisticated. Atiyah proved that if M is any compact manifold and $E \rightarrow M$ is a real vector bundle, then $\text{Th}(E \rightarrow M)$ is Spanier-Whitehead dual to $\text{Th}(-E - T_M \rightarrow M)$, where T_M is the tangent bundle to M (for Thom spaces of negative bundles, see Section ??). This is called ‘Atiyah duality’.

Recall from Example ?? that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \cong \text{Th}((n-k)L \rightarrow \mathbb{R}P^{k-1})$. Also, for $M = \mathbb{R}P^{k-1}$ one has $T_M = kL - 1$ in $KO^0(M)$ (Example ??). So Atiyah Duality

gives that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ is Spanier-Whitehead dual to

$$\mathrm{Th}\left(\begin{array}{c} -(n-k)L-T \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) = \mathrm{Th}\left(\begin{array}{c} -(n-k)L-kL+1 \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) = \mathrm{Th}\left(\begin{array}{c} -nL+1 \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) = \Sigma \mathrm{Th}\left(\begin{array}{c} -nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right).$$

If r_k is the order of $L-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$ then the last spectrum may be interpreted as

$$\begin{aligned} \Sigma \mathrm{Th}\left(\begin{array}{c} -nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) &\simeq \Sigma \Sigma^{-sr_k} \mathrm{Th}\left(\begin{array}{c} sr_k - nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) \simeq \Sigma^{1-sr_k} \mathrm{Th}\left(\begin{array}{c} sr_k L - nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) \\ &= \Sigma^{1-sr_k} \mathrm{Th}\left(\begin{array}{c} (sr_k - n)L \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) \\ &\simeq \Sigma^{1-sr_k} \left[\mathbb{R}P^{sr_k - n + k - 1} / \mathbb{R}P^{sr_k - n - 1} \right], \end{aligned}$$

where s is any integer sufficiently large so that $sr_k - n - 1 \geq 0$ and the last line is again by Example ???. We have therefore proven:

The Spanier-Whitehead dual of $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ is (up to suspension) $\mathbb{R}P^{sr_k - n + k - 1} / \mathbb{R}P^{sr_k - n - 1}$, where s is any integer such that $sr_k - n - 1 \geq 0$.

Step 6: A direct consequence of the previous statement is that

$$\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \text{ splits off its top cell (stably)}$$

if and only if

$$\mathbb{R}P^{sr_k - n + k - 1} / \mathbb{R}P^{sr_k - n - 1} \text{ splits off its bottom cell (stably),}$$

where $s \gg 0$ as above.

Step 7: Using step 6 we can add a condition onto the list of implications from step 1. Namely, we have (iv) $\Rightarrow$ (v) where the latter is

$$(v) \quad \mathbb{R}P^{sr_k - un + k - 1} / \mathbb{R}P^{sr_k - un - 1} \text{ splits off its bottom cell (stably), for } u \gg 0 \text{ and } s \gg 0.$$

The Freudenthal suspension theorem says that $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$ is a bijection if Y is r -connected and $\dim X \leq 2r$. By iterating, one finds that under these hypotheses $[X, Y]$ agrees with the set of stable maps $[\Sigma^\infty, \Sigma^\infty Y]$. In particular, $[\mathbb{R}P^a / \mathbb{R}P^{b-1}, S^b]$ is “stable” if $a \leq 2(b-1)$. So if $k+1 \leq sr_k - un$ then $\mathbb{R}P^{sr_k - un + k - 1} / \mathbb{R}P^{sr_k - un - 1}$ splitting off its bottom cell stably implies that it splits off its bottom cell unstably. Since the inequality will be satisfied for s large enough, we have (v) $\Rightarrow$ (vi) where the latter is

$$(vi) \quad \mathbb{R}P^{sr_k - un + k - 1} / \mathbb{R}P^{sr_k - un - 1} \text{ splits off its bottom cell (unstably), for } u \gg 0 \text{ and } s \gg 0. \text{ (Note that the bound on } s \text{ will depend on } u.)$$

Step 8: Adams calculated $\widetilde{KO}(\mathbb{R}P^a)$ for all a , together with the Adams operations on these groups. He used this knowledge, together with Step 2 above, to prove the following:

For any $m \geq 0$, $\mathbb{R}P^{m+\rho(m)+1} / \mathbb{R}P^{m-1}$ does not split off its bottom cell unstably.

Step 9: Completion of the proof.

Proof of Theorem 7.1. Suppose there are $k-1$ vector fields on S^{n-1} . Then by Step 7 the space $\mathbb{R}P^{sr_k-un+k-1}/\mathbb{R}P^{sr_k-un-1}$ stably splits off its bottom cell for any u and s such that $un+2 > 2k$ and $sr_k-un-1 \geq 0$. By choosing s large enough so that $sr_k-un-1 \geq k$ we can in fact get an unstable splitting. Choose u to be odd, and choose s to be a multiple of $2n$. Set $m = sr_k-un$, and note that m is an odd multiple of n ; consequently, we have $\rho(m) = \rho(n)$.

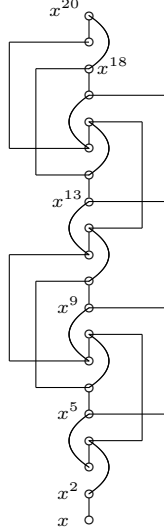
We have that $\mathbb{R}P^{m+k-1}/\mathbb{R}P^{m-1}$ splits off its bottom cell. Therefore $\mathbb{R}P^{m+u}/\mathbb{R}P^{m-1}$ splits off its bottom cell for any $u \leq k-1$. By Step 8 this implies that $k-1 < \rho(m)+1$, or equivalently that $k-1 \leq \rho(m) = \rho(n)$. So there are at most $\rho(n)$ vector fields on S^{n-1} . $\square$

The missing pieces from our outline are: **Step 2, Step 5, and Step 8**. We now fill in the details for these steps, one by one—but not quite in the above order. We save Atiyah duality for last, only because the other two pieces belong more to the same theme.

7.4. Steenrod operations and stunted projective spaces (steps 2 and 3).

Let $x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ be the nonzero element. The Steenrod operations on $\mathbb{R}P^\infty$ are easily computed from the facts $Sq^1(x) = x^2$, $Sq^i(x) = 0$ for $i > 1$, and the Cartan formula. We leave this as an exercise for the reader. Figure 7.4.1 shows the Sq^1 , Sq^2 , and Sq^4 operations on $H^*(\mathbb{R}P^{20}; \mathbb{Z}/2)$. The Sq^1 operations are depicted as vertical lines, the Sq^2 operations as curved lines, and the Sq^4 s as “offset vertical lines”. For example, one can read off of the diagram that $Sq^1(x^5) = x^6$, $Sq^2(x^{10}) = x^{12}$, and $Sq^4(x^{10}) = 0$ (in the latter case because the diagram does not have a Sq^4 emanating from the x^{10} class).

FIGURE 7.4.1. Steenrod algebra structure on $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$



The pattern of Sq^{2^r} operations in $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is very simple. The first Sq^{2^r} operation occurs on x^{2^r} and thereafter they follow the pattern of “ 2^r on, 2^r off”: for example, starting at x^4 and moving up the diagram we have four Sq^4 operations that are nonzero, and then the next four are all zero. This is captured by the formula

$$\text{Sq}^{2^r}(x^a) = \begin{cases} x^{a+2^r} & \text{if } a \geq 2^r \text{ and } a \equiv 2^r, 2^r + 1, \dots, 2^{r+1} - 1 \pmod{2^{r+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Of course for $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ the formulas must be truncated to account for the fact that classes above dimension n are not present.

To see how Steenrod operations give obstructions to stable splittings, consider $\mathbb{R}P^9/\mathbb{R}P^6$. Its cohomology has a Sq^2 connecting the class in degree 7 to the class in degree 9. Suppose the projection onto the top cell $p: \mathbb{R}P^9/\mathbb{R}P^6 \rightarrow S^9$ has a splitting χ in the stable homotopy category. Then the composite

$$H^*(S^9) \xrightarrow{p^*} H^*(\mathbb{R}P^9/\mathbb{R}P^6) \xrightarrow{\chi^*} H^*(S^9)$$

is an isomorphism. Write x_i for the generator in $H^i(\mathbb{R}P^9/\mathbb{R}P^6)$, so that in this notation we have $\text{Sq}^2(x_7) = x_9$. Necessarily we must have $\chi^*(x_7) = 0$, therefore $0 = \text{Sq}^2(\chi^*x_7) = \chi^*(\text{Sq}^2x_7) = \chi^*(x_9)$. But x_9 is in the image of p^* , so this is a contradiction.

Clearly this kind of argument will work for any $\mathbb{R}P^n/\mathbb{R}P^{n-b}$ where we have a nontrivial cohomology operation hitting the top class. Based on this, it is now easy to prove the following:

Proposition 7.5. *Write $n + 1 = 2^s \cdot \text{odd}$. If $\mathbb{R}P^n/\mathbb{R}P^{n-b}$ splits off its top cell stably then $b \leq 2^s$.*

Proof. If n is even then in $H^*(\mathbb{R}P^\infty)$ there is a Sq^1 hitting the class in degree n , and this operation will be present in $H^*(\mathbb{R}P^n/\mathbb{R}P^{n-b})$ as long as $b > 1$. So the top cell can not split off in this case. In other words, if $n + 1 = 2^0 \cdot (\text{odd})$ then splitting of the top cell can only happen if $b \leq 2^0$.

Similarly, if $n = 4e + 1$ then in $H^*(\mathbb{R}P^\infty)$ there is a Sq^2 hitting the class in degree n . This Sq^2 will be present in $H^*(\mathbb{R}P^n/\mathbb{R}P^{n-b})$ as long as $b > 2$, and again we find that under this criterion the top cell can not split off. So $n + 1 = 2^1 \cdot (\text{odd})$ implies splitting of the top cell can only happen if $b \leq 2^1$.

The same style of argument continues. If $n = 2^r e + (2^{r-1} - 1)$ then there is a $\text{Sq}^{2^{r-1}}$ hitting our class in degree n , and this obstructs the splitting of the top cell as long as $b > 2^{r-1}$. Rephrased, this says that if $n + 1 = 2^{r-1}(2e + 1)$ then the splitting does not exist when $b > 2^{r-1}$. Replacing $r - 1$ with s , we have the desired result. $\square$

Corollary 7.6. *If $n = 2^s \cdot \text{odd}$ then there are at most $2^s - 1$ independent vector fields on S^{n-1} .*

Proof. If there are $k - 1$ vector fields on S^{n-1} then $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1}$ splits off its top cell for all $u \gg 0$. Choose a u that is odd, so that $un = 2^s \cdot \text{odd}$. By Proposition 7.5 we conclude that $k \leq 2^s$. $\square$

The upper bounds provided by Corollary 7.6 agree with the Hurwitz-Radon lower bounds when $s \leq 3$. Note that these few cases cover quite a bit more than one first might think. For example, for spheres of dimension less than 50 it follows

that the Hurwitz-Radon number of vector fields is the maximum possible for all but three cases, namely the spheres S^{15} , S^{31} , and S^{47} (multiples of 16 minus one). Corollary 7.6 yields that there exist at most 15 vector fields on S^{15} , 31 on S^{31} , and 15 on S^{47} , whereas the Hurwitz-Radon construction only gives 8 vector fields on S^{15} and S^{47} , and 9 vector fields on S^{31} . This demonstrates that our bounds for the spheres S^{16e-1} are still far away from our goal.

7.7. Adams’s Theorem and KO -theory (Step 8). Next we move to the piece that finally cracked the proof, namely the following theorem of Adams [Adams2]:

Theorem 7.8. *Let $m \geq 1$. Then the space $\mathbb{R}P^{m+\rho(m)+1}/\mathbb{R}P^{m-1}$ does not split off its bottom cell in the unstable homotopy category.*

Before embarking on the proof let’s do a small piece of analysis around $\rho(m)$ that will be useful. Recall that if $m = 2^{4a+b} \cdot \text{odd}$ then $\rho(m) = 8b + 2^a - 1$. Where does this come from? If $\varphi(x)$ is the number of integers s in the range $0 < s \leq x$ such that s is equivalent to 0, 1, 2, or 4 modulo 8, the expression $8b + 2^a - 1$ comes up when we try to invert this function:

Lemma 7.9. *Given a positive integer $Q = 4b + a$, with $0 \leq a \leq 3$, the largest x such that $\varphi(x) = Q$ is $x = 8b + 2^a - 1$.*

Proof. Let “ φ -count” stand for counting the integers that are congruent to 0, 1, 2, or 4 modulo 8. Every cycle of 8 consecutive integers contributes 4 to the φ -count, and so for $\varphi(x) \geq 4b$ we would need $x \geq 8b$. The cases $a = 0, 1, 2, 3$ can now be analyzed by hand: for $a = 0$ we have $x = 8b$; for $a = 1$ we have $x = 8b + 1$; for $a = 2$ we have $x = 8b + 3$; and for $a = 3$ we have $x = 8b + 7$. So in general the largest x such that $\varphi(x) = 4b + a$ is $x = 8b + 2^a - 1$. $\square$

Now we prove Adams’s theorem:

Proof of Theorem 7.8. Write $m = 2^u \cdot \text{odd}$, and consider $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ for $N \geq m$. Note that there is a Sq^{2^u} operation on $H^*(\mathbb{R}P^N/\mathbb{R}P^{m-1}; \mathbb{Z}/2)$ connecting the generator in degree m to the generator in degree $m + 2^u$. This proves that the bottom cell does not split off when $N \geq m + 2^u$. This settles the theorem in the case $u \leq 3$, as here $\rho(m) = 2^u - 1$ and so $m + 2^u = m + \rho(m) + 1$.

We will next do a similar argument—but using KO -theory—in the case $u \geq 4$. Actually, the KO -theory argument only uses $u \geq 3$ so for the sake of pedagogy let us just make this weaker assumption.

Since $m \equiv 0 \pmod{4}$ we know by Theorem 6.19(b) that there is a short exact sequence

$$(7.10) \quad \begin{array}{ccccccc} 0 \leftarrow \widetilde{KO}(\mathbb{R}P^m/\mathbb{R}P^{m-1}) & \leftarrow & \widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^{m-1}) & \leftarrow & \widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^m) & \leftarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ & & \mathbb{Z} & & \mathbb{Z}/(2^f) & & \end{array}$$

where f is a certain integer we will recall in a moment. The sequence is (of course) split-exact as a sequence of abelian groups. Each of the groups has Adams operations on it, and the maps are compatible with these operations. If we let $\mathcal{B} = \mathbb{Z}[\psi^2, \psi^3, \psi^5, \dots]$, then (7.10) is an exact sequence of $\mathcal{B}$ -modules (note that

$\mathcal{B}$ is just the monoid ring $\mathbb{Z}[\mathbb{N}]$ from Section 5). If $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell then this extension is split in the category of $\mathcal{B}$ -modules. Note that it is at this step where it is important that we are working in the unstable homotopy category, since the Adams operations are not stable. We will next algebraically analyze when such a splitting exists.

As a $\mathcal{B}$ -module the group $\widetilde{KO}(\mathbb{R}P^m/\mathbb{R}P^{m-1})$ is $\mathbb{Z}(\frac{m}{2})$, meaning that each ψ^k acts as multiplication by $k^{\frac{m}{2}}$. It will be convenient to set $r = \frac{m}{2}$. Also, we know by Theorem 6.19(a) that on $\widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^m)$ the operation ψ^k acts as zero when k is even, and the identity when k is odd. We also have determined the action of the ψ^k 's on the middle group, but let us ignore this for the moment and consider the situation in a bit more generality.

Let A be any abelian group, and let $A[2]$ be the $\mathcal{B}$ -module whose underlying abelian group is A and where ψ^k acts as zero when k is even, and the identity when k is odd. Consider an extension of $\mathcal{B}$ -modules

$$0 \leftarrow \mathbb{Z}(r) \leftarrow E \leftarrow A[2] \leftarrow 0.$$

Let g be an element of E that maps onto a generator for $\mathbb{Z}(r)$. Then we can write $\psi^k g = k^r g + \alpha_k$ for unique elements $\alpha_k \in A$, and the relations $\psi^k \psi^l = \psi^l \psi^k$ show that we must have $(k^r - \psi^k)\alpha_l = (l^r - \psi^l)\alpha_k$. This is equivalent to

$$(7.11) \quad \begin{cases} k^r \alpha_l = (l^r - 1)\alpha_k & \text{whenever } k \text{ is even and } l \text{ is odd,} \\ k^r \alpha_l = l^r \alpha_k & \text{whenever } k \text{ and } l \text{ are both even,} \\ (k^r - 1)\alpha_l = (l^r - 1)\alpha_k & \text{whenever } k \text{ and } l \text{ are both odd.} \end{cases}$$

So the extension E is determined by the elements $\alpha_k \in A$, for $k \geq 2$, satisfying the above equations.

To analyze when the sequence is split, let $\tilde{g}$ denote the image of g in $\mathbb{Z}(r)$. A splitting would send $\tilde{g}$ to an element $g + a$, for some $a \in A$. Since $\psi^k(\tilde{g}) = k^r \tilde{g}$ we find that

$$k^r(g + a) = \psi^k(g + a) = k^r g + \alpha_k + \begin{cases} 0 & \text{if } k \text{ is even,} \\ a & \text{if } k \text{ is odd.} \end{cases}$$

Rearranging to solve for α_k , we obtain

$$\alpha_k = \begin{cases} k^r a & \text{if } k \text{ is even,} \\ (k^r - 1)a & \text{if } k \text{ is odd.} \end{cases}$$

So these are the properties of an (α_k) sequence that are equivalent to the extension $0 \leftarrow \mathbb{Z}(r) \leftarrow E \leftarrow A[2] \leftarrow 0$ being split. Notice that such sequences are in some sense the “trivial” solutions of the relations (7.11).

We make one more general comment before returning to our specific situation. Let $a \in A$ and consider the sequence defined by

$$(7.12) \quad \alpha_k = \begin{cases} \frac{1}{2}k^r \cdot a & k \text{ even} \\ \frac{1}{2}(k^r - 1) \cdot a & k \text{ odd.} \end{cases}$$

Note that the fractions multiplying a are in fact integers. This sequence defines a valid extension E , and if a is not a multiple of 2 in A then the extension doesn't “look” split. Precisely, if A is torsion-free and $a \notin 2A$, then the extension is clearly nonsplit. We will see that the case where A is torsion is a bit more subtle.

Now let us return to the extension in (7.10). Here $A = \mathbb{Z}/(2^f)$, where $f = \varphi(N) - \varphi(m)$. Since m is a multiple of 8 (and this is the first place where we use this assumption), f also equals $\varphi(N - m)$.

In Theorem 6.24 we previously computed the action of the Adams operations on $\widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^{m-1})$. The corresponding α -sequence is precisely the one given by (7.12), where a is a generator for $\mathbb{Z}/(2^f)$. So the extension is split if and only if there exists a $B \in \mathbb{Z}$ such that

$$\frac{1}{2}k^r \cdot a = k^r \cdot (Ba) \quad (k \text{ even}), \quad \frac{1}{2}(k^r - 1) \cdot a = (k^r - 1) \cdot (Ba) \quad (k \text{ odd})$$

for all k . Phrased differently, these say that

$$2^f \text{ divides } \begin{cases} \frac{1}{2}k^r(1 - 2B) & \text{if } k \text{ is even,} \\ \frac{1}{2}(k^r - 1)(1 - 2B) & \text{if } k \text{ is odd.} \end{cases}$$

Since $1 - 2B$ is always odd, this is equivalent to

$$2^{f+1} \text{ divides } \begin{cases} k^r & k \text{ even,} \\ k^r - 1 & k \text{ odd.} \end{cases}$$

Let $\nu(x)$ denote the 2-adic valuation of the integer x , and let us restate what we have now shown:

If $8|m$ and $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell, then

$$\varphi(N - m) + 1 \leq \min\{r, \nu(3^r - 1), \nu(5^r - 1), \nu(7^r - 1), \nu(9^r - 1), \dots\}$$

where $r = m/2$.

Here we have used $\nu(2^r) = r$ and have also left out $\nu(k^r)$ for even integers $k > 2$, as these numbers are at least r and hence irrelevant for the minimum.

Our next task is to consider the numbers $\nu(3^r - 1)$, and for these we refer to Lemma 7.14 below. Since in our case r is even one has $\nu(3^r - 1) = \nu(r) + 2$. Note that as $r \geq 4$ one has $\nu(r) + 2 \leq r$, and so the first term in the above minimum is irrelevant.

We could proceed to analyze the terms $\nu(k^r - 1)$ for odd $k > 3$, which is not hard, but in fact we have done enough to conclude the proof already. We have shown that

$$(7.13) \quad \begin{aligned} &\text{If } 8|m \text{ and } \mathbb{R}P^N/\mathbb{R}P^{m-1} \text{ splits off its bottom cell, then} \\ &\varphi(N - m) + 1 \leq \nu(r) + 2 = \nu(m) + 1, \text{ or simply } \varphi(N - m) \leq \nu(m). \end{aligned}$$

So our task is to find the largest x for which $\varphi(x) = \nu(m)$. But this is handled by Lemma 7.9: if $\nu(m) = 4b + a$ with $0 \leq a \leq 3$ then the largest such x is $8b + 2^a - 1 = \rho(m)$.

Putting everything together, if $\varphi(N - m) \leq \nu(m)$ then $N - m \leq \rho(m)$. So if $N \geq m + \rho(m) + 1$ then $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ cannot split off its bottom cell. $\square$

Lemma 7.14. *If r is even then $\nu(3^r - 1) = \nu(r) + 2$. If r is odd then $\nu(3^r - 1) = 1$.*

Proof. If r is odd then modulo 4 we have $3^r = (-1)^r = -1$, so $3^r - 1 \equiv 2 \pmod{4}$. This proves that $\nu(3^r - 1) = 1$.

If $r = 2^f \cdot u$ where u is odd, we prove by induction on f that $\nu(3^r - 1) = f + 2$. The base case is $f = 1$, and here we use $3^r - 1 = 3^{2u} - 1 = (3^u - 1)(3^u + 1)$. We know

$\nu(3^u - 1) = 1$ by the preceding paragraph. Modulo 4 one has $3^u + 1 = (-1)^u + 1 = 0$, but modulo 8 one has $3^u + 1 = 4$. So $\nu(3^u + 1) = 2$, which confirms that $\nu(3^r - 1) = 3$.

For the inductive step, if $r = 2^{f+1}u$ where u is odd then write

$$3^r - 1 = (3^{2^f u} - 1)(3^{2^f u} + 1).$$

By induction we know $\nu(3^{2^f u} - 1) = f + 2$. Modulo 4 we have $3^{2^f u} + 1 = ((-1)^{2^f})^u + 1 = 1 + 1 = 2$. So $\nu(3^{2^f u} + 1) = 1$, hence $\nu(3^r - 1) = f + 3$. $\square$

Regarding the number-theoretic lemma above, we again refer the reader to the “Lifting-the-Exponent” results described in Exercise 5.17.

Remark 7.15. It is intriguing that a number-theoretic analysis of $\nu(3^r - 1)$ was the ultimate step in both the Hopf invariant one problem and the vector fields on spheres problem. To my knowledge, there is no a priori reason to suspect any connection between these two problems.

Exercise 7.16. If k is any odd number, prove that $\nu(k^r - 1) \geq \nu(r) + 2$ when r is even. This confirms that the terms for $k > 3$ were irrelevant for the minimum considered in the proof of Theorem 7.8.

Example 7.17. To demonstrate Theorem 7.8, consider $m = 576 = 2^6 \cdot 9$ and refer back to (7.13). Starting strictly above 576, we mark off numbers until we have exceeded a φ -count of $\nu(m) = 6$: at that point we know what we cannot have an unstable splitting of the bottom cell. In the following sequence, the numbers contributing to the φ -count have boxes around them:

$$576 \mid \boxed{577}, \boxed{578}, 579, \boxed{580}, 581, 582, 583, \boxed{584}, \boxed{585}, \boxed{586}, 587, \boxed{588}$$

So $\mathbb{R}P^{588}/\mathbb{R}P^{575}$ does not split off its bottom cell, and consequently the same is true for $\mathbb{R}P^N/\mathbb{R}P^{575}$ whenever $N \geq 588$. Note, of course, that $\rho(576) = 11$ and $588 = 576 + 11 + 1$. The point, however, is that one does not need to remember the awkward formula for $\rho(m)$; the procedure is simply to count past m until the φ -count exceeds $\nu(m)$.

7.18. Atiyah duality (Step 5). This is the final piece. The material in this section will complete our proof of Theorem 7.1.

Consider the space $\mathbb{R}P^9/\mathbb{R}P^4$. Its cohomology is shown below in the diagram on the left:



We obtained the picture on the right by simply turning the left diagram upside down; is this also the cohomology of a space? It is easy to see that the answer is yes: the right diagram is $H^*(\mathbb{R}P^{10}/\mathbb{R}P^5)$. This turns out to be a general phenomenon, first discovered by Atiyah. And the kind of ‘duality’ we are seeing actually takes place at a deeper level than just that of cohomology. It is essentially a geometric duality, taking place inside of the stable homotopy category.

The stable homotopy category is symmetric monoidal: the monoidal product is the smash $E, F \mapsto E \wedge F$, and the unit is the sphere spectrum S . It is also *closed*

symmetric monoidal, meaning that there exist function objects $E, F \mapsto \mathcal{F}(E, F)$ and a natural adjunction

$$\mathrm{Hom}_{\mathrm{Ho}(Sp)}(E, \mathcal{F}(X, Y)) \cong \mathrm{Hom}_{\mathrm{Ho}(Sp)}(E \wedge X, Y).$$

Spanier-Whitehead duality has to do with the functor $X \mapsto DX = \mathcal{F}(X, S)$. This functor preserves cofiber sequences and it sends the n -sphere $S^n = \Sigma^\infty S^n$ to the $(-n)$ -sphere S^{-n} . So if a certain spectrum X is built from cells in dimensions 0 through n , the spectrum DX is built from cells in dimensions $-n$ through 0.

For nice enough spectra X one has the property that $D(D(X)) \simeq X$; such spectra are called **dualizable**. All finite cell complexes are dualizable. One can prove that $H_i(DX) \cong H^{-i}(X)$, and so with field coefficients $H^i(DX; F) \cong H^{-i}(X; F)$. Colloquially speaking, $H^*(DX)$ is obtained by ‘taking $H^*(X)$ and turning it upside down’. This even works with the Steenrod algebra structure if one is a little careful.

The essentials of Spanier-Whitehead duality were known long before the details of the stable homotopy category had all been worked out (particularly the details behind the smash product). Here is the main result for finite complexes:

Proposition 7.19. *Let X be a finite cell complex that is embedded in S^n as a subcomplex (of some chosen cell structure on S^n). Then*

$$\Sigma^{n-1}D(\Sigma^\infty X) \simeq \Sigma^\infty(S^n - X).$$

Example 7.20. Let us check the above proposition in some very easy examples. We use the form $D(X) \simeq \Sigma^{-n+1}(S^n - X)$.

- (a) $X = S^0$. Then $S^n - S^0 \simeq S^{n-1}$, and so we find $D(S^0) \simeq \Sigma^{-n+1}S^{n-1} \simeq S^0$. The 0-sphere is self-dual.
- (b) Let $X = S^{n-1}$, embedded as the equator in S^n . The complement $S^n - X$ is S^0 (up to homotopy), so we have $D(S^{n-1}) \simeq \Sigma^{-n+1}S^0 = S^{-(n-1)}$. Again, this is as expected.
- (c) Let $X = T$ be the torus. The attaching map for the top cell is stably null, so stably we have $X \simeq S^1 \vee S^1 \vee S^2$. The standard embedding $T \hookrightarrow \mathbb{R}^3 \hookrightarrow S^3$ has complement consisting of two solid tori minus their boundaries: so $S^3 - X \simeq S^1 \amalg S^1$. But $\Sigma^\infty(S^1 \amalg S^1) \simeq S^1 \vee S^1 \vee S^0$ (think about suspending the two spaces), so $D(X) \simeq \Sigma^{-2}(S^1 \vee S^1 \vee S^0) \simeq S^{-1} \vee S^{-1} \vee S^{-2}$.

For some classical references on Spanier-Whitehead duality, see [Adams5, Chapter III.5] and [Switzer, Chapter 14].

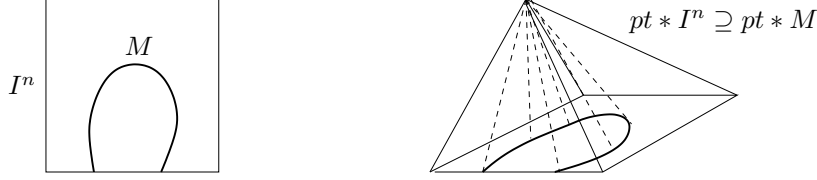
We can now state Atiyah’s main theorem. The proof is taken directly from [Atiyah2]. See Section ?? for a discussion of Thom spaces of virtual bundles.

Theorem 7.21 (Atiyah Duality). *Let M be a compact, smooth manifold.*

- (a) *If M has boundary then $D(M/\partial M) \simeq \mathrm{Th}(-T_M)$ where $T_M \rightarrow M$ is the tangent bundle.*
- (b) *Now assume that M is closed, and let $E \rightarrow M$ be a real vector bundle. Then*

$$D(\mathrm{Th} E) \simeq \mathrm{Th}(-E - T_M).$$

Proof. For (a), embed M into I^n (nicely) in such a way that ∂M maps into $I^{n-1} \times \{0\}$. (See [Atiyah2] for details). Consider the join $pt * I^n$, which is a pyramid; its boundary is homeomorphic to S^n . Refer to the following picture for an example:



Consider the subcomplex

$$X = M \cup (pt * \partial M) \subseteq \partial(pt * I^n) \cong S^n.$$

We have $M/\partial M \simeq M \cup (pt * \partial M)$, and so

$$D(M/\partial M) \simeq D(X) \simeq \Sigma^{-n+1}(S^n - X).$$

Projection away from pt gives a deformation retraction $S^n - X \xrightarrow{\sim} I^n - M$. Next observe $I^n - M \simeq I^n - U$, where U is a tubular neighborhood of M in I^n . Finally, notice that since I^n is contractible we have that $I^n/(I^n - U)$ is a model for $\Sigma(I^n - U)$ (up to homotopy). Putting everything together, we have

$$\begin{aligned} D(M/\partial M) &\simeq \Sigma^{-n+1}(I^n - U) \simeq \Sigma^{-n}(I^n/(I^n - U)) = \Sigma^{-n} \text{Th}(N_{I^n/M}) \\ &\simeq \text{Th}(N_{I^n/M} - \underline{n}). \end{aligned}$$

Now use that $T_M \oplus N_{I^n/M} \cong \underline{n}$.

For (b) use that $\text{Th}(E) \cong Di(E)/S(E)$, where $Di(E)$ is the disk bundle and $S(E)$ is the sphere bundle of E . The disk bundle is a compact manifold with boundary $S(E)$, so by (a) one has

$$D(\text{Th } E) = D(Di(E)/S(E)) \simeq \text{Th}(-T_{Di(E)}).$$

If $\pi: Di(E) \rightarrow M$ is the bundle map then it is easy to see that $T_{Di(E)} \cong \pi^*(E \oplus T_M)$. Since π is a homotopy equivalence,

$$\text{Th}(-T_{Di(E)}) = \text{Th}(-\pi^*(E \oplus T_M)) \simeq \text{Th}(-(E \oplus T_M)).$$

This finishes the proof. $\square$

We next apply what we just learned to stunted projective spaces. Recall from Example ?? that all stunted projective spaces are Thom spaces:

$$\mathbb{R}P^{n-1}/\mathbb{R}P^{k-1} \cong \text{Th} \left(\begin{array}{c} (n-k)L \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array} \right).$$

Recall as well that the tangent bundle to $\mathbb{R}P^{k-1}$ satisfies $T \oplus 1 \cong kL$ (Example ??). Using these two facts, Atiyah Duality now gives that

$$D(\mathbb{R}P^{n-1}/\mathbb{R}P^{k-1}) \simeq \text{Th} \left(\begin{array}{c} -(n-k)L - T \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array} \right) = \text{Th} \left(\begin{array}{c} -nL + 1 \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array} \right) \simeq \Sigma \text{Th} \left(\begin{array}{c} -nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array} \right).$$

Let r_{k-1} be the additive order of $[L] - 1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$. Then $r_{k-1}L \cong \underline{r_{k-1}}$ (stably), and hence for any $s \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Th}\left(\begin{array}{c} -nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) &\simeq \Sigma^{-sr_{k-1}} \mathrm{Th}\left(\begin{array}{c} -nL+sr_{k-1} \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) \simeq \Sigma^{-sr_{k-1}} \mathrm{Th}\left(\begin{array}{c} -nL+sr_{k-1}L \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) \\ &\simeq \Sigma^{-sr_{k-1}} \mathrm{Th}\left(\begin{array}{c} (sr_{k-1}-n)L \\ \downarrow \\ \mathbb{R}P^{k-1} \end{array}\right) \\ &\simeq \Sigma^{-sr_{k-1}} \left[\mathbb{R}P^{sr_{k-1}-n+k-1} / \mathbb{R}P^{sr_{k-1}-n-1} \right]. \end{aligned}$$

In the last line we imagine s chosen to be large enough so that $sr_{k-1} - n - 1 \geq 0$. Putting everything together, we have proven the following:

Corollary 7.22. *Let r_{k-1} be the additive order of $[L] - 1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$. Then there is a stable homotopy equivalence*

$$D(\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}) \simeq \Sigma^{1-sr_{k-1}} \left[\mathbb{R}P^{sr_{k-1}-n+k-1} / \mathbb{R}P^{sr_{k-1}-n-1} \right]$$

where s is any integer such that $sr_{k-1} - n - 1 \geq 0$.

Example 7.23. Let us consider the Spanier-Whitehead dual of $\mathbb{R}P^9/\mathbb{R}P^4$, as in the beginning of this section. Relative to our above discussion, $n = 10$ and $k = 5$. By Theorem 6.15 we know $\widetilde{KO}(\mathbb{R}P^4) \cong \mathbb{Z}/8$, so the order of $[L] - 1$ is 8. The above corollary gives

$$D(\mathbb{R}P^9/\mathbb{R}P^4) \simeq \Sigma^{1-8s} \left[\mathbb{R}P^{8s-6} / \mathbb{R}P^{8s-11} \right]$$

for any s where the right-hand-side makes sense. The smallest choice is $s = 2$, giving $D(\mathbb{R}P^9/\mathbb{R}P^4) \simeq \Sigma^{-15}(\mathbb{R}P^{10}/\mathbb{R}P^5)$.

7.24. Summary. Quite a bit happened in this section. Make sure you understand the following points:

- The existence of k vector fields on S^{n-1} implies that a certain infinite family of stunted projective space splits off their top cell stably.
- By Atiyah duality, this implies that a related infinite family of stunted projective spaces split off their bottom cell stably.
- The Freudenthal suspension theorem shows that “large enough” members of this family will split off their bottom cell unstably.
- Such an unstable splitting would give a split-exact sequence of KO -groups with Adams operations, where one of the groups is cyclic of order 2^e where e is a φ -number.
- By an algebraic analysis, Adams showed that such an extension can only exist if a certain number-theoretic condition involving the φ -numbers is satisfied. Analysis of that condition leads to the resolution of the problem.

8. THE IMMERSION PROBLEM FOR $\mathbb{R}P^n$

Let M be a compact, n -dimensional, smooth manifold. It is a classical theorem of Whitney from the 1940s that M can be immersed in $\mathbb{R}^{2n-1}$ and embedded into $\mathbb{R}^{2n}$ [Whitney2, Whitney3].¹ An even deeper result, proved by R. Cohen [CohenR]

¹This is not an easy result! Although there are modern textbook proofs for Whitney’s earlier results where the dimension on the Euclidean spaces is increased by one [GP], [Hartshorne], there are not many modern sources for Whitney’s stronger version.

in 1985, says that M can be immersed in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the binary expansion of n . As a general result this is known to be the best possible, but for specific choices of M one could conceivably do better. Define the **immersion dimension** (resp. the **embedding dimension**) of M to be the smallest k such that M immerses (resp., embeds) into $\mathbb{R}^k$.

In general, determining the immersion and embedding dimensions of a given manifold seem to be difficult problems. Over the last 60+ years they have been extensively studied, particularly for the manifold $\mathbb{R}P^n$. The problem tends to involve two distinct components. As one aspect, clever geometric constructions are used to produce immersions (or embeddings) and therefore upper bounds on the immersion dimension. For lower bounds one must prove *non-immersion* results, and this is usually done by making use of some sort of homotopical invariants. Over the years the problem for $\mathbb{R}P^n$ has been used as a testing ground for every new homotopical technique to come along.

Our intent here is not to give a complete survey of this problem, as this would take far too long. We will be content to give a small taste, entirely concentrating our focus on some easily-obtained lower bounds in the case of $\mathbb{R}P^n$. The methods involve Stiefel-Whitney classes (in singular cohomology) and some related characteristic classes in KO -theory.

8.1. A short survey. Before jumping into our analysis, let us give some sense of what is known about the problem. The following table shows the current knowledge (as of this writing) about the immersion and embedding dimensions for $\mathbb{R}P^n$ when $n \leq 24$:

TABLE 8.2. Immersion and embedding dimensions for $\mathbb{R}P^n$

$\mathbb{R}P^n$	2	3	4	5	6	7	8	9	10	11	12	13
imm. dim.	3	4	7	7	7	8	15	15	16	16	18	22
emb. dim.	4	5	8	9	[9, 11]	[9, 12]	16	17	17	[17, 18]	[18, 21]	[22, 23]

$\mathbb{R}P^n$	14	15	16	17	18	19	20	21	22	23	24
imm.	22	22	31	31	32	32	34	38	38	38	[38, 39]
emb.	[22, 23]	[23, 24]	32	33	33	[33, 34]	[34, 37]	39	39	39	[39, 42]

In the table, entries in brackets are given when the exact answer is not known. For example, the embedding dimension of $\mathbb{R}P^6$ is only known to lie in the interval $[9, 11]$. $\mathbb{R}P^6$ definitely embeds into $\mathbb{R}^{11}$ and does not embed into $\mathbb{R}^8$ —but it is not known if $\mathbb{R}P^6$ embeds into $\mathbb{R}^9$ or $\mathbb{R}^{10}$. In comparison, we know much more about the immersion problem; the smallest unknown case is $\mathbb{R}P^{24}$.

The above data on the immersion and embedding dimensions was taken from a table compiled by Don Davis [Davis2]. Davis's table contains substantially more data, covering (roughly speaking) $\mathbb{R}P^n$ for $n = 2^e + d$ with $d \leq 63$.

One of the earliest results is explained in Milnor-Stasheff [MS, Theorem 4.8]: if $n = 2^r$ then the immersion dimension of $\mathbb{R}P^n$ equals $2n - 1$, showing that the Whitney upper bound is sharp in this case. Peterson [Peterson] proved that if $n = 2^r$ then the embedding dimension equals $2n$, again showing that the Whitney bound is sharp here. We recount the Milnor-Stasheff argument in Corollary 8.11

below. In general, if $n = 2^r + d$ for $0 \leq d < 2^r$ and d is relatively small, then one can expect the immersion and embedding dimensions to be $2^{r+1} + x$ where x is a known quantity or one that is tightly constrained. The following theorem encompasses most of what is known for $d \leq 10$:

Theorem 8.3. *Write $n = 2^i + d$ where $0 \leq d < 2^i$. Then the immersion dimension of $\mathbb{R}P^n$ equals $2^{i+1} + e$ and the embedding dimension equals $2^{i+1} + f$ where the following is known:*

d	0	1	2	3	4	5	6	7	8	9	10
i	≥ 1	≥ 2	≥ 3	≥ 3	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4
e	-1	-1	0	0	2	6	6	6	[6, 7]	14	14
f	0	1	1	[1, 2]	[2, 5]	7	7	7	[7, 10]	[14, 15]	[14, 16]

For example, for $\mathbb{R}P^{40}$ we write $40 = 2^5 + 8$ and consult the $d = 8$ column. It tells us that the immersion dimension is either $2^6 + 6$ or $2^6 + 7$ (70 or 71), and that the embedding dimension lies between $2^6 + 7$ and $2^6 + 10$ (71 through 74).

The above theorem is not credited because it represents the combined work over many years of several authors. Much credit should be given to Davis, who has brought all the results together and given complete references. The above theorem is just the first few lines of the table [Davis2].

8.4. Stiefel-Whitney techniques. Suppose that M is a compact manifold of dimension n , and that M is immersed in $\mathbb{R}^{n+k}$. The immersion has a normal bundle ν , and there is an isomorphism of bundles $T_M \oplus \nu \cong \underline{n+k}$. Taking total Stiefel-Whitney classes of both sides gives

$$w(T_M) \cdot w(\nu) = w(T_M \oplus \nu) = w(\underline{n+k}) = 1.$$

Recall that the total Stiefel-Whitney class of a bundle E is $w(E) = 1 + w_1(E) + w_2(E) + \dots$. Because the zero-coefficient is 1 we can formally invert this expression, and because $H^*(M)$ is zero in sufficiently large degrees this formal inverse actually makes sense as an element of $H^*(M)$. So we can feel free to write $w(E)^{-1}$, and we obtain

$$w(T_M)^{-1} = w(\nu).$$

We don't know anything about ν except its rank, which is equal to k . This guarantees that $w(\nu)$ does not have any terms of degree larger than k , and so we obtain the following simple result [MS, material preceding Theorem 4.8]:

Proposition 8.5. *Let M be a compact smooth manifold of dimension n . If M immerses in $\mathbb{R}^{n+k}$ then $w(T_M)^{-1}$ vanishes in degrees larger than k .*

Let us apply this proposition to $\mathbb{R}P^n$. Here we have the identity $T_{\mathbb{R}P^n} \oplus 1 = (n+1)L$ (Example ??) and so

$$w(T_{\mathbb{R}P^n}) = w(T_{\mathbb{R}P^n} \oplus 1) = w((n+1)L) = w(L)^{n+1} = (1+x)^{n+1},$$

where x denotes the generator for $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. Taking inverses gives

$$w(T_{\mathbb{R}P^n})^{-1} = (1+x)^{-(n+1)} = \sum_{i=0}^{\infty} \binom{-(n+1)}{i} x^i.$$

We can rewrite the coefficient of x^i as

$$\binom{-(n+1)}{i} = (-1)^i \frac{(n+1) \cdot (n+2) \cdots (n+i)}{i!} = (-1)^i \binom{n+i}{i}.$$

Putting everything together we obtain the following:

Corollary 8.6. *If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then $\binom{n+i}{i}$ is even for $k < i \leq n$.*

The above corollary yields very concrete non-immersion results, but to obtain these we need to be good at checking when binomial coefficients are even. Historically, topologists got pretty good at this because of the presence of binomial coefficients in the Adem relations. A key result is the following:

Lemma 8.7. *Let $n_j n_{j-1} \dots n_0$ be the base 2 representation for n ; that is, each $n_j \in \{0, 1\}$ and $n = \sum n_i 2^i$. Similarly, let $k_j k_{j-1} \dots k_0$ be the base 2 representation for k . Then*

$$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \pmod{2}.$$

Proof. The result follows easily from four points: (i) $\binom{2n}{i}$ is even if i is odd; (ii) $\binom{2n}{2k} \equiv \binom{n}{k} \pmod{2}$; (iii) $\binom{2n+1}{2k} \equiv \binom{2n}{2k} \pmod{2}$; and (iv) $\binom{2n+1}{2k+1} \equiv \binom{n}{k} \pmod{2}$. The four cases correspond to the four possibilities for n_0 and k_0 , and set us up for a straightforward induction.

For (i) and (ii), imagine a column A of the numbers 1 through n and a second “mirror” column B containing the same entries. If i is odd, the i -element subsets of the two columns taken together may be partitioned into two classes: those which contain more elements from column A than column B , and those which contain more elements from column B . The operation of “switch entries between the two columns” gives a bijection between these two classes, thereby showing that $\binom{2n}{i}$ is even.

For (ii), note that the i -element subsets can be partitioned into groups determined by the number of elements from column A . This gives rise to the formula

$$\binom{2n}{2k} = \binom{n}{0} \binom{n}{2k} + \binom{n}{1} \binom{n}{2k-1} + \dots + \binom{n}{2k-1} \binom{n}{1} + \binom{n}{2k} \binom{n}{0}.$$

The terms on the right-hand-side are symmetric and so can be grouped in pairs, except for the middle term which is $\binom{n}{k}^2$. So working mod 2 we have

$$\binom{2n}{2k} \equiv \binom{n}{k}^2 \equiv \binom{n}{k}.$$

For (iii) just use Pascal’s identity $\binom{2n+1}{2k} = \binom{2n}{2k} + \binom{2n}{2k-1}$ together with (i). For (iv) use $\binom{2n+1}{2k+1} = \binom{2n}{2k+1} + \binom{2n}{2k}$ together with (i) and (ii). $\square$

Example 8.8. To determine if $\binom{20}{9}$ is even then we note that 20 is 10100 in base 2, and 9 is 1001. Using the above lemma we compute

$$\binom{20}{9} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 = 0.$$

So $\binom{20}{9}$ is even.

Notice that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1$, whereas $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. So in computations like the one above, the final answer is even if and only if $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ appears at least once in the product—that is, if n has a certain bit turned “off” and the corresponding bit of k is “on”. In particular, the following three statements are now obvious:

- (i) If $n = 2^r$ then $\binom{n}{i}$ is even for all i in the range $0 < i < n$.
- (ii) If $n = 2^r - 1$ then $\binom{n}{i}$ is odd for all $i \leq n$.
- (iii) $\binom{2n}{n}$ is always even.

The point for the first two is that 2^r has all of its bits turned off except for the r th, whereas $2^r - 1$ has all of its bits turned on. For statement (iii) just consider the smallest bit of n that is turned on, and note that the corresponding bit is off in $2n$.

Corollary 8.6 is most often used in the form below. The proof is immediate from Corollary 8.6.

Corollary 8.9. *Fix $n \geq 2$, and let k be the largest integer such that $k \leq n$ and $\binom{n+k}{k}$ is odd. Then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{n+k-1}$.*

Note that k will be strictly less than n , as $\binom{2n}{n}$ is always even; this conforms with the Whitney immersion result. As an application of the above corollary, suppose we want to immerse $\mathbb{R}P^{10}$. We start with

$$\binom{20}{10} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

We have 2^{10} dividing the numerator and 2^8 dividing the denominator. Start removing factors from the left, one by one from the numerator and denominator simultaneously, watching what happens to the number of 2's in each. The fraction does not become odd until we are looking at $\binom{15}{5}$. So $k = 5$ in Corollary 8.9 and the conclusion is that $\mathbb{R}P^{10}$ does not immerse into $\mathbb{R}P^{14}$.

The above process is cumbersome, and with a little investigation it is not hard to produce a shortcut.

Proposition 8.10. *Write $n = 2^i + d$ where $0 \leq d < 2^i$. Then the largest k in the range $0 \leq k \leq n$ such that $\binom{n+k}{k}$ is odd is $k = 2^i - d - 1$.*

Proof. Note that if $j = 2^i - d - 1$ then $n + j = 2^{i+1} - 1$ and so $\binom{n+j}{j}$ is certainly odd. We must show that $\binom{n+j}{j}$ is even for j in the range $2^i - d \leq j \leq n$. This is the kind of analysis that is perhaps best left to the reader, but we will give a sketch. Suppose to the contrary that j is in this range and $\binom{n+j}{j}$ is odd. Let $e = j - (2^i - d - 1)$, so that $n + j = (2^i + d) + e + (2^i - d - 1) = e + (2^{i+1} - 1)$. Let the smallest bit of e that is turned on be the r th bit; this is also the smallest bit of $n + j$ that is turned off. Since $\binom{n+j}{n}$ is odd, this bit must be also off in n . So we have proven

- The r th bit is off in n and on in e .

If we write $e = e' + 2^r$, then $n + j = e' + (2^r + 2^{i+1} - 1)$. The term in parentheses has all bits off from the r th through the i th, and so the bits of $n + j$ agree with the bits of e' (and of e) from the $(r+1)$ st through the i th. Since $\binom{n+j}{n}$ is odd, every bit of n that is turned on in this range must also be turned on in $n + j$ —and therefore also in e . We have thus shown that

- All bits greater than the r th that are on in n are also on in e .

The two bulleted facts together show that $e > n$, which is not allowed since $e \leq j \leq n$. $\square$

Corollary 8.11. *If $2^i \leq n < 2^{i+1}$ then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{2^{i+1}-2}$. In particular, if $n = 2^i$ then the immersion dimension of $\mathbb{R}P^n$ equals $2n - 1$.*

Proof. The first line is immediate from Corollary 8.9 and Proposition 8.10. The second statement follows from the first together with the Whitney theorem saying that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-1}$. $\square$

Remark 8.12. Note that Corollary 8.11 is really about the particular case $n = 2^i$. The non-immersions for the other n in the given range immediately follow from this one, since an immersion of $\mathbb{R}P^n$ would restrict to one for $\mathbb{R}P^{2^i}$.

Let us now change gears just a bit and consider embeddings. We can also use characteristic classes to give obstructions in this setting. The key result is the following:

Proposition 8.13. *Suppose M is a compact smooth n -manifold that is embedded in $\mathbb{R}^{n+k}$. Then $w_k(\nu) = 0$ where ν is the normal bundle.*

Proof. Choose a metric on ν and let $S(\nu)$ be the sphere bundle. If $p: S(\nu) \rightarrow M$ denotes the projection map, then clearly $p^*\nu$ splits off a trivial bundle: $p^*\nu = \underline{1} \oplus E$ for some rank $k-1$ bundle E on $S(\nu)$. The Whitney formula then gives $0 = w_k(p^*\nu) = p^*(w_k(\nu))$.

The proof will be completed by showing that $p^*: H^*(M; \mathbb{Z}/2) \rightarrow H^*(S(\nu); \mathbb{Z}/2)$ is injective. Let U be a tubular neighborhood of M in $\mathbb{R}^{n+k}$, arranged so that its closure $\bar{U}$ is homeomorphic to the disk bundle of ν . Write $\partial\bar{U}$ for the boundary, which is homeomorphic to $S(\nu)$. We have the long exact sequence

$$\cdots \rightarrow H^i(\bar{U}, \partial\bar{U}) \rightarrow H^i(\bar{U}) \rightarrow H^i(\partial\bar{U}) \rightarrow \cdots$$

where all cohomology groups have $\mathbb{Z}/2$ -coefficients. The projection $\bar{U} \rightarrow M$ is a homotopy equivalence, so our map p^* is isomorphic to $H^i(\bar{U}) \rightarrow H^i(\partial\bar{U})$. We can verify that this is injective by checking that the previous map in the long exact sequence is zero. We look only at $i > 0$, as the $i = 0$ case is trivial.

To this end, consider the inclusion of pairs $(\bar{U}, \partial\bar{U}) \subseteq (\mathbb{R}^{n+k}, \mathbb{R}^{n+k} - M)$ and the maps below from the two long exact sequences:

$$\begin{array}{ccc} H^i(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} - M) & \longrightarrow & H^i(\mathbb{R}^{n+k}) \\ \cong \downarrow & & \downarrow \\ H^i(\bar{U}, \partial\bar{U}) & \longrightarrow & H^i(\bar{U}) \end{array}$$

The left vertical map is an isomorphism by excision, and the group in the upper right corner is zero. So the bottom horizontal map is zero, as we desired. $\square$

Exercise 8.14. We didn't use much about $H^*(-; \mathbb{Z}/2)$ in the above proof. It looks like we used $H^i(pt) = 0$ for $i > 0$, but in fact we didn't even use that. Adapt the above proof to show that if $\mathcal{E}$ is any cohomology theory then $\mathcal{E}^*(M) \rightarrow \mathcal{E}^*(S(\nu))$ is an embedding. [Hint: Used reduced cohomology.]

Remark 8.15. In the case where M is orientable, Whitney proved in [Whitney1, Theorem on p. 113] that there is necessarily a non-vanishing section of the normal bundle. This immediately implies that the mod 2 Euler class of the normal bundle vanishes, i.e. that $w_k(\nu) = 0$. This gives some added geometric motivation for Proposition 8.13, though it doesn't recover the entire result. Whitney also proved [loc. cit] that for any embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$ the normal bundle cannot admit a non-vanishing section, so this argument doesn't generalize to the non-orientable case even though we still know that the top Stiefel-Whitney class is zero there. See Exercise??? for further discussion.

Compare the next result to Proposition 8.5:

Corollary 8.16. *Let M be a compact smooth n -manifold. If M embeds into $\mathbb{R}^{n+k}$ then $w(T_M)^{-1}$ vanishes in degrees k and larger.*

Proof. We saw in the proof of Proposition 8.5 that $w(T_M)^{-1} = w(\nu)$. Since ν has rank k , this forced the Stiefel-Whitney classes to vanish in degrees larger than k . But now Proposition 8.13 also gives us the vanishing in degree k . $\square$

Corollary 8.17. *If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $\binom{n+i}{i}$ is even for $k \leq i \leq n$.*

Proof. Same as for Corollary 8.6. $\square$

Corollary 8.18. *Suppose $2^i \leq n < 2^{i+1}$. Then $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{2^{i+1}-1}$. In particular, if $n = 2^i$ then the embedding dimension of $\mathbb{R}P^n$ equals $2n$.*

Proof. Write $n = 2^i + d$. Let k be the largest integer in the range $0 \leq k \leq n$ such that $\binom{n+k}{k}$ is odd. Then Corollary 8.17 shows that $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{n+k}$. But Proposition 8.10 identifies $k = 2^i - d - 1$, and so $n + k = 2^{i+1} - 1$. This proves the first statement. The second statement is then a consequence of the first together with the Whitney theorem that $\mathbb{R}P^n$ embeds into $\mathbb{R}^{2n}$. $\square$

To gauge the relative strength of Corollaries 8.11 and 8.18, see Table 8.19 and compare to Table 8.2. One gets a clear sense of how far algebraic topology has progressed since the early days of Stiefel-Whitney classes!

TABLE 8.19. Stiefel-Whitney lower bounds for the immersion and embedding dimensions of $\mathbb{R}P^n$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
imm $\geq$	3	4	7	7	7	8	15	15	15	15	15	15	15	16	31	31	31
emb $\geq$	4	4	8	8	8	8	16	16	16	16	16	16	16	16	32	32	32

8.20. K -theoretic techniques. One can readily imagine adapting the basic approach from the last section by replacing the Stiefel-Whitney classes with characteristic classes taking values in some other cohomology theory. Atiyah [Atiyah3] pursued this idea using KO -theory, and certain constructions of Grothendieck provided the appropriate theory of characteristic classes. In this way he obtained some new non-immersion and non-embedding theorems. We explain this work next.

In Section ?? we described Chern classes in complex K -theory, which were defined using the $\tilde{\gamma}$ -operations. One can basically repeat those constructions in KO -theory verbatim to obtain KO -characteristic classes for real vector bundles. Since the context is a little different, we give a very quick review here.

Let X be a bund-compact space. For an element $x \in \widetilde{KO}(X)$ define

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \frac{t}{1-t}[\lambda^1 x] + \left(\frac{t}{1-t}\right)^2[\lambda^2 x] + \cdots$$

Note that $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$. Define $\gamma^i(x)$ to be the coefficient of t^i in $\gamma_t(x)$. So we have

$$\gamma_k(x+y) = \sum_{i+j=k} \gamma^i(x)\gamma^j(y).$$

Note also that $\gamma_t(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t} = 1 + t + t^2 + \cdots$

For a vector bundle E over X define

$$\tilde{\gamma}_t(E) = \gamma_t(E - \underline{\text{rank}} E) = \frac{\gamma_t(E)}{\gamma_t(\underline{\text{rank}} E)} = \frac{\gamma_t(E)}{\gamma_t(1)^{\text{rank } E}} = \gamma_t(E) \cdot (1-t)^{\text{rank } E}.$$

One should think of this as just being a renormalization of the γ_t construction; note that $\tilde{\gamma}_t(E) = 1$ if E is a trivial bundle. Observe that we still have the analog of the Whitney formula:

$$\begin{aligned} \tilde{\gamma}_t(E \oplus F) &= \gamma_t(E \oplus F - \underline{\text{rank}}(E + F)) = \gamma_t((E - \underline{\text{rank}} E) + (F - \underline{\text{rank}} F)) \\ &= \gamma_t(E - \underline{\text{rank}} E) \gamma_t(F - \underline{\text{rank}} F) \\ &= \tilde{\gamma}_t(E) \tilde{\gamma}_t(F). \end{aligned}$$

If L is a line bundle then

$$\tilde{\gamma}_t(L) = \gamma_t(L) \cdot (1-t) = \left[1 + \frac{t}{1-t}[L]\right] \cdot (1-t) = 1 - t + t[L] = 1 + t([L] - 1).$$

So $\tilde{\gamma}_1(L) = [L] - 1$ and $\tilde{\gamma}_i(L) = 0$ for $i > 1$.

Finally, we observe that if E is a rank k bundle then $\tilde{\gamma}_i(E) = 0$ for all $i > k$. This is because

$$\tilde{\gamma}_t(E) = \lambda_{\frac{t}{1-t}}(E)(1-t)^k = \sum_{i=0}^{\infty} \left(\frac{t}{1-t}\right)^i [\Lambda^i E] \cdot (1-t)^k = \sum_{i=0}^k t^i (1-t)^{k-i} [\Lambda^i E].$$

The final expression is a polynomial in t of degree at most k .

Compare the following result to Proposition 8.5 and Corollary 8.16.

Proposition 8.21. *Let M be a compact smooth n -manifold. If M immerses into $\mathbb{R}^{n+k}$ then the power series $\tilde{\gamma}_t(T_M)^{-1}$ vanishes in degrees larger than k . If M embeds into $\mathbb{R}^{n+k}$ then $\tilde{\gamma}_t(T_M)^{-1}$ vanishes in degrees k and larger.*

Proof. The proofs are the same as before. If M immerses into $\mathbb{R}^{n+k}$ then $T_M \oplus \nu \cong \underline{n+k}$ where ν is the normal bundle. So $\tilde{\gamma}_t(T_M) \tilde{\gamma}_t(\nu) = \tilde{\gamma}_t(T_M \oplus \nu) = \tilde{\gamma}_t(\underline{n+k}) = 1$, and then $\tilde{\gamma}_t(T_M)^{-1} = \tilde{\gamma}_t(\nu)$. But since ν has rank k we have $\tilde{\gamma}_i(\nu) = 0$ for $i > k$.

For the second part of the proposition we need to prove that if M embeds into $\mathbb{R}^{n+k}$ then $\tilde{\gamma}_k(\nu) = 0$. The proof is essentially the same as for Proposition 8.13, taking into account Exercise 8.14. $\square$

For the following proposition, recall from Definition 6.16 that $\varphi(n)$ denotes the number of integers s such that $0 < s \leq n$ and s is congruent to 0, 1, 2, or 4 modulo 8.

Corollary 8.22 (Atiyah).

- (a) If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then $2^{\varphi(n)-j+1}$ divides $\binom{n+j}{j}$ for $k < j \leq \varphi(n)$.
- (b) If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $2^{\varphi(n)-j+1}$ divides $\binom{n+j}{j}$ for $k \leq j \leq \varphi(n)$.

Proof. Recall that $1 \oplus T_{\mathbb{R}P^n} \cong (n+1)L$, as in Example ?? . We get

$$\tilde{\gamma}_t(T_{\mathbb{R}P^n}) = \tilde{\gamma}_t(T_{\mathbb{R}P^n} \oplus 1) = \tilde{\gamma}_t((n+1)L) = \tilde{\gamma}_t(L)^{n+1} = (1+t\lambda)^{n+1}$$

where $\lambda = [L] - 1 \in \widetilde{KO}^0(\mathbb{R}P^n)$. So

$$\tilde{\gamma}_t(T_{\mathbb{R}P^n})^{-1} = (1+t\lambda)^{-(n+1)} = \sum_{j=0}^{\infty} \binom{-(n+1)}{j} \lambda^j \cdot t^j = \sum_{j=0}^{\infty} (-1)^j \binom{n+j}{j} \lambda^j \cdot t^j.$$

If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then by Proposition 8.21 $\binom{n+j}{j}\lambda^j = 0$ in $\widetilde{KO}^0(\mathbb{R}P^n)$ for all $j > k$. If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $\binom{n+j}{j}\lambda^j = 0$ for all $j \geq k$.

Now we recall from Theorem 6.15 that $\widetilde{KO}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^{\varphi(n)})$ and that λ is a generator. Also recall that $\lambda^2 = -2\lambda$, so $\lambda^j = (-2)^{j-1}\lambda$. The desired conclusions follow immediately. $\square$

Corollary 8.22 is best used in the following form. Let $\sigma(n)$ denote the largest value of j in the range $1 \leq j \leq \varphi(n)$ for which $\binom{n+j}{j}$ is not divisible by $2^{\varphi(n)+1-j}$; if no such j exists then set $\sigma(n) = 0$ by default. Then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{n+\sigma(n)-1}$ and does not embed into $\mathbb{R}^{n+\sigma(n)}$.

For some values of n the result of Corollary 8.22 is stronger than what we obtained from Corollaries 8.6 and 8.18, and for some values of n it is weaker. We demonstrate some examples:

Example 8.23. For the question of immersions of $\mathbb{R}P^8$, we have $\varphi(8) = 4$ and $\sigma(8) = 4$. So Atiyah's result (8.22) gives that $\mathbb{R}P^8$ does not immerse into $\mathbb{R}^{11}$. The Stiefel-Whitney classes, however, gave the better result that $\mathbb{R}P^8$ does not immerse into $\mathbb{R}^{14}$.

In contrast, for $\mathbb{R}P^{15}$ we have $\varphi(15) = 7$ and $\sigma(15) = 4$. So Atiyah's result tells us that $\mathbb{R}P^{15}$ does not immerse into $\mathbb{R}^{18}$. The method of Stiefel-Whitney classes (8.6) gives no information in this case.

Table 8.24 shows the lower bounds for the immersion dimension of $\mathbb{R}P^n$ obtained from Stiefel-Whitney techniques versus the KO -theoretic techniques. The reader will notice that for $2^i \leq n < 2^{i+1}$, the Stiefel-Whitney bounds are significantly better near the lower end of the interval whereas the KO -theoretic bounds are better near the upper end.

TABLE 8.24. Lower bounds for the immersion dimension of $\mathbb{R}P^n$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
S-W	3	4	7	7	7	8	15	15	15	15	15	15	15	16	31	31	31
KO	3	4	7	7	7	8	12	13	15	15	17	17	19	19	24	25	27

n	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
S-W	31	31	31	31	31	31	31	31	31	31	31	31	32	63	63
KO	27	31	31	31	31	34	35	38	39	40	41	42	43	48	49

As an example when n is much larger, the Stiefel-Whitney classes give no information on the immersion dimension of $\mathbb{R}P^{255}$. By contrast, the Atiyah result gives that the immersion dimension is at least 375. For $n = 2^i - 1$ the improvement of the Atiyah bound over the Hopf bound is on the order of $\frac{n}{2}$.

8.25. Immersions and geometric dimension. So far we have used various characteristic classes to give lower bounds on the immersion/embedding dimensions for $\mathbb{R}P^n$. To close this section we will show how to produce upper bounds for the *immersion* dimension, via a geometric result of Hirsch that translates this into a bundle-theoretic problem. The central tool again ends up being KO -theory. Using

these methods we will completely determine the immersion dimension of $\mathbb{R}P^n$ for $n \leq 9$.

Let $E \rightarrow X$ be a vector bundle. Define the **virtual dimension** of E by the formula

$$\text{v. dim } E = \min\{k \mid E \cong F \oplus (\text{rank } E) - k \text{ for some bundle } F \text{ of rank } k\}.$$

The virtual dimension is the smallest k such that E is isomorphic to a stabilized rank k bundle. If X is paracompact Hausdorff then by Corollary ?? we can also write

$$\text{v. dim } E = \text{rank } E - \max\{j \mid E \text{ has } j \text{ independent sections}\}.$$

Note that the virtual dimensions of E and $E \oplus \underline{1}$ might be different; the latter might be smaller than the former. With this in mind we can also introduce the **stable virtual dimension**:

$$\begin{aligned} \text{sv. dim } E &= \min\{\text{v. dim}(E \oplus r) \mid r \geq 0\} \\ &= \min\{\text{rank } F \mid F \text{ is a bundle that is stably equivalent to } E\}. \end{aligned}$$

Finally, we introduce the following related concept. For X compact Hausdorff and $\alpha \in \widetilde{KO}^0(X)$ define the **geometric dimension** of α to be

$$\text{g. dim } \alpha = \min\{k \mid \alpha + k = [F] \text{ for some vector bundle } F \text{ on } X\}$$

(note that we need the hypothesis on X so that $KO^0(X)$ coincides with the Grothendieck group of bundles, hence such a k always exists).

The above three concepts are related as follows:

Proposition 8.26. *Let $E \rightarrow X$ be a vector bundle, where X is compact Hausdorff.*

- (a) $\text{g. dim}(E - \underline{\text{rank } E}) = \text{sv. dim}(E) \leq \text{v. dim}(E)$.
- (b) *If X is a finite CW-complex and $\text{rank } E > \dim X$ then $\text{sv. dim}(E) = \text{v. dim}(E)$.*

Proof. For (a) only the first equality requires proof. This equality is almost a tautology: for any integer $d \geq 0$ we have

$$\begin{aligned} \text{sv. dim}(E) \leq d &\iff \text{there exists a bundle } F \text{ of rank } d \text{ such that } E \cong_{st} F \\ &\iff \exists F \text{ of rank } d \text{ such that } [E] - \underline{\text{rank } E} = [F] - \underline{d} \text{ in } \widetilde{KO}^0(X) \\ &\iff \exists F \text{ such that } [E] - \underline{\text{rank } E} + \underline{d} = [F] \text{ in } KO^0(X) \\ &\iff \text{g. dim}(E - \underline{\text{rank } E}) \leq d. \end{aligned}$$

For the second equivalence, see the discussion in Section ??. The desired equality follows immediately.

For (b), let $r = \text{sv. dim}(E)$, $k = \text{rank}(E)$, and note that $r \leq k$. Then there exists a rank r bundle F such that E and F are stably equivalent: $E \oplus \underline{N} \cong F \oplus (\underline{N} + k - r)$ for some $N > 0$. Since $\text{rank}(E) > \dim X$ we can cancel the $\underline{N}$ factors to get $E \cong F \oplus (\underline{k} - r)$, by Proposition ??. Hence $\text{v. dim}(E) \leq \text{rank}(F) = r = \text{sv. dim}(E)$. $\square$

The following result of Hirsch [Hirsch1], and its corollary, translate the immersion problem into a purely homotopy-theoretic question. This is the key to why immersions are better understood than embeddings.

Theorem 8.27 (Hirsch). *Let M be a compact smooth manifold of dimension n . For $k \geq 1$ the following statements are equivalent:*

- (a) M can be immersed in $\mathbb{R}^{n+k}$

- (b) *There exists a bundle F of rank k such that $T_M \oplus F$ is trivial.*
(c) *There exists an O_n -equivariant map $\text{Fr}(T_M) \rightarrow V_n(\mathbb{R}^{n+k})$, where $\text{Fr}(T_M)$ is the bundle of n -frames in T_M .*

Observe that (a) implies (b) by taking F to be the normal bundle of the immersion. Also, if $\phi: T_M \oplus F \rightarrow \underline{n+k}$ is an isomorphism then any n -frame in T_M yields an n -frame in $\mathbb{R}^{n+k}$ by applying ϕ ; thus, one gets an equivariant map $\text{Fr}(T_M) \rightarrow V_n(\mathbb{R}^{n+k})$. This shows (b) implies (c). So the content of the above theorem is really in (c) $\Rightarrow$ (a), and this is what was proven by Hirsch via geometric arguments: see [Hirsch1, Theorem 6.1 (taking $r = 0$ there)], though the context is really in [Hirsch1, Theorem 5.7] taking $K = \emptyset$ and $L = M$ there. Hirsch actually showed much more, essentially proving that homotopy classes of immersions from M to $\mathbb{R}^{n+k}$ are in bijective correspondence with equivariant homotopy classes of maps $\text{Fr}(T_M) \rightarrow V_n(\mathbb{R}^{n+k})$. We will not give Hirsch's proof here, but we will use the following corollary of his result. This corollary first appeared in [Atiyah3, Proposition 3.2] and in [Sanderson2, Theorem 2.1].

Corollary 8.28. *Let $k \geq 1$, and let M be a compact smooth manifold of dimension n . Then M immerses in $\mathbb{R}^{n+k}$ if and only if $\text{g. dim}(n - T_M) \leq k$.*

Proof. We have already seen the ‘only if’ direction when we obtained obstructions to immersions: if an immersion exists then $\underline{n+k} \cong T_M \oplus \nu$ where ν is the normal bundle, therefore $n - T_M = \nu - k$ and hence $\text{g. dim}(n - T_M) = \text{g. dim}(\nu - k) \leq k$.

For the other direction, assume $\text{g. dim}(n - T_M) \leq k$. So there exists a rank k vector bundle F such that $n - T_M + k = F$ in $KO^0(M)$. This implies $n + k = T_M + F$ in $KO^0(M)$, which in turn yields that $\underline{n+k+N} \cong T_M \oplus F \oplus \underline{N}$ for some $N \geq 0$ (by the analog of Proposition ??). Since the rank of $T_M \oplus F$ is larger than $\dim M$, it follows by Proposition ?? that we can cancel the $\underline{N}$ on both sides to get $\underline{n+k} \cong T_M \oplus F$. Then by Theorem 8.27 we know that M immerses into $\mathbb{R}^{n+k}$. $\square$

Remark 8.29. Note in particular that if M is parallelizable then M immerses into $\mathbb{R}^{n+1}$ (taking $k = 1$ in Corollary 8.28, since $k = 0$ is not allowed).

We now specialize again to the case of $M = \mathbb{R}P^n$. Here we have

$$n - T_{\mathbb{R}P^n} = (n+1) - (1 \oplus T_{\mathbb{R}P^n}) = (n+1) - (n+1)L = (n+1)(1-L).$$

Proposition 8.30. *For $n \leq 8$ the immersion dimension of $\mathbb{R}P^n$ is as given in Table 8.2.*

Proof. Given the lower bounds given by Stiefel-Whitney classes (see Table 8.24), we only have to demonstrate the required immersions. The fact that $\mathbb{R}P^8$ immerses in $\mathbb{R}^{15}$ is a special case of Whitney's classical theorem. Both $\mathbb{R}P^3$ and $\mathbb{R}P^7$ have trivial tangent bundles (using the related fact for S^3 and S^7), and so by Remark 8.29 they immerse into $\mathbb{R}^4$ and $\mathbb{R}^8$, respectively.

For $\mathbb{R}P^6$ we must calculate the geometric dimension of $7(1-L) = -7(L-1)$. But $\varphi(6) = 3$, and so $\widehat{KO}(\mathbb{R}P^6) \cong \mathbb{Z}/8$. Hence $8(L-1) = 0$, and so $-7(L-1) = L-1$. The geometric dimension of $L-1$ is clearly at most 1, and so by Corollary 8.28 $\mathbb{R}P^6$ immerses into $\mathbb{R}^7$. Of course $\mathbb{R}P^4$ and $\mathbb{R}P^5$ therefore also immerse into $\mathbb{R}^7$.

A similar argument works to show that $\mathbb{R}P^2$ immerses into $\mathbb{R}^3$ (or one can just construct the immersion geometrically). $\square$

The reader should note why the above result stopped with $\mathbb{R}P^8$. For $\mathbb{R}P^9$ one finds that the immersion problem boils down to determining the geometric dimension of $-10(L-1) = 22(L-1)$ (here we used that $\varphi(9) = 5$ and so $\widetilde{KO}^0(\mathbb{R}P^9) = \mathbb{Z}/32$). The precise value of this geometric dimension is far from clear. We will close this section by analyzing it completely, following Sanderson [Sanderson2]. However, we take a short detour to illustrate some general principles.

Recall that for general n we have $n - T_{\mathbb{R}P^n} = (n+1)(1-L)$. We would like to interpret the geometric dimension of this class as being a stable virtual dimension, but for this we would need to be looking at a positive multiple of $L-1$ rather than $1-L$. There are two ways to get ourselves into this position. The first, which we have already seen, proceeds by recalling that $L-1$ has order $2^{\varphi(n)}$ in $\widetilde{KO}(\mathbb{R}P^n)$. So we can write

$$n - T_{\mathbb{R}P^n} = -(n+1)(L-1) = (2^{\varphi(n)} - (n+1))(L-1)$$

and hence

$$\begin{aligned} \text{g. dim}(n - T_{\mathbb{R}P^n}) &= \text{g. dim}\left([2^{\varphi(n)} - (n+1)]L - [2^{\varphi(n)} - (n+1)]\right) \\ &= \text{sv. dim}\left([2^{\varphi(n)} - (n+1)]L\right) \end{aligned}$$

(the last equality is by Proposition 8.26(b)). The second approach is from Sanderson [Sanderson1, Lemma 2.2]:

Proposition 8.31. *For the bundle $L \rightarrow \mathbb{R}P^n$, the statement $\text{g. dim}(a(L-1)) \leq b$ is equivalent to $\text{g. dim}((b-a)(L-1)) \leq b$, for any $a, b \in \mathbb{Z}$ with $b \geq 0$.*

Proof. It suffices to prove the implication in one direction, by symmetry. So suppose $\text{g. dim}(a(L-1)) \leq b$. This implies that $a(L-1) + b = E$ in $KO^0(\mathbb{R}P^n)$, for some rank b bundle E . Multiply by L to get $a(1-L) + bL = E \otimes L$, and then rearrange to find $(b-a)(L-1) + b = E \otimes L$. This yields that $\text{g. dim}((b-a)(L-1)) \leq b$. $\square$

Corollary 8.32. *Let $L \rightarrow \mathbb{R}P^n$ be the tautological bundle. For $k > 0$ the following statements are equivalent:*

- (1) $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$,
- (2) $\text{g. dim}(-(n+1)(L-1)) \leq k$,
- (3) $\text{g. dim}((n+k+1)(L-1)) \leq k$,
- (4) $\text{g. dim}([2^{\varphi(n)} - (n+1)](L-1)) \leq k$.

Proof. The equivalence (1) $\iff$ (2) comes from Corollary 8.28, and (2) $\iff$ (3) is by Proposition 8.31. Finally, (2) $\iff$ (4) is true because $2^{\varphi(n)}(L-1) = 0$ in $\widetilde{KO}^0(\mathbb{R}P^n)$. $\square$

Part (3) of the above result, which is the part that comes from Proposition 8.31, will actually not be needed in the remainder of this section. But we record it here for later use (see Proposition 9.16).

We close this section by settling the immersion problem for $\mathbb{R}P^9$:

Proposition 8.33 (Sanderson). *The immersion dimension of $\mathbb{R}P^9$ equals 15.*

This result is from [Sanderson2, Theorem 5.3]. Sanderson proves much more than this, for example that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-3}$ whenever n is odd. He also proved that $\mathbb{R}P^{11}$ immerses into $\mathbb{R}^{16}$, which ends up being the best result for both $\mathbb{R}P^{11}$ and $\mathbb{R}P^{10}$.

Proof. The lower bound of 15 is given by Stiefel-Whitney classes, as in Table 8.24. So we only need to prove that $\mathbb{R}P^9$ immerses into $\mathbb{R}^{15}$. By Corollary 8.32 this is equivalent to $\text{g. dim}(22(L-1)) \leq 6$, and is also equivalent to $\text{g. dim}(16(L-1)) \leq 6$. The proof below works for both statements, but for specificity we just prove the former. Note that what we must prove is equivalent to $\text{sv. dim}(22L) \leq 6$, by Proposition 8.26. We will outline the steps for this, and then give more details afterwards.

Step 1: There exists a rank 4 complex bundle E on $\mathbb{R}P^9$ such that $r_{\mathbb{R}}E$ is stably equivalent to $22L$ (recall that $r_{\mathbb{R}}E$ denotes the real bundle obtained from E by forgetting the complex structure).

Step 2: The bundle $E|_{\mathbb{R}P^8}$ has a nonzero section s .

Step 3: The bundle $r_{\mathbb{R}}E|_{\mathbb{R}P^8}$ has a field of (real) 2-frames.

Step 4: For $N \gg 0$ the field of real $(2+N)$ -frames of $(r_{\mathbb{R}}E \oplus N)|_{\mathbb{R}P^8}$ may be extended over $\mathbb{R}P^9$.

Step 5: $\text{sv. dim}(22L) = \text{sv. dim}(r_{\mathbb{R}}E) \leq 6$, hence $\mathbb{R}P^9$ can be immersed into $\mathbb{R}^{15}$.

We now justify each of these steps. For step 1 use that $r_{\mathbb{R}}: \tilde{K}(\mathbb{R}P^n) \rightarrow \widetilde{KO}(\mathbb{R}P^n)$ has image equal to $\langle 2(L-1) \rangle$, by Theorems 6.15 and 6.18. Since $22(L-1)$ is therefore in the image, there is a complex bundle E on $\mathbb{R}P^9$ such that $r_{\mathbb{R}}E$ is stably equivalent to $22L$. The bundle E is represented by a map $\mathbb{R}P^9 \rightarrow BU$, and such a map necessarily factors up to homotopy through $BU(4)$ by Proposition ??.

For step 2 we again proceed by obstruction theory. We have the sphere bundle $S(E|_{\mathbb{R}P^8}) \rightarrow \mathbb{R}P^8$ with fiber S^7 , and all the obstruction groups vanish except for the last one: $H^8(\mathbb{R}P^8; \pi_7 S^7)$. The coefficients are untwisted because the complex structure gives a canonical orientation to each fiber. This final obstruction class is the same as the Euler class of $r_{\mathbb{R}}E$, or equivalently the top Chern class of E . Since $H^8(\mathbb{R}P^8; \mathbb{Z}) = \mathbb{Z}/2$ it will be sufficient to compute the mod 2 reduction of this class, which is the top Stiefel-Whitney class $w_8(r_{\mathbb{R}}E)$. Now we use that $r_{\mathbb{R}}E$ is stably isomorphic to $22L$, so the total Stiefel-Whitney class is $w(r_{\mathbb{R}}E) = w(22L) = w(L)^{22} = (1+x)^{22}$. Hence $w_8(r_{\mathbb{R}}E) = \binom{22}{8}x^8$. Since $\binom{22}{8}$ is even, the obstruction class vanishes and we indeed have a nonzero section.

Step 3 is trivial: the sections s and is give the field of real 2-frames.

Step 4 is obstruction theory yet again. If $F = r_{\mathbb{R}}E \oplus N$ then we have a (partial) section of $V_{N+2}(F) \rightarrow \mathbb{R}P^9$ defined over $\mathbb{R}P^8$, and we must extend this to all of $\mathbb{R}P^9$. The obstruction lies in $H^9(\mathbb{R}P^9; \pi_8(V_{N+2}(\mathbb{R}^{N+8})))$. But the homotopy group in the coefficients has been computed [Whitehead2, BP] and is known to be zero for all $N \geq 3$ (see also Exercise 8.34 below for hints to this calculation).

Step 5 is now immediate: $r_{\mathbb{R}}E \oplus N$ has rank $8+N$ and has $2+N$ independent sections, hence $\text{v. dim}(r_{\mathbb{R}}E \oplus N) \leq 6$. So $\text{sv. dim}(r_{\mathbb{R}}E) \leq 6$ as well. Since $r_{\mathbb{R}}E$ is stably equivalent to $22L$, $\text{sv. dim}(22L) = \text{sv. dim}(r_{\mathbb{R}}E) \leq 6$. $\square$

Exercise 8.34. Compute the homotopy group needed in the above proof via the following steps:

- (a) Use the fibration sequences $V_k(\mathbb{R}^n) \rightarrow V_{k+1}(\mathbb{R}^{n+1}) \rightarrow S^n$ to deduce that $\pi_8 V_{N+2}(\mathbb{R}^{N+8}) \cong \pi_8 V_4(\mathbb{R}^{10})$ for all $N \geq 3$.
- (b) Using the cell structure from Proposition ?? reduce the problem to that of computing $\pi_8(\mathbb{R}P^9/\mathbb{R}P^5)$.

- (c) Attempt to inductively compute $\pi_8(\mathbb{R}P^n/\mathbb{R}P^5)$ using the long exact homotopy sequences

$$\cdots \rightarrow \pi_8(\mathbb{R}P^n/\mathbb{R}P^5) \rightarrow \pi_8(\mathbb{R}P^{n+1}/\mathbb{R}P^5) \rightarrow \pi_8(\mathbb{R}P^{n+1}/\mathbb{R}P^5, \mathbb{R}P^n/\mathbb{R}P^5) \rightarrow \cdots$$

for $n \geq 6$. The attaching map for the top cell gives a map of pairs $(D^{n+1}, S^n) \rightarrow (\mathbb{R}P^{n+1}/\mathbb{R}P^5, \mathbb{R}P^n/\mathbb{R}P^5)$. Verify that the Blakers-Massey theorem implies that the induced map

$$\pi_{r-1}(S^n) \cong \pi_r(D^{n+1}, S^n) \rightarrow \pi_r(\mathbb{R}P^{n+1}/\mathbb{R}P^5, \mathbb{R}P^n/\mathbb{R}P^5)$$

is an isomorphism for $r \leq n+4$. So we can apply this to $r = 8$ and our long exact homotopy sequence becomes

$$\cdots \rightarrow \pi_8(S^n) \rightarrow \pi_8(\mathbb{R}P^n/\mathbb{R}P^5) \rightarrow \pi_8(\mathbb{R}P^{n+1}/\mathbb{R}P^5) \rightarrow \pi_7(S^n) \rightarrow \cdots$$

where the first map is induced by $S^n \rightarrow \mathbb{R}P^n \rightarrow \mathbb{R}P^n/\mathbb{R}P^5$.

- (d) Check that $\mathbb{R}P^7/\mathbb{R}P^5 \simeq S^7 \vee S^6$ (think about the attaching map of the 7-cell), so that $\pi_8(\mathbb{R}P^7/\mathbb{R}P^5) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with the generators corresponding to the maps η and η^2 (consult the Hilton-Milnor theorem).
- (e) By using the action of the Steenrod operations on $H^*(\mathbb{R}P^8/\mathbb{R}P^5; \mathbb{Z}/2)$, make some deductions about the attaching map for the 8-cell and use this to analyze the long exact sequence from (c). Conclude that $\pi_8(\mathbb{R}P^8/\mathbb{R}P^5) = \mathbb{Z}/2$ with the generator represented by the composite $S^8 \xrightarrow{\eta} S^7 \hookrightarrow S^7 \vee S^6 = \mathbb{R}P^7/\mathbb{R}P^5 \hookrightarrow \mathbb{R}P^8/\mathbb{R}P^5$.
- (f) Finally, repeat the strategy from (e) to analyze the attaching map of the 9-cell in $\mathbb{R}P^9/\mathbb{R}P^5$ and deduce that $\pi_8(\mathbb{R}P^9/\mathbb{R}P^5) = 0$.

The arguments in this section naturally suggest the following problem, which is open:

Problem: Compute the geometric dimension of $k([L] - 1) \in \widetilde{KO}(\mathbb{R}P^n)$, for any given k and n .

Over the years this problem has been extensively studied by Adams, Davis, Gitler, Lam, Mahowald, Randall, and many others. Just a few of the many references are [GM1, GM2, Adams6, Astey, DGM, LR1, LR2, BDM]. See also Section 9.15 for a bit more discussion.

8.35. Summary. In this section we obtained two sets of non-immersion/non-embedding results, one using Stiefel-Whitney classes in mod 2 singular cohomology and the other using the $\tilde{\gamma}$ classes in KO -theory. We also translated the immersion problem into a question about geometric dimension of reduced bundles, and for $\mathbb{R}P^n$ we completely solved this question for $n \leq 9$. As we have said before, this is far from the whole story—in fact it is just the very tip of a large and interesting iceberg. We refer the reader to the references cited in [Davis2] for other pieces of the story, as well as to the survey paper [Davis1].

9. THE SUMS-OF-SQUARES PROBLEM AND BEYOND

This section is in some ways an epilogue to the previous one. In the last section we started with a geometric problem, that of immersing $\mathbb{R}P^n$ into Euclidean space. We then used cohomology theories and characteristic classes to obtain necessary conditions for such an immersion to exist: we obtained two sets of conditions, one

from mod 2 singular cohomology and one from KO -theory. In the present section we start with an *algebraic* problem, one that at first glance seems completely unrelated to immersions. It is the problem of finding sums-of-squares formulas in various dimensions, which we encountered already back in Section ?? (we will review the problem below). Once again we will use cohomology theories to obtain necessary conditions for the existence of such formulas. The surprise is that these conditions are basically the same as the ones that arose in the immersion problem! This is because both problems lead to the same homotopy-theoretic situation involving bundles over real projective space.

In theory the present section could be read completely independently of the last one. But because the underlying homotopy-theoretic problem is the same, we refer to the previous section for many details of its analysis.

9.1. Review of the basic problem. Recall from Section ?? that a **sums-of-squares formula of type $[r, s, n]$** (over $\mathbb{R}$) is a bilinear pairing $\phi: \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^n$ with the property that

$$(9.2) \quad |\phi(x, y)|^2 = |\phi(x)|^2 \cdot |\phi(y)|^2$$

for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. If we write $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$ then $\phi(x, y) = (z_1, \dots, z_n)$ where each z_i is a bilinear expression in the x 's and y 's. Formula (9.2) becomes

$$(x_1^2 + \dots + x_r^2) \cdot (y_1^2 + \dots + y_s^2) = z_1^2 + \dots + z_n^2.$$

We will also refer to this as an “ $[r, s, n]$ -formula” for short. Note that if an $[r, s, n]$ -formula exists then one trivially has $[i, j, k]$ -formulas for any $i \leq r$, $j \leq s$, and $k \geq n$.

For what values of r , s , and n does an $[r, s, n]$ -formula exist? This is the **sums-of-squares** problem. Said differently, given a specific r and s what is the smallest value of n for which an $[r, s, n]$ -formula exists? Call this number $r * s$. As with the immersion problem, there are two aspects here. One is the problem of constructing sums-of-squares formulas, thereby giving upper bounds for $r * s$; the other is the problem of finding necessary conditions for their existence, thereby giving lower bounds. The latter is the part that involves topology.

The sums-of-squares formulas that everyone knows are the ones coming from the multiplications on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. These have types $[1, 1, 1]$, $[2, 2, 2]$, $[4, 4, 4]$, and $[8, 8, 8]$. Hurwitz proved that an $[n, n, n]$ formula only exists when $n \in \{1, 2, 4, 8\}$, and the Hurwitz-Radon theorem generalizes this:

Theorem 9.3 (Hurwitz-Radon). *A sums-of-squares formula of type $[r, n, n]$ exists if and only if $r \leq \rho(n) + 1$, where $\rho(n)$ is the Hurwitz-Radon number from Definition ??.*

This “Hurwitz-Radon theorem” is really the same as the one on vector fields on spheres that we saw as Theorem ??. The proof is the same as well:

Proof. Write $n = (\text{odd}) \cdot 2^{a+4b}$ with $0 \leq a \leq 3$. Recall from Theorem ?? that an $[r, n, n]$ -formula exists if and only if there exists a Cl_{r-1} -module structure on $\mathbb{R}^n$. We saw in Section ?? that representations of Cl_{r-1} only exist on vector spaces

whose dimension is a multiple of $2^{\varphi(r-1)}$. Thus, we have the chain of equivalences

$$\begin{aligned}
 \text{an } [r, n, n]\text{-formula exists} &\iff \text{there exists a } \text{Cl}_{r-1}\text{-module structure on } \mathbb{R}^n \\
 &\iff 2^{\varphi(r-1)} \mid n \\
 &\iff \varphi(r-1) \leq a+4b \\
 &\iff r-1 \leq 2^a+8b-1 = \rho(n).
 \end{aligned}$$

For the last equivalence note that $\varphi(2^a+8b-1) = \varphi(2^a-1)+4b = a+4b$, where the first equality is the 8-fold quasi-periodicity of φ and the second is just a calculation for $0 \leq a \leq 3$. Moreover, 2^a+8b-1 is the *largest* number whose φ -value is $a+4b$; by periodicity this can again be checked just for $b=0$ and $0 \leq a \leq 3$. So in general we have $\varphi(s) \leq a+4b = \varphi(2^a+8b-1)$ if and only if $s \leq 2^a+8b-1$; this is what is used in the final equivalence. $\square$

Remark 9.4. Note that if there exist formulas of type $[r, s_1, n_1]$ and $[r, s_2, n_2]$ then there is also a formula of type $[r, s_1+s_2, n_1+n_2]$ (by distributivity). This says that

$$r * (s_1 + s_2) \leq r * s_1 + r * s_2.$$

Also notice that $r * s \leq (r+a) * (s+b)$ whenever $a, b \geq 0$, because a formula of type $[r+a, s+b, n]$ automatically yields one of type $[r, s, n]$ by plugging in zeros for a of the x 's and b of the y 's.

The classical identities show that $2 * 2 = 2$, $4 * 4 = 4$, and $8 * 8 = 8$, and it is trivial that $n * 1 = n$. Using these together with the observations of the previous paragraph, one can obtain upper bounds on $r * s$. For example, $3 * 10 \leq 12$ because

$$3 * 10 \leq 3 * 8 + 3 * 2 \leq 8 * 8 + 4 * 4 = 8 + 4 = 12.$$

Exercise 9.5. Use the above methods to show $9 * 9 \leq 25$ and try to convince yourself that this is the best possible upper bound obtainable using these methods.

Table 9.5 shows what is known about $r * s$ for small values of r and s . For $r \leq 8$ the values completely agree with the upper bounds obtained by the methods of Remark 9.4. As soon as one hits $r = 9$ there is quite a bit of difference, though.

TABLE 9.5. Values of $r * s$

$r \backslash s$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16	18
3		4	4	7	8	8	8	11	12	12	12	15	16	16	16	19
4			4	8	8	8	8	12	12	12	12	16	16	16	16	20
5				8	8	8	8	13	14	15	16	16	16	16	16	21
6					8	8	8	14	14	16	16	16	16	16	16	22
7						8	8	15	16	16	16	16	16	16	16	23
8							8	16	16	16	16	16	16	16	16	24
9								16	16	[16,17]	??					
10									16	[16,17]						
11										17						

To justify the numbers in the above table, one thing we have to do is produce lower bounds for $r * s$. For example, we have to explain why there do not exist

formulas of type [5, 10, 13]. Almost all of the known lower bounds come from topological methods; we will describe some of these next.

9.6. Lower bounds via topology. Here is the key result that shows how a sums-of-squares formula gives rise to something homotopy-theoretic:

Proposition 9.7. *If an $[r, s, n]$ -formula exists then there exists a rank $n - r$ bundle E on $\mathbb{R}P^{s-1}$ such that $rL \oplus E \cong \underline{n}$ (here rL is the direct sum of r copies of L).*

Proof. Let $\phi: \mathbb{R}^r \otimes \mathbb{R}^s \rightarrow \mathbb{R}^n$ be the map giving the sums-of-squares formula. If $u \in \mathbb{R}^s$ is a unit vector, check that $\phi(e_1, u), \dots, \phi(e_r, u)$ is an orthonormal frame in $\mathbb{R}^n$. This is an easy consequence of the sums-of-squares identity; it is an exercise, but see the proof of Corollary ?? if you get stuck. In this way we obtain a map $f: S^{s-1} \rightarrow V_r(\mathbb{R}^n)$. Compose with the projection $V_r(\mathbb{R}^n) \rightarrow \text{Gr}_r(\mathbb{R}^n)$ and then note that the map factors to give $F: \mathbb{R}P^{s-1} \rightarrow \text{Gr}_r(\mathbb{R}^n)$. Precisely, given a line in $\mathbb{R}^s$ spanned by a vector u its image under F is the r -plane spanned by $\phi(e_1, u), \dots, \phi(e_r, u)$.

Let η be the tautological r -plane bundle on $\text{Gr}_r(\mathbb{R}^n)$. We claim that $F^*\eta = rL$. This follows from the commutative diagram

$$\begin{array}{ccc} rL & \xrightarrow{\tilde{F}} & \eta \\ \downarrow & & \downarrow \\ \mathbb{R}P^{s-1} & \xrightarrow{F} & \text{Gr}_r(\mathbb{R}^n) \end{array}$$

where the top map is described as follows. Given r points on the same line $\langle u \rangle$ we write them as $\lambda_1 u, \dots, \lambda_r u$ and then send them to the element $\lambda_1 \phi(e_1, u) + \dots + \lambda_r \phi(e_r, u)$ on the r -plane $F(\langle u \rangle)$. One readily checks that this does not depend on the choice of u ; in fact, we could just say that points $z_1, \dots, z_r$ on a common line ℓ are sent to the element $\phi(e_1, z_1) + \dots + \phi(e_r, z_r)$ on $F(\ell)$. The fact that $\phi(e_1, u), \dots, \phi(e_r, u)$ are orthonormal (hence independent) shows that $\tilde{F}$ is injective on fibers, hence an isomorphism on fibers. This shows that $F^*\eta \cong rL$.

We have now done all the hard work. To finish, just recall that η sits inside a short exact sequence $0 \rightarrow \eta \rightarrow \underline{n} \rightarrow Q \rightarrow 0$ where Q is the standard quotient bundle. This sequence is split because $\text{Gr}_r(\mathbb{R}^n)$ is compact. Pulling back along F now gives $rL \oplus F^*Q \cong \underline{n}$, as desired. $\square$

The following result was originally proven independently by Hopf [Hopf] and Stiefel [Stiefel]; Stiefel's method is the one we follow here.

Corollary 9.8 (Hopf-Stiefel). *If an $[r, s, n]$ -formula exists then the following two equivalent conditions hold:*

- (1) $\binom{r+i-1}{i}$ is even for $n - r < i < s$;
- (2) $\binom{n}{i}$ is even for $n - r < i < s$.

Proof. By Proposition 9.7 we know that $rL \oplus E \cong \underline{n}$ for some rank $n - r$ bundle on $\mathbb{R}P^{s-1}$. Applying total Stiefel-Whitney classes gives $w(rL)w(E) = w(rL \oplus E) = w(\underline{n}) = 1$, or $w(E) = w(rL)^{-1} = w(L)^{-r}$. So $w(L)^{-r}$ vanishes in degrees larger than $n - r$. But $w(L) = 1 + x$ where x is the generator for $H^1(\mathbb{R}P^{s-1}; \mathbb{Z}/2)$, and the coefficient of x^i in $(1 + x)^{-r}$ is $\binom{-r}{i} = \binom{r+i-1}{i}$ (recall that we are working modulo 2). So $\binom{r+i-1}{i}$ is even for $n - r < i < s$.

The equivalence of the conditions in (1) and (2) follows at once from the lemma below, taking $k = n - r + 1$ and $u = r + s - n - 2$ (note that conditions (1) and (2) are both vacuous unless $u \geq 0$). $\square$

Lemma 9.9. *For any non-negative integers n , k , and u , the following $\mathbb{Z}$ -linear spans are the same inside of $\mathbb{Z}$ (that is, the lists on the two sides generate the same ideal):*

$$\mathbb{Z}\left\langle \binom{n}{k}, \binom{n}{k+1}, \dots, \binom{n}{k+u} \right\rangle = \mathbb{Z}\left\langle \binom{n}{k}, \binom{n+1}{k+1}, \dots, \binom{n+u}{k+u} \right\rangle.$$

Consequently, an integer is a common divisor of the first set of binomial coefficients if and only if it is a common divisor of the second set.

Proof. Taking first differences and using Pascal's identity shows (via multiple iterations) that

$$\begin{aligned} \mathbb{Z}\left\langle \binom{n}{k}, \binom{n+1}{k+1}, \dots, \binom{n+u}{k+u} \right\rangle &= \mathbb{Z}\left\langle \binom{n}{k}, \binom{n}{k+1}, \binom{n+1}{k+2}, \binom{n+2}{k+3}, \dots, \binom{n+u-1}{k+u} \right\rangle \\ &= \mathbb{Z}\left\langle \binom{n}{k}, \binom{n}{k+1}, \binom{n}{k+2}, \binom{n+1}{k+3}, \dots, \binom{n+u-2}{k+u} \right\rangle \\ &= \dots \\ &= \mathbb{Z}\left\langle \binom{n}{k}, \binom{n}{k+1}, \dots, \binom{n}{k+u} \right\rangle. \end{aligned}$$

$\square$

Example 9.10. Does a formula of type $[10, 10, 15]$ exist? If it did, statement (2) of Corollary 9.8 would imply that $\binom{15}{i}$ is even for $5 < i < 10$. But $\binom{15}{6}$ is odd.

The full power of the numerical conditions in Corollary 9.8 is subtle, and one really needs a computer to thoroughly investigate them. But the following consequence represents much of the information buried in those conditions:

Corollary 9.11. *If $r + s > 2^k$ then $r * s \geq 2^k$.*

Proof. We must show that if $r + s > 2^k$ then formulas of type $[r, s, 2^k - 1]$ do not exist. If they did, the Hopf-Stiefel conditions would imply that $\binom{2^k-1}{i}$ is even for i in the range $2^k - 1 - r < i < s$. But $\binom{2^k-1}{i}$ is odd no matter what i is, so the conditions are only consistent if the range is empty—or equivalently, if $2^k - 1 - r \geq s - 1$. The hypothesis $r + s > 2^k$ guarantees that this is not the case. $\square$

For example, sums-of-squares formulas of type $[16, 17, n]$ must all have $n \geq 32$.

For $r \leq 8$ the Hopf-Stiefel lower bounds for $r * s$ turn out to exactly match the upper bounds obtained via the constructive methods of Remark 9.4. So this justifies the numbers in Table 9.5 for the range $r \leq 8$.

9.12. K-theoretic techniques. We can also analyze the implications of Proposition 9.7 using KO -theory. This was first done by Yuzvinsky [Yuzvinsky]. Note that one could also use complex K -theory here, but KO -theory gives stronger results: the point is that $\widetilde{K}^0(\mathbb{R}P^m)$ and $\widetilde{KO}^0(\mathbb{R}P^m)$ are almost the same, but for certain values of m the latter group is slightly bigger (by a factor of 2).

Corollary 9.13 (Yuzvinsky). *If an $[r, s, n]$ -formula exists then the following two equivalent conditions hold:*

(1) $2^{\varphi(s-1)-i+1}$ divides $\binom{r+i-1}{i}$ for $n - r < i \leq \varphi(s - 1)$;

(2) $2^{\varphi(s-1)-i+1}$ divides $\binom{n}{i}$ for $n-r < i \leq \varphi(s-1)$.

Proof. Proposition 9.7 gives that $rL \oplus E \cong \underline{n}$ for some bundle E on $\mathbb{R}P^{s-1}$. We again use characteristic classes, but this time the $\tilde{\gamma}$ classes in KO -theory. We find that $\tilde{\gamma}_t(E) = \tilde{\gamma}_t(L)^{-r}$, and so $\tilde{\gamma}_t(L)^{-r}$ must vanish in degrees larger than $n-r$. Recall that $\tilde{\gamma}_t(L) = 1 + t\lambda$ where $\lambda = [L] - 1$, and so the coefficient of t^i in $\tilde{\gamma}_t(L)^{-r}$ is $\binom{-r}{i}\lambda^i = \pm \binom{r+i-1}{i}2^{i-1}\lambda$. Here we have used $\lambda^2 = -2\lambda$.

Recalling that $\widetilde{KO}^0(\mathbb{R}P^{s-1}) \cong \mathbb{Z}/(2^{\varphi(s-1)})$, we find that $2^{\varphi(s-1)}$ divides $2^{i-1}\binom{r+i-1}{i}$ for all $i > n-r$. This statement only has content for $i \leq \varphi(s-1)$, and thus we obtain the condition in (1).

The equivalence of (1) and (2) is an instance of the following general observation: the sequence of conditions

$$2^N \mid \binom{n}{k}, 2^{N-1} \mid \binom{n}{k+1}, 2^{N-2} \mid \binom{n}{k+2}, \dots, 2^{N-j} \mid \binom{n}{k+j}$$

is equivalent to the sequence of conditions

$$2^N \mid \binom{n}{k}, 2^{N-1} \mid \binom{n+1}{k+1}, 2^{N-2} \mid \binom{n+2}{k+2}, \dots, 2^{N-j} \mid \binom{n+j}{k+j}.$$

This follows at once by applying Lemma 9.9 multiple times, with $i = 1, i = 2, \dots, i = j$. $\square$

The Hopf-Stiefel conditions are symmetric in r and s , but this is not true for the KO -theoretic conditions in the above proposition. For example, applying the conditions yields no information on $[3, 6, n]$ -formulas beyond $n \geq 5$ (which is trivial), whereas applying the conditions to $[6, 3, n]$ -formulas yields $n \geq 8$. One must therefore apply the conditions to both $[r, s, n]$ and $[s, r, n]$ to get the best information.

Example 9.14. Like we saw for the immersion problem, in some dimensions the KO -theoretic conditions are stronger than the Hopf-Stiefel conditions—and in some dimensions they are weaker. Neither result is strictly stronger than the other.

For example, the Hopf-Stiefel conditions show that $4 * 5 \geq 8$ whereas the KO -conditions only show $4 * 5 = 5 * 4 \geq 7$. The smallest dimension for which the KO -theoretic conditions are stronger is when $r = 10$ and $s = 15$. The Hopf-Stiefel conditions rule out the existence of $[10, 15, 15]$ -formulas, but not $[10, 15, 16]$. The KO -conditions rule out $[15, 10, 16]$, however, and therefore also $[10, 15, 16]$ by symmetry.

To pick a larger example, the Hopf-Stiefel conditions show that $127 * 127 \geq 128$ but the KO -conditions show that $127 * 127 \geq 184$. The KO -conditions seem to give their greatest power when r and s are slightly less than a power of 2.

9.15. Other problems. The reader will probably have noticed that the above methods and results for the existence of sums-of-squares formulas are almost exactly the same as the ones in the last section on the existence of immersions for real projective spaces. It is natural to wonder if there is a closer connection between these two problems. There is! In fact there turns out to be a whole family of interconnected problems circling this area. We will explain this next.

A careful look at Proposition 9.7 shows that one can make the argument work with something much weaker than a sums-of-squares formula. Specifically, all we needed was a map $f: \mathbb{R}^r \otimes \mathbb{R}^s \rightarrow \mathbb{R}^n$ such that $f(x \otimes y) = 0$ only when $x = 0$ or $y = 0$. Such a bilinear map is usually called **nonsingular**. Given such a map and a nonzero $u \in \mathbb{R}^r$, the elements $\phi(u, e_1), \dots, \phi(u, e_r)$ are necessarily linearly independent—and this is really all that was needed in the proof of Proposition 9.7.

We can replace our sums-of-squares problem with the following: given r and s , for what values of n does there exist a nonsingular bilinear map of type $[r, s, n]$? The topological obstructions we found for sums-of-squares formulas are of course still valid in this new context.

The existence of nonsingular bilinear maps turns out to be related to the immersion problem for real projective spaces. More than this, both problems are connected to a number of similar questions that have been intently studied by algebraic topologists since the 1940s. Many of these problems were originally raised by Hopf [Hopf]. This material takes us somewhat away from our main theme of K -theory, but it seems worthwhile to tell a bit of this story since we have encountered it.

To start with, let us introduce the following classes of statements:

- SS $[r, s, n]$: there exists a sums-of-squares formula of type $[r, s, n]$
- NS $[r, s, n]$: there exists a nonsingular bilinear map $\mathbb{R}^r \otimes \mathbb{R}^s \rightarrow \mathbb{R}^n$
- RR $[r, s, n]$: there exist $n \times s$ matrices $A_1, \dots, A_r$ with the property that every nonzero linear combination of them has rank s
- T $[r, s, n]$: the tangent bundle $T_{\mathbb{R}P^n}$ has s independent sections when restricted to $\mathbb{R}P^r$
- IS $[r, s, n]$: the bundle $nL \rightarrow \mathbb{R}P^r$ has s independent sections
- GD $[r, s, n]$: over $\mathbb{R}P^r$ one has $\text{g. dim}(n(L - 1)) \leq n - s$
- ES $[r, s, n]$: the ‘first-vector’ map $p_1: V_s(\mathbb{R}^{n+1}) \rightarrow S^n$ has a $\mathbb{Z}/2$ -equivariant section over the subspace $S^r \subseteq S^n$. Here $\mathbb{Z}/2$ acts antipodally on $\mathbb{R}^{n+1}$, and both $V_s(\mathbb{R}^{n+1})$ and S^n get the induced action.
- AX $[r, s, n]$: there exists an ‘axial map’ $\mathbb{R}P^r \times \mathbb{R}P^s \rightarrow \mathbb{R}P^n$; this is a map with the property that the restrictions $\mathbb{R}P^r \times \{*\} \rightarrow \mathbb{R}P^n$ and $\{*\} \times \mathbb{R}P^s \rightarrow \mathbb{R}P^n$ are homotopic to linear embeddings, for some choice of basepoints in $\mathbb{R}P^r$ and $\mathbb{R}P^s$
- IM $[r, n]$: $\mathbb{R}P^r$ immerses into $\mathbb{R}P^n$
- VF $[k, n]$: there exist k independent vector fields on S^n .

The acronyms are mostly self-evident, except for a few: RR stands for ‘rigid rank’, IS for ‘independent sections’, and ES for ‘equivariant sections’. For the axial maps, note that if $n > r$ then $[\mathbb{R}P^r, \mathbb{R}P^n] \rightarrow [\mathbb{R}P^r, \mathbb{R}P^\infty] = H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ is an isomorphism and therefore there are precisely two homotopy classes $\mathbb{R}P^r \rightarrow \mathbb{R}P^n$, represented by the constant map and the linear inclusion. So when $n > r$ and $n > s$ (which is the only case of interest) the axial map condition is equivalent to saying that the two restrictions are non-null.

The above statements are closely interrelated, as the next result demonstrates. We should point out that very little from this result will be needed in our subsequent discussion. We are including it because most of the claims are easy to prove, and because the various statements get used almost interchangeably (often without much explanation) in the literature on the immersion problem.

Proposition 9.16.

- (a) $\text{GD}[r, s, n] \iff \text{GD}[r, -n, -s]$
- (b) $\text{IM}[r, n] \iff \text{GD}[r, -(n+1), -(r+1)] \iff \text{GD}[r, r+1, n+1]$
- (c) $T[n, k, n] \Rightarrow \text{VF}[k, n]$

(d) One has the following implications:

$$\begin{array}{ccccc}
 \text{SS}[r, s, n] & \xrightarrow{1} & \text{NS}[r, s, n] & \xleftrightarrow{2} & \text{RR}[r, s, n] & & \text{ES}[r-1, s, n-1] \\
 & & & & \downarrow 3 & \swarrow & \downarrow 6 \\
 \text{GD}[r-1, s, n] & \xleftarrow{5} & \text{IS}[r-1, s, n] & \xleftarrow{4} & \text{T}[r-1, s-1, n-1] & & \text{AX}[r-1, s-1, n-1]
 \end{array}$$

(e) If $r < n$ then implication 4 is reversible, and if $r \leq n$ then 5 is reversible.

(f) If $r < n$ and $r \leq 2(n-s)$ then implication 6 in (d) is also reversible.

Proof. Part (a) is Proposition 8.31, and part (b) is Corollary 8.32; we have seen these already. Part (c) follows from the fact that $T_{S^n} = p^*T_{\mathbb{R}P^n}$ where $p: S^n \rightarrow \mathbb{R}P^n$ is the projection.

For part (d), the first implication is obvious. The others we treat one by one.

NS $[r, s, n] \iff \mathbf{RR}[r, s, n]$: The equivalence follows from adjointness, as bilinear maps $f: \mathbb{R}^r \otimes \mathbb{R}^s \rightarrow \mathbb{R}^n$ correspond bijectively to linear maps $F: \mathbb{R}^r \rightarrow \text{Hom}(\mathbb{R}^s, \mathbb{R}^n)$. The linear map F is specified by the $n \times s$ matrices $F(e_1), \dots, F(e_r)$. It is easy to verify that f is nonsingular if and only if all nontrivial linear combinations of these matrices have rank s .

RR $[r, s, n] \Rightarrow \mathbf{T}[r-1, s-1, n-1]$: First note that the tangent bundle of $\mathbb{R}P^n$ is the collection of pairs (x, v) such that $x, v \in \mathbb{R}^n$, $|x| = 1$, and $x \cdot v = 0$, modulo the identifications $(x, v) \sim (-x, -v)$. Secondly, note that since **NS** $[r, s, n]$ is symmetric in r and s the same is true for **RR** $[r, s, n]$. The condition **RR** $[s, r, n]$ says that we have $n \times r$ matrices $A_1, \dots, A_s$ such that every nontrivial linear combination has rank r . For any $x \in \mathbb{R}^r - 0$ it follows that $A_1x, \dots, A_sx$ are independent: for if $\sum t_i A_i x = 0$ then $\sum t_i A_i$ has rank less than r , which only happens when all $t_i = 0$. Note that there is a matrix $P \in GL_n(\mathbb{R})$ such that the columns of $A_1 P$ are the standard basis $e_1, \dots, e_r$, so by replacing each A_i with $A_i P$ we can just assume these are the columns of A_1 .

For every $x \in S^{r-1} \subseteq \mathbb{R}^r \subseteq \mathbb{R}^n$ consider the independent vectors $A_1x, A_2x, A_3x, \dots, A_sx$. Note that $A_1x = x$. Let $u_i(x) = A_i x - (A_i x \cdot x)x$. Then $u_2(x), \dots, u_r(x)$ are independent, and orthogonal to x . Since $u_i(-x) = -u_i(x)$ for each i , these give us $s-1$ independent sections of $T_{\mathbb{R}P^{n-1}}$ defined over $\mathbb{R}P^{r-1}$.

T $[r-1, s-1, n-1] \Rightarrow \mathbf{IS}[r-1, s, n]$: This follows from the bundle isomorphism $T_{\mathbb{R}P^{n-1}} \oplus 1 \cong nL$ (see Example ??).

IS $[r-1, s, n] \Rightarrow \mathbf{GD}[r-1, s, n]$: Trivial.

T $[r-1, s-1, n-1] \iff \mathbf{ES}[r-1, s, n-1]$: First note that the frame bundle $V_{s-1}(T_{\mathbb{R}P^{n-1}})$ is homeomorphic to $V_s(\mathbb{R}^n)/\pm 1$ in an evident way. Under this homeomorphism the projection $V_{s-1}(T_{\mathbb{R}P^{n-1}}) \rightarrow \mathbb{R}P^{n-1}$ corresponds to the first-vector map $V_s(\mathbb{R}^n)/\pm 1 \rightarrow S^{n-1}/\pm 1$. So **T** $[r-1, s-1, n-1]$ is equivalent to the latter bundle having a section over $S^{r-1}/\pm 1$. But then consider the diagram

$$\begin{array}{ccc}
 V_s(\mathbb{R}^n) & \longrightarrow & V_s(\mathbb{R}^n)/\pm 1 \\
 \downarrow & & \downarrow \\
 S^{n-1} & \longrightarrow & S^{n-1}/\pm 1
 \end{array}$$

where the two horizontal maps are 2-fold covering spaces. This is a pullback square. It is easy to see that the right vertical map has a section defined over $S^{r-1}/\pm 1$ if and only if the left vertical map has a $\mathbb{Z}/2$ -equivariant section defined over S^{r-1} .

ES $[r-1, s, n-1] \Rightarrow$ **AX** $[r-1, s-1, n-1]$: Note that there is an evident map $V_s(\mathbb{R}^n) \rightarrow \mathcal{J}op(S^{s-1}, S^{n-1})$ that sends a frame $v_1, \dots, v_s$ to the map $(a_1, \dots, a_s) \mapsto a_1 v_1 + \dots + a_s v_s$. So a section $\chi: S^{r-1} \rightarrow V_s(\mathbb{R}^n)$ gives by composition a map $S^{r-1} \rightarrow \mathcal{J}op(S^{s-1}, S^{n-1})$, and then by adjointness a map $g: S^{r-1} \times S^{s-1} \rightarrow S^{n-1}$. The fact that χ was a section of the first-vector map shows that $g(x, e_1) = x$ for all $x \in S^{r-1}$. Also, it is clear that $g(e_1, -)$ is a linear inclusion $S^{s-1} \hookrightarrow S^{n-1}$ (there is nothing special about e_1 here). The $\mathbb{Z}/2$ -equivariance of χ shows that $g(-x, y) = -g(x, y)$, and the similar identity $g(x, -y) = -g(x, y)$ is trivial. So g descends to give a map $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$, and this map is axial.

For (e), the reversibility of both implications is governed by stability theory of vector bundles. For example, assume **IS** $[r-1, s, n]$. Then $nL \cong s \oplus E$ for some bundle E of rank $n-s$, where we are working over $\mathbb{R}P^{r-1}$. Recall that $j^*T_{\mathbb{R}P^{n-1}} \oplus 1 \cong nL$, where $j: \mathbb{R}P^{r-1} \hookrightarrow \mathbb{R}P^{n-1}$ is the inclusion. So $j^*T_{\mathbb{R}P^{n-1}} \oplus 1 \cong s \oplus E \cong 1 \oplus ((s-1) \oplus E)$. Since we are working over $\mathbb{R}P^{r-1}$ and $r-1 < n-1$, we can cancel the 1 on both sides to get $j^*T_{\mathbb{R}P^{n-1}} \cong (s-1) \oplus E$ (see Proposition ??). This says that **T** $[r-1, s-1, n-1]$ holds. The reversibility of implication 5 is very similar, and is left to the reader.

Part (f), on the reversibility of implication 6, is the only part of the proposition that is not elementary. Let $\mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ denote the space of $\mathbb{Z}/2$ -equivariant maps, where the spheres have the antipodal action. This is a subspace of the usual function space $\mathcal{J}op(S^{s-1}, S^{n-1})$. Note that the space of equivariant maps has a $\mathbb{Z}/2$ -action, given by composing (or equivalently, precomposing) a given map with the antipodal map. James [James1, Theorem 2.1] shows that the evident map $\mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \rightarrow \mathcal{J}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1})$ is a principal $\mathbb{Z}/2$ -bundle with respect to the above action.

We will also have need of the evaluation map $\text{ev}: \mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \rightarrow S^{n-1}$ sending $h \mapsto h(e_1)$. James [James1, Theorem 2.3] shows that this is also a fibration.

Note that the Stiefel manifold $V_s(\mathbb{R}^n)$ is a subspace of $\mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ in the evident way, and that the following diagram commutes:

$$\begin{array}{ccc} V_s(\mathbb{R}^n) & \xrightarrow{\quad} & \mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \\ & \searrow p_1 & \swarrow \text{ev} \\ & S^{n-1} & \end{array}$$

If all of the spaces are given the evident sign action of $\mathbb{Z}/2$, all of the maps are equivariant. The key to the rest of the argument is that $V_s(\mathbb{R}^n) \hookrightarrow \mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ is $[2(n-s)-1]$ -connected: see James [James1, Theorem 6.5], or [HH, Lemma 1.1]), or Exercise 9.24 below. It follows that if $r \leq 2(n-s)-1$ then any map $S^r \rightarrow \mathcal{J}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ can be factored up to homotopy through $V_s(\mathbb{R}^n)$. Moreover, if the map is equivariant this can be done in the category of $\mathbb{Z}/2$ -equivariant pointed spaces over S^{n-1} (the only hard part here is the $\mathbb{Z}/2$ -equivariance, and for this one uses that S^{r-1} has an equivariant cell decomposition made from free $\mathbb{Z}/2$ -cells). See Exercise 9.19 below for details.

Suppose $f: \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \rightarrow \mathbb{R}P^{n-1}$ is an axial map. By the homotopy extension property we can assume that $f(*, -)$ and $f(-, *)$ are both equal to the canonical embeddings, where $*$ = $[1: 0: 0: \dots]$ is our chosen basepoint. The axial map f gives a map $F: \mathbb{R}P^{r-1} \rightarrow \mathcal{Top}(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1})$ by adjointness. Regard the target as pointed by the canonical embedding j , and note that F is a pointed map.

Covering space theory gives that there is a unique map $\tilde{F}: S^{r-1} \rightarrow \mathcal{Top}_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ such that $\tilde{F}(e_1)$ is the canonical embedding $S^{s-1} \hookrightarrow S^{n-1}$ and such that the left square below commutes:

$$\begin{array}{ccccc} S^{r-1} & \xrightarrow{\tilde{F}} & \mathcal{Top}_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) & \xrightarrow{\text{ev}} & S^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}P^{r-1} & \xrightarrow{F} & \mathcal{Top}(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1}) & \xrightarrow{\text{ev}} & \mathbb{R}P^{n-1}. \end{array}$$

Note that $\text{ev} \circ F$ is the standard inclusion, so covering space theory implies that $\text{ev} \circ \tilde{F}$ is also the standard inclusion $S^{r-1} \hookrightarrow S^{n-1}$. Therefore $\tilde{F}(e_1) \neq \tilde{F}(-e_1)$, and so we must instead have $\tilde{F}(-e_1) = A\tilde{F}(e_1)$, where A denotes the antipodal action. Then $\tilde{F}$ and $A\tilde{F}A$ agree at e_1 and are both liftings of F , hence they are equal. This says that $\tilde{F}$ is $\mathbb{Z}/2$ -equivariant.

If $r-1 \leq 2(n-s)-1$ then we can now factor $\tilde{F}$, equivariantly and over S^{n-1} , though the space $V_s(\mathbb{R}^n)$. In this way one produces the required equivariant section of the map $V_s(\mathbb{R}^n) \rightarrow S^{n-1}$. $\square$

Remark 9.17. We have included the above proposition because it is very useful as a reference. However, it should be pointed out that there is something slightly deceptive about part (d). Some of the implications are more obvious than the long chains would suggest. For example, $\text{NS}[r, s, n] \Rightarrow \text{AX}[r-1, s-1, n-1]$ is a very easy argument of one or two lines. Likewise, $\text{NS}[r, s, n] \Rightarrow \text{GD}[r-1, s, n]$ is just the argument in Proposition 9.7. The picture in (d) is useful in showing *all* the relations at once, but it makes some of the statements seem more distant than they really are.

The reader will have noticed that Proposition 9.16 encodes several things that we have seen before. Some are transparently familiar, like parts (a) and (b). A less transparent example is

$$\text{SS}[r, n, n] \iff \text{SS}[n, r, n] \Rightarrow \text{T}[n-1, r-1, n-1] \Rightarrow \text{VF}[r-1, n-1].$$

This was the content of Corollary ???. In contrast, here is a similarly-obtained implication that we have *not* seen yet:

$$\text{NS}[r, r, n] \Rightarrow \text{GD}[r-1, r, n] \iff \text{IM}[r-1, n-1].$$

From this we learn that immersion results for real projective space can be obtained by demonstrating the existence of nonsingular bilinear maps. This approach was successfully used by K.Y. Lam in [Lam1]. We briefly sketch his method simply to give the basic idea; for details the reader may consult [Lam1] and similar papers.

Recall that $\mathbb{O}$ denotes the octonions. Consider the map $f: \mathbb{O}^2 \times \mathbb{O}^2 \rightarrow \mathbb{O}^3$ given by

$$f((u_1, u_2), (x_1, x_2)) = (u_1x_1 - \bar{x}_2u_2, x_2u_1 + u_2\bar{x}_1, u_2x_2 - x_2u_2).$$

With a little work one can prove that this is nonsingular. Also, it is a general fact about the octonions that for any $a, b \in \mathbb{O}$ the commutator $[a, b] = ab - ba$

is imaginary. So the image of f actually lies in the 23-dimensional subspace of $\mathbb{O}^3$ where the real part of the third coordinate vanishes. So f gives a nonsingular bilinear map of type [16, 16, 23]. This shows that $\mathbb{R}P^{15}$ immerses into $\mathbb{R}^{22}$. By restricting f to appropriate subspaces Lam also obtained nonsingular bilinear maps of types [11, 11, 17], [13, 13, 19], and [10, 10, 16], thereby proving that $\mathbb{R}P^{10}$ immerses into $\mathbb{R}^{16}$, $\mathbb{R}P^{12}$ immerses into $\mathbb{R}^{18}$, and $\mathbb{R}P^9$ immerses into $\mathbb{R}^{15}$. Amazing stuff!

The literature on all of these problems is enormous, so that it is not practical to give a complete list of references. But a few that can serve as a starting point are [AGJ], [DGM], [James2], [Lam1], [Milgram], ????

9.18. Summary. In this section we examined the sums-of-squares problem, and saw how characteristic classes can be used to obtain lower bounds for the numbers $r * s$. Use of Stiefel-Whitney classes in singular cohomology yielded the Hopf-Stiefel lower bounds on $r * s$, whereas the use of the γ -classes in KO -theory gave the Yuzvinsky lower bounds. This story is very similar to the one for immersions of $\mathbb{R}P^n$ discussed in Section 8, and in fact the sums-of-squares problem is closely connected to this immersion problem. We closed the section by exploring the relations between these and a host of similar problems.

◦ Exercises ◦

Exercise 9.19. It is a fact that $X \rightarrow Y$ is an N -equivalence if and only if for every $k \leq N$, every solid-arrow diagram

$$(9.20) \quad \begin{array}{ccc} \partial D^k & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ D^k & \longrightarrow & Y \end{array}$$

admits a lifting that makes the upper triangle commute and the lower triangle commute up to a homotopy rel ∂D^k . See [DI2], for example. Call this a “relative-homotopy-lifting-property” (RHLP).

- (a) Prove that if (A_2, A_1) is a relative cell complex of dimension at most N , then $X \rightarrow Y$ also has the RHLP with respect to (A_2, A_1) . In particular, if A is a cell complex with $\dim A \leq N$ then $[A, X] \rightarrow [A, Y]$ is surjective.
- (b) Now suppose that $X \twoheadrightarrow B$ and $Y \twoheadrightarrow B$ are fibrations, and $X \rightarrow Y$ is a map of spaces over B . Prove that any diagram of form (9.20) of spaces over B has the analogous lifting property over B . Then prove the analog of (a) for pairs over B .
- (c) Suppose that X , Y , and B are all $\mathbb{Z}/2$ -spaces and that all of the maps are equivariant. If S^r has the antipodal $\mathbb{Z}/2$ -action and $r \leq n$, prove that any equivariant map $S^r \rightarrow Y$ over B can be lifted, up to homotopy, to an equivariant map $S^r \rightarrow X$. This provides the missing details for a piece of the proof of Proposition 9.16. [Hint: Use the standard equivariant cell decomposition on S^r .]

Exercise 9.21.

- (a) Prove that $\text{IM}[n, n+k] \Rightarrow \text{AX}[n, n+k]$ and that the second arrow is reversible if $n < 2k$.

- (b) Use the following procedure from [AGJ] to give a geometric proof of the first implication. If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then there is map of bundles

$$\begin{array}{ccc} T_{\mathbb{R}P^n} & \xrightarrow{F} & T_{\mathbb{R}^{n+k}} \xlongequal{\quad} \underline{n+k} \\ \downarrow & & \downarrow \\ \mathbb{R}P^n & \longrightarrow & \mathbb{R}^{n+k} \end{array}$$

that is an injection on the fibers. Add a trivial line bundle to both sides, and then take the induced map of projective bundles:

$$\mathbb{P}(T_{\mathbb{R}P^n} \oplus \underline{1}) \xrightarrow{\tilde{F}} \mathbb{P}(\underline{n+k+1}).$$

The bundle $T_{\mathbb{R}P^n} \oplus \underline{1}$ is trivializable, so the domain can be identified with $\mathbb{R}P^n \times \mathbb{R}P^n$. The codomain is $\mathbb{R}P^{n+k} \times \mathbb{R}^{n+k}$, which projects down to $\mathbb{R}P^{n+k}$. Show that the resulting map $\mathbb{R}P^n \times \mathbb{R}P^n \rightarrow \mathbb{R}P^{n+k}$ is axial.

- (c) Project: When $k > 0$ the paper [AGJ] actually proves that $\text{IM}[n, n+k] \iff \text{AX}[n, n, n+k]$ without the restriction $n < 2k$ that we saw in (a). Go look into how they did it.

Exercise 9.22. Consider the composition $F: \mathbb{R}P^n \times \mathbb{R}P^n \rightarrow \mathbb{R}P^\infty \times \mathbb{R}P^\infty \rightarrow \mathbb{R}P^\infty$ where the last map is the unique axial map on $\mathbb{R}P^\infty$, i.e. the one that represents the class $x \otimes 1 + 1 \otimes x \in H^1(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2)$.

- (a) Prove that $\text{AX}[n, n, M]$ holds if and only if F factors through $\mathbb{R}P^M \hookrightarrow \mathbb{R}P^\infty$.
(b) Prove that $\text{AX}[n, n, 2n]$ holds.
(c) Using obstruction theory (or otherwise), prove that $\text{AX}[n, n, 2n-1]$ holds.
[Note: What do (b) and (c) tell you about immersions of $\mathbb{R}P^n$?]

Exercise 9.23. Fill in the proof of the reversibility of implication 5 in Proposition 9.16(e).

Exercise 9.24. Let $1 \leq k \leq n$. An orthonormal k -frame in $\mathbb{R}^n$ induces an isometric linear embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ and hence a map $S^{k-1} \rightarrow S^{n-1}$. This map of spheres is $\mathbb{Z}/2$ -equivariant. In the proof of Proposition 9.16 we needed that the induced map $V_k(\mathbb{R}^n) \rightarrow \mathcal{Top}_{\mathbb{Z}/2}(S^{k-1}, S^{n-1})$ is $[2(n-k)-1]$ -connected. Prove this by induction on k . [Hint: Consider the map of fibration sequences

$$\begin{array}{ccccc} S^{n-k} & \longrightarrow & V_k(\mathbb{R}^n) & \xrightarrow{p} & V_{k-1}(\mathbb{R}^n) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & \mathcal{Top}_{\mathbb{Z}/2}(S^{k-1}, S^{n-1}) & \xrightarrow{q} & \mathcal{Top}_{\mathbb{Z}/2}(S^{k-2}, S^{n-1}). \end{array}$$

If $j: S^{k-2} \hookrightarrow S^{n-1}$ is the standard inclusion, prove that $F = q^{-1}(j)$ is homeomorphic to the space of maps $D^{k-1} \rightarrow S^{n-1}$ which are equal to j on the boundary. Then prove $F \simeq \Omega^{k-1}S^{n-1}$ and that $S^{n-k} \rightarrow F$ is (up to homotopy) the adjoint to the suspension map. Use the Freudenthal suspension theorem to bound the connectivity of $S^{n-k} \rightarrow F$.]

Exercise 9.25. Say that a map $f: S^r \times S^s \rightarrow S^n$ is **skew** if $f(-x, y) = f(x, -y) = -f(x, y)$ for all $x \in S^r$ and $y \in S^s$. Let $\text{SK}[r, s, n]$ be the statement that a skew map of this form exists. Prove that $\text{NS}[r+1, s+1, n+1] \Rightarrow \text{SK}[r, s, n] \iff \text{AX}[r, s, n]$.

Exercise 9.26 (Connection to Hopf invariant one). It is a well-known fact that the existence of a pairing $f: S^{n-1} \times S^{n-1} \rightarrow S^{n-1}$ such that $f|_{S^{n-1} \times *}$ and $f|_{* \times S^{n-1}}$ both have degree one implies that there exists a map in $\pi_{2n-1}(S^n)$ of Hopf invariant one. See [MT, Chapter 4 exercises]. Using this, prove that

- (1) $\text{NS}[n, n, n] \Rightarrow n \in \{1, 2, 4, 8\}$,
- (2) $\text{AX}[n, n, n] \Rightarrow n \in \{0, 1, 3, 7\}$,
- (3) $\text{SK}[n, n, n] \Rightarrow n \in \{0, 1, 3, 7\}$.

Explain why $k > 0$ is needed in Exercise 9.21(c), but why the scope of that restriction is fairly small.

Exercise 9.27. Let X^I denote the space of paths $I \rightarrow X$, and let $\text{ev}: X^I \rightarrow X \times X$ send a path to its pair of endpoints. Then ev is a fibration with fiber ΩX . A section of ev is a continuous choice, for every $a, b \in X$, of a path from a to b . This situation comes up in motion planning for robotics, where X is a configuration space of positions and one wants a schema for the robot to move from any position to any other one.

- (a) Prove that if ev has a section then X is contractible.
- (b) Farber [Farber] defines the **topological complexity** $\text{TC}(X)$ to be smallest size of an open cover of $X \times X$ such that a section of ev exists over every piece of the cover. For $X = S^n$, define $U_1 = \{(a, b) \mid b \neq -a\}$ and $U_2 = \{(a, b) \mid a \neq b\}$. Use this cover to prove that $\text{TC}(S^n) = 2$ when n is odd. [Note: It turns out that $\text{TC}(S^n) = 3$ then n is even.]
- (c) Project: The paper [FTY] proves the following:
 - (1) $\text{TC}(\mathbb{R}P^n)$ equals the smallest k such that $\text{SK}[n, n, k-1]$ holds (see Exercise 9.25).
 - (2) Except for $n = 1, 3, 7$, $\text{TC}(\mathbb{R}P^n)$ is exactly one more than the immersion dimension of $\mathbb{R}P^n$.
 Go study these results.

Part 2. An introduction to index theory

10. K -THEORY AND FREDHOLM OPERATORS, PART I

The model for $\mathbb{Z} \times BU$ that we have mostly used in this book, in terms of a colimit of Grassmannians, is in many ways the most convenient representing space for the K^0 functor. But it is not the only representing space, and for connections between K -theory and geometry it is helpful to understand several different models and how they interact. In this section and the next we introduce the Atiyah-Jänich model of Fredholm operators on a Hilbert space. While this model is quite interesting and useful, the cost is that we need to review some of the basics from functional analysis.

For readers who are more inclined towards algebra and geometry and who might perhaps feel intimidated by the appearance of analysis here, let us stress that the parts of functional analysis that we need are the ones that are intimately tied to learning how to “do geometry” in infinite dimensions. Keep this in mind as the fundamental motivation. We mention [Conway] and [Pedersen] as good introductory texts that go deeper than the brief survey we give here.

10.1. The language of functional analysis. A complex Hilbert space is a $\mathbb{C}$ -vector space V with a Hermitian inner product such that the associated metric topology is complete. Standard examples are any $\mathbb{C}^n$ with its usual Hermitian product, or the set of ℓ^2 -functions $\ell^2(S)$ on a set S with $\langle f, g \rangle = \sum_{s \in S} f(s)\overline{g(s)}$, or $L^2(X, \mu; \mathbb{C})$ for any measure space (X, μ) with $\langle f, g \rangle = \int_X f(x)\overline{g(x)} d\mu$. An isomorphism of Hilbert spaces is a unitary linear bijection. Unfortunately, one frequently needs to interact with linear homeomorphisms that are not unitary and it is perfectly natural to also refer to these as “isomorphisms”. We will try to be careful to use “unitary isomorphism” or “isometric isomorphism” when the isomorphism preserves the inner product.

A *basis* for a Hilbert space is an orthonormal subset whose span is dense. Note that a basis in this sense need not be a vector space basis, which is awkward. When one wants to talk about a vector space basis one says “Hamel basis”.

Example 10.2. Let $H = \ell^2(\mathbb{N})$ and let $e_i: \mathbb{N} \rightarrow \mathbb{C}$ be the function that sends $i \mapsto 1$ and is zero elsewhere. Then $\{e_1, e_2, \dots\}$ is an orthonormal basis for H (every ℓ^2 -sequence is the limit of its finite truncations) but not a Hamel basis.

A **separable** Hilbert space is a Hilbert space with a countable basis. This includes finite-dimensional Hilbert spaces. Although it is not quite standard, let us define an **∞ -separable Hilbert space** to be a Hilbert space with a countably-infinite basis. It is a theorem that all ∞ -separable Hilbert spaces are unitary isomorphic, and in particular are unitary isomorphic to $\ell^2(\mathbb{N})$. The Hilbert spaces $L^2([0, 1])$ and $L^2(\mathbb{R}^n)$ are ∞ -separable.

A **Banach space** is a topological vector space with a norm such that the associated norm topology is complete. So every Hilbert space is a Banach space, though the converse is not true. Completeness implies that if $\{x_n\}$ is a sequence of elements in a Banach space such that $\sum_n \|x_n\| < \infty$, then $\sum_n x_n$ converges (because the sequence of partial sums is Cauchy).

If V and V' are Banach spaces, a linear map $T: V \rightarrow V'$ is called **bounded** if there exists a $C > 0$ such that $\|Tv\| \leq C\|v\|$ for all $v \in V$. A linear map is

bounded if and only if it is continuous—this an easy exercise, or see [Pedersen, 2.1.2]. Topologists would probably prefer to just say “continuous” everywhere, but analysts prefer to use “bounded”. For a given T , the smallest such C that makes the above inequality true is called the **operator norm** of T . One can also write $\|T\| = \sup_{\|x\|=1} \|Tx\|$, and on a Hilbert space with orthonormal basis $\{e_1, e_2, \dots\}$ one has

$$(10.3) \quad \|T\| = \sup_{\sum_i |a_i|^2 = 1} \left\| \sum a_i T(e_i) \right\|.$$

The set $\mathcal{B}(V, V')$ of bounded linear maps is itself a Banach space, and composition $\mathcal{B}(V', V'') \times \mathcal{B}(V, V') \rightarrow \mathcal{B}(V, V'')$ is continuous.

All linear functionals $x \mapsto \langle x, u \rangle$ are bounded, as this follows from the Cauchy-Schwartz inequality $|\langle x, u \rangle| \leq \|x\| \cdot \|u\|$. The Riesz representation theorem says that every bounded linear functional is of this form.

Example 10.4. Here is an example of a linear isomorphism that is not bounded. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i \geq 1}$. Extend this to a Hamel basis $\mathcal{B}$. Define $T: H \rightarrow H$ by $T(e_n) = ne_n$ for all n , and $T(b) = b$ for $b \in \mathcal{B} - \{e_i\}_{i \geq 1}$.

Remark 10.5. We will use “operator” as synonymous with “linear map”. Some authors instead use the term to mean “bounded linear map”, so one has to discern the intention from context. All of the operators we use will be bounded, though.

When dealing with topological vector spaces the term “subspace” becomes ambiguous, because it can refer to either a topological subspace or a linear subspace. Most of the time it will be clear from context which we mean, but when there is possible confusion we will try to stress “linear subspace” when that is the intention.

A **Banach algebra** is a Banach space that is also a $\mathbb{C}$ -algebra and where the norm satisfies $\|xy\| \leq \|x\| \cdot \|y\|$ for all x, y , which in particular ensures that the multiplication is continuous. The Banach space $\mathcal{B}(V) = \mathcal{B}(V, V)$ is a Banach algebra under composition. In a Banach algebra, if $\|x\| < 1$ then $\sum_{n=0}^{\infty} x^n$ converges; it follows formally that what it converges to must be a multiplicative inverse for $1 - x$. Said differently, the open unit ball around the identity consists entirely of invertible elements. It follows that the subset of invertible elements in a Banach algebra is always open. Consequently, the set of isomorphisms $\mathcal{B}(V, V')^{iso}$ is open in $\mathcal{B}(V, V')$: if this set is nonempty, use a choice of isomorphism to relate everything back to $\mathcal{B}(V, V)$.

Just as being an isomorphism is an open condition in $\mathcal{B}(V, V')$, so is the property of being surjective. But unlike the finite-dimensional case, the property of being injective is *not* an open condition. See Exercise 10.33.

Let H be an ∞ -separable Hilbert space with orthonormal basis $e_1, e_2, \dots$, and let H' be another Hilbert space. Consider the map $\mathcal{B}(H, H') \rightarrow \prod_{i=1}^{\infty} H'$ that sends a transformation T to the tuple $(T(e_1), T(e_2), \dots)$. This is injective because the subspace $\mathbb{C}\langle e_1, e_2, \dots \rangle$ is dense in H . It is not necessarily surjective, though, because not every linear transformation is bounded. The image consists of the so-called **Bessel sequences**, which can be described by a concrete norm inequality: $(f_1, f_2, \dots)$ is Bessel if and only if there is a $C > 0$ such that

$$(10.6) \quad \sum_{k=1}^{\infty} |\langle x, f_k \rangle|^2 < C \|x\|^2, \quad \text{for all } x \in H'.$$

See [Christensen, Theorem 3.2.3]. We will sometimes define maps T by giving the values on an orthonormal basis, and in all our applications this norm inequality will be straightforward. For example, setting $T(e_n) = \frac{1}{n}e_n$ characterizes a unique bounded linear operator on H since (10.6) is readily verified with $C = 1$.

Example 10.7. Let $(\lambda_1, \lambda_2, \dots)$ be a sequence of complex numbers. By (10.6) there exists a bounded operator $D_\lambda: H \rightarrow H$ satisfying $D(e_i) = \lambda_i e_i$ for all i if and only if the set of norms $|\lambda_i|$ is bounded above. We call these *diagonal operators* relative to the basis $\{e_i\}$.

A linear subspace of an ∞ -separable Hilbert space need not be closed—for example, take the span of any orthonormal basis. However, the kernel of any bounded linear operator must be closed by continuity, and conversely any closed subspace is the kernel of a bounded linear operator. Similarly, if $V \subseteq H$ is a subspace then $V^\perp$ must be closed, by the continuity of the maps $\langle -, v \rangle$. In general one has $(V^\perp)^\perp = \overline{V}$, and so $(V^\perp)^\perp = V$ only when V is closed. Of course any finite-dimensional subspace is closed. Be warned that the image of a bounded operator need not be closed:

Example 10.8. Let $H = \ell^2(\mathbb{N})$ and let $e_0, e_1, \dots$ be the standard orthonormal basis of H (see Example 10.2). Define $T: H \rightarrow H$ by $T(e_0) = 0$ and $T(e_k) = \frac{1}{k}e_k$ for $k \geq 1$. Then T is clearly bounded. One might think that $\text{coker } T$ is one-dimensional, but one must remember that we are dealing with H and not $\mathbb{C}^\infty$. For example, the element $z = \sum_{k \geq 1} \frac{1}{k}e_k$ is in H but is not in $\text{im } T$, since the only element that could be a preimage would be $\sum_{k \geq 1} e_k$. Since all of the finite sums $\sum_{k=1}^N \frac{1}{k}e_k$ do belong to $\text{im } T$, this shows that $\text{im } T$ is not closed.

If $V \subseteq H$ is a closed subspace then orthogonal projection $\pi_V: H \rightarrow V$ is defined by letting $\pi_V(x)$ be the vector on V that is closest to x (the existence of such a point follows from completeness of H and closure of V). It then follows that $V^\perp = \ker \pi_V$ and that $V \oplus V^\perp = H$. Note that $V^\perp$ is also closed, and $\pi_{V^\perp} = \text{id} - \pi_V$. The map $H/V \rightarrow V^\perp$ induced by $\pi_{V^\perp}$ is an isomorphism. Also note that all projections onto nonzero closed subspaces have norm equal to 1: this follows from $|x|^2 = |\pi_V(x)|^2 + |\pi_{V^\perp}(x)|^2$, so that $|\pi_V(x)| \leq |x|$ for all x , with equality when $x \in V$.

If H and H' are Hilbert spaces then any bounded linear map $T: H \rightarrow H'$ has an adjoint $T^*: H' \rightarrow H$ that is also bounded, uniquely characterized by $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in H, y \in H'$. The composites TT^* and T^*T are self-adjoint, and any self-adjoint operator is diagonalizable in the sense that there is an orthonormal basis for H consisting of eigenvectors. The $*$ -operator $*$: $\mathcal{B}(H) \rightarrow \mathcal{B}(H)$ is continuous and makes $\mathcal{B}(H)$ into a C^* -algebra.

Exercise 10.9. Let $T: H \rightarrow H'$ be a bounded operator.

- Prove that $\ker T^* = (\text{im } T)^\perp$ and $\ker T = (\text{im } T^*)^\perp$.
- Let P be orthogonal projection onto $\ker T$. Prove that $T^*T + P$ is an isomorphism. [Hint for surjectivity: if A is a diagonalizable transformation, then $\text{im } A^2 = \text{im } A$.]

One can talk about real Hilbert spaces, Banach spaces, etcetera, by making the evident adjustments. It is still true that any two ∞ -separable real Hilbert spaces are orthogonally isomorphic. In fact, the theory also works in the quaternionic setting.

10.10. Spectral theory. In the finite-dimensional case, spectral theory refers to the understanding of self-maps $V \rightarrow V$ via eigenvalues and eigenvectors, diagonalization, and Jordan canonical form. The infinite-dimensional case contains many subtleties, but as a first approximation it is the same general idea. We will only recount a small portion of the general theory, see [Pedersen, Chapter 4] and [Conway, Chapters VII and IX] for deeper treatments.

Let $T: H \rightarrow H$ be a bounded operator. The **spectrum** $\sigma(T) \subseteq \mathbb{C}$ is the set of λ such that $\lambda I - T$ is not invertible. This includes all eigenvalues—that is, λ such that $\ker(\lambda I - T) \neq 0$ —but also λ for which $\lambda I - T$ is injective but not surjective. The latter are a feature of the infinite-dimensional setting.

It is a theorem that $\sigma(T)$ is always nonempty and compact (hence closed and bounded). The operator from Example 10.8 has spectrum $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$, showing that the spectrum need not be discrete. If λ is an isolated point of $\sigma(T)$ then λ is necessarily an eigenvalue of T , but the converse does not hold.

A bounded operator T is said to be **normal** if $T^*T = TT^*$. This includes all diagonalizable, self-adjoint, skew-adjoint, and unitary operators. In some sense the normal operators are the ones for which spectral theory works the nicest.

Suppose that λ is an isolated point in the spectrum of a bounded operator T . Let Γ be a small circle around λ in the complex plane whose interior does not contain any other elements of the spectrum of T . We get a map $S^1 \rightarrow \mathcal{B}(H)^*$ by $z \mapsto (zI - T)^{-1}$, and we can integrate this to get an element of $\mathcal{B}(H)$:

$$P_T = \frac{1}{2\pi i} \int_{\Gamma} (zI - T)^{-1} dz.$$

This turns out to be an orthogonal projection—called the **Riesz projection**—onto a certain subspace $\mathcal{R}_T$ called the **Riesz spectral subspace**. When T is diagonalizable and λ is an eigenvalue then $\mathcal{R}_T$ is simply the λ -eigenspace of T . In the finite-dimensional case, $\mathcal{R}_T = \bigcup_n \ker(\lambda I - T)^n$ —the T -invariant subspace associated to the Jordan blocks with eigenvalue λ . In general, $\mathcal{R}_T$ is the maximal T -invariant subspace on which the (restriction of the) operator T has spectrum $\{\lambda\}$.

More generally, if the interior of Γ contains multiple elements of $\sigma(T)$ then the Riesz projector P_T is orthogonal projection onto the sum of the Riesz spectral subspaces for the different elements.

For a general introduction to these kinds of Riesz integrals in the setting of a Banach algebra (such as $\mathcal{B}(H)$), see [Conway, Section VII.4].

One important theme in spectral theory is understanding to what extent the spectral decomposition of an operator varies continuously. We will need the following piece:

Proposition 10.11. *Fix a simple closed contour Γ in $\mathbb{C}$. Let $\mathcal{N}_{\Gamma} \subseteq \mathcal{B}(H)$ be the subset of normal operators T such that Γ does not meet $\sigma(T)$. Then $T \rightarrow P_T$ is a continuous map $\mathcal{N}_{\Gamma} \rightarrow \mathcal{B}(H)$.*

See Exercise 10.37 for a proof. To parse the content of this result, there are two points to focus on. First, if Γ intersects $\sigma(T)$ then of course P_T is not even defined, since $(zI - T)$ will fail to be invertible. But another way of thinking about the issues is that if we consider a family of operators where a point of the spectrum moves from being outside Γ to inside Γ then we get a jump in the corresponding

direct sum of spectral subspaces, and these are the discontinuities we need to worry about.

Corollary 10.12. *Fix $\lambda \in \mathbb{C}$ and $k \geq 1$. Let $\mathcal{D}(\lambda, k) \subseteq \mathcal{B}(H)$ be the subspace of diagonalizable operators for which λ is an isolated point of the spectrum and $\dim \text{Eig}_\lambda = k$. Then the map $T \mapsto \text{Eig}_\lambda(T)$ is a continuous map $\mathcal{D}(\lambda, k) \rightarrow \text{Gr}_k(H)$.*

Proof. Let $S \in \mathcal{D}(\lambda, k)$. Choose a circle Γ that surrounds λ but whose interior contains no other points from $\sigma(S)$. Let M be the maximum value of $\|zI - S\|^{-1}$ on Γ . If T is another bounded operator then we can write

$$zI - T = [I - (T - S)(zI - S)^{-1}](zI - S).$$

We claim the bracketed expression is invertible for T close to S and z on Γ . If $\|T - S\| < \frac{1}{M}$ and z is on Γ then $\|(T - S)(zI - S)^{-1}\| \leq \|T - S\| \cdot \|(zI - S)^{-1}\| < 1$, and this implies that $I - (T - S)(zI - S)^{-1}$ is invertible. Hence $zI - T$ is invertible. That is, if $\|T - S\| < \frac{1}{M}$ then Γ does not intersect $\sigma(T)$. So the open set $\mathcal{D}(\lambda, k) \cap B(S, \frac{1}{M})$ is contained in $\mathcal{N}_\Gamma$, and by Proposition 10.11 the map $T \mapsto P_T$ is continuous here.

The operator P_T is the orthogonal projection onto the sum of Riesz spectral subspaces for the elements of $\sigma(T)$ inside Γ . These elements consist of λ and perhaps others—a multiple eigenvalue at λ can deform into an eigenvalue of smaller multiplicity and other eigenvalues. But the requirement that $\dim \text{Eig}_\lambda(T) = k$ for $T \in \mathcal{D}(\lambda, k)$ ensures this does not happen. So for T in our neighborhood the only element of $\sigma(T)$ inside Γ is λ , and thus P_T is projection onto $\text{Eig}_\lambda(T)$. $\square$

One can consider more general versions of the above corollary, where the operators are not necessarily diagonalizable, but we will not need them.

10.13. Polar decomposition. If $T: H \rightarrow H'$ is a bounded operator, then T^*T is self-adjoint, hence unitarily diagonalizable with non-negative real eigenvalues. If $\{e_i\}$ is an orthonormal eigenbasis for T^*T with associated eigenvalues $r_i \geq 0$, we can define a new bounded operator denoted $(T^*T)^{1/2}$ that has the same eigenbasis and eigenvalues $\sqrt{r_i}$ (use Example 10.7). A little work shows this is a continuous function of T . Since $(T^*T)^{1/2}$ is obviously still self-adjoint we can compute

$$\langle (T^*T)^{1/2}x, (T^*T)^{1/2}x \rangle = \langle (T^*T)x, x \rangle = \langle Tx, Tx \rangle.$$

That is, $\|(T^*T)^{1/2}x\| = \|Tx\|$ for all x . So the stretching/shrinking done by T is all done by $(T^*T)^{1/2}$ as well.

In fact, for any continuous function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ we can construct a bounded operator $f(T^*T)$ that has the same eigenbasis as T^*T but where the eigenvalues are the $f(r_i)$. This is sometimes useful, though we don't need it at this exact moment.

It is an amazing fact that any bounded operator $T: H \rightarrow H$ can be broken canonically into two parts: the first is a diagonalizable operator that does all of the stretching/shrinking (including collapsing to zero), and the second is an operator that is almost unitary. To make this precise, say that an operator $Q: H \rightarrow H'$ is a **partial isometry** if $Q: (\ker Q)^\perp \hookrightarrow H'$ is an isometric embedding. The polar decomposition theorem [Pedersen, Theorem 3.2.17] says that an arbitrary bounded operator T can be uniquely written as $T = Q_T(T^*T)^{1/2}$ where Q_T is a partial isometry with $\ker Q_T = \ker T$.

When T is invertible so is T^*T , and hence so is $(T^*T)^{1/2}$. So here Q_T is an actual isometry and we can write $Q_T = T(T^*T)^{-1/2}$. So for T invertible Q_T is a continuous function of T .

10.14. General linear groups and Grassmannians. For a Hilbert space H , define the general linear group $GL(H)$ to be the units $\mathcal{B}(H)^* \subseteq \mathcal{B}(H)$, with the subspace topology. This is the space of isomorphisms from H to itself, meaning bounded linear bijections (the inverse is automatically bounded, see [Pedersen, Corollary 2.2.5]). We will need the following important result about these topological groups:

Theorem 10.15 (Kuiper's Theorem). *Let H be an ∞ -separable Hilbert space (real or complex or quaternionic). Then $GL(H) = \mathcal{B}(H)^*$ is contractible.*

Kuiper's proof involves a very concrete, multi-step deformation procedure, but it takes several pages and the ideas don't surface anywhere else for us. So we skip the proof and refer the reader to [Kuiper]. But note that the analogous result for $GL(\mathbb{C}^\infty) = \text{colim}_n GL(\mathbb{C}^n)$ with the colimit topology is false, since this space is homotopy equivalent to the infinite unitary group U and this has nonvanishing homotopy groups. So the use of H with the norm topology in place of $\mathbb{C}^\infty$ with the colimit topology is important here.

Kuiper's theorem has several immediate and useful consequences. Define the unitary group $U(H) \subseteq \mathcal{B}(H)^*$ to be the subgroup of elements T such that $T^*T = \text{id}$. The usual Gram-Schmidt argument still works in this infinite-dimensional setting to show that $\mathcal{B}(H)^*$ deformation retracts down to $U(H)$, hence $U(H)$ is also contractible. Here we do it via the polar decomposition: $\mathcal{B}(H)^* \rightarrow U(H)$ is given by $T \mapsto Q_T$, and $(T, t) \mapsto T((1-t)I + t(T^*T)^{-1/2})$ is a homotopy to the identity. Here it is important that T^*T is diagonalizable with positive real eigenvalues, so that $(1-t)I + t(T^*T)^{-1/2}$ is invertible for all $t \in [0, 1]$.

The subspace $V_k(H) \subseteq H^k$ of orthonormal k -frames in H has a transitive action of $U(H)$, and the stabilizer of $(e_1, \dots, e_k)$ is $U((\mathbb{C}^k)^\perp)$, so we can identify $V_k(H) \cong U(H)/U((\mathbb{C}^k)^\perp)$. But the latter sits in the fiber sequence $U((\mathbb{C}^k)^\perp) \rightarrow U(H) \rightarrow U(H)/U((\mathbb{C}^k)^\perp)$, and Kuiper's theorem says that the first two spaces are contractible. So $V_k(H)$ is a weakly contractible space. It also has a free action of $U(k)$, hence the quotient $\text{Gr}_k(H)$ is a model for $BU(k)$. Said differently, the evident map $\text{Gr}_k(\mathbb{C}^\infty) \rightarrow \text{Gr}_k(H)$ is a weak homotopy equivalence. (For a naive proof of this result, see Exercise 10.40).

Kuiper's theorem also says something interesting about bundle theory. Define a *Hilbert bundle* on a space X to be a family of vector spaces that is locally isomorphic to $X \times H$ and where the transition functions are given by continuous maps into $\mathcal{B}(H)^*$ with its norm topology (see Section 11.13 for a more detailed discussion). Hilbert bundles on paracompact Hausdorff spaces are then classified by maps into the classifying space $B[U(H)]$, but this is contractible since $U(H)$ is. So all Hilbert bundles on paracompact Hausdorff spaces are trivializable.

While we are on the subject of Grassmannians, let us note the following useful result:

Proposition 10.16. *The map $\text{Gr}_k(H) \rightarrow \mathcal{B}(H)$ given by $V \mapsto \pi_V$ is continuous, as is the map $\text{Gr}_k(H) \rightarrow \mathcal{B}(H)$ given by $V \mapsto \pi_{V^\perp}$. Both are homeomorphisms onto their image. In particular, one can regard $\text{Gr}_k(H)$ either as a quotient space*

of $V_k(H)$ or as the space of orthogonal projections of rank k , equipped with the norm topology.

Proof. Since $\pi_{V^\perp} = \text{id} - \pi_V$, it suffices to prove the results just for π_V .

Consider the map $H^k \rightarrow \mathcal{B}(H)$ that sends a tuple $(v_1, \dots, v_k)$ to the operator $T(x) = \sum_{i=1}^k \langle x, v_i \rangle v_i$. This is readily checked to be continuous. It restricts to a map $T': V_k(H) \rightarrow \mathcal{B}(H)$, and $T'(v_1, \dots, v_k) = \pi_{\langle v_1, \dots, v_k \rangle}$. Since $V_k(H) \rightarrow \text{Gr}_k(H)$ is a quotient map and T' respects the quotient relation, it induces a continuous map $\text{Gr}_k(H) \rightarrow \mathcal{B}(H)$.

To prove that this map is a homeomorphism onto its image, we will construct local inverses. Fix a k -dimensional subspace V . The intuition is that for subspaces very close to V we can use orthogonal projection to identify them with V , and via these identifications a fixed k -frame for V induces k -frames on all nearby subspaces. This allows us to locally construct a map into $V_k(H)$. To make these ideas rigorous, let $\mathcal{O}_V \subseteq \mathcal{B}(H)$ be the set of π_W such that $\|\pi_W - \pi_V\| < 1$. For $\pi_W \in \mathcal{O}_V$ one can show that $\pi_W: V \rightarrow W$ is an isomorphism (see Exercise ???).

Fix an orthonormal basis $v_1, \dots, v_k$ of V . Applying π_W then gives a basis for W , and we can apply Gram-Schmidt to make it orthonormal. This defines a continuous map $\mathcal{O}_V \rightarrow V_k(H)$, and the composite $\mathcal{O}_V \rightarrow V_k(H) \rightarrow \text{Gr}_k(H)$ is the inverse we were looking for. We can also say this via explicit formulas. Let $T_W = \pi_W|_V: V \rightarrow W$. The polar decomposition gives $T_W = Q_W(T_W^* T_W)^{1/2}$ and $Q_W = T_W(T_W^* T_W)^{-1/2}$, where Q_W is an isometry. Our map $\mathcal{O}_V \rightarrow V_k(H)$ sends W to $(Q_W v_1, \dots, Q_W v_k)$, and clearly this is continuous. $\square$

For Banach spaces V and V' , a bounded operator $T: V \rightarrow V'$ is called **compact** if for every bounded subset $S \subseteq V$ one has that $\overline{T(S)}$ is compact. Since the behavior of linear operators is invariant under scaling, this is equivalent to each of the following two statements:

- $\overline{T(B)}$ is closed where $B \subseteq V$ is the closed unit ball.
- There is an open set $U \subseteq V$ containing the origin such that $T(U)$ is contained in a compact subset of V' .

When V and V' are Hilbert spaces, a bounded operator is compact if and only if it is the limit of a sequence of finite-rank operators [Conway, Theorem II.4.4].

Here are two useful properties of compact operators T :

- (1) If $\{e_i\}$ is an orthonormal basis for H then $\lim_{n \rightarrow \infty} \|Te_n\| = 0$.
- (2) For any $\lambda \neq 0$, the λ -eigenspace of T is finite-dimensional.

For the first, see Exercise 10.35. The second property follows from the first: if $\text{Eig}_\lambda(T)$ is infinite-dimensional then choose orthonormal bases $\{f_i\}$ and $\{g_i\}$ for Eig_λ and $(\text{Eig}_\lambda)^\perp$. Put them together to make an orthonormal basis for H that contradicts (1).

Passing from an operator T to $T + K$ where K is finite-rank is often regarded as a small perturbation, and likewise for the limiting case where K is compact. So the moral is that it is useful to regard the compact operators as “small”. One can study invariants of operators that are unchanged by compact perturbations by algebraically modding out by compact operators. So let $\mathcal{K}(V) \subseteq \mathcal{B}(V)$ be the subspace of compact operators. This is in fact a two-sided ideal, so one may form the quotient algebra $\mathcal{A}(V) = \mathcal{B}(V)/\mathcal{K}(V)$. This is called the **Calkin algebra**, and

it is again a Banach algebra. We will only use this when V is an ∞ -separable Hilbert space. Let $\pi: \mathcal{B}(V) \twoheadrightarrow \mathcal{A}(V)$ be the canonical projection.

10.17. Back to K -theory. Suppose one wants to define a K -theory group based on bundles of Hilbert spaces on X . Because Hilbert spaces are infinite-dimensional, the Eilenberg swindle implies that the naive Grothendieck group construction yields the zero group. But as we have seen in the algebraic setting (e.g. Section ??) one can side-step the Eilenberg swindle by instead using bounded chain complexes of such bundles satisfying a finiteness hypotheses on the homology. And we have seen that one can often get the same group by only looking at length-one complexes. This leads us to consider complexes of Hilbert bundles $0 \rightarrow E_1 \rightarrow E_0 \rightarrow 0 \rightarrow 0$ having the property that on each fiber the kernel and cokernel are finite-dimensional. By Kuiper's theorem there are trivializations $E_1 \cong X \times H$ and $E_0 \cong X \times H$, so morally speaking this complex has the same data as a map $T: X \rightarrow \mathcal{B}(H)$ such that each T_x has finite-dimensional kernel and cokernel (but see Section 11.13 for a discussion of some subtleties here). Such elements T_x of $\mathcal{B}(H)$ are called *Fredholm operators*. Let us now pause and develop the background we need on these objects.

10.18. Fredholm operators. Let H and H' be ∞ -separable Hilbert spaces. A bounded operator $T: H \rightarrow H'$ is called **Fredholm** if $\dim \ker(T) < \infty$ and $\dim \operatorname{coker}(T) < \infty$. Sometimes sources include the requirement that $\operatorname{im}(T)$ is a closed set, but that is implied by the finite-dimensionality conditions (see Exercise 10.34). The space of Fredholm operators from H to H' is defined to be the subspace $\operatorname{Fred}(H, H') \subseteq \mathcal{B}(H, H')$. When $H = H'$ we write $\operatorname{Fred}(H)$ instead of $\operatorname{Fred}(H, H)$. All isomorphisms are Fredholm, so $\mathcal{B}(H, H')^{\operatorname{iso}} \subseteq \operatorname{Fred}(H, H')$.

Example 10.19. Let $H = \ell^2(\mathbb{N})$.

- (1) Let $\operatorname{sh}_{-k}: H \rightarrow H$ send $(x_0, x_1, x_2, \dots) \mapsto (x_k, x_{k+1}, \dots)$. This operator has k -dimensional kernel and zero cokernel.
- (2) Let $\operatorname{sh}_k: H \rightarrow H$ send $(x_0, x_1, \dots) \mapsto (0, 0, \dots, 0, x_0, x_1, \dots)$ where the x_0 appears in the k th coordinate. Here the kernel is zero and the cokernel is k -dimensional.
- (3) By composing operators of types (a) and (b) one can make Fredholm operators with any prescribed dimensions for the kernel and cokernel.
- (4) The following non-example shows some of the subtleties that appear. Let T be the operator from Example 10.8: $T(e_0) = 0$ and $T(e_k) = \frac{1}{k}e_k$ for $k > 0$. Then $\ker(T) = \mathbb{C}e_0$. It might feel like $\operatorname{coker}(T)$ is one-dimensional, but as we saw in that example it is not. In fact we saw that $\operatorname{im}(T)$ is not closed, hence T cannot be Fredholm.

It is a theorem that $T: H \rightarrow H'$ is Fredholm if and only if it is invertible modulo compact operators—that is, if there is a bounded operator $S: H' \rightarrow H$ such that $ST = \operatorname{id}_H + K_1$ and $TS = \operatorname{id}_{H'} + K_2$ where K_1 and K_2 are compact (this is readily proven using Exercise 10.9). It follows that adding a compact operator to a Fredholm operator is still Fredholm. In particular, if T is Fredholm and K is compact then the entire line $T + tK$ for $t \in \mathbb{R}$ lies in space of Fredholm operators. This will be frequently used to construct straight-line homotopies.

If we start with $q: \mathcal{B}(H) \rightarrow \mathcal{A}(H)$ and let $\mathcal{A}(H)^*$ denote the group of units in $\mathcal{A}(H)$, then $\operatorname{Fred}(H) = \pi^{-1}(\mathcal{A}(H)^*)$. From this definition it is clear that $\operatorname{Fred}(H)$

is closed under composition, though it is also easy to see that directly (see Section 10.26 below). So $\text{Fred}(H)$ has a continuous product making it into a topological monoid. Consequently, for any space X the set $[X, \text{Fred}(H)]$ is a monoid. In fact, one can say more. The map $\mathcal{B}(H) \rightarrow \mathcal{A}(H)$ is a Serre fibration (see Exercise 10.39) with fiber $\mathcal{K}(H)$, and so restricting the base to $\mathcal{A}(H)^*$ gives the fibration sequence $\mathcal{K}(H) \rightarrow \text{Fred}(H) \rightarrow \mathcal{A}(H)^*$. But $\mathcal{K}(H)$ is contractible, via the homotopy $(K, t) \mapsto tK$. So $\text{Fred}(H) \rightarrow \mathcal{A}(H)^*$ is a weak homotopy equivalence, and therefore $[X, \text{Fred}(H)]$ is actually a group. (For a more naive proof of this fact, see Exercise 10.41.)

Our goal is the following:

Theorem 10.20 (Atiyah-Jänich). *Let H be an ∞ -separable complex Hilbert space. For compact Hausdorff spaces X , there is a natural isomorphism of groups*

$$[X, \text{Fred}(H)] \rightarrow K(X).$$

Replacing H with an ∞ -separable real or quaternionic Hilbert space yields analogous isomorphisms involving $KO(X)$ and $KSp(X)$.

This theorem was proven independently in [Jänich] and [Atiyah5, Appendix]. We will spend the rest of the section giving the proof. In fact, with a little extra work one also gets the following:

Theorem 10.21. *The spaces $\mathbb{Z} \times BU$ and $\text{Fred}(H)$ are weakly equivalent.*

We will prove this in Section 11.5.

Remark 10.22. The topological spaces $\mathbb{Z} \times BU_{gr}$, $\text{Fred}(H)$, and $\mathcal{A}(H)^*$ are all weakly equivalent. Here are some advantages and disadvantages of these models:

- $\text{Fred}(H)$ comes with an evident associative multiplication, whereas $\mathbb{Z} \times BU_{gr}$ does not. The space $\mathcal{A}(H)^*$ even comes with a natural group multiplication.
- The spaces $\text{Fred}(H)$ and $\mathcal{A}(H)^*$ are metrizable, as they come to us as subspaces of Banach spaces. The space $\mathbb{Z} \times BU_{gr}$ is not metrizable, as it is not even first-countable (see Exercise ??).
- The space $\mathbb{Z} \times BU_{gr}$ comes with a CW-structure (Exercise ??), whereas $\text{Fred}(H)$ and $\mathcal{A}(H)^*$ do not.

Remark 10.23. We have stated the Atiyah-Jänich theorem for compact Hausdorff spaces because this is the classical form, and it is in this setting that $K(X)$ is the Grothendieck group of vector bundles. But Theorem 10.21 says $\text{Fred}(H) \simeq \mathbb{Z} \times BU$, and so for arbitrary semi-cofibrant spaces if we define $K(X) = [X, \mathbb{Z} \times BU]$ then we still have $[X, \text{Fred}(H)] \cong K(X)$. In other words, for the correct definition of $K(X)$ the Atiyah-Jänich theorem is true for all semi-cofibrant spaces. This in turn allows one to describe $K(X)$ in terms of a Grothendieck-group like construction—not the Grothendieck group of vector bundles, but instead a Grothendieck group of Fredholm complexes. See Proposition 11.16.

Before proceeding further we will need a couple more technical results about Fredholm operators:

Proposition 10.24. *Fix $k \geq 1$ and let $\text{Fred}(H, H')_k \subseteq \text{Fred}(H, H')$ be the subspace of Fredholm operators having k -dimensional kernel. Then the map $\text{Fred}(H, H')_k \rightarrow \text{Gr}_k(H)$ given by $T \mapsto \ker T$ is continuous.*

This might seem intuitively obvious, but be warned that if we replace $\text{Fred}(H, H')$ by $\mathcal{B}(H, H')$ then the result is not true anymore (see Exercise 10.38). So some nontrivial analysis is needed for the result.

Proof. For $T \in \text{Fred}(H, H')_k$ form the self-adjoint operator T^*T . Then T^*T is diagonalizable with non-negative eigenvalues, and the 0-eigenspace is $\ker(T^*T) = \ker T$. Note that since T is Fredholm so is T^* , and therefore also T^*T . So T^*T is both self-adjoint and Fredholm, and this implies 0 is an isolated point of the spectrum of T^*T (see Exercise 10.36). The result now follows from Corollary 10.12. $\square$

Recall from Proposition 10.16 that the maps $\text{Gr}_k(H) \rightarrow \mathcal{B}(H)$ given by $V \mapsto \pi_V$ and $V \mapsto \pi_{V^\perp}$ are continuous. The first lands in the subspace $\mathcal{K}(H)$, and the second lands in $\text{Fred}(H)$. Combining these with the previous result yields the following:

Corollary 10.25. *Fix $k \geq 1$.*

- (a) *The map $\text{Fred}(H, H')_k \rightarrow \mathcal{K}(H)$ sending $T \mapsto \pi_{\ker T}$ is continuous, as is the map $\text{Fred}(H, H')_k \rightarrow \text{Fred}(H)$ sending $T \mapsto \pi_{(\ker T)^\perp}$.*
- (b) *If $W \subseteq H'$ is finite-dimensional then the maps $T \mapsto \pi_{T^{-1}W} \in \mathcal{K}(H)$ and $T \mapsto \pi_{(T^{-1}W)^\perp} \in \text{Fred}(H)$ are also continuous when the domain is restricted to those $T \in \text{Fred}(H, H')$ such that $\dim T^{-1}W = k$.*

Proof. Part (a) is immediate from Propositions 10.24 and 10.16. For (b), observe that $T^{-1}W$ is the kernel of the composite $H \xrightarrow{T} H' \twoheadrightarrow H'/W$. Then use the continuous function $\text{Fred}(H, H') \rightarrow \text{Fred}(H, H'/W)$ together with what has already been proven. $\square$

10.26. The index of a Fredholm operator. Taking $X = pt$ in the Atiyah-Jänich theorem yields the statement $\pi_0(\text{Fred}(H)) \cong \mathbb{Z}$. We start our journey towards a proof by first understanding this special case. It says that there is an integer-valued invariant of Fredholm operators that is unchanged under 1-parameter deformations (paths). What could this be? The evident integers attached to a Fredholm operator T are $\dim \ker(T)$ and $\dim \text{coker}(T)$, but we know from the finite-dimensional case that those dimensions can jump: our standard example is the multiplication by t map $\mathbb{R} \xrightarrow{t} \mathbb{R}$ as t ranges in $[-1, 1]$. But we also know that when V and W are finite-dimensional and $T: V \rightarrow W$ then the exact sequence

$$0 \longrightarrow \ker(T) \longrightarrow V \xrightarrow{T} W \longrightarrow \text{coker}(T) \longrightarrow 0$$

shows that $\dim \ker(T) - \dim \text{coker}(T) = \dim V - \dim W$. So in a family of operators T between finite-dimensional vector spaces, every time there is a jump in $\dim \ker(T)$ there must be a corresponding jump in $\dim \text{coker}(T)$. The same principal will apply to operators $H \rightarrow H'$ as long as we can convince ourselves that the “jumps” can each be controlled within some finite-dimensional setting. Ninety percent of the work in proving the Atiyah-Jänich theorem amounts to developing techniques to reduce questions about $\text{Fred}(H, H')$ to situations that are controlled by finite-dimensional subspaces.

The above is motivation for defining the **index** of a Fredholm operator $T: H \rightarrow H'$ by the formula

$$\text{ind}(T) = \dim \ker(T) - \dim \text{coker}(T).$$

We have seen in Example 10.19 that every integer is the index of some operator T . If $S: H' \rightarrow H''$ is Fredholm then a simple diagram shows that ST is Fredholm and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$:

$$(10.27) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{\begin{bmatrix} \text{id} \\ T \end{bmatrix}} & H \oplus H' & \xrightarrow{[T - \text{id}]} & H' \longrightarrow 0 \\ & & \downarrow T & & \downarrow \begin{bmatrix} 0 & \text{id} \\ ST & 0 \end{bmatrix} & & \downarrow S \\ 0 & \longrightarrow & H' & \xrightarrow{\begin{bmatrix} \text{id} \\ S \end{bmatrix}} & H' \oplus H'' & \xrightarrow{[-S \text{ id}]} & H'' \longrightarrow 0 \end{array}$$

Here the vertical maps are all Fredholm operators, and the horizontal rows are exact sequences of vector spaces. The middle vertical map is $ST \oplus \text{id}_H$, so the kernel and cokernel are isomorphic to those of ST . The long exact sequence for homology gives

$$0 \rightarrow \ker T \rightarrow \ker ST \rightarrow \ker S \rightarrow \text{coker } T \rightarrow \text{coker } ST \rightarrow \text{coker } S \rightarrow 0,$$

and from this the desired conclusions are immediate.

We will see below that the index is constant on path components and gives a group isomorphism $\pi_0 \text{Fred}(H) \cong \mathbb{Z}$, but to prove this we need some more tools.

Remark 10.28. In certain contexts it feels more natural to define the index as $\text{ind}(T) = \dim \text{coker}(T) - \dim \ker(T)$, which is the negative of the standard definition. In essence the index is a kind of Euler characteristic, and the difference comes from whether one regards $H \xrightarrow{T} H$ as a chain complex with the target H in degree 0 or as a cochain complex with the domain H in degree 0. For good or bad, the common definition of $\text{ind}(T)$ was set up according to the *cohomological* conventions. We will stick with that, though with occasional sighs of regret.

In proving the Atiyah-Jänich theorem there is a choice to be made: one can put most of the focus on kernels, or one can put most of the focus on cokernels. Jänich does the former, and Atiyah does the latter. We follow Jänich in putting our emphasis on kernels, but we also incorporate some of Atiyah's clever touches.

Let $V \subseteq H$ and $W \subseteq H'$ be finite-dimensional. Define $\mathcal{J}_{V,W} \subseteq \mathcal{B}(H, H')$ to be the set of all T such that the composite $V^\perp \hookrightarrow H \xrightarrow{T} H' \rightarrow H'/W$ is an isomorphism. This condition implies that $W \rightarrow \text{coker}(T)$ is surjective and $\ker T \cap V^\perp = 0$, so that $\pi_V: \ker T \rightarrow V$ is injective; hence T must be Fredholm. Thus, $\mathcal{J}_{V,W} \subseteq \text{Fred}(H, H')$. But $\mathcal{J}_{W,V}$ is also open in $\mathcal{B}(H, H')$, because it is the preimage under $\mathcal{B}(H, H') \rightarrow \mathcal{B}(V^\perp, H'/W)$ of the open set $\mathcal{B}(V^\perp, H'/W)^{\text{iso}}$. In a moment we will see that the $\mathcal{J}$ -sets give an open cover of $\text{Fred}(H, H')$.

Note that the defining condition for $\mathcal{J}_{V,W}$ is equivalent to $\pi_{W^\perp} \circ T: V^\perp \rightarrow W^\perp$ being an isomorphism, and we will often use it in that form.

Let $T \in \text{Fred}(H, H')$ and let $W \subseteq H'$ be finite-dimensional. Since $\ker(T)$ is finite-dimensional, $T^{-1}(W)$ is also finite-dimensional. Consider the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(T) & \longrightarrow & H & \xrightarrow{T} & H' & \longrightarrow & H'/\text{im}(T) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \ker(T) & \longrightarrow & T^{-1}(W) & \xrightarrow{T} & W & \longrightarrow & W/(W \cap \text{im}(T)) \longrightarrow 0. \end{array}$$

By algebra the right vertical map is injective, and is surjective if and only if $\text{im}(T) + W = H'$. Assuming this condition, it follows that

$$\text{ind}(T) = \dim T^{-1}(W) - \dim W.$$

So in this context the index of T reduces to the index of a finite-dimensional transformation.

Moreover, one readily checks that the condition $W + \text{im } T = H'$ implies that the composite $(T^{-1}W)^\perp \hookrightarrow H \xrightarrow{T} H' \rightarrow H'/W$ is an isomorphism. So $T \in \mathcal{J}_{T^{-1}W, W}$, and this proves that the sets $\mathcal{J}$ -sets cover $\text{Fred}(H, H')$.

Motivated by the above, for any finite-dimensional subspace $W \subseteq H'$ define $\mathcal{U}_W \subseteq \text{Fred}(H, H')$ to be the set of operators T such that $\text{im}(T) + W = H'$. This condition is equivalent to saying that the composite $H \xrightarrow{T} H' \rightarrow H'/W$ is surjective, and therefore $\mathcal{J}_{V, W} \subseteq \mathcal{U}_W$ for all finite-dimensional $V \subseteq H$. Since $T \in \mathcal{U}_W$ implies $T \in \mathcal{J}_{T^{-1}W, W}$, we have $\mathcal{U}_W = \bigcup_{T \in \mathcal{U}_W} \mathcal{J}_{T^{-1}W, W}$. So each $\mathcal{U}_W$ is open. The $\mathcal{U}_W$ sets give another open cover of $\text{Fred}(H, H')$, slightly more coarse than the covering by $\mathcal{J}$ -sets. Note that if $W \subseteq W'$ then $\mathcal{U}_W \subseteq \mathcal{U}_{W'}$.

Proposition 10.29. *Let $V \subseteq H$ and $W \subseteq H'$ be finite-dimensional.*

- (a) *For $T \in \mathcal{B}(H, H')$ the following conditions are equivalent:*
- (1) $T \in \mathcal{J}_{V, W}$.
 - (2) $V^\perp \cap T^{-1}W = 0$ and $H' = T(V^\perp) + W$.
 - (3) $H = V^\perp \oplus T^{-1}W$ and $H' = T(V^\perp) \oplus W$.
 - (4) $H' = \text{im } T + W$ and the composite $T^{-1}W \hookrightarrow H \xrightarrow{\pi_V} V$ is an isomorphism.
- (b) *For every $T \in \mathcal{J}_{V, W}$ one has $\text{ind}(T) = \dim V - \dim W$.*

Proof. Part (a) is just basic linear algebra and will be left to the reader. For (b), we know that if $T \in \mathcal{J}_{V, W}$ then $T \in \mathcal{U}_W$ and so $\text{ind}(T) = \dim T^{-1}W - \dim W$. Now use condition (4) from (a) to see that $\dim T^{-1}W = \dim V$. $\square$

Corollary 10.30. *Let H and H' be Hilbert spaces.*

- (a) *The index function $\text{ind}: \text{Fred}(H, H') \rightarrow \mathbb{Z}$ is continuous. In particular, it is constant on path components.*
- (b) *If X is compact then any map $X \rightarrow \text{Fred}(H, H')$ factors through $\mathcal{U}_W$ for some finite-dimensional W .*
- (c) *Let $\mathcal{E}_W = \{(T, x) \mid T \in \mathcal{U}_W, x \in T^{-1}(W)\} \subseteq \mathcal{U}_W \times H$. Then the projection map $\pi_1: \mathcal{E}_W \rightarrow \mathcal{U}_W$ is a vector bundle.*
- (d) *Assume $W \subseteq W' \subseteq H$ are both finite-dimensional, and let $j: \mathcal{U}_W \hookrightarrow \mathcal{U}_{W'}$ be the inclusion. Then $j^*\mathcal{E}_{W'} \cong \mathcal{E}_W \oplus \underline{(W' - W)}$, where $W' - W$ is the orthogonal complement of W in W' .*

Proof. Part (a) follows from the fact that the $\mathcal{J}_{V, W}$ sets are open and that the index is constant on them. For (b), since $\{\mathcal{U}_W\}$ is an open cover of $\text{Fred}(H, H')$ and X is compact its image is contained in a finite union $\bigcup_{i=1}^k \mathcal{U}_{W_k}$. But this union is in turn contained in $\mathcal{U}_{W_1 + \dots + W_k}$.

For (c), let $S \in \mathcal{U}_W$ and let $\mathcal{E}_{S^{-1}W, W} \subseteq \mathcal{E}_W$ be the preimage of the open neighborhood $\mathcal{J}_{S^{-1}W, W} \subseteq \mathcal{U}_W$. For $T \in \mathcal{U}_{S^{-1}W, W}$ the composite $T^{-1}W \hookrightarrow H \xrightarrow{\pi_{S^{-1}W}} S^{-1}W$ is an isomorphism, which yields the trivialization $\mathcal{E}_{S^{-1}W, W} \cong \mathcal{U}_{S^{-1}W, W} \times S^{-1}W$.

Finally, to prove (d) let $T \in \mathcal{U}_W$ and start with the orthogonal direct sum decomposition $T^{-1}W' = T^{-1}W \oplus (T^{-1}W' - T^{-1}W)$ (where again, if $B \subseteq A$ then

$A - B$ denotes the orthogonal complement of B in A). If $\pi_{W'-W}: W' \rightarrow W' - W$ is orthogonal projection, then one readily checks by linear algebra that

$$T^{-1}W' - T^{-1}W \xrightarrow{\pi_{W'-W} \circ T} W' - W$$

is an isomorphism. The desired bundle isomorphism follows immediately. $\square$

By Corollary 10.30 the index map induces $\text{ind}: \pi_0 \text{Fred}(H) \rightarrow \mathbb{Z}$. We know this to be a group homomorphism. As a warm-up to the full Atiyah-Jänich theorem let us prove the following special case:

Proposition 10.31. *The index map $\text{ind}: \pi_0 \text{Fred}(H) \rightarrow \mathbb{Z}$ is an isomorphism.*

Proof. The Fredholm operator sh_{-1} from Example 10.19 has index 1, so the index map (which is a group homomorphism) is surjective. For injectivity, suppose $T \in \text{Fred}(H)$ has index 0. We know we can choose a finite-dimensional $W \subseteq H$ such that $W + \text{im}(T) = H$, and then $\pi_{W^\perp}T: (T^{-1}W)^\perp \rightarrow W^\perp$ is an isomorphism.

We can write

$$T = \text{id} \circ T = (\pi_{W^\perp} + \pi_W) \circ T = \pi_{W^\perp}T + \pi_W T.$$

In the rightmost term, $\pi_{W^\perp}T$ is the composite of Fredholm operators and thus Fredholm, whereas $\pi_W T$ is compact. The path $t \mapsto [\pi_{W^\perp}T] + t(\pi_W T)$ shows that $T \sim \pi_{W^\perp}T$ in $\pi_0 \text{Fred}(H)$. But $\text{ind}(T) = \dim T^{-1}W - \dim W$, so if this is zero we can choose an isomorphism $f: T^{-1}W \rightarrow W$. The operator $f\pi_{T^{-1}W}$ is compact and so the path $t \mapsto [\pi_{W^\perp}T] + tf\pi_{T^{-1}W}$ shows that $T \sim \pi_{W^\perp}T \sim \pi_{W^\perp}T + f\pi_{T^{-1}W}$ in $\pi_0 \text{Fred}(H)$. But $\pi_{W^\perp}T + f\pi_{T^{-1}W}$ can be regarded as the direct sum of two isomorphisms, hence is itself an isomorphism. So it lies in $\mathcal{B}(H)^* \subseteq \text{Fred}(H)$, and Kuiper's theorem says that $\mathcal{B}(H)^*$ is contractible. Hence we have a path to id , and therefore $T = \text{id}$ in $\pi_0 \text{Fred}(H)$. $\square$

The moral of the above proof is that Fredholm operators can be understood as having a big piece that is an isomorphism and a small (finite-dimensional) piece where other stuff happens. But on finite-dimensional pieces one can homotop the map to any other map, using compact operators and straight-line paths. The proof of the Atiyah-Jänich theorem will use the same techniques, the only complication being that one needs to apply the arguments to *families* of Fredholm operators. So we will need to work a little harder to understand how the “pieces” vary in families, and to ensure continuity. We take this up in the next section.

◦ Exercises ◦

Exercise 10.32. Let V be a Banach space and let $M \subseteq V$ be a closed subspace.

- (a) Define a norm on V/M by $\|\bar{v}\| = \inf_{m \in M} \|v + m\|$. Show that this makes V/M into a Banach space, and the projection $V \rightarrow V/M$ is bounded. Moreover, the norm topology on V/M is the quotient topology.
- (b) Suppose M is finite-dimensional. Prove that $V \rightarrow V/M$ sends closed linear subspaces to closed linear subspaces.
- (c) Now let $T: V \rightarrow W$ be a bounded operator between Banach spaces such that T is surjective with finite-dimensional kernel. Prove that T sends closed linear subspaces to closed linear subspaces. (Use the Open Mapping Theorem to reduce to the previous parts).

Exercise 10.33. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i \geq 1}$.

- (a) Define $T: H \rightarrow H$ by $T(e_k) = \frac{1}{k}e_k$ for all $k \geq 1$. Define $T_n = T - \frac{1}{n}\pi_{\langle e_n \rangle}$. Verify that $\{T_n\}$ converges to T , T is injective, but none of the T_n are injective. So the set of injective maps is not open in $\mathcal{B}(H)$.
- (b) Suppose $S: H \rightarrow H'$ is a bounded linear surjection. Show that $S: (\ker S)^\perp \rightarrow H'$ is an isomorphism, and let R be an inverse. So $SR = \text{id}_{H'}$. Use the identity $TR = I + (T - S)R$ to show that if $\|T - S\|$ is small enough then T is also surjective. Conclude that the set of surjections is open in $\mathcal{B}(H, H')$.

[Note: It is also true that if V and V' are Banach spaces then the set of surjections is open in $\mathcal{B}(V, V')$, but the proof is more difficult in that context.]

Exercise 10.34. Prove that if $T: H \rightarrow H'$ is a Fredholm operator then $\text{im}(T)$ is a closed subspace. [Hint: If $M \subseteq H'$ is a finite-dimensional subspace that surjects onto $\text{coker}(T)$ then consider $H \oplus M \rightarrow H'$ and apply the previous exercise.]

Exercise 10.35. Let $T: H \rightarrow H'$ be a compact operator and let $\{e_i\}_{i \geq 1}$ be an orthonormal basis for H . Prove that $\lim_n \|Te_n\| = 0$ via the following steps:

- (a) If 0 is not the limit, then there exists $\epsilon > 0$ and a subsequence e_{n_k} such that $\|Te_{n_k}\| > \epsilon$ for all k .
- (b) There is a further subsequence e_{n_ℓ} such that Te_{n_ℓ} converges, say to an element x .
- (c) For a general element $z \in H'$, use $\langle Te_i, z \rangle = \langle e_i, T^*z \rangle$ to show that $\langle x, z \rangle = 0$.
- (d) Conclude that $x = 0$, and derive a contradiction.

[Note: It turns out that an operator is compact if and only if for *every* orthonormal basis $\{e_i\}$ one has $\lim_n \|Te_n\| = 0$.]

Exercise 10.36. Let $A: H \rightarrow H$ be a self-adjoint bounded operator and suppose that $\text{im}(A)$ is closed.

- (a) Explain why we have an orthogonal direct sum decomposition $\text{im}(A) \oplus \ker(A) = H$.
- (b) Prove that the restricted map $A: \text{im}(A) \rightarrow \text{im}(A)$ is an isomorphism.
- (c) Use the inverse of the map in (b) to conclude that there exists a $C > 0$ such that $\|Ay\| \geq C\|y\|$ for all $y \in (\ker A)^\perp$.
- (d) Prove that A has no eigenvalues in the set $(-C, C) - \{0\}$. (So if 0 is in the spectrum of A , it is an isolated point of the spectrum.)

Exercise 10.37. Fix a closed, simple contour Γ in $\mathbb{C}$. Let S be a normal bounded operator such that $\sigma(S)$ does not meet Γ . Let M be the maximum value of $\|(zI - S)^{-1}\|$ for z on Γ . Suppose T is also a normal bounded operator such that $\sigma(T)$ does not meet Γ .

- (a) For z on Γ prove that

$$(zI - T)^{-1} - (zI - S)^{-1} = (zI - T)^{-1}(T - S)(zI - S)^{-1}.$$

Use this to get the estimate $\|P_T - P_S\| \leq \frac{\|T - S\|}{2\pi} \cdot M \cdot \int_\Gamma \|(zI - T)^{-1}\| dz$.

- (b) Using $zI - T = [I - (T - S)(zI - S)^{-1}](zI - S)$ prove that if $\|T - S\| < \frac{1}{2M}$ then $\|(zI - T)^{-1}\| < 2M$, for z on Γ . [Consult the proof of Corollary 10.12.]
- (c) Prove that $\|P_T - P_S\|$ can be made arbitrarily small by restricting $\|T - S\|$, and thereby prove the continuity of $T \mapsto P_T$ that is asserted in Proposition 10.11.

Exercise 10.38. Let $H = \ell^2(\mathbb{N})$ and let $\mathcal{B}(H)_1 \subseteq \mathcal{B}(H)$ be the subset of operators having a 1-dimensional kernel, with the subspace topology. We will show that $\mathcal{B}(H)_1 \rightarrow \text{Gr}_1(H)$ given by $S \mapsto \ker S$ is not continuous.

Let $T \in \mathcal{B}(H)$ be the operator from Example 10.19(4). For $n \geq 1$ define T_n by

$$T_n(e_i) = \begin{cases} T(e_i) & \text{if } i \notin \{0, n\}, \\ \frac{1}{n}e_0 & \text{if } i = 0, \\ 0 & \text{if } i = n. \end{cases}$$

- (a) Verify that $T_n \in \mathcal{B}(H)$ for all n .
- (b) Compute that $\|T_n - T\| = \frac{1}{n}$ and so $\lim_n T_n = T$ in $\mathcal{B}(H)$.
- (c) Explain why this shows that $\mathcal{B}(H)_1 \rightarrow \text{Gr}_1(H)$ given by $S \mapsto \ker S$ is not continuous.
- (d) Show that none of the operators T_n are Fredholm.

Exercise 10.39.

- (a) Let X and Y be topological groups and let $f: X \rightarrow Y$ be a continuous group homomorphism that is surjective. Suppose that f has a continuous section s . Prove that f is a Hurewicz fibration, i.e., that f has the homotopy lifting property for all spaces W . [Hint: Given a homotopy lifting diagram, applying s gives a lifting that doesn't necessarily make everything commute. Use the group structure to modify it appropriately.]
- (b) The so-called “Bartle-Graves theorem” says that a surjective bounded linear map between Banach spaces always has a continuous section. See [Michael, Proposition 7.2] for the result stated in this way. Conclude that the projection $\mathcal{B}(H) \twoheadrightarrow \mathcal{A}(H)$ is a Hurewicz fibration, hence also a Serre fibration.

Exercise 10.40. In this exercise we will give a naive proof that $\text{Gr}_k(\mathbb{C}^\infty) \rightarrow \text{Gr}_k(H)$ is a weak homotopy equivalence.

- (a) For each finite-dimensional $Z \subseteq H$ with $\dim Z \geq k$, let $\text{Gr}_k(H)_Z \subseteq \text{Gr}_k(H)$ be the subspace of k -planes M such that $M \cap Z^\perp = 0$ (equivalently, $\pi_Z(M)$ is k -dimensional). We have $\text{Gr}_k(Z) \subseteq \text{Gr}_k(H)_Z$, and $M \mapsto \pi_Z(M)$ gives a retraction. Prove this $\text{Gr}_k(Z)$ is a strong deformation retract of $\text{Gr}_k(H)_Z$.
- (b) Suppose that X is compact and $f: X \rightarrow \text{Gr}_k(H)$ is continuous. Prove that there exists a finite-dimensional $Z \subseteq H$ such that the image of f lands in $\text{Gr}_k(H)_Z$. [Hint: For each $x \in X$ show there is a neighborhood U_x such that $f(y) \cap f(x)^\perp = 0$ for all $y \in U_x$. Use compactness to get a finite subcover, and then find a closed $W \subseteq H$ of finite codimension such that $f(x) \cap W = 0$ for all $x \in X$.]
- (c) Prove that $\text{Gr}_k(\mathbb{C}^\infty) \rightarrow \text{Gr}_k(H)$ induces an isomorphism on all homotopy groups. [Hint: This is almost immediate from the preceding parts, though some care is required regarding basepoints.]

Exercise 10.41. We proved that $\text{Fred}(H) \rightarrow \mathcal{A}(H)^*$ is a Serre fibration with contractible fiber, and as a consequence one gets that $[X, \text{Fred}(H)]$ is a group whenever X is semi-cofibrant. But the fact that $\text{Fred}(H) \rightarrow \mathcal{A}(H)^*$ is a Serre fibration is non-trivial. In this exercise we give a more elementary argument that $[X, \text{Fred}(H)]$ is a group that works whenever X is compact.

- (a) Prove that any map $T: X \rightarrow \text{Fred}(H)$ is homotopic to one where $\ker T$ and $\ker T^*$ are vector bundles on X .

- (b) Assume now that $\ker T$ is a vector bundle. Consider the map $X \rightarrow \text{Fred}(H)$ given by $x \mapsto T_x^* T_x + \pi_{\ker T_x}$. Verify that this is continuous and that its image lands in $\mathcal{B}(H)^*$. Use this to show that $T^* \circ T = 0$ in $[X, \text{Fred}(H)]$.
- (c) Explain why $T^* \circ T = 0$ now comes for free, and deduce that $[X, \text{Fred}(H)]$ is a group.

11. K -THEORY AND FREDHOLM OPERATORS, PART 2

The previous section introduced the space $\text{Fred}(H)$, the index of a Fredholm operator, and the statement of the Atiyah-Jänich theorem that for X compact Hausdorff there is a natural “index isomorphism” $\text{ind}_X: [X, \text{Fred}(H)] \rightarrow K^0(X)$. We also developed some general techniques for dealing with $\text{Fred}(H)$. In this section we prove the Atiyah-Jänich theorem and also upgrade it to a weak homotopy equivalence between the spaces $\text{Fred}(H)$ and $\mathbb{Z} \times BU$. This implies there is an isomorphism $[X, \text{Fred}(H)] \cong K^0(X)$ (representable K -theory) for X semi-cofibrant, even when X is not compact. We then use this to re-interpret $[X, \text{Fred}(H)]$ as a Grothendieck group of Fredholm maps between Hilbert bundles.

11.1. Proof of the Atiyah-Jänich theorem. Throughout this section let X be a compact Hausdorff space. A map $f: X \rightarrow \text{Fred}(H)$ factors through some $\mathcal{U}_W$ where $W \subseteq H$ is finite-dimensional, so define $\alpha(f) = f^* \mathcal{E}_W - \underline{W} \in K^0(X)$. By Corollary 10.30(d) choosing a larger W gives the same $\alpha(f)$, and this implies that $\alpha(f)$ is independent of W : if we had chosen another W' , comparing both W and W' to $W + W'$ shows the independence. The map $\alpha: \mathcal{T}op(X, \text{Fred}(H)) \rightarrow K(X)$ is clearly natural in X . So in the diagram

$$\begin{array}{ccc} \mathcal{T}op(X \times I, \text{Fred}(H)) & \xrightarrow{\alpha} & K(X \times I) \\ \text{ev}_0 \downarrow \quad \downarrow \text{ev}_1 & & \text{ev}_0 \downarrow \quad \downarrow \text{ev}_1 \\ \mathcal{T}op(X, \text{Fred}(H)) & \xrightarrow{\alpha} & K(X) \end{array}$$

the two squares obtained by choosing both ev_0 or both ev_1 each commute. Homotopy invariance of the K functor implies $\text{ev}_0 = \text{ev}_1$ on the right, therefore $\alpha \text{ev}_0 = \text{ev}_0 \alpha = \text{ev}_1 \alpha = \alpha \text{ev}_1$ and we get an induced map $[X, \text{Fred}(H)] \rightarrow K(X)$. This is also called the *index map*, but in this context we will write it as ind_X .

Lemma 11.2. *The map ind_X is a group homomorphism.*

Proof. Let $S, T: X \rightarrow \text{Fred}(H)$. We will first prove that $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$ in the case where T is pointwise surjective. Choose a finite-dimensional $W \subseteq H$ such that S maps into $\mathcal{U}_W$. Recall this means $W + \text{im } S_x = H$ for all $x \in X$. Since T is pointwise-surjective, $\text{im}(ST)_x = \text{im } S_x$ for all $x \in X$, hence ST also maps into $\mathcal{U}_W$. Therefore

$$\text{ind}_X(T) = [\ker T], \quad \text{ind}_X(S) = [S^{-1}W] - [\underline{W}], \quad \text{ind}_X(ST) = [(ST)^{-1}W] - [\underline{W}].$$

By inspection we have a short-exact sequence of vector bundles $0 \rightarrow \ker T \rightarrow (ST)^{-1}W \xrightarrow{T} S^{-1}W \rightarrow 0$. So

$$\text{ind}_X(ST) = [\ker T] + [S^{-1}W] - [\underline{W}] = \text{ind}_X(T) + \text{ind}_X(S).$$

(It might be informative to look back at (10.27) and see how this argument connects to that diagram, especially the surjectivity of T).

Choose an orthonormal basis $\{e_i\}_{i \geq 0}$ of H . Recall the shift operators sh_{-m} and sh_m from Example 10.19. In the remainder of the proof we will use a few times that $\text{sh}_m \circ \text{sh}_{-m}$ and $\text{sh}_{-m} \circ \text{sh}_m$ are path-connected to id_H in $\text{Fred}(H)$. These can be proven directly, but they also follow from Proposition 10.31 using that the index is zero.

We next claim that for $m > 0$ and general T one has $\text{ind}_X(T \circ \text{sh}_{-m}) = \text{ind}_X(T) + m = \text{ind}_X(\text{sh}_{-m} \circ T)$. This is an easy exercise. As a consequence, note that we have

$$\text{ind}_X(S) = \text{ind}_X(S \circ \text{sh}_m \circ \text{sh}_{-m}) = \text{ind}_X(S \circ \text{sh}_m) + m$$

and so $\text{ind}_X(S \circ \text{sh}_m) = \text{ind}_X(S) - m$.

Finally, we deal with the general case. Given T , we know there is a finite-dimensional W such that T maps into $\mathcal{U}_W$. So $W + \text{im } T_x = H$ for all $x \in H$. But $W \subseteq \langle e_0, \dots, e_{m-1} \rangle$ for some m , and it follows immediately that $\text{sh}_{-m} \circ T_x$ is surjective for every x .

Since $\text{sh}_{-m} \circ T$ is pointwise surjective we can use what has already been prove to write

$$\begin{aligned} \text{ind}_X(S \circ T) &= \text{ind}_X(S \circ \text{sh}_m \circ \text{sh}_{-m} \circ T) = \text{ind}_X((S \circ \text{sh}_m) \circ (\text{sh}_{-m} \circ T)) \\ &= \text{ind}_X(S \circ \text{sh}_m) + \text{ind}_X(\text{sh}_{-m} \circ T) \\ &= \text{ind}_X(S) - m + (m + \text{ind}_X(T)) \\ &= \text{ind}_X(S) + \text{ind}_X(T) \end{aligned}$$

where the third and fourth equalities were justified earlier in the proof. $\square$

Surjectivity of ind_X follows from the next lemma:

Lemma 11.3. *Let $E \rightarrow X$ be a vector bundle. Then there is a map $T: X \rightarrow \text{Fred}(H)$ that is pointwise surjective and has $\ker T \cong E$.*

Proof. Since X is compact Hausdorff, we can embed E in a trivial bundle $\underline{n}$. Let $P: \underline{n} \rightarrow \underline{n}$ be orthogonal projection onto E , which we regard as a continuous map $P: X \rightarrow M_{n \times n}(\mathbb{C})$. Define $T: X \rightarrow \mathcal{B}(H \otimes \mathbb{C}^n)$ by

$$T_x(v \otimes w) = [v \otimes (\text{id} - P_x)w] + [\text{sh}_{-1}(v) \otimes P_x w].$$

We can also write this as $T = [\text{id} \otimes (\text{id} - P)] + [\text{sh}_{-1} \otimes P]$, which makes it clear that T is continuous. If we decompose $H \otimes \mathbb{C}^n = (H \otimes E_x) \oplus (H \otimes E_x^\perp)$ then the operator T_x splits as $(\text{sh}_{-1} \otimes \text{id}) \oplus (\text{id} \otimes \text{id})$. So clearly T_x is surjective with kernel $\langle e_0 \rangle \otimes E_x$, and in particular T_x is Fredholm. Restating what was just shown, T is pointwise surjective and $\ker T$ is isomorphic to E .

To get a map from X into $\text{Fred}(H)$ —rather than $\text{Fred}(H \otimes \mathbb{C}^n)$ —just choose an isomorphism of Hilbert spaces $H \cong H \otimes \mathbb{C}^n$.

The clever construction of T is from [Atiyah5, Proof of Proposition A.6]. $\square$

Corollary 11.4. *The map ind_X is surjective.*

Proof. Let $E \rightarrow X$ be a vector bundle, and let $T: X \rightarrow \text{Fred}(H)$ be a map guaranteed by Lemma 11.3. Since T is pointwise-surjective we can take $W = 0$ in the definition of $\alpha(T)$, in which case $\alpha(T) = [\ker T] \in K(X)$. But $\ker(T) \cong E$. So every vector bundle is in the image of ind_X . Since ind_X is a group map and the classes of vector bundles generate $K(X)$ as a group, the map is surjective. $\square$

The last piece of the Atiyah-Jänich theorem is injectivity:

Proof of Theorem 10.20 (the Atiyah-Jänich Theorem). Let $T: X \rightarrow \text{Fred}(H)$ and assume $\text{ind}_X(T) = 0$. Pick $W \subseteq H$ finite-dimensional such that T maps into $\mathcal{U}_W$, so that $\text{ind}_X(T) = [T^{-1}W] - [W]$. Since this is zero in $K(X)$, there exists $n > 0$ and a bundle isomorphism $f: T^{-1}W \oplus \underline{n} \rightarrow W \oplus \underline{n}$.

Write $\pi = \pi_{W^\perp}$, orthogonal projection onto $W^\perp$. For each $x \in X$ consider the operator

$$\begin{array}{ccc} (T_x^{-1}W)^\perp \oplus T_x^{-1}W \oplus \mathbb{C}^n & \xrightarrow{\pi T_x \oplus f_x} & W^\perp \oplus W \oplus \mathbb{C}^n \\ \parallel & & \parallel \\ H \oplus \mathbb{C}^n & & H \oplus \mathbb{C}^n \end{array}$$

where the vertical identifications are the evident ones. The map πT_x is an isomorphism, so the horizontal map is an isomorphism—in particular, it is Fredholm. We claim that this describes a map $S: X \rightarrow \text{Fred}(H \oplus \mathbb{C}^n)$, but let us be clear about why S is continuous. Let $\pi_{(T^{-1}W)^\perp}$ denote the map $X \rightarrow \text{Fred}(H \oplus \mathbb{C}^n)$ given by $x \mapsto \pi_{(T_x^{-1}W)^\perp}$. This is continuous by Corollary 10.25(b), since $T^{-1}W$ is a vector bundle and therefore $T_x^{-1}W$ has locally constant rank. Likewise, let $\pi_{T^{-1}W \oplus \mathbb{C}^n}$ be the map $X \rightarrow \mathcal{K}(H \oplus \mathbb{C}^n)$ given by $x \mapsto \pi_{T_x^{-1}W \oplus \mathbb{C}^n}$. This is continuous by the same result. Then we can write

$$S = [\pi_{W^\perp} \circ T \circ \pi_{(T^{-1}W)^\perp}] + [f \circ \pi_{T^{-1}W \oplus \mathbb{C}^n}]$$

and this describes a continuous map $X \rightarrow \text{Fred}(H \oplus \mathbb{C}^n)$. In this decomposition of S , the first term is a map $S_0: X \rightarrow \text{Fred}(H \oplus \mathbb{C}^n)$ and the second is a map $S_1: X \rightarrow \mathcal{K}(H \oplus \mathbb{C}^n)$. The homotopy $S_0 + tS_1$ therefore makes sense as a map $X \times I \rightarrow \text{Fred}(H \oplus \mathbb{C}^n)$ and shows $S \simeq S_0$.

Each S_x is an isomorphism, so S factors through the subspace $\mathcal{B}(H \oplus \mathbb{C}^n)^* \subseteq \text{Fred}(H \oplus \mathbb{C}^n)$. Kuiper's theorem says that this subspace is contractible, so $S \simeq c_{\text{id}}$ (the constant map with value id). Hence $S_0 \simeq c_{\text{id}}$, and we can now forget about S .

Finally, consider the map $a: \text{Fred}(H) \rightarrow \text{Fred}(H \oplus \mathbb{C}^n)$ given by $P \mapsto P \oplus 0$ and the map $b: \text{Fred}(H \oplus \mathbb{C}^n) \rightarrow \text{Fred}(H)$ given by $b(Q) = \pi_H \circ Q \circ i_H$ where $i_H: H \hookrightarrow H \oplus \mathbb{C}^n$ is the inclusion. Then $ba = \text{id}$ (also $ab \simeq \text{id}$, but we will not need this). Note that $S_0 = a(\pi_{W^\perp} \circ T \circ \pi_{(T^{-1}W)^\perp})$, therefore

$$\pi_{W^\perp} \circ T \circ \pi_{(T^{-1}W)^\perp} = b(S_0) = b(c_{\text{id}}) = c_{\text{id}}$$

in $[X, \text{Fred}(H)]$. Finally, we use that

$$T = \pi_{W^\perp} T + \pi_W T = [\pi_{W^\perp} T \pi_{(T^{-1}W)^\perp}] + \pi_{W^\perp} T \pi_{T^{-1}W} + \pi_W T$$

with the first term being a map $X \rightarrow \text{Fred}(H)$ and the second two being maps $X \rightarrow \mathcal{K}(H)$. By the usual straight-line homotopy we get $T \simeq \pi_{W^\perp} T \pi_{(T^{-1}W)^\perp}$, and we have therefore shown $T = c_{\text{id}}$ in $[X, \text{Fred}(H)]$. That is, T is the zero element.

Since our focus on details has made this proof a little clunky, let us wrap up with a brief summary. To start we have

$$T \simeq \begin{bmatrix} (T^{-1}W)^\perp \oplus T^{-1}W \\ \downarrow \pi T \quad \quad \downarrow 0 \\ W^\perp \quad \oplus \quad W \end{bmatrix}$$

by homotoping compact pieces to zero. Now add on $\text{id}_{\mathbb{C}^n}$ and observe

$$T \oplus \text{id}_{\mathbb{C}^n} \simeq \pi T \oplus 0 \oplus \text{id}_{\mathbb{C}^n} \simeq \pi T \oplus 0 \oplus 0 \simeq \pi T \oplus f \simeq c_{\text{id}} = c_{\text{id}_H} \oplus \text{id}_{\mathbb{C}^n},$$

most of the steps by homotoping compact pieces but the final step using Kuiper's theorem because $\pi T \oplus f$ is an isomorphism. Now apply b to everything to get $T \simeq c_{\text{id}_H}$. $\square$

11.5. The weak equivalence of $\mathbb{Z} \times BU$ and $\text{Fred}(H)$. The Atiyah-Jänich theorem certainly suggests that $\text{Fred}(H) \simeq \mathbb{Z} \times BU$, and it almost proves it—but to make that work we need an actual map, or at least a map in $\text{Ho}(\mathcal{T}op)$. Constructing such a map is more delicate than it first appears. For example, there are evident maps $\theta_{k,N}: \text{Gr}_k(\mathbb{C}^N) \rightarrow \text{Fred}(H)$ obtained by sending a subspace V to $\pi_{V^\perp}$. One might expect that these can be used to cook up a map $\mathbb{Z} \times BU \rightarrow \text{Fred}(H)$. But $\pi_{V^\perp} = \text{id} - \pi_V$ and π_V is compact, so in fact $\theta_{k,N}$ is homotopic to the constant map by the straight-line homotopy $\text{id} - t\pi_V$. So we need to be more clever than this.

We will construct our comparison map as the following zig-zag:

$$(11.6) \quad \begin{array}{ccc} V_{\text{Fred}}(H) & \xrightarrow{\alpha} & \text{Fred}(H_+) \\ q \downarrow \simeq & & \\ \text{Gr}_{ft}(\mathbb{C}^\infty) & \xrightarrow{a} & \text{Gr}_{\text{Fred}}(H). \end{array}$$

Here $\mathbb{C}^\infty = \mathbb{C}_+^\infty \oplus \mathbb{C}_-^\infty$, where $\mathbb{C}_+^\infty$ has orthonormal basis $\{e_k \mid k \geq 1\}$ and $\mathbb{C}_-^\infty$ has orthonormal basis $\{e_{-k} \mid k \geq 1\}$. The space $\text{Gr}_{ft}(\mathbb{C}^\infty)$ is the Grassmannian of finite-type subspaces as defined in Exercise ???. For $m \leq m'$ define $\mathbb{C}_{[m,m']}$ to be the subspace spanned by the e_k for $m \leq k \leq m'$ and $k \neq 0$. Also define H_+ and H_- to be the Hilbert spaces arising as completions of $\mathbb{C}_+^\infty$ and $\mathbb{C}_-^\infty$, and let $H = H_+ \oplus H_-$. Let $H_{\geq m}$ be the closure of $\mathbb{C}_{[m,\infty)}$. Let π_+ be orthogonal projection $H \rightarrow H_+$, and likewise for π_- .

- (1) Let $\text{Gr}_\infty(H) \subseteq \mathcal{B}(H)$ be the subset of orthogonal projections onto closed, infinite-dimensional subspaces, equipped with the norm topology. In practice we will identify each projection with its image and thereby regard the elements of $\text{Gr}_\infty(H)$ as subspaces of H .
- (2) Define the Stiefel space $V_\infty(H) \subseteq \mathcal{B}(H_+, H)$ to be the subspace of isometric embeddings $A: H_+ \hookrightarrow H$. Note that the image of such an embedding is necessarily closed: this follows from completeness and the fact that if $\{Ax_n\}$ is a Cauchy sequence then so is $\{x_n\}$.
- (3) Define $q: V_\infty(H) \rightarrow \text{Gr}_\infty(H)$ to send an isometric embedding A to the orthogonal projection $\pi_{\text{im } A}$. This orthogonal projection is also equal to AA^* , and so q is continuous. For every $W \in \text{Gr}_\infty(H)$ the fiber $q^{-1}(W)$ is homeomorphic to $U(H_+)$, and is therefore contractible by Kuiper's theorem.
- (4) The map q is a fiber bundle (see Proposition 11.9 below). Since the fibers are contractible, q is therefore a weak homotopy equivalence.
- (5) Define $\text{Gr}_{\text{Fred}}(H) \subseteq \text{Gr}_\infty(H)$ to be the subspace consisting of those $W \subseteq H$ such that $\pi_+: W \rightarrow H_+$ is Fredholm. Let $V_{\text{Fred}}(H) \subseteq V_\infty(H)$ be the subspace of those $A: H_+ \rightarrow H$ such that the composite $\pi_+ A$ is Fredholm. The map $q: V_{\text{Fred}}(H) \rightarrow \text{Gr}_{\text{Fred}}(H)$ is the restriction of our previous q , and hence is also a fiber bundle with contractible fiber.
- (6) Let α be the map that sends $A: H_+ \rightarrow H$ to $\pi_+ \circ A: H_+ \rightarrow H_+$. This is continuous as a map $\mathcal{B}(H_+, H) \rightarrow \mathcal{B}(H_+, H_+)$, and it sends the subspace $V_{\text{Fred}}(H)$ into $\text{Fred}(H_+)$.

- (7) The map a sends a finite-type subspace $V \subseteq \mathbb{C}^\infty$ to its closure in H , or really the orthogonal projection onto its closure. Continuity comes down to checking that $\text{Gr}_k(\mathbb{C}_{[-m, m-1]}) \rightarrow \text{Gr}_\infty(H)$ given by $V \mapsto \pi_{(V \oplus H_{\geq m})}$ is continuous, but as $\pi_{(V \oplus H_{\geq m})} = \pi_V + \pi_{H_{\geq m}}$ this is obvious.

At this point we have justified the diagram in (11.6) and therefore have produced a map $\text{Gr}_{ft}(\mathbb{C}^\infty) \rightarrow \text{Fred}(H_+)$ in $\text{Ho}(\mathcal{Top})$. Recall from Exercise ?? that we have a standard isomorphism $[X, \text{Gr}_{ft}(\mathbb{C}^\infty)] \rightarrow K^0(X)$ for X compact Hausdorff.

Proposition 11.7. *Let X be cofibrant and compact Hausdorff. Then the composite*

$$K^0(X) \cong [X, \text{Gr}_{ft}(\mathbb{C}^\infty)] \longrightarrow [X, \text{Fred}(H_+)] \xrightarrow{\text{ind}_X} K^0(X)$$

is the identity.

Proof. Any element of $K^0(X)$ may be written as $[E] - m$ for some $m \geq 0$ and some vector bundle $E \rightarrow X$. Choose a classifying map $f: X \rightarrow \text{Gr}_k(\mathbb{C}_{[-N, -1]})$ for E , where N is some positive integer. Use the inclusion $\mathbb{C}_{[-N, -1]} \subseteq \mathbb{C}_{[-N, m]}$ to regard this as a map $f: X \rightarrow \text{Gr}_k(\mathbb{C}_{[-N, m]})$. Write $F: X \rightarrow \text{Gr}_{ft}(\mathbb{C}^\infty)$ for the map $F(x) = f(x) \oplus \mathbb{C}_{\geq(m+1)}$.

We know (see Exercise ??) that the map $[X, \text{Gr}_{ft}(\mathbb{C}^\infty)] \rightarrow K^0(X)$ sends F to $[f^*\eta_k] - m = [E] - m$.

Next, consider the composite $X \xrightarrow{F} \text{Gr}_{ft}(\mathbb{C}^\infty) \rightarrow \text{Gr}_{Fred}(H)$. This sends x to $f(x) \oplus H_{\geq(m+1)}$. Since X is cofibrant and $V_{Fred}(H) \rightarrow \text{Gr}_{Fred}(H)$ is a Serre fibration, our map can be lifted to $V_{Fred}(H)$. This amounts to continuously choosing isometries $A_x: H_+ \rightarrow f(x) \oplus H_{\geq(m+1)}$ (the exact choices will turn out not to matter). If X were only semi-cofibrant we could only lift up to homotopy, which would cause difficulties in the rest of the argument.

The assignment $x \mapsto T_x = \pi_+ A_x$ is a map $T: X \rightarrow \text{Fred}(H_+)$, and we must now compute $\text{ind}_X(T)$. By construction we have $\mathbb{C}_{[1, m]} + \text{im } T_x = H_+$ for all x , and so $\text{ind}_X = [T^{-1}\mathbb{C}_{[1, m]}] - m$. But $A_x: T_x^{-1}\mathbb{C}_{[1, m]} \rightarrow f(x)$ is an isomorphism, so $T^{-1}\mathbb{C}_{[1, m]} \cong E$. Thus, $\text{ind}_X(T) = [E] - m$ and we are done. $\square$

Note that one of the obstacles in the above proof is that we do not know a priori that $[X, \text{Gr}_{ft}(\mathbb{C}^\infty)] \rightarrow [X, \text{Fred}(H)]$ is a group map. This is why the mechanics of the proof are a little more awkward than one might wish.

The following corollary is immediate:

Corollary 11.8. *The map $\text{Gr}_{ft}(\mathbb{C}^\infty) \rightarrow \text{Fred}(H)$ is a weak homotopy equivalence.*

Proof. By Proposition 11.7 together with Theorem 10.20, the map induces bijections on π_0 and on $[S^n, -]$ (unpointed homotopy classes) for every $n \geq 1$. Such a map is a weak homotopy equivalence provided that π_1 of each component of the domain and codomain is abelian. This is clear for the identity component of $\text{Fred}(H)$ because it is a topological monoid, and the fact that $\pi_0 \text{Fred}(H)$ is a group shows that all of the components have the same homotopy type.

For the domain $\text{Gr}_{ft}(\mathbb{C}^\infty) \simeq \mathbb{Z} \times BU$, we know this is a loop space (and hence has abelian π_1) by Bott periodicity—but that is cracking a walnut with a sledgehammer. We can instead use the Schubert cell decomposition to see directly that $\pi_1 \cong \pi_1(\mathbb{C}P^\infty) = 0$, which is abelian (and in the real case we would get $\pi_1 \cong \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$). $\square$

We have considered the chain $\text{Gr}_{ft}(\mathbb{C}^\infty) \rightarrow \text{Gr}_{Fred}(H) \subseteq \text{Gr}_\infty(H)$, but in fact there are many intermediate spaces one could consider that are also models for $\mathbb{Z} \times BU$. Exploring this would take us too far afield, but let us at least list several of them. A bounded operator T is called *Hilbert-Schmidt* provided that for some (equivalently, any) orthonormal basis $\{e_i\}$ one has $\sum \|Te_i\|^2 < \infty$. The class of Hilbert-Schmidt operators sits in between the finite rank and compact operators, and is convenient because the Hilbert-Schmidt operators form a Hilbert space. Consider the following increasing sequence of Grassmannian spaces:

- (1) Closed subspaces $V \subseteq H$ such that $H_{\geq m} \subseteq V \subseteq H_{\geq m'}$ for some $m' \leq m$.
- (2) Closed subspaces $V \subseteq H$ such that $\pi_V - \pi_+$ has finite rank.
- (3) Closed subspaces $V \subseteq H$ such that $\pi_V - \pi_+$ is Hilbert-Schmidt (equivalently, $\pi_+ : V \rightarrow H_+$ is Fredholm and $\pi_- : V \rightarrow H_-$ is Hilbert-Schmidt).
- (4) Closed subspaces $V \subseteq H$ such that $\pi_V - \pi_+$ is compact (equivalently, $\pi_+ : V \rightarrow H_+$ is Fredholm and $\pi_- : V \rightarrow H_-$ is compact).
- (5) Closed subspaces $V \subseteq H$ such that $\pi_+ : V \rightarrow H_+$ is Fredholm.

Each of these has a corresponding Stiefel-space that sits over it and that maps into $\text{Fred}(H)$, and all of the evident projections and inclusions are weak homotopy equivalences. The space in (3) is studied in [PrSe, Chapter 7], and the space in (4) appears in [SW]. A unified approach to these spaces and others is given in [AM].

There is still one technical detail from our argument that has not been completely justified, that we have saved for the end:

Proposition 11.9. *The map $q: V_\infty(H) \rightarrow \text{Gr}_\infty(H)$ is a fiber bundle.*

Proof. The proof is the Hilbert-space analog of the standard argument: for an infinite-dimensional closed subspace $W \subseteq H$ there is an open neighborhood in the Grassmannian where every subspace is the graph of a map $L: W \rightarrow W^\perp$, such a subspace has a canonical isometry with W , and this gives the local trivialization. In the finite-dimensional case we would construct our neighborhood as the set of subspaces having trivial intersection with $W^\perp$, but in the infinite-dimensional case this is not an open condition. So the only subtlety is in finding the correct analog.

In more detail, for a fixed W let $\mathcal{O}_W \subseteq \text{Gr}_\infty(H)$ be the open subset of orthogonal projections π_V such that $|\pi_V - \pi_W| < 1$. It takes a little work, but this is equivalent to the condition that $\pi_W : V \rightarrow W$ is an isomorphism (see Exercise 11.19). This latter condition is in turn manifestly equivalent to saying that $V = \{w + Aw \mid w \in W\}$ for some $A \in \mathcal{B}(W, W^\perp)$ —i.e., that V is the graph of A . Just define $A(w)$ to be $x - w$ where $x \in V$ is the unique element with $\pi_W(x) = w$.

For $V \in \mathcal{O}_W$ let T_V be the inverse of $\pi_W : V \rightarrow W$. This is an isomorphism $W \rightarrow V$, so we can consider its polar decomposition $T_V = Q_T \circ (T_V^* T_V)^{1/2}$ with Q_T an isometry. Then $Q_T = T_V (T_V^* T_V)^{-1/2}$, and this varies continuously with V .

Fix an isometry $\phi_0: H_+ \rightarrow W$. Then we get a continuous bijection $U(H_+) \times \mathcal{O}_W \rightarrow q^{-1}\mathcal{O}_W$ by $(\phi, V) \mapsto Q_V \phi_0 \phi$. It is an easy exercise to write down the inverse and see it is continuous, so this is our local trivialization. $\square$

Exercise 11.10. This exercise concerns the homotopy type of $\text{Gr}_\infty(H)$, which we will see is NOT that of $\mathbb{Z} \times BU$.

- (a) Note that there is an evident action of $U(H)$ on the closed subspaces of H . Prove that two closed subspaces V and W are in the same $U(H)$ -orbit if and only if $(\dim V, \dim V^\perp) = (\dim W, \dim W^\perp)$.

- (b) Show that the $U(H)$ -orbit of V in the space of orthogonal projections is homeomorphic to $U(H)/[U(V) \times U(V^\perp)]$.
- (c) If V is infinite-dimensional, use Kuiper's theorem to show that the homotopy type of $U(H)/[U(V) \times U(V^\perp)]$ is $BU(V^\perp)$.
- (d) Conclude that $\mathrm{Gr}_\infty(H) \simeq \coprod_{0 \leq \dim V^\perp \leq \infty} BU(V^\perp) \simeq \left(\coprod_{k=0}^\infty BU(k) \right) \amalg *$.
- (e) Prove that our map $\mathrm{Gr}_{ft}(\mathbb{C}^\infty) \rightarrow \mathrm{Gr}_\infty(H)$ lands entirely in the component corresponding to $\dim V^\perp = \infty$.

11.11. Changing the Hilbert space. Before tackling a Grothendieck-group model for $K^0(X)$ we need to do a little more prep work. Suppose that H and H' are two Hilbert spaces, and let $\phi: H \rightarrow H'$ be an isomorphism. Then we get a map $\phi_*: \mathrm{Fred}(H) \rightarrow \mathrm{Fred}(H')$ (in fact, a homeomorphism) by $T \mapsto \phi T \phi^{-1}$. But even more, the homotopy class of this map does not depend on ϕ : by Kuiper's theorem $\mathcal{B}(H, H')^{iso}$ is contractible, hence any two choices of ϕ are connected by a path. So in the homotopy category, any two spaces $\mathrm{Fred}(H)$ and $\mathrm{Fred}(H')$ may be canonically identified in this manner.

Now let H_{odd} and H_{ev} be the closed subspaces of H having orthonormal bases $\{e_{2i-1}\}_{i \geq 1}$ and $\{e_{2i}\}_{i \geq 1}$, respectively. So $H = H_{odd} \oplus H_{ev}$. Fix the isomorphism $\phi_{odd}: H \rightarrow H_{odd}$ sending e_i to e_{2i-1} for all i , and if $T: H \rightarrow H$ write $T_{odd} = \phi_{odd} T \phi_{odd}^{-1}$. Write id_{ev} for the identity map on H_{ev} .

Proposition 11.12. *The map $\mathrm{Fred}(H) \rightarrow \mathrm{Fred}(H)$ that sends $T \mapsto T_{odd} \oplus \mathrm{id}_{ev}$ is a weak homotopy equivalence. Even more, in the homotopy category it is equal to the identity map.*

Proof. The first statement is fairly straightforward, but the second statement involves a clever trick.

Since $\mathrm{Fred}(H)$ is a topological monoid it has abelian π_1 . So a map $\mathrm{Fred}(H) \rightarrow \mathrm{Fred}(H)$ is a weak homotopy equivalence provided it induces an isomorphism on unpointed homotopy classes $[X, \mathrm{Fred}(H)] \rightarrow [X, \mathrm{Fred}(H)]$ for every compact Hausdorff space X (in fact, it is enough just to know it when $X = *$ and when X is a sphere).

Consider the diagram

$$\begin{array}{ccc}
 [X, \mathrm{Fred}(H)] & \xrightarrow{\quad} & [X, \mathrm{Fred}(H)] \\
 & \searrow \scriptstyle \cong & \downarrow \scriptstyle \cong \text{ind}_X \\
 & & K^0(X).
 \end{array}$$

Once we prove it commutes, the desired result follows immediately from the Atiyah-Jänich isomorphism. But if $T: X \rightarrow \mathrm{Fred}(H)$ we know by Corollary 10.30 that there is a finite-dimensional $W \subseteq H$ such that $W + \mathrm{im}(T_x) = H$ for all $x \in X$. Then $\mathrm{ind}_X(T) = [T^{-1}W] - [W]$. Let $S = T_{odd} + \mathrm{id}_{ev}$ and let $V = \phi_{odd}(W) \subseteq H_{odd}$. It is immediate that $V + \mathrm{im}(S_x) = H$ for all x , and so $\mathrm{ind}_X(S) = [S^{-1}V] - [V]$. But the bundles $S^{-1}V$ and $T^{-1}W$ are clearly isomorphic, so we are done. Thus, our map is a weak homotopy equivalence.

The map $\mathrm{Fred}(H) \rightarrow \mathrm{Fred}(H \oplus H)$ sending $T \mapsto T \oplus \mathrm{id}$ is isomorphic to the one we just considered, and so this is also a weak homotopy equivalence.

Now we come to the clever trick. Let $H \oplus H$ have orthonormal basis $e_1, e_2, \dots$ for the first factor and $f_1, f_2, \dots$ for the second factor. Let $H \oplus H \oplus H$ have orthonormal

basis $\{a_i\}$ for the first factor, $\{b_i\}$ for the second, and $\{c_i\}$ for the third. Define two isomorphisms $F, G: H \oplus H \rightarrow H \oplus H \oplus H$ by the following tables:

i	1	2	3	4	$\dots$	i	1	2	3	4	$\dots$
$F(e_i)$	a_1	a_2	a_3	a_4		$G(e_i)$	a_1	b_1	a_2	b_2	
$F(f_i)$	b_1	c_1	b_2	c_2		$G(f_i)$	c_1	c_2	c_3	c_4	

For $T \in \text{Fred}(H)$ consider $T \oplus \text{id} \oplus \text{id} \in \text{Fred}(H \oplus H \oplus H)$. Observe that $(F_*)^{-1}(T \oplus \text{id} \oplus \text{id}) = T \oplus \text{id}$ and $(G_*)^{-1}(T \oplus \text{id} \oplus \text{id}) = [T_{\text{odd}} \oplus \text{id}_{\text{ev}}] \oplus \text{id}$. Since $F_*^{-1} \simeq G_*^{-1}$, the two maps $T \mapsto T \oplus \text{id}$ and $T \mapsto [T_{\text{odd}} \oplus \text{id}_{\text{ev}}] \oplus \text{id}$ are homotopic.

Consider the diagram

$$\text{Fred}(H) \rightrightarrows \text{Fred}(H) \longrightarrow \text{Fred}(H \oplus H)$$

where the two maps on the left are $T \rightarrow T$ and $T \rightarrow [T_{\text{odd}} \oplus \text{id}_{\text{ev}}]$, and where the map on the right is $S \rightarrow S \oplus \text{id}$. We have just shown that the two composites are homotopic. But earlier in the proof we showed that the right map is a weak homotopy equivalence and therefore an isomorphism in the homotopy category. It follows that the two maps on the left are equal in the homotopy category. $\square$

11.13. Hilbert bundles and $K^0(X)$. If $K^0(X)$ denotes representable K -theory $[X, \mathbb{Z} \times BU]$ (for X semi-cofibrant), then by Corollary 11.8 we have an isomorphism $[X, \text{Fred}(H)] \cong K^0(X)$ for all semi-cofibrant spaces X . This can be used to give a Grothendieck-group type description of $K^0(X)$ that fits in with other things we have seen, and we explain this next. We warn the reader in advance that this new description of $K^0(X)$ is going to be somewhat tautological, but it nevertheless leads to some interesting issues.

Define a **Hilbert bundle** on a space X to mean a trivial bundle $X \times H$ for some ∞ -separable Hilbert space H . It is important here that the Hilbert space can vary, so that for example $X \times H$ and $X \times (H \oplus H)$ are both Hilbert bundles under this definition. As usual, let us write $\underline{H}$ for the trivial bundle $X \times H$. The focus on trivial bundles seems unnatural at first, but we will shortly explain why we stick to this setting.

Given a continuous map $F: X \rightarrow \mathcal{B}(H, H')$ we get an associated bundle map $\underline{H} \rightarrow \underline{H}'$ that equals F_x on the fiber over x . One must check that $X \times H \rightarrow X \times H'$ is actually continuous for the product topologies, but this is a routine exercise. However, the reverse does not work: It is possible to have bundle maps $X \times H \rightarrow X \times H'$ for which the associated map $X \rightarrow \mathcal{B}(H, H')$ is *not* continuous. The following gives an example:

Example 11.14. Let $X = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\} \subseteq [0, 1]$. This is a good test space for questions about convergence, since maps $X \rightarrow Y$ are precisely convergent sequences in Y . Let H have orthonormal basis $\{e_i\}_{i \geq 1}$. Define $F: X \times H \rightarrow X \times H$ fiberwise by $F_0 = 0$ and $F_{\frac{1}{n}}$ is orthogonal projection onto $\mathbb{C}e_n$. This is a continuous map: both X and H are metric spaces so it suffices to check that convergent sequences are preserved by F , and this boils down to the fact that $\lim_n \langle x, e_n \rangle = 0$ for any x . The associated map $X \rightarrow \mathcal{B}(H)$ is *not* continuous, though: one has $|F_{\frac{1}{n}} - F_0| = 1$ for all n , so the sequence of projections does not converge to F_0 in the norm topology.

For similar strange behavior, consider $A: X \times H \rightarrow X \times H$ given fiberwise by $A_0 = 0$, $A_{\frac{1}{n}}(v) = \langle v, e_n \rangle e_1$. This is not so different from the previous example and one again sees that the associated map $X \rightarrow \mathcal{B}(H)$ is not continuous, but here one

also sees something else. The fiberwise adjoint is $A_0^* = 0$, $A_{\frac{1}{n}}^*(v) = \langle v, e_1 \rangle e_n$, and here A^* is not even continuous as a map $X \times H \rightarrow X \times H$. The sequence $(\frac{1}{n}, e_1)$ converges to $(0, e_1)$ but is not convergent upon applying A^* . So here we have a bundle map where the fiberwise adjoint is *not* a bundle map.

There are also examples where the map is a fiberwise isomorphism. Consider the bundle map given by $B_{\frac{1}{n}} = \text{id} + A_{\frac{1}{n}}$ for $n > 1$ and $B_0 = B_1 = \text{id}$. Since $A_{\frac{1}{n}}^2 = 0$ for $n > 1$, the map B is a fiberwise isomorphism—and even a bundle isomorphism. But the associated map $X \rightarrow \mathcal{B}(H)$ is again not continuous, since $|B_{\frac{1}{n}} - B_0| = 1$ for all n .

While we are considering these examples, it is worth also thinking about one more. The bundle map $\text{id} - F$ is surjective on the fiber over 0, but it is not fiberwise surjective in any neighborhood of 0. This is another indication that the naive notion of bundle map is not a good one when dealing with infinite-dimensional Hilbert space fibers.

The lesson of the above examples is that continuity with respect to the product topology, when dealing with Hilbert space fibers, is simply not as nice as one would like. For the moment we fix this by defining the problem away. Let us *define* the notion of “Hilbert bundle map” $\underline{H} \rightarrow \underline{H}'$ to mean a bundle map for which the associated map $X \rightarrow \mathcal{B}(H, H')$ is continuous. With this definition, say that a bundle map is **Fredholm** if the map on each fiber is Fredholm.

An isomorphism between two Hilbert bundles E and E' is a map of Hilbert bundles $E \rightarrow E'$ which is an isomorphism on each fiber. Since the inverse map $\mathcal{B}(H, H')^{\text{iso}} \rightarrow \mathcal{B}(H', H)^{\text{iso}}$ is continuous, this definition is equivalent to being an isomorphism in the category of Hilbert bundles.

Given two maps of Hilbert bundles $E \rightarrow J$ and $E' \rightarrow J'$, we say these are isomorphic if there are isomorphisms $E \rightarrow E'$ and $J \rightarrow J'$ making the evident diagrams commute. As a bundle map $E \rightarrow J$ can also be thought of as a chain complex of length one, we will often use the phrase “length one Fredholm complex” when talking about Fredholm bundle maps. Unless otherwise indicated, the intention is always that such complexes are concentrated in homological degrees 0 and 1.

Definition 11.15. Define $K_{\text{Hilb}}(X)$ by taking the free abelian group on symbols $[\phi] = [E \xrightarrow{\phi} F]$ where ϕ is a (length one) Fredholm complex over X and quotienting out the following relations:

- (1) $[\phi] = [\phi']$ if ϕ is isomorphic to ϕ' .
- (2) $[E \rightarrow F] = 0$ whenever $E \rightarrow F$ is acyclic (that is, a fiberwise isomorphism).
- (3) $[E \rightarrow F] + [E' \rightarrow F'] = [E \oplus E' \rightarrow F \oplus F']$.
- (4) If $\phi: E \rightarrow F$ is a Fredholm complex over $X \times I$, then $[\phi]_{X \times 0} = [\phi]_{X \times 1}$.

Compare this to Definition ??.

Let H be a fixed ∞ -separable Hilbert space. Note that any continuous map $T: X \rightarrow \text{Fred}(H)$ gives an associated Fredholm complex $T: \underline{H} \rightarrow \underline{H}$. Denote this complex by $[T]_H$. This gives us a map of sets $[X, \text{Fred}(H)] \rightarrow K_{\text{Hilb}}(X)$.

The following result and proof are adapted from [Segal3, Section 5]:

Proposition 11.16. Let H be a fixed ∞ -separable Hilbert space and let X be any semi-cofibrant space. Then the natural map $[X, \text{Fred}(H)] \rightarrow K_{\text{Hilb}}(X)$ is an isomorphism of groups. Consequently, we have $K^0(X) \cong K_{\text{Hilb}}(X)$.

Proof. In this proof, “complex” always means chain complex of length one concentrated in homological degrees 0 and 1. Take the set of isomorphism classes of Fredholm complexes over X and impose the following equivalence relation: $E \sim E'$ if and only if there exist acyclic complexes A and A' together with a Fredholm complex J on $X \times I$ and isomorphisms $E \oplus A \cong J|_{X \times 0}$ and $E \oplus A' \cong J|_{X \times 1}$. (Exercise: Check that this is actually an equivalence relation.) Let $\mathcal{S}_X$ be the set of equivalence classes.

Direct sum of complexes induces a monoid structure on $\mathcal{S}_X$. For $T: X \rightarrow \text{Fred}(H)$, the assignment $T \mapsto [T]_H$ gives a map of sets $\theta: [X, \text{Fred}(H)] \rightarrow \mathcal{S}_X$. This is surjective, since any ∞ -separable Hilbert space is isometrically isomorphic to H .

We claim that θ is a homomorphism of monoids. To see this, let $S, T: X \rightarrow \text{Fred}(H)$. Consider the map $\begin{bmatrix} I & 0 \\ 0 & ST \end{bmatrix}: X \rightarrow \text{Fred}(H \oplus H)$ and observe the following homotopies:

- (i) $\begin{bmatrix} I & 0 \\ 0 & ST \end{bmatrix} \simeq \begin{bmatrix} T & I \\ ST & 0 \end{bmatrix}$ using that $\begin{bmatrix} I & 0 \\ 0 & ST \end{bmatrix} = \begin{bmatrix} T & I \\ ST & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & I \\ I & -T \end{bmatrix}$. The second factor is a map $X \rightarrow \mathcal{B}(H \oplus H)^*$ and so is homotopic to the identity by Kuiper’s theorem.
- (ii) $\begin{bmatrix} T & I \\ ST & 0 \end{bmatrix} \simeq \begin{bmatrix} T & I \\ 0 & -S \end{bmatrix}$ using the homotopy $\begin{bmatrix} I & 0 \\ -tS & I \end{bmatrix} \begin{bmatrix} T & I \\ ST & 0 \end{bmatrix}$.
- (iii) $\begin{bmatrix} T & I \\ 0 & -S \end{bmatrix} \simeq \begin{bmatrix} T & 0 \\ 0 & -S \end{bmatrix}$ using the homotopy $\begin{bmatrix} T & tI \\ 0 & -S \end{bmatrix}$.
- (iv) $\begin{bmatrix} T & 0 \\ 0 & -S \end{bmatrix} \simeq \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$ since the constant map $-I$ is path-connected to I in $\mathcal{B}(H)^*$ (which is contractible by Kuiper’s theorem).

Putting everything together, we have that $\begin{bmatrix} I & 0 \\ 0 & ST \end{bmatrix} \simeq \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$ as maps $X \rightarrow \text{Fred}(H \oplus H)$. Taking the associated Fredholm complexes we get that $[ST]_H \sim [T]_H \oplus [S]_H$, since $[I]_H$ is acyclic. This is what we needed.

Injectivity of θ takes a little thought, but is almost tautological. Suppose that $S, T: X \rightarrow \text{Fred}(H)$ and $[S]_H \sim [T]_H$. Then there exist acyclic complexes A and A' and a Fredholm complex J on $X \times I$ such that $J|_{X \times 0} \cong [S]_H \oplus A$ and $J|_{X \times 1} \cong [T]_H \oplus A'$. By adding $[\text{id}]_H$ to everything in sight if necessary we can assume that neither A nor A' is the zero complex. The complexes A and A' , being acyclic and nonzero, are isomorphic to $[\text{id}]_H$. So J can be considered as giving us a homotopy $\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \simeq \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$ of maps $X \rightarrow \text{Fred}(H \oplus H)$, or equivalently $S_{\text{odd}} + \text{id}_{\text{ev}} \simeq T_{\text{odd}} + \text{id}_{\text{ev}}$ as maps $X \rightarrow \text{Fred}(H)$. But we know from Proposition 11.12 that $S \simeq S_{\text{odd}} + \text{id}_{\text{ev}}$ and $T \simeq T_{\text{odd}} + \text{id}_{\text{ev}}$, so $S \simeq T$. This completes the proof that $[X, \text{Fred}(H)] \rightarrow \mathcal{S}_X$ is an isomorphism.

We have the evident set map $\mathcal{S}_X \rightarrow K_{\text{Hilb}}(X)$, and this is a map of monoids. But from the isomorphism $\theta: [X, \text{Fred}(H)] \rightarrow \mathcal{S}_X$ we know that $\mathcal{S}_X$ is a group, and so the universal property of $K_{\text{Hilb}}(X)$ gives a group map $K_{\text{Hilb}}(X) \rightarrow \mathcal{S}_X$ that sends each Fredholm complex to its class in $\mathcal{S}_X$. The fact that these are inverses is immediate. $\square$

Again, note that the actual content of Proposition 11.16 is quite small. We are simply reinterpreting $[X, \text{Fred}(H)]$ as a Grothendieck group in a rather trivial way. But this line of thought leads to some questions:

- (1) We can try to generalize the notion of Hilbert bundle. A plausible choice is to look at locally trivial bundles E of Hilbert spaces having the extra property that there is an open cover $\{U_i\}$ of the base and trivializations $E|_{U_i} \cong U_i \times H$ over each U_i such that the transition maps $U_i \cap U_j \rightarrow \mathcal{B}(H)^*$ are continuous in the norm topology. This is the correct notion if we want bundle theory to

connect to the theory of principle $U(H)$ -bundles. But since $U(H)$ is contractible by Kuiper's theorem, with this new definition all bundles over paracompact Hausdorff spaces are trivializable. So from a certain perspective it is not more general than what we already did, though in practice bundles might arise that are not "obviously trivial" and we would then be in a position to accomodate these.

- (2) Even with this expanded definition of Hilbert bundle, one still runs into the issue of whether a bundle map is required to be locally given by continuous functions into $\mathcal{B}(H)$ —it is the same problem as before, where one can have maps of trivial bundles that do not arise from continuous maps into $\mathcal{B}(H)$. One can force this requirement into the category, but it would be better if we didn't have to do that.
- (3) In the paper [Segal3], Segal allows for quite general bundles whose fibers are Banach spaces. He defines a notion of compact bundle map, which extend to chain complexes levelwise. For bounded chain complexes of such bundles (of arbitrary length) he defines Fredholm complexes to be those where the identity is chain homotopic to a compact map. For X paracompact Hausdorff it is shown that $[X, \text{Fred}(H)]$ coincides with a Grothendieck group of these generalized Fredholm complexes, made from relations similar to what we used above. He could have used only complexes of length one, but as we have seen before it is in some sense more natural to use arbitrary bounded complexes (for example, the tensor product of complexes works out more naturally).

The arguments in [Segal3] require some subtle pointset topology and analysis, so we will not pursue these here. But it is worth knowing that this paper exists. One nice piece of the story is the result that when X is compact Hausdorff every generalized Fredholm complex is chain homotopy equivalent to a chain complex of finite-rank vector bundles, which explains why the K -group reduces to the familiar one in this case.

◦ Exercises ◦

Exercise 11.17. Let X be compact Hausdorff and let $T: X \rightarrow \text{Fred}(H)$ be a map. Suppose there exist maps $f: X \rightarrow \text{Gr}_{k_0}(\mathbb{C}^N)$ and $g: X \rightarrow \text{Gr}_{k_1}(\mathbb{C}^M)$ such that for every $x \in X$ the following are true:

- $T(f(x)) \subseteq g(x)$, and
- the composite $f(x)^\perp \xrightarrow{T} H \xrightarrow{\pi_{(g(x)^\perp)}} g(x)^\perp$ is an isomorphism.

Prove that $\text{ind}_X(T) = [f] - [g] \in K(X)$, where we have written $[f]$ as shorthand for $[f^* \eta_{k_0}]$ and likewise for g . [Hint: ???]

Exercise 11.18. Write $H = H_+ \oplus H_-$ as usual. Define $A: H_+ \rightarrow H_-$ by $A(e_n) = \frac{1}{n}e_{-n}$, and let $W = \Gamma(A)$. Let $X_W \subseteq \text{Gr}_\infty(H)$ be the set of subspaces K such that $K \cap W = 0$. We claim that X_W is not open in $\text{Gr}_\infty(H)$.

- (a) Check that the operator A is bounded, hence W is a closed subspace of H .
- (b) Set $w_n = (e_n, \frac{1}{n}e_{-n}) \in W$ and let N_n be the closure of the span $\langle e_1, e_2, \dots, e_{n-1}, w_n, e_{n+1}, e_{n+2}, \dots \rangle$. Prove that $|\pi_{N_n} - \pi_{H_+}| = \frac{1}{n}$, so that $\{\pi_{N_n}\}$ converges to π_{H_+} in $\text{Gr}_\infty(H)$.

- (c) Check that $H_+ \in X_W$, but $N_n \notin X_W$ for all n . Conclude that X_W cannot be open.

[Comments: The point in this example is that even though $W \cap H_+ = 0$, the subspaces still become infinitely-close in a certain sense. For example, the lines $\langle w_n \rangle$ in W converge to a line in H_+ . This cannot happen in the finite-dimensional setting, where if two subspaces intersect trivially then there is a positive angle between them.]

Exercise 11.19. Let W and V be subspaces of H and assume $|\pi_V - \pi_W| < 1$. This exercise will show that $\pi_V: W \rightarrow V$ is an isomorphism.

- (a) First show that if $x \in V^\perp$ then $|\pi_W(x)| < |x|$. Conclude that $W \cap V^\perp = 0$, and by symmetry $V \cap W^\perp = 0$.
- (b) Let $C = |\pi_V - \pi_W|$, and recall $C < 1$. Prove that if $w \in W$ then $|\pi_V(w) - w| \geq (1 - C)|w|$. (In operator-theoretic language, the operator $\pi_V: W \rightarrow V$ is “bounded below”.) [Hint: Use the reverse triangle inequality $|x - y| \geq |y| - |x|$.]
- (c) If $\{x_i\}$ is a sequence in W such that $\{\pi_V(x_i)\}$ is Cauchy, use (b) to prove that $\{x_i\}$ is Cauchy. Conclude that $\pi_V(W) \subseteq V$ is closed.
- (d) Let K be the orthogonal complement of $\pi_V(W)$ in V . If $z \in K$ prove that $z \in W^\perp$, hence $z = 0$. [Hint: Use that π_V is self-adjoint.]
- (e) Conclude that $\pi_V(W) = V$, hence $\pi_V: W \rightarrow V$ is an isomorphism.
- (f) Prove that $|\pi_{V^\perp} - \pi_{W^\perp}| = |\pi_V - \pi_W|$ and so conclude that $\pi_{V^\perp}: W^\perp \rightarrow V^\perp$ is also an isomorphism.
- (g) Prove that $|\pi_V - \pi_W| < 1$ is equivalent to the condition that $\pi_V: W \rightarrow V$ and $\pi_{V^\perp}: W^\perp \rightarrow V^\perp$ are both isomorphisms.

Exercise 11.20. Let $W \in \text{Gr}_\infty(H)$ and let $\mathcal{O}_W \subseteq \text{Gr}_\infty(H)$ be the open set of projections P such that $\|P - \pi_W\| < 1$. Show that $\mathcal{B}(W, W^\perp) \rightarrow \mathcal{O}_W$ given by $A \mapsto \Gamma(A)$ is a homeomorphism. [Hint: Show that the projection matrix for $\Gamma(A)$ is $\begin{bmatrix} (I + A^*A)^{-1} & (I + A^*A)^{-1}A^* \\ A(I + A^*A)^{-1} & A(I + A^*A)^{-1}A \end{bmatrix}$, where the block diagonal matrix is with respect to $H = W \oplus W^\perp$. Likewise, show that the inverse $\mathcal{O}_W \rightarrow \mathcal{B}(W, W^\perp)$ is given by $P \mapsto \pi_{W^\perp}P|_W(\pi_W P|_W)^{-1}$.]

12. ???

13. COMPLEX-ORIENTED COHOMOLOGY THEORIES, AGAIN

As a prelude to the Atiyah-Singer theorem, in this section we explore further constructions one can perform in the context of complex-oriented cohomology theories. In particular, this will lead to invariants for any compact manifold (not necessarily complex).

13.1. Cohomology with compact support. Let $\mathcal{E}$ be any cohomology theory. Defines the **$\mathcal{E}$ -theory with compact support** by

$$\mathcal{E}_{cs}^n(X) = \text{colim}_{K^{cpt} \subseteq X} \mathcal{E}^n(X, X - K).$$

Let us be clear about the maps in the colimit: if $K_0 \subseteq K_1$ then $X - K_1 \subseteq X - K_0$ and we have the induced map $\mathcal{E}^n(X, X - K_0) \rightarrow \mathcal{E}^n(X, X - K_1)$. Also, observe that the maps $\mathcal{E}^n(X, X - K) \rightarrow \mathcal{E}^n(X)$ induce a natural comparison map $\mathcal{E}_{cs}^n(X) \rightarrow \mathcal{E}^n(X)$. When X is compact this is an isomorphism, as X is a terminal object in the system of K 's.

Observe that $\mathcal{E}_{cs}^*(X)$ is naturally a left (and right) module over $\mathcal{E}^*(X)$: this is induced from the pairings $\mathcal{E}^p(X) \otimes \mathcal{E}^q(X, X - K) \rightarrow \mathcal{E}^{p+q}(X, X - K)$. The map $\mathcal{E}_{cs}^*(X) \rightarrow \mathcal{E}^*(X)$ is a map of $\mathcal{E}^*(X)$ -modules.

The $\mathcal{E}_{cs}^*$ -groups are functorial, but only for weakly proper maps (see Definition ??): a map $f: X \rightarrow Y$ is weakly proper if for every compact set $K \subseteq Y$ the preimage $f^{-1}K \subseteq X$ is also compact. We have a map of pairs $f: (X, X - f^{-1}K) \rightarrow (Y, Y - K)$ and therefore $f^*: \mathcal{E}^n(Y, Y - K) \rightarrow \mathcal{E}^n(X, X - f^{-1}K) \rightarrow \mathcal{E}_{cs}^n(X)$. These maps are compatible under the colimit system and induce $f^*: \mathcal{E}_{cs}^n(Y) \rightarrow \mathcal{E}_{cs}^n(X)$. This is a map of $\mathcal{E}^*(Y)$ -modules. It will be useful to remember that every closed inclusion is weakly proper.

The pairings $\mathcal{E}^p(X, X - K) \otimes \mathcal{E}^q(Y, Y - L) \rightarrow \mathcal{E}^{p+q}(X \times Y, X \times Y - (K \times L))$ pass to the colimits to give $\mathcal{E}_{cs}^p(X) \otimes \mathcal{E}_{cs}^q(Y) \rightarrow \mathcal{E}_{cs}^{p+q}(X \times Y)$. When X is Hausdorff the diagonal $X \hookrightarrow X \times X$ is closed, hence weakly proper, and therefore the external products induce an internal product on $\mathcal{E}_{cs}^*(X)$. But one should not think of this as a ring, as there is typically no unit element—in fact, often $\mathcal{E}_{cs}^0(X) = 0$. As an intuitive picture it is better to regard $\mathcal{E}_{cs}^*(X)$ as an “ideal” for $\mathcal{E}^*(X)$, though this is not technically correct because the map $\mathcal{E}_{cs}^*(X) \rightarrow \mathcal{E}^*(X)$ need not be an injection.

Let $\rho: U \hookrightarrow X$ be an open inclusion. For any compact set $L \subseteq U$ we have the excision isomorphism $\mathcal{E}^n(X, X - L) \rightarrow \mathcal{E}^n(U, U - L)$, and the inverses of these assemble to give a map $\rho_*: \mathcal{E}_{cs}^n(U) \rightarrow \mathcal{E}_{cs}^n(X)$. These are called “extension maps”. In the following, it will be useful to always use the symbol ρ for open inclusions.

Exercise 13.2. When $i: U \hookrightarrow X$ is an open inclusion check that

$$\begin{array}{ccc} \mathcal{E}_{cs}^*(U) & \xrightarrow{\rho_*} & \mathcal{E}_{cs}^*(X) \\ \downarrow & & \downarrow \\ \mathcal{E}^*(U) & \xleftarrow{\rho^*} & \mathcal{E}^*(X) \end{array}$$

is commutative, ρ_* is a map of $\mathcal{E}^*(X)$ -modules, and when X is Hausdorff ρ_* preserve products.

The following is a useful compatibility statement between the extension and pullback maps:

Proposition 13.3. *Suppose given a commutative square*

$$\begin{array}{ccc} U & \xrightarrow[\text{open}]{\rho} & X \\ f \uparrow & & \uparrow g \\ V & \xrightarrow[\text{open}]{\rho} & Y \end{array}$$

where the vertical maps are weakly proper and the horizontal maps are open inclusions. Also assume that $V \rightarrow U \times_X Y$ is surjective. Then $g^*\rho_* = \rho_*f^*$ as maps $\mathcal{E}_{cs}^*(U) \rightarrow \mathcal{E}_{cs}^*(Y)$.

Proof. Let $K \subseteq U$ be compact. The hypothesis that $V \rightarrow U \times_X Y$ is surjective implies that $g^{-1}K = f^{-1}K$. So we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}^*(U, U - K) & \xleftarrow[\cong]{\rho^*} & \mathcal{E}^*(X, X - K) \\ f^* \downarrow & & \downarrow g^* \\ \mathcal{E}^*(V, V - f^{-1}K) & \xleftarrow[\cong]{\rho^*} & \mathcal{E}^*(Y, Y - f^{-1}K). \end{array}$$

The maps ρ_* are defined using the inverses of the ρ^* maps, and from here the result is immediate. $\square$

The above definitions are in some sense the most natural ones, though they can be a bit intimidating. A simplification is available for locally compact spaces that are reasonably nice. Suppose X is locally compact, and recall the one-point compactification X^+ . When $K \subseteq X$ is compact consider the natural maps

$$\mathcal{E}^n(X, X - K) \xleftarrow[\cong]{} \mathcal{E}^n(X^+, X^+ - K) \longrightarrow \mathcal{E}^n(X^+, \infty)$$

where the first is an isomorphism by excision. These yield a natural map $\mathcal{E}_{cs}^n(X) \rightarrow \mathcal{E}^n(X^+, \infty)$, and in good cases this is an isomorphism:

Proposition 13.4. *Suppose there are a cofinal collection of compact sets $K \subseteq X$ with the property that $X^+ - K$ is contractible. Then $\mathcal{E}_{cs}^n(X) \rightarrow \mathcal{E}^n(X^+, \infty)$ is an isomorphism.*

Proof. For $K \subseteq X$ compact consider the long exact sequence

$$\cdots \rightarrow \mathcal{E}^n(X^+, X^+ - K) \rightarrow \mathcal{E}^n(X^+, \infty) \rightarrow \mathcal{E}^n(X^+ - K, \infty) \rightarrow \cdots$$

If $X^+ - K$ is contractible then the groups $\mathcal{E}^*(X^+ - K, \infty)$ are all zero, and so $\mathcal{E}^n(X^+, X^+ - K) \rightarrow \mathcal{E}^n(X^+, \infty)$ is an isomorphism. Now pass to the colimit. $\square$

When X satisfies the hypothesis of the above proposition we will say that “ X^+ is locally contractible at ∞ ”.

Example 13.5. For the space $\mathbb{R}^k$ we have the cofinal system of compact subsets $L_N = [-N, N]^k$, and $S^k - L_N$ is contractible. The above result implies that $\mathcal{E}_{cs}^*(\mathbb{R}^k) \cong \tilde{\mathcal{E}}^*(S^k)$.

It is worth giving the usual warning that the groups $\mathcal{E}_{cs}^*(X)$ are not homotopy invariants of X . For instance, the above example shows that $\mathcal{E}_{cs}^*(\mathbb{R}^1)$ is usually not isomorphic to $\mathcal{E}_{cs}^*(pt)$.

Exercise 13.6.

- (a) If $f: X \rightarrow Y$ is weakly proper, check that there is an induced map $f^+: X^+ \rightarrow Y^+$.
- (b) If $U \subseteq X$ is open, check that there is a natural continuous surjection $X^+ \rightarrow U^+$. Check that applying $\mathcal{E}^*(-)$ to the above maps is compatible with the functorial maps f^* and ρ_* for $\mathcal{E}_{cs}^*$.

13.7. Generalized Thom homomorphisms. If $E \rightarrow X$ is a vector bundle (real or complex) and X is compact, then we have seen before ??? that $\text{Th } E \cong E^+$. So in this case $\mathcal{E}_{cs}^*(E) \cong \tilde{\mathcal{E}}^*(\text{Th } E)$. When X is only locally compact we still have a comparison map, but it is not necessarily an isomorphism. To identify the map, choose a metric on E and identify E (up to homeomorphism) with $D^\circ E$, the interior of the disk bundle DE . For every compact set $K \subseteq D^\circ E$ consider the maps

$$\mathcal{E}^n(D^\circ E, D^\circ E - K) \xleftarrow{\cong} \mathcal{E}^n(DE, DE - K) \longrightarrow \mathcal{E}^n(DE, SE),$$

where the left map is an excision isomorphism. These assemble to give $\mathcal{E}_{cs}^n(E) \rightarrow \mathcal{E}^n(DE, SE)$. Here is an example where this fails to be an isomorphism:

Example 13.8. Consider $X = \mathbb{R}$, and for convenience let $E = \underline{1}$ be the trivial real vector bundle over X . Then $E \cong \mathbb{R}^2$ and so $\mathcal{E}_{cs}^*(E) \cong \tilde{\mathcal{E}}^*(S^2)$. But $\text{Th } E \cong (\mathbb{R} \times D^1)/(\mathbb{R} \times \partial D^1) \simeq S^1$, so $\tilde{\mathcal{E}}^*(\text{Th } E) \cong \tilde{\mathcal{E}}^*(S^1)$.

Now assume that $\mathcal{E}$ is complex-oriented, and let $p: E \rightarrow X$ be a complex vector bundle of rank n . The Thom class $\mathcal{U}_E \in \mathcal{E}^{2n}(DE, SE)$ will not necessarily lift into $\mathcal{E}_{cs}^{2n}(E)$, but we can still use this class to construct a Thom homomorphism $th_E: \mathcal{E}_{cs}^*(X) \rightarrow \mathcal{E}_{cs}^{*+2n}(E)$. If $\alpha \in \mathcal{E}^i(X, X - K)$ then $p^*\alpha \in \mathcal{E}^i(E, E - p^{-1}K)$, so the product $(p^*\alpha) \cdot \mathcal{U}_E$ lies in $\mathcal{E}^{i+2n}(E, (E - 0) \cup (E - p^{-1}K)) = \mathcal{E}^{i+2n}(E, E - \zeta K)$ (here ζ is the zero-section). The usual commutative diagrams show that as an element of $\mathcal{E}_{cs}^{i+2n}(E)$ this does not depend on the choice of K .

The square

$$\begin{array}{ccc} \mathcal{E}_{cs}^i(X) & \xrightarrow{th_E} & \mathcal{E}_{cs}^{i+2n}(E) \\ \downarrow & & \downarrow \\ \mathcal{E}^i(X) & \xrightarrow{p^*(-) \cup \mathcal{U}_E} & \mathcal{E}^{i+2n}(DE, SE) \end{array}$$

is readily checked to be commutative. When X is compact the two vertical maps are isomorphisms.

Theorem 13.9. *Let $E \rightarrow X$ be a complex vector bundle of rank n , where X is paracompact and Hausdorff. Then $th_E: \mathcal{E}_{cs}^*(X) \rightarrow \mathcal{E}_{cs}^{*+2n}(E)$ is an isomorphism.*

Proof. ??? □

13.10. The $\mathcal{E}$ -index of a compact manifold. The following constructions are from [AS1]. Let Y be a (real) differentiable manifold and let $j: M \hookrightarrow Y$ be the inclusion of a compact submanifold of codimension c . Let $M \subseteq N$ be a tubular neighborhood, so that we have a map $N \rightarrow M$ making N into a real vector bundle. On the level of tangent bundles we then have $TM \subseteq TN \subseteq TY$, and TN is a tubular neighborhood of TM . As a normal bundle $TN \rightarrow TM$ we have $TN \cong \pi^*(N \oplus N)$ where $\pi: TM \rightarrow M$ is the bundle projection. In particular, we can put a complex structure on TN by letting the multiplication by i take the first copy of N to the second. Let $i: TN \hookrightarrow TY$ be the (open) inclusion.

Consider the composite

$$\mathcal{E}_{cs}^n(TM) \xrightarrow{th_{TN}} \mathcal{E}_{cs}^{n+2c}(TN) \xrightarrow{j_*} \mathcal{E}_{cs}^{n+2c}(TY).$$

The usual arguments show that this does not depend on the choice of tubular neighborhood N . Denote the composite as $j_!$.

When $Y = \mathbb{R}^N$ we can add in one more step. In this case $TY = \mathbb{R}^N \times \mathbb{R}^N$ is a complex vector space by letting multiplication by i take the first copy to the second. Regarding this as a complex bundle over a point, we can consider the chain

$$\mathcal{E}_{cs}^n(TM) \xrightarrow{j_!} \mathcal{E}_{cs}^{n+2c}(TY) \xleftarrow{th_{TY}} \mathcal{E}^{n+2c-2N}(pt).$$

The second map is an isomorphism. The resulting map

$$\mathcal{E}_{cs}^*(TM) \longrightarrow \mathcal{E}^{*-2\dim M}(pt)$$

is called the **$\mathcal{E}$ -index map** and we will write it as $\text{ind}_M^{\mathcal{E}}$. It does not depend on the choice of Y or the choice of the embedding. Since M is compact we have $\mathcal{E}_{cs}^*(TM) \cong \tilde{\mathcal{E}}^*(\text{Th } M)$ and so we may recast this as a map $\tilde{\mathcal{E}}^*(\text{Th } M) \rightarrow \mathcal{E}^{*-2\dim M}(pt)$.

Here are some easy properties of these $\mathcal{E}$ -index maps:

- (1) $\text{ind}_{pt}^{\mathcal{E}} = \text{id}$.
- (2) Suppose M' is a compact differentiable manifold and $j: M \hookrightarrow M'$ is a submanifold. Then the diagram

$$\begin{array}{ccc} \mathcal{E}_{cs}^*(TM) & \xrightarrow{j_!} & \mathcal{E}_{cs}^{*+2(\dim M' - \dim M)}(TM') \\ & \searrow \text{ind}_M^{\mathcal{E}} & \swarrow \text{ind}_{M'}^{\mathcal{E}} \\ & \mathcal{E}^{*-2\dim M}(pt) & \end{array}$$

is commutative.

- (3) Suppose $f: M \rightarrow M'$ is a diffeomorphism. Then

$$\begin{array}{ccc} \mathcal{E}_{cs}^*(TM') & \xrightarrow{(Tf)^*} & \mathcal{E}_{cs}^*(TM) \\ & \searrow \text{ind}_{M'}^{\mathcal{E}} & \swarrow \text{ind}_M^{\mathcal{E}} \\ & \mathcal{E}^{*-2\dim M}(pt) & \end{array}$$

is commutative.

- (4) If $j: pt \hookrightarrow M$ then $\text{ind}_M^{\mathcal{E}}(j_!(1)) = 1$.
- (5) If M and M' are compact manifolds then $\text{ind}_{M \times M'}^{\mathcal{E}}(a \cdot b) = \text{ind}_M^{\mathcal{E}}(a) \cdot \text{ind}_{M'}^{\mathcal{E}}(b)$ whenever $a \in \mathcal{E}_{cs}^*(TM)$ and $b \in \mathcal{E}_{cs}^*(TM')$.
- (6) If TM has a complex structure then the composite

$$\mathcal{E}^*(M) \xrightarrow{th_{TM}} \mathcal{E}_{cs}^*(TM) \xrightarrow{\text{ind}_M^{\mathcal{E}}} \mathcal{E}^{*-2\dim M}(pt)$$

is the pushforward map $\pi_!$, where $\pi: M \rightarrow pt$.

14. K -THEORY WITH COMPACT SUPPORT

Let $E_{\bullet}$ be a bounded chain complex of vector bundles on a space X . Say that $E_{\bullet}$ has **compact support** if there is a compact set $L \subseteq X$ such that $(E_x)_{\bullet}$ is exact for every $x \in X - L$ (that is, $E_{\bullet}$ is supported on L).

Definition 14.1. Define $\mathcal{K}_{cs}(X)$ by taking the free abelian group generated by bounded, compactly-supported chain complexes of complex vector bundles on X and quotienting by the following relations:

- (1) $[E'] + [E''] = [E' \oplus E'']$,
- (2) $[E] = 0$ if E_x is exact for all $x \in X$,

(3) $[J|_{X \times 0}] = [J|_{X \times 1}]$ if J is a bounded, compactly-supported chain complex of vector bundles on $X \times I$.

For (3) we need to know that $J|_{X \times 0}$ is compactly-supported, but if $K \subseteq X \times I$ is compact then so is $K \cap (X \times \{0\})$ since $X \times \{0\}$ is closed in $X \times I$.

This definition exactly parallels the definition of the relative $\mathcal{K}$ -groups from ???. We clearly have natural maps $\mathcal{K}(X, X - L) \rightarrow \mathcal{K}_{cs}(X)$ for $L \subseteq X$ compact, and the following is almost immediate:

Proposition 14.2. *The natural map $\operatorname{colim}_{L^{cpt} \subseteq X} \mathcal{K}(X, X - L) \rightarrow \mathcal{K}_{cs}(X)$ is an isomorphism.*

Proof. The natural map clearly exists and is surjective. Define a map in the other direction as follows: for any compactly-supported chain complex E choose a compact L on which it is supported and send $[E] \in \mathcal{K}_{cs}(X)$ to the image in the colimit of the element $[E]_L \in \mathcal{K}(X, X - L)$. This doesn't depend on the choice of L for the usual reason: if L' is another choice, one can compare both classes to the one obtained from using $L \cup L'$.

Relations (1) and (2) are satisfied in the colimit because they are satisfied in each individual group. The only subtlety is in checking relation (3). Suppose J is a chain complex on $X \times I$ that is supported on some compact subset $L \subseteq X \times I$. Let πL be the image of L under the projection $X \times I \rightarrow X$. The image of a compact set is compact, so πL is compact. Since $L \subseteq (\pi L) \times I$, the complex J is supported on $(\pi L) \times I$. Therefore $[J|_{X \times 0}] = [J|_{X \times 1}]$ in $\mathcal{K}(X, X - \pi L)$, and so relation (3) is satisfied.

We have now shown that we get a map of groups $\mathcal{K}_{cs}(X) \rightarrow \operatorname{colim}_{L^{cpt} \subseteq X} \mathcal{K}(X, X - L)$, and it is immediate that this is a two-sided inverse for our original map. $\square$

Recall from Theorem ?? that there is a natural transformation $\chi: \mathcal{K}(X, A) \rightarrow K^0(X, A)$. Proposition 14.2 implies that this induces a corresponding map on the compactly-supported groups:

$$\begin{array}{ccc} \operatorname{colim}_{L^{cpt} \subseteq X} \mathcal{K}(X, X - L) & \xrightarrow{\cong} & \mathcal{K}_{cs}(X) \\ \downarrow \chi & & \downarrow \chi \\ \operatorname{colim}_{L^{cpt} \subseteq X} K^0(X, X - L) & \xrightarrow{\cong} & K_{cs}^0(X). \end{array}$$

The induced map χ is not always an isomorphism, but it is in many cases of interest:

Proposition 14.3. *Assume that X is homotopically compact and that there is a cofinal system of compact sets $L \subseteq X$ such that $X - L$ is also homotopically compact. Then $\chi: \mathcal{K}_{cs}(X) \rightarrow K_{cs}^0(X)$ is an isomorphism.*

Proof. Immediately from Theorem ??. $\square$

The hypotheses of the above proposition are not so easy to check, but mostly we will just need the existence of χ and not that it is an isomorphism. The space $\mathbb{C}^n$ is one example where the hypotheses are clearly satisfied, since we may use the closed disks centered at the origin as our cofinal system of compact sets.

Part 3. Bott periodicity

Bott periodicity is a deep result, and we will spend several sections exploring it from different perspectives. What do I mean by “deep”? There are many instances in mathematics where, once you learn to look at a result in just the right way, you say to yourself “I could have guessed that.” This doesn’t mean it is easy to prove, but at least you have understood the result at some intuitive level. Then there are other results where, no matter how you look at it, you can’t quite get to that “I could have guessed this” spot. For me, Bott periodicity is in this latter category.

A simpler example I like to mention of a “deep” result is the Pythagorean Theorem. Although I know several proofs of this fact—some of them quite simple!—down in my bones I suppose I have never felt like I truly know why the universe has to be that way. For me (sadly) the Pythagorean Theorem remains an unenlightening computation.

Of course, once one has the Pythagorean Theorem it becomes an absolutely crucial tool for exploring geometry, used at almost every turn in the road. It becomes part of one’s thinking, and is a fundamental technique by which one understands more complicated results in geometry. There is a sense in which one gets to “understand” the Pythagorean Theorem just by learning to use it so well. Bott periodicity is also kind of like this. There are by now many different proofs, and once one has the result it opens entire worlds—one of these is the world of K -theory, that we have spent this whole book exploring. So one approach is just to accept Bott periodicity the same way as one accepts the Pythagorean Theorem, as a fundamental computation that will get used over and over again, and not worry too much about whether one could have “guessed” it. But I can’t help feeling that one day we might have a deeper understanding.

While I do not know a way to “guess” the Bott periodicity theorems, I want to offer a very rough picture that the reader can keep in their head as a first approximation to the story. First, it is a basic fact in homotopy theory that $\Omega BG \simeq G$ for any topological group G ; so we get $\Omega(\mathbb{Z} \times BU) \simeq U$ for free. Bott periodicity comes from a calculation of ΩU , or of $\Omega U(n)$ for large n . For (at the moment) unexplained reasons we will replace this with the homotopy equivalent space $\Omega_{I,-I} U(n)$ of paths from I to $-I$. If $H \subseteq \mathbb{C}^n$ is a complex subspace, then we can make an easy example of such a path by multiplying vectors in H by $e^{\pi i t}$ and vectors in $H^\perp$ by $e^{-\pi i t}$, as t goes from 0 to 1. So we are doing a (gradual) complex rotation on H and the opposite rotations on $H^\perp$. There are, of course, many types of paths on $U(n)$ other than these simple ones—but the fundamental calculation of Bott is that as n gets large these simple paths represent more and more of the homotopy type of $\Omega_{I,-I} U(n)$. So in a large range that grows with n we have that $\Omega_{I,-I} U(n)$ is weakly equivalent to the space of all H , which is the Grassmannian $\mathrm{Gr}_\bullet(\mathbb{C}^n) = \coprod_k \mathrm{Gr}_k(\mathbb{C}^n)$. This space has $n+1$ components, but in the stabilization process (which has both k and n going to infinity) it becomes a $\mathbb{Z}$ ’s worth of components and we get $\mathbb{Z} \times BU$. Thus we arrive at the Bott equivalence $\Omega U \simeq \mathbb{Z} \times BU$.

There are, of course, lots of questions to answer here. Why do these simple H -paths constitute a large range of the homotopy type of $\Omega_{I,-I} U(n)$? What exactly is going on in the stabilization process? (It is trickier than you might expect!) But the above paragraph does an adequate job of giving a very quick picture of what Bott periodicity is about.

What about periodicity for the orthogonal group O ? Here one looks at the space $\Omega_{I,-I}O(2n)$ and observes that for every orthogonal complex structure J on $\mathbb{R}^{2n}$ (an orthogonal transformation such that $J^2 = -I$) the formula $t \mapsto e^{\pi t J}$ gives a very simple path from I to $-I$. Note that the condition $J^2 = -I$ gives that $e^{sJ} = \cos(s)I + \sin(s)J$. The Bott arguments show that the homotopy type of $\Omega_{I,-I}O(2n)$ is approximated in a large range by the space of these “simple” paths—i.e., by the space of all such J . This is the space of complex structures on $\mathbb{R}^{2n}$. As one iterates this procedure one finds that paths of complex structures (from a chosen J to $-J$) are approximated by spaces of quaternionic structures, and very quickly one finds the real Clifford algebras appearing. The eightfold periodicity of the Clifford algebras reappears in the analysis of these iterated loop spaces, so that $\Omega^8 O$ turns out to be O again. And that—in a nutshell—is real Bott periodicity. It is a complicated nutshell, but we will spend the next several sections filling in the details.

To help the reader navigate the next several sections we provide the following historical survey:

- (1) After Bott’s initial work on the periodicity theorem, Clifford algebras were brought into the picture in joint work of Bott and Shapiro. After Shapiro’s early death, Atiyah joined the project and this led to [ABS]. This paper develops Clifford algebras and connects them to Bott periodicity and K -theory, but does not connect them to the Morse-theoretic arguments that Bott originally used (which are taken as a black box). [Milnor6, Chapter 24] explains that connection, although the *explicit* mention of Clifford algebras is confined to one sentence.
- Our Sections 15 and 16 recount Bott’s approach to the iterated loop spaces of U and O , essentially taking the approach of [Milnor6] for O but incorporating Clifford algebras more explicitly as a key part of the story.
- (2) To construct K -theory spectra one really wants **de-loopings** of U and O , not the loop spaces. Periodicity implies that the iterated loop spaces are, in fact, also de-loopings, but approaching things this way feels a bit unnatural. Karoubi showed how one can use the full bigraded family of Clifford algebras—again building on top of Bott’s Morse-theoretic arguments—to produce the desired deloopings and to construct natural models of the K -theory spectra. Section 17 tells this part of the story.
- (3) As one works with Bott periodicity, it quickly becomes clear that it is desirable to have explicit generators for $\pi_* U$ and $\pi_* O$, as well as explicit formulas for the periodicity isomorphisms. In the original approach of Bott these things are buried fairly deep, and extracting them is where Clifford algebras and their modules yield their big payoffs. We cover this in Sections 18 and 19, combining several perspectives. Section 18 is devoted to producing an explicit matrix formula for the Bott periodicity isomorphism for U . Doing the same for O requires us to step back and develop more of the Clifford theory, which we undertake in Section 19. This section covers most of the key results from [ABS] that produce the algebraic groups $\mathcal{A}_*$ (defined entirely in terms of Clifford algebras) and the isomorphisms $\mathcal{A}_* \rightarrow \pi_* O$. These isomorphisms still require the output of the original Bott Morse-theoretic arguments.
- (4) The most natural manifestation of algebraic Bott periodicity is in the theory of $\mathbb{Z}/2$ -graded real division algebras, due to [Wall]. From this perspective the

number 8 has a nice explanation: it comes from a combination of the $\mathbb{Z}/2$ -grading (TWO pieces), the number of (ungraded) real division algebras with center $\mathbb{R}$ (of which there are TWO, namely $\mathbb{R}$ and $\mathbb{H}$), and the fact that up to squares there are exactly TWO units in $\mathbb{R}^*$, namely 1 and -1 . These three TWOs combine into the $2^3 = 8$ -fold periodicity, which is captured in the super Brauer group of $\mathbb{R}$ being $\mathbb{Z}/8$.

Clifford algebras interact with this story in that they are $\mathbb{Z}/2$ -graded central simple algebras over $\mathbb{R}$ (where central and simple are taken in the $\mathbb{Z}/2$ -graded sense), and therefore Morita equivalent to central $\mathbb{Z}/2$ -graded division algebras. The 8-fold periodicity in the Morita types of Clifford algebras is exactly reflecting the super Brauer group of $\mathbb{R}$.

Finally, as one shifts perspective from π_*U and π_*O to $K^*(pt)$ and $KO^*(pt)$, one finds that the $\mathbb{Z}/2$ -graded Clifford algebras are more useful than their ungraded analogs. This finally connects back to some of the work in [ABS].

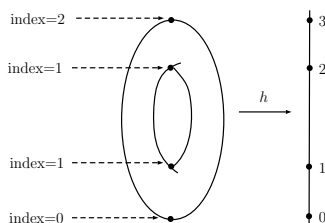
Section 20 covers all of these aspects, culminating in a concrete set of generators and relations for the coefficient groups K^* and KO^* .

- (5) In the above telling of the story, we never leave the shadow of Bott's original Morse-theoretic approach to periodicity. It is worth pointing out that there *are* other approaches. Some of these are discussed in Section ????

15. PERIODICITY VIA BOTT-MORSE THEORY

Bott's original approach to the periodicity theorems was through Morse theory and the analysis of geodesics on Lie groups and their homogeneous spaces. Later approaches have tried to eliminate as much of the Morse and Lie theory as possible, but I'm not sure how helpful this is in the end. There is something important hidden in the Bott-Morse arguments that is worth understanding, so this is where we will begin our journey.

15.1. Brief tour of Morse theory. Let X be a smooth manifold, and let $h: X \rightarrow \mathbb{R}$ be a smooth function with nondegenerate critical points—this means that the Hessian matrix of second partial derivatives has full rank at these points. Such critical points are isolated (by the Taylor approximation), and to each one we can associate an **index**: geometrically, this is the number of independent directions on the manifold where the function h is *decreasing*. Algebraically, the index is the number of negative eigenvalues of the Hessian matrix. The standard example is the height function on the torus, as depicted in the following picture:



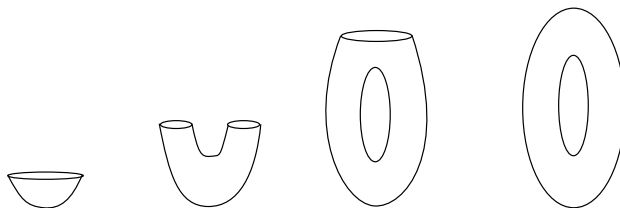
In Morse theory one looks at the spaces $X_{\leq c} = h^{-1}((-\infty, c])$ and studies the changes as c increases. The homotopy type of $X_{\leq c}$ changes precisely when c is a critical value, and the change is equivalent to a cell attachment where the dimension

of the cell is the index of the corresponding critical point. So in the example from the picture we get

$$\begin{aligned} c < 0: & \quad X_c \simeq \emptyset \\ 0 \leq c < 1: & \quad X_c \simeq pt \\ 1 \leq c < 2: & \quad X_c \simeq e^0 \cup e^1 = S^1 \\ 2 \leq c < 3: & \quad X_c \simeq e^0 \cup e^1 \cup e^1 = S^1 \vee S^1 \\ 3 \leq c < \infty: & \quad X_c \simeq e^0 \cup e^1 \cup e^1 \cup e^2 = X. \end{aligned}$$

Figure 15.1.1 depicts samples from the four nonempty stages.

FIGURE 15.1.1. Stages in the Morse filtration of the torus



Morse showed that when X is a Riemannian manifold one can apply similar techniques to study the loop space ΩX , taking h to be a certain “energy functional”. The space ΩX is no longer a finite-dimensional manifold but with some care the techniques still work (this is similar to what happens in the classical calculus of variations). It will be useful to modify the loop space slightly, though. Choose $P, Q \in X$ and let $\Omega_{P,Q}X$ denote the space of paths which begin at P and end at Q . When $P = Q$ we will simplify the notation to $\Omega_P X$. If γ is any fixed path from Q to P , post-composition with γ and γ^{-1} gives the maps in a homotopy equivalence $\Omega_{P,Q}X \simeq \Omega_P X$.

We next need to restrict to the subspace $\Omega_{P,Q}^{sm}X \hookrightarrow \Omega_{P,Q}X$ consisting of piecewise C^∞ paths. The inclusion can be shown to be a homotopy equivalence [Milnor6, Theorem 17.1]. We define the “energy functional” $E: \Omega_{P,Q}^{sm}(X) \rightarrow \mathbb{R}$ by

$$E(\gamma) = \int_0^1 \left| \frac{d\gamma}{dt} \right|^2 dt$$

where the norm of $\frac{d\gamma}{dt}$ is of course defined via the Riemannian metric on X . The details of E are not so important here, but the crucial fact is that the critical points of E are the geodesics on X . The index of a geodesic is the dimension of a space of certain kinds of “small variations”. We will demonstrate this concept with the example of $X = S^n$ for $n \geq 2$.

The main observation about the sphere is that there is a unique minimal geodesic between points P and Q except when they are antipodal. In the antipodal case, there is a whole family of minimal geodesics—and that family is precisely an S^{n-1} , parameterized by the midpoint of the geodesic (where it intersects the equator).

First consider the case where P and Q are generically positioned. Without loss of generality we assume P is the north pole, and Q is any point other than $-P$. There is exactly one minimal geodesic from P to Q , and there are no “variations” on this

geodesic—so it has index 0. Next, there is the geodesic from P to Q going around the sphere in the opposite direction, passing through $-P$. The part of the geodesic going from P to $-P$ allows for “variation”, though, as we discussed above—the dimension of such variations is $n - 1$, so that is the index of this geodesic.

Continuing on, the next longest geodesic from P to Q is the one that takes the short path from P to Q but then continues all the way around the sphere—passing through $-P$ and P —before once again returning to Q . For this geodesic we get variations for the portion from P to $-P$, but then also for the portion from $-P$ back to P —so that is $(n - 1) + (n - 1)$ dimensions of variation. Proceeding in this way, one finds that the geodesics from P to Q have indices 0, $n - 1$, $2(n - 1)$, $3(n - 1)$, and so forth. So Morse theory ends up telling us that

$$\Omega_{P,Q}S^n \simeq e^0 \cup e^{n-1} \cup e^{2(n-1)} \cup \dots$$

Exercise 15.2. If $n \geq 3$ then the above decomposition yields a simple computation of $H^*(\Omega S^n)$ (as groups). Compare this to other approaches you might know, such as via the Serre spectral sequence.

As we described above, Morse’s original techniques assume that the critical points of h are isolated. In more general settings one might instead have that the inverse image of a critical value is a subspace of positive dimension. Bott extended Morse theory so that it could be applied in these settings; while it still gives a decomposition of the space X , the components are not just cells—they will be disk bundles of vector bundles over the critical subspaces instead. As an example, consider now the space $\Omega_{P,-P}S^n$. We don’t have a single minimal geodesic from P to $-P$ like we did before—we have a whole family of minimal geodesics, forming the space S^{n-1} . This is our index 0 piece of the decomposition. The other geodesics all have index at least $2(n - 1)$, so even without analyzing them in detail we find that

$$(*) \quad \Omega_{P,-P}S^n \simeq S^{n-1} \cup (\text{cells of dimension at least } 2(n - 1)).$$

Even this limited knowledge is substantial; for example, it is enough to prove the Freudenthal Suspension Theorem! Fix a minimal-length geodesic γ from P to $-P$, and consider the map $f: S^{n-1} \rightarrow \Omega_{P,-P}S^n$ that sends a point x to the path “follow the geodesic from P to $-P$ that passes through x , then follow γ^{-1} back to P ”. This is essentially the map $S^{n-1} \rightarrow \Omega S^n$ that is adjoint to the identity $\Sigma S^{n-1} \rightarrow S^n$ (one should contract the image of γ to identify S^n with the reduced suspension of S^{n-1}). The Bott-Morse decomposition of $(*)$ tells us that f induces isomorphisms on π_i for $i \leq 2n - 4$ and an epimorphism for $i = 2n - 3$, which implies that the suspension map $\pi_i(S^{n-1}) \rightarrow \pi_{i+1}(S^n)$ has the same behavior. This is the Freudenthal Suspension Theorem.

Remark 15.3. Consider now the case of S^1 . There are two minimal geodesics from a point P to $-P$, corresponding to the two ways of going around the circle. So the space of minimal geodesics is S^0 , and we have the decomposition

$$\Omega_{P,-P}S^1 \simeq S^0 \cup (\text{other cells}).$$

Note that $\Omega_{P,-P}S^1 \simeq \mathbb{Z}$, so this decomposition gives an accurate approximation but only on two of the components. The issue here is that we have non-minimal geodesics that also have index zero, and so give rise to 0-cells. In fact, *all* of the geodesics have index 0, and the $\mathbb{Z}$ is exactly parameterizing the different geodesics.

This is not just an idle example. In our work below we will often encounter situations where ΩX is not connected and the minimal geodesics only tell us about some of the components. Of course in a loop space all of the components are homotopy equivalent, so as soon as we understand one we understand them all. But the existence of non-minimal geodesics of small index (in particular, index 0) will sometimes get in the way of us being able to specify how close an approximation the space of minimal geodesics is to ΩX .

15.4. Applying Bott-Morse theory to the unitary group. We will consider the group $U(n)$ and attempt to analyze geodesics from I to $-I$. Let us recall how the basic Lie theory works. For A in the tangent space at the identity, the path $t \mapsto e^{tA}$ is a geodesic in $U(n)$ whose tangent vector at $t = 0$ is A . The condition that e^{tA} is unitary is $I = e^{tA}(e^{tA})^\dagger = e^{tA}e^{tA^\dagger}$. Applying $\frac{d}{dt}\big|_{t=0}$ gives the equation $0 = A + A^\dagger$. Conversely, this equation implies that A and $A^\dagger$ commute and therefore $e^{tA}(e^{tA})^\dagger = e^{tA}e^{tA^\dagger} = e^{t(A+A^\dagger)} = e^0 = I$; that is, $e^{tA} \in U(n)$. So the tangent space $T_I U(n)$ consists precisely of the skew-Hermitian matrices.

The maximal torus in $U(n)$ has rank n , and the standard one consists of the diagonal matrices. The tangent space of this torus is the subspace D_{skH} of skew-Hermitian diagonal matrices. We identify this with $\mathbb{R}^n$ via $(r_1, \dots, r_n) \mapsto \text{diag}(ir_1, \dots, ir_n)$. Every element of $T_I U(n)$ is conjugate (via the adjoint action of $U(n)$, which is literally matrix conjugation) to an element of D_{skH} .

Finding geodesics from I to $-I$ is the same as finding all skew-Hermitian matrices A for which $t \mapsto e^{tA}$ passes through $-I$ when $t = 1$. By conjugating, we can assume that A is diagonal. Therefore $A = \text{diag}(k_1\pi i, \dots, k_n\pi i)$ for some odd integers k_j . One can check that the minimal geodesics are the ones for which all the k_j 's are 1 or -1 . Our space of minimal geodesics is the set of $U(n)$ -conjugates of these. Since $U(n)$ contains the permutation matrices, we reduce to thinking about the conjugates of the matrices $D_r = \text{diag}(\pi i, \dots, \pi i, -\pi i, \dots, -\pi i)$ where πi occurs r times and $-\pi i$ occurs $n - r$ times. The stabilizer of D_r is readily checked to be $U(r) \times U(n - r)$, and so our space of minimal geodesics is

$$\coprod_{r=0}^n U(n)/(U(r) \times U(n-r)) \cong \coprod_{r=0}^n \text{Gr}_r(\mathbb{C}^n).$$

We can also think about this isomorphism in terms of eigenspaces. We are looking at skew-Hermitian matrices whose only eigenvalues are πi and $-\pi i$. Because they are skew-Hermitian, the eigenspaces must be orthogonal—hence knowing one eigenspace determines the other. So the conjugacy classes are determined by the πi -eigenspace.

In this way we have produced a map

$$\coprod_{r=0}^n \text{Gr}_r(\mathbb{C}^n) \xrightarrow{\beta} \Omega_{I, -I} U(n)$$

that sends a subspace $H \subseteq \mathbb{C}^n$ to the geodesic $t \mapsto e^{tA}$ where A is the unique linear transformation $\mathbb{C}^n \rightarrow \mathbb{C}^n$ which is multiplication by πi on H and multiplication by $-\pi i$ on $H^\perp$. This map β is called the **Bott map**. The appearance of the disjoint union in the domain is a little annoying at first, but in retrospect something like this is expected. Recall that $\pi_1 U(n) \cong \mathbb{Z}$, with the isomorphism induced by the determinant map $\det: U(n) \rightarrow S^1$. So $\pi_0(\Omega U(n)) \cong \mathbb{Z}$, meaning that $\Omega U(n)$ has

countably many path components. Our map β is touching $n+1$ of these components. How do we understand which components? We use $\det: U(n) \rightarrow S^1$ to induce a map $\Omega_{I,-I}U(n) \rightarrow \Omega_{1,(-1)^n}S^1$, and this is a bijection on path components. The path components of the target can be indexed by integers k , with one possible correspondence having k associated to the path $t \mapsto e^{t\pi i(2k-n)}$. If $H \subseteq \mathbb{C}^n$ has dimension r then the map “multiplication by $e^{\pi it}$ on H and multiplication by $e^{-\pi it}$ on $H^\perp$ ” has determinant $e^{\pi i(2r-n)}$ and therefore $\beta(H)$ is in the path component of $\Omega_{I,-I}U(n)$ labelled by r .

The appearance of the disjoint union causes some issues in our Bott-Morse arguments, as in Remark 15.3. We will come back to this, but for the moment let us take an approach that gets us around that issue. Assume $n = 2k$ and consider only the portion $\text{Gr}_k(\mathbb{C}^{2k}) \rightarrow \Omega_{I,-I}U(2k)$ of the Bott map. The image consists of conjugates of the geodesic $t \mapsto \text{diag}(e^{\pi it}, \dots, e^{\pi it}, e^{-\pi it}, \dots, e^{-\pi it})$ where $e^{\pi it}$ and $e^{-\pi it}$ both appear k times. Such matrices (and their conjugates) all have determinant equal to 1, and so these are geodesics lying in $SU(2k)$. The loop space of $SU(2k)$ is path-connected, so we can save some trouble by just working here. We have found that the space of minimal geodesics from I to $-I$ is the space $\text{Gr}_k(\mathbb{C}^{2k})$.

With a little effort one can analyze the non-minimal geodesics in $SU(2k)$ and calculate their index: for the geodesic corresponding to the matrix $A = \text{diag}(\pi i r_1, \dots, \pi i r_{2k})$ (with all r_i odd and $\sum r_i = 0$) the index turns out to be

$$\text{index} = \sum_{r_i > r_j} (r_i - r_j - 2).$$

We refer the reader to [Milnor6, End of Section 23] or [Bott1, Section 7.1] for this calculation (the notation in the former is closer to our treatment). When half the r 's are 1 and the other half -1 , this gives index 0—as expected for minimal geodesics. In all other cases one can readily prove that the index is at least $2k+2$ (a sequence giving this minimum is 3 followed by $k-2$ copies of 1, followed by $k+1$ copies of -1). So Bott-Morse theory gives us that

$$\Omega_{I,-I}SU(2k) \simeq \text{Gr}_k(\mathbb{C}^{2k}) \cup (\text{cells of dimension } (2k+2) \text{ and higher}).$$

From here we get that $\pi_i \Omega SU(2k) \cong \pi_i \text{Gr}_k(\mathbb{C}^{2k})$ for $i \leq 2k$. But then for each $i \geq 2$ we can choose a large enough k and argue that

$$\begin{aligned} (15.5) \quad \pi_i U &= \pi_i U(2k) = \pi_i SU(2k) = \pi_{i-1} \Omega SU(2k) \cong \pi_{i-1} \text{Gr}_k(\mathbb{C}^{2k}) \\ &= \pi_{i-1} \text{Gr}_k(\mathbb{C}^\infty) \\ &= \pi_{i-1} BU(k) \\ &= \pi_{i-2}(\Omega BU(k)) \\ &= \pi_{i-2}(U(k)) \\ &= \pi_{i-2}U. \end{aligned}$$

In the fifth isomorphism we have used that $\text{Gr}_k(\mathbb{C}^\infty)$ is obtained from $\text{Gr}_k(\mathbb{C}^{2k})$ by adding $(2k+2)$ -cells and higher, which follows from the standard cell decompositions for the Grassmannians. So we have now proven the most primitive version of Bott periodicity, namely the isomorphisms $\pi_t U \cong \pi_{t+2} U$ for $t \geq 0$.

We can perform the stabilization at the level of spaces rather than after taking homotopy groups, but this involves a little care. The issue is that we are dealing

with paths from I to $-I$, and these do not map to similar paths under the inclusion $SU(2n) \hookrightarrow SU(2n+2)$. So we have to change back to paths from I to I .

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{C}^n$. Let $M^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$, where the basis for the first copy of $\mathbb{C}^n$ is denoted $e_1^+, \dots, e_n^+$ and the basis for the second copy is $e_1^-, \dots, e_n^-$. From now on when we write $U(2n)$ we mean the group of unitary automorphisms of M^{2n} , and $SU(2n) \subseteq U(2n)$ is the subgroup of elements having determinant one. Let $\gamma: I \rightarrow SU(2n)$ be the path from I to $-I$ given by

$$\gamma(t) = \text{diag}(e^{-\pi it}, \dots, e^{-\pi it}, e^{\pi it}, \dots, e^{\pi it})$$

(with n copies of $e^{-\pi it}$ and n copies of $e^{\pi it}$). For any other path α from I to $-I$, write $\alpha \odot \gamma$ for the path $t \mapsto \alpha(t)\gamma(t)$, which is a loop at I . This gives a homeomorphism $\Gamma_n: \Omega_{I, -I}SU(2n) \rightarrow \Omega_I SU(2n)$, where $\Gamma_n(\alpha) = \alpha \odot \gamma$.

Now consider the following diagram:

$$\begin{array}{ccccc} \text{Gr}_n(M^{2n}) & \xrightarrow{\beta} & \Omega_{I, -I}(SU(2n)) & \xrightarrow{\Gamma_n} & \Omega_I SU(2n) \\ j_n \downarrow & & g \downarrow & & \downarrow \\ \text{Gr}_{n+1}(M^{2(n+1)}) & \xrightarrow{\beta} & \Omega_{I, -I}(SU(2(n+1))) & \xrightarrow{\Gamma_{n+1}} & \Omega_I SU(2(n+1)). \end{array}$$

Here j_n sends a subspace $W \subseteq M^{2n}$ to $W \oplus \langle e_{n+1}^+ \rangle \subseteq M^{2(n+1)}$. For the map g , if $u: [0, 1] \rightarrow SU(2n)$ is a path from I to $-I$ we define $(gu)(t) = u(t) \oplus \text{diag}(e^{\pi it}, e^{-\pi it})$. The right vertical map is induced by the standard inclusion of $SU(2n)$ into $SU(2(n+1))$, namely $A \mapsto A \oplus \text{id}$.

We need to check that the diagram commutes. Commutativity of the right-most square is easy. Commutativity of the left square follows after one remembers that the map β sends a subspace H to the path which at time t scales H by $e^{\pi it}$ and scales $H^\perp$ by $e^{-\pi it}$. Note that j_n can be thought of as “adding e_{n+1}^+ to H and adding e_{n+1}^- to $H^\perp$ ”.

Taking vertical colimits (as $n \rightarrow \infty$) in the above diagram, the first and third columns yield a map $\text{colim}_n \text{Gr}_n(M^{2n}) \rightarrow \Omega_I SU$. This also goes under the name “the Bott map”, and will be denoted β_∞ (sometimes we will drop the subscript, though). Since the horizontal composites $\Gamma_n \circ \beta$ become arbitrarily highly connected as n goes to ∞ , the map β_∞ is a weak equivalence. It only remains to understand the homotopy type of the colimit in the domain. For this, consider the doubly infinite diagram

$$\begin{array}{ccccccc} \text{Gr}_1(\mathbb{C}_+^1 \oplus \mathbb{C}_-^1) & \longrightarrow & \text{Gr}_1(\mathbb{C}_+^1 \oplus \mathbb{C}_-^2) & \longrightarrow & \text{Gr}_1(\mathbb{C}_+^1 \oplus \mathbb{C}_-^3) & \longrightarrow & \cdots \implies \text{Gr}_1(\mathbb{C}^\infty) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Gr}_2(\mathbb{C}_+^2 \oplus \mathbb{C}_-^1) & \longrightarrow & \text{Gr}_2(\mathbb{C}_+^2 \oplus \mathbb{C}_-^2) & \longrightarrow & \text{Gr}_2(\mathbb{C}_+^2 \oplus \mathbb{C}_-^3) & \longrightarrow & \cdots \implies \text{Gr}_2(\mathbb{C}^\infty) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

The horizontal maps are induced by inclusions $\mathbb{C}_+^p \oplus \mathbb{C}_-^q \hookrightarrow \mathbb{C}_+^p \oplus \mathbb{C}_-^{q+1}$, whereas the vertical maps send a subspace $W \subseteq \mathbb{C}_+^p \oplus \mathbb{C}_-^q$ to the subspace $W \oplus \langle e_{p+1}^+ \rangle \subseteq \mathbb{C}_+^{p+1} \oplus \mathbb{C}_-^q$. The space $\text{colim}_n \text{Gr}_n(M^{2n})$ is the colimit along the diagonal of the above diagram, and so is canonically isomorphic to the space obtained by first

taking horizontal colimits and then vertical colimits. That is, $\operatorname{colim}_n \operatorname{Gr}_n(M^{2n}) \simeq \operatorname{Gr}_\infty(\mathbb{C}^\infty) \simeq BU$.

So our Bott map is an equivalence $\beta_\infty: BU \xrightarrow{\simeq} \Omega_I SU$, and then for $i \geq 0$ we have

$$\pi_i U = \pi_i(\Omega BU) = \pi_{i+1}(BU) \xrightarrow[\beta_\infty]{\simeq} \pi_{i+1}(\Omega SU) = \pi_{i+2} SU = \pi_{i+2} U.$$

As the last topic of this section we describe an approach to understanding ΩU that handles all of the components at once, rather than just focusing on the $\Omega SU = (\Omega U)_0$ component. This amounts to just a small modification of what we have already seen.

Write $\mathbb{C}^{p,q}$ as an abbreviation for $\mathbb{C}^p \oplus \mathbb{C}^q$ where the first summand has basis $e_1^+, \dots, e_p^+$ and the second summand has basis $e_1^-, \dots, e_q^-$. Let $\mathbb{C}_+^p$ denote the first summand and $\mathbb{C}_-^q$ the second. For $p \leq p'$ and $q \leq q'$ we have the evident inclusions $\mathbb{C}^{p,q} \subseteq \mathbb{C}^{p',q'}$. Write $\operatorname{Gr}_\bullet \mathbb{C}^{p,q}$ for $\coprod_k \operatorname{Gr}_k(\mathbb{C}^{p,q})$. Let us also write $U(p,q)$ for the group of unitary automorphisms of $\mathbb{C}^{p,q}$, though note that there is an evident identification $U(p,q) \cong U(p+q)$.

Let $\gamma: [0,1] \rightarrow U(p,q)$ be the path $\gamma(t) = e^{-\pi i t}|_{\mathbb{C}_+^p} \oplus e^{\pi i t}|_{\mathbb{C}_-^q}$. That is, $\gamma(t)$ multiplies positive basis elements by $e^{-\pi i t}$ and negative basis elements by $e^{\pi i t}$. For a subspace $H \subseteq \mathbb{C}^{p,q}$, let $\Phi_H: [0,1] \rightarrow U(p,q)$ be the path where $\Phi_H(t)$ multiplies elements of H by $e^{\pi i t}$ and elements of $H^\perp$ by $e^{-\pi i t}$. Define $\beta_H(t) = \Phi_H(t) \cdot \gamma(t)$. Then β_H is a loop based at I , so that we have a map $\beta: \operatorname{Gr}_\bullet \mathbb{C}^{p,q} \rightarrow \Omega_I U(p+q)$. The following property will be important to us:

(**) For all t , the linear map $\beta_H(t)$ is the identity when restricted to $H \cap \mathbb{C}_+^p$ or to $H^\perp \cap \mathbb{C}_-^q$.

For $H \in \operatorname{Gr}_k(\mathbb{C}^{p,q})$ the unitary transformation $\Phi_H(t)$ has determinant equal to $e^{k\pi i t} \cdot e^{-(p+q-k)\pi i t} = e^{(2k-p-q)\pi i t}$. The transformation $\gamma(t)$ has determinant $e^{-p\pi i t} \cdot e^{q\pi i t} = e^{(q-p)\pi i t}$. So $\det \beta_H(t) = e^{2(k-p)\pi i t}$. Therefore β maps $\operatorname{Gr}_k(\mathbb{C}^{p,q})$ into the component $[\Omega_I U(p,q)]_{k-p}$. For this reason, let us label the components of $\operatorname{Gr}_\bullet(\mathbb{C}^{p,q})$ so that Gr_k is the “ $k-p$ component”. So the components of $\operatorname{Gr}_\bullet(\mathbb{C}^{p,q})$ are labelled as $-p, -(p-1), \dots, q-1, q$.

We now consider the following lattice diagram, consisting of two kinds of stabilizations:

$$(15.6) \quad \begin{array}{ccccc} \operatorname{Gr}_\bullet \mathbb{C}^{1,1} & \longrightarrow & \operatorname{Gr}_\bullet \mathbb{C}^{1,2} & \longrightarrow & \operatorname{Gr}_\bullet \mathbb{C}^{1,3} \longrightarrow \dots \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Gr}_\bullet \mathbb{C}^{2,1} & \longrightarrow & \operatorname{Gr}_\bullet \mathbb{C}^{2,2} & \longrightarrow & \operatorname{Gr}_\bullet \mathbb{C}^{2,3} \longrightarrow \dots \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Gr}_\bullet \mathbb{C}^{3,1} & \longrightarrow & \operatorname{Gr}_\bullet \mathbb{C}^{3,2} & \longrightarrow & \operatorname{Gr}_\bullet \mathbb{C}^{3,3} \longrightarrow \dots \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \end{array}$$

The horizontal maps consist of the evident inclusions $\operatorname{Gr}_k(\mathbb{C}^{p,q}) \hookrightarrow \operatorname{Gr}_k(\mathbb{C}^{p,q+1})$ (“add e_{q+1}^- to the complement of the subspace H ”), whereas the vertical maps consist

of the inclusions $\mathrm{Gr}_k(\mathbb{C}^{p,q}) \hookrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}^{p+1,q})$ (“add e_{p+1}^+ to the subspace H ”). Note that both horizontal and vertical maps respect the labels on the components—that is, in all cases the component with label x is sent to the component with the same label x . The colimit across the top row is a disjoint union $\coprod_{k \geq 0} \mathrm{Gr}_k(\mathbb{C}^\infty)$, with components labelled -1 and higher. Taking colimits across each row gives the diagram

$$\begin{array}{ccccccc}
 \cdots & \boxed{-2} & & \boxed{-1} & & \boxed{0} & & \boxed{1} & & \cdots \\
 & & & & & & & & & \\
 & & & \mathrm{Gr}_0(\mathbb{C}^\infty) & \amalg & \mathrm{Gr}_1(\mathbb{C}^\infty) & \amalg & \mathrm{Gr}_2(\mathbb{C}^\infty) & \amalg & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & \mathrm{Gr}_0(\mathbb{C}^\infty) & \amalg & \mathrm{Gr}_1(\mathbb{C}^\infty) & \amalg & \mathrm{Gr}_2(\mathbb{C}^\infty) & \amalg & \mathrm{Gr}_3(\mathbb{C}^\infty) & \amalg & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \ddots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
 \end{array}$$

where in the first row the components are labelled with -1 and higher, in the second row with -2 and higher, and so on. So the colimit of the entire diagram is $\mathbb{Z} \times BU$.

Now consider the corresponding lattice of unitary groups:

$$\begin{array}{ccccccc}
 U(1,1) & \longrightarrow & U(1,2) & \longrightarrow & U(1,3) & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 U(2,1) & \longrightarrow & U(2,2) & \longrightarrow & U(2,3) & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 U(3,1) & \longrightarrow & U(3,2) & \longrightarrow & U(3,3) & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \vdots & & \vdots & & \vdots & &
 \end{array}$$

Here both horizontal and vertical maps are induced by the inclusions $\mathbb{C}^{p,q} \hookrightarrow \mathbb{C}^{p',q'}$. The colimit of the diagram is $U(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)$ and therefore a model of the group U .

The Bott maps β give a map of diagrams from the array of Grassmannians to Ω_I applied to the array of unitary groups. One readily checks that property (**) above implies that all the squares commute. Taking colimits gives us the map $\beta_\infty: \mathbb{Z} \times BU \rightarrow \Omega_I U$. The Bott-Morse arguments imply that taking colimits in each row gives us a map that is a weak equivalence on the components that are hit, and then taking the colimit in the vertical direction allows us to hit all of the components. So our map $\mathbb{Z} \times BU \rightarrow \Omega_I U$ is a weak equivalence.

Exercise 15.7. Check that the maps $\mathrm{Gr}_\bullet \mathbb{C}^{p,q} \rightarrow \Omega_I U(p,q)$ really do assemble to give a map of diagrams between our two infinite lattices. This involves checking that two types of squares commute.

15.8. Geodesics on the orthogonal group. Analyzing geodesics on $O(n)$ is fairly similar to what we did for $U(n)$, but just different enough so that the space

of minimal geodesics is not a Grassmannian anymore. This immediately dashes the chances for the kind of 2-fold periodicity we saw before. In a moment we will see precisely what the space of minimal geodesics is, but before diving into the analysis let us talk about the big picture.

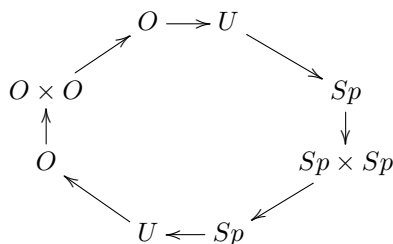
The space of minimal geodesics turns out to be another Lie-theoretic space: not a Lie group, but a Riemannian submanifold of a Lie group having the property that geodesics that start tangent to the submanifold never leave it (like the equator of a sphere). So one can apply the Bott technique *again* to analyze its loop space. In this manner one gets yet another space of minimal geodesics (technically the minimal geodesics on the space of minimal geodesics), which again ends up Lie-theoretic in nature. So one continues in this process, and after seven times something magical happens in that one finally encounters the real Grassmannian—so that after looping one more time one gets back to the orthogonal group.

Part of what is challenging in navigating this argument is keeping track of the eight spaces and their various relations to each other. So let us spend some time on this topic before diving into details.

The key players are the orthogonal, unitary, and symplectic groups: $O(n)$, $U(n)$, and $Sp(n)$. These are the space of inner-product preserving linear automorphisms of $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$. As matrix groups they consist of $n \times n$ matrices A of (real, complex, quaternionic) numbers such that $A^\dagger A = I$ where $(-)^{\dagger}$ is the conjugate-transpose. The inclusions $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$ therefore induce standard inclusions of groups $O(n) \subseteq U(n) \subseteq Sp(n)$.

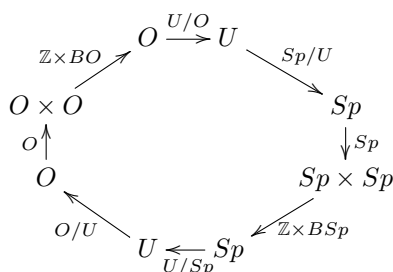
However, we can also think of $\mathbb{H}$ as $\mathbb{C}^2$ or $\mathbb{C}$ as $\mathbb{R}^2$. When we do this, every symplectic map $\mathbb{H}^n \rightarrow \mathbb{H}^n$ is a unitary map $\mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$, and every unitary map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ is an orthogonal map $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. So we get inclusions $Sp(n) \hookrightarrow U(2n)$ and $U(n) \hookrightarrow O(2n)$.

Now let us discard the indices by passing to the direct limit groups O , U , and Sp . The following “clock” shows the four inclusions we have just mentioned and includes four extra maps we will explain presently:



Here the inclusions $O \rightarrow O \times O$ and $Sp \rightarrow Sp \times Sp$ are the diagonals. The inclusion $O \times O \rightarrow O$ comes from writing $\mathbb{R}^\infty = \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ (think of separating the even and odd basis elements), and then sending a pair of orthogonal transformations (A, B) to the transformation that does A on the first copy of $\mathbb{R}^\infty$ and B on the second. The inclusion $Sp \times Sp \rightarrow Sp$ is defined similarly.

Okay, we are almost done. Now go back to the above clock and label each arrow $H \hookrightarrow G$ with the corresponding quotient G/H , with the provisos that $(G \times G)/G = G$ and $G/(G \times G) = \mathbb{Z} \times BG$ (the former is a true quotient, but for the moment the latter is just a convention—see Exercise 15.9 and Remark 15.12). The resulting figure is called the “real Bott periodicity clock”:



What Bott's arguments will show is that applying loops to a label on an arrow gives the label on the preceding arrow:

$$\begin{array}{ll}
 \Omega(\mathbb{Z} \times BO) \simeq O & \longrightarrow \Omega(\mathbb{Z} \times BSp) \simeq Sp \\
 \Omega O \simeq O/U & \\
 \Omega(O/U) \simeq U/Sp & \\
 \Omega(U/Sp) \simeq \mathbb{Z} \times BSp & \longleftarrow \Omega Sp \simeq Sp/U \\
 & \Omega(Sp/U) \simeq U/O \\
 & \Omega(U/O) \simeq \mathbb{Z} \times BO.
 \end{array}$$

Note that we can now get the homotopy groups for O (or really, any of the arrow labels) by simply knowing π_0 of all the spaces involved. We know that O has two path components, whereas U and Sp both have only one. So O/U still has two components, whereas U/O , U/Sp , and Sp/U are all connected. Starting with $\mathbb{Z} \times BO$ and moving counterclockwise, the π_0 values are therefore $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0 , $\mathbb{Z}$, 0 , 0 , $\mathbb{Z}$. Voila!

Exercise 15.9. Let $\Delta_G \subseteq G \times G$ be the diagonal subgroup. Prove that as left $(G \times G)$ -sets there is an isomorphism $(G \times G)/\Delta_G \cong G$, where the action on G is $(x, y).g = xgy^{-1}$.

Exercise 15.10. Use the above octagon to work out the homotopy groups of U/O , O/U , and all of the other spaces in the Bott periodicity clock. Even better, work out a method for directly reading these off of the Bott periodicity clock that only requires remembering the number of components of O , U , and Sp .

Remark 15.11. One can remember how to construct the Bott periodicity clock as follows. Write down the groups O , U , Sp in order (which is the order of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$) and then on the other side of the clock write them in reverse order. Between the two O 's write $O \times O$, and between the two Sp 's write $Sp \times Sp$. It doesn't matter which direction you draw the arrows, as long as they all point in the same direction. The formula $\Omega(\mathbb{Z} \times BO) \simeq O$ will tell you which way the loop-relations go. If you were stuck on a desert island and needed to remember the spaces in the Ω -spectrum for KO , this is a way to do so.

Remark 15.12. It will take us a while to explain the meaning of the "quotient" $G/(G \times G) = \mathbb{Z} \times BG$, but here is a hint of a justification. Consider $O(n) \times O(n) \hookrightarrow O(2n)$, where the former is the evident subgroup of block diagonal matrices. The quotient is $O(2n)/(O(n) \times O(n)) \cong \text{Gr}_n(\mathbb{R}^{2n})$. Stabilizing with n gives a familiar model of the space BO . This doesn't explain the extra $\mathbb{Z}$ factor, but it does (sort of) give a way of rationalizing the mnemonic $G/(G \times G) = \mathbb{Z} \times BG$.

Here are the precise statements that come out of the Bott-Morse argument with minimal geodesics:

$$\begin{aligned}
\Omega O(2n) &\simeq O(2n)/U(n) \cup (\text{cells of dimension } \geq 2n-2) \\
\Omega[O(4n)/U(2n)] &\simeq U(2n)/Sp(n) \cup (\text{cells of dimension } \geq 4n-2) \\
\Omega_0[U(4n)/Sp(2n)] &\simeq Sp(2n)/(Sp(n) \times Sp(n)) \cup (\text{cells of dimension } \geq 4n+4) \\
\Omega Sp(n) &\simeq Sp(n)/U(n) \cup (\text{cells of dimension } \geq 2n+2) \\
\Omega[Sp(2n)/U(n)] &\simeq U(n)/O(n) \cup (\text{cells of dimension } \geq n+1) \\
\Omega_0[U(2n)/O(2n)] &\simeq O(2n)/(O(n) \times O(n)) \cup (\text{cells of dimension } \geq n+1)
\end{aligned}$$

Here Ω_0 refers to the basepoint component of the loop space, whereas the actual loop space has $\pi_0 = \mathbb{Z}$ (coming from $\pi_1 U(n) = \mathbb{Z}$). This is the extra $\mathbb{Z}$ we were missing in Remark 15.12.

Now that we have the lay of the land, let us take a look at the analysis of $\Omega_{I,-I}O(2n)$ (note that we need the dimension to be even in order for I and $-I$ to be in the same path component). The tangent space of $O(2n)$ at the identity consists of matrices A such that e^{At} lies in $O(2n)$ for all t . So we get $I = e^{At}(e^{At})^T = e^{(A+A^T)t}$ and this implies that $A + A^T = 0$. So the tangent space consists of the skew-symmetric matrices. We take as our maximal torus $T \subseteq O(2n)$ the set of rotation matrices of the form $R\theta_1 \oplus \cdots \oplus R\theta_n$ where $R\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. The corresponding tangent vectors are matrices of the form $D(r_1, \dots, r_n) := r_1 J \oplus \cdots \oplus r_n J$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $r_1, \dots, r_n \in \mathbb{R}$. Lie theory tells us that every matrix $A \in T_I O(2n)$ is conjugate to one of the above $D(r_1, \dots, r_n)$ matrices, but in fact we can do a little better. In $O(2)$ the matrix J is conjugate to $-J$, and so we can assume that all $r_i \geq 0$. (We can also use permutation matrices to ensure that $r_1 \geq r_2 \geq \cdots \geq r_n$, but will not need this).

Exercise 15.13. Check that J is conjugate to $-J$ in $O(2)$.

Our next task is to determine which matrices $A \in T_I O(2n)$ have the property that $t \mapsto e^{tA}$ is a path from I to $-I$. Since $e^A = Pe^{D(\bar{x})}P^{-1}$, this is the condition that $e^{D(\bar{x})} = -I$. But note that $e^{rJ} = \cos(r)I + \sin(r)J$, so the previous condition is equivalent to saying that each r_i is a (positive) odd multiple of π . It turns out (and is not surprising) that the *minimal* geodesics are the ones where $r_i = \pi$ for all i . The space of minimal geodesics is then homeomorphic to the $O(2n)$ -orbit space of $\pi D(1, \dots, 1)$ under conjugation. If we think of $D(1, \dots, 1)$ as giving a complex structure on $\mathbb{R}^{2n}$ (its square is $-I$, after all), the stabilizer of $D(1, \dots, 1)$ is the subgroup of orthogonal transformations that commute with this complex structure—and this is precisely $U(n)$. So our space of minimal geodesics is $O(2n)/U(n)$.

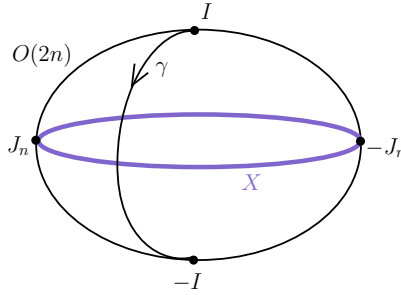
The final task is to analyze the indices of the non-minimal geodesics. A calculation shows that for the geodesics corresponding to $\pi \cdot D(k_1, \dots, k_n)$, with the k_i odd positive integers, the index is

$$(n-1) \left[\sum_i k_i - n \right] + \sum_{k_i > k_j} (k_i - k_j - 2)$$

(see [Bott1, Section 8] or [Milnor6, Section 24] for this). The minimal case $(1, \dots, 1)$ gives index 0 (as expected), and the next largest case will be $(3, 1, \dots, 1)$ which gives index $2n - 2$. So all non-minimal indices have index at least $2n - 2$, and therefore the Bott-Morse theory gives that

$$\Omega O(2n) \simeq O(2n)/U(n) \cup (\text{cells of dimension } \geq 2n - 2).$$

Now we will use the same techniques to analyze $\Omega[O(2n)/U(n)]$. The first crucial observation is that $O(2n)/U(n)$ —our space of minimal geodesics from the above argument—can be identified with a certain subspace X of $O(2n)$. Here is a picture to keep in mind ($J_n = J^{\oplus(n)}$):



The geodesics γ on $O(2n)$ starting at I are of the form $t \mapsto e^{tA}$, and therefore $\gamma(1) = \gamma(\frac{1}{2})^2$. So if the geodesic goes from I to $-I$, then its midpoint $\gamma(\frac{1}{2})$ squares to $-I$ —that is, it gives a complex structure on $\mathbb{R}^{2n}$. This gives us our candidate for X :

$$X = CS_{2n} = \{J \in O(2n) \mid J^2 = -I\}.$$

This is the space of orthogonal complex structures on $\mathbb{R}^{2n}$, and it will play the role of the “equator” in the above picture. Let $\mathbb{J}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\mathbb{J}_n = \mathbb{J}_1^{\oplus(n)}$ denote the standard complex structures. The following diagram of homeomorphisms will be important for us:

$$\begin{array}{ccc} \{2n \times 2n \text{ skew-symm. } A \text{ with eigenvalues } \pm \pi i\} & \xrightarrow{A \mapsto e^{\frac{1}{2}A} = \frac{1}{\pi}A} & CS_{2n} \\ \downarrow A \mapsto (t \mapsto e^{tA}) & \nearrow \gamma \mapsto \gamma(\frac{1}{2}) & \\ \{\text{minimal geodesics } I \rightarrow -I \text{ on } O(2n)\} & & \end{array}$$

The vertical map is a homeomorphism because geodesics are determined by their tangent vector at I . For the horizontal map, note that if A is skew-symmetric with eigenvalues $\pm \pi i$ then there is a $P \in O(2n)$ such that $PAP^{-1} = \pi \mathbb{J}_n$. Then

$$e^{\frac{1}{2}A} = e^{\frac{\pi}{2}P^{-1}\mathbb{J}_nP} = P^{-1}e^{\frac{\pi}{2}\mathbb{J}_n}P = P^{-1}\mathbb{J}_nP = \frac{1}{\pi}A.$$

The map $A \mapsto \frac{1}{\pi}A$ is clearly a homeomorphism. The diagonal map is therefore also a homeomorphism, by commutativity of the diagram.

All of the maps in the diagram are compatible with conjugation by $O(2n)$. As we have remarked before, the space in the upper left corner is a single $O(2n)$ -orbit: every such matrix A is conjugate to $\pi \mathbb{J}$. So in CS_{2n} every element is conjugate to

$\mathbb{J}$. The stabilizer of $\mathbb{J}$ is—by definition—the unitary group $U(n) \subseteq O(2n)$. So we have $CS_{2n} \cong O(2n)/U(n)$.

We next want to compute the space of minimal geodesics in CS_{2n} from $\mathbb{J}$ to $-\mathbb{J}$. However, we first need to deal with the fact that CS_{2n} has two components (corresponding to the two components in $O(2n)$). The path components are represented by $\mathbb{J}$ and $-\mathbb{J}_1 \oplus \mathbb{J}_1^{\oplus(n-1)}$, and changing the sign on two of the $\mathbb{J}_1$ blocks doesn't change the path component. So $\mathbb{J}$ and $-\mathbb{J}$ are in the same path component precisely when n is even. From now on we make this extra assumption.

Exercise 15.14. Find a reflection in $O(2)$ that conjugates $\mathbb{J}_1$ to $-\mathbb{J}_1$, and use this to verify the assertions of the previous paragraph.

Observe that $T_{\mathbb{J}}O(2n) = \mathbb{J} \cdot T_I O(2n) = \{\mathbb{J}A \mid A \text{ skew-symmetric}\}$. Given a tangent vector $\mathbb{J}A \in T_{\mathbb{J}}O(2n)$, the corresponding geodesic is $t \mapsto \mathbb{J}e^{tA}$. For what values of A does this geodesic lie in CS_{2n} ? Expanding $e^{tA} = I + tA + \dots$ gives

$$-I = \mathbb{J}e^{tA}\mathbb{J}e^{tA} = \mathbb{J}^2 + t(\mathbb{J}A\mathbb{J} + \mathbb{J}^2A) + (\text{higher order terms in } t)$$

and we obtain the condition $\mathbb{J}A\mathbb{J} + \mathbb{J}^2A = 0$, or $\mathbb{J}A = -A\mathbb{J}$. Conversely, if we have this anti-commuting condition then since $\mathbb{J} = -\mathbb{J}^{-1}$ we can argue that

$$\mathbb{J}e^{tA} \cdot \mathbb{J}e^{tA} = -\mathbb{J}e^{tA}\mathbb{J}^{-1}e^{tA} = -e^{t\mathbb{J}A\mathbb{J}^{-1}}e^{tA} = -e^{-tA}e^{tA} = -I.$$

This computation shows that

$$T_{\mathbb{J}}CS_{2n} = \{\mathbb{J}A \mid A \text{ skew-symmetric and } \mathbb{J}A = -A\mathbb{J}\}.$$

From this we find that the geodesics in CS_{2n} from $\mathbb{J}$ to $-\mathbb{J}$ all come by applying left-multiplication-by $\mathbb{J}$ to *certain* geodesics in $O(2n)$ from I to $-I$ —namely, ones of the form $t \mapsto e^{tA}$ where A anti-commutes with $\mathbb{J}$. The *minimal* such geodesics will again be where A has eigenvalues $\pm\pi i$, since that is the same condition for minimality in $O(2n)$.

We now get the following diagram of homeomorphisms:

$$\begin{array}{ccc} \left\{ \begin{array}{l} 2n \times 2n \text{ skew-sym } A \text{ with } \mathbb{J}A = -A\mathbb{J} \\ \text{and eigenvalues } \pm\pi i \end{array} \right\} & \xrightarrow{A \mapsto \mathbb{J}e^{\frac{1}{2}A} = \frac{1}{\pi}\mathbb{J}A} & \left\{ \begin{array}{l} L \in O(2n), L^2 = -I, \\ \mathbb{J}L = -L\mathbb{J} \end{array} \right\} \\ A \mapsto (t \mapsto \mathbb{J}e^{tA}) \Big| \cong \downarrow & \nearrow \gamma \mapsto \gamma(\frac{1}{2}) & \\ \{\text{minimal geodesics } \mathbb{J} \rightarrow -\mathbb{J} \text{ in } CS_{2n}\} & & \end{array}$$

We have already seen that the vertical map is a homeomorphism, and the horizontal map is clearly injective. So we only need to check that the horizontal map is surjective. But given an L in the target, we set $A = -\pi\mathbb{J}L$ and readily check that A has the required properties to be a preimage.

The space in the upper right corner of the above diagram can be described as the set of orthogonal quaternionic structures on $\mathbb{R}^{2n}$ (remember that n is even) that extend the fixed complex structure $\mathbb{J}$. Recalling that $U(n) \subseteq O(2n)$ is the subgroup of automorphisms that commute with $\mathbb{J}$, conjugation by $U(n)$ acts on all the spaces in the above diagram. By Lemma 15.15 below, the $U(n)$ -action is transitive. For a fixed L , the stabilizer of L is precisely $Sp(\frac{n}{2})$. So the orbit of L is $U(n)/Sp(\frac{n}{2})$,

and this is our space of minimal geodesics. After some index computations that we omit, the Morse-Bott arguments show that when n is even

$$\Omega(O(2n)/U(n)) \simeq U(n)/Sp(\frac{n}{2}) \cup (\text{cells of dimension } \geq 2n - 2).$$

One can now proceed, as Bott did, to use the same techniques to analyze $\Omega[U(2n)/Sp(n)]$, and so on. At each stage something very similar happens, though the patterns are not perhaps totally manifest at first. This is where Clifford algebras enter the story, as they give a nice way of organizing what is happening in this process. We will pick this thread up in the next section.

The following result from linear algebra is a leftover piece from our above analysis:

Lemma 15.15. *Consider $\mathbb{C}^n$ with its standard Hermitian product $\langle -, - \rangle$. For $x, y \in \mathbb{C}^n$ let $x \cdot y = \operatorname{Re}(\langle x, y \rangle)$ be the associated real inner product. Suppose $L: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is conjugate-linear (i.e. $L(ix) = -iL(x)$), orthogonal (preserves the dot product), and $L^2 = -I$. Then n is even and there exists a unitary map $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $P^{-1}LP$ equals the map $j: \mathbb{C}^n \rightarrow \mathbb{C}^n$ where*

$$j(e_k) = \begin{cases} e_{k+1} & \text{if } k \text{ is odd,} \\ -e_{k-1} & \text{if } k \text{ is even.} \end{cases}$$

Proof. First note that the Hermitian inner product is recovered from the dot product by the formula $\langle x, y \rangle = (x \cdot y) + i(x \cdot iy)$. This follows at once from the formula $\langle x, iy \rangle = -i\langle x, y \rangle$. Consequently, we obtain

$$\begin{aligned} \langle Lx, Ly \rangle &= (Lx \cdot Ly) + i(Lx \cdot iLy) = (x \cdot y) + i(Lx \cdot -L(iy)) \\ &= (x \cdot y) - i(x \cdot iy) \\ &= \overline{\langle x, y \rangle}. \end{aligned}$$

Equivalently, $\langle x, y \rangle = \overline{\langle Lx, Ly \rangle}$. As a corollary we find that $\langle x, Lx \rangle = 0$ for all x , since

$$\langle x, Lx \rangle = \overline{\langle Lx, L^2x \rangle} = -\overline{\langle Lx, x \rangle} = -\langle x, Lx \rangle.$$

Let a_1 denote any unit vector in $\mathbb{C}^n$, and let $b_1 = L(a_1)$. Note that b_1 is also a unit vector, since L is orthogonal. Moreover, $\langle a_1, b_1 \rangle = \langle a_1, La_1 \rangle = 0$. Consequently, the vectors a_1, b_1 are $\mathbb{C}$ -linearly independent and $\mathbb{C}\langle a_1, b_1 \rangle$ is stable under L .

Let $H = \mathbb{C}\langle a_1, b_1 \rangle^\perp$, the orthogonal complement with respect to the Hermitian inner product. Then L restricts to a map $H \rightarrow H$, and we proceed by induction. In this way we establish that n is even and we produce a unitary basis $a_1, b_1, \dots, a_r, b_r$ ($n = 2r$) where $L(a_k) = b_k$. Let $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the transformation given by $e_{2k-1} \mapsto a_k$, $e_{2k} \mapsto b_k$. Then $P^{-1}LP$ has the desired form, and since $\underline{a}, \underline{b}$ is a unitary basis the transformation P is unitary. $\square$

Summary. In this section we developed the Morse-theoretic approach to proving complex Bott periodicity, and we went through the initial stages for proving real Bott periodicity. It is an informative exercise to try to complete the argument for the real case by hand on one's own, perhaps consulting [Bott1] or [Milnor6] for some of the details. Our task in the next few sections will be to develop some machinery for handling the bookkeeping in these arguments, with the goal of improving our understanding of the bigger picture.

Exercise 15.16. Knowing the groups $\pi_*(U/O)$ and $\pi_*(O/U)$, analyze the long exact sequences for the fibrations $O \rightarrow U \rightarrow U/O$ and $U \rightarrow O \rightarrow O/U$ to determine what all of the maps are. Verify that

$$\pi_i(O) \rightarrow \pi_i(U) \text{ is } \begin{cases} \text{multiplication by } \pm 2 & \text{when } i \equiv 3 \pmod{8}, \\ \text{multiplication by } \pm 1 & \text{when } i \equiv 7 \pmod{8}, \\ \text{zero} & \text{otherwise.} \end{cases}$$

Likewise,

$$\pi_i(U) \rightarrow \pi_i(O) \text{ is } \begin{cases} \text{the projection } \mathbb{Z} \rightarrow \mathbb{Z}/2 & \text{when } i \equiv 1 \pmod{8}, \\ \text{multiplication by } \pm 1 & \text{when } i \equiv 3 \pmod{8}, \\ \text{multiplication by } \pm 2 & \text{when } i \equiv 7 \pmod{8}, \\ \text{zero} & \text{otherwise.} \end{cases}$$

Compute the induced maps $\pi_i(\mathbb{Z} \times BO) \rightarrow \pi_i(\mathbb{Z} \times BU)$ and $\pi_i(\mathbb{Z} \times BU) \rightarrow \pi_i(\mathbb{Z} \times BO)$ and thereby compute the maps $c: KO^*(pt) \rightarrow K^*(pt)$ and $r: K^*(pt) \rightarrow KO^*(pt)$ when $* \leq 0$. Think about what information would be needed to extend to $* > 0$.

Exercise 15.17. In our investigation of Bott periodicity we had to use the two homotopy-equivalent spaces $\Omega_{I,-I}SU(2n)$ and $\Omega_I SU(2n)$. Working more generally, let G be a topological group with identity e and let $x \in G$.

- For any path f from x to e , check that $\Omega_{e,x}G \rightarrow \Omega_e G$ given by post-concatenation with f is continuous.
- Let $\gamma: I \rightarrow G$ be a path from e to x , and let γ^{-1} denote the path $t \mapsto \gamma(t)^{-1}$ from e to x^{-1} . For $\lambda \in \Omega_{e,x}G$, let $\lambda \odot \gamma^{-1}$ be the path $t \mapsto \lambda(t)\gamma^{-1}(t)$. Check that this defines a continuous map $\Omega_{e,x}G \rightarrow \Omega_e G$, and further that this map is homotopic to post-concatenation with the path $t \mapsto x \cdot \gamma(t)^{-1}$. [Hint: In $I^2 \subseteq \mathbb{R}^2$ consider the diagonal path as well as the path with the same endpoints that moves counterclockwise around the boundary. Since I^2 is convex, these paths are homotopic relative to the endpoints via the straight-line homotopy. Use this fact to avoid having to write down any horrible formulas.]

Exercise 15.18. Let $X = \text{Gr}_n(\mathbb{R}^{2n})$. For $M \in X$ the tangent space $T_M X$ is naturally identified with $\text{Hom}(M, M^\perp)$ (?). To any linear map $A: M \rightarrow M^\perp$ we have the adjoint $A^*: M^\perp \rightarrow M$, determined by $\langle Am, m' \rangle = \langle m, A^*m' \rangle$ for $m \in M, m' \in M^\perp$. We equip $T_M X$ with the *Hilbert-Schmidt inner product* given by $\langle A, B \rangle = \text{tr}(A^*B)$. This gives the Riemannian structure on X .

Verify the following:

- For $A \in T_M X$ the corresponding geodesic $\exp(tA)$ is given by $\exp(tA) = e^{tK_A}(M)$ where $K_A = \begin{bmatrix} 0 & -A^* \\ A & 0 \end{bmatrix}$ where the block-diagonal structure is with respect to the decomposition $\mathbb{R}^{2n} = M \oplus M^\perp$. [One way is to use the identification $\text{Gr}_n(\mathbb{R}^{2n}) \cong O(2n)/[O(n) \times O(n)]$.]
- The transformation A^*A is self-adjoint (hence diagonalizable) and has nonnegative real eigenvalues. So for any function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ we can define $f(A^*A)$ to be the transformation with the same eigenspaces as A^*A but where the eigenvalues are obtained by applying f to those of A^*A . For example, we have the transformation $\sqrt{A^*A}$. Using $f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}$ (defined to equal 1 at $x = 0$) gives

a transformation $\frac{\sin(\sqrt{A^*A})}{\sqrt{A^*A}}$. For $t \in [0, 1]$ figure out how we can also make sense of $\frac{\sin(t\sqrt{A^*A})}{\sqrt{A^*A}}$.

- (c) Determine formulas for K_A^m and use them to prove that

$$e^{tK_A} = \begin{bmatrix} \cos(t\sqrt{A^*A}) & 0 \\ 0 & \cos(t\sqrt{AA^*}) \end{bmatrix} + K_A \cdot \begin{bmatrix} \frac{\sin(t\sqrt{A^*A})}{\sqrt{A^*A}} & 0 \\ 0 & \frac{\sin(t\sqrt{AA^*})}{\sqrt{AA^*}} \end{bmatrix}.$$

- (d) Prove that $e^{tK_A}(M) = \left\{ \left(\cos(t\sqrt{A^*A})x, A \cdot \frac{\sin(t\sqrt{A^*A})}{\sqrt{A^*A}}x \right) \mid x \in M \right\}$.
- (e) For convenience let us now reparameterize our geodesics as $\gamma_A(t) = e^{\frac{\pi}{2}tA}$. Prove that $\gamma_A(1) = M^\perp$ if and only if the eigenvalues of A^*A are all squares of odd integers. In particular, observe that this implies that $A: M \rightarrow M^\perp$ is an isomorphism.
- (f) Convince yourself that $\gamma_A|_{[0,1]}$ is a minimal geodesic from M to $M^\perp$ if and only if $A^*A = I$, that is to say A is orthogonal. So the space of minimal geodesics from M to $M^\perp$ coincides with the space of orthogonal transformations $M \rightarrow M^\perp$.
- (g) When A is an orthogonal transformation, verify that the midpoint $\gamma_A(\frac{1}{2})$ is precisely the graph of A .

Exercise 15.19 (The symplectic group). The symplectic group can be a bit confusing because there are two very different ways of looking at it. We will define two groups $Sp(n)_{\mathbb{H}} \subseteq M_{n \times n}(\mathbb{H})$ and $Sp(n)_{\mathbb{C}} \subseteq M_{2n \times 2n}(\mathbb{C})$ which will turn out to be isomorphic.

- (a) Let V be a (left) vector space over $\mathbb{H}$ equipped with a Hermitian inner product $\langle -, - \rangle$. This means it is linear in the second variable and satisfies the usual conjugate-symmetry, but in the first variable the rule is then forced to be

$$\langle qv, w \rangle = \langle v, w \rangle \bar{q}.$$

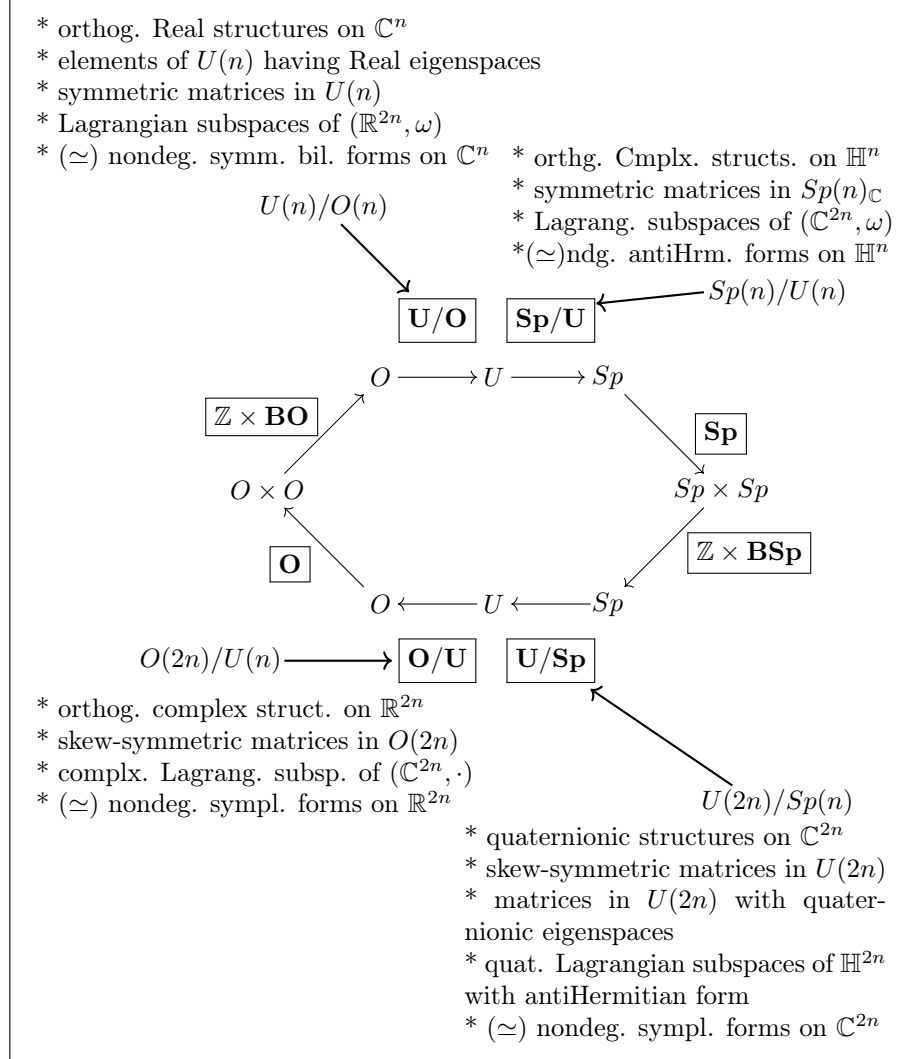
Regard elements of $\mathbb{H}^n$ as column vectors, as usual. With these conventions the standard Hermitian form on $\mathbb{H}^n$ is $\langle v, w \rangle = w^T \bar{v}$. (It would be more convenient to have the first variable be the linear one, but we instead use the conventions that are most common in the complex setting).

Define $Sp(V) \subseteq GL_{\mathbb{H}}(V)$ to be the subgroup of $\mathbb{H}$ -linear transformations T such that $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$.

It is possible to represent linear transformations by matrices, but since $\mathbb{H}$ is noncommutative one must be careful how this is done. Linear maps with respect to the left action correspond to *right* multiplication of matrices. Also be warned that $(AB)^T \neq B^T A^T$, though check that $(AB)^\dagger = B^\dagger A^\dagger$.

For $A \in M_{n \times n}(\mathbb{H})$, let $\mathcal{T}_A: \mathbb{H}^n \rightarrow \mathbb{H}^n$ be defined by $\mathcal{T}_A(v) = (v^T A^\dagger)^T$. Check that this is a linear transformation, $\mathcal{T}_A \circ \mathcal{T}_B = \mathcal{T}_{AB}$, and for the standard basis $\{e_k\}$ one has that $\mathcal{T}_A(e_k)$ equals the conjugate of the k th column of A . Check that the assignment $\mathcal{T}_A \mapsto A$ is an isomorphism from $Sp(\mathbb{H}^n)$ to the group

FIGURE 15.18.1. The mixed-type spaces in real Bott periodicity



$Sp(n)_{\mathbb{H}}$ of matrices $A \in M_{n \times n}(\mathbb{H})$ such that $A^\dagger A = I$, and that this condition implies $AA^\dagger = I$ as usual.

- (b) Inside of $Sp(n)_{\mathbb{H}}$ one sees $U(n)$ as the evident subgroup of matrices with complex entries. In a basis-free approach, check that $U(n)$ is the centralizer of the element “right-multiplication-by- i ” in $Sp(\mathbb{H}^n)$.
- (c) Let ${}_{\mathbb{C}}V$ denote V but thought of as a $\mathbb{C}$ -vector space. Write

$$\langle v, w \rangle = \langle v, w \rangle_{\mathbb{C}} + \Omega(v, w)j$$

where both $\langle v, w \rangle_{\mathbb{C}}$ and $\Omega(v, w)$ are in $\mathbb{C}$. Then the first is a complex Hermitian form on ${}_{\mathbb{C}}V$ and the second is a complex symplectic form (complex linear in

both variables and alternating). Observe the basic relations

$$\langle jv, w \rangle_{\mathbb{C}} = \Omega(v, w), \quad \Omega(jv, w) = -\langle v, w \rangle_{\mathbb{C}}.$$

Check that $Sp(V)$ is equal to the following:

- The set of $\mathbb{C}$ -linear maps ${}_{\mathbb{C}}V \rightarrow {}_{\mathbb{C}}V$ that are unitary for $\langle -, - \rangle_{\mathbb{C}}$ and are j -linear.
 - The set of $\mathbb{C}$ -linear maps ${}_{\mathbb{C}}V \rightarrow {}_{\mathbb{C}}V$ that are unitary and also preserve the symplectic form Ω .
- (d) Show that for any conjugate-linear map $\mathcal{F}: \mathbb{C}^k \rightarrow \mathbb{C}^k$ there is a unique $k \times k$ matrix F such that $\mathcal{F}(v) = F\bar{v}$ for all $v \in \mathbb{C}^k$. One needs to be a little careful in how these matrices interact with those that represent linear transformations. Check that if $\mathcal{T}$ is a $\mathbb{C}$ -linear transformation represented by the matrix T , then $\mathcal{T} \circ \mathcal{F}$ is represented by TF but $\mathcal{F} \circ \mathcal{T}$ is represented by $F\bar{T}$.
- (e) Let $\mathcal{J}: \mathbb{H}^n \rightarrow \mathbb{H}^n$ be the left-multiplication-by- j map. As a map $\mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ this is conjugate-linear. Check that this can be represented by either the matrix $J = \text{diag}\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$ or $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$, depending on how one orders the complex basis $e_1, \dots, e_n, je_1, \dots, je_n$. Check that a matrix $Q \in GL_{2n}(\mathbb{C})$ represents an $\mathbb{H}$ -linear map if and only if $QJ = JQ$.
- (f) Because dealing with quaternionic matrices can be a little dicey, sometimes people avoid this by using only the versions of $Sp(V)$ from the previous part. We next discuss the matrix versions of these. If we take $V = \mathbb{H}^n$ with the standard Hermitian form then ${}_{\mathbb{C}}V = \mathbb{C}^{2n}$. Choose the basis $e_1, \dots, e_n, je_1, \dots, je_n$ where $e_1, \dots, e_n$ are the standard quaternionic basis for V . Then left multiplication by j is represented (in the sense of (e)) by the matrix $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$.

The form Ω is given by $\Omega(v, w) = \langle jv, w \rangle_{\mathbb{C}} = \langle J\bar{v}, w \rangle_{\mathbb{C}} = v^T J^T w$. So the condition $\Omega(Av, Aw) = \Omega(v, w)$ for all v and w is equivalent to $A^T J^T A = J^T$, or $J = A^T J A$. Thus the set of maps ${}_{\mathbb{C}}V \rightarrow {}_{\mathbb{C}}V$ that are unitary and preserve Ω can be represented as the matrix group

$$Sp(n)_{\mathbb{C}} = \{A \in U(2n) \mid A^T J A = J\}.$$

In particular, we have that $Sp(n)_{\mathbb{H}} \cong Sp(\mathbb{H}^n) \cong Sp(n)_{\mathbb{C}}$. Also check that the condition $A^T J A = J$ can be rewritten (for $A \in U(2n)$) as $AJ = J\bar{A}$, and so is equivalent to condition that the transformation represented by A is j -linear. We already saw this equivalence back in (c).

- (g) Verify that a matrix lies in $Sp(n)_{\mathbb{C}}$ if and only if it has the form $\begin{bmatrix} A & -\bar{C} \\ C & \bar{A} \end{bmatrix}$ where $AA^\dagger + \bar{C}C^T = I$ and $AC^\dagger - \bar{C}A^T = 0$. Then check that the natural identification between $Sp(n)_{\mathbb{C}}$ and $Sp(n)_{\mathbb{H}}$ sends

$$\begin{bmatrix} A & -\bar{C} \\ C & \bar{A} \end{bmatrix} \mapsto \bar{A} - Cj.$$

Note that this identification does not send symmetric matrices to symmetric matrices.

- (h) In $Sp(n)_{\mathbb{C}}$ note the element $\begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$. Check that the centralizer of this element is the subgroup of matrices $\begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}$ where $A \in U(n)$, and hence is an isomorphic copy of $U(n)$. This is the analog of the subgroup from (b).

Exercise 15.20 (The space $U(n)/O(n)$).

- (a) Let V be an n -dimensional complex vector space, and suppose we have a real inner product on the underlying real vector space. Prove that there is a bijection between the following two sets:

- $\{\text{real subspaces } N \subseteq V \text{ such that } iN = N^\perp\}$
- $\{\text{conjugate-linear orthogonal maps } \sigma: V \rightarrow V \text{ such that } \sigma^2 = \text{id}\}.$

If we topologize the first as a subspace of $\text{Gr}_n(V)$ and the second as a subspace of $O(V)$, check that your bijection is a homeomorphism.

Following terminology of [Atiyah4], we call a choice of σ (or of the corresponding N) an **orthogonal Real** structure on V . Note that the standard model of $\mathbb{C}^n$ comes equipped with a standard such structure, specified by $\mathbb{R}^n \subseteq \mathbb{C}^n$ or by taking σ to be complex-conjugation.

- (b) Prove that $U(n)$ acts by conjugation on the sets introduced in (a), the action is transitive, and the stabilizer of the canonical structure is $O(n)$. So $U(n)/O(n)$ is homeomorphic to the space of orthogonal Real structures on $\mathbb{C}^n$.
- (c) Given a complex vector space V with a Real structure, say that an element $v \in V$ is Real if it is fixed by the conjugation. Likewise, say that a complex subspace of V is Real if it is closed under the conjugation. Prove that a complex subspace is Real if and only if it has a basis consisting of Real vectors.
- (d) Now fix $\mathbb{C}^n$ with its standard Real structure. Prove that the following three subsets of $U(n)$ are the same:
- The set of $A \in U(n)$ whose eigenspaces are Real subspaces.
 - The set of matrices PDP^{-1} where $P \in O(n)$ and $D \in U(n)$ is diagonal.
 - The set $U(n)_{\text{sym}}$ of symmetric matrices in $U(n)$.
- (e) Consider the action of $U(n)$ on $U(n)_{\text{sym}}$ given by $P.A = PAP^T$. Prove that this action is transitive, and that the stabilizer of the identity matrix is $O(n)$. So $U(n)/O(n)$ is homeomorphic to the subspace of $U(n)$ described in (d). [Hint for transitivity: Using the previous part, it suffices to show that any diagonal $D \in U(n)$ can be written as QQ^T for some $Q \in U(n)$.]
- (f) Equip $\mathbb{C}^n$ with the standard Hermitian inner product $\langle -, - \rangle$. Our conventions are that this is linear in the second variable and conjugate-linear in the first. Separating into real and imaginary parts, write

$$\langle v, w \rangle = q(v, w) + i\omega(v, w)$$

with $q(v, w), \omega(v, w) \in \mathbb{R}$. Then q is a Euclidean inner product and ω is a symplectic nondegenerate bilinear form on the underlying space $\mathbb{R}^{2n}$. Check that $\omega(v, w) = q(iv, w)$. Focusing on $(\mathbb{R}^{2n}, \omega)$, a subspace $W \subseteq \mathbb{R}^{2n}$ is called **isotropic** if $\omega(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. A maximal isotropic subspace is called a **Lagrangian subspace**. It is not hard to see that all such subspaces have dimension n . The subspace of $\text{Gr}_n(\mathbb{R}^{2n})$ consisting of the Lagrangian subspaces is called the **Lagrangian Grassmannian** of $(\mathbb{R}^{2n}, \omega)$, as introduced in [Arnold1]. Prove that the Lagrangian subspaces are precisely those real subspaces $N \subseteq \mathbb{R}^{2n}$ such that $iN = N^\perp$, where the perpendicular complement is with respect to the Euclidean form q . Conclude from (a) that the Lagrangian Grassmannian is homeomorphic to $U(n)/O(n)$.

Exercise 15.21 (The space $U(2n)/Sp(n)$).

- (a) A quaternionic structure on $\mathbb{C}^{2n}$ is a map $\mathcal{J}: \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ that is conjugate-linear and satisfies $\mathcal{J}^2 = -\text{id}$. Such a map makes $\mathbb{C}^{2n}$ into an $\mathbb{H}$ -module via the usual action of i and having j act as $\mathcal{J}$. An **orthogonal quaternionic structure** is

one where the map $\mathcal{J}$ preserves the Euclidean inner product. We topologize the set of orthogonal quaternionic structures as a subspace of $O(4n)$. Check that conjugation by elements of $U(2n)$ acts transitively on the space of orthogonal quaternionic structures, and the stabilizer of the standard structure is precisely $Sp(n)$. So $U(2n)/Sp(n)$ is the space of orthogonal quaternionic structures on $\mathbb{C}^{2n}$.

- (b) Show that the following two sets are the same:
- The set of $A \in U(2n)$ such that all eigenspaces are quaternionic.
 - The set of $A \in U(2n)$ that can be written as QDQ^{-1} for $Q \in Sp(n)$ and $D = \text{diag}\{d_1 I_2, \dots, d_n I_2\}$ for some $d_i \in \mathbb{C}$.
- Then show that there is a bijection between this set and the set of skew-symmetric elements $U(2n)_{\text{skew-symm}}$, given by $X \mapsto XJ$.
- (c) For the sets appearing in the previous part, give them the subspace topology from $U(2n)$. Let $U(2n)$ act on $U(2n)_{\text{skew-symm}}$ by $Q \cdot X = QXQ^T$. Prove that this action is transitive and the stabilizer of J is $Sp(n)$, and thereby conclude that all of the spaces from the previous part are homeomorphic to $U(2n)/Sp(n)$. [Hint for transitivity: If $A \in U(2n)_{\text{skew-symm}}$ then by (d) we can write $AJ = QDQ^{-1}$ as above. Go from there.]
- (d) The space $\mathbb{H}^{2n}$ comes equipped with a standard anti-Hermitian form. Define the map $t: \mathbb{H}^{2n} \rightarrow \mathbb{H}^{2n}$ by $t(x, y) = (-y, x)$, and note that $t^2 = -\text{id}$. For $v, w \in \mathbb{H}^{2n}$ define $b(v, w) = \langle v, tw \rangle$. Then b is conjugate-linear in the first variable and linear in the second, and

$$b(w, v) = \langle w, tv \rangle = \overline{\langle tv, w \rangle} = \overline{\langle -v, tw \rangle} = -\overline{b(v, w)}.$$

So b is anti-Hermitian. Say that a quaternionic subspace $N \subseteq \mathbb{H}^{2n}$ is **Lagrangian** if it is maximal isotropic for b . Such subspaces all have dimension n , and so form a subspace of $\text{Gr}_n^{\mathbb{H}}(\mathbb{H}^{2n})$. Show that an n -dimensional subspace N is Lagrangian if and only if $t(N) = N^{\perp_{\text{Hermit.}}}$, which is also equivalent to $\mathbb{H}^{2n} = N \oplus t(N)$.

- (e) Let $G \subseteq Sp(2n)$ be the subgroup of g such that $gt = tg$. Since $t^2 = -\text{id}$ and t is complex-linear, $\mathbb{H}^{2n}$ is the orthogonal direct sum of the $+i$ and $-i$ eigenspaces of t . Call the first W_+ and the second W_- . Every $g \in G$ preserves these, and so restriction gives a group map $G \rightarrow U(W_+)$. Prove that this is an isomorphism, and so $G \cong U(2n)$.
- (f) The group G acts in the natural way on the space Λ of quaternionic Lagrangian subspaces of $\mathbb{H}^{2n}$. Show that the action is transitive, and the stabilizer of $\mathbb{H}^n \oplus 0$ is equal to $Sp(n)$. Conclude that $\Lambda \cong U(2n)/Sp(n)$.

Exercise 15.22 (The space $O(2n)/U(n)$). A complex structure on $\mathbb{R}^{2n}$ is a linear map $J: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $J^2 = -\text{id}$. An **orthogonal complex structure** is such a J that is also an orthogonal transformation, where $\mathbb{R}^{2n}$ is equipped with the standard inner product. We equip the space CS_{2n} of orthogonal complex structures on $\mathbb{R}^{2n}$ with the subspace topology induced from $O(2n)$. By the *standard complex structure* we mean the map J represented by the matrix $\text{diag}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})$.

- (a) Check that $CS_{2n} \subseteq O(2n)$ equals the set of skew-symmetric orthogonal matrices.
- (b) Check that $O(2n)$ acts on CS_{2n} by conjugation. Prove that the action is transitive and the stabilizer of the standard complex structure is $U(n)$, thus concluding that $CS_{2n} \cong O(2n)/U(n)$.

- (c) Equip $\mathbb{C}^{2n}$ with the standard Hermitian form and the standard conjugation operator. Define $b(v, w) = \langle \bar{v}, w \rangle$. This is a symmetric, nondegenerate bilinear form on $\mathbb{C}^{2n}$ (it is the analog of the dot product, with $b(v, w) = \sum_i v_i w_i$).
- (d) Check that $O(2n) \subseteq U(2n)$ is the subgroup of unitary transformations having the property that $Av = A\bar{v}$, and that $b(Av, Aw) = b(v, w)$ for all $v, w \in \mathbb{C}^{2n}$.
- (e) A complex subspace of $\mathbb{C}^{2n}$ that is maximal isotropic with respect to b will be called **Lagrangian**. Define N_0 to be the span of the vectors $e_{2j-1} + ie_{2j}$, $1 \leq j \leq n$. Check that this is Lagrangian. If N is a Lagrangian subspace prove that $\bar{N} = N^\perp$, where the $\perp$ is with respect to the Hermitian metric.
- (f) Let $G \subseteq O(2n)$ be the set of $g \in G$ such that $g(N_0) = N_0$. Prove that the restriction map $G \rightarrow U(N_0)$ is an isomorphism. Define an element $t \in O(2n)$ by $t(x) = x$ if $x \in N_0$ and $t(x) = -x$ if $x \in \bar{N}_0$. Check that G is the centralizer of t in $O(2n)$, and therefore can be identified with the standard inclusion $U(n) \subseteq O(2n)$ for the complex structure on $\mathbb{C}^{2n}$ defined by t .
- (g) Let $\Lambda \subseteq \text{Gr}_n(\mathbb{C}^{2n})$ be the subspace of Lagrangians. The subgroup $O(2n) \subseteq U(2n)$ acts on Λ in the evident way. Prove that this action is transitive, and conclude that $\Lambda \cong O(2n)/U(n)$.

Exercise 15.23 (The space $Sp(n)/U(n)$).

- (a) Let V be an n -dimensional quaternionic vector space with Hermitian inner product. Consider the following sets:
 - Complex subspaces $W \subseteq V$ such that $W \perp jW$.
 - Complex linear involutions $\sigma: V \rightarrow V$ that are unitary and anti-commute with j : $\sigma(jv) = -j\sigma(v)$.
 Prove that there is a bijection between these sets sending σ to $\ker(\sigma - I)$. Check that if one regards the first set as a subspace of $\text{Gr}_n^{\mathbb{C}}(V)$ and the second as a subspace of $U(V)$, then the bijection is a homeomorphism.

We will call such a structure (in either of the equivalent forms) an **orthogonal Complex structure** on V . When $V = \mathbb{H}^n$, we take $U = \mathbb{C}^n \subseteq \mathbb{H}^n$ as the standard orthogonal Complex structure. The associated involution is the conjugation-by- i map $C_i(v) = ivi^{-1}$.
- (b) The group $Sp(n)$ acts on the space of orthogonal Complex structures on $\mathbb{H}^n$: in the evident manner on subspaces U , and by conjugation on involutions σ . Show that the action is transitive and the stabilizer of the standard structure is $U(n)$, so that $Sp(n)/U(n)$ is homeomorphic to the space of orthogonal Complex structures.
- (c) Check that a $T \in Sp(n)$ satisfies $T \circ C_i = C_i \circ T$ if and only if $T(vi) = T(v)i$ for all $v \in \mathbb{H}^n$. In the matrix formulation, check that this is equivalent to $A \in Sp(n)$ satisfying $iA^\dagger = A^\dagger i$, which is equivalent to the entries of A all being complex numbers. That is, the subgroup of $Sp(n)$ of elements commuting with C_i is equal to $U(n)$.
- (d) More generally, let σ be a fixed orthogonal Complex structure on $\mathbb{H}^n$, with the associated subspace $W = \ker(\sigma - I)$. Prove that the subgroup of all $T \in Sp(n)$ that commute with σ is isomorphic to the unitary group $U(W)$.
- (e) Consider $\mathbb{H}^n = \mathbb{C}^{2n}$ with the standard complex symplectic form Ω . Let $\Lambda \subseteq \text{Gr}_n(\mathbb{C}^{2n})$ be the associated Lagrangian Grassmannian. Recall from Exercise 15.19(c) that $Sp(n)$ can be regarded as the unitary transformations of $\mathbb{C}^{2n}$ that preserve Ω . So $Sp(n)$ acts on Λ in the natural way. Prove that the action is transitive and the stabilizer of $\mathbb{C}^n \oplus 0$ is $U(n)$, so that $Sp(n)/U(n) \cong \Lambda$.

- (f) Let S denote the set of symmetric matrices in $Sp(n)_{\mathbb{C}}$. The group $Sp(n)$ acts on S by $PA = PAP^T$. The stabilizer of the identity is the subgroup H consisting of P such that $PP^T = I$, or equivalently $P = \bar{P}$. So this is the subgroup of $O(2n)$ whose elements commute with J , which is a copy of $U(n)$. Find an element $B \in Sp(n)_{\mathbb{C}}$ that conjugates our canonical $U(n) \subseteq Sp(n)_{\mathbb{C}}$ to H , thereby showing $Sp(n)/U(n) \cong Sp(n)/H$.
- (g) Prove that the action of $Sp(n)_{\mathbb{C}}$ on S is transitive, and conclude that $Sp(n)/U(n) \cong S$. [This is a little challenging, so here is one way to approach it. Given $A \in S$, define $\mathcal{C}: \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ by $\mathcal{C}(v) = A\bar{v}$. Then $\mathcal{C}$ is a conjugate-linear involution; let W be the fixed set, which has real dimension equal to $2n$. Check that $\mathcal{C}$ commutes with J , so J preserves W . Find a real basis for W such that if B is the $2n \times 2n$ matrix whose columns are this basis then $B \in Sp(n)_{\mathbb{C}}$ and $BB^T = A$.]

16. BOTT PERIODICITY AND CLIFFORD ALGEBRAS

Our goal in this section is to give a systematic approach to real Bott periodicity. The core of the arguments are still via Bott-Morse theory, but Clifford algebras are used to help with the bookkeeping for the spaces that appear.

Recall from Section ?? the real Clifford algebras $Cl_r = \mathbb{R}\langle e_1, \dots, e_n \rangle / (e_i^2 = -1, e_i e_j = -e_j e_i)$. We previously denoted these Cl_r^+ , but here we will drop the $+$ since we will not need to use the negative Clifford algebras. The Clifford algebras exhibit the pattern $Cl_{r+8} \cong Cl_r(16)$, where recall that $T(n)$ is shorthand for the matrix algebra $M_{n \times n}(T)$. Table 16.1 lists Cl_r for $0 \leq r \leq 8$. By inspection the module theory of these algebras is very easy to understand, in each case the module category being semisimple. The table also lists the irreducible modules for each Cl_r . Finally, in the last column we list the groups from the Bott periodicity clock, as discussed in the previous section. We have not yet explained the connection of these groups to the Clifford algebra story, but the table certainly demonstrates that *something* interesting is going on. In this section we will explain what it is.

TABLE 16.1. Clifford algebras and their modules

r	Cl_r	Irreducible modules	Bott group
0	$\mathbb{R}$	$\mathbb{R}$	O
1	$\mathbb{C}$	$\mathbb{C}$	U
2	$\mathbb{H}$	$\mathbb{H}$	Sp
3	$\mathbb{H} \times \mathbb{H}$	$\mathbb{H}_+, \mathbb{H}_-$	$Sp \times Sp$
4	$\mathbb{H}(2)$	$\mathbb{H}^2$	Sp
5	$\mathbb{C}(4)$	$\mathbb{C}^4$	U
6	$\mathbb{R}(8)$	$\mathbb{R}^8$	O
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	$\mathbb{R}_+^8, \mathbb{R}_-^8$	$O \times O$
8	$\mathbb{R}(16)$	$\mathbb{R}^{16}$	O

Let M be a Cl_r -module. It will be useful to assume that all of our modules are equipped with a real inner product, denoted $\langle -, - \rangle$, with respect to which all of the e_i -multiplications are orthogonal. Any module can be equipped with such a structure: choose any inner product on the underlying vector space and then average it with respect to multiplication by the 2^r monomials in the e_i 's.

Remark 16.2. For the bulk of our work from now on, all Clifford modules will be assumed to be “orthogonal” in the above sense—even when not said explicitly. The restriction to considering these orthogonal modules is not strictly necessary, but it has the effect of producing spaces that relate to the compact groups $O(n)$ rather than the non-compact groups $GL_n(\mathbb{R})$. See Remark 16.19 for another perspective.

Write $O(M)$ for the set of all orthogonal transformations of the underlying inner product space of M . For each $k \leq r+1$, define

$$\mathcal{E}x_k(M) = \{J \in O(M) \mid e_1, \dots, e_{k-1}, J \text{ gives a } \text{Cl}_k\text{-module structure on } M\}.$$

Equip this set with the subspace topology from $O(M)$. This is the space of Cl_k -extension structures of the underlying Cl_{k-1} -structure of M .

Note that $\mathcal{E}x_1(M)$ is the space of orthogonal complex structures on M , and $\mathcal{E}x_2(M)$ is the space of (orthogonal) quaternionic structures where the given e_1 action is multiplication by i . We saw both of these spaces coming up in our arguments from the last section. It is natural to set $\mathcal{E}x_0(M) = O(M)$. Note that we have the sequence of subsets

$$\mathcal{E}x_0(M) \supseteq \mathcal{E}x_1(M) \supseteq \mathcal{E}x_2(M) \supseteq \dots$$

since if $e_1, \dots, e_{k-1}, J$ satisfy the Clifford relations then so do $e_1, \dots, e_{k-2}, J$.

The action of e_k on M gives a natural basepoint in $\mathcal{E}x_k(M)$, for each $k \leq r$. We also have the antipodal element $-e_k \in \mathcal{E}x_k(M)$. If the Cl_k -structure extends to a Cl_{k+1} -structure (e.g., if $k < r$) then e_k and $-e_k$ are in the same path component of $\mathcal{E}x_k(M)$: to see this, check that $t \mapsto \cos(\pi t)e_k + \sin(\pi t)e_{k+1}$ describes a path in $\mathcal{E}x_k(M)$. We will attempt to use Bott-Morse theory to approximate $\Omega_{e_k, -e_k} \mathcal{E}x_k(M)$ by the space of minimal geodesics.

The tangent space $T_l O(M)$ may be identified with the set of linear maps $A: M \rightarrow M$ that are skew-symmetric in the sense that $\langle Ax, y \rangle = -\langle x, Ay \rangle$ for all $x, y \in M$. Then a little legwork shows that

$$T_{e_k} \mathcal{E}x_k(M) = \{e_k \cdot A \mid A \text{ is skew-symmetric, } Ae_k = -e_k A, \text{ and } A \text{ commutes with } e_1, \dots, e_{k-1}\}.$$

One way to at least check the plausibility of this claim is to take a geodesic $t \mapsto e^{tA}$ in $O(M)$, left-multiply by e_k , and then see what conditions on A will result in the geodesic $t \mapsto e_k e^{tA}$ lying in $\mathcal{E}x_k(M)$. The above conditions fall out immediately.

Given such a tangent vector $e_k A$, the corresponding geodesic $t \mapsto e_k e^{tA}$ arrives at $-e_k$ when $t = 1$ precisely when the eigenvalues of A are odd multiples of πi . The *minimal* geodesics from e_k to $-e_k$ correspond to those A where the eigenvalues are $\pm \pi i$. We obtain the following diagram of homeomorphisms:

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{skew-sym } A \text{ with } e_k A = -A e_k, \text{ } A \\ \text{commutes with } e_1, \dots, e_{k-1}, \text{ and has} \\ \text{eigenvalues } \pm \pi i \end{array} \right\} & \xrightarrow{A \mapsto e_k e^{\frac{1}{2}A} = \frac{1}{\pi} e_k A} & \mathcal{E}x_{k+1}(M) \\ \downarrow A \mapsto (t \mapsto e_k e^{tA}) & \nearrow \gamma \mapsto \gamma(\frac{1}{2}) & \\ \{\text{minimal geodesics } e_k \rightarrow -e_k \text{ in } \mathcal{E}x_k(M)\} & & \end{array}$$

Exercise 16.3. Verify that for A as in the upper left corner one has $e^{\frac{1}{2}A} = \frac{1}{\pi}A$ (use diagonalization). Then check that the top horizontal map is a homeomorphism, so that the diagonal map is a homeomorphism as well.

Let $\beta: \mathcal{E}x_{k+1}(M) \rightarrow \Omega_{e_k, -e_k} \mathcal{E}x_k(M)$ be the inverse to the diagonal map in the above diagram. In retrospect, β is easy to describe. It sends an element $\tilde{e}_{k+1}$ to the path $t \mapsto e_k e^{-\pi t e_k \tilde{e}_{k+1}}$, which is equal to the path

$$t \mapsto \cos(\pi t)e_k + \sin(\pi t)\tilde{e}_{k+1}$$

that we have seen before. The Bott-Morse arguments (which we omit) show that β is an n_k -equivalence for a certain integer n_k that goes to infinity as $\dim M \rightarrow \infty$. So we next stabilize, though before doing so we need a brief aside.

16.4. Finite codimension subspaces.

Exercise 16.5. Let $\mathcal{U}$ be an inner product space having countably infinite dimension (over either $\mathbb{R}$ or $\mathbb{C}$). Consider the following conditions on a subspace $V \subseteq \mathcal{U}$:

- (1) $V^\perp$ is finite-dimensional.
- (2) $\mathcal{U}/V$ is finite-dimensional.
- (3) $V = W^\perp$ for some finite-dimensional subspace $W \subseteq \mathcal{U}$.

Prove that $(3) \Rightarrow (2) \Rightarrow (1)$, but that neither implication is reversible. Show that $\text{Span}\{e_j - e_{j+1} \mid j \geq 1\} \subseteq \mathbb{R}^\infty$ is a counterexample to $(2) \Rightarrow (3)$. Identify $\mathbb{R}^\infty$ with $\mathbb{R}[t]$, and e_j with t^j . Show that the span of all even powers of $(1+t)$ gives a counterexample to $(1) \Rightarrow (2)$. [Hint for the last part: if $p(t) \in V^\perp$, construct a related polynomial that vanishes at every even integer.]

In the context of Exercise 16.5, say that a subspace $V \subseteq \mathcal{U}$ is of **orthogonally finite codimension** if it satisfies condition (3). We will also write “o-finite codimension”, or even just “ofc”, for short. Be warned that papers in the literature sometimes say “finite codimension” when they really need this stronger notion.

Exercise 16.6. Let $\mathcal{U}$ be an inner product space of countably infinite dimension.

- (a) Suppose $V \subseteq W \subseteq \mathcal{U}$ are subspaces, and that $W \subseteq \mathcal{U}$ is ofc. Prove that $V \subseteq W$ is ofc if and only if $V \subseteq \mathcal{U}$ is ofc.
- (b) Let $V \subseteq \mathcal{U}$ and $W \subseteq \mathcal{U}$. If both are ofc, prove that $V \cap W \subseteq \mathcal{U}$ is ofc.
- (c) Let $V \subseteq \mathcal{U}$ be ofc. Let $F: \mathcal{U} \rightarrow \mathcal{U}$ be an orthogonal transformation such that $F^n = \text{id}$ for some n . Prove that there is an ofc subspace $W \subseteq V$ such that $F(W) = W$.
- (d) Suppose that $\mathcal{U}$ is an (orthogonal) Cl_n -module. If $V \subseteq \mathcal{U}$ is ofc, prove that there is an ofc subspace $W \subseteq V$ such that W is a Cl_n -submodule of $\mathcal{U}$.

16.7. Stabilization. Observe that if M' is another Cl_r -module then we have natural maps $\mathcal{E}x_k(M) \rightarrow \mathcal{E}x_k(M \oplus M')$ sending $\tilde{e}_k$ to $\tilde{e}_k \oplus e_k$. That is, we use the “new” e_k -structure on the M summand and the “old” e_k -structure (the one from the Cl_r -structure) on the M' summand.

Let $\mathbb{M} = \{M_i\}$ be a sequence of finite-dimensional representations of Cl_r , and let $\mathcal{E}x_k(\mathbb{M})$ be the colimit of the directed system

$$\mathcal{E}x_k(M_1) \rightarrow \mathcal{E}x_k(M_1 \oplus M_2) \rightarrow \mathcal{E}x_k(M_1 \oplus M_2 \oplus M_3) \rightarrow \cdots \quad \Rightarrow \mathcal{E}x_k(\mathbb{M}).$$

If we also write $\mathbb{M}$ for the infinite sum $\bigoplus_{i=1}^{\infty} M_i$, then $\mathcal{E}x_k(\mathbb{M})$ can be alternatively described—as a set—as

$$\mathcal{E}x_k(\mathbb{M}) = \{\tilde{e}_k \in O_{\omega}(\mathbb{M}) \mid e_1, \dots, e_{k-1}, \tilde{e}_k \text{ is a } \text{Cl}_k\text{-structure on } \mathbb{M} \text{ and } \tilde{e}_k \text{ agrees with } e_k \text{ on a subspace of } \mathfrak{o}\text{-finite codimension}\}$$

where $O_{\omega}(\mathbb{M})$ is the group of orthogonal maps $\mathbb{M} \rightarrow \mathbb{M}$. Note the subtlety here that $O_{\omega}(M) \neq O(M)$, where $O(M) = \text{colim}_k O(\bigoplus_{i=1}^k M_i)$. Elements of $O(M)$ are equal to the identity on a subspace of finite codimension, and so we do not have $\tilde{e}_k \in O(M)$. But use of $O_{\omega}(M)$ here leads to problems with getting the topology correct. To resolve this, consider the following diagram:

$$\begin{array}{ccccccc} \mathcal{E}x_k(M_1) & \xrightarrow{\quad} & \mathcal{E}x_k(M_1 \oplus M_2) & \xrightarrow{\quad} & \cdots & \Longrightarrow & \mathcal{E}x_k(\mathbb{M}) \\ \downarrow & & \downarrow & & & & \downarrow \\ O(M_1) & \xrightarrow{\oplus e_k} & O(M_1 \oplus M_2) & \xrightarrow{\oplus e_k} & \cdots & & \\ \downarrow (-e_k) \cdot & & \downarrow (-e_k) \cdot & & & & \downarrow \\ O(M_1) & \xrightarrow{\oplus \text{id}} & O(M_1 \oplus M_2) & \xrightarrow{\oplus \text{id}} & \cdots & \Longrightarrow & O(\mathbb{M}). \end{array}$$

The vertical maps $(-e_k) \cdot$ were chosen because they convert the $\oplus(e_k)$ into $\oplus(\text{id})$. Note that these vertical maps are homeomorphisms. Using this diagram, we can write

$$\mathcal{E}x_k(\mathbb{M}) = \{w \in O(\mathbb{M}) \mid e_1, \dots, e_{k-1}, e_k w \text{ is a } \text{Cl}_k\text{-structure on } \mathbb{M}\}$$

and now this is an identity of topological spaces when we give the right-hand-side the subspace topology from $O(\mathbb{M})$.

Stabilizing the Bott maps here is a little tricky, because the map $j: \mathcal{E}x_k(M) \rightarrow \mathcal{E}x_k(M \oplus M')$ sends $e_k \mapsto e_k \oplus e_k$ but $-e_k \mapsto (-e_k) \oplus e_k \neq -(e_k \oplus e_k)$. That is, $j(-e_k) \neq -j(e_k)$. So it does not send paths $e_k \rightarrow -e_k$ to paths $e_k \rightarrow -e_k$. Some of the tricks we used in the previous section—in the case of U and O —to change from paths to loops don't readily solve the problem here, essentially because $E_k(M)$ doesn't have a multiplication on it. The following seems to be the best we can do.

Let $k \leq r-1$ and consider the system

$$\begin{array}{ccccccc} \mathcal{E}x_{k+1}(M_1) & \longrightarrow & \mathcal{E}x_{k+1}(M_1 \oplus M_2) & \longrightarrow & \mathcal{E}x_{k+1}(M_1 \oplus M_2 \oplus M_3) & \longrightarrow & \cdots \\ \beta \downarrow & & \beta \downarrow & & \beta \downarrow & & \\ \Omega_{e_k}(\mathcal{E}x_k(M_1)) & \longrightarrow & \Omega_{e_k}(\mathcal{E}x_k(M_1 \oplus M_2)) & \longrightarrow & \Omega_{e_k}(\mathcal{E}x_k(M_1 \oplus M_2 \oplus M_3)) & \longrightarrow & \cdots \end{array}$$

The horizontal maps are all induced by the inclusions, but we need to explain the Bott maps β . In each setting, let γ denote the map $t \mapsto \cos(\pi t)e_k + \sin(\pi t)e_{k+1}$, and regard this as the canonical path from e_k to $-e_k$ in $\mathcal{E}x_k(M)$ (whatever M is). Define the Bott map β to send $\tilde{e}_{k+1}$ to the concatenation of paths

$$\beta(\tilde{e}_{k+1}) = \gamma^{-1} * (t \mapsto \cos(\pi t)e_k + \sin(\pi t)\tilde{e}_{k+1})$$

($f * g$ means the path that travels g first and then follows with f). Note that this path moves from e_k to $-e_k$ via $\tilde{e}_{k+1}$, and then back to e_k via e_{k+1} (this is the γ^{-1} part).

The squares in the above diagram do not commute. Given an $\tilde{e}_{k+1}$, the two loops one gets by going around a square in the two different ways agree on the first part

of the module, but disagree on the summand that is being added on. On this last summand one loop is constant, whereas the other loop follows the canonical path and then its inverse. So the squares commute up to homotopy, and in fact up to a very specific homotopy (the homotopy showing that a path followed by its inverse is null). It follows that we get a map from the telescope (or homotopy colimit) of the top sequence to the colimit of the bottom sequence. Since the telescope collapses down to the honest colimit, let us display all of this via the diagram

$$\mathcal{E}x_{k+1}(\mathbb{M}) \xleftarrow[\simeq]{\pi} \mathrm{Tel}_n \mathcal{E}x_{k+1}(\mathbb{M}_n) \xrightarrow{\beta_\infty} \mathrm{colim}_n \Omega_{e_k} \mathcal{E}x_k(\mathbb{M}_n) \xrightarrow[\simeq]{} \Omega_{e_k} \mathcal{E}x_k(\mathbb{M}).$$

Since the Bott maps $\mathcal{E}x_{k+1}(\mathbb{M}_n) \rightarrow \Omega_{e_k} \mathcal{E}x_k(\mathbb{M}_n)$ are equivalences in a range that goes to infinity with n , the map β_∞ is a weak equivalence. So we have shown that in the sequence of spaces

$$\mathcal{E}x_r(\mathbb{M}), \mathcal{E}x_{r-1}(\mathbb{M}), \dots, \mathcal{E}x_0(\mathbb{M})$$

each space is equivalent to loops on the one to the right, by a very specific zig-zag.

To eliminate the need to stop at $\mathcal{E}x_r$, we do the following. Start with $\mathbb{M} = \mathbb{R}^\infty$, then group the summands by twos and give them the usual complex structure. Then group the $\mathbb{C}$ -summands by twos and give the groupings the usual quaternionic structure. Take the first $\mathbb{H}$ and give it the Cl_3 -structure of $\mathbb{H}_+$, the second $\mathbb{H}$ and give it the Cl_3 -structure of $\mathbb{H}_-$, alternating back and forth forever. Continue inductively in the evident manner, so that in the end $\mathbb{R}^\infty$ is equipped with a Cl_∞ -structure. Then we have the spaces $\mathcal{E}x_k(\mathbb{M})$ for all k , and our previous arguments give specific equivalences (or really, a zig-zag of equivalences) between each space and loops on the one to its right:

$$\dots, \mathcal{E}x_2(\mathbb{M}), \mathcal{E}x_1(\mathbb{M}), \mathcal{E}x_0(\mathbb{M}) \quad \mathcal{E}x_{k+1}(\mathbb{M}) \xrightarrow[\simeq]{} \Omega \mathcal{E}x_k(\mathbb{M}).$$

In this way we have obtained ‘algebraic’ models for all of the iterated loop spaces of $\mathcal{E}x_0(\mathbb{M}) \simeq O$.

Note that one can do these same constructions using the complex Clifford modules, thereby obtaining a sequence of algebraic models for the iterated loop spaces of U .

It remains to identify the homotopy types of the spaces $\mathcal{E}x_k(M)$ with something familiar (and similarly for the direct limit $\mathcal{E}x_k(\mathbb{M})$, of course). To this end, define the group of Clifford automorphisms

$$CA_k(M) = \{f \in O(M) \mid f \text{ is a } \mathrm{Cl}_k\text{-linear automorphism}\}.$$

Note that

$$O(M) = CA_0(M) \supseteq CA_1(M) \supseteq CA_2(M) \supseteq \dots$$

as well as the evident identifications $CA_1(M) = U(M)$ and $CA_2(M) = Sp(M)$. The group $CA_{k-1}(M)$ acts on $\mathcal{E}x_k(M)$ by conjugation: $f \cdot \tilde{e}_k$ is the composite $f \circ \tilde{e}_k \circ f^{-1}: M \rightarrow M$. The stabilizer of the point e_k is $CA_k(M)$, so we get the injective map

$$(16.8) \quad CA_{k-1}(M)/CA_k(M) \longrightarrow \mathcal{E}x_k(M), \quad [f] \mapsto f e_k f^{-1}.$$

We have seen these maps before in the cases $k = 1$ and $k = 2$, where they were actually homeomorphisms. For example, the case $k = 1$ is the identification of $O(M)/U(M)$ with the space of complex structures on M . But they are not always homeomorphisms, as the following example shows.

Example 16.9. Consider the Cl_3 -module $M = \mathbb{H}$ where the action of e_1 and e_2 is that of i and j , respectively, and e_3 acts as e_1e_2 . We will examine

$$CA_2(M)/CA_3(M) \hookrightarrow \mathcal{E}_{\mathbf{x}_3}(M).$$

An easy exercise shows that there are exactly two e_3 -structures extending the Cl_2 -action, namely $e_3 = \pm e_1e_2$. [The action of $e_1e_2e_3$ squares to 1 and is $\mathbb{H}$ -linear, so it is either id or $-\text{id}$]. So $\mathcal{E}_{\mathbf{x}_3}(M)$ consists of exactly two points.

The group $CA_2(M)$ is just $Sp(1)$, the group of unit quaternions, and $CA_3(M)$ is the same group since if a map respects the e_1 - and e_2 -actions on M then it also must respect e_1e_2 . So $CA_2(M)/CA_3(M)$ is a single point. The image in $\mathcal{E}_{\mathbf{x}_3}(M)$ consists of the e_3 -structure $e_3 = e_1e_2$.

For a similar example where the spaces are bigger, let $M = \mathbb{H}_+ \oplus \mathbb{H}_-$ where e_3 acts on the first factor as e_1e_2 and on the second factor as $-e_1e_2$. In this case it turns out that

$$\mathcal{E}_{\mathbf{x}_3}(M) = * \amalg S^4 \amalg *.$$

The first point is the structure $e_3 = e_1e_2$, and the last point is the structure $e_3 = -e_1e_2$. The S^4 is really $\mathbb{H}P^1$, and coincides with e_3 -structures that are equal to e_1e_2 on a 1-dimensional $\mathbb{H}$ -subspace V and equal to $-e_1e_2$ on $V^\perp$.

The group $CA_2(M)$ is $Sp(2)$, and the subgroup $CA_3(M)$ is $Sp(1) \times Sp(1)$: for the latter, an $\mathbb{H}$ -linear map $f: M \rightarrow M$ commutes with the e_3 -structure if and only if it preserves the subspaces $\mathbb{H}_+$ and $\mathbb{H}_-$. The map $CA_2(M)/CA_3(M) \rightarrow \mathcal{E}_{\mathbf{x}_3}(M)$ is a homeomorphism onto the second component, but of course is not surjective. (We will soon see that a better way to think about $\mathcal{E}_{\mathbf{x}_3}(M)$ is via the formula $\mathcal{E}_{\mathbf{x}_3}(M) = \text{Gr}_2^{\mathbb{H}}(M) \amalg \text{Gr}_1^{\mathbb{H}}(M) \amalg \text{Gr}_0^{\mathbb{H}}(M)$, as the e_3 -structure is entirely specified by the quaternionic subspace where e_3 acts as e_1e_2).

The above example reveals the key issue. Sometimes the Cl_{k-1} -module M admits different e_k -structures that lead to non-isomorphic Cl_k -modules, and when this happens the space $\mathcal{E}_{\mathbf{x}_k}(M)$ has extra summands. In fact, we know the group $CA_{k-1}(M)$ acts on $\mathcal{E}_{\mathbf{x}_k}(M)$ and therefore $\mathcal{E}_{\mathbf{x}_k}(M)$ will be a disjoint union of orbits, each of which is a homogeneous space for $CA_{k-1}(M)$. Two e_k -structures (call them e_k and e'_k) are in the same orbit if and only if there is a Cl_k -isomorphism between M and M' . Thus, we obtain the following:

Proposition 16.10. *Let $\mathcal{J}(M)$ be the set of isomorphism classes of Cl_k -module structures on M extending the given Cl_{k-1} -structure. For each element of $\mathcal{J}(M)$ choose a specific e_k -structure representing it, and let $CA_k(M; e_k) \subseteq CA_{k-1}(M)$ be the subgroup of automorphisms that preserve this structure. Then there is a homeomorphism*

$$\coprod_{\mathcal{J}(M)} CA_{k-1}(M)/CA_k(M; e_k) \xrightarrow{\cong} E_k(M)$$

sending the coset $f \cdot CA_k(M; e_k)$ to fe_kf^{-1} , for all $f \in CA_{k-1}(M)$.

Exercise 16.11. Prove the above proposition.

Corollary 16.12. *Let M be a finite-dimensional Cl_k -module. When $k \not\equiv 3 \pmod{4}$ the map $CA_{k-1}(M)/CA_k(M; e_k) \rightarrow \mathcal{E}_{\mathbf{x}_k}(M)$ is a homeomorphism.*

Proof. Taking a look back at Table 16.1, when $k \not\equiv 3 \pmod{4}$ the isomorphism type of a Cl_k -module is completely determined by its dimension. So in those cases the set $\mathcal{J}(M)$ will be a singleton. $\square$

16.13. Understanding Clifford module structures. Let M be a real vector space. Giving a Cl_2 -structure on M is the same as giving M the structure of an $\mathbb{H}$ -vector space, so we understand that fairly concretely. What does one need to specify in order to extend this to a Cl_3 -structure? Rather than consider e_3 , it is helpful to look instead at the element $y = e_1 e_2 e_3$. Note that giving the Cl_3 -structure is the same as specifying left-multiplication by y . But $y^2 = 1$, so the eigenvalues of (multiplication by) y are ± 1 ; so M will decompose as the orthogonal sum of the two eigenspaces V_+ and V_- . Since y commutes with e_1 and e_2 (check this), each of V_+ and V_- is closed under the $\mathbb{H}$ -action. One can readily reverse this reasoning and see that specifying V_+ is *equivalent* to specifying $y \cdot$ (note that $V_- = V_+^\perp$). So extending the Cl_2 -structure to a Cl_3 -structure is equivalent to giving an $\mathbb{H}$ -subspace V of M .

If the Cl_3 -structure extends to a Cl_4 -structure then e_4 —which must anticommute with $e_1 e_2 e_3$ —must send V_+ to V_- and vice versa. So such an extension can only occur when $\dim V_+$ is exactly half of $\dim M$, and when this condition is satisfied giving the e_4 -structure is equivalent to specifying an $\mathbb{H}$ -linear isomorphism $V_+ \rightarrow V_-$.

One can repeat the above style of argument to inductively analyze what is required to extend a Cl_r -structure to a Cl_{r+1} -structure. There is some slight cleverness in finding an analog of y in each case, but essentially the first thing one tries always works. Table 16.13 and Figure 16.14.1 summarize the results, and we leave it to the reader to work out the details as an exercise (but we give some hints). When some Clifford module structures do not extend to the next row, we indicate this—so if there is no notation, all extend. Also, we write $A^{\perp B}$ for the orthogonal complement of A inside of B .

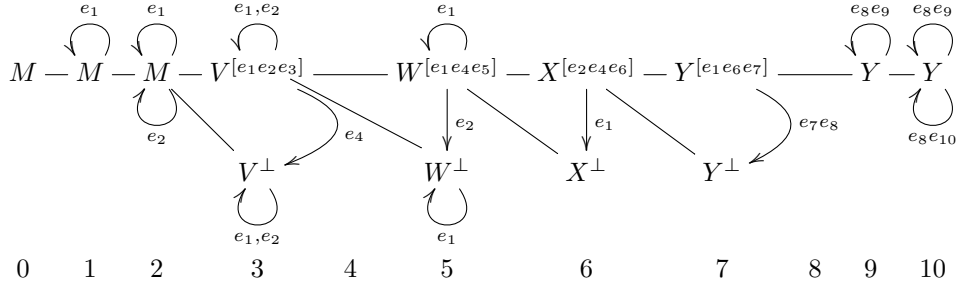
TABLE 16.13. Giving Clifford module structures

r	Equivalent description of a Cl_r -structure on M
0	$\mathbb{R}$ -module (extends only when $\dim$ is even)
1	$\mathbb{C}$ -module (extends only when $\dim_{\mathbb{C}}$ is even)
2	$\mathbb{H}$ -module
3	$\mathbb{H}$ -module together with $\mathbb{H}$ -submodule $V \subseteq M$ (extends only when $\dim V$ is half of $\dim M$) [V is the $+1$ eigenspace of $e_1 e_2 e_3$]
4	above together with an $\mathbb{H}$ -linear isomorphism $V \rightarrow V^\perp$ (always extends) [the isomorphism is multiplication by e_4]
5	above together with a $\mathbb{C}$ -submodule $W \subseteq V$ such that $e_2 W = W^{\perp V}$ [W is the $+1$ -eigenspace of $e_1 e_4 e_5$]
6	above together with a real subspace $X \subseteq W$ such that $e_1 X = X^{\perp W}$ [X is the $+1$ -eigenspace of $e_2 e_4 e_6$]
7	above together with a real subspace $Y \subseteq X$ (extends only when $\dim Y = \frac{1}{2} \dim X$) [Y is the $+1$ -eigenspace of $e_1 e_6 e_7$]
8	above together with an $\mathbb{R}$ -linear isomorphism $Y \rightarrow Y^{\perp X}$ (extends when $\dim Y$ is even) [the isomorphism is multiplication by $e_7 e_8$]
9	above together with a complex structure on Y [given by $e_8 e_9$] (extends when $\dim_{\mathbb{C}} Y$ is even)
10	above together with a quaternionic structure on Y extending the $\mathbb{C}$ -structure [given by $e_8 e_{10}$]

Remark 16.14. One can start to see the 8-fold quasi-periodicity in this chart. For bookkeeping purposes imagine a row -1 with a purely fictitious e_{-1} operation. Then in row $8n + r$ for $0 \leq r \leq 7$ we take the statement in row r and change every word $e_{i_1} \cdots e_{i_s}$ to $e_{8n} e_{8n+i_1} \cdots e_{8n+i_s}$. (One can also imagine e_0 appearing in the rows where $n = 0$, but with the convention that $e_0 = 1$).

Figure 16.14.1 below gives another way of keeping track of what is happening here. Columns 0 through n show the deconstruction of a Cl_n -module M in terms of the spaces $M \supseteq V \supseteq W \supseteq X \supseteq Y$. (Warning: This is obvious once you see it, but take note that the map from V to $V^\perp$ only shows up at level 4, not at level 3.)

FIGURE 16.14.1. Deconstruction of a Clifford module



Exercise 16.15. Justify all of the lines in Table 16.13. The following notes—also indexed by r to match the table—give short guides:

- (3) $(e_1e_2e_3)^2 = 1$, so the eigenvalues of $e_1e_2e_3$ are ± 1 and M is the orthogonal sum of the eigenspaces. Since $e_1e_2e_3$ commutes with e_1 and e_2 , these eigenspaces are closed under the $\mathbb{H}$ -action.
- (4) e_4 anticommutes with $e_1e_2e_3$, and so sends the $+1$ -eigenspace V to the -1 -eigenspace $V^\perp$. [In particular, an e_4 -structure exists only when V and $V^\perp$ have the same dimension].
- (5) $(e_1e_4e_5)^2 = 1$ and $e_1e_4e_5$ commutes with $e_1e_2e_3$.
- (6) $e_2e_4e_6$ squares to 1 and commutes with $e_1e_2e_3$, $e_1e_4e_5$.
- (7) $e_1e_6e_7$ squares to 1 and commutes with $e_1e_2e_3$, $e_1e_4e_5$, and $e_2e_4e_6$.
- (8) e_7e_8 anticommutes with $e_1e_6e_7$, but commutes with the three triples listed in the previous line.

Continue far enough that you believe the quasi-periodicity. (The following might be helpful. Say that a monomial in the e_i 's is even/odd depending on the number of terms, and say that two monomials overlap if they have common factors in their shortest presentation. Prove that an even monomial commutes with any other monomial that doesn't overlap with it, and that if A and B are odd monomials that do not overlap with e_i then $[e_iA, e_iB] = [A, B]$.)

Based on the above work, we can identify the spaces $\mathcal{E}_{X_k}(M)$ for all k . To help with this, the following result will be useful:

Proposition 16.16. *One has the following homeomorphisms:*

$$\begin{aligned} \{\text{orthogonal complex structures on } \mathbb{R}^{2n}\} &\cong O(2n)/U(n) \\ \{\text{rank } n \text{ real subspaces } H \text{ of } \mathbb{C}^n \text{ s.t. } iH = H^\perp\} &\cong U(n)/O(n) \\ \{\text{orthogonal } \mathbb{H}\text{-structures on } \mathbb{C}^{2n} \text{ extending the complex structure}\} &\cong U(2n)/Sp(n) \\ \{\text{rank } n \text{ complex subspaces } H \text{ of } \mathbb{H}^n \text{ s.t. } jH = H^\perp\} &\cong Sp(n)/U(n) \end{aligned}$$

Proof. The first and the third have similar proofs. For the third, $U(2n)$ acts on the space of quaternionic structures by conjugation. By Lemma 15.15 the action is transitive. The stabilizer of a fixed quaternionic structure is precisely $Sp(n)$, so this completes the identification.

For the second homeomorphism, observe that $U(n)$ acts on the left space. We will show the action is transitive and calculate the stabilizer of $\mathbb{R}^n$. Let V and W be two rank n real subspaces of $\mathbb{C}^n$ such that $iV = V^\perp$ and $iW = W^\perp$. Note that $\mathbb{C}^n = V \oplus iV = W \oplus iW$. Choose orthonormal $\mathbb{R}$ -bases for each of V and W , and note that these are necessarily unitary $\mathbb{C}$ -bases for $\mathbb{C}^n$. Therefore the $\mathbb{C}$ -linear transformation $L: \mathbb{C}^n \rightarrow \mathbb{C}^n$ sending the first basis to the second is unitary. This proves that $U(n)$ acts transitively on the given space. The stabilizer of the subspace $\mathbb{R}^n \subseteq \mathbb{C}^n$ is precisely $O(n)$. The proof for the fourth homeomorphism is identical. $\square$

Remark 16.17. For a Hermitian inner product space V , a real subspace H such that $iH = H^\perp$ is sometimes called a “real structure” on V . Likewise, if V is a quaternionic inner product space then a complex subspace $H \subseteq V$ such that $jH = H^\perp$ is called a “complex structure” on V . Using this terminology, the results of Proposition 16.16 take on a nice symmetry.

Proposition 16.18. *Let M be a Cl_k -module, with $\dim_{\mathbb{R}} M = n$. Then one has the following identifications:*

k	$\mathcal{E}x_k(M)$	conditions	$CA_k(M)$
0	$O(n)$		$O(n)$
1	$O(n)/U(\frac{n}{2})$	n is even	$U(\frac{n}{2})$
2	$U(\frac{n}{2})/Sp(\frac{n}{4})$	n is a multiple of 4	$Sp(\frac{n}{4})$
3	$\text{Gr}_{\bullet}^{\mathbb{H}}(\mathbb{H}^{\frac{n}{4}})$	n is a multiple of 4	$Sp(V) \times Sp(V^\perp)$
4	$Sp(\frac{n}{8})$	n is a multiple of 8	$Sp(V) = Sp(\frac{n}{8})$
5	$Sp(\frac{n}{8})/U(\frac{n}{8})$	n is a multiple of 8	$U(W) = U(\frac{n}{8})$
6	$U(\frac{n}{8})/O(\frac{n}{8})$	n is a multiple of 8	$O(U) = O(\frac{n}{8})$
7	$\text{Gr}_{\bullet}^{\mathbb{R}}(\mathbb{R}^{\frac{n}{8}})$	n is a multiple of 8	$O(T) \times O(T^{\perp U})$
8	$O(\frac{n}{16})$	n is a multiple of 16	$O(T) = O(\frac{n}{16})$

Here $\text{Gr}_{\bullet}^F(N)$ denotes the space of all F -subspaces of N , and so is a disjoint union with components indexed by the possible F -dimensions of such subspaces. The spaces V , W , U , and T are as in Table 16.13. Note that the space $\mathcal{E}x_k(M)$ only depends on the Cl_{k-1} -structure of M .

Proof. The identification of the $\mathcal{E}x_k(M)$ spaces follows immediately from the information in Table 16.13. The $k = 1, 2, 5, 6$ cases use Proposition 16.16, whereas

$k = 3, 7$ are self-evident. For $k = 4$ one notes that the space of $\mathbb{H}$ -linear isomorphisms $V \rightarrow V^\perp$ is (when nonempty) a torsor for the group $Sp(V)$ via precomposition. Similar for $k = 8$.

The identification of the groups in $CA_k(M)$ is also mostly immediate from Table 16.13, with the exceptions of $k = 4, 8$. For $k = 4$, the automorphisms of V and $V^\perp$ from $CA_3(M)$ must be compatible with the isomorphism $e_4 \cdot : V \rightarrow V^\perp$, which is equivalent to saying that the automorphism of $V^\perp$ is determined by the one on V . So $CA_4(M) = Sp(V)$ and the map $CA_4(M) \rightarrow CA_3(M)$ is $f \mapsto (f, x \mapsto -e_4 f(e_4 x))$. The analysis for $k = 8$ is entirely similar. $\square$

Taking the results of Proposition 16.18 and stabilizing, we can now complete the identification of the iterated loop spaces of O . In the following chart we start with the rightmost O and taking loops moves us to the left:

$$\begin{array}{cccccccccc} \cdots & \mathcal{E}_{x_8} \mathbb{M} & \mathcal{E}_{x_7} \mathbb{M} & \mathcal{E}_{x_6} \mathbb{M} & \mathcal{E}_{x_5} \mathbb{M} & \mathcal{E}_{x_4} \mathbb{M} & \mathcal{E}_{x_3} \mathbb{M} & \mathcal{E}_{x_2} \mathbb{M} & \mathcal{E}_{x_1} \mathbb{M} & \mathcal{E}_{x_0} \mathbb{M} \\ \cdots & O & \mathbb{Z} \times BO & U/O & Sp/U & Sp & \mathbb{Z} \times BSp & U/Sp & O/U & O \end{array}$$

The only slightly subtle point in the stabilization is what happens for $k = 3, 7$. But here one gets a bigraded lattice as in (15.6) and the analysis exactly follows what was done there. See Section 17.6 below, though, for a less ad hoc approach to stabilization.

Remark 16.19 ($*$ -algebras and modules). Throughout this section we always concentrated on orthogonal Clifford module structures. This was not absolutely necessary, but is convenient because it produces objects that connect to the compact groups O_n , U_n , and Sp_n rather than $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, and $GL_n(\mathbb{H})$. But the constant insistence on orthogonal actions can feel a little clunky.

Here is some different language that accomplishes the same thing. Let M be a finite-dimensional real inner product space, and let $\text{End}_{\mathbb{R}}(M)$ be the algebra of $\mathbb{R}$ -linear endomorphisms. Equip $\text{End}_{\mathbb{R}}(M)$ with the involution $f \mapsto f^\dagger$. Recall that the adjoint $f^\dagger$ is the linear map uniquely characterized by $\langle f^\dagger x, y \rangle = \langle x, fy \rangle$, and in terms of matrices with respect to an orthonormal basis this just amounts to taking the transpose.

The pair $(\text{End}_{\mathbb{R}}(M), \dagger)$ is a prototypical example of a $*$ -algebra, which is an $\mathbb{R}$ -algebra A equipped with a conjugation $*$: $A \rightarrow A$ that is an $\mathbb{R}$ -linear anti-involution; the “anti” part is the identity $(xy)^* = y^*x^*$. As it turns out, the Clifford algebras come with natural structures of $*$ -algebras where one sets $e_i^* = -e_i$ (note that this determines the $*$ -operation on all other elements). We say that M is a $*$ -module for Cl_k if the action map $\rho: \text{Cl}_k \rightarrow \text{End}_{\mathbb{R}}(M)$ is a map of $*$ -algebras; in addition to being a module in the ordinary sense this amounts to the extra condition that $\rho(e_i^*) = \rho(e_i)^\dagger$ for all i . But because $e_i^* = -e_i = e_i^{-1}$, this is the condition that $\rho(e_i)^{-1} = \rho(e_i)^\dagger$ —equivalently, $\rho(e_i)$ is orthogonal in the sense of preserving the inner product.

The upshot is that the “orthogonal Clifford modules” we have been considering can also be described as the $*$ -modules for the Clifford $*$ -algebras. There is nothing deep here—it is just a shift of language—but the theory of $*$ -algebras and $*$ -modules can feel a bit less ad hoc, and connects to the vast theory of C^* -algebras in analysis.

Exercise 16.20. For the $*$ -operation on Cl_n defined in Remark 16.19, check that $(e_{i_1} \cdots e_{i_k})^* = (-1)^{\binom{k+1}{2}} e_{i_1} \cdots e_{i_k}$ when the i_j are all distinct.

17. THE CLIFFORD MENAGERIE

One thing that is missing from our story so far is a “natural” construction of the K -theory spectra K and KO . Clifford algebras have led us to models for the iterated loop spaces on U and O , but to write down a spectrum one needs iterated deloopings of $\mathbb{Z} \times BU$ and $\mathbb{Z} \times BO$. The periodicity theorems allow us to recast the iterated loops on O as iterated deloopings of $\mathbb{Z} \times BO$, but this approach feels a bit unsatisfying.

The next remark will at first seem unconnected, but now that Clifford algebras have been allowed in the door it turns out there is much more one can do with them. Instead of the singly-indexed family of Clifford algebras we have studied so far, there is a natural bigraded family—this comes from the usual bigraded family of nondegenerate quadratic forms over the real numbers. Karoubi showed how to use this full spectrum of Clifford algebras to produce the required deloopings and the associated models of K -theory spectra. The current section recounts this story.

17.1. The bigraded real Clifford algebras. The Clifford algebra construction is something that can be applied to any vector space equipped with a quadratic form, over any field. Given (V, q) we can define the Clifford algebra as a quotient of the tensor algebra $T(V)$:

$$\mathrm{Cl}(V, q) = T(V) / \langle v \otimes v = -q(v) \mid v \in V \rangle.$$

Over the real numbers every nondegenerate quadratic form is isomorphic to one of the canonical types

$$q(x_1, \dots, x_{p+q}) = x_1^2 + \dots + x_p^2 - (x_{p+1}^2 + \dots + x_{p+q}^2)$$

for integers $p, q \geq 0$. The Clifford algebra associated to this form will be denoted $\mathrm{Cl}_{p,q}$. Note that $\mathrm{Cl}_{p,0}$ is the algebra that we have previously been calling Cl_p , whereas $\mathrm{Cl}_{0,q}$ is an algebra we saw in Section ?? and denoted Cl_q^- there.

One can readily check that $\mathrm{Cl}_{p,q}$ is isomorphic to the quotient of the tensor algebra $\mathbb{R}\langle e_1, \dots, e_p, f_1, \dots, f_q \rangle$ by the relations saying that $e_i^2 = -1$, $f_j^2 = 1$, and any two distinct e - or f -variables anti-commute with each other. We will often find this model particularly convenient.

Computing the $\mathrm{Cl}_{p,q}$ algebras explicitly is just as easy as the computation of the $\mathrm{Cl}_k^\pm$ algebras that we did in Section ??. We can do it with only one additional fact:

Proposition 17.2. *For any $p, q \geq 0$ there is an isomorphism of $\mathbb{R}$ -algebras $\mathrm{Cl}_{p+1,q+1} \rightarrow \mathrm{Cl}_{p,q}(2)$ given by $e_{p+1} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $f_{q+1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $e_i \mapsto \begin{bmatrix} 0 & e_i \\ e_i & 0 \end{bmatrix}$, $f_j \mapsto \begin{bmatrix} 0 & f_j \\ f_j & 0 \end{bmatrix}$ for $i \leq p$ and $j \leq q$.*

The proof is straightforward once one has the above formulas for the isomorphism, but before giving the proof let us explain where those formulas come from. Let N be a $\mathrm{Cl}_{p+1,q+1}$ -module. The difference between $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{p+1,q+1}$ lies in the two generators e_{p+1} and f_{q+1} . Since $f_{q+1}^2 = 1$ the underlying vector space of N splits as $V_+ \oplus V_-$, where the summands are the $+1$ and -1 eigenspaces for f_{q+1} . The other e 's and f 's anticommute with f_{q+1} and so they map V_+ to V_- and vice versa. But we can use e_{p+1} to identify V_+ and V_- , and in that way we can think of the generators of $\mathrm{Cl}_{p,q}$ as operating on just one of these— V_+ , say. More precisely, define

$\tilde{e}_i = e_i e_{p+1}$ and $\tilde{f}_j = f_j e_{p+1}$ and check that these elements define an action of $\text{Cl}_{p,q}$ on V_+ . Set $M = V_+$ with this action. Then $N = M \oplus e_{p+1}M$, and if we identify the underlying vector space with $M \oplus M$ in the evident way then f_{q+1} acts as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, e_{p+1} acts as the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, e_i acts as $\begin{bmatrix} 0 & \tilde{e}_i \\ \tilde{e}_i & 0 \end{bmatrix}$, and similarly for the f_j . Since it is easy to get confused about signs here, let us justify the nonzero entries in the matrix for e_i as an example: $e_i m = e_{p+1} e_i e_{p+1} m = e_{p+1}(\tilde{e}_i m)$ and $e_i(e_{p+1} m) = \tilde{e}_i m$.

Proof of Proposition 17.2. It is routine to check that the given matrices satisfy the defining relations for the Clifford algebras, so that we do indeed get a map of $\mathbb{R}$ -algebras. Taking products of the given matrices readily shows that we get all matrices of forms

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \quad \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}, \quad \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$$

where x is a product of e 's and f 's. Such matrices generate $\text{Cl}_{p,q}(2)$ as a vector space, and so our map is surjective. Both the domain and codomain have dimension 2^{p+q+2} as vector spaces, and so the map is injective as well. $\square$

Morita theory tells us that the module categories of $\text{Cl}_{p,q}$ and $\text{Cl}_{p,q}(2) \cong \text{Cl}_{p+1,q+1}$ are equivalent, though we basically proved this by hand in our above discussion. But let us record the result for future reference:

Proposition 17.3. *There is an equivalence of categories $\text{Cl}_{p,q} - \text{Mod} \rightarrow \text{Cl}_{p+1,q+1} - \text{Mod}$ that sends a module M to $M \oplus M$ with the action of the Clifford generators given by the formulas from Proposition 17.2.*

Recall that we already computed the algebras $\text{Cl}_{p,0}$ and $\text{Cl}_{0,q}$. Proposition 17.2 then lets us fill out the table in Figure 17.3.1.

FIGURE 17.3.1. The Clifford algebras $\text{Cl}_{p,q}$

$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(32) \times \mathbb{H}(32)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128) \times \mathbb{R}(128)$	$\mathbb{R}(256)$
$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(16) \times \mathbb{H}(16)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \times \mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
$\mathbb{H}(4)$	$\mathbb{H}(8) \times \mathbb{H}(8)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \times \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
$\mathbb{H}(4) \times \mathbb{H}(4)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \times \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32) \times \mathbb{H}(32)$
$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \times \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \times \mathbb{H}(16)$	$\mathbb{H}(32)$
$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \times \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \times \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
$\mathbb{R}(2)$	$\mathbb{R}(2) \times \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \times \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
$\mathbb{R} \times \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \times \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \times \mathbb{R}(16)$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \times \mathbb{R}(8)$	$\mathbb{R}(16)$
$\boxed{0} \xrightarrow{p} \boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	

Figure 17.3.1 shows the algebras $\text{Cl}_{p,q}$ where p is on the horizontal axis and q is on the vertical. Observe from the table that if we write $\text{Cl}_{p,q} \cong A(n)$ where n is maximal possible, then the “root” A only depends on $p - q \bmod 8$. This is a

combination of the $(1, 1)$ -periodicity from Proposition 17.2, the 8-fold periodicity of the $\text{Cl}_{p,0}$ and $\text{Cl}_{0,q}$ algebras, and a certain kind of ‘duality’ between the $\text{Cl}_{p,0}$ and $\text{Cl}_{0,q}$ families (they have the same roots but in reverse order). A deeper explanation of what is going on here will come when we discuss the Brauer group of real superalgebras; see Section 20.7.

Given an orthogonal $\text{Cl}_{p,q}$ -module M , let us now write $\mathcal{E}x_{p,q}^{(0,1)}(M)$ for the space of orthogonal $\text{Cl}_{p,q+1}$ -extension structures on M . That is, an element of this space is an $f_{q+1} \in O(M)$ such that $f_{q+1}^2 = \text{id}_M$ and such that f_{q+1} anti-commutes with the action of the generators of $\text{Cl}_{p,q}$ on M . We will likewise write $\mathcal{E}x_{p,q}^{(1,0)}(M)$ for the space of orthogonal $\text{Cl}_{p+1,q}$ -extension structures on M . The spaces $E_{p,0}^{(1,0)}(M)$ were studied in Section 16 and there called $\mathcal{E}x_{p+1}(M)$.

The following result is almost an immediate consequence of Proposition 17.3.

Proposition 17.4 ((1,1)-periodicity). *For any $\text{Cl}_{p,q}$ -module M there are homeomorphisms $\mathcal{E}x_{p,q}^\epsilon(M) \xrightarrow{\cong} \mathcal{E}x_{p+1,q+1}^\epsilon(M \oplus M)$, where ϵ is either $(1, 0)$ or $(0, 1)$ and $M \oplus M$ is given the $\text{Cl}_{p+1,q+1}$ -structure described in Proposition 17.3.*

Proof. A linear map $M \oplus M \rightarrow M \oplus M$ is of the form $(x, y) \mapsto (Ax + By, Cx + Dy)$ for linear maps $A, B, C, D: M \rightarrow M$. The key observation is that such a map anti-commutes with the actions of e_{p+1} and f_{q+1} if and only if $A = D = 0$ and $B = C$. This is an easy computation. Based on this, we define the map

$$\mathcal{E}x_{p,q}^{(0,1)}(M) \xrightarrow{\cong} \mathcal{E}x_{p+1,q+1}^{(0,1)}(M \oplus M)$$

to send an f_{q+1} -structure f to $(x, y) \mapsto (fy, fx)$, which is readily checked to be an f_{q+2} -structure. The above key observation shows that any f_{q+2} -structure on $M \oplus M$ is of this form, and the inverse map is readily constructed.

The exact same proof works for $E^{(1,0)}$ in place of $E^{(0,1)}$. $\square$

Remark 17.5. We have set up the definition of $\mathcal{E}x_{p,q}^\epsilon$ (where ϵ is $(1, 0)$ or $(0, 1)$) so that it involves *orthogonal* Clifford modules structures. Although this is convenient, it also leads to some issues. Dropping the orthogonality conditions yields spaces that are different but that turn out to have the same homotopy type, analogous to $GL_n(\mathbb{R})$ and $O(n)$.

One motivation for dropping the orthogonality condition is to generalize the setup as follows. Let $R \subseteq S$ be an extension of $\mathbb{R}$ -algebras and let $\{s_i\}$ be a set of elements that generates S as an R -algebra. If M is an R -module one can define a space $\mathcal{E}x_{R \rightarrow S}(M)$ of S -module extensions of M . We regard this as a subset of $\prod_i \text{Hom}_{\mathbb{R}}(M, M)$ by recording the action of each s_i , and we give $\mathcal{E}x_{R \rightarrow S}(M)$ the subspace topology induced by the product topology. (Note that in most applications the $\{s_i\}$ will be a finite collection and M will be finite-dimensional over $\mathbb{R}$, so this product is just a finite-dimensional Euclidean space).

One nice benefit of this generalization is that certain results become entirely algebraic. For example, Morita theory shows that the evident map

$$\mathcal{E}x_{R \rightarrow S}(M) \longrightarrow \mathcal{E}x_{R(n) \rightarrow S(n)}(M^{\oplus n})$$

is a homeomorphism. To see this, observe that we can construct a map in the other direction as follows. Given an $S(n)$ -extension structure on $M^{\oplus n}$, give M the

S -extension structure defined by

$$s.m = \pi_1 \left(\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} m \\ 0 \end{bmatrix} \right)$$

(showing the case $n = 2$ for brevity). One of the compositions is then clearly the identity, and demonstrating that the other is the identity is a nice exercise in algebra.

For another result along these lines, consider the diagonal map $R \rightarrow R \times R$. A generator for the target is $f = (1, 0)$, and in fact the target can be identified with $R[f]/(f^2 - 1)$. Giving an R -module M the structure of $R[f]/(f^2 - 1)$ -algebra is the same as specifying an idempotent R -linear map $M \rightarrow M$. But by linear algebra any such map is diagonalizable with possible eigenvalues 1 and -1 , so this is the same as giving two R -submodules M_+ and M_- of M such that $M = M_+ \oplus M_-$ (the eigenspace decomposition of multiplication by f). In this way we identify $\mathcal{E}x_{R \rightarrow R \times R}(M)$ with a subspace of $\text{Gr}_\bullet(M) \times \text{Gr}_\bullet(M)$, where $\text{Gr}_\bullet(M) = \coprod_k \text{Gr}_k(M)$. It is the space of R -module splittings of M .

As a final example, consider $R \times R$ regarded as the subring of diagonal matrices in the matrix algebra $M_{2 \times 2}(R) = R(2)$. Let $f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and write $\text{diag}(a, b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Observe that $f^2 = 1$ and $f \cdot \text{diag}(a, b) = \text{diag}(b, a) \cdot f$ for all $a, b \in R$. Moreover, f generates $R(2)$ as an $R \times R$ -algebra.

Every $R \times R$ -module has the form $M \oplus N$ where M and N are R -modules and $(a, b) \cdot (m, n) = (am, bn)$; so $R \times R$ acts on M via the first coordinate and N via the second. We claim that extending this to an $R(2)$ -structure is the same as specifying an R -linear isomorphism $\phi: M \rightarrow N$. Given such a map, define $f \cdot (m, n) = (\phi^{-1}(n), \phi(m))$ and check that this defines an $R(2)$ -structure. In the other direction, given an $R(2)$ structure one recovers ϕ by $\phi(m) = \pi_2(f \cdot \begin{bmatrix} m \\ 0 \end{bmatrix})$ and it is routine to check that ϕ is an isomorphism.

We conclude that $\mathcal{E}x_{R \times R \rightarrow R(2)}(M \oplus N)$ is the space of R -linear isomorphisms from M to N .

While we will not develop this theory in complete detail, sometimes we will go back and forth between the “orthogonal” situation and the “algebraic” situation when convenient and without much comment.

17.6. Stabilization. Stabilization works as follows. We want to have maps $\mathcal{E}x_{p,q}^\epsilon(M) \rightarrow \mathcal{E}x_{p,q}^\epsilon(M \oplus N)$, where ϵ stands for either $(1, 0)$ or $(0, 1)$. In the latter case, for example, the map needs to take an f_{q+1} -structure on M to one on $M \oplus N$: but while it is clear what to do on the M factor, for N we would need to be given an f_{q+1} -structure to start with. So let us do the following. Let $H_1, \dots, H_r$ be a complete list of irreducible $\text{Cl}_{p,q+1}$ -modules. We know from our identification of the Clifford algebras that r will always be 1 or 2, but that is not important. For each i we obtain maps

$$\theta_i: \mathcal{E}x_{p,q}^{(0,1)}(M) \rightarrow \mathcal{E}x_{p,q}^{(0,1)}(M \oplus H_i)$$

as we described above: $f_{q+1} \mapsto (f_{q+1}, f_{q+1}^{H_i})$ where $f_{q+1}^{H_i}$ comes from the $\text{Cl}_{p,q+1}$ -structure on H_i . We can then form an infinite r -dimensional lattice where the nodes are indexed by the modules $H_1^{\epsilon_1} \oplus \dots \oplus H_r^{\epsilon_r}$ and we have the maps $\theta_1, \dots, \theta_r$ leaving each node. Define the colimit of this diagram to be $\mathcal{E}x_{p,q}^{(0,1)}(\underline{H})$.

Remark 17.7. It was not important that the H_i themselves be irreducible, but rather that every irreducible representation be contained as a summand of some H_i . As long as we have this condition then our lattice will be cofinal in the one above, and so will have the same colimit. This observation will be important a bit later. Since under these hypotheses the colimit does not depend on the specific choice of the H_i 's, we will also write $\mathcal{E}x_{p,q}^{(0,1)}(\infty)$ instead of $\mathcal{E}x_{p,q}^{(0,1)}(\underline{H})$.

If we also use $\underline{H}$ to denote the infinite $\text{Cl}_{p,q+1}$ -module $\bigoplus_i H_i^{\oplus \infty}$, then we can think of the set $\mathcal{E}x_{p,q}^{(0,1)}(\underline{H})$ as the collection of orthogonal f_{q+1} -structures on $\underline{H}$ having the property that $f_{q+1} - f_{q+1}^{\underline{H}}$ has finite rank—or equivalently, the two structures f_{q+1} and $f_{q+1}^{\underline{H}}$ agree on a subspace of finite codimension. To get the topology correct we use a slightly different formulation. Define $O(\underline{H})$ to be $\text{colim}_{r,s} O(\bigoplus_{i=1}^r H_i^{\oplus s})$. Then $\mathcal{E}x_{p,q}^{(0,1)}(\underline{H})$ may be identified with the subspace of $O(\underline{H})$ consisting of elements A such that $f_{q+1}^{\underline{H}} A$ is an f_{q+1} -structure on $\underline{H}$. (Recall that we saw this issue before, back in Section 16).

Observe that the same approach works to define the stabilized spaces $\mathcal{E}x_{p,q}^{(1,0)}(\infty)$ except that here the H 's need to be irreducible representations of $\text{Cl}_{p+1,q}$.

Example 17.8. Let us consider $\mathcal{E}x_{0,0}^{(0,1)}(\infty)$. Here $\text{Cl}_{0,1} \cong \mathbb{R} \times \mathbb{R}$, and so we take $H_1 = \mathbb{R}_+$ and $H_2 = \mathbb{R}_-$ where the actions are $(a,b).r = ar$ and $(a,b).r = br$, respectively. Giving an f_1 -structure on a vector space V is the same as giving a decomposition $V = V_+ \oplus V_-$ where V_+ is the $+1$ -eigenspace of f_1 and V_- is the -1 -eigenspace. This explains the notation $\mathbb{R}_+$ and $\mathbb{R}_-$. Note that when V has an inner product and we are specifying an orthogonal f_1 -structure then we only need to give V_+ , as V_- will be the orthogonal complement.

To ease notation let us just write $\mathcal{E}x_{0,0}^{(0,1)} = G$. By the above remarks, $G(V)$ is the space of subspaces $V_+ \subseteq V$ —that is, $G(V) = \coprod_{k=0}^{\dim V} \text{Gr}_k(V)$. To stabilize we form the bigraded array

$$\begin{array}{ccccccc}
 G(0) & \longrightarrow & G(\mathbb{R}_+) & \longrightarrow & G(\mathbb{R}_+^2) & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 G(\mathbb{R}_-) & \longrightarrow & G(\mathbb{R}_+ \oplus \mathbb{R}_-) & \longrightarrow & G(\mathbb{R}_+^2 \oplus \mathbb{R}_-) & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 G(\mathbb{R}_-^2) & \longrightarrow & G(\mathbb{R}_+ \oplus \mathbb{R}_-^2) & \longrightarrow & G(\mathbb{R}_+^2 \oplus \mathbb{R}_-^2) & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \vdots & & \vdots & & \vdots & &
 \end{array}$$

and $G(\underline{H})$ is the colimit. Since our model for G tracks the positive eigenspace of f_1 , the horizontal maps

$$G(\mathbb{R}_+^a \oplus \mathbb{R}_-^b) \longrightarrow G(\mathbb{R}_+^{a+1} \oplus \mathbb{R}_-^b)$$

send $W \mapsto W \oplus e_{a+1}$ (here e_{a+1} denotes an element of the standard basis, not a Clifford algebra generator) whereas the vertical maps

$$G(\mathbb{R}_+^a \oplus \mathbb{R}_-^b) \longrightarrow G(\mathbb{R}_+^a \oplus \mathbb{R}_-^{b+1})$$

send W to W . Note that in this way we recover the same bigraded diagram that we found by ad hoc methods back in Section 15.4. Our conclusion is that $\mathcal{E}x_{(0,0)}^{(0,1)}(\infty) \cong \mathbb{Z} \times BO$.

17.9. The Bott maps.

If M is a $\text{Cl}_{p,q+1}$ -module then there is a map

$$\beta: E_{p,q+1}^{(0,1)}(M) \longrightarrow \Omega_{f_{q+1}, -f_{q+1}} E_{p,q}^{(0,1)}(M)$$

that sends f_{q+2} to the path $t \mapsto \cos(\pi t)f_{q+1} + \sin(\pi t)f_{q+2}$. Similarly, if M is a $\text{Cl}_{p+1,q}$ -module then we have the map

$$\beta: E_{p+1,q}^{(1,0)}(M) \longrightarrow \Omega_{e_{p+1}, -e_{p+1}} E_{p,q}^{(1,0)}(M)$$

that sends e_{p+2} to the map $t \mapsto \cos(\pi t)e_{p+1} + \sin(\pi t)e_{p+2}$. The Bott arguments show that these are equivalences in a range that goes to infinity as $\dim M$ increases.

Passing to colimits is mostly straightforward. We can choose a collection of finitely-generated $\text{Cl}_{p,q+2}$ -modules $H_1, \dots, H_r$ such that every irreducible is a summand of some H_i . But even more, we can ensure that when we restrict to the $\text{Cl}_{p,q+1}$ -structure it is still true that every irreducible $\text{Cl}_{p,q+1}$ -module is a summand of some H_i . This is essentially automatic because of the way the Clifford modules work, but it is worth a moment's thought. We get a commutative lattice of Bott maps indexed by the modules $H_1^{a_1} \oplus \dots \oplus H_r^{a_r}$, and passing to the colimits gives a weak equivalence

$$(17.10) \quad \mathcal{E}x_{p,q+1}^{(0,1)}(\infty) \xrightarrow{\simeq} \Omega \mathcal{E}x_{p,q}^{(0,1)}(\infty).$$

Note that we are using Remark 17.7 here, to know that the colimit in the target is what we want it to be. We are also using that Ω commutes with filtered colimits, due to the compactness of S^1 .

A similar construction where $\underline{H}$ is a system of $\text{Cl}_{p+1,q}$ -modules gives a weak equivalence

$$\mathcal{E}x_{p+1,q}^{(1,0)}(\infty) \xrightarrow{\simeq} \Omega \mathcal{E}x_{p,q}^{(1,0)}(\infty).$$

17.11. The periodicity theorems and the representing spectrum for K -theory.

The Bott theorems give us equivalences $\mathcal{E}x_{p,q+1}^{(0,1)}(\infty) \xrightarrow{\simeq} \Omega \mathcal{E}x_{p,q}^{(0,1)}(\infty)$, and $(1,1)$ -periodicity gives us homeomorphisms $\mathcal{E}x_{p,q+1}^{(0,1)}(\infty) \cong \mathcal{E}x_{p-1,q}^{(0,1)}(\infty)$. Putting these together gives weak equivalences

$$\mathcal{E}x_{p-1,q}^{(0,1)}(\infty) \xrightarrow{\simeq} \Omega \mathcal{E}x_{p,q}^{(0,1)}(\infty).$$

Fixing q therefore gives us an Ω -spectrum

$$K_{(q)}^{\mathbb{R}} = \left[\mathcal{E}x_{0,q}^{(0,1)}(\infty), \quad \mathcal{E}x_{1,q}^{(0,1)}(\infty), \quad \mathcal{E}x_{2,q}^{(0,1)}(\infty), \quad \dots \right]$$

These spectra are transparently 8-fold periodic by the 8-fold periodicity of the Clifford algebras, together with Morita theory. The spectrum $K_{(0)}^{\mathbb{R}}$ is more typically called KO . Observe that the $(1,1)$ -periodicity in the table of Clifford algebras $\text{Cl}_{p,q}$ shows that the spaces in the spectrum for $K_{(q)}^{\mathbb{R}}$ are obtained from those in $K_{(q+1)}^{\mathbb{R}}$ by shifting to the left one spot. That is,

$$K_{(q)}^{\mathbb{R}} = \Sigma K_{(q+1)}^{\mathbb{R}}.$$

This can also be written as $K_{(q)}^{\mathbb{R}} = \Sigma^{-1} K_{(q-1)}^{\mathbb{R}}$, which yields

$$K_{(q)}^{\mathbb{R}} \simeq \Sigma^{-q} K_{(0)}^{\mathbb{R}} \simeq \Sigma^{-q} KO.$$

Bott's original arguments were about the equivalences $\mathcal{E}x_{p+1,0}^{(1,0)}(\infty) \xrightarrow{\simeq} \Omega \mathcal{E}x_{p,0}^{(1,0)}(\infty)$. In our current indexing we have $\mathcal{E}x_{-1,0}^{(1,0)}(\infty) \cong O \cong \mathcal{E}x_{-1,0}^{(0,1)}(\infty)$.

Now consider the first two rows of our table of Clifford algebras—that is, the Clifford algebras $Cl_{p,0}$ and $Cl_{p,1}$. The diagram below shows these together with the (horizontal) inclusions $Cl_{p,0} \hookrightarrow Cl_{p+1,0}$ and the (vertical) inclusions $Cl_{p,0} \hookrightarrow Cl_{p,1}$.

$$\begin{array}{cccccccccccc}
 \mathbb{R} \times \mathbb{R} & \rightarrow & \mathbb{R}(2) & \rightarrow & \mathbb{C}(2) & \rightarrow & \mathbb{H}(2) & \rightarrow & \mathbb{H}(2) \times \mathbb{H}(2) & \rightarrow & \mathbb{H}(4) & \rightarrow & \mathbb{C}(8) & \rightarrow & \mathbb{R}(16) & \rightarrow & \mathbb{R}(16) \times \mathbb{R}(16) \\
 \eta \uparrow & & \zeta \uparrow & & \epsilon \uparrow & & \delta \uparrow & & \gamma \uparrow & & \beta \uparrow & & \alpha \uparrow & & \theta \uparrow & & \eta \uparrow \\
 \mathbb{R} & \xrightarrow{\alpha} & \mathbb{C} & \xrightarrow{\beta} & \mathbb{H} & \xrightarrow{\gamma} & \mathbb{H} \times \mathbb{H} & \xrightarrow{\delta} & \mathbb{H}(2) & \xrightarrow{\epsilon} & \mathbb{C}(4) & \xrightarrow{\zeta} & \mathbb{R}(8) & \xrightarrow{\eta} & \mathbb{R}(8) \times \mathbb{R}(8) & \xrightarrow{\theta} & \mathbb{R}(16) \\
 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8
 \end{array}$$

The maps with the same label—for example, the two maps labelled α —are not identical, but are related to each other in the evident way. Recall that the $E^{(1,0)}$ -spaces involve the difference across a horizontal map, whereas the $E^{(0,1)}$ -spaces involve the difference across a vertical map. Making use of Morita invariance, the above diagram yields the unexpected relation $\mathcal{E}x_{0,0}^{(1,0)}(\infty) \cong \mathcal{E}x_{6,0}^{(0,1)}(\infty)$, and more generally

$$(*) \quad \mathcal{E}x_{6-p,0}^{(1,0)}(\infty) \cong \mathcal{E}x_{p,0}^{(0,1)}(\infty)$$

for $0 \leq p \leq 6$ together with $\mathcal{E}x_{7,0}^{(1,0)}(\infty) \cong \mathcal{E}x_{7,0}^{(0,1)}(\infty)$. This is another manifestation of Bott periodicity, although the exact connection might not be immediately clear. Recall that in the following sequence each space is loops on the one immediately after it:

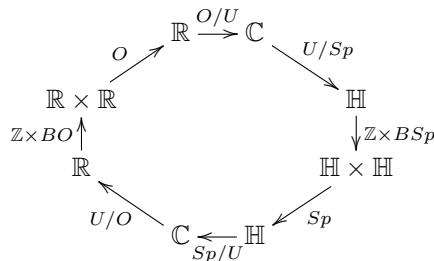
$$\dots \quad \mathcal{E}x_{2,0}^{(1,0)} \quad \mathcal{E}x_{1,0}^{(1,0)} \quad \mathcal{E}x_{0,0}^{(1,0)} \quad O \quad E_{0,0}^{(0,1)} \quad \mathcal{E}x_{1,0}^{(0,1)} \quad \mathcal{E}x_{2,0}^{(0,1)} \quad \dots$$

(we are now leaving out the ∞ notation for brevity). The $\mathcal{E}x_{7,0}^{(1,0)}$ and $\mathcal{E}x_{7,0}^{(0,1)}$ terms are coincident with the O in the 8-fold pattern, and one can now see that $(*)$ is also part of this pattern. Although $(*)$ might at first feel like a 6-fold phenomenon, it is secretly 8-fold because of the two $(0,0)$ terms and the extra O .

Here is a quick summary of our explicit descriptions of the various E -spaces. When R and S are successive Clifford algebras the space $\mathcal{E}x_{R \rightarrow S}$ often takes the form $G(R)/G(S)$ for appropriate groups $G(R)$ and $G(S)$ that depend only on the ‘roots’ of the Clifford algebras. For example, $\mathcal{E}x_{\mathbb{R} \rightarrow \mathbb{C}} \cong O/U$. In addition, $\mathcal{E}x_{R \rightarrow R \times R}$ is the Grassmannian $\mathbb{Z} \times BG(R)$, where $G(R)$ is that same group; e.g., $\mathcal{E}x_{\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}} \cong \mathbb{Z} \times BO$. And $\mathcal{E}x_{R \times R \rightarrow R(2)}$ is $G(R)$. Using these basic facts, one readily identifies the above sequence of spaces as

$$\dots \quad \mathbb{Z} \times BSp \quad U/Sp \quad O/U \quad \boxed{O} \quad \mathbb{Z} \times BO \quad U/O \quad Sp/U \quad Sp \quad \mathbb{Z} \times BSp \dots$$

We can remember all of this with another version of the Bott periodicity clock:



There are two ways to read this clock. Starting with $\mathbb{R} \rightarrow \mathbb{C}$ and going around clockwise, these are the maps $\text{Cl}_{p,0} \rightarrow \text{Cl}_{p+1,0}$ (but remember we only record the roots) and the labels on the arrows are the $\mathcal{E}x_{p,0}^{(1,0)}$ spaces. Reading this way, loops on each label gives the label on the next arrow. This is the left half of the above sequence. Alternatively, we can start with $\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ and regard this as $\text{Cl}_{0,0} \rightarrow \text{Cl}_{0,1}$ (the first vertical map in the above array), and then going counterclockwise gives the successive maps $\text{Cl}_{p,0} \rightarrow \text{Cl}_{p,1}$ (up to roots, of course). In this version the labels on the arrows are the $\mathcal{E}x_{p,0}^{(0,1)}$ spaces, and traveling counterclockwise gives *deloopings*. This is the right half of the above sequence.

Remark 17.12 (Symmetric spaces). A **symmetric space** is a Riemannian manifold M having the property that for every $p \in M$ there is an isometry $I_p: M \rightarrow M$ that fixes p and where the derivative acts on $T_p M$ as $-Id$. The map I_p is called an “inversion” about p . A simple example of a symmetric space is S^1 , where for I_p we take the reflection in the diameter of the circle that passes through p . More generally, every Lie group G equipped with a bi-invariant metric is a symmetric space: inversion at the identity e is the map $g \mapsto g^{-1}$, and inversions at other points can be obtained by conjugation (e.g. $I_g(x) = g \cdot I_e(g^{-1}x)$). However, there exist symmetric spaces that are not Lie groups.

If G is a connected Lie group and $\sigma \in \text{Aut}(G)$ is an involution, let H be a subgroup of $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ that contains the connected component of the identity. Then G/H is a symmetric space: σ induces $G/H \rightarrow G/H$ given by $gH \mapsto \sigma(g)H$, and one readily checks that $D_{eH}\sigma$ is $-Id$. Inversions at other points are obtained using the group action. As one example, consider $G = U(n)$ with σ complex conjugation, and $H = O(n) = G^\sigma$. Then the symmetric space is $U(n)/O(n)$. It is not immediately obvious, but it turns out that every symmetric space arises as such a G/H .

There is a complete classification of symmetric spaces, due to Elie Cartan. Similar to the classification of compact Lie groups, there are a handful of infinite families and then several isolated exceptional cases. The infinite families turn out to all be related to Clifford algebras—in fact, they are precisely the spaces that come up in Bott periodicity of the unitary and orthogonal groups.

We will not need to use symmetric spaces in the rest of this book, but it is good to know a little about how they fit into the story of Bott periodicity. A standard reference for symmetric spaces and their classification is [Helgason].

Exercise 17.13. Let $G_{p,q} \subseteq \text{Cl}_{p,q}$ be the multiplicative subgroup generated by all the e_i and f_j elements. As a set this consists of all monomials in the e 's and f 's (there are 2^{p+q} of these) and their negatives. So $G_{p,q}$ has order 2^{p+q+1} . You may have noticed that in our work with the Clifford algebras and their modules we never used any elements of $\text{Cl}_{p,q}$ outside of the set $G_{p,q}$. One can therefore tell a version of the story that only uses these groups and their representations, not the Clifford algebras. In this problem we explore this, following the paper [LS].

- (a) Let C , K , D , and Q be the groups $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_8 (dihedral group), and Q_8 (quaternion group). Check that $G_{1,0} \cong C$, $G_{0,1} \cong K$, $G_{2,0} \cong Q$, and $G_{0,2} \cong G_{1,1} \cong D$.
- (b) Prove that $G_{p,q}$ is the group generated by symbols $m, e_1, \dots, e_p, f_1, \dots, f_q$ subject to the relations

$$m^2 = 1, \quad e_i^2 = m, \quad f_j^2 = 1, \quad e_i f_j = m f_j e_i,$$

$$e_i e_j = m e_j e_i \quad (i \neq j), \quad f_i f_j = m f_j f_i \quad (i \neq j), \quad m f_j = f_j m.$$

Also prove that the relations $m f_j = f_j m$ are consequences of the others except in the one case $(p, q) = (0, 1)$.

- (c) Let $\mathbb{Z}/2 = \{1, m\}$. Define an **m -group** to be a group G with a homomorphism $\mathbb{Z}/2 \rightarrow G$ whose image is in the center. We will write m both for the element of $\mathbb{Z}/2$ and its image in G . If G_1 and G_2 are m -groups, define $G_1 \ddot{\times} G_2$ to be the quotient of $G_1 \times G_2$ by the normal subgroup $\langle (m, m) \rangle$. Prove that

$$G_{p,q} \ddot{\times} G_{s,t} \cong \begin{cases} G_{p+t, q+s} & \text{if } p+q \text{ even and } p + \frac{p+q}{2} \text{ odd,} \\ G_{p+s, q+t} & \text{if } p+q \text{ even and } p + \frac{p+q}{2} \text{ even,} \\ G_{p,q} \ddot{\times} G_{t,s} & \text{if } p+q \text{ odd and } p + \frac{p+q-1}{2} \text{ is odd.} \end{cases}$$

[Note: This is the hardest part of the exercise. If you get stuck, take the result as given and do the parts below. Or just do the cases $p+q \leq 2$, which are enough for our purposes.]

- (d) Prove that
- (i) $C \ddot{\times} G_{p,q} \cong C \ddot{\times} G_{q,p}$.
 - (ii) $G_{p+2,q} \cong Q \ddot{\times} G_{q,p}$.
 - (iii) $G_{p,q+2} \cong D \ddot{\times} G_{q,p}$.
 - (iv) $G_{p+1,q+1} \cong D \ddot{\times} G_{p,q}$.
- (e) Using (d), build a table analogous to Figure 17.3.1 showing all of the groups $G_{p,q}$. Use GH as an abbreviation for $G \ddot{\times} H$, and show that the table can be made so that every entry has one of the forms D^i , $D^i C$, $D^i Q$, $D^i K$, $D^i QK$. [Hint: Note that $GH = HG$, $Q^2 = D^2$, and $CD = CQ$.]
- (f) Prove that $G_{p+4,q} \cong DQ G_{p,q} \cong G_{p,q+4}$. Then prove the analog of Bott periodicity: $G_{p+8,q} \cong D^4 G_{p,q} \cong G_{p,q+8}$.
- (g) If G is an m -group, define the group algebra $\mathbb{R}_m[G]$ to be the quotient of the usual group algebra $\mathbb{R}[G]$ by the relation $m = -1$. An **m -representation** is a module over $\mathbb{R}_m[G]$. Check that $\mathbb{R}_m[G_{p,q}] = \text{Cl}_{p,q}$ (this is almost tautologous), and so m -representations of $G_{p,q}$ are the same as $\text{Cl}_{p,q}$ -modules.

Remark: The paper [LS] calls the groups $G_{p,q}$ the **Clifford-Littlewood-Eckmann groups**, or CLE groups for short. The $G_{p,0}$ groups were introduced much earlier by Eckmann [Eckmann1] in relation to the Hurwitz-Radon theorem. See both [LS] and [Eckmann2] for further results and applications.

18. PINNING DOWN BOTT PERIODICITY

In (15.5) we proved the periodicity isomorphisms $\pi_i U \cong \pi_{i+2} U$ for all $i \geq 0$, but we have not as yet produced explicit generators for these groups. Relatedly, the periodicity isomorphism itself is difficult to unravel—it is not as if we have a concrete formula for it. These things are buried in the Bott-Morse arguments, but it takes a bit to tease them out. That is our goal in the next two sections, for both the unitary and orthogonal groups. We will explore some down-to-earth constructions that yield very specific formulas, but will also see how this information can be encoded and abstracted with the continued help from Clifford algebras and their modules.

18.1. Matrix formulas and the Bott suspension map. We will start by investigating some relations between $U(n)$ and $U(2n)$. The identification $\mathbb{C}^{2n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n$ yields some easy connections. We will eventually approach this in very categorical ways, but let us begin by taking a track where everything is very concrete.

Observe that whenever $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$ then we have the element

$$E_{\alpha, \beta} = \begin{bmatrix} \alpha I_n & -\bar{\beta} I_n \\ \beta I_n & \bar{\alpha} I_n \end{bmatrix} \in SU(2n)$$

(exercise: check that the determinant is 1, perhaps by row-reducing). Identifying S^3 with the unit sphere in $\mathbb{C}^2$, we have described a map $S^3 \rightarrow SU(2n)$. When $n = 1$ this is the standard isomorphism $S^3 \xrightarrow{\cong} SU(2)$. For $n > 1$ we can think of E as arising as a composite $S^3 \rightarrow SU(2) \rightarrow SU(2n)$ (the second map is basically tensoring with $Id_{\mathbb{C}^n}$, by the way).

If $A \in U(n)$ then we also get the element $F_A = \begin{bmatrix} I_n & 0 \\ 0 & A \end{bmatrix} \in U(2n)$. By conjugating, we obtain elements

$$G_{\alpha, \beta, A} = F_A E_{\alpha, \beta} F_A^{-1} = \begin{bmatrix} \alpha I_n & -\bar{\beta} A^\dagger \\ \beta A & \bar{\alpha} I \end{bmatrix} \in SU(2n).$$

Here we have written $A^\dagger$, but we could also have written A^{-1} since they are equal. This formula describes a map $G: S^3 \times U(n) \rightarrow SU(2n)$.

Observe that

(18.2)

$$G_{\alpha, \beta, A} = \operatorname{Re}(\alpha) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \operatorname{Im}(\alpha) \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + \operatorname{Re}(\beta) \begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix} + \operatorname{Im}(\beta) \begin{bmatrix} 0 & iA^\dagger \\ iA & 0 \end{bmatrix}$$

where each of the four matrices appearing in the linear combination is in $SU(2n)$. The coefficients of our linear combination are a point in S^3 , and we have seen that any such “unital” linear combination gives a matrix in $SU(2n)$. This surprising property turns out to be very useful and rare, and we will see in Section 19.7 that such situations always come from Clifford algebra representations. In fact you might notice that the four matrices in our linear combination can be used to give a quaternionic structure on $\mathbb{C}^{2n}$ and that $\mathbb{H} \cong \operatorname{Cl}_2(\mathbb{R})$ (the four matrices gives the action of 1, i , j , and $-k$).

For what is about to happen we will not need the full S^3 parameter space, and it will be enough to restrict down to a 2-disk. But there is not a canonical choice for such a disk, and this leads to a bit of a mess where different viewpoints call for different choices. Most of the work in this section amounts to navigating this.

Exercise 18.3. Verify that an element $X \in U(2n)$ anticommutes with $\begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$ if and only if it has the form $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where $B, C \in U(n)$. If in addition $X^2 = -I$ verify that $B = -C^\dagger$ (and note that this forces $X \in SU(2n)$). So if one went looking for quaternionic structures extending $\begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}$ then one would quickly discover the above formulas.

We will only need the portion of G defined on S_{upp}^2 , the upper hemisphere of S^2 , regarded as sitting in S^3 as the points $(x, y, z, 0)$ with $x, y, z \in \mathbb{R}$ and $z \geq 0$. In relation to the complex coordinates used earlier, $\alpha = x + iy$ and $z = \text{Re}(\beta)$. Let G' denote the map $S_{\text{upp}}^2 \times U(n) \rightarrow SU(2n)$ given by

$$(18.4) \quad (x, y, z, A) \mapsto xI + y \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + z \begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix}.$$

When $z = 0$ the output does not depend on A , which says that G' respects the quotient relation $(u, A) \sim (u, B)$ for all $u \in \partial S_{\text{upp}}^2$ and $A, B \in U(n)$. So we obtain an induced map

$$\mathcal{B}: (S_{\text{upp}}^2 \times U(n))/\sim \longrightarrow SU(2n).$$

As we will shortly explain, the domain is a model for $\Sigma^2 U(n)$. So we will also write this as $\mathcal{B}: \Sigma^2 U(n) \rightarrow SU(2n)$. We call $\mathcal{B}$ the “Bott suspension map”.

To understand the claim about the domain, if Λ is a space with a boundary $\partial\Lambda$ write $D_\Lambda W = (\Lambda \times W)/\sim$ where the quotient relation is $(u, w) \sim (u, w')$ for $u \in \partial\Lambda$ and $w, w' \in W$. Observe that $D_I W$ is the standard model for ΣW . The construction $D_{I^2} W$ is almost $D_I(D_I W)$, but not quite. There is a canonical surjection $D_{I^2} W \rightarrow D_I(D_I W)$ that squashes two intervals down to distinct points (the two cone points in $\Sigma^2 W$), and therefore is a homotopy equivalence for reasonable spaces W . Our space $(S_{\text{upp}}^2 \times U(n))/\sim$ is just another way of looking at $D_{I^2} U(n)$. See also Exercise 18.13(g) below for another perspective.

Exercise 18.5. For $W = S^r$ check that a homeomorphism $S^{2+r} \rightarrow (S_{\text{upp}}^2 \times S^r)/\sim$ is given by

$$(u, x) \mapsto \left((u_1, u_2, |x|), \frac{x}{|x|} \right)$$

for $(u, x) \in \mathbb{R}^2 \times \mathbb{R}^{r+1}$ such that $|u|^2 + |x|^2 = 1$. Here the image point has the evident interpretation when $|x| = 0$. [Suggestion: It is easier to work with the inverse.]

Exercise 18.6. Verify that the square

$$\begin{array}{ccc} (S_{\text{upp}}^2 \times U(n))/\sim & \xrightarrow{\mathcal{B}} & SU(2n) \\ \downarrow \text{id} \times j & & \downarrow j \\ (S_{\text{upp}}^2 \times U(n+1))/\sim & \xrightarrow{\mathcal{B}} & SU(2n+2) \end{array}$$

commutes up to homotopy, where each j is the standard inclusion. If your homotopy is basepoint-preserving, great. If not, use the group structure on $SU(2n+2)$ to show how to easily modify it to make it basepoint-preserving.

Remark 18.7. Notice that we made a choice in (18.4) to use the matrix $\begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix}$ from (18.2) rather than $\begin{bmatrix} 0 & iA^\dagger \\ iA & 0 \end{bmatrix}$. It turns out that this choice doesn't matter. We can replace $\begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix}$ with the matrix $\begin{bmatrix} 0 & -\bar{s}A^\dagger \\ sA & 0 \end{bmatrix}$ for any $s \in S^1$ and obtain a map

G'_s . Taking a path in S^1 from 1 to s gives a homotopy between our G' and any other G'_s . The formulas corresponding to G_1 , G_{-1} , G_i , and G_{-i} are all commonly found in the literature, with different sources making different choices.

Remark 18.8. Another choice commonly found in the literature is to switch the locations of A and $A^\dagger$ in formula (18.4). This time the switch does alter the homotopy type of the maps G' and $\mathcal{B}$, but in a predictable way: it essentially precomposes them with the automorphism of $U(n)$ given by $A \mapsto A^\dagger$. We will not have to deal with this issue, but want to alert the reader to differences that might be encountered in other sources. The profusion of different choices here makes the literature a bit of a nightmare.

Given $f: S^r \rightarrow U(n)$ let $\mathcal{B}_*\Sigma^2(f)$ denote the composite

$$S^{2+r} \xlongequal{\quad} (S_{\text{upp}}^2 \times S^r)/\sim \xrightarrow{\text{id} \times f} (S_{\text{upp}}^2 \times U(n))/\sim \xrightarrow{\mathcal{B}} SU(2n)$$

where the first isomorphism is the one from Exercise 18.5. The assignment $f \mapsto \mathcal{B}_*\Sigma^2(f)$ gives a map of groups $\pi_r U(n) \rightarrow \pi_{r+2} SU(2n)$, as it is the composite

$$\pi_r U(n) \xrightarrow{\Sigma^2} \pi_{r+2}(\Sigma^2 U(n)) = \pi_{r+2}((S_{\text{upp}}^2 \times U(n))/\sim) \xrightarrow{\mathcal{B}_*} \pi_{r+2} SU(2n).$$

This also explains the symbology $\mathcal{B}_*\Sigma^2$.

In coordinates $(u, x) \in \mathbb{R}^2 \times \mathbb{R}^{r+1}$ with $|u|^2 + |x|^2 = 1$ we have

$$(\mathcal{B}_*\Sigma^2 f)(u, x) = u_1 I + u_2 \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + |x| \begin{bmatrix} 0 & -f(\frac{x}{|x|})^\dagger \\ f(\frac{x}{|x|}) & 0 \end{bmatrix}.$$

Note that when $|x| = 0$ the last term is interpreted to be the zero matrix. Sometimes it is useful to absorb u_1 and u_2 into a single complex coordinate α . Here we identify S^{2+r} with $S(\mathbb{C} \times \mathbb{R}^{r+1})$ and then write

$$(18.9) \quad (\mathcal{B}_*\Sigma^2 f)(\alpha, x) = \begin{bmatrix} \alpha I & -|x|f(\frac{x}{|x|})^\dagger \\ |x|f(\frac{x}{|x|}) & \bar{\alpha} I \end{bmatrix} = G_{\alpha, |x|, f(\frac{x}{|x|})}.$$

We regard $(1, 0)$ as the basepoint of $S(\mathbb{C} \times \mathbb{R}^{r+1})$ and note that $(\mathcal{B}_*\Sigma^2 f)(1, 0) = Id$.

The following result is a very concrete version of Bott periodicity:

Theorem 18.10. *For $r < 2n$ the map $\mathcal{B}_*\Sigma^2: \pi_r U(n) \rightarrow \pi_{r+2} SU(2n)$ is an isomorphism. The stabilization square*

$$\begin{array}{ccc} \pi_r U(n) & \xrightarrow{j_*} & \pi_r U(n+1) \\ \mathcal{B}_*\Sigma^2 \downarrow & & \mathcal{B}_*\Sigma^2 \downarrow \\ \pi_{r+2} SU(2n) & \xrightarrow{j_*} & \pi_{r+2} SU(2n+2) \end{array}$$

is commutative for all r and n , and the colimit over n is precisely the stable Bott isomorphism $\pi_r U \rightarrow \pi_{r+2} SU$ from (15.5).

Note that the commutativity of the square follows from Exercise 18.6, so the real content is in the other claims. The proof is not deep, and consists mostly of just carefully comparing our formula for $\mathcal{B}_*\Sigma^2$ to the maps that came up in the Bott-Morse arguments. But this is easier said than done and does take some care. The remainder of the section will be focused on this proof, but very little of what

goes into this proof will be used later. So readers who are more interested in seeing applications of Theorem 18.10 should feel free to jump ahead to Section 19.

Formulas for $\mathcal{B}_*\Sigma^2$ similar to ours, and isomorphisms such as in Theorem 18.10, appear explicitly in [PR, Section 1] and in [Fomenko, Section 23.1]. See also [CR]. Formulas such as (18.9) can also be found in [Eckmann2] and [Eckmann3]. It seems likely that Theorem 18.10 would have been known by experts much earlier than any of these listed papers, but I do not know of a reference.

Exercise 18.11. (Highly recommended.) Use the isomorphisms in Theorem 18.10 together with the long exact sequences for $SU(r-1) \hookrightarrow SU(r) \rightarrow S^{2r-1}$ to prove that for all $k \leq 2n-1$ one has

$$\pi_k(U(n)) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

That is to say, the Bott periodic groups in $\pi_*U(n)$ persist up through $*$ = $2n-1$. Also prove that $\pi_{2n}U(n)$ (the first non-periodic group) is always cyclic. [Suggestion: Construct a table of the groups $\pi_rSU(n)$, start filling it out, and consider the different ways you have for propagating information from one cell of the table to another.]

The rest of this section is devoted to the proof of Theorem 18.10. While the argument might at first seem long and complicated, we want to stress that all of the steps are fairly simple. The proof looks more intimidating than it really is.

18.12. Joins. If X and Y are spaces then the join $X*Y$ is defined to be the quotient space $(X \times I \times Y)/\sim$ where the equivalence relation is generated by $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$ for all $x, x' \in X$ and $y, y' \in Y$. Intuitively one thinks of the join as the space of lines from points in X to points in Y . It will be convenient to adopt the notation

$$(x, t, y) = (1-t)[x] + t[y],$$

so that $X * Y = \{a[x] + b[y] \mid x \in X, y \in Y, a, b \in I, a + b = 1\}$.

Note that $S^0 * Y$ is the (unreduced) suspension of Y . We will often give the points of S^0 names like $\pm P$, in which case points of $S^0 * Y$ will be written $a[\pm P] + (1-a)[y]$ where $y \in Y$ and $a \in I$.

When X and Y are pointed CW-complexes the homotopy type of $X * Y$ coincides with $\Sigma(X \wedge Y)$. We will not need this general fact, but it is good to know; one source for a proof is [DI3, Appendix B].

The following exercise establishes several basic facts we will need about the behavior of the join:

Exercise 18.13. Justify all of the following claims.

(a) For the unit circle $S^1 \subseteq \mathbb{C}$, a homeomorphism $S^1 \cong S^0 * S^0 = \{\pm 1\} * \{\pm i\}$ is given by

$$\alpha \mapsto \operatorname{Re}(\alpha)^2[\operatorname{sgn}(\operatorname{Re} \alpha)] + \operatorname{Im}(\alpha)^2[\operatorname{sgn}(\operatorname{Im} \alpha)i]$$

where $\operatorname{sgn}(r)$ is $+1$ when $r > 0$ and -1 when $r < 0$ (the value when $r = 0$ doesn't actually matter for the above formula).

- (b) The inverse of the map in (a) is given by $a[\pm 1] + b[\pm i] \mapsto (\pm\sqrt{a}) \pm (\sqrt{b} \cdot i)$, where the first $\pm$ follows the sign on 1 and the second follows the sign on i . Verify that

$$a[\pm 1] + b[\pm i] \mapsto \frac{1}{\sqrt{a^2+b^2}}[\pm a \pm bi]$$

also gives a homeomorphism $S^0 * S^0 \rightarrow S^1$, and that this map is homotopic to the first one.

In fact, suppose $\gamma: I \rightarrow \mathbb{R}^2$ is any homeomorphism onto the first quadrant of the unit circle ($x, y \geq 0$) such that $\gamma(0) = (0, 1)$ and $\gamma(1) = (1, 0)$. Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Then

$$a[\pm 1] + b[\pm i] \mapsto \pm\gamma_1(a) \pm \gamma_2(a) \cdot i$$

is a homeomorphism and is homotopic to the above maps.

There is a moral here: *because joins are made up of straight lines, maps out of them can often be represented by many different formulas that are the same up to homotopy.*

- (c) For the standard spheres $S^r \subseteq \mathbb{R}^{r+1}$ and $S^s \subseteq \mathbb{R}^{s+1}$, there is a homeomorphism $S^{r+s+1} \cong S^r * S^s$ given by

$$(x, y) \mapsto |x|^2 \left[\frac{x}{|x|} \right] + |y|^2 \left[\frac{y}{|y|} \right]$$

where $x \in \mathbb{R}^{r+1}$, $y \in \mathbb{R}^{s+1}$, and $|x|^2 + |y|^2 = 1$.

- (d) When X , Y , and Z are compact Hausdorff the triple join $X * (Y * Z)$ is homeomorphic to the quotient space obtained from the set of formal sums

$$a[x] + b[y] + c[z]$$

where $a, b, c \geq 0$ and $a + b + c = 1$ (topologized as $X \times Y \times Z \times \Delta^2$) subject to the quotient relation generated by $0[x] + b[y] + c[z] \sim 0[x'] + b[y] + c[z]$ and the two analogous relations corresponding to $b = 0$ and $c = 0$. [Note: This result is non-obvious because of the need to commute products with the quotienting processes. Possibly the hypotheses that the spaces are compact and Hausdorff are unnecessarily strong, but they allow one to use adjunction arguments and other simplifying techniques. If $Q = (X \times Y \times Z \times \Delta^2)/\sim$ start by producing a map $X \times (Y \times Z \times I) \times I \rightarrow Q$, then get $(Y \times Z \times I)/\sim \rightarrow Q^{X \times I}$, and continue to arrive at a map $X * (Y * Z) \rightarrow Q$.]

- (e) A homeomorphism $S(\mathbb{C} \times \mathbb{R}^{r+1}) \cong S^0 * S^0 * S^r$ is given by

$$(\alpha, x) \mapsto \operatorname{Re}(\alpha)^2 [\operatorname{sgn}(\operatorname{Re}(\alpha))] + \operatorname{Im}(\alpha)^2 [\operatorname{sgn}(\operatorname{Im}(\alpha))i] + |x|^2 \left[\frac{x}{|x|} \right].$$

- (f) $S^1 * X \cong (D^2 \times X)/\sim$ where the equivalence relation has $(a, x) \sim (a, x')$ for all $a \in \partial D^2$ and $x, x' \in X$. One such homeomorphism is given by $t[\alpha] + (1-t)[x] \mapsto (t\alpha, x)$, and another (homotopic) one is given by $t[\alpha] + (1-t)[x] \mapsto (\sqrt{t} \cdot \alpha, x)$. [This does not need X to be compact Hausdorff, though the proof is easier in that case. Likewise for the next part.]

- (g) There is a homeomorphism

$$S^0 * S^0 * X \rightarrow (S_{\text{upp}}^2 \times X)/\sim$$

given by $a[\pm M] + b[\pm N] + c[x] \mapsto ((\pm_M \sqrt{a}, \pm_N \sqrt{b}, \sqrt{c}), x)$, where our model of the double join is the one from (d). Here we are using S_{upp}^2 to line up with discussion from the beginning of the section, but note that we could just as well use D^2 instead (but getting slightly different formulas).

- (h) Suppose given a map $f: Z \rightarrow \Omega_{P,Q}W$, and recall that the target denotes the space of paths that begin at P and end at Q . Check that one gets an “adjoint” $\text{adj}(f): S^0 * Z \rightarrow W$ via the formula

$$(1-t)[\pm N] + t[z] \mapsto \begin{cases} [f(z)](\frac{t}{2}) & \text{for } + \\ [f(z)](1 - \frac{t}{2}) & \text{for } -, \end{cases}$$

having the property that $[+N] \rightarrow P$ and $[-N] \rightarrow Q$. Geometrically, the canonical path in $S^0 * Z$ from N to $-N$ that passes through a point $z \in Z$ is being sent to the path $f(z)$ in W .

Show that if Z has a basepoint ζ and $\gamma = f(\zeta)$ then the following diagram commutes

$$\begin{array}{ccc} \pi_r(Z, \zeta) & \xrightarrow{f_*} & \pi_r(\Omega_{P,Q}W, \gamma) \\ \Sigma \downarrow & & \cong \downarrow \text{Adj} \\ \pi_{r+1}(S^0 * Z, [+N]) & \xrightarrow{\text{adj}(f)_*} & \pi_{r+1}(W, P) \end{array}$$

where Adj sends a homotopy element $\alpha: S^r \rightarrow \Omega_{P,Q}W$ to $\text{adj}(\alpha): S^0 * S^r \rightarrow W$ and $S^0 * S^r$ has the north pole $[+N]$ as basepoint. As part of this, check that Σ and Adj are well-defined, group maps, and that Adj is an isomorphism. [For Adj , one approach for dealing with some of the unpleasant technicalities is to get $\Omega_{P,P}W$ into the picture via your favorite homotopy equivalence $\Omega_{P,Q}W \rightarrow \Omega_{P,P}W$.]

Using Exercise 18.13(g) we now see that the Bott suspension map $\mathcal{B}$ can be reinterpreted as the map $\mathcal{B}: S^0 * S^0 * U(n) \rightarrow U(2n)$ given by the formula

$$a[\pm M] + b[\pm N] + c[A] \mapsto \pm_M \sqrt{a}I + \pm_N \sqrt{b} \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + \sqrt{c} \begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix}.$$

18.14. Proof of Theorem 18.10.

Recall the Bott map $\beta: \text{Gr}_n(\mathbb{C}_+^n \oplus \mathbb{C}_-^n) \rightarrow \Omega_{I,-I}SU(2n)$ sending a subspace H to the path β_H given by

$$\beta_H(t) = [\text{the transformation that is } \cdot e^{\pi it} \text{ on } H \text{ and } \cdot e^{-\pi it} \text{ on } H^\perp].$$

For stabilization we need to pass to $\Omega_I SU(2n)$ instead, so let σ be the path from I to $-I$ where

$$\sigma(t) = [\text{the transformation that is } \cdot e^{-\pi it} \text{ on } \mathbb{C}_+^n \text{ and } \cdot e^{\pi it} \text{ on } \mathbb{C}_-^n].$$

We use the homeomorphism $\Omega_{I,-I}SU(2n) \rightarrow \Omega_I SU(2n)$ that sends a path λ to the loop $t \mapsto \lambda(t) \cdot \sigma(t)$ (see Exercise 15.17), and will write $\tilde{\beta}$ for the composite

$$\text{Gr}_n(\mathbb{C}_+^n \oplus \mathbb{C}_-^n) \xrightarrow{\beta} \Omega_{I,-I}SU(2n) \longrightarrow \Omega_I SU(2n).$$

The proof centers on analyzing the large diagram in Figure 18.14.1. This diagram is off-putting, but it is easier to navigate than one might think. Start with the following:

- The sequence of boxed terms represents one version of the Bott isomorphism, with $\tilde{\beta}$ the map that came to us from Bott-Morse theory.

- Along the top edge of the diagram we have the stable version of the Bott isomorphism from (15.5), and the six rectangles above the boxed terms all commute trivially (we will say more below). This upper part of the diagram is just about stablization, and can largely be ignored.
- The “action” in the diagram lies in the two pentagons at the bottom. This is where the Bott isomorphism will be compared to $\mathcal{B} \circ \Sigma^2$.

FIGURE 18.14.1. Comparing Bott maps

$$\begin{array}{ccccccc}
\pi_r U & \xrightarrow[\cong]{c} & \text{Vect}^{st}(S^{r+1}) & \xleftarrow{\dots\dots\dots} & \pi_{r+1} \text{Gr}_\infty(\mathbb{C}^{2\infty}) & \xrightarrow[\cong]{\tilde{\beta}_*} & \pi_{r+1} \Omega_I SU = \pi_{r+2} SU \\
\uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \uparrow \\
C_k \pi_r U_k & \xrightarrow[\cong]{c} & C_k \text{Vect}_k(S^{r+1}) & \xleftarrow{\cong} & C_k \pi_{r+1} \text{Gr}_k(\mathbb{C}_+^k \oplus \mathbb{C}_-^\infty) & & \\
\uparrow [r \leq 2n-1] & & \uparrow & & \uparrow [r \leq 2n-1] & & \uparrow [r < 4n-2] \\
\boxed{\pi_r U_n} & \xrightarrow[\cong]{c} & \boxed{\text{Vect}_n(S^{r+1})} & \xleftarrow{\cong} & \boxed{\pi_{r+1} \text{Gr}_n(\mathbb{C}_+^n \oplus \mathbb{C}_-^\infty)} & & \\
\searrow \Sigma & & & & \uparrow [r \leq 2n-1] & & \\
& & \pi_{r+1} \Sigma U_n & \xrightarrow{\Theta_*} & \boxed{\pi_{r+1} \text{Gr}_n(\mathbb{C}_+^n \oplus \mathbb{C}_-^n)} & \xrightarrow[\cong]{\tilde{\beta}_*, r \leq 2n-1} & \boxed{\pi_{r+1} \Omega_I SU_{2n}} = \boxed{\pi_{r+2} SU_{2n}} \\
& & \searrow \Sigma & & \uparrow & & \uparrow \\
& & & & \pi_{r+2}(S^0 * S^0 * U_n) & \xrightarrow{\mathcal{B}_*} &
\end{array}$$

One part of this diagram—the map Θ_* —has not been explained yet, but we will get to it shortly. We instead begin with several remarks to help the reader parse the above behemoth:

- (1) The composition across the top row is the stable form of the Bott periodicity isomorphism. The dotted arrow is explained in (3) below. The space $\text{Gr}_\infty(\mathbb{C}^{2\infty})$ is the construction $\text{colim}_k \text{Gr}_k(\mathbb{C}_+^k \oplus \mathbb{C}_-^\infty)$ that we encountered in Section 15. Recall that this space is a model for BU .
- (2) C_k is shorthand for colim_k , and we have written U_k as shorthand for $U(k)$.
- (3) The dotted arrow in the top row is dotted because it is not naturally occurring; rather, it is defined to be the composite of the other isomorphisms in that square. All other maps in the diagram are the evident ones.
- (4) Recall that the maps $\text{Gr}_n(\mathbb{C}_+^n \oplus \mathbb{C}_-^\infty) \rightarrow \text{Gr}_{n+1}(\mathbb{C}_+^{n+1} \oplus \mathbb{C}_-^{n+1})$ send an n -plane H to $H \oplus \langle e_{n+1,+} \rangle$. That is, both $e_{n+1,+}$ and $e_{n+1,-}$ are added to the ambient space, but only the former is added to the subspace. Going to the colimit can be broken up as first adding in all of the negative basis vectors to the ambient space, and then adding the positive basis vectors one by one into both the ambient space and the subspace. This is what is going on at the top of the central column. We are doing this because $\text{Gr}_k(\mathbb{C}_+^k \oplus \mathbb{C}_-^\infty)$ is a model for $BU(k)$ and hence classifies rank k vector bundles.
- (5) The maps labelled c are given by the clutching construction. The inequalities in brackets are ranges where the given maps are isomorphisms. There are

three sources of these: the fiber sequences $U_k \hookrightarrow U_{k+1} \hookrightarrow S^{2k+1}$, the fact that $\mathrm{Gr}_n(\mathbb{C}^\infty)$ is obtained from $\mathrm{Gr}_n(\mathbb{C}^{2n})$ by attaching cells of dimension $2n+2$ and higher, and the Bott-Morse arguments giving the connectivity of the lower $\tilde{\beta}$ map.

- (6) The four squares in the upper left obviously commute.
- (7) The maps $\tilde{\beta}$ send a subspace H to the path of transformations

$$t \mapsto (e^{\pi i t} \text{ on } H, e^{-\pi i t} \text{ on } H^\perp) \cdot (e^{-\pi i t} \text{ on } \mathbb{C}_+^n, e^{\pi i t} \text{ on } \mathbb{C}_-^n)$$

where the $\cdot$ denotes multiplication of matrices or composition of transformations. If we denote this path $\tilde{\beta}_H$, notice that for $x \in H \cap \mathbb{C}_+^n$ or $x \in H^\perp \cap \mathbb{C}_-^n$ we have $(\tilde{\beta}_H)_t(x) = x$ for all t . In particular, this explains why $\tilde{\beta}$ extends to the colimit. This is the commutativity of the leftmost of the two upper right rectangles. The commutativity of the rightmost one is self-evident.

At this point we have analyzed the top portion of the diagram, made up of the six rectangles. It is difficult to directly relate the composition across the bottom of these rectangles to the $\mathcal{B}_* \circ \Sigma^2$ composite because some of the maps go in the “wrong” direction. We will correct for this by introducing the map $\Theta: \Sigma U(n) \rightarrow \mathrm{Gr}_n(\mathbb{C}^{2n})$, which helps connect the two pieces. This map is in some sense a geometric construct underlying the isomorphism $\pi_r U \cong \pi_{r+1} BU$.

Let $\mathcal{G}: U(n) \rightarrow \mathrm{Gr}_n(\mathbb{C}^{2n})$ be the map that sends a matrix A to its graph $\mathcal{G}A = \{(x, Ax) \mid x \in \mathbb{C}^n\}$. There is an extension of $\mathcal{G}$ to a map $\Sigma U(n) \rightarrow \mathrm{Gr}_n(\mathbb{C}^{2n})$ that deforms $\mathcal{G}A$ to each of the two “coordinate axes” in $\mathbb{C}^n \oplus \mathbb{C}^n$. Precisely, we have $\Theta: S^0 * U(n) \rightarrow \mathrm{Gr}_n(\mathbb{C}^{2n})$ given by

$$\begin{aligned} (1-t)[N] + t[A] &\mapsto \{(x, tAx) \mid x \in \mathbb{C}^n\} \\ (1-t)[-N] + t[A] &\mapsto \{(tx, Ax) \mid x \in \mathbb{C}^n\}. \end{aligned}$$

Note that $[N] \mapsto \mathbb{C}^n \oplus 0$ and $[-N] \mapsto 0 \oplus \mathbb{C}^n$.

Pulling back the tautological bundle η on $\mathrm{Gr}_n(\mathbb{C}^{2n})$ along Θ gives a rank n bundle on $\Sigma U(n)$. As we have seen in Section ??, vector bundles on such a suspension are characterized by a clutching function $U(n) \rightarrow U(n)$, and we claim that in this case the clutching function is the identity. To see this, observe that the formulas

$$\begin{aligned} (1-t)[N] + t[A] &\mapsto [(e_1, tAe_1), \dots, (e_n, tAe_n)] \\ (1-t)[-N] + t[A] &\mapsto [(tA^{-1}e_1, e_1), \dots, (tA^{-1}e_n, e_n)] \end{aligned}$$

give trivializing sections over the upper and lower cones, respectively. When $t = 1$ the matrix that writes the first basis in terms of the second basis is precisely A (just look in the second coordinates).

Continuing now with our analysis of the big diagram:

- (8) The trapezoidal pentagon in the lower left commutes because, given $f: S^r \rightarrow U(n)$, the vector bundle on S^{r+1} constructed with this clutching function is precisely the pullback in the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\quad \quad \quad} & \eta_n & & \\ \downarrow & & \downarrow & & \\ \Sigma S^r & \xrightarrow{\Sigma f} & \Sigma U(n) & \xrightarrow{\Theta} & \mathrm{Gr}_n(\mathbb{C}^{2n}). \end{array}$$

This is due to our construction of Θ , and the fact that $\Theta^*\eta_n$ is the bundle on $\Sigma U(n)$ whose clutching function is the identity.

- (9) We observe now that the theorem will follow immediately once we have proven the commutativity of the bottom pentagon (which looks more like a triangle in the diagram), using that the indicated maps are isomorphisms in the given ranges.

To prove commutativity of the pentagon we need to unpack the Bott map β . It will be convenient to break down what is happening into two steps: given a subspace $H \in \text{Gr}_n(\mathbb{C}^{2n})$ we form the transformation T_H of $\mathbb{C}^{2n}$ that is multiplication by 1 on H and multiplication by -1 on $H^\perp$, and then β_H is the path $t \mapsto e^{\pi i t T_H}$. In fact, the first step is really a homeomorphism between $\text{Gr}_n(\mathbb{C}^{2n})$ and the subspace $X \subseteq U(2n)$ consisting of matrices C such that $C^2 = I$ and the $+1$ and -1 eigenspaces of C have the same dimension (recall that if $C^2 = I$ then the matrix is automatically diagonalizable). From now on we will move back and forth between $\text{Gr}_n(\mathbb{C}^{2n})$ and X at will via this homeomorphism. For example, the Bott map $\beta: \text{Gr}_n(\mathbb{C}^{2n}) \rightarrow \Omega_{I,-I} SU(2n)$ is also the map $\beta: X \rightarrow \Omega_{I,-I} SU(2n)$ given by

$$(18.15) \quad C \mapsto [t \mapsto e^{\pi i t C} = \cos(\pi t)I + \sin(\pi t)iC].$$

We next want to re-interpret our map $\Theta: S^0 * U(n) \rightarrow \text{Gr}_n(\mathbb{C}^{2n})$ (see above) as a map $\Theta': S^0 * U(n) \rightarrow X$. This involves finding matrices in X whose 1-eigenspaces are the given subspaces of $\mathbb{C}^{2n}$. A little legwork/inspiration suggests that to have (x, tAx) be an eigenvector one should look at unitary matrices of the form $Q = \begin{bmatrix} rI & sA^\dagger \\ \bar{s}A & -\bar{r}I \end{bmatrix}$ for $r, s \in \mathbb{C}$, and we leave the reader to check that

- Q is unitary precisely when $|r|^2 + |s|^2 = 1$.
- When Q is unitary, $Q^2 = I$ if and only if $r \in \mathbb{R}$.
- When Q is unitary and $Q^2 = I$ and $t \in [0, 1]$, Q has $\{(x, tAx) \mid x \in \mathbb{C}^n\}$ inside its 1-eigenspace precisely when $r = \frac{1-t^2}{1+t^2}$ and $s = \frac{2t}{1+t^2}$. In this case the -1 -eigenspace contains $\{(tA^{-1}x, -x) \mid x \in \mathbb{C}^n\}$, therefore the two eigenspaces actually equal the ones given and $Q \in X$.
- When Q is unitary and $Q^2 = I$ and $t \in [0, 1]$, Q has $\{(tx, Ax) \mid x \in \mathbb{C}^n\}$ as its 1-eigenspace precisely when $r = \frac{t^2-1}{1+t^2}$ and $s = \frac{2t}{1+t^2}$ (just change t to t^{-1} in the previous bullet point). Again, $Q \in X$ here.

These are routine exercises. The upshot is that the formula for the map $\Theta': S^0 * U(n) \rightarrow X$ is

$$(18.15) \quad (1-t)[\pm N] + t[A] \mapsto \begin{bmatrix} \pm rI & sA^\dagger \\ sA & \mp rI \end{bmatrix} = \pm r \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + s \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix}$$

where

$$r = \frac{1-t^2}{1+t^2}, \quad s = \frac{2t}{1+t^2}.$$

Note that $r^2 + s^2 = 1$ here, for all values of t .

The above formulas for r and s are a little awkward, so we make the following observation. All that is important here is that the assignment $t \mapsto (r, s)$ maps $[0, 1]$ to the first quadrant of the unit circle, with $0 \mapsto (1, 0)$ and $1 \mapsto (0, 1)$. All such maps are homotopic, and using any such map to give the coefficients for r and s in the above formula gives a map $S^0 * U(n) \rightarrow X$ that is homotopic to the specific Θ' above. For example, one could use $(r, s) = (\sqrt{1-t}, \sqrt{t})$ or $(r, s) = (\cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t))$, and these lead to more pleasant-looking formulas.

Finally, we consider the composite

$$S^0 * U(n) \xrightarrow{\Theta'} X \xrightarrow{\beta} \Omega_{I,-I} SU(2n).$$

Combining (18.15) and (18.15) we find that this is given by the formula

$$(1-t)[\pm N] + t[A] \mapsto \left[u \mapsto \cos(\pi u)I + \sin(\pi u) \cdot i \cdot \left[\pm r(t) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + s(t) \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} \right] \right].$$

We again take note that the coefficients $r_2(u) = \cos(\pi u)$ and $s_2(u) = \sin(\pi u)$ can be varied somewhat. All that is important is that $u \mapsto (r_2(u), s_2(u))$ maps I to the top of the unit circle and sends 0 to $(1, 0)$ and 1 to $(-1, 0)$. All such paths are homotopic, so we can change the coefficients in the above formula to any such $r_2(u)$ and $s_2(u)$ without changing the overall homotopy class of the map. In a moment it will also be convenient that $u = \frac{1}{2}$ is mapped to $(0, 1)$, so we will also insist on that.

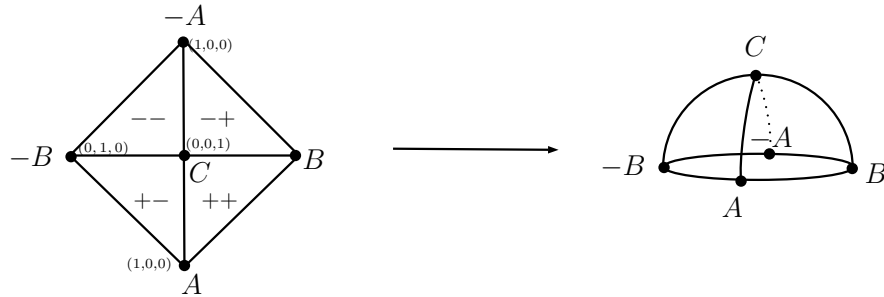
Taking the “adjoint” of the above composite, as in Exercise 18.13(h), gives the map $\text{adj}(\beta\Theta'): S^0 * S^0 * U(n) \rightarrow SU(2n)$ whose formula is

$$(*) \quad a[\pm M] + b[\pm N] + c[A] \mapsto r_2\left(\frac{1 \mp Ma}{2}\right) I \pm_N s_2\left(\frac{1 \mp Ma}{2}\right) r\left(\frac{c}{1-a}\right) \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} \\ + s_2\left(\frac{1 \mp Ma}{2}\right) s\left(\frac{c}{1-a}\right) \begin{bmatrix} 0 & iA^\dagger \\ iA & 0 \end{bmatrix}.$$

Recall that the $\frac{1 \mp Ma}{2}$ term is mapping the “suspension coordinate” given by a to a “path coordinate” that goes from 0 to 1, as shown in the following picture:

$$\begin{array}{ccc} [+M] \ (a=1) & \longrightarrow & 0 \\ \downarrow & & \\ (a=0) & \longrightarrow & \frac{1}{2} \\ \uparrow & & \\ [-M] \ (a=1) & \longrightarrow & 1. \end{array}$$

The coefficients in $(*)$ are off-putting, but what is happening here is that we have four maps from $\Delta^2 = \{(a, b, c) \in I^3 \mid a + b + c = 1\}$ to S_{upp}^2 (corresponding to the four choices of signs) that patch together to give the map shown in the following picture:



The diamond is really $\Delta^2 \subseteq \mathbb{R}^3$ together with its three images under the different reflections $(a, b, c) \mapsto (\pm a, \pm b, c)$. The points labelled $\pm A$, $\pm B$, C in the diamond map to the corresponding points in the sphere, with indicated lines mapping onto the indicated arcs of great circles. Each triangle in the diamond is identified with

our standard Δ^2 by having the $\pm A$ vertex map to $(1, 0, 0)$, the $\pm B$ vertex map to $(0, 1, 0)$, and the C vertex map to $(0, 0, 1)$ (notice that, conveniently, these are also related to the coordinates of their images in the sphere).

While this map from the diamond to the upper half-sphere is not too hard to understand from a geometric perspective, the algebraic formula in $(*)$ is unpleasant. As we have seen before, it can largely be ignored: all that is important here is that the map sends the boundary of the diamond homeomorphically onto the equator and the four corners to the four corresponding equatorial poles—all such maps are homotopic. So we can instead use, for example, $(\pm a, \pm b, c) \mapsto (\pm\sqrt{a}, \pm\sqrt{b}, \sqrt{c})$ if we prefer. This shows that $\text{adj}(\beta\Theta')$ is homotopic to the map

$$(**) \quad a[\pm M] + b[\pm N] + c[A] \mapsto \pm_M \sqrt{a} I \pm_N \sqrt{b} \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + \sqrt{c} \begin{bmatrix} 0 & iA^\dagger \\ iA & 0 \end{bmatrix}.$$

It then follows that we have a homotopy-commutative diagram

$$(18.16) \quad \begin{array}{ccc} S^0 * S^0 * U(n) & \xrightarrow{\text{adj}(\beta\Theta')} & SU(2n) \\ & \searrow \cong & \nearrow \mathcal{B} \\ & (S^2_{\text{upp}} \times U(n)) / \sim & \end{array}$$

where the isomorphism is the one from Exercise 18.13(g), namely $a[\pm M] + b[\pm N] + cA \mapsto ((\pm_M \sqrt{a}, \pm_N \sqrt{b}, \sqrt{c}), A)$. This requires one more comment, because our formula from $(**)$ does not exactly match the formula for $\mathcal{B}$ from (18.4) due to the presence of the i 's in the final matrix. However, by Remark 18.7 $\mathcal{B}$ is homotopic to the version with the i 's.

Exercise 18.17. For those who prefer algebra to geometry, check that the functions $r(t) = \sqrt{1-t}$, $s(t) = \sqrt{t}$, and

$$r_2(u) = \begin{cases} \sqrt{1-2u} & \text{if } 0 \leq u \leq \frac{1}{2}, \\ -\sqrt{2u-1} & \text{if } \frac{1}{2} \leq u \leq 1 \end{cases}, \quad s_2(u) = \sqrt{1-|2u-1|}$$

satisfy our various requirements and turn formula $(*)$ exactly into formula $(**)$.

We can now complete the last step of the proof:

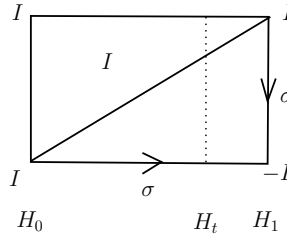
- (10) To see commutativity of the bottom pentagon in our large diagram we expand the picture as follows:

$$\begin{array}{ccccccc} \pi_{r+1} \Sigma U_n & \xrightarrow{\Theta_*} & \pi_{r+1} \text{Gr}_n(\mathbb{C}^{2n}) & \xrightarrow{\tilde{\beta}_*} & \pi_{r+1} \Omega_I SU_{2n} & \xlongequal{\quad} & \pi_{r+2} SU_{2n} \\ & \searrow \Theta'_* & \uparrow \cong & & \uparrow K_* & \nearrow \text{Adj} & \uparrow \\ & & \pi_{r+1} X & \xrightarrow{\beta_*} & \pi_{r+1} \Omega_{I,-I} SU_{2n} & \xrightarrow[\cong]{\text{adj}(\beta\Theta')_*} & \pi_{r+2} SU_{2n} \\ & & & & & & \uparrow \mathcal{B}_* \\ & & & & & & \pi_{r+2} (S^0 * S^0 * U_n) \end{array}$$

Σ

Here $K: \Omega_{I,-I} SU(2n) \rightarrow \Omega_I SU(2n)$ is the map that right-multiplies a loop by the canonical path $\sigma: t \mapsto \text{diag}(e^{-\pi it}, \dots, e^{-\pi it}, e^{\pi it}, \dots, e^{\pi it})$, and Adj is the map that sends $S^{r+1} \rightarrow \Omega_{I,-I} SU_{2n}$ to its adjoint $S^{r+2} \cong S^0 * S^{r+1} \rightarrow SU_{2n}$ (see Exercise 18.13(h)) where the basepoint of the domain is the north pole. The upper left triangle and the upper rectangle commute by

definition. The upper right triangle is checked to commute as follows. Let H_t be the homotopy from the constant path at I to σ rel the initial point shown in the following diagram:



(on the top triangle this is constant and on the lower one it is—looked at the right way—the homotopy showing that σ concatenated with its reverse is homotopic to the constant path). If $\alpha: S^{r+1} \rightarrow \Omega_{I,-I} SU_{2n}$ then $\text{Adj}(\alpha): S^0 * S^{r+1} \rightarrow SU_{2n}$ is the map that for each $x \in S^{r+1}$ sends the path “suspension of x ” (from north to south pole) to $\alpha(x)$. Let $\text{Adj}(\alpha)_t: S^0 * S^{r+1} \rightarrow SU_{2n}$ be the similar map that sends the path “suspension of x ” to $\alpha(x) \cdot H_t$, where the multiplication is pointwise, using the group structure on SU_{2n} . This gives a homotopy, pointed at the north pole, from $\text{Adj}(\alpha)_0 = \text{Adj}(\alpha)$ to $\text{Adj}(\alpha)_1 = K_*(\alpha)$.

The map $\text{adj}(\beta\Theta')$ is the “adjoint” of the composite $\beta\Theta'$, and the bottom pentagon containing this map commutes by Exercise 18.13(h). Finally, the maps $\text{adj}(\beta\Theta')$ and $\mathcal{B}$ are homotopic by (18.16), and hence induce the same map on homotopy groups.

Summary: The main goal of this section was the very concrete version of Bott periodicity for U given by Theorem 18.10. The proof was a longish chase through the Bott periodicity map that comes out of Morse theory, but it was mostly tedious rather than having any groundbreaking ideas. Attempting to do something similar for O , however, presents some real challenges. To do this in a navigable manner will require us to step back and develop some more structure to help organize all that is happening.

19. PINNING DOWN BOTT PERIODICITY, PART 2

In the last section we uncovered a very down-to-earth version of Bott periodicity. We produced a family of maps $\mathcal{B}: (S^2_{\text{upp}} \times U(n))/\sim \rightarrow SU(2n)$, with the domain a model for $\Sigma^2 U(n)$ and given by very explicit matrix formulas. Then we proved in Theorem 18.10 that the composition

$$\pi_r U(n) \xrightarrow{\Sigma^2} \pi_{r+2} \Sigma^2 U(n) \xrightarrow{\mathcal{B}_*} \pi_{r+2} SU(2n)$$

is an isomorphism for $r < 2n$ and is compatible with the Bott map β . The composition is also denoted $\mathcal{B}$. This theorem opens the door to several questions:

- (1) We can now produce explicit generators for the groups $\pi_* U$ by starting with known generators in low dimensions and applying $\mathcal{B}$. What do these generators look like?
- (2) The generators of $\pi_* U$ translate into generators for $\pi_*(BU) = K^{-*}(pt)$. Can we use these explicit descriptions to understand the ring structure?

- (3) The construction of the map $\mathcal{B}$ was somewhat ad hoc. How do we better understand it?
- (4) How do we generalize all of this to O instead of U ?

Many of these questions are connected to each other. Our goal in this section is to answer them as best as we can.

19.1. Generators for π_*U . Let us use $\mathcal{B}$ to produce some interesting elements in the homotopy of the unitary groups. Start with the standard generator $f_1 \in \pi_1 U(1)$, namely the map $(x, y) \in S^1 \mapsto x + iy$. Then $\mathcal{B}f_1$ is the map $S^3 \rightarrow SU(2)$ that sends (z, x, y) to $\begin{bmatrix} z & -x + iy \\ x + iy & \bar{z} \end{bmatrix}$. Note that this could also be described using only complex coordinates, as

$$f_3: (z, w) \mapsto \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix},$$

which is the standard isomorphism from S^3 to $SU(2)$. Applying $\mathcal{B}$ one more time gives the map $f_5 \in \pi_5 SU(4)$ with formula

$$(19.2) \quad f_5: (u, z, w) \mapsto \begin{bmatrix} u & 0 & -\bar{z} & -\bar{w} \\ 0 & u & w & -z \\ z & -\bar{w} & \bar{u} & 0 \\ w & \bar{z} & 0 & \bar{u} \end{bmatrix}.$$

Clearly it is a triviality to keep writing these down in higher and higher dimensions to produce explicit formulas for $f_{2n+1} \in \pi_{2n+1} SU(2^n)$.

This procedure produces generators for $\pi_{2s+1}U$ that come from $\pi_{2s+1}U(2^s)$. But the stability isomorphisms shows that in fact a generator can be lifted all the way to $\pi_{2s+1}U(s+1)$. This is a substantial difference! For example, the above technique lets us write down an explicit lifting of the generator for $\pi_{15}U$ to an element of $\pi_{15}U(128)$, when in fact we know that a lifting actually exists into $\pi_{15}U(8)$.

One thing that is notable about the generators produced by Theorem 18.10 is that they are *linear*. For example, f_3 is given by the formula

$$(19.3) \quad (z_0, z_1, w_0, w_1) \mapsto z_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + w_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + w_1 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Liftings of these generators into smaller $U(n)$'s tend not to be linear anymore (in fact, we will see in Corollary 19.9 that they are *never* linear). For example, here is a lifting of f_5 into $\pi_5 SU(3)$:

$$(19.4) \quad (u, z, w) \mapsto \begin{bmatrix} u - \bar{w}z & \bar{w}^2 & -\bar{z} - \bar{w}\bar{u} \\ z^2 & u + z\bar{w} & w - z\bar{u} \\ w + \bar{u}z & \bar{z} - \bar{u}\bar{w} & \bar{u}^2 \end{bmatrix}.$$

How did we get this map? There is an interesting technique from [Lundell], later refined and simplified in [PR], that we will now describe. Let $X \subseteq SU(d)$ be the subspace of matrices A having $a_{dd} = 0$. These are unitary transformations having the property that $A(e_d)$ lies in $\langle e_d \rangle^\perp$. Set $b = A(e_d)$. Under these circumstances we can follow A with the rotation in the plane $\langle e_d, b \rangle$ that rotates b back to e_d and e_d to $-b$. In this way we deform the original transformation into one that fixes e_d , and therefore lives in $SU(d-1)$. Here is a more precise statement:

Proposition 19.5. *There is a homotopy $H: X \times I \rightarrow SU(d)$ where H_0 is the inclusion and H_1 factors through $SU(d-1)$, given by*

$$H_t: \begin{bmatrix} E & b \\ c & 0 \end{bmatrix} \mapsto \begin{bmatrix} E - bc \sin(\frac{\pi}{2}t) & b \cos(\frac{\pi}{2}t) \\ c \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \end{bmatrix}.$$

Proof. On one level the proof is just a computation: one simply checks that the given formula always gives a matrix in $SU(d)$, and then the stated properties are immediate. Checking that the formula gives a unitary matrix is routine algebra, though verifying that the determinant is 1 is a little trickier.

The result is clearer when we interpret the given formula as post-composing the original unitary transformation $A = \begin{bmatrix} E & b \\ c & 0 \end{bmatrix}$ with a rotation. The fact that A is unitary implies that $E(e_1), \dots, E(e_{d-1})$ are orthogonal to $A(e_d) = b$. Write R_t for the rotation that fixes $\langle e_d, b \rangle^\perp$ and sends $e_d \mapsto \cos(\frac{\pi}{2}t)e_d - \sin(\frac{\pi}{2}t)b$ and $b \mapsto \sin(\frac{\pi}{2}t)e_d + \cos(\frac{\pi}{2}t)b$. Then R_t clearly lies in $SU(d)$. The composite $R_t A$ has the following behavior:

$$\begin{aligned} \text{For } 1 \leq i \leq d-1: \quad e_i &\mapsto Ee_i + c_i e_d \mapsto Ee_i + c_i (\cos(\frac{\pi}{2}t)e_d - \sin(\frac{\pi}{2}t)b) \\ e_d &\mapsto b \mapsto \cos(\frac{\pi}{2}t)b + \sin(\frac{\pi}{2}t)e_d. \end{aligned}$$

Note that this exactly reproduces the matrix formula in the statement of the proposition. $\square$

If a map $f: S^r \rightarrow SU(d)$ factors through X , Proposition 19.5 gives an explicit way to homotop it down into $SU(d-1)$. The condition about factoring through X seems rather specialized, but anytime there is a fixed matrix entry which vanishes for the entire image of f then there is a way to move that entry to the (d, d) -spot via rotations, so that in fact we land back inside of X . For example, start with the formula for f_5 in (19.2) and note the 0 in the $(3, 4)$ -entry. Left-multiply with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ 0 & 0 & \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{bmatrix}$$

to get a homotopy between the f_5 -map and the similar map where the last two rows have been switched and the (new) third row negated. Now we have the zero in the $(4, 4)$ -entry, and applying the formula from Proposition 19.5 spits out (19.4).

Exercise 19.6. Start with the formula for f_5 in (19.2) and apply $\mathcal{B}$ to get a linear element $f_7 \in \pi_7 SU(8)$. You will notice that the formula has many zeros in it. Convince yourself that after using the above method to reduce to an element in $\pi_7 SU(7)$ with quadratic entries one can apply the method *again* to reduce to an element in $\pi_7 SU(6)$ with quartic entries. Challenge: For this example, how long can one continue this process?

19.7. Linear homotopy classes. The formula of (19.3) is very reminiscent of similar formulas we saw when talking about vector fields on spheres and the Hurwitz-Radon problem, as well as in the (related) theory of Clifford modules. This is, of course, no surprise since Clifford modules are somehow fundamental to Bott periodicity. But let us tease this out a bit more from an elementary perspective.

Say that a map $S^r \rightarrow U(n)$ is **linear** if it has the form

$$(x_0, \dots, x_r) \mapsto x_0 A_0 + x_1 A_1 + \dots + x_r A_r$$

for some $A_0, \dots, A_r \in U(n)$. Note that given a linear map like this we can always left-multiply by A_0^{-1} and obtain a related map where $A_0 = Id$. From now on we will always assume that linear maps have been normalized in this way.

Proposition 19.8. *The elements $A_0, \dots, A_r \in U(n)$, with $A_0 = Id$, give a linear map if and only if $A_i^2 = -I$ and $A_i A_j = -A_j A_i$ for all $1 \leq i < j \leq r$. That is, the matrices $A_1, \dots, A_r$ give a unitary representation of the complex Clifford algebra $Cl_r(\mathbb{C})$ on $\mathbb{C}^n$. [And the same result holds with $U(n)$ replaced by $O(n)$ and the complex Clifford algebras replaced by the real ones.]*

Proof. The “if” direction is a simple verification that we leave to the reader. For the “only if” direction, assume we have a linear map $S^r \rightarrow U(n)$. Then, in particular, $x_0 I + x_1 A_1 \in U(n)$ whenever $x_0^2 + x_1^2 = 1$. So

$$I = (x_0 I + x_1 A_1)(x_0 I + x_1 A_1^\dagger) = (x_0^2 + x_1^2)I + x_0 x_1 (A_1 + A_1^\dagger),$$

and since this holds whenever $x_0^2 + x_1^2 = 1$ we conclude that $A_1 + A_1^\dagger = 0$. Then $A_1^{-1} = A_1^\dagger = -A_1$, so $A_1^2 = -I$. Similarly, $A_i^2 = -I$ for $i \geq 1$.

By the same kind of analysis one uses that $x_i A_i + x_j A_j \in U(n)$ whenever $x_i^2 + x_j^2 = 1$ to conclude that $A_i A_j^\dagger + A_j A_i^\dagger = 0$. Then use that $A_i^\dagger = A_i^{-1} = -A_i$ (and similarly for A_j) to conclude that A_i and A_j anti-commute. $\square$

We will call a collection of matrices $A_1, \dots, A_r$ satisfying the above conditions a set of **Hurwitz-Radon matrices**, or equivalently a set of **Clifford matrices**.

Corollary 19.9. *There exists a linear map $S^r \rightarrow U(n)$ if and only if n is a multiple of $2^{\lfloor \frac{r}{2} \rfloor}$. Likewise, there is a linear map $S^r \rightarrow O(n)$ if and only if n is a multiple of $2^{\sigma(r)}$ where $\sigma(r) = \#\{s \mid 0 < s \leq r \text{ and } s \equiv 0, 1, 2, \text{ or } 4 \pmod{8}\}$.*

Proof. This follows immediately from the representation theory of the Clifford algebras. All of the irreducible modules for Cl_r have the same dimension: $2^{\lfloor \frac{r}{2} \rfloor}$ in the complex case, and $2^{\sigma(r)}$ in the real case. Since the representation theory is semisimple, the dimension of any representation will be a sum of the dimensions of the irreducibles. $\square$

By just checking that the dimensions line up, we see from Corollary 19.9 that the linear maps $S^r \rightarrow U(n)$ produced by the $\mathcal{B}$ map—as in the discussion after Theorem 18.10—are the best possible. For example, the linear map $S^5 \rightarrow SU(4)$ cannot be lifted to a linear map into $SU(3)$, even though it does lift to a non-linear map into $SU(3)$ as we have seen. (We have not yet considered the orthogonal case but will do so shortly!)

We now understand that a unitary $Cl_r(\mathbb{C})$ -action on $\mathbb{C}^n$ gives a linear map $f: S^r \rightarrow U(n)$. If the action extends to a $Cl_{r+1}(\mathbb{C})$ -action then f extends to $S^{r+1} \rightarrow U(n)$, which means that f was null-homotopic (because the inclusion of S^r into S^{r+1} is null). We want to systemize this way of passing from Clifford modules to linear homotopy classes in order to get a better handle on it.

Let G_r be the Grothendieck group of finite-dimensional unitary $Cl_r(\mathbb{C})$ -modules. We can also make this construction for $Cl_r(\mathbb{R})$, and when necessary we will write

$G_r(\mathbb{C})$ and $G_r(\mathbb{R})$ to distinguish between the complex and real cases. The G_r groups are all free abelian with generators corresponding to isomorphism classes of irreducibles, since the representation theory of the Clifford algebras is semisimple.

The inclusion $\text{Cl}_r(\mathbb{C}) \hookrightarrow \text{Cl}_{r+1}(\mathbb{C})$ gives rise to a forgetful map from $\text{Cl}_{r+1}(\mathbb{C})$ -modules to $\text{Cl}_r(\mathbb{C})$ -modules, which in turn gives a map of abelian groups $G_{r+1} \rightarrow G_r$. Let $\mathcal{A}_r$ denote the cokernel. Our next goal will be to construct group homomorphisms

$$\mathcal{A}_r(\mathbb{C}) \rightarrow \pi_r U, \quad \mathcal{A}_r(\mathbb{R}) \rightarrow \pi_r O.$$

Let M be a finite-dimensional unitary $\text{Cl}_r(\mathbb{C})$ -module (recall this means that M comes equipped with a Hermitian inner product and that the Clifford generators act as unitary transformations). By picking a unitary basis $\epsilon_1, \dots, \epsilon_n$ for M we can represent each Clifford generator as acting by an element of $U(n)$, and thereby get a linear map $f_M: S^r \rightarrow U(n)$ as above. This depends on our choice of basis, though. Another choice of basis would change this to $P \cdot f_M \cdot P^{-1}$, where $P \in U(n)$ is the change-of-basis matrix. Fortunately, $U(n)$ is connected: a choice of path from Id to P will give a basepoint-preserving homotopy from f_M to $P f_M P^{-1}$, and so we find that the element $f_M \in \pi_r U(n)$ is independent of the choice of basis.

The analogous argument in the real case requires an extra step, because of the fact that the groups $O(n)$ are not connected. The matter again reduces to comparing a map $f_M: S^r \rightarrow O(n)$ to a conjugate $P f_M P^{-1}$ where $P \in O(n)$. There is a path in $O(n)$ from P to either Id or R , where R is the reflection $\text{diag}(-1, 1, 1, \dots, 1)$, and as before such a path gives a homotopy. So we reduce to comparing f_M to $R f_M R^{-1}$. We can re-interpret conjugation by R as conjugation by $-R$, and if n is odd then $-R$ has determinant 1 and so is path-connected to Id . So if n is odd then we are okay. But if n is even then we can compose with the inclusion $O(n) \hookrightarrow O(n+1)$ to reduce to the odd case. So the conclusion here is that, while $f_M \in \pi_r O(n)$ might depend on the choice of basis, the image in $\pi_r O$ does not.

Proposition 19.10. $f_{M \oplus N} = f_M + f_N$ in $\pi_r U$ (or $\pi_r O$ for the real case). Consequently, we obtain maps of abelian groups $G_r(\mathbb{C}) \rightarrow \pi_r(U)$ and $G_r(\mathbb{R}) \rightarrow \pi_r(O)$ as well as induced maps $\mathcal{A}_r(\mathbb{C}) \rightarrow \pi_r(U)$ and $\mathcal{A}_r(\mathbb{R}) \rightarrow \pi_r(O)$.

Proof. This boils down to checking that if $f, g: S^r \rightarrow U(n)$ then the product $fg: x \mapsto f(x)g(x)$ is homotopic to

$$f \oplus g: x \mapsto \begin{bmatrix} f(x) & 0 \\ 0 & g(x) \end{bmatrix}$$

when regarded as maps $S^r \rightarrow U(2n)$ (in the first case, via the inclusion $U(n) \hookrightarrow U(2n)$). That statement in turn follows from the claim that the two maps $U(n) \rightarrow U(2n)$ given by

$$(19.11) \quad A \mapsto \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad A \mapsto \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$$

are homotopic. To justify this claim, consider the path in $U(2n)$ given by

$$t \mapsto \begin{bmatrix} \cos(t)I & -\sin(t)I \\ \sin(t)I & \cos(t)I \end{bmatrix}$$

from $t = 0$ to $t = \frac{\pi}{2}$. Left-multiplication by this path shows that the identity on $U(2n)$ is homotopic to the map $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}$. Right-multiplication by

the path gives a similar homotopy between the identity and a map that involves switching column blocks instead of row blocks. Applying these two homotopies in succession gives the desired homotopy between the two maps of (19.11).

Check O case and basepoints. □

At this point we have constructed maps of graded groups $\mathcal{A}_*(\mathbb{C}) \rightarrow \pi_*(U)$ and $\mathcal{A}_*(\mathbb{R}) \mapsto \pi_*O$. We want to next understand how periodicity of the two sides interact. In each case, periodicity stems from a certain kind of multiplication that we now explain.

Let us first observe a clever “doubling” construction that one can apply to Clifford modules. If M is a Cl_r -module then consider the real vector space $M \oplus M$. This comes equipped with the map $f = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ which has $f^2 = -I$. Check that f anticommutes with $\begin{bmatrix} 0 & A \\ uA & 0 \end{bmatrix}$ (u a scalar constant) if and only if $u = 1$, and that f anticommutes with $\begin{bmatrix} B & 0 \\ 0 & uB \end{bmatrix}$ if and only if $u = -1$. This gives us a couple of ways of regarding $M \oplus M$ as a Cl_{r+1} -module, via the formulas $e_i \mapsto \begin{bmatrix} 0 & e_i \\ e_i & 0 \end{bmatrix}$, $e_{r+1} \mapsto f$ or via the formulas $e_i \mapsto \begin{bmatrix} e_i & 0 \\ 0 & -e_i \end{bmatrix}$, $e_{r+1} \mapsto f$.

Suppose given a Cl_r -module M and a Cl_s -module N . Can we merge these two actions into something like a Cl_{r+s} -module? Write the generators of Cl_r as $e_1, \dots, e_r$ and write the generators of Cl_s as $g_{r+2}, \dots, g_{r+s+1}$ (the strange indexing will make more sense in a moment). We need to construct a new module where the e ’s and g ’s act simultaneously and anticommute. If we had said “commute” rather than “anticommute”, there would be a standard solution: take $M \otimes N$ and consider the actions given by $e_i \otimes \text{id}_N$ and $\text{id}_M \otimes g_j$. To get anticommutativity instead we can take the double $(M \otimes N) \oplus (M \otimes N)$ and intertwine with the f -map in the following way:

$$\begin{aligned} e_i &\mapsto \begin{bmatrix} e_i \otimes \text{id}_N & 0 \\ 0 & -e_i \otimes \text{id}_N \end{bmatrix} & \text{for } 1 \leq i \leq r \\ e_{r+1} &\mapsto \begin{bmatrix} 0 & \text{id}_M \otimes \text{id}_N \\ -\text{id}_M \otimes \text{id}_N & 0 \end{bmatrix} \\ e_j &\mapsto \begin{bmatrix} 0 & -\text{id}_M \otimes g_j \\ -\text{id}_M \otimes g_j & 0 \end{bmatrix} & \text{for } r+2 \leq j \leq r+s+1. \end{aligned}$$

Note that we actually get a Cl_{r+s+1} -action in this way, not just a Cl_{r+s} -action. Write $M \otimes_c N$ for $(M \otimes N) \oplus (M \otimes N)$ with this Cl_{r+s+1} -action. We will call it the “Clifford tensor product”.

For a more algebraic perspective on this construction, see Exercise 19.32.

Remark 19.12. If the Clifford algebra Cl_r is thought of as containing generators $e_0, e_1, \dots, e_r$ with $e_0 = 1$, then the generators for the two Clifford algebras Cl_r and Cl_s merge to give

$$e_0, \dots, e_r, e_{r+1}, \dots, e_{r+s+2}$$

with e_{r+i} corresponding to e_i in Cl_s . In particular, e_{r+1} corresponds to the e_0 in Cl_s . Keeping this in mind helps parse the formulas in the definition of $M \otimes_c N$.

Remark 19.13. There were choices made of where to put the signs in the matrices defining the Clifford action on the tensor product. One is of course free to multiply any of the matrices by -1 and this produces a different module. The sign choices for the first set of matrices seem reasonably natural, but the signs that appear in e_{r+1} and in the last set of matrices are not at all intuitive.

For now we will just say that the sign choices come from taking a more natural construction that we will encounter later and deconstructing it. See Section 20.16.

One thing that is not at all obvious is that our choice of signs in the definition of the Clifford tensor product makes the following true:

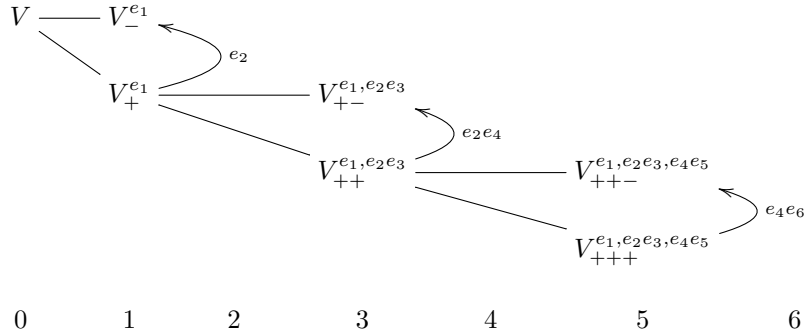
Proposition 19.14. *The Clifford tensor product is associative up to isomorphism: if M is a Cl_s -module, N is a Cl_t -module, and W is a Cl_u -module, then $(M \otimes_c N) \otimes_c W \cong M \otimes_c (N \otimes_c W)$ as $\text{Cl}_{s+t+u+2}$ -modules.*

The above result will be a simple consequence of work we do in later sections, see Exercise 20.23. Another approach, from a more hands-on perspective, is in Exercise 19.33.

The Clifford tensor product induces a product on the groups G_* , with the unfortunate grading $G_s \otimes G_t \rightarrow G_{s+t+1}$ (if this really bothers you, keep heart because we will eventually repair this oddity). By Proposition 19.14 this product is associative, and so it makes $\bigoplus_s G_s$ into a rng (a ring without identity element). It likewise induces pairings $\mathcal{A}_s \otimes \mathcal{A}_t \rightarrow \mathcal{A}_{s+t+1}$. We want to use this extra information to help understand the structure of the $\mathcal{A}_*$ groups. To this end, we need notation for talking about the irreducible Cl_n -modules before we go to the next stage of computing tensor products. We already saw this in the real case back in Figure 16.14.1, but let us give a quick tour of the complex version.

When n is even there is a unique (up to isomorphism) irreducible Cl_n -module, so there are no issues there: let us just denote it F_n . But when n is odd there are two non-isomorphic irreducibles, which we will call $F_{n,+}$ and $F_{n,-}$. In order to explain what we mean by these, let us recall how Clifford modules work via the following diagram:

FIGURE 19.14.1. Deconstruction of a $\text{Cl}_n(\mathbb{C})$ -module



Here are the instructions for parsing this:

- A Cl_0 -module is just a complex vector space V .
- To make V into a Cl_1 -module one must give the action of e_1 . But since $e_1^2 = -1$, the action will be diagonalizable and will split V into the $+i$ and $-i$ eigenspaces, denoted $V_+^{e_1}$ and $V_-^{e_1}$ (sometimes we will drop the superscripts for convenience).

- If the Cl_1 -action extends to a Cl_2 -action then since e_2 anti-commutes with e_1 it will have to interchange the two eigenspaces V_+ and V_- . In fact, multiplication by e_2 will have to give an isomorphism $V_+ \rightarrow V_-$ and so the Cl_1 -action extends if and only if V_+ and V_- have the same dimension.
- If the Cl_2 -action extends to a Cl_3 -action, then since e_2e_3 commutes with e_1 it will preserve each of the V_+ and V_- eigenspaces. But $(e_2e_3)^2 = -1$ so it will decompose V_+ (for example) into $+i$ and $-i$ eigenspaces, denoted here V_{++} and V_{+-} . Note that there will be a similar decomposition of V_- , but we don't need to include that because we obtain it from the V_+ -decomposition via multiplication by e_2 .
- To extend to a Cl_4 -action, the element e_2e_4 commutes with e_1 and so will have to preserve V_+ ; but it anticommutes with e_2e_3 and so will interchange V_{++} and V_{+-} and give an isomorphism between them.
- At the next stage, e_4e_5 commutes with e_1 and e_2e_3 and so will preserve the V_{++} eigenspace, and since $(e_4e_5)^2 = -1$ it splits it into $+i$ and $-i$ eigenspaces.
- At this point the pattern is clear: at each stage one writes down a certain element e_je_k which commutes with the previously chosen elements, or else anticommutes with the one immediately preceding it. In the first case the element splits a previous eigenspace into two, whereas in the second case it gives an isomorphism between the two previous eigenspaces.
- Note that the diagram has been set up so that columns 0 through n show the information that constitutes a Cl_n -module, with the understanding that the curved isomorphisms all lie in the column containing their label. And when we say “show”, of course the diagram only shows the splittings for the $V_{+,\dots,+}$ eigenspaces, since the splittings of the other eigenspaces can be obtained from these.

One can reverse the above decomposition process to *construct* Clifford modules very explicitly. Briefly, one starts with some chosen information at level n and then lets this freely generate the rest of the module by moving to the left in the diagram. For example, a Cl_3 -module is determined by the choice of V_{++} and V_{+-} . Let $F_{3,+}$ be the module where $V_{++} = \mathbb{C}$ and $V_{+-} = 0$ (and $F_{3,-}$ the opposite). Note that these two spaces determine V_+ , as it is their direct sum, which then determines V_- (as it is e_2V_+), which then determines V . So for example, we see that V is 2-dimensional in this case. For a Cl_4 -module we would specify V_{++} , then set $V_{+-} = e_2e_4V_{++}$, and then continue as above—this produces a 4-dimensional Cl_4 -module, and it is clear that when we restrict to Cl_3 we get $F_{3,+} \oplus F_{3,-}$.

Exercise 19.15. The module $F_{3,+}$ can be described by letting x be a generator of V_{++} , so that $e_1x = e_2e_3x = ix$. Check that x, e_2x gives a basis for V and that $e_3x = -ie_2x$. Verify that $e_1e_2e_3$ acts on $F_{3,+}$ as multiplication by -1 . Then do this for $F_{3,-}$ and show that $e_1e_2e_3$ acts as the identity instead.

Next do something similar for $F_{5,+}$: if x is a generator for V_{++++} show that x, e_2x, e_4x , and e_2e_4x form a basis for V , and compute the e_j multiplications on these basis elements for all j . Check that $e_1e_2e_3e_4e_5$ acts on $F_{5,+}$ as multiplication by $-i$. Likewise, check that for $F_{5,-}$ the action is by $+i$.

It is often the case that one has a Cl_{2n+1} -module M and by a dimension count one knows it is isomorphic to either $F_{2n+1,+}$ or $F_{2n+1,-}$; but how do we decide which one? The above exercise gives a clue. Note that $(e_1e_2 \cdots e_{2n+1})^2 = (-1)^{n+1}$

and so multiplication by $e_1 e_2 \cdots e_{2n+1}$ will have eigenvalues $\pm i^{n+1}$ (that is, ± 1 when n is odd and $\pm i$ when n is even).

Proposition 19.16. *The element $e_1 e_2 \cdots e_{2n+1}$ acts as multiplication by i^{n+1} on $F_{2n+1,+}$ and as multiplication by $-i^{n+1}$ on $F_{2n+1,-}$.*

Proof. Set $V = F_{2n+1,+}$ and let $\omega = e_1 e_2 \cdots e_{2n+1}$. If x is a generator for $V_{++\dots+}$ then a $\mathbb{C}$ -basis for V is given by the set of elements $e_{2j_1} e_{2j_2} \cdots e_{2j_k} x$ for $j_1 < j_2 < \cdots < j_k$ (exercise). Each Clifford generator e_r commutes with ω , therefore if we prove $\omega x = \lambda x$ for some $\lambda \in \mathbb{C}$ then α acts by multiplication by λ on all of V .

Now we just observe that $\omega = (1 \cdot e_1)(e_2 e_3)(e_4 e_5) \cdots (e_{2n} e_{2n+1})$ and each pair in parentheses acts on x as multiplication by i . So $\omega x = i^{n+1} x$, as desired.

In the case of $V = F_{2n+1,-}$ we have $x \in V_{++\dots-}$ and so each pair in the above decomposition of ω acts on x as multiplication by i except for the final pair, which acts as multiplication by $-i$. $\square$

With these considerations in mind we can proceed to analyze some tensor products:

Proposition 19.17. *One has isomorphisms*

- (i) $F_0 \otimes_{\mathbb{C}} F_{2n} \cong F_{2n+1,+} \oplus F_{2n+1,-} \cong F_{2n} \otimes_{\mathbb{C}} F_0$
- (ii) $F_0 \otimes_{\mathbb{C}} F_{2n+1,+} \cong F_{2n+2} \cong F_{2n+1,+} \otimes_{\mathbb{C}} F_0$
- (iii) $F_0 \otimes_{\mathbb{C}} F_{2n+1,-} \cong F_{2n+2} \cong F_{2n+1,-} \otimes_{\mathbb{C}} F_0$
- (iv) $F_{1,+} \otimes_{\mathbb{C}} F_{2n+1,\pm} \cong F_{2n+3,\mp}$
- (v) $F_{1,-} \otimes_{\mathbb{C}} F_{2n+1,\pm} \cong F_{2n+3,\pm}$.

Before proving all of these isomorphisms let us give the immediate consequence:

Corollary 19.18. *The group $\mathcal{A}_{2n+1}$ is generated by the element $(F_{1,+})^n$ (or alternatively, by $(F_{1,-})^n$).*

Proof. We know that $\mathcal{A}_{2n+1}$ is generated by $F_{2n+1,+}$, and that $[F_{2n+1,+}] = -[F_{2n+1,-}]$. So it suffices to observe that Proposition 19.17(iv) shows that $F_{2n+1,-}$ is isomorphic to the n -fold Clifford tensor product of $F_{1,-}$ with itself, or $[F_{2n+1,-}] = [F_{1,-}]^n$. $\square$

Proof of Proposition 19.17. Note that F_{2n} has dimension 2^n as a $\mathbb{C}$ -vector space, as does $F_{2n+1,\pm}$. Statements (ii) and (iii) are easy, since even Clifford algebras have exactly one irreducible and so the isomorphism class of a module is determined by its dimension over $\mathbb{C}$. So, for example, $F_0 \otimes_{\mathbb{C}} F_{2n+1,+}$ and F_{2n+2} both have dimension 2^{n+1} and hence are isomorphic.

For $F_0 \otimes_{\mathbb{C}} F_{2n}$, the dimension over $\mathbb{C}$ is 2^{n+1} and so the module can only be isomorphic to one of $F_{2n+1,+} \oplus F_{2n+1,+}$, $F_{2n+1,+} \oplus F_{2n+1,-}$, and $F_{2n+1,-} \oplus F_{2n+1,-}$. But note that the Cl_0 -action on F_0 extends to a Cl_1 -action, and so the Cl_{2n+1} -action on $F_0 \otimes_{\mathbb{C}} F_{2n}$ extends to a Cl_{2n+2} -action. That is, the tensor product is in the image of $G_{2n+2} \rightarrow G_{2n+1}$. This is enough to identify it is as the second of the above options.

Finally, we tackle part (iv) (the proof for (v) is similar). A dimension calculation shows that the Cl_{2n+3} -module $F_{1,+} \otimes_{\mathbb{C}} F_{2n+1,+}$ must be isomorphic to either $F_{2n+3,+}$ or $F_{2n+3,-}$, so we use Proposition 19.16 to decide which one. The action of the Clifford generators on the tensor product is given by the matrices

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -e_1 \\ -e_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -e_2 \\ -e_2 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -e_{2n+1} \\ -e_{2n+1} & 0 \end{bmatrix}$$

and the product of these matrices is $\omega = \begin{bmatrix} -ie_1 \cdots e_{2n+1} & 0 \\ 0 & -ie_1 \cdots e_{2n+1} \end{bmatrix}$. But $e_1 \cdots e_{2n+1}$ acts on $F_{2n+1,+}$ as multiplication by i^{n+1} , so the action of ω on $F_{1,+} \otimes_c F_{2n+1,+}$ is multiplication by $-i^{n+2}$. This shows we must have $F_{1,+} \otimes_c F_{2n+1,+} \cong F_{2n+3,-}$, and the analysis of $F_{1,+} \otimes_c F_{2n+1,-}$ is identical except for the sign change. $\square$

We are in the midst of analyzing $\mathcal{A}_* \rightarrow \pi_* U$, and we have now introduced and understood a rng structure on the domain. The same types of formulas lead to a rng structure on the target. Indeed, for $f: S^s \rightarrow U(n)$ and $g: S^t \rightarrow U(q)$ define $f * g: S^{s+t+1} \rightarrow U(2nq)$ to be given by

$$(19.19) \quad (f * g)(u, x) = \begin{bmatrix} |u|f(\hat{u}) \otimes I & I \otimes |x|g(\hat{x})^\dagger \\ -I \otimes |x|g(\hat{x}) & |u|f(\hat{u})^\dagger \otimes I \end{bmatrix}$$

for $(u, x) \in \mathbb{R}^{s+1} \oplus \mathbb{R}^{t+1}$ such that $|u|^2 + |x|^2 = 1$. Here we have written $\hat{u} = \frac{u}{|u|}$ etc, and are interpreting the matrix as a transformation of $(\mathbb{C}^n \otimes \mathbb{C}^q) \oplus (\mathbb{C}^n \otimes \mathbb{C}^q)$. One readily computes that the transformation is unitary.

It is clear that $f * g$ only depends on the homotopy classes $f \in \pi_s U(n)$ and $g \in \pi_t U(q)$, since for example a homotopy from f to f' could be essentially substituted into the above formula to give a homotopy from $f * g$ to $f' * g$. If f (respectively, g) is the constant map at the identity then $f * g$ is null-homotopic, since the definition can be extended to pairs $(u, x) \in S(\mathbb{R}^{s+2} \oplus \mathbb{R}^{t+1})$ (respectively, $S(\mathbb{R}^{s+1} \oplus \mathbb{R}^{t+2})$). From this it immediately follows that the definition of $f * g$ is compatible with stabilization, so that it induces a pairing $\pi_s U \times \pi_t U \rightarrow \pi_{s+t+1} U$. To check bilinearity use the model of addition in $\pi_* U$ which is induced from block sum of matrices. See Exercise 19.30 for more details.

The definitions have been set up so that it follows immediately that $\mathcal{A}_* \rightarrow \pi_* U$ is a rng map (and similarly for the orthogonal case).

Exercise 19.20.

- Verify that $\mathcal{A}_* \rightarrow \pi_* U$ is a rng homomorphism.
- Show that $F_{1,+}$ is sent to the homotopy element represented by the map $f: S^1 \rightarrow U(1)$ given by $(x, y) \mapsto (x + iy)$.
- Verify that $f * (-): \pi_t U(n) \rightarrow \pi_{t+2} U(2n)$ equals the operation $\mathcal{B}_* \Sigma^2$ from Theorem 18.10. [Hint: Comparing to formula (18.9) there is a sign discrepancy, but see Remark 18.7 for how to resolve it.]

A short summary of our situation is:

- From algebraic computation we know all of the groups $\mathcal{A}_*$, the ring structure, and the fact that multiplication by the generator in $\mathcal{A}_1$ gives periodicity isomorphisms.
- If we had reason to know that $\mathcal{A}_* \rightarrow \pi_* U$ is an isomorphism of groups then we would deduce Bott periodicity for $\pi_* U$, but unfortunately we have no a priori explanation of that fact.
- We do know the groups $\pi_* U$ by the Bott-Morse arguments from previous sections. By Exercise 19.20, the content of Theorem 18.10 shows that in the rng $\pi_* U$ multiplication by the generator in degree 1 gives a periodicity isomorphism.
- It then *follows* from algebra that $\mathcal{A}_* \rightarrow \pi_* U$ is an isomorphism in all degrees. We record this important fact for later use:

Theorem 19.21. *The map $\mathcal{A}_* \rightarrow \pi_* U$ is an isomorphism of graded rngs.*

Proof. Follows from Theorem 18.10 by the above reasoning. $\square$

Let us next take a look at the orthogonal case, starting by determining the ring structure on $G_*(\mathbb{R})$. Much of the hard work can be avoided by using the complexification map $G_*(\mathbb{R}) \rightarrow G_*(\mathbb{C})$ (which is a ring homomorphism) together with what we already know about the codomain, but we still need to get our hands dirty in a few cases.

We know that $\text{Cl}_{4n+3}(\mathbb{R})$ has two non-isomorphic irreducible representations, and once again we need to know a quick way to distinguish them. Based on our experience from the complex case, we know what to look for.

Proposition 19.22. *In $\text{Cl}_{4n+3}(\mathbb{R})$ let $\omega = e_1 e_2 \cdots e_{4n+3}$. On an irreducible representation ω must act as multiplication by 1 or -1 , and both actually occur. So the two irreducible representations are distinguished by this property.*

Proof. Let W be an irreducible representation of Cl_{4n+3} . One checks that $\omega^2 = 1$, and so W splits into the $+1$ and -1 -eigenspaces for W . But ω commutes with all of the e_i 's, and so these eigenspaces will actually be Cl_{4n+3} -submodules. Since W was irreducible, one of them is zero and the other is all of W .

Given a representation of a Clifford algebra Cl_k , we can make a new representation by changing the sign on the e_1 -action. It is clear that this preserves irreducibles, since it preserves direct sums and doing it twice gives back the original representation. So if we have an irreducible action of Cl_{4n+3} where ω acts as multiplication by $u \in \{1, -1\}$, this procedure gives another irreducible where ω acts by $-u$. $\square$

To work out the rng structure on $G_*(\mathbb{R})$ we need to assign some names to elements. Let $x = \mathbb{R}$ be the irreducible representation of $\text{Cl}_0(\mathbb{R})$, and let y_+ and y_- be the irreducible representations of $\text{Cl}_3(\mathbb{R})$ where $\omega_3 = e_1 e_2 e_3$ acts as multiplication by $+1$ and by -1 , respectively. Likewise, let z_+ and z_- be the irreducible representations of $\text{Cl}_7(\mathbb{R})$ defined similarly. Finally, let g_5 and g_6 denote the unique irreducible representations of $\text{Cl}_5(\mathbb{R})$ and $\text{Cl}_6(\mathbb{R})$.

In processing the next result, remember that if M and N are Clifford representations then $\dim(M \otimes_c N) = 2 \dim M \cdot \dim N$.

Proposition 19.23. *Positive generators of $G_*(\mathbb{R})$ and $G_*(\mathbb{C})$ up through degree eight, together with the behavior of the complexification map on these generators, are as shown in Figure 19.23.1. The bracketed formulas on the left and right give relations in the two rngs, and in addition both rngs are commutative.*

Numbers in parentheses indicate the dimension of the module over the underlying ground field. Δ indicates that a generator maps to the sum of the two shown generators, 2 indicates that a generator is sent to twice the indicated generator of the target, and in all other cases the shown generators map to the shown generators. Finally, $(z_+)^2 = (z_-)^2$ and multiplication by z_+ is an isomorphism $G_i(\mathbb{R}) \rightarrow G_{i+8}(\mathbb{R})$ for all i .

Proof. The relations in $G_*(\mathbb{C})$ were already established by Proposition 19.17. The information there shows that the elements a , b_+ , and b_- all commute, and they give

$$(b_+)^n = \begin{cases} F_{2n-1,+} & \text{if } n \text{ is odd,} \\ F_{2n-1,-} & \text{if } n \text{ is even,} \end{cases} \quad (b_+)^{n-1} b_- = \begin{cases} F_{2n-1,-} & \text{if } n \text{ is even,} \\ F_{2n-1,+} & \text{if } n \text{ is odd.} \end{cases}$$

FIGURE 19.23.1. $G_*(\mathbb{R})$ and $G_*(\mathbb{C})$

	$G_*(\mathbb{R})$	$G_*(\mathbb{C})$
	$x(1) \longrightarrow$	$a(1)$
	$x^2(2) \xrightarrow{\Delta}$	$b_+, b_-(1) \quad [a^2 = b_+ + b_-]$
	$x^3(4) \xrightarrow{2}$	$ab_+(2) \quad [ab_- = ab_+]$
$[x^4 = y_+ + y_-]$	$y_+, y_-(4) \xrightarrow{2,2}$	$b_+^2, b_+b_-(2) \quad [b_-^2 = b_+^2]$
$[xy_+ = xy_-]$	$xy_+(8) \xrightarrow{2}$	$ab_+^2(4)$
$[x^2y_+ = 2g_5]$	$g_5(8) \xrightarrow{\Delta}$	$b_+^3, b_+^2b_-(4)$
$[xg_5 = 2g_6]$	$g_6(8) \longrightarrow$	$ab_+^3(8)$
$[y_+^2 = y_-^2 = 4z_-, y_+y_- = 4z_+]$	$z_-, z_+(8) \longrightarrow$	$b_+^4, b_+^3b_-(8)$
$[xz_- = xz_+]$	$xz_-(16) \longrightarrow$	$ab_+^4(16).$

So the elements listed in the table generate the groups in each dimension. Let us also recall for later use that ω_{2n-1} acts on $F_{2n-1,\pm}$ as multiplication by $\pm i^n$.

Modules over Cl_1 and Cl_2 are determined by their dimension, so one readily sees that x^2 and x^3 are the irreducibles there. If c denotes complexification then we start with $c(x) = a$ and then get $c(x^2) = a^2 = b_+ + b_-$ and $c(x^3) = a^3 = a(b_+ + b_-) = 2ab_+$.

Recall that $y_{\pm}$ has the property that $e_1e_2e_3$ acts as ± 1 . Since x^2 is $\mathbb{C}$ with e_1 acting as i , $x^4 = x^2 \otimes_c x^2$ is $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ with Cl_3 -structure given by the matrices

$$\begin{bmatrix} i \otimes 1 & 0 \\ 0 & -i \otimes 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \otimes 1 \\ -1 \otimes 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \otimes i \\ -1 \otimes i & 0 \end{bmatrix}.$$

So the product $e_1e_2e_3$ acts as $\begin{bmatrix} -i \otimes i & 0 \\ 0 & -i \otimes i \end{bmatrix}$, which has a 4-dimensional eigenspace for eigenvalue $+1$ and a 4-dimensional eigenspace for eigenvalue -1 (exercise). So $x^4 = y_+ + y_-$.

The complexification $c(y_+)$ has dimension 8, and so it is a sum of two of the irreducibles. But $c(y_+)$ has $e_1e_2e_3$ acting as 1, the action on $b_+ = F_{3,-}$ is by 1, and on $b_+b_- = F_{3,+}$ is by -1 . So $c(y_+) = 2b_+^2$. Likewise, $c(y_-) = 2b_+b_-$.

Both xy_+ and xy_- must be the generator of $G_4(\mathbb{R})$ for dimension reasons, and $c(xy_+) = a(2b_+^2) = 2ab_+^2$.

The irreducible Cl_5 -module g_5 has dimension 8 over $\mathbb{R}$, so dimension arguments show $x^2y_+ = 2g_5$. Since $c(x^2y_+) = 2a^2b_+^2 = 2b_+^3 + 2b_+^2b_-$, we conclude that $c(g_5) = b_+^3 + b_+^2b_-$. The same kind of reasoning shows that $xg_5 = 2g_6$ and $c(g_6) = ab_+^3$.

The module z_+ has dimension 8 and $\omega_7 = e_1e_2 \cdots e_7$ acting as $+1$, so $c(z_+)$ has those same properties. But ω_7 acts on $b_+^4 = F_{7,-}$ and -1 , and on $b_+^3b_- = F_{7,+}$ as $+1$. So $c(z_+) = b_+^3b_-$, and likewise $c(z_-) = b_+^4$. Computing $c(xy_6) = b_+^4 + b_+^3b_-$

then yields that $xy_6 = z_+ + z_-$, and similar reasoning shows $y_+^2 = y_-^2 = 4z_-$ and $y_+y_- = 4z_+$.

We know the dimensions of the generators for $G_s(\mathbb{R})$ for all s , and a dimension count shows that multiplication by z_- (or z_+) must map generators to generators in all degrees except possibly those congruent to 3 modulo 4. In those exceptional degrees one must give a bit more argument. If M is a Cl_r -module then the Clifford generators act on $M \otimes_c z_-$ via the matrices

$$\begin{bmatrix} e_1 I & 0 \\ 0 & -e_1 I \end{bmatrix}, \dots, \begin{bmatrix} e_r I & 0 \\ 0 & -e_r I \end{bmatrix}, \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \begin{bmatrix} 0 & -Ie_1 \\ -Ie_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -Ie_7 \\ -Ie_7 & 0 \end{bmatrix},$$

where we have written $e_1 I$ instead of $e_1 \otimes I$ for typographical reasons. The product of these matrices is

$$\omega_{r+8} = \begin{bmatrix} -\omega_r \otimes \omega_7 & 0 \\ 0 & -\omega_r \otimes \omega_7 \end{bmatrix}.$$

So if ω_r acts with eigenvalue λ on M , then ω_{r+8} acts with eigenvalue λ on $M \otimes_c z_-$. From this it immediately follows that multiplication by z_- is an isomorphism $G_{4s+3}(\mathbb{R}) \rightarrow G_{4s+11}(\mathbb{R})$ for all $s \geq 0$. The similar analysis of the behavior of ω_{r+8} on $M \otimes_c z_+$ reveals that $(z_-)^2 = (z_+)^2$. $\square$

Exercise 19.24. The above proposition does not list all the relations in $G_*(\mathbb{R})$, but the rest are easy to work out. For example, prove that $y_+g_5 = 2x^2z_+$.

Corollary 19.25. *The rng $\mathcal{A}_*(\mathbb{R})$ is generated by $x \in \mathcal{A}_1(\mathbb{R})$, $y \in \mathcal{A}_3(\mathbb{R})$, and $z \in \mathcal{A}_7(\mathbb{R})$ subject to the relations*

$$2x = 0, \quad x^3 = 0, \quad xy = 0, \quad y^2 = 4z$$

together with commutativity.

Proof. Take $y = y_+$ and $z = z_+$. The rest is left to the reader. $\square$

Now consider the square

$$\begin{array}{ccc} \mathcal{A}_*(\mathbb{R}) & \longrightarrow & \mathcal{A}_*(\mathbb{C}) \\ \downarrow & & \downarrow \\ \pi_* O & \longrightarrow & \pi_* U. \end{array}$$

We know all of the groups involved. We also know the top maps by Proposition 19.23, we know the bottom maps by Exercise 15.16, and we know the right vertical map is an isomorphism by Theorem 19.21. Thinking it through, this immediately yields that $\mathcal{A}_*(\mathbb{R}) \rightarrow \pi_* O$ is an isomorphism except possibly in the cases where $* \in \{0, 1\} \pmod{8}$. It only takes a little more legwork to cover these cases as well:

Theorem 19.26. *The map $\mathcal{A}_*(\mathbb{R}) \rightarrow \pi_* O$ is an isomorphism of graded rngs.*

Proof. We already know this is a rng map. As remarked above, the comparison to the complex case shows immediately that the map is an isomorphism in degrees not congruent to 0 or 1 mod 8. It remains to check these final cases. For this, consider instead the square

$$\begin{array}{ccc} \mathcal{A}_*(\mathbb{C}) & \longrightarrow & \mathcal{A}_*(\mathbb{R}) \\ \downarrow & & \downarrow \\ \pi_* U & \xrightarrow{r} & \pi_* O. \end{array}$$

Here the top map is induced by restriction of scalars along the evident maps of algebras $\mathrm{Cl}_i(\mathbb{R}) \rightarrow \mathrm{Cl}_i(\mathbb{C})$. It is a quick check that this square commutes. When $*$ is equivalent to 1 mod 8 the square takes the form

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \\ \cong \downarrow & & \downarrow \\ \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/2. \end{array}$$

The left vertical map is an isomorphism by Theorem 19.21 and the bottom horizontal map is projection by Exercise 15.16. It follows at once that the right vertical map must be nonzero, hence an isomorphism.

It remains to analyze the case where $*$ $\equiv$ 0 mod 8. We claim this follows from the other cases, the rng structure on $\mathcal{A}_*(\mathbb{R})$, and the fact that $\mathcal{A}_*(\mathbb{R}) \rightarrow \pi_*O$ is a rng homomorphism. Call this map ϕ . In degree $8n+1$ this is a map $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2$, so we only need show it is nonzero. The group $\mathcal{A}_{8n+1}(\mathbb{R})$ is generated by $\beta^n x$, and the group $\mathcal{A}_{8n+2}(\mathbb{R})$ is generated by $\beta^n x^2$. We already know that $\phi(\beta^n x^2) \neq 0$ by the previous analysis, and so $\phi(\beta^n x) \cdot \phi(x) \neq 0$. Therefore $\phi(\beta^n x) \neq 0$, and we are done. $\square$

Remark 19.27. The above result demonstrates a useful principle: often some hard work regarding real K -theory can be avoided by instead pulling the information from the complex case via the two maps relating them. It is almost as if the quadruple (O, U, c, r) should be regarded as one object. This technique comes up often enough that it is worth keeping near the top of one's toolbox.

Theorem 19.26 now lets us (finally) give the real analog of Theorem 18.10:

Corollary 19.28. *Let $\beta_{\mathbb{R}}$ be a generator for $\pi_7 O \cong \mathbb{Z}$, for example the generator arising from either of the fundamental representations of $\mathrm{Cl}_7(\mathbb{R})$ on $\mathbb{R}^8$. Then the maps*

$$\pi_i O \xrightarrow{\beta_{\mathbb{R}} \cdot (-)} \pi_{i+8} O$$

are isomorphisms for every $i \geq 0$.

Proof. Immediate from Theorem 19.26. $\square$

Let us also explicitly describe the generators for $\pi_* O$ in low degrees. These are obtained directly from $\mathcal{A}_*(\mathbb{R}) \rightarrow \pi_* O$.

$i = 0$: $S^0 \rightarrow O(1) = \{+1, -1\}$, basepoint $\mapsto 1$ and nonbasepoint $\mapsto -1$

$i = 1$: $S^1 \rightarrow O(2)$, $\underline{a} = (a_0, a_1) \mapsto$ multiplication by $a_0 + ia_1$ on $\mathbb{C}$

$i = 3$: $S^3 \rightarrow O(4)$, $\underline{a} \mapsto$ left multiplication by $a_0 + a_1 i + a_2 j + a_3 k$ on $\mathbb{H}$.

$i = 7$: $S^7 \rightarrow O(8)$, $\underline{a} \mapsto$ left multiplication by $\underline{a}$ on $\mathbb{O}$.

The last line takes a little explanation. The octonions can be constructed by starting with $\mathbb{R}^8$ and defining products of basis elements by explicit formulas. Let the basis elements be denoted $e_0, \dots, e_7$ with $e_0 = 1$. The products satisfy $e_i^2 = -1$ for $i \geq 1$ and $e_i e_j = -e_j e_i$ for $1 \leq i < j$. It turns out that one gets a map of algebras $\mathrm{Cl}_7 \rightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{O})$ by sending e_i to left multiplication by e_i . This would

be immediate if $\mathbb{O}$ were associative, but lack of associativity forces one to work a little harder. The key observation is that $e_i(e_jx) = -e_j(e_ix)$ for all $x \in \mathbb{O}$ and for all distinct $i, j > 0$. This property can be checked by brute force from the defining relations of $\mathbb{O}$, or deduced with less trouble from the Cayley-Dickson construction of $\mathbb{O}$. (Note that we are not claiming $e_i(e_jx) = (e_ie_j)x$, which would give associativity.) Since $\dim_{\mathbb{R}} \mathbb{O} = 8$, the only possibility is that $\mathbb{O}$ is one of the two irreducible representations of Cl_7 . It doesn't matter which one it is, as they both give generators for $\mathcal{A}_7(\mathbb{R})$.

Remark 19.29. We want to stress that the main theorems of this section have used the fact that we previously computed π_*U and π_*O , as well as the comparison maps c and r between them, via the Morse-theoretic arguments. Results like Corollary 19.28 are an appealing and informative repackaging of Bott periodicity, but in our current telling of the story they do not offer an independent proof.

Theorem 19.26 almost brings to a close our long journey over the last several sections. We have at this point computed the groups π_*U and π_*O together with their ring structure, and moreover have seen how Clifford algebras give rise to a nice algebraic model for these rings via $\mathcal{A}_*(\mathbb{C})$ and $\mathcal{A}_*(\mathbb{R})$. The only thing that remains is for us to connect all of this back to K -theory.

A slight complaint is that the ring structures on $\mathcal{A}_*$ and π_*U (resp. $\mathcal{A}_*(\mathbb{R})$ and π_*O) feel a bit strange and ad hoc. As we connect back to K -theory we will be able to fix this and understand the true origins of those structures. This will be our task in the next section.

◦ Exercises ◦

Exercise 19.30. This exercise concerns the operation that takes $f \in \pi_s U(n)$ and $g \in \pi_t U(q)$ and produces $f * g \in \pi_{s+t+1} U(2nq)$.

- (a) Let f be the constant map at the identity. Then formula (19.19) for $(f * g)(u, x)$ decomposes into blocks of the form

$$(19.31) \quad \begin{bmatrix} |u|I_q & |x|g(\hat{x})^\dagger \\ -|x|g(\hat{x}) & |u|I_q \end{bmatrix}.$$

Treating this block as a map $P: S^{s+t+1} \rightarrow U(2q)$, produce an explicit formula for a homotopy $H: S^{s+t+1} \times I \rightarrow U(2nq)$ showing that P is null-homotopic.

[Hint: Formula (19.31) makes sense for $(u, x) \in S(\mathbb{R}^{s+2} \times \mathbb{R}^{t+1})$. When $u \in \mathbb{R}^{s+1}$ there is a canonical path from (u, x) to $(e_{s+2}, 0)$.]

- (b) Use (a) to prove that if f is the constant map then $f * g$ is null.
(c) Repeat (a) and (b) for the case where g is the constant map, but f is not.
(d) Prove that $f * g$ yields a well-defined operation $\pi_s U \otimes \pi_t U \rightarrow \pi_{s+t+1} U$. [Hint: Use ?????]

For the following exercises it will be convenient to write Cl_{s+1+t} for the Clifford algebra on generators $e_1, \dots, e_s, e_{s+1}, e_{s+2}, \dots, e_{s+1+t}$, with the idea that $e_1, \dots, e_s$ form a distinguished collection of size s , e_{s+1} is distinguished as a single element, and $e_{s+2}, \dots, e_{s+1+t}$ form a distinguished collection of size t . Expressions like $\text{Cl}_{s+1+t+1+u}$ will be interpreted likewise: it just means that the Clifford generators are grouped into subsets of the given sizes.

Exercise 19.32. Let M be a Cl_s -module and N be a Cl_t -module. Regard Cl_s as a subalgebra of Cl_{s+1+t} in the usual way, and let $k: \text{Cl}_t \hookrightarrow \text{Cl}_{s+1+t}$ send $e_i \mapsto e_{s+1+i}e_{s+1}$ for $1 \leq i \leq t$. Check that k is an inclusion of algebras and that it gives rise to an embedding $\text{Cl}_s \otimes \text{Cl}_t \hookrightarrow \text{Cl}_{s+1+t}$.

Verify that an isomorphism $M \otimes_c N \longrightarrow \text{Cl}_{s+1+t} \otimes_{(\text{Cl}_s \otimes \text{Cl}_t)} (M \otimes N)$ is given by sending w in the first summand of $M \otimes_c N$ to $1 \otimes w$ and w in the second summand to $-e_{s+1} \otimes w$.

Exercise 19.33. Continuing the previous exercise, let W be a Cl_u -module. Show that $(M \otimes_c N) \otimes_c W$ may be identified with $\text{Cl}_{s+1+t+1+u} \otimes_{(\text{Cl}_s \otimes \text{Cl}_t \otimes \text{Cl}_u)} (M \otimes N \otimes W)$ for a certain embedding $\text{Cl}_s \otimes \text{Cl}_t \otimes \text{Cl}_u \hookrightarrow \text{Cl}_{s+1+t+1+u}$. Likewise, show that $M \otimes_c (N \otimes_c W)$ may be identified with $\text{Cl}_{s+1+t+1+u} \otimes_{(\text{Cl}_s \otimes \text{Cl}_t \otimes \text{Cl}_u)} (M \otimes N \otimes W)$ for another embedding $\text{Cl}_s \otimes \text{Cl}_t \otimes \text{Cl}_u \hookrightarrow \text{Cl}_{s+1+t+1+u}$. Check that the two embeddings are the same and thereby prove associativity of $\otimes_c$. [This takes some care, but requires nothing tricky.]

20. GRADED CLIFFORD MODULES AND K -THEORY

The homotopy groups of U and BU are the same but with a shift. We have seen that the groups $\pi_* U$ are intimately related to Clifford modules, but it turns out that the groups $\pi_* BU$ are better described using *graded* Clifford modules. This might seem odd: given that the two sets of groups are essentially the same, why not just use the ungraded Clifford modules to describe both? But the algebra of graded modules turns out to be a better fit for the algebra of $\pi_* BU$, most notably the ring structure.

20.1. Gradings. A vector space V is said to be $\mathbb{Z}/2$ -**graded** if it is equipped with a decomposition $V = V_0 \oplus V_1$ into an “even” part and an “odd” part, labelled by 0 and 1. The degree is also called the parity in this context. If V and W are two $\mathbb{Z}/2$ -graded vector spaces then $\text{Hom}(V, W)$ becomes $\mathbb{Z}/2$ -graded by the formulas

$$\underline{\text{Hom}}_0(V, W) = \text{Hom}(V_0, W_0) \oplus \text{Hom}(V_1, W_1),$$

$$\underline{\text{Hom}}_1(V, W) = \text{Hom}(V_0, W_1) \oplus \text{Hom}(V_1, W_0).$$

So the even maps are parity-preserving, and the odd maps are parity-reversing. There is a corresponding tensor product, called the **graded tensor product**:

$$(V \hat{\otimes} W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \quad (V \hat{\otimes} W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

Exercise 20.2. Check that these definitions yield a natural adjunction isomorphism $\underline{\text{Hom}}(V \hat{\otimes} W, Z) \cong \underline{\text{Hom}}(V, \underline{\text{Hom}}(W, Z))$.

For any ungraded vector space V , write $V[j]$ ($j = 0, 1$) for V regarded as a $\mathbb{Z}/2$ -vector space concentrated entirely in degree j . Note the natural isomorphism $V[j] \hat{\otimes} W[k] \cong (V \otimes W)[j+k]$, where $j+k$ is interpreted modulo 2. From now on we will stop saying “modulo 2” when it is clear from context. Also note that $\mathbb{R}[0]$ is the unit for the graded tensor product.

A $\mathbb{Z}/2$ -graded algebra A is a $\mathbb{Z}/2$ -graded vector space equipped with a unit $\eta: \mathbb{R}[0] \rightarrow A$ and a multiplication map $\mu: A \hat{\otimes} A \rightarrow A$ that is associative and unital. In down to earth terms, A comes with a grading $A = A_0 \oplus A_1$ where the unit lives in degree 0 and the product of two homogenous elements adds the degree. One likewise defines $\mathbb{Z}/2$ -graded modules over A in the expected way.

If A and B are $\mathbb{Z}/2$ -graded algebras then $A \hat{\otimes} B$ inherits a $\mathbb{Z}/2$ -graded algebra structure, but this happens in more than one way: there are some choices regarding signs. The choice that turns out to be most useful for us is the one following the Koszul sign rule, namely

$$(20.3) \quad (a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2.$$

The maps $A \rightarrow A \hat{\otimes} B$ and $B \rightarrow A \hat{\otimes} B$ given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ are maps of algebras, the even elements of A and B map to elements that commute with each other, whereas the odd elements of A and B map to elements that anti-commute with each other. If M is an A -module and N is a B -module then $M \hat{\otimes} N$ becomes an $A \hat{\otimes} B$ -module via the analogous formula.

Exercise 20.4. If A and B are $\mathbb{Z}/2$ -graded algebras check that there is an isomorphism of $\mathbb{Z}/2$ -graded algebras $A \hat{\otimes} B \rightarrow B \hat{\otimes} A$ given by $a \otimes b \mapsto (-1)^{|a| \cdot |b|} b \otimes a$ whenever a and b are homogeneous.

Clifford algebras are naturally $\mathbb{Z}/2$ -graded. If (V, q) is a vector space with quadratic form then the tensor algebra $T(V)$ is $\mathbb{N}$ -graded and the relations $v^2 = -q(v)$ is even; so the quotient is $\mathbb{Z}/2$ -graded. So $\text{Cl}(V, q)_0$ consists of products of an even number of vectors in V , and $\text{Cl}(V, q)_1$ consists of products of an odd number. We will write $\text{Cl}(V, q)^{\text{grd}}$ when we need to accentuate that we are thinking of the Clifford algebra as $\mathbb{Z}/2$ -graded.

The biggest advantage of introducing these $\mathbb{Z}/2$ -gradings is that the graded tensor product gives us the most natural way of relating small Clifford algebras to bigger ones:

Proposition 20.5. *There is a unique map of graded algebras*

$$\text{Cl}(V, q_V) \hat{\otimes} \text{Cl}(W, q_W) \longrightarrow \text{Cl}(V \oplus W, q_V \oplus q_W)$$

that sends $v \otimes 1$ to $(v, 0)$ for $v \in V$ and $1 \otimes w$ to $(0, w)$ for $w \in W$, and moreover this map is an isomorphism.

Proof. Left as an exercise. $\square$

Remark 20.6. When applied to the standard Clifford algebras for the sum-of-squares form this becomes the isomorphism $\text{Cl}_{s+t} \longrightarrow \text{Cl}_s \hat{\otimes} \text{Cl}_t$ given by

$$e_i \mapsto \begin{cases} e_i \otimes 1 & \text{if } 1 \leq i \leq s, \\ 1 \otimes e_{i-s} & \text{if } s+1 \leq i \leq s+t. \end{cases}$$

Using this, we see that if M is a graded Cl_s -module and N is a graded Cl_t -module then $M \hat{\otimes} N$ becomes a $\text{Cl}_s \hat{\otimes} \text{Cl}_t$ -module and therefore a graded Cl_{s+t} -module via the above isomorphism.

After the development of supersymmetry in physics, and the filtering of some of those ideas into mathematics, $\mathbb{Z}/2$ -graded objects of all types started to be described as “super” objects—super vector spaces, super algebras, etc. We will occasionally use this language, but mostly will stick to the more prosaic “ $\mathbb{Z}/2$ -graded”.

20.7. Super division algebras. The content of this section will at first seem to be an unmotivated digression into more algebra, but with fairly little work we will find that it gives us perhaps the best viewpoint on where Bott periodicity truly comes from. A super division algebra is a super-algebra in which every homogeneous

element is invertible. In analogy with the fact that there are only a small number of real division algebras ($\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$), it is reasonable to suspect that there will also be a small number of real super division algebras. And in fact that is correct—there are precisely eight of them. Not only that, but these structures form a group—the super Brauer group of $\mathbb{R}$ —which turns out to be $\mathbb{Z}/8$. This perspective turns out to explain much of what we have been observing on an *ad hoc* basis in our tour of real Bott periodicity.

We begin with some general foundations. If A is a $\mathbb{Z}/2$ -graded algebra then A^{op} denotes the same underlying graded vector space but with the product $a \odot b = (-1)^{|a| \cdot |b|} ba$. If M is a right A -module then $\text{End}_A(M)$ is $\underline{\text{Hom}}_A(M, M)$ with the multiplication given by composition. To clarify this a bit, note that $\text{Hom}_A(M, N)$ may be canonically identified with $\text{Hom}_A(\Pi M, \Pi N)$ in the evident way (no sign changes come into play here). We have $\text{End}_A(M)_0 = \text{Hom}_A(M, M) = \text{Hom}_A(\Pi M, \Pi M)$ and $\text{End}_A(M)_1 = \text{Hom}_A(\Pi M, M) = \text{Hom}_A(M, \Pi M)$. Given $f, g \in \text{Hom}_A(\Pi M, M)$ we interpret fg as either the composition $\Pi(f) \circ g \in \text{Hom}_A(\Pi M, \Pi M)$ or $f \circ \Pi(g) \in \text{Hom}_A(M, M)$, both of which specify the same element in $\text{End}_A(M)_0$.

Exercise 20.8. For each $a \in A$ let $\lambda_a: A \rightarrow A$ be left multiplication by a . Check that $a \mapsto \lambda_a$ gives an isomorphism of algebras $A \cong \text{End}_A(A)$.

For $p, q \geq 0$ define $A^{p|q} = A^{\oplus p} \oplus (\Pi A)^{\oplus q}$. This is the free A -module with p even generators and q odd generators. Define $A(p|q) = \text{End}_A(A^{p|q})$. These are the $\mathbb{Z}/2$ -graded analogs of matrix algebras. There are evident isomorphisms

- $A^{a|b} \hat{\otimes} A^{p|q} \cong A^{ap+bq|aq+bp}$.
- $A(p|q) \cong \mathbb{R}(p|q) \hat{\otimes}_{\mathbb{R}} A$.
- $A(p|q)(a|b) \cong A(pa + qb|pb + qa)$.

Exercise 20.9. Let A be an R -algebra, let M be an R -module, and let N be an A -module. All of this is taking place in the $\mathbb{Z}/2$ -graded setting. There is a canonical map

$$\theta: \text{End}_R(M) \hat{\otimes} \text{End}_A(N) \longrightarrow \text{End}_{R \otimes_R A}(M \hat{\otimes}_R N), \quad f \otimes g \mapsto f \otimes g.$$

- (a) Check that θ is a homomorphism of graded rings.
- (b) Check that the map θ is an isomorphism when M is finitely-generated and free, meaning that M is a finite direct sum of copies of R and ΠR .
- (c) If A is an $\mathbb{R}$ -algebra note that $A^{p|q} = \mathbb{R}^{p|q} \hat{\otimes}_{\mathbb{R}} A$. Use (b) to conclude that $\mathbb{R}(p|q) \hat{\otimes}_{\mathbb{R}} A \cong A(p|q)$.
- (d) Use (b) for $\text{End}_{\mathbb{R}}(\mathbb{R}^{p|q}) \hat{\otimes}_{\mathbb{R}} \text{End}_{\mathbb{R}}(\mathbb{R}^{a|b}) \longrightarrow \text{End}_{\mathbb{R}}(\mathbb{R}^{p|q} \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{a|b})$ to conclude that $\mathbb{R}(p|q) \hat{\otimes}_{\mathbb{R}} \mathbb{R}(a|b) \cong \mathbb{R}(pa + qb|pb + qa)$, and deduce the analogous result with $\mathbb{R}$ replaced by A .

We will often find ourselves using the operation $A \mapsto A(1|1)$. Note that iterating this operation k times gives $A \mapsto A(2^{k-1}|2^{k-1})$. We will use the abbreviation $A((n))$ for $A(n|n)$.

A $\mathbb{Z}/2$ -graded algebra A is a **super division algebra** if ρ_a is a bijection for every homogeneous element $a \in A$. The **supercenter** of a $\mathbb{Z}/2$ -graded algebra A is the subalgebra of A_0 consisting of elements that commute with everything in A . Let F be a field. Call a super division algebra over F **central** if its supercenter is precisely F . Let $\text{Br}_{\text{sup}}(F)$ be the set of isomorphism classes of finite-dimensional central super division algebras over F . This set can be equipped with a monoid structure

defined as follows. Given finite-dimensional central super division F -algebras A and B , the algebra $A \hat{\otimes} B$ is isomorphic to $D(p|q)$ for a unique central superdivision algebra D : one defines $[A] \cdot [B] = [D]$ in $\text{Br}_{\text{sup}}(F)$. This makes $\text{Br}_{\text{sup}}(F)$ into an abelian group, called the **super Brauer group** of F (it is abelian by Exercise 20.4).

Remark 20.10. A **graded two-sided ideal** of a super algebra A is a graded subspace $I \subseteq A$ that is a two-sided ideal in the usual sense. A super algebra is said to be **simple** if the only graded two-sided ideals are 0 and A . A **central simple** F -superalgebra is an F -superalgebra A that is simple and whose supercenter is F . One can prove that every central simple F -superalgebra has the form $D(p|q)$ for some central division superalgebra over F and some $p, q \geq 0$. One can also define $\text{Br}_{\text{sup}}(F)$ to be the isomorphism classes of graded central simple F -algebras modulo the equivalence relation generated by $A \sim A(p|q)$ for every $p, q \geq 0$. The (graded) tensor product of two central simple F -superalgebras is another central simple F -superalgebra, and this preserves the equivalence relation; so it induces a product on $\text{Br}_{\text{sup}}(F)$.

With a little work one can classify all of the finite-dimensional super division algebras over $\mathbb{R}$. This was first done in [Wall], which also set up the basic theory of super division algebras and the super Brauer group (there called the graded Brauer group). See also [Deligne] for a more modern treatment of the foundations, which we followed above.

In the following discussion we will stop saying “finite-dimensional” although it is always in the background. If A is a super division algebra over $\mathbb{R}$ note that A_0 will be an (ungraded) division algebra over $\mathbb{R}$, and so will be either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. These give all of the graded division algebras where $A_1 = 0$. If $f \in A_1$ is nonzero then right multiplication by f will give an isomorphism $A_0 \rightarrow A_1$. So our algebra will take the form

$$A = A_0 \oplus A_0 \cdot f$$

and will be determined by two other pieces of information: for every $x \in A_0$ one must have $fx = \phi(x)f$ for some unique $\phi(x) \in A_0$, and $f^2 = \lambda$ for some $\lambda \in A_0$. Note that the product on A is specified by the product on A_0 and the two rules

$$fa = \phi(a)f, \quad (af)(bf) = a(fb)f = a\phi(b)f^2 = a\phi(b)\lambda.$$

The function $\phi: A_0 \rightarrow A_0$ must be additive and unital, and for $\mathbb{R}$ to be in the center of A it must be $\mathbb{R}$ -linear.

Exercise 20.11. Prove that A is associative if and only if the following hold:

- (i) $\phi: A_0 \rightarrow A_0$ is an $\mathbb{R}$ -algebra isomorphism,
- (ii) $\phi(\lambda) = \lambda$,
- (iii) $\phi^2(a)\lambda = \lambda a$ for all $a \in A_0$.

[Hint for (ii) and (iii): Think about $\lambda f = f^2 \cdot f$ and $\lambda a = f^2 a$.]

The upshot is that super division algebras over $\mathbb{R}$ with $A_1 \neq 0$ are determined (not necessarily uniquely) by the data (ϕ, λ) subject to the above requirements. Note that we can always replace f with rf for some $r \in \mathbb{R} - \{0\}$, which will replace λ by $r^2\lambda$ while describing the same graded algebra.

For $A_0 = \mathbb{R}$ we must have A_0 acting centrally on f , and up to squares we have only two possibilities: $f^2 = 1$ and $f^2 = -1$. So there are two possible algebras here:

$$\mathbb{R}_+ = \mathbb{R}[f]/(f^2 + 1) \quad \text{and} \quad \mathbb{R}_- = \mathbb{R}[f]/(f^2 - 1).$$

There is some choice in how to name these algebras (one could imagine switching which is called $\mathbb{R}_+$ and $\mathbb{R}_-$), and the above names were chosen because they work well with the connections to Clifford algebras (see below).

For $A_0 = \mathbb{C}$ we can have either ϕ equal to id or complex conjugation. In the first case we can rescale f by any complex number to reduce to the case $f^2 = 1$ (this uses commutativity of $\mathbb{C}$), so our unique isomorphism class of division algebra is represented by

$$\mathbb{C}_+ = \mathbb{C}[f]/(f^2 - 1).$$

The alternative is to have ϕ equal to complex conjugation, so we will have $zf = f\bar{z}$ for all $z \in \mathbb{C}$. Since λ must be fixed by complex conjugation, up to scaling by squares of real numbers we again have the two possibilities $f^2 = \pm 1$. So our two possibilities are

$$\overline{\mathbb{C}}_+ = \mathbb{C} \oplus \mathbb{C}.f, \quad [f^2 = -1, zf = f\bar{z}] \quad \text{and} \quad \overline{\mathbb{C}}_- = \mathbb{C} \oplus \mathbb{C}.f, \quad [f^2 = 1, zf = f\bar{z}]$$

where in each case z ranges over all complex numbers.

For $A_0 = \mathbb{H}$, this is a central simple division algebra over $\mathbb{R}$ and so the Skolem-Noether theorem says that all automorphisms are inner. That is, ϕ must be given by $\phi(x) = qxq^{-1}$ for some $q \in \mathbb{H}$. The change of variable $g = q^{-1}f$ leads to the new commutation rule

$$ga = q^{-1}fa = q^{-1}(qaq^{-1})f = a(q^{-1}f) = ag.$$

So we reduce to the case where $\phi = \text{id}$.

Property (iii) from Exercise 20.11 then says $a\lambda = \lambda a$ for all $a \in \mathbb{H}$, and so $\lambda \in \mathbb{R}$. By rescaling our generator we reduce to $\lambda = \pm 1$, as before. So the super division algebras with $A_0 = \mathbb{H}$ are all isomorphic to one of

$$\mathbb{H}_+ = \mathbb{H}[f]/(f^2 + 1) \quad \text{and} \quad \mathbb{H}_- = \mathbb{H}[f]/(f^2 - 1).$$

Exercise 20.12. Prove that f and $-f$ are the only two odd elements of $\mathbb{H}_+$ that commute with all even degree elements and square to -1 (and similarly for $\mathbb{H}_-$). Use this to show that $\mathbb{H}_+$ and $\mathbb{H}_-$ are not isomorphic.

So the complete list of $\mathbb{Z}/2$ -graded division algebras over $\mathbb{R}$ consists of the following ten algebras:

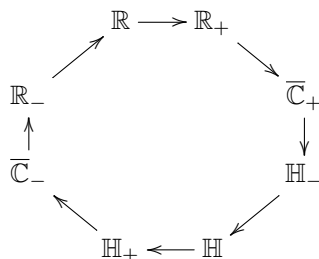
$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{R}_+, \mathbb{R}_-, \mathbb{C}_+, \overline{\mathbb{C}}_+, \overline{\mathbb{C}}_-, \mathbb{H}_+, \mathbb{H}_-.$$

If we restrict to the ones whose center is exactly $\mathbb{R}$ we get down to eight, namely

$$\mathbb{R}, \mathbb{H}, \mathbb{R}_+, \mathbb{R}_-, \overline{\mathbb{C}}_+, \overline{\mathbb{C}}_-, \mathbb{H}_+, \mathbb{H}_-.$$

The super Brauer group over $\mathbb{R}$ will then be one of $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, and $(\mathbb{Z}/2)^3$, and we can determine which one by analyzing the elements of order 2. Recall that the inverse of A in the Brauer group is A^{op} . One readily checks that $(\mathbb{R}_+)^{op} = \mathbb{R}_-$ and analogously for $\overline{\mathbb{C}}_{\pm}$ and $\mathbb{H}_{\pm}$, but $\mathbb{H}^{op} \cong \mathbb{H}$ (via conjugation). So the only element of order 2 is $\mathbb{H}$, and the super Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8$.

If we set $x = \mathbb{R}_+$ then it is easy to compute the graded tensor product and see that $x^2 = \overline{\mathbb{C}}_+$. Since $x^2 \notin \{\mathbb{R}, \mathbb{H}\}$ this shows that x does not have order 2 or 4, hence it must have order 8 and be a generator. It immediately follows that the group structure is as depicted in this circle:



(the arrows depict multiplication by $\mathbb{R}_+$). The only things we used to get this picture are our knowledge of the inverses, the fact that $x^2 = \overline{\mathbb{C}}_+$, and the self-evident fact that $\mathbb{R}_+ \hat{\otimes} \mathbb{H} = \mathbb{H}_+$. Everything else is forced from these.

In this version of the Bott periodicity clock we can interpret the reflective symmetry about the line joining $\mathbb{R}$ and $\mathbb{H}$ as representing the inverses in the group. The subgroup $\{\mathbb{R}, \mathbb{H}\}$ is really the classical (ungraded) Brauer group $\text{Br}(\mathbb{R})$.

Exercise 20.13. Check the calculation $\mathbb{R}_+ \hat{\otimes} \mathbb{R}_+ \cong \overline{\mathbb{C}}_+$ and also prove directly that $\mathbb{R}_+ \hat{\otimes} \overline{\mathbb{C}}_+ \cong \mathbb{H}_-$. [For the latter, one way is to find in $\mathbb{R}_+ \hat{\otimes} \overline{\mathbb{C}}_+$ the two elements that are analogs of the ones highlighted in Exercise 20.12].

If we look instead at graded division algebras over $\mathbb{C}$ there are just two of them: $\mathbb{C}$ and $\mathbb{C}_+$. The super Brauer group is $\mathbb{Z}/2$ in this case.

Remark 20.14. One might wonder if there is a way we could have predicted the appearance of the number 8. Wall showed in [Wall] that for a field k the super Brauer group $\text{Br}_{\text{sup}}(k)$ is an iterated extension of the three groups $\mathbb{Z}/2 = \text{Br}_{\text{sup}}(\overline{k})$ (this is really just tracking the $\mathbb{Z}/2$ -grading), $k^*/(k^*)^2$, and $\text{Br}(k)$. For $k = \mathbb{R}$ all three of these groups are $\mathbb{Z}/2$, leading to a super Brauer group of order 8. See also [Deligne, Lemma 3.7] for an exposition of Wall's result.

Let us now connect this story back to Clifford algebras. Our original collection of Clifford algebras $\text{Cl}_{p,0}$, if we regard them as graded, are just the tensor powers of $\text{Cl}_{1,0} = \mathbb{R}_+$. We now recognize this as a generator for the super Brauer group, and so the division algebras underlying the tensor powers—what we once called the “roots” of the Clifford algebras—will go through an 8-fold periodicity.

If we instead look at the Clifford algebras $\text{Cl}_{0,q}$ (again as super algebras), then these are tensor powers of $\text{Cl}_{0,1} = \mathbb{R}_-$. Since this is the inverse of $\mathbb{R}_+$ in the super Brauer group, we will see the eight-fold periodicity appearing in the reverse order. Finally, the $\text{Cl}_{p,q}$ algebras will correspond to $(\mathbb{R}_+)^p \cdot (\mathbb{R}_-)^q = (\mathbb{R}_+)^{p-q}$ in the super Brauer group, and so we get the familiar 8-fold periodicity with respect to $p - q$.

Figure 20.14.1 shows the graded Clifford algebras $\text{Cl}_{p,q}^{\text{grd}}$. Note that the names are different from what we saw in Figure 17.3.1, but the underlying algebras are the same. For example, the graded algebras $\mathbb{R}_+$ and $\overline{\mathbb{C}}_+$ become—after forgetting the grading— $\mathbb{C}$ and $\mathbb{H}$.

For ease of reference we also provide the following table:

Exercise 20.15. Justify the entries in the bottom row of Figure 20.14.2

20.16. Graded versus ungraded. The previous section demonstrated the importance of considering $\mathbb{Z}/2$ -graded algebras. It turns out there are interesting ways

$\mathbb{R}((8))$	$\mathbb{R}_+((8))$	$\overline{\mathbb{C}}_+((8))$	$\mathbb{H}_-((8))$	$\mathbb{H}((16))$	$\mathbb{H}_+((16))$	$\overline{\mathbb{C}}_-((32))$	$\mathbb{R}_-((64))$	$\mathbb{R}((128))$
$\mathbb{R}_+((4))$	$\overline{\mathbb{C}}_+((4))$	$\mathbb{H}_-((4))$	$\mathbb{H}((8))$	$\mathbb{H}_+((8))$	$\overline{\mathbb{C}}_-((16))$	$\mathbb{R}_-((32))$	$\mathbb{R}((64))$	$\mathbb{R}_+((64))$
$\overline{\mathbb{C}}_+((2))$	$\mathbb{H}_-((2))$	$\mathbb{H}((4))$	$\mathbb{H}_+((4))$	$\overline{\mathbb{C}}_-((8))$	$\mathbb{R}_-((16))$	$\mathbb{R}((32))$	$\mathbb{R}_+((32))$	$\overline{\mathbb{C}}_+((32))$
$\mathbb{H}_-((1))$	$\mathbb{H}((2))$	$\mathbb{H}_+((2))$	$\overline{\mathbb{C}}_-((4))$	$\mathbb{R}_-((8))$	$\mathbb{R}((16))$	$\mathbb{R}_+((16))$	$\overline{\mathbb{C}}_+((16))$	$\mathbb{H}_-((16))$
$\mathbb{H}((1))$	$\mathbb{H}_+((1))$	$\overline{\mathbb{C}}_-((2))$	$\mathbb{R}_-((4))$	$\mathbb{R}((8))$	$\mathbb{R}_+((8))$	$\overline{\mathbb{C}}_+((8))$	$\mathbb{H}_-((8))$	$\mathbb{H}((16))$
$\mathbb{H}_+$	$\overline{\mathbb{C}}_-((1))$	$\mathbb{R}_-((2))$	$\mathbb{R}((4))$	$\mathbb{R}_+((4))$	$\overline{\mathbb{C}}_+((4))$	$\mathbb{H}_-((4))$	$\mathbb{H}((8))$	$\mathbb{H}_+((8))$
$\overline{\mathbb{C}}_-$	$\mathbb{R}_-((1))$	$\mathbb{R}((2))$	$\mathbb{R}_+((2))$	$\overline{\mathbb{C}}_+((2))$	$\mathbb{H}_-((2))$	$\mathbb{H}((4))$	$\mathbb{H}_+((4))$	$\overline{\mathbb{C}}_-((8))$
$\mathbb{R}_-$	$\mathbb{R}((1))$	$\mathbb{R}_+((1))$	$\overline{\mathbb{C}}_+((1))$	$\mathbb{H}_-((1))$	$\mathbb{H}((2))$	$\mathbb{H}_+((2))$	$\overline{\mathbb{C}}_-((4))$	$\mathbb{R}_-((8))$
$\mathbb{R}$	$\mathbb{R}_+$	$\overline{\mathbb{C}}_+$	$\mathbb{H}_-$	$\mathbb{H}((1))$	$\mathbb{H}_+((1))$	$\overline{\mathbb{C}}_-((2))$	$\mathbb{R}_-((4))$	$\mathbb{R}((8))$
$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$

FIGURE 20.14.1. Graded Clifford algebras $\text{Cl}_{p,q}^{grd}$

Super division algebra:	$\mathbb{R}$	$\mathbb{R}_+$	$\overline{\mathbb{C}}_+$	$\mathbb{H}_-$	$\mathbb{H}$	$\mathbb{H}_+$	$\overline{\mathbb{C}}_-$	$\mathbb{R}_-$
Underlying algebra:	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{R}(2)$	$\mathbb{R} \times \mathbb{R}$

FIGURE 20.14.2. Super division algebras and their underlying (ungraded) forms

of moving back and forth between graded and ungraded Clifford algebras and their modules, which we now describe in detail.

We begin by describing a functor $D: \text{Cl}_s\text{-Mod} \rightarrow \text{Cl}_{s+1}^{grd}\text{-Mod}$, which will turn out to be part of an equivalence of categories. We will give three descriptions of D , all of which are equivalent. However, first there is an annoying technical issue we need to deal with.

We have the standard inclusion of Clifford algebras $\text{Cl}_s \hookrightarrow \text{Cl}_{s+1}$, where the latter has the extra generator e_{s+1} . For reasons that will shortly become clear it will be better for us to consider the analogous map where we add a generator at the beginning rather than at the end. Let us write Cl_{1+s} for the Clifford algebra on generators $e_0, e_1, \dots, e_s$, and $j: \text{Cl}_s \hookrightarrow \text{Cl}_{1+s}$ will be the inclusion of the subalgebra generated by $e_1, \dots, e_s$. Of course there is some sophistry in this notation, since as numbers $1+s = s+1$ but as algebras $\text{Cl}_{1+s} \neq \text{Cl}_{s+1}$. The notation is still useful, one just has to be careful.

In addition to the inclusion $j: \text{Cl}_s \hookrightarrow \text{Cl}_{1+s}$ there is a second useful inclusion: define $\phi: \text{Cl}_s \rightarrow \text{Cl}_{1+s}$ by sending $e_i \mapsto e_i e_0$ for $1 \leq i \leq s$. One readily checks that the defining relations of Cl_s are satisfied and that the resulting map is an injection. If Cl_s is regarded as a graded algebra that is concentrated in even degree, then ϕ is a map of graded algebras $\text{Cl}_s \hookrightarrow \text{Cl}_{1+s}^{grd}$.

Unless otherwise noted, when we regard Cl_s as a subalgebra of Cl_{1+s} it is through the map j .

We now give three constructions of the functor $D: \text{Cl}_s\text{-Mod} \rightarrow \text{Cl}_{1+s}^{grd}\text{-Mod}$. Let M be a Cl_s -module.

Version #1: Define DM to be the graded vector space with $(DM)_0 = (DM)_1 = M$. If $m \in M$ write $m_{[0]}$ for $m \in (DM)_0$ and $m_{[1]}$ for $m \in (DM)_1$. Define the $\text{Cl}_{1+s}^{\text{grd}}$ -structure on DM by

$$\begin{aligned} e_i(m_{[j]}) &= (e_i m)_{[j+1]}, \quad \text{for } 1 \leq j \leq s, \\ e_0(m_{[j]}) &= (-1)^j m_{[j+1]} \end{aligned}$$

for all $m \in M$ and $j \in \{0, 1\}$.

Version #2: Define $DM = \text{Cl}_{1+s} \otimes_{\text{Cl}_s} M$. Give this a grading by setting

$$(DM)_0 = \{1 \otimes m - e_0 \otimes m \mid m \in M\}, \quad (DM)_1 = \{1 \otimes m + e_0 \otimes m \mid m \in M\}.$$

Version #3: Define $DM = (\text{Cl}_{1+s})_\phi \otimes_{\text{Cl}_s} M$, where the notation means that Cl_{1+s} is regarded as a (right) Cl_s -module via ϕ . Here

$$(DM)_0 = \{1 \otimes m \mid m \in M\}, \quad (DM)_1 = \{e_0 \otimes m \mid m \in M\}.$$

Said differently, $DM = (\text{Cl}_{1+s}^{\text{grd}})_\phi \otimes_{\text{Cl}_s} M$ and it inherits its grading from $\text{Cl}_{1+s}^{\text{grd}}$.

Exercise 20.17. Show that the three definitions of DM are isomorphic as graded modules over $\text{Cl}_{1+s}^{\text{grd}}$.

Proposition 20.18. *The following diagram is commutative*

$$\begin{array}{ccc} \text{Cl}_s\text{-Mod} & \xrightarrow{U} & \text{Cl}_{s-1}\text{-Mod} \\ \downarrow D & & \downarrow D \\ \text{Cl}_{1+s}^{\text{grd}}\text{-Mod} & \xrightarrow{U} & \text{Cl}_{1+(s-1)}^{\text{grd}}\text{-Mod}, \end{array}$$

where the U maps are forgetful functors induced by the standard inclusions j .

Proof. This is particularly obvious using Version #1, though it is easy enough using any of the other definitions as well. $\square$

Remark 20.19. For the above proposition it is very important that when passing from Cl_s to $\text{Cl}_{1+s}^{\text{grd}}$ the new generator was positioned *before* the others. The result would not be true with $\text{Cl}_{s+1}^{\text{grd}}$ instead, unless we changed the maps j .

We will show that D is an equivalence of categories by explicitly constructing an inverse. Recall the map $\phi: \text{Cl}_s \rightarrow \text{Cl}_{1+s}^{\text{grd}}$, which is a map of graded algebras when the domain is regarded as being concentrated in even degree. If W is a $\text{Cl}_{1+s}^{\text{grd}}$ -module then define $\Phi(W)$ to be W_0 but regarded as a Cl_s -module via ϕ . Explicitly, this means the Cl_s -action has e_i acting on W_0 as left multiplication by $e_i e_0$. Then Φ is a functor $\text{Cl}_{1+s}^{\text{grd}}\text{-Mod} \rightarrow \text{Cl}_s\text{-Mod}$.

Proposition 20.20. *The functors $D: \text{Cl}_s\text{-Mod} \rightleftarrows \text{Cl}_{1+s}^{\text{grd}}\text{-Mod}: \Phi$ are an equivalence of categories.*

Proof. Using version #1 of the D functor, the relation $\Phi D = Id$ is self-evident. For W a $\text{Cl}_{1+s}^{\text{grd}}$ -module we need to produce a natural isomorphism $h: D\Phi(W) \rightarrow W$. This will consist of maps $h_i: W_0 \rightarrow W_i, i \in \{0, 1\}$. Let h_0 be the identity, and let h_1 be left multiplication by e_0 . One readily checks that this is a map of $\text{Cl}_{1+s}^{\text{grd}}$ -modules, and it is transparently an isomorphism. $\square$

We have seen before that giving a direct-sum decomposition for a vector space W is equivalent to giving an endomorphism $f: W \rightarrow W$ with $f^2 = 1$: the direct summands correspond to the $+1$ and -1 -eigenspaces of f . An operator $e: W \rightarrow W$ interchanges the two summands if and only if it anticommutes with f . From this it is readily checked that giving a Cl_s^{grd} -module structure on a vector space W is equivalent to giving a $\text{Cl}_{s,1}$ -module structure on W . This is completely functorial as well:

Proposition 20.21. *There is an equivalence of categories $\text{Cl}_s^{\text{grd}}\text{-Mod} \rightleftarrows \text{Cl}_{s,1}\text{-Mod}$.*

Proof. Immediate. $\square$

In light of the above proposition, the equivalence from Proposition 20.20 is just a manifestation of the Morita equivalence between Cl_s and $\text{Cl}_{s+1,1}$ that we saw in Proposition 17.3.

We can now give a nice explanation of the Clifford tensor product of modules that we introduced back in ???:

Proposition 20.22. *Let M be a Cl_s -module and N be a Cl_t -module. Then DM is a $\text{Cl}_{1+s}^{\text{grd}}$ -module and DN is a $\text{Cl}_{1+t}^{\text{grd}}$ -module, therefore $DM \hat{\otimes} DN$ is a $\text{Cl}_{1+s}^{\text{grd}} \hat{\otimes} \text{Cl}_{1+t}^{\text{grd}}$ -module. Use the standard identification $\text{Cl}_{1+s}^{\text{grd}} \otimes \text{Cl}_{1+t}^{\text{grd}} = \text{Cl}_{1+s+1+t}^{\text{grd}}$ to regard $DM \hat{\otimes} DN$ as a $\text{Cl}_{1+s+1+t}^{\text{grd}}$ -module. Then $\Phi(DM \hat{\otimes} DN)$ is a Cl_{s+1+t} -module, and it precisely coincides with $M \otimes_c N$.*

Proof. For convenience write $W = \Phi(DM \hat{\otimes} DN)$. Let us use the model $DM = (\text{Cl}_{1+s}^{\text{grd}})_{\phi} \otimes_{\text{Cl}_s} M$. The graded pieces can be written as $1 \otimes M$ and $e_0 \otimes M$. Therefore $W = (1 \otimes M \otimes 1 \otimes N) \oplus (e_0 \otimes M \otimes e_0 \otimes N)$, which we identify with $(M \otimes N) \oplus (M \otimes N)$ in the evident way.

Now we just write down formulas. For example, the e_i action on W is the $e_i e_0$ action on $DM \hat{\otimes} DN$, and if $1 \leq i \leq s$ then $e_i e_0$ means $(e_i \otimes 1)(e_0 \otimes 1) = e_i e_0 \otimes 1$. Then compute that

$$\begin{aligned} (e_i e_0 \otimes 1) \otimes (1 \otimes m \otimes 1 \otimes n) &= e_i e_0 \otimes m \otimes 1 \otimes n = 1 \otimes e_i m \otimes 1 \otimes n, \\ (e_i e_0 \otimes 1) \otimes (e_0 \otimes m \otimes e_0 \otimes n) &= -e_i \otimes m \otimes e_0 \otimes n \\ &= -e_0 e_i e_0 m \otimes e_0 \otimes n \\ &= -e_0 \otimes e_i m \otimes e_0 \otimes n. \end{aligned}$$

So in the $(M \otimes N) \oplus (M \otimes N)$ model the action of e_i is by $\begin{bmatrix} e_i^M \otimes \text{id}_N & 0 \\ 0 & -e_i^M \otimes \text{id}_N \end{bmatrix}$, which coincides with the definition of $M \otimes_c N$.

The calculations are similar for the other e 's, but since it is easy to get confused by signs we sketch the details. The e_{s+1} -action on W is the $e_{s+1} e_0$ action on $DM \hat{\otimes} DN$, and $e_{s+1} e_0 = (1 \otimes e_0)(e_0 \otimes 1) = -e_0 \otimes e_0$. Then use that

$$\begin{aligned} -(e_0 \otimes e_0)(1 \otimes m \otimes 1 \otimes n) &= -e_0 \otimes m \otimes e_0 \otimes n, \\ -(e_0 \otimes e_0)(e_0 \otimes m \otimes e_0 \otimes n) &= 1 \otimes m \otimes 1 \otimes n \end{aligned}$$

(note that an extra minus sign appears in the second formula because e_0 and $e_0 \otimes m$ both have degree 1). So the matrix for the e_{s+1} -action is $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

Finally, for $s+2 \leq i \leq s+t+2$ the e_i -action on W is the $e_i e_0$ -action on $DM \hat{\otimes} DN$, and $e_i e_0 = (1 \otimes e_{i-s-1})(e_0 \otimes 1) = -e_0 \otimes e_{i-s-1}$. Write $j = i - s - 1$

and then use

$$\begin{aligned}
 -(e_0 \otimes e_j)(1 \otimes m \otimes 1 \otimes n) &= -e_0 \otimes m \otimes e_j \otimes n \\
 &= -e_0 \otimes m \otimes e_0 e_j e_0 \otimes n = -e_0 \otimes m \otimes e_0 \otimes e_j n \\
 -(e_0 \otimes e_j)(e_0 \otimes m \otimes e_0 \otimes n) &= -1 \otimes m \otimes e_j e_0 \otimes n \\
 &= -1 \otimes m \otimes 1 \otimes e_j n.
 \end{aligned}$$

So the e_i -action is given by the matrix $\begin{bmatrix} 0 & -\text{id}_M \otimes e_j^N \\ -\text{id}_M \otimes e_j^N & 0 \end{bmatrix}$. $\square$

The Clifford tensor product seemed somewhat ad hoc when it first appeared. We can now see it as transporting the (very natural) graded tensor product across the equivalences provided by Proposition 20.20. Also, when we introduced the Clifford tensor product we remarked that its behavior with respect to the Clifford indexing was unfortunate. Notice that shifting focus to the graded tensor product fixes this issue.

Exercise 20.23. Prove that the Clifford tensor product $\otimes_c$ is associative up to natural isomorphism, by deducing this from the associativity of the graded tensor product.

Define G_s^{grd} to be the Grothendieck group of finitely-generated Cl_s^{grd} -modules. The D functor gives us isomorphisms $D: G_s \rightarrow G_{1+s}^{grd}$, for all s . We know $\bigoplus_s G_s$ is a rng via the Clifford tensor product, and $\bigoplus_s G_s^{grd}$ is a ring via the graded tensor product. Our above work shows the following:

Corollary 20.24. *The map $D: \bigoplus_s G_s \rightarrow \bigoplus_s G_s^{grd}$ is a rng homomorphism. In fact, it is an inclusion whose image is the ideal of positive-degree elements.*

Proof. Let M be a Cl_s -module and N be a Cl_t -module. Then

$$D(M \otimes_c N) \cong D\Phi(DM \hat{\otimes} DN) \cong DM \hat{\otimes} DN$$

as $\text{Cl}_{1+s+1+t}^{grd}$ -modules, where the first isomorphism uses Proposition 20.22 and the second is from the (D, Φ) equivalence. $\square$

Recall that in the case of ungraded modules we defined $\mathcal{A}_s$ to be the cokernel of the restriction of scalars map $U: G_{s+1} \rightarrow G_s$. We make the analogous definition of $\mathcal{A}_s^{grd}$. By Proposition 20.18 the D -functor yields an induced map $D: \mathcal{A}_s \rightarrow \mathcal{A}_{1+s}^{grd}$, and this is again an isomorphism. We record the following:

Corollary 20.25. *The map $D: \bigoplus_s \mathcal{A}_s \rightarrow \bigoplus_s \mathcal{A}_s^{grd}$ is a rng homomorphism, an injection, and the image is the set of positive-degree elements.*

Proof. Immediate. $\square$

Using graded modules gives us another useful tool that has not been mentioned yet. Recall that if M is a Cl_s^{grd} -module then ΠM denotes M but with the parity shifted: $(\Pi M)_i = M_{i+1}$.

Proposition 20.26. *The induced map $\Pi: G_s^{grd}(\mathbb{C}) \rightarrow G_s^{grd}(\mathbb{C})$ is the identity when s is even, and interchanges the two positive generators when s is odd. Likewise, the induced map $\Pi: G_s^{grd}(\mathbb{R}) \rightarrow G_s^{grd}(\mathbb{R})$ is the identity when $s \not\equiv 0 \pmod{4}$, and interchanges the two positive generators otherwise.*

Proof. Clearly Π must take irreducibles to irreducibles. In the non-exceptional cases there is a unique isomorphism class of irreducible module, and thus the action of Π is trivial. So the only work is in checking the exceptional cases where there are two isomorphism classes of irreducibles.

We give the argument in the real case, with the complex case being similar. Let $s = 4k$. Recall from ??? that the irreducible modules over Cl_{s-1} are distinguished by the action of $\omega = e_1 \cdots e_{s-1}$, which will be multiplication by either $+1$ or -1 . Let M be an irreducible Cl_{s-1} -module where ω acts as $+1$. We will compute the corresponding actions of ω on $\Phi(DM)$ and on $\Phi(\Pi(DM))$.

We already know $M \cong \Phi(DM)$ and so the action of ω on $\Phi(DM)$ is $+1$. But let us also compute it by brute force, so that we understand what changes when we move to $\Pi(DM)$. On $\Phi(DM)$ we have ω acting as $(e_1 e_0)(e_2 e_0) \cdots (e_{s-1} e_0)$, and we can write

$$(e_1 e_0)(e_2 e_0) \cdots (e_{s-1} e_0) = (-1)^{1+2+\cdots+(s-1)} e_0^{s-1} e_1 \cdots e_{s-1}.$$

Because $4|s$ the sign is positive and $e_0^{s-1} = -e_0$, so the whole expression is $-e_0 e_1 \cdots e_{s-1}$. When we think of this acting on $(DM)_0$, the $e_1 \cdots e_{s-1}$ terms take us into degree one and then $-e_0$ takes us back to degree 0 via the identity. So we are recovering the ω -action on M , as expected.

When we change from DM to $\Pi(DM)$, the e_i for $i > 0$ all act the same but the e_0 action changes to $-e_0$. So the ω action picks up an extra $(-1)^{s-1}$ from the product $(e_1 e_0) \cdots (e_{s-1} e_0)$. Since $4|s$, this sign is -1 . So ω acts on $\Phi(\Pi(DM))$ as -1 .

We have therefore proven that M and ΠM become different irreducibles after application of Φ , so they are the two positive generators of $G_s^{grd}(\mathbb{R})$. $\square$

Exercise 20.27. Think through the details of the above argument in the complex case, where for $\text{Cl}_{2k-1}(\mathbb{C})$ the ω -action on irreducibles is multiplication by $\pm i^k$ (see Proposition 19.16).

20.28. The coefficient rings of K and KO . At this point in the book we have developed a number of ways to access the groups $K^*(pt)$ and $KO^*(pt)$, together with their ring structure, but we have not as yet carefully verified the consistency of these different approaches. We finally turn to this issue now. The work will focus on the following large diagram (note the use of red and blue colors, which we will discuss shortly):

$$\begin{array}{ccccc}
 \pi_{s-1}O & \xleftarrow{\cong_R} & \mathcal{A}_{s-1} & \xrightarrow{D} & \mathcal{A}_s^{grd} \\
 \downarrow \text{clutch} \cong & \searrow 1 & \downarrow 2 & \swarrow 3 & \downarrow 4 \\
 KO_{st}^0(S^s) & \xleftarrow{\text{clutch} \cong} & L^1(D^s, \partial D^s) & \xrightarrow{\cong} & KO_{cplx}^0(D^s, \partial D^s) \\
 \downarrow \cong & & \uparrow \pi^* & & \downarrow \pi^* \cong_R \\
 \pi_s BO & \xrightarrow{\cong} & \widetilde{KO}^0(S^s) & \xleftarrow{\chi} & L^1(S^s, *) & \xrightarrow{\cong} & KO_{cplx}^0(S^s, *) \\
 & & & \searrow \chi & & & \\
 & & & \cong_R & & &
 \end{array}$$

The following several paragraphs explain how to parse this diagram. Note that we have drawn the diagram for KO^* , but the diagram for K^* is completely analogous.

All of the maps in this diagram turn out to be isomorphisms, but the arrows are drawn in the direction where the maps are most natural to define. For example, the two maps labelled “clutch” are clutching constructions. The vertical one is induced by the construction that takes a map $f: S^{s-1} \rightarrow O(n)$ to the rank n bundle on S^s having this as its clutching function. The inverse of this map feels a bit less natural, as it can only be constructed by choosing trivializations of a bundle on the upper and lower hemispheres (though to be fair, even the original map involves a choice of representing element f for the homotopy class).

The horizontal clutching map sends a map $g: E_1 \rightarrow E_0$ of bundles on D^s , exact on ∂D^s , to the bundle on S^s obtained by taking E^1 on the upper hemisphere, E_0 on the lower hemisphere, and then using g as the clutching function.

The two maps labelled χ sends a complex of vector bundles E_* (possibly just of length one) to $\sum_i (-1)^i [E_i]$. The maps labelled π^* are the pullbacks induced by the projection $\pi: (D^s, \partial D^s) \rightarrow (S^s, *)$.

Taking the direct sum over s gives a graded ring in each spot of the diagram. The red spots are the ones where we can describe this ring structure with a precise algebraic formula; e.g., on $K_{cplx}^0(D^s, \partial D^s)$ it is given by the external tensor product of chain complexes (note that this uses the canonical isomorphisms $D^s \times D^t \cong D^{s+t}$). The blue terms are ones where we have a ring structure for formal topological reasons, but it is much less clear how to actually compute it. For example, The external tensor product $K^0(S^s) \otimes K^0(S^t) \rightarrow K^0(S^s \times S^t)$ can be shown to send $\tilde{K}^0(S^s) \otimes \tilde{K}^0(S^t)$ into the image of $\tilde{K}^0(S^s \wedge S^t)$ in the target, and in this way defines an element of $\tilde{K}^0(S^s \wedge S^t)$. This is the ring structure on $\bigoplus_s \tilde{K}^0(S^s)$. But while it is true that the external tensor product is a “formula” that we understand, re-interpreting it as an element of $\tilde{K}^0(S^s \wedge S^t)$ is not formulaic—there is no evident way of digesting the formal difference of bundles on $S^s \times S^t$ into a formal difference of bundles on $S^s \wedge S^t$.

The terms that are neither red nor blue are ones where there is no reasonable ring structure to consider on the direct sum over s . That is to say, one can artificially put a ring structure by transplanting the ones we know across the isomorphisms, but the result is not useful.

One can start to see some of the difficulties in navigating elements of this diagram. Although there is a very direct map from $\pi_{*-1}O$ to $\tilde{KO}^0(S^*)$, it is challenging to relate the algebraic multiplication on the former to the topological multiplication on the latter. To compare these two it ends up being easiest to go the long way around the diagram, as we will soon see.

Maps labelled $\cong_R$ are ring isomorphisms. Note that the rightmost π_* is a ring isomorphism by naturality and excision, whereas the bottom χ is a ring isomorphism by Theorem ??.

The maps labelled 1–4 are defined as follows:

- (1) Given $f: S^{s-1} \rightarrow O(n)$, for $x \in D^s$ define $\hat{f}(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\hat{f}(x) = |x|f(\frac{x}{|x|})$ with the understanding that this is the zero map when $x = 0$. Send f to the map of bundles $\underline{n} \rightarrow \underline{n}$ that is multiplication by $\hat{f}(x)$ over the point $x \in D^s$.

- (2) Identify $\mathbb{R}^s$ with the subspace of Cl_{s-1} spanned by $1, e_1, \dots, e_n$. Given a Cl_{s-1} -module M , send this to the map of trivial bundles $\underline{M} \rightarrow \underline{M}$ on D^s that over $x \in D^s$ is multiplication by x .
- (3) Regard $\mathbb{R}^s$ as a subspace of Cl_s in the usual way, i.e. as the span of the elements $e_1, \dots, e_s$. Given a graded Cl_s -module M , send this to the map of trivial bundles $\underline{M}_1 \rightarrow \underline{M}_0$ on D^s which over $x \in D^s$ is multiplication by x .
- (4) Same as (3).

All of the triangles in the top part of the diagram are readily checked to commute. All of the maps labelled as isomorphisms are known to be so by previous results in the text. It then follows from commutativity of the diagram that the maps 1–4 are also isomorphisms.

The bottom right rectangle is clearly commutative, as is the triangle at the very bottom of the diagram. Commutativity of the bottom left rectangle takes a moment's thought. Take an element $x = [\alpha: E_1 \rightarrow E_0] \in L^1(S^s, *)$. By adding a complement to E_1 to both the domain and codomain, we can assume that E_1 is free. The fact that α is an isomorphism over the basepoint implies that E_0 and E_1 have the same rank, say n . We have $\chi(x) = [E_0] - [E_1] = [E_0] - n$. Going the other way around the square, we consider π^*E_0 and π^*E_1 on D^s and clutch them together using the isomorphism α . Using that E_1 is trivial, the resulting vector bundle is clearly isomorphic to E_0 . So going the other way around the square gives $[E_0] - n$ again, hence the square commutes.

That the top horizontal maps are ring homomorphisms has been verified in Theorem 19.21 and Proposition ???. It remains to check this property for the vertical map:

Proposition 20.29. *The map $\mathcal{A}_s^{grd}(\mathbb{R}) \rightarrow KO_{cplx}^0(D^s, \partial D^s)$ is a ring isomorphism, and likewise for the complex case.*

Proof. Note that the “isomorphism” part follows from the commutativity of the big diagram, so the only question is whether the map is a ring homomorphism. The main issue here is that the tensor product on graded modules is $\mathbb{Z}/2$ -graded whereas the one on chain complexes is $\mathbb{Z}$ -graded, so one must work a little to fit these together. The key step is the process described in ??? for folding a chain complex (in this case of length two) down into a length-one chain complex—this is the argument we used to prove that $KO_{cplx}^0(X, A)$ is isomorphic to $L^1(X, A)$.

Let M be a Cl_s^{grd} -module and N be a Cl_t^{grd} -module. Recall that $\underline{M}_i$ denotes the trivial bundle on D^s with fiber M_i . Let $\theta(M)$ denote the chain complex of bundles over D^s that has $\underline{M}_i$ in degree i and where the differential is left multiplication by x on the fiber over x . Similarly, $\theta(N)$ is the corresponding chain complex of bundles on D^t .

It will be useful to use x to represent elements of D^s , y to represent elements of D^t , and $w = (x, y)$ to represent the corresponding element of $D^s \times D^t$. Note that $|w|^2 = |x|^2 + |y|^2$.

Let $\alpha_M: \underline{M}_0 \rightarrow \underline{M}_1$ be the map that is left multiplication by $\underline{x}$ on the fiber over x . This gives a contracting homotopy for the map of chain complexes $\theta(M) \rightarrow \theta(M)$

that is multiplication by $-|x|^2$:

$$\begin{array}{ccc} \underline{M}_1 & \xrightarrow{-|x|^2} & \underline{M}_1 \\ x \downarrow & \nearrow \alpha_M & \downarrow x \\ \underline{M}_0 & \xrightarrow{-|x|^2} & \underline{M}_0. \end{array}$$

Consider the following map of complexes of bundles on $D^s \times D^t$ (exact away from $(0,0)$), where the domain is $(\theta M) \otimes_{cplx} (\theta N)$ (appearing on the left):

$$\begin{array}{ccc} M_1 \otimes N_1 & \xrightarrow{\text{id}} & M_1 \otimes N_1 \\ \downarrow & & \downarrow 2|w|^2 \\ M_0 \otimes N_1 \oplus M_1 \otimes N_0 & \xrightarrow{-\alpha_M \otimes 1 + 1 \otimes \alpha_N} & M_1 \otimes N_1 \\ \downarrow & & \\ M_0 \otimes N_0 & & \end{array}$$

For ease of typography we are simplifying $\underline{M}$ and $\underline{N}$ to just M and N , and also writing $\otimes$ for the external tensor product (to avoid confusion with the graded tensor product).

Let Γ denote the desuspension of the mapping cone of the above map, recalling that the desuspension involves changing the signs on all the differentials. Explicitly, Γ is the following chain complex concentrated in degrees 2, 1, and 0 (for typographical reasons we have omitted the tensor symbols between M_i and N_j):

$$M_1 N_1 \xrightarrow{\begin{bmatrix} d \otimes 1 \\ -1 \otimes d \\ -1 \otimes 1 \end{bmatrix}} M_0 N_1 \oplus M_1 N_0 \oplus M_1 N_1 \xrightarrow{\begin{bmatrix} 1 \otimes d & d \otimes 1 & 0 \\ \alpha_M \otimes 1 & -1 \otimes \alpha_N & -2|w|^2 \end{bmatrix}} M_0 N_0 \oplus M_1 N_1.$$

If E is a vector bundle let $D(E)$ be the chain complex that has E in degrees 0 and 1 and where the differential is the identity. There are short exact sequences

$$0 \longrightarrow \theta(M \hat{\otimes} N) \longrightarrow \Gamma \longrightarrow \Sigma D(M_1 \otimes N_1) \longrightarrow 0$$

and

$$0 \longrightarrow X \longrightarrow \Gamma \longrightarrow \theta M \otimes_{cplx} \theta N \longrightarrow 0$$

where X has $M_1 \otimes N_1$ in degrees 0 and 1 and the differential is multiplication by $-2|w|^2$. These exact sequences are more or less self-evident as soon as one goes looking for them (remember that Σ changes the sign on the differentials in addition to shifting them up). Note that every complex that appears in these sequences is exact on $(D^s \times D^t) - \{(0,0)\}$.

The short exact sequences give us the K -theory relations

$$[\theta(M \hat{\otimes} N)] = [\Gamma] = [X] + [\theta M \otimes_{cplx} \theta N]$$

in $KO^0(D^s \times D^t, D^s \times D^t - \{(0,0)\})$. As the final step we observe that the differential in X can be homotoped to one that is exact, via the homotopy $-2|w|^2 - t$, $0 \leq t \leq 1$. So X represents 0 in K -theory. This completes the argument for the real case.

The same exact argument works in the complex case. Just note that we are using the chain of inclusions $D^s \subseteq \mathbb{R}^s \subseteq \mathbb{C}^s \subseteq \text{Cl}_s(\mathbb{C})$, and so if $x \in D^s$ then it is still true that $x^2 = -|x|^2$ in $\text{Cl}_s(\mathbb{C})$. $\square$

Corollary 20.30. *For all $n \geq 1$, the group $K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$ (which is isomorphic to $\mathbb{Z}$) is generated by the Koszul complex $K(z_1, \dots, z_n)$.*

Proof. The inclusion $(D^{2n}, D^{2n} - 0) \hookrightarrow (\mathbb{R}^{2n}, \mathbb{R}^{2n} - 0) = (\mathbb{C}^n, \mathbb{C}^n - 0)$ induces an isomorphism $K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \rightarrow K^0(D^{2n}, D^{2n} - 0)$, which we will use implicitly for the rest of the proof. So Proposition 20.29 gives us a ring isomorphism $\bigoplus_s A_{2s}^{grd}(\mathbb{C}) \rightarrow \bigoplus_s K^0(\mathbb{C}^s, \mathbb{C}^s - 0)$.

Recall that $\bigoplus_s A_{2s}^{grd}(\mathbb{C})$ is the graded polynomial ring $\mathbb{Z}[M]$ where M is either of the two irreducibles over $\text{Cl}_2^{grd}(\mathbb{C})$. We will show that $K^0(\mathbb{C}^1, \mathbb{C}^1 - 0)$ is generated by the Koszul complex $K(z_1)$. Since $K(z_1, \dots, z_n)$ is the n -fold tensor power of $K(z_1)$, our ring isomorphism will then immediately imply that $K(z_1, \dots, z_n)$ generates $K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$.

To get our hands on $\text{Cl}_2^{grd}(\mathbb{C})$ -modules we use the equivalence of categories $D: \text{Cl}_1\text{-Mod} \rightarrow \text{Cl}_2^{grd}\text{-Mod}$ from Proposition 20.20. Here $\text{Cl}_1(\mathbb{C}) = \mathbb{C}[e]/(e^2 = -1)$ and one of the irreducible modules is $N = \mathbb{C}$ with e acting as multiplication by $-i$. Then $M = D(N)$ is one of the irreducible $\text{Cl}_2^{grd}(\mathbb{C})$ -modules. It has $M_0 = M_1 = \mathbb{C}$, $e_1: M_1 \rightarrow M_0$ is $-\text{id}$ and $e_2: M_1 \rightarrow M_0$ is multiplication by $-i$. Then $\theta(M)$ is the complex of bundles $0 \rightarrow \underline{M}_1 \rightarrow \underline{M}_0 \rightarrow 0$ where over $(x, y) \in D^2$ the map is multiplication by $xe_1 + ye_2$, or as maps $\mathbb{C} \rightarrow \mathbb{C}$ this is multiplication by $-x - iy$. As a complex of bundles this is isomorphic to $K(z_1)$, which is the similar complex having multiplication by $x + iy$ over (x, y) (use the identity map in degree 0 and $-\text{id}$ in degree 1). So $K(z_1)$ generates $K^0(D^2, D^2 - 0)$, or equivalently $K^0(\mathbb{C}, \mathbb{C} - 0)$. $\square$

20.31. Understanding the KO -theory of a point. The ring $\bigoplus_n K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$ is a polynomial ring $\mathbb{Z}[\beta]$ where β is represented by the Koszul complex on one variable. The same thing happens in algebraic K -theory over any field F . But the analogous result is *not* true for $\bigoplus_n KO^0(\mathbb{R}^n, \mathbb{R}^n - 0)$. In fact, the Bott computations of KO^* show that it cannot possibly be true. It is informative, though, to spend some time unravelling precisely what is different here.

It is indeed true that $KO^0(\mathbb{R}^1, \mathbb{R}^1 - 0)$ is generated by the Koszul complex on $\mathbb{R}^1$. We will write this as $K(x)$ here. But the group it generates is a $\mathbb{Z}/2$, not a $\mathbb{Z}$. That's the first difference. The group $KO^0(\mathbb{R}^2, \mathbb{R}^2 - 0)$ is again generated by the Koszul complex on $\mathbb{R}^2$, but once we get to $KO^0(\mathbb{R}^3, \mathbb{R}^3 - 0)$ the group is zero. So these are the two differences we wish to explain: why $\mathbb{Z}/2$ rather than $\mathbb{Z}$, and why does the third power vanish? Fairly explicit reasons can be given for both.

We have seen quite generally ??? that $2[K(x)] = [K(x^2)]$ in $KO^0(\mathbb{R}^1, \mathbb{R}^1 - 0)$. That is, twice the class of the Koszul complex is the Koszul complex on x^2 . But the deformation $x^2 + t$ shows that $K(x^2)$ can be deformed to $K(x^2 + 1)$, and the latter is exact on all of $\mathbb{R}^1$. This is the reason for the $\mathbb{Z}/2$. Notice that this argument does not work over the complex numbers, since the complexes $K(x^2 + t)$ fail to be exact for all $x \in \mathbb{C}^n - 0$.

We have to work a bit harder to give an underlying geometric reason that the Koszul complex on $\mathbb{R}^3$ represents zero in $KO^0(\mathbb{R}^3, \mathbb{R}^3 - 0)$. Of course Bott tells us that the group is zero, but the goal is to give an independent explanation. Ideally we would give an explicit deformation of $K(x, y, z)$ into a complex that is exact, but that seems to be tricky. Instead we will use a K -theoretic argument to first replace $K(x, y, z)$ by a different model.

Let $x \in \mathbb{R}^n$. On the exterior algebra $\Lambda^*(\mathbb{R}^n)$ we have the Koszul differential d_x , which decreases degree, as well as the operator $x \wedge (-)$ which increases degree.

Define $T_x = d_x + (x \wedge -)$, which can be regarded both as a map $\Lambda^{odd} \mathbb{R}^n \rightarrow \Lambda^{ev} \mathbb{R}^n$ and as $\Lambda^{ev} \mathbb{R}^n \rightarrow \Lambda^{odd} \mathbb{R}^n$. The square of this operator is:

$$\begin{aligned} T_x(T_x \omega) &= d_x(d_x \omega + x \wedge \omega) + x \wedge (d_x \omega + x \wedge \omega) = d_x(x \wedge \omega) + x \wedge d_x \omega \\ &= d_x(x) \omega - x \wedge d_x \omega + x \wedge d_x \omega \\ &= d_x(x) \omega = |x|^2 \omega. \end{aligned}$$

In particular, as long as $x \neq 0$ the operator T_x is invertible.

Write $(\Lambda^*(\mathbb{R}^n), T)$ for the two-term chain complex

$$0 \leftarrow \Lambda^{odd} \mathbb{R}^n \xrightarrow{T_x} \Lambda^{ev} \mathbb{R}^n \longrightarrow 0$$

of bundles on $\mathbb{R}^n$ (concentrated in degrees 0 and 1), where on the fiber over x the map is T_x . By the preceding paragraph this defines an element in $KO^0(\mathbb{R}^n, \mathbb{R}^n - 0)$. The exact same construction in the complex case gives an element in $K^0(\mathbb{R}^n, \mathbb{R}^n - 0)$. It is important to note, though, that we do not get an element of $K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$: for $x \in \mathbb{C}^n$ we can have $\sum x_i^2 = 0$ even when $x \neq 0$.

Proposition 20.32. *The element $(\Lambda^*(\mathbb{R}^n), T)$ is equal to the standard Koszul complex as elements of $KO^0(\mathbb{R}^n, \mathbb{R}^n - 0)$.*

Proof. ??? □

The pair $(\Lambda^*(\mathbb{R}^n), T)$ is actually part of a graded Clifford module structure, as this follows immediately from $(T_x)^2 = |x|^2$. To be precise, $\Lambda^*(\mathbb{R}^n)$ is a module over $Cl_{0,n}^{grd}$ where $x \in \mathbb{R}^n$ acts as T_x . When $n = 3$ we can use the Clifford structure to deform the complex into one that is exact on all of $\mathbb{R}^n$. Set $\omega = e_1 e_2 e_3$. Then ω commutes with each e_i and therefore with any $x \in \mathbb{R}^n$. Also $\omega^2 = -1$ (recall $e_i^2 = 1$ for each i). For $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ consider the formula

$$(x + t\omega)(x - t\omega) = |x|^2 - t^2 \omega^2 = |x|^2 + t^2.$$

So $x + t\omega$ is invertible as long as $(x, t) \neq (0, 0)$. The complex $(\Lambda^*(\mathbb{R}^n), T + t\omega)$ therefore gives a deformation of $(\Lambda^*(\mathbb{R}^n), T)$ (at $t = 0$) to a complex (at $t = 1$) which is exact on all of $\mathbb{R}^n$.

Corollary 20.33. *For $n \geq 3$ the Koszul complex on $\mathbb{R}^n$ equals zero in $KO^0(\mathbb{R}^n, \mathbb{R}^n - 0)$.*

Proof. Using the ring structure on $KO^0(\mathbb{R}^n, \mathbb{R}^n - 0)$, and the fact that the Koszul complex on $\mathbb{R}^n$ is the n th tensor power of the Koszul complex on $\mathbb{R}^1$, it suffices to prove this for $n = 3$. This was done by the above deformation. □

Again, we already knew that $KO^0(\mathbb{R}^3, \mathbb{R}^3 - 0) = 0$ by the Bott calculation of KO^* . But the above deformation helps us see what is going on behind the scenes.

Exercise 20.34. Try to apply the above deformation argument to $n = 1$ or $n = 2$. Convince yourself that $n = 3$ is the smallest case where it works.

Exercise 20.35. It is possible to use a different sign convention when making $\Lambda^* \mathbb{R}^n$ into a Clifford module. For $x \in \mathbb{R}^n$ define $S_x(\alpha) = -d_x(\alpha) + x \wedge \alpha$. Prove that $S_x^2 = -|x|^2$, and so S_x is again an isomorphism when $x \neq 0$. Using these operators $\Lambda^* \mathbb{R}^n$ is a $Cl_{n,0}^{grd}$ -module. Prove that the chain complex of bundles $(\Lambda^* \mathbb{R}^n, S)$ on $\mathbb{R}^n$ is isomorphic to the complex $(\Lambda^* \mathbb{R}^n, T)$ that we used above.

20.36. Clifford structures on exterior products. Let V be a n -dimensional real vector space with an inner product $\langle -, - \rangle$. There is an induced inner product on the each vector space $\Lambda^k V$. To define, start with

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det[\langle v_i, w_j \rangle]$$

and extend linearly. By convention we set $\langle \alpha, \gamma \rangle = 0$ if $\alpha \in \Lambda^k V$ and $\gamma \in \Lambda^l V$ and $k \neq l$, and so we can regard the inner product as defined on all of $\Lambda^* V$. Observe that if $e_1, \dots, e_n$ is an orthonormal basis for V then $\{e_{i_1} \wedge \cdots \wedge e_{i_r} \mid i_1 < \cdots < i_r\}_{0 \leq r \leq n}$ is an orthonormal basis for $\Lambda^* V$.

Because we have a nondegenerate form on $\Lambda^* V$, we can define elements by specifying what their inner products are with all elements of $\Lambda^* V$. More precisely, if Ω is a linear functional on $\Lambda^* V$ then there is a unique $\omega \in \Lambda^* V$ with the property that $\Omega(\alpha) = \langle \alpha, \omega \rangle$ for all $\alpha \in \Lambda^* V$.

Suppose $\theta \in \Lambda^k V$. One defines the **interior product operator** (also called the **contraction operator**) ι_θ by the adjunction formula

$$\langle \theta \wedge \alpha, \gamma \rangle = \langle \alpha, \iota_\theta(\gamma) \rangle.$$

Observe that $\iota_{\theta_1 \wedge \theta_2} = \iota_{\theta_2} \circ \iota_{\theta_1}$ and, in particular, $(\iota_\theta)^2 = 0$.

We claim that for $u_1, \dots, u_k \in V$ one has

$$(20.37) \quad \iota_v(u_1 \wedge \cdots \wedge u_k) = \sum_{j=1}^k (-1)^{j-1} \langle v, u_j \rangle u_1 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_k.$$

That is to say, ι_v is really the same as the Koszul differential d_v that we have encountered before. To see this, start with

$$\begin{aligned} \langle w_1 \wedge \cdots \wedge w_{k-1}, \iota_v(u_1 \wedge \cdots \wedge u_k) \rangle &= \langle v \wedge w_1 \wedge \cdots \wedge w_{k-1}, u_1 \wedge \cdots \wedge u_k \rangle \\ &= \det \begin{bmatrix} \langle v, u_1 \rangle & \cdots & \langle v, u_k \rangle \\ \langle w_1, u_1 \rangle & \cdots & \langle w_1, u_k \rangle \\ \vdots & & \vdots \\ \langle w_{k-1}, u_1 \rangle & \cdots & \langle w_{k-1}, u_k \rangle \end{bmatrix} \end{aligned}$$

and just do a cofactor expansion for the determinant along the first row.

From (20.37) one readily checks that

$$\iota_v(\alpha \wedge \gamma) = \iota_v(\alpha) \wedge \gamma + (-1)^{|\alpha|} \alpha \wedge \iota_v(\gamma)$$

whenever α and γ are homogeneous. That is, ι_v acts as a derivation with respect to the exterior product.

Now suppose that V has a chosen orientation. This is just a fixed unit vector $\omega \in \Lambda^n V$. The **Hodge star operator** is a linear map $*$: $\Lambda^* V \rightarrow \Lambda^* V$ that sends each $\Lambda^k V$ to $\Lambda^{n-k} V$, and is characterized uniquely by the equation

$$\alpha \wedge (*\gamma) = \langle \alpha, \gamma \rangle \cdot \omega$$

for all $\gamma \in \Lambda^k V$ and $\alpha \in \Lambda^{n-k} V$. If $e_1, \dots, e_n$ is an oriented orthonormal basis for V and $\{i_1 < \cdots < i_k\} \cup \{j_1 < \cdots < j_{n-k}\}$ is a partition of n into two sets, then

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$$

where the sign is the unique one that makes the equation

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge \pm(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) = e_1 \wedge \cdots \wedge e_n$$

hold. If $\alpha \in \Lambda^k V$ then $*(\alpha) = (-1)^{k(n-k)} \alpha$.

We have the canonical inclusion $V \hookrightarrow \text{Cl}(V)$. We obtain an induced map $\Lambda^k V \rightarrow \text{Cl}(V)$ by combining Clifford multiplication with the usual antisymmetrization formula:

$$v_1 \wedge \cdots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma} (-1)^{\sigma} v_{\sigma(1)} \cdots v_{\sigma(k)}.$$

If $e_1, \dots, e_n$ is an orthonormal basis for V then this sends

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}.$$

Therefore our map $\Lambda^* V \rightarrow \text{Cl}(V)$ is an isomorphism of vector spaces, since it sends a basis to a basis. Since this map is canonical, we will implicitly identify the two underlying vector spaces from now on. Note that $\Lambda^* V \rightarrow \text{Cl}(V)$ is of course NOT a ring homomorphism: on the left we have $v^2 = 0$ and on the right we have $v^2 = -|v|^2$.

The action of $\text{Cl}(V)$ on itself now translates into an action of $\text{Cl}(V)$ on $\Lambda^* V$. For $v \in V$ and $\gamma \in \Lambda^* V$ the formula for this action is

$$v \cdot \gamma = v \wedge \gamma - \iota_v(\gamma).$$

Both sides are linear in the two variables, so it suffices to check it where $v = e_i$ and $\gamma = e_{j_1} \wedge \cdots \wedge e_{j_k}$. There are two cases, depending on whether i appears among the j 's or not, and the formula is readily verified in both cases. Going forward we will refer to this as the canonical action of $\text{Cl}(V)$ on $\Lambda^* V$.

Proposition 20.38. *Let V be an inner product space of dimension n . Then as a module over $\text{Cl}(V)$, $\Lambda^* V$ is $2^{n-\varphi(n)}$ times the unique irreducible when $n \not\equiv 3 \pmod{4}$. When $n \equiv 3 \pmod{4}$ one instead has that $\Lambda^* V$ is $2^{n-\varphi(n)-1}$ times the sum of the two irreducibles.*

Proof. Of course this is really a result about $\text{Cl}(V)$ as a module over itself. When $n \not\equiv 3 \pmod{4}$ there is a unique irreducible and it has dimension $2^{\varphi(n)}$. Since $\text{Cl}(V)$ has dimension 2^n , the desired result is immediate.

Now assume that $n \equiv 3 \pmod{4}$. It will suffice to show that both irreducible representations appear in $\text{Cl}(V)$ with the same multiplicity. As usual fix an orthonormal basis for V and let $\omega = e_1 \cdots e_n$. The irreducibles are distinguished by whether ω acts on them as $+1$ or as -1 . Let $W_+ \subseteq \text{Cl}(V)$ be the set of elements x such that $\omega x = x$, and let W_- be those elements where $\omega x = -x$.

Let $f: \text{Cl}(V) \rightarrow \text{Cl}(V)$ be any automorphism having parity -1 . For example, one can take f to be the map that sends $e_1 \rightarrow -e_1$ and fixes $e_2, \dots, e_n$. We claim f takes W_+ to W_- and vice versa (and likewise for f^{-1}). For if $\omega x = x$ then $f(x) = f(\omega x) = f(\omega)f(x) = -\omega f(x)$. This shows that W_+ and W_- have the same dimension, and from there the result is immediate. $\square$

Exercise 20.39. Let V be oriented with orientation class $\omega \in \Lambda^n V$. Prove that the Hodge $*$ -operator is connected to Clifford multiplication via the formula

$$\alpha \cdot \omega = (-1)^{\lfloor \frac{k+1}{2} \rfloor} \cdot (*\alpha)$$

for $\alpha \in \Lambda^k V$.

20.40. The Hermitian version. Now let V be a complex vector space with a Hermitian inner product $\langle -, - \rangle$. Our convention is that this is linear in the second variable and conjugate-linear in the first. Let $b(v, w) = \operatorname{Re}\langle v, w \rangle$ be the underlying real inner product, so that $\langle v, w \rangle = b(v, w) + b(iv, w)i$. Note as well that $\langle v, w \rangle = \langle iv, iw \rangle$ and so $b(v, w) = b(iv, iw)$.

Let ${}_{\mathbb{R}}V$ denote V but regarded as a real vector space. If $J: V \rightarrow V$ denotes the multiplication-by- i map, then $(\operatorname{id} \otimes J)^2 = -\operatorname{id}$ on $\mathbb{C} \otimes {}_{\mathbb{R}}V$ and therefore splits $\mathbb{C} \otimes {}_{\mathbb{R}}V$ into the $+i$ and $-i$ -eigenspaces. The former is $\{1 \otimes v - i \otimes iv \mid v \in V\}$ and the latter is $\{1 \otimes v + i \otimes iv \mid v \in V\}$, which as complex vector spaces are transparently identified with V and $\bar{V}$. It will be useful to throw in a scaling factor, so that $v \in V$ is identified with $\frac{1}{2}(1 \otimes v - i \otimes iv)$, and $\tilde{v} \in \bar{V}$ is identified with $\frac{1}{2}(1 \otimes \tilde{v} + i \otimes i\tilde{v})$. We will use this canonical identification $\mathbb{C} \otimes {}_{\mathbb{R}}V = V \oplus \bar{V}$ implicitly from now on.

The real inner product b on ${}_{\mathbb{R}}V$ induces a complex symmetric bilinear form $b_{\mathbb{C}}$ on $\mathbb{C} \otimes {}_{\mathbb{R}}V = V \oplus \bar{V}$. Let us be clear about what this is. For $v, w \in V$ we have

$$\begin{aligned} b_{\mathbb{C}}(v, w) &= b\left(\frac{1}{2}(1 \otimes v - i \otimes iv), \frac{1}{2}(1 \otimes w - i \otimes iw)\right) \\ &= \frac{1}{4} \left[b(v, w) - ib(iv, w) - ib(v, iw) - b(iv, iw) \right] = 0. \end{aligned}$$

Likewise, for $\tilde{v}, \tilde{w} \in \bar{V}$ one has $b_{\mathbb{C}}(\tilde{v}, \tilde{w}) = 0$. But

$$\begin{aligned} b_{\mathbb{C}}(\tilde{w}, v) &= b\left(\frac{1}{2}(1 \otimes \tilde{w} + i \otimes i\tilde{w}), \frac{1}{2}(1 \otimes v - i \otimes iv)\right) \\ &= \frac{1}{4} \left[2b(\tilde{w}, v) - ib(\tilde{w}, iv) + ib(i\tilde{w}, v) \right] \\ &= \frac{1}{2} (b(\tilde{w}, v) + ib(i\tilde{w}, v)) \\ &= \frac{1}{2} \langle \tilde{w}, v \rangle. \end{aligned}$$

Since $b_{\mathbb{C}}$ is symmetric we can also write this as

$$b_{\mathbb{C}}(v, \tilde{w}) = \frac{1}{2} \langle \tilde{w}, v \rangle = \frac{1}{2} \overline{\langle v, \tilde{w} \rangle}.$$

The upshot is that in $V \oplus \bar{V}$ the two summands are orthogonal, and the pairing between one summand and the other is almost the original form $\langle -, - \rangle$. The $\frac{1}{2}$ factor is a little unsightly, but we will live with it. Some sources use different normalization factors: if we had $\frac{1}{\sqrt{2}}$ in our original isomorphism instead of $\frac{1}{2}$ that would remove the scaling factor on the form, though it would introduce other factors later on.

By our previous remarks we have a canonical vector space isomorphism

$$\Lambda^*({}_{\mathbb{R}}V) \rightarrow \operatorname{Cl}({}_{\mathbb{R}}V).$$

Tensoring with $\mathbb{C}$ now gives an induced isomorphism

$$\Lambda^*(V \oplus \bar{V}) = \mathbb{C} \otimes \Lambda^*({}_{\mathbb{R}}V) \rightarrow \mathbb{C} \otimes \operatorname{Cl}({}_{\mathbb{R}}V)$$

(note that the exterior product on the left is happening over $\mathbb{C}$). The action of $\mathbb{C} \otimes \operatorname{Cl}({}_{\mathbb{R}}V)$ on the target gives an action on the domain, and just as in the real case we would like to write a formula for that in terms of exterior calculus.

20.41. Symmetries of Clifford algebras. Clifford algebras have so many symmetries that the situation can feel overwhelming. Any orthogonal transformation of f of V is the restriction of a unique automorphism $\operatorname{Cl}(f)$ of $\operatorname{Cl}(V)$. There are also the inner automorphisms $x \mapsto \alpha x \alpha^{-1}$ for any $\alpha \in \operatorname{Cl}(V)^*$.

Exercise 20.42. Recall the distinguished element $\omega = e_1 \cdots e_n \in \text{Cl}_n$.

- (a) Prove that the center of Cl_n is $\mathbb{R}$ when n is even and is the subalgebra $\mathbb{R}\langle 1, \omega \rangle$ when n is odd.
- (b) Prove that the supercenter of Cl_n^{grd} coincides with the center of Cl_n .
- (c) Suppose n is odd. Prove that an automorphism $f: \text{Cl}_n \rightarrow \text{Cl}_n$ must send ω to either ω or $-\omega$, and prove that both possibilities actually occur. Define the **parity** (or **orientation**) of an automorphism to be $+1$ or -1 in the two cases. Check that inner automorphisms all have parity one.
- (d) An automorphism $f: \text{Cl}_n \rightarrow \text{Cl}_n$ induces a map $f^*: G_n \rightarrow G_n$ on the Grothendieck group of modules, sending a module M to its restriction of scalars along f (the same module but with the action $x.m = f(x)m$). Prove that f^* is the identity when $n \not\equiv 3 \pmod{4}$, and that when $n \equiv 3 \pmod{4}$ the situation depends on the parity: parity one maps induces the identity, whereas parity -1 maps switch the two positive generators.

There is also a canonical anti-automorphism σ of $\text{Cl}(V)$ given by $\sigma(v_1 \cdots v_k) = v_k \cdots v_1$ for $v_i \in V$. But what turns out to be most useful is to combine σ with the antipodal map on V , which we call the **transpose** operator and denote $x \rightarrow x^T$. That is to say,

$$(v_1 \cdots v_k)^T = (-v_k)(-v_{k-1}) \cdots (-v_1) = (-1)^k v_k \cdots v_1.$$

The transpose is the unique anti-automorphism that sends $v \mapsto -v$ for $v \in V$. When $e_1, \dots, e_n$ is an orthonormal basis of V let us record that

$$(e_{i_1} \cdots e_{i_k})^\sigma = (-1)^{\binom{k}{2}} e_{i_1} \cdots e_{i_k}, \quad (e_{i_1} \cdots e_{i_k})^T = (-1)^{\binom{k+1}{2}} e_{i_1} \cdots e_{i_k}.$$

Why work with the transpose rather than with σ ? We will see that certain constructions work out more cleanly with the extra signs built in, but we can also motivate it by the following observation. Suppose $\text{Cl}(V)$ acts on $\mathbb{R}^n$ orthogonally, and for each $x \in \text{Cl}(V)$ let A_x denote matrix giving the action of x . For an element $v \in V$ of norm 1, A_v is orthogonal and $(A_v)^2 = -I$, which shows that $(A_v)^T = (A_v)^{-1} = -A_v = A_{(v^T)}$. It follows that for any $x \in \text{Cl}(V)$ one has $A_{(x^T)} = (A_x)^T$.

The transpose allows us to naturally change right $\text{Cl}(V)$ -modules into left $\text{Cl}(V)$ -modules (of course one can also do this with any anti-automorphism). For example, let M be a (left) $\text{Cl}(V)$ -module. Then $M^* = \text{Hom}(M, \mathbb{R})$ is naturally a right $\text{Cl}(V)$ -module, via the formula

$$(\phi \cdot x)(m) = \phi(x \cdot m)$$

with $\phi \in M^*$, $m \in M$, and $x \in \text{Cl}(V)$. This construction satisfies $\phi(xy) = (\phi x)y$, as needed for a right module. But we can recast M^* as a left module via

$$(x \cdot \phi)(m) = \phi(x^T \cdot m).$$

For the most part we will want to stay inside the world of left Clifford modules, and so when we write M^* the intention is always that it is equipped with the above left Clifford action.

Note that when $\text{Cl}(V)$ is regarded as $\mathbb{Z}/2$ -graded then the transpose preserves the grading, and so if M is a graded $\text{Cl}(V)^{grd}$ -module then we can define the dual M^* by the same formulas, with the grading $(M^*)_i = (M_i)^*$ for $i = 0, 1$.

Exercise 20.43.

- (a) The assignments $M \mapsto M^*$ induce group maps $G_n \rightarrow G_n$ and $G_n^{grd} \rightarrow G_n^{grd}$. Check that these are equal to the identity, for all n .
- (b) Recall the functor $D: \text{Cl}_n\text{-Mod} \rightarrow \text{Cl}_{1+n}^{grd}\text{-Mod}$. Prove that this satisfies $D(M^*) \cong (DM)^*$, for all Cl_n -modules M .
- (c) Check that the statements in the above two parts would not be true if we had defined the module structure on duals using σ rather than the transpose.

20.44. More connections. The following works over any ground field not of characteristic 2. Let V be any vector space, and equip $V \oplus V^*$ with the bilinear form where

$$\langle v + \phi, v' + \phi' \rangle = \frac{1}{2}(\phi(v') + \phi'(v)).$$

Note that $\langle v, v' \rangle = 0 = \langle \phi, \phi' \rangle$ for every $v, v' \in V$ and $\phi, \phi' \in V^*$. We can form the Clifford algebra $\text{Cl}(V \oplus V^*)$, which will contain Λ^*V and $\Lambda^*(V^*)$ as subalgebras. We claim this acts naturally on Λ^*V via the formula

$$(v + \phi) \cdot \alpha = v \wedge \alpha - \iota_\phi(\alpha).$$

Indeed, just compute that

$$(v + \phi) \cdot ((v + \phi)\alpha) = -v \wedge \iota_\phi(\alpha) - \iota_\phi(v \wedge \alpha) = -\iota_\phi(v)\alpha = -\phi(v)\alpha$$

and $\langle v + \phi, v + \phi \rangle = \phi(v)$.

If V comes equipped with a nondegenerate symmetric bilinear form b , then this gives an isomorphism $V \rightarrow V^*$ by $v \mapsto b(v, -)$. Let us just write $v^* = b(v, -)$, for brevity. The map $V \rightarrow V \oplus V^*$ that sends $v \mapsto (v, v^*)$ is then an isometric embedding, since

$$\langle (v, v^*), (w, w^*) \rangle = \frac{1}{2}(v^*(w) + w^*(v)) = \frac{1}{2}(b(v, w) + b(w, v)) = b(v, w).$$

So we get an embedding for Clifford algebras $\text{Cl}(V) \hookrightarrow \text{Cl}(V \oplus V^*)$, and in this way an action of $\text{Cl}(V)$ on Λ^*V . The formula for this action is

$$v \cdot \alpha = v \wedge \alpha - \iota_{v^*}\alpha$$

but note that the contraction term could also be written as $\iota_v(\alpha)$ using our earlier conventions.

Now suppose that V is a complex vector space equipped with a Hermitian form $h(-, -)$. Our convention is that this is complex-linear in the second variable, and conjugate-linear in the first variable. So we get an isomorphism $\bar{V} \rightarrow V^*$ by $v \mapsto h(v, -)$. The above construction then makes Λ^*V into a module over $\text{Cl}(V \oplus \bar{V})$. We can also identify $V \oplus \bar{V} = \mathbb{C} \otimes_{\mathbb{R}} V$. Specifically, $v \in V$ maps to $Av = \frac{1}{2}(1 \otimes v - i \otimes iv)$ and $\tilde{v} \in \bar{V}$ maps to $Bv = \frac{1}{2}(1 \otimes v + i \otimes iv)$. Note the identity $1 \otimes v = Av + Bv$. Said differently, the element $v \in {}_{\mathbb{R}}V$ corresponds to the element $v + \tilde{v} \in V \oplus \bar{V}$.

So we now have an action of $\mathbb{C} \otimes \text{Cl}({}_{\mathbb{R}}V)$ on Λ^*V . The formula for this action is

$$v \cdot \alpha = (v + \tilde{v}) \cdot \alpha = v \wedge \alpha - \iota_{\tilde{v}}(\alpha).$$

Note that here the contraction is with respect to the Hermitian inner product $\langle v, - \rangle$, and so this is *not* the Koszul differential d_v . If $\{e_i\}$ is an orthonormal basis for V and $v = \sum v_i e_i$, then ι_v can be identified with the Koszul differential $d_{\tilde{v}}$ where $\tilde{v} = \sum_i \bar{v}_i e_i$. This identification is not natural, though.

Proposition 20.45. *Let $V = \mathbb{C}^n$ with its standard Hermitian inner product. As elements of $K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$ the complex $[\Lambda^*\mathbb{C}^n, z \wedge (-) - \iota_z(-)]$ is equal to $(-1)^n$ times the standard Koszul complex.*

Proof. For $z \in \mathbb{C}^n$ define the operator W_z by the formula $\langle W_z(\alpha), \gamma \rangle = \langle \alpha, d_z(\gamma) \rangle$. Then the Koszul complex is equal to $[\Lambda^* \mathbb{C}^n, W_z + d_z]$ by our previous argument (??). Since $d_z = \iota_{\bar{z}}$ we get $W_z(\alpha) = \bar{z} \wedge \alpha$.

The complex in the statement of the proposition may be written as $[\Lambda^* \mathbb{C}^n, W_{\bar{z}} - d_{\bar{z}}]$. As we have seen before (???), this is isomorphic to the complex $[\Lambda^* \mathbb{C}^n, W_{\bar{z}} + d_{\bar{z}}]$. So this complex becomes the one from the preceding paragraph after pullback along the map $(\mathbb{C}^n, \mathbb{C}^n - 0) \rightarrow (\mathbb{C}^n, \mathbb{C}^n - 0)$ that sends $z \mapsto \bar{z}$. Restricted to the sphere $S(\mathbb{C}^n)$ this map has degree $(-1)^n$, as there are n sign changes among the real coordinates. So the induced map on K -theory is $(-1)^n$, which completes the proof. $\square$

21. SPECTRAL PROOFS OF BOTT PERIODICITY

Up until now we have discussed essentially two proofs of Bott periodicity: the original proof of Bott using Morse theory, which was outlined in Sections 15 and 16, and the Atiyah-Bott proof of *complex* periodicity via the K -functor, that was sketched in Section ?? . In this section and the next our goal is to describe some of the other approaches in the literature.

The classical spectral theorem says that every unitary matrix is diagonalizable, with orthogonal eigenspaces. There is a similar result for orthogonal matrices, though it is a bit more involved. There are several papers that get at Bott periodicity by tying it to these spectral decompositions: one line of attack was developed in the series of papers [McDuff1], [AP], [Behrens1, Behrens2], whereas a different approach was developed in [Harris] and [Giffen]. We will see that these are really the same approach, though, just using slightly different technology.

21.1. Homotopy fiber sequences and quasifibrations. To demonstrate that a space X is weakly equivalent to ΩY , one technique is to produce a homotopy fiber sequence $X \rightarrow E \rightarrow Y$ where $E \simeq *$. The easiest way to spot a homotopy fiber sequence is if $E \rightarrow Y$ is a fibration and X is the actual fiber, but fibrations are somewhat rare in nature. Dold and Thom [DT] developed a theory of *quasifibrations* that also works for this purpose. The definition is simple: a map $E \rightarrow Y$ is a quasifibration if the canonical maps from its fibers to its homotopy fibers are all weak equivalences. This condition is not so easy to check, but the paper [DT] developed some techniques for doing so. Note that every fibration is a quasifibration, but not vice versa: the vertical projection map from the letter $\mathbb{L}$ to its base is a counterexample to the converse.

So one way to prove complex Bott periodicity is to produce a quasifibration $E \rightarrow U$ where the fiber is $\mathbb{Z} \times BU$, and likewise for the loop relations that make up real Bott periodicity. The challenge is to produce the quasifibration, which often involves some geometric ingenuity. An argument along these lines was first proposed in [McDuff1], where the complex version was outlined. That argument was flushed out in [AP], and later simplified and expanded to the real case in [Behrens1, Behrens2].

Theorem 21.2 (Dold and Thom). *Let $p: E \rightarrow B$ be a map. Assume that B is a T_1 -space and that it comes equipped with a filtration $\emptyset = B_{-1} \subseteq B_0 \subseteq B_1 \subseteq \cdots$ by closed subsets such that $B = \bigcup_i B_i$ with the colimit topology. Set $E_i = p^{-1}(B_i)$. Suppose that*

(1) *p is a fibration over $B_i - B_{i-1}$, for all i .*

- (2) For each $i \geq 0$ there exists an open neighborhood $B_i \subseteq N_i$ in B_{i+1} and a deformation retraction from N_i down to B_i : that is, a homotopy $h: N_i \times I \rightarrow N_i$ such that $h_0 = \text{id}$, $h_1(N_i) \subseteq B_i$, and $(h_1)|_{B_i} = \text{id}_{B_i}$.
- (3) For each $i \geq 0$ there exists a deformation retraction H from $p^{-1}(N_i)$ down to E_i that covers h , and for every $y \in N_i$ the map $H_1: p^{-1}(y) \rightarrow p^{-1}(h_1(y))$ is a weak equivalence.

Then $E \rightarrow B$ is a quasifibration.

The above result is a combination of [DT, Theorems 2.2, 2.10, and 2.15]. See also [Hatcher1, Lemma 4K.3]. Some of the criteria can be weakened a bit, but this version will suffice for our purposes. It is useful to have a name for the maps $H_1: p^{-1}(y) \rightarrow p^{-1}(h_1(y))$, and we will call them **binding maps**.

21.3. Spectral decompositions. Now we come to the key ideas. For every unitary transformation $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$, there are unique eigenvalues $\lambda_j \in S^1$ and orthogonal eigenspaces $V_j \subseteq \mathbb{C}^n$ such that $T = \sum_j \lambda_j \pi_{V_j}$, where here π_W denotes orthogonal projection onto W . This is the usual finite-dimensional spectral theorem. If we define $A = \sum_j \frac{1}{2\pi i} \log(\lambda_j) \pi_{V_j}$ then since each $\log(\lambda_j) \in i\mathbb{R}$ this is a Hermitian matrix, and $e^{2\pi i A} = T$. In fact, up to the choice of $\log(\lambda_j)$ (which is only defined up to multiples of $2\pi i$) A is the unique Hermitian matrix with $e^{2\pi i A} = T$. We can narrow down the possibilities for A by requiring that the eigenvalues (which must be real) lie in $[0, 1]$, though this still leaves some ambiguity for the case $\lambda_j = 1$. Here the eigenspace V_{λ_j} must be orthogonally split into the 0- and 1-eigenspaces of A .

Define E_n to be the set of Hermitian $n \times n$ matrices with eigenvalues in $[0, 1]$, topologized as a subspace of $M_{n \times n}(\mathbb{C})$. Note that this is contractible, via the homotopy $A \mapsto tA$. Define $p_n: E_n \rightarrow U_n$ by $p(A) = e^{2\pi i A}$. For an element $T = \sum_j \lambda_j \pi_{V_j} \in U_n$, the space $p^{-1}(T)$ consists of Hermitian transformations of the form

$$\sum_{\lambda_j \neq 1} \frac{1}{2\pi i} \log(\lambda_j) \pi_{V_j} + 1 \cdot \pi_{W_1} + 0 \cdot \pi_{W_0}$$

where each $\log(\lambda_j) \in [0, 1]$, $W_1 \subseteq \ker(T - I)$, and W_0 is the orthogonal complement of W_1 in $\ker(T - I)$. The only choice here was W_1 , and in this way we see that $p_n^{-1}(T) \cong \text{Gr}_\bullet(\ker(T - I))$, where as usual $\text{Gr}_\bullet(V) = \coprod_k \text{Gr}_k(V)$.

Next we stabilize, and the evident way to do this is via the usual inclusions $U_n \hookrightarrow U_{n+1}$ ($T \mapsto T \oplus \text{id}$) and $E_n \hookrightarrow E_{n+1}$ given by $A \mapsto A \oplus 0$. These definitions commute with the p maps. If we let $\tilde{E} = \text{colim}_n E_n$ then we obtain a map $\tilde{p}: \tilde{E} \rightarrow U$, where $\tilde{E}$ is weakly contractible (since all the E_n were). For $T \in U$ the fiber $\tilde{p}^{-1}(T)$ is homeomorphic to $\text{Gr}_\bullet(\ker(T - I)) \cong \text{Gr}_\bullet(\mathbb{C}^\infty)$. We are starting to see the connection between U and Grassmannians, but note that $\tilde{p}$ cannot possibly be a quasifibration: if it were, then we would have $\Omega U \simeq \text{Gr}_\bullet(\mathbb{C}^\infty)$ which we know is not the case. This $\tilde{p}$ is a good example of a map where all of the fibers are homeomorphic but where the map is not a quasifibration.

The bit of ingenuity is to stabilize in a different way. Let $\mathbb{C}_+^n$ and $\mathbb{C}_-^n$ denote two copies of $\mathbb{C}^n$, the first with orthonormal basis $e_1^+, \dots, e_n^+$ and the second with orthonormal basis $e_1^-, \dots, e_n^-$. Set $\mathbb{S}\mathbb{C}^n = \mathbb{C}_+^n \oplus \mathbb{C}_-^n$ (the S is for ‘super’), and note the evident inclusions $\mathbb{S}\mathbb{C}^n \hookrightarrow \mathbb{S}\mathbb{C}^{n+1}$. Let $E(\mathbb{S}\mathbb{C}^n)$ be the space of Hermitian transformations with eigenvalues in $[0, 1]$, and let $U(\mathbb{S}\mathbb{C}^n)$ denote the usual unitary

group. Let $p_n: E(\mathbb{S}\mathbb{C}^n) \rightarrow U(\mathbb{S}\mathbb{C}^n)$ be the map $p_n(A) = e^{2\pi i A}$. Define $E(\mathbb{S}\mathbb{C}^n) \hookrightarrow E(\mathbb{S}\mathbb{C}^{n+1})$ to be the map

$$A \mapsto A \oplus (1 \cdot \pi_{\langle e_{n+1}^+ \rangle}) \oplus (0 \cdot \pi_{\langle e_{n+1}^- \rangle}).$$

Note that writing the last term is unnecessary because the coefficient is zero, but including it shows the symmetry: we are adding vectors of eigenvalue 1 and eigenvalue 0 simultaneously.

If W is a finite-dimensional vector space let us write $(\mathbb{Z} \times BU)(W) = \text{colim}_n \text{Gr}_\bullet(W \oplus \mathbb{S}\mathbb{C}^n)$, where the transition maps

$$\text{Gr}_\bullet(W \oplus \mathbb{S}\mathbb{C}^n) \rightarrow \text{Gr}_\bullet(W \oplus \mathbb{S}\mathbb{C}^{n+1})$$

send a subspace N to $N \oplus \langle e_{n+1}^+ \rangle$. These colimits were studied back in Exercise ??, where we saw that $(\mathbb{Z} \times BU)(W)$ is homeomorphic to the space $\mathbb{Z} \times BU$ and for any inclusion $W \hookrightarrow W'$ the evident inclusion $(\mathbb{Z} \times BU)(W) \hookrightarrow (\mathbb{Z} \times BU)(W')$ is a weak homotopy equivalence.

Set $E_\infty = \text{colim}_n E(\mathbb{S}\mathbb{C}^n)$, $U_\infty = \text{colim}_n U(\mathbb{S}\mathbb{C}^n)$, and $p = \text{colim}_n p_n$. Then U_∞ is a model for the infinite unitary group and E_∞ is contractible. If $T \in U_\infty$ then for some n we can write $T = T_n \oplus \text{id}_{(\mathbb{S}\mathbb{C}^n)^\perp}$ where $T_n \in U(\mathbb{S}\mathbb{C}^n)$. One readily checks that $p^{-1}(T) \cong (\mathbb{Z} \times BU)(\ker(T_n - I))$. In particular, all of the fibers of p are homeomorphic to $\mathbb{Z} \times BU$. It remains to prove that p is a quasifibration. We only outline the details of this argument.

For simplicity now write $E = E_\infty$ and $U = U_\infty$.

- (i) For $T \in U$ let us call the eigenvalues in $S^1 - \{1\}$ the **proper** eigenvalues, and let $\text{Eig}_{\text{prop}}(T)$ be the direct sum of the eigenspaces for all of the proper eigenvalues. This space is also equal to $\ker(T - I)^\perp$.
- (ii) Let $B_n \subseteq U$ be the subspace of all T such that $\dim \text{Eig}_{\text{prop}}(T) \leq n$. Then $B_n \cap U_N = \{T \in U_N \mid \dim \text{Eig}_1(T) \leq 2N - n\}$, which is closed in U_N . So B_n is closed in U , and $U = \bigcup_n B_n$ as topological spaces (for the topology part of this, use that $U_N = \bigcup_{k=1}^{2N} (B_k \cap U_N)$).
- (iii) Let $\alpha: I \rightarrow I$ be any map that is homotopic to the identity and sends the interval $[0, \frac{1}{3}]$ to 0 and the interval $[\frac{2}{3}, 1]$ to 1. Let $\alpha': S^1 \rightarrow S^1$ be the map $e^{2\pi i t} \mapsto e^{2\pi i \alpha(t)}$, for $t \in [0, 1]$.

The maps α and α' can be used to induce maps on E and on U (or perhaps subspaces of such) that apply α (resp., α') to the eigenvalues in the spectral decomposition of a map.

- (iv) Let $[e^{1/3 \cdot 2\pi i}, e^{2/3 \cdot 2\pi i}] \subseteq S^1$ denote the evident counterclockwise arc from the left endpoint to the second. Let $N_n \subseteq B_{n+1}$ be the subspace of all T such that the total dimension of the eigenspaces for $\lambda \in (e^{1/3 \cdot 2\pi i}, e^{2/3 \cdot 2\pi i})$ is at most n . Note that N_n is open and $B_n \subseteq N_n$. Applying α' to the eigenvalues gives a deformation retraction $N_n \rightarrow B_n$, and applying α to the eigenvalues gives a deformation retraction $p^{-1}(N_n) \rightarrow p^{-1}(B_n)$ that covers it.
- (v) The binding maps associated to the above deformation retraction are the “ $\oplus W$ ” maps $(\mathbb{Z} \times BU)(V) \rightarrow (\mathbb{Z} \times BU)(V \oplus W)$, which are weak homotopy equivalences by Exercise ??(d).
- (vi) The final step is to check that when restricted to each $X_n = B_n - B_{n-1}$, p is a Serre fibration. It suffices to show that $p_N^{-1}(X_n \cap U(\mathbb{S}\mathbb{C}^N)) \rightarrow X_n \cap U(\mathbb{S}\mathbb{C}^N)$ is a fibration for $N \gg 0$, since any cubical homotopy lifting diagram for p

will (by compactness of cubes) factor through one of these finite-dimensional versions.

The association $T \mapsto \text{Eig}_1(T)$ is continuous on $Y_n = X_n \cap U(\mathbb{S}\mathbb{C}^N)$, since on this set the dimension of Eig_1 is constant (it equals $2N - n$). Similarly, the association $A \mapsto (\text{Eig}_1(A), \text{Eig}_0(A) \oplus \text{Eig}_1(A))$ is continuous on $p_N^{-1}(Y_n)$. It follows that we have a pullback diagram

$$\begin{array}{ccc} p_N^{-1}(Y_n) & \longrightarrow & \{(U, V) \mid U \subseteq V \subseteq \mathbb{S}\mathbb{C}^N\} \\ \downarrow & & \downarrow \pi_2 \\ Y_n & \longrightarrow & \text{Gr}_\bullet(\mathbb{S}\mathbb{C}^N). \end{array}$$

This reflects the fact that the only difference in information between a point in Y_n and a point in the fiber over it is that the Eig_1 eigenspace has been split into two orthogonal pieces, the Eig_0 and Eig_1 -eigenspaces of the corresponding Hermitian matrix. The map π_2 is a large disjoint union of maps $\text{Gr}_{j \leq k}(\mathbb{C}^{2N}) \rightarrow \text{Gr}_k(\mathbb{C}^{2N})$, and these are fibrations: for example, they can be written as quotient maps

$$U(2N)/(U(j) \times U(k-j) \times U(2N-k)) \rightarrow U(2N)/U(k) \times U(2N-k)$$

where $U(j) \times U(k-j) \times U(2N-k) \subseteq U(k) \times U(2N-k)$ is the evident inclusion of subgroups. Since a pullback of a fibration is a fibration, this concludes the proof.

For full details behind these steps, see [Behrens1]. Beware that it is very easy to make mistakes with point-set topology when dealing with some of the spaces that appear in these arguments.

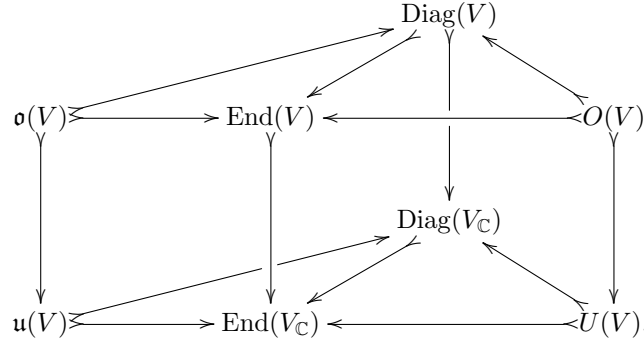
Remark 21.4. In the end, the above proof is perhaps not so different in spirit than the original Bott proof. The Morse theory technology has been traded for the technology of quasifibrations.

Exercise 21.5. Return to the map $\tilde{p}: \tilde{E} \rightarrow U$ that was obtained by our naive stabilization. Use the above method to try to prove that $\tilde{p}$ is a quasifibration. Show that the binding maps are the inclusions $\text{Gr}_\bullet(V \oplus \mathbb{C}^\infty) \rightarrow \text{Gr}_\bullet(V \oplus W \oplus \mathbb{C}^\infty)$, and the proof fails because these are not weak equivalences.

To adapt this style of argument to real Bott periodicity, the best way is to involve Clifford algebras again. This is done in [Behrens2]. If $\mathcal{U}$ is a complete universe for Cl_n , recall from Section 16 that $\mathcal{E}_{\mathbf{x}_n}$ denotes the space of Clifford extension structures on $\mathcal{U}$ that agree with e_n on a space of $\mathfrak{o}$ -finite codimension. Behrens shows how to produce quasifibrations $E_n(\mathcal{U}) \rightarrow \mathcal{E}_{\mathbf{x}_n}(\mathcal{U})$ with fiber $\mathcal{E}_{\mathbf{x}_{n+1}}(\mathcal{U})$ and having $E_n(\mathcal{U}) \simeq *$, thus showing $\Omega \mathcal{E}_{\mathbf{x}_n}(\mathcal{U}) \simeq \mathcal{E}_{\mathbf{x}_{n+1}}(\mathcal{U})$. The proof that the $\mathcal{E}_{\mathbf{x}_n}(\mathcal{U})$ are the spaces that appear in real Bott periodicity was given in Proposition 16.18. After a short prelude we will give a quick tour of Behrens's argument.

For a finite-dimensional inner product space V define $O(V)$ to be the set of isometries of V and $\mathfrak{o}(V)$ to be the tangent space of $O(V)$ at the identity (think skew-symmetric matrices). Then $\mathfrak{o}(V)$ acts on V and satisfies $\langle Av, v' \rangle = \langle v, -Av' \rangle$ for $v, v' \in V$, $A \in \mathfrak{o}(V)$. So we have two embeddings $\mathfrak{o}(V) \hookrightarrow \text{End}(V)$ and $O(V) \hookrightarrow \text{End}(V)$ (the latter factors through the automorphisms, of course). Let $\sigma(A)$ be the set of eigenvalues of A acting on $V \otimes \mathbb{C}$. By the skew self-adjointness, $\sigma(A) \subseteq i\mathbb{R}$.

Give $V_{\mathbb{C}} = V \otimes \mathbb{C}$ the induced Hermitian inner product, and write $U(V_{\mathbb{C}})$ and $\mathfrak{u}(V_{\mathbb{C}})$ for the unitary group and its Lie algebra. We again have embeddings of both of these into $\text{End}(V_{\mathbb{C}})$, and spectral theory tells us that the images are contained in the subset $\text{Diag}(V_{\mathbb{C}})$ of diagonalizable transformations having orthogonal eigenspaces. Let $\text{Diag}(V)$ be the intersection of $\text{End}(V) \hookrightarrow \text{End}(V_{\mathbb{C}})$ and $\text{Diag}(V_{\mathbb{C}}) \hookrightarrow \text{End}(V_{\mathbb{C}})$. As a subset of $\text{Diag}(V_{\mathbb{C}})$ these are the elements such that the eigenvalues appear in conjugate pairs and the eigenspace for λ is the complex conjugate of the eigenspace for $\bar{\lambda}$. The following diagram, with all maps inclusions, shows the big picture:



The image of $\mathfrak{o}(V)$ in $\text{Diag}(V)$ is the subset of transformations whose complex eigenvalues are in $i\mathbb{R}$. The image of $O(V)$ in $\text{Diag}(V)$ consists of transformations whose complex eigenvalues are in S^1 . If $f \in O(V)$ satisfies $f^2 = -I$ then the eigenvalues are $\pm i$, so its image in $\text{Diag}(V)$ is also in the image of $\mathfrak{o}(V)$. We can therefore regard such f as simultaneously belonging to both $\mathfrak{o}(V)$ and $O(V)$.

We also have the exponential map $\mathbb{R} \times \mathfrak{o}(V) \rightarrow O(V)$, $(t, A) \mapsto e^{tA}$. Note that the eigenspaces for e^{tA} are the same as those for A , but the corresponding eigenvalues are changed by $\lambda \mapsto e^{t\lambda}$. So for example, if $f \in O(V)$ satisfies $f^2 = -I$ then we have $e^{\pi f} = -I$.

After the above prelude we can get down to business:

- (1) Let M be an orthogonal Cl_{n+1} -module. Define

$$E_n(M) = \{A \in \mathfrak{o}(M) \mid \sigma(A) \subseteq [-i, i], e_n A = -A e_n, e_j A = A e_j \text{ for } j < n\}.$$

The products like $e_n A$ are taken in $\text{End}(M)$. The space $E_n(M)$ is contractible by the homotopy $A \mapsto tA$, and we define $p: E_n(M) \rightarrow \mathcal{E}x_n(M)$ by $p(A) = -e^{\pi A} e_n$. The reader should check that this is indeed an element of $\mathcal{E}x_n(M)$.

- (2) The element $e_n e_{n+1} \in O(M)$ squares to $-I$, and is readily checked to belong to $E_n(M)$. We have $p(e_n e_{n+1}) = -(-I)e_n = e_n$.
- (3) Let $f_n \in \mathcal{E}x_n(M)$. It is easy to check that $\ker(e_n - f_n)$ is a Cl_n -submodule of M . Slightly less trivial is that there is a homeomorphism

$$p^{-1}(f_n) \xrightarrow{\cong} \mathcal{E}x_{n+1}(\ker(e_n - f_n)), \quad A \mapsto A e_n.$$

If $p(A) = f_n$ then $-e^{\pi A} e_n = f_n$ and so $e^{\pi A} = f_n e_n$. Then compute that

$$(A e_n)(A e_n) = -A^2 e_n^2 = A^2.$$

But on $\ker(e_n - f_n)$, the transformation $f_n e_n$ acts as $-I$, and therefore A has eigenvalues $\pm i$. Hence $A^2 = -I$ on $\ker(e_n - f_n)$. The fact that $A e_n$

anticommutes with the e_j for $j \leq n$ is immediate from $A \in E_n(M)$. This much gives us the injection $p^{-1}(f_n) \hookrightarrow \mathcal{E}x_{n+1}(\ker(e_n - f_n))$. For surjectivity, if $g \in \mathcal{E}x_{n+1}(\ker(e_n - f_n))$ then $g^2 = -I$ and so $\ker(e_n - f_n) \otimes \mathbb{C} = N_+ \oplus N_-$ where these are $+i$ and $-i$ -eigenspaces for g . The spectral decomposition for $f_n e_n$ consists of $\ker(e_n - f_n)$ (eigenvalue -1) together with eigenspaces V_λ in $\ker(e_n - f_n)^\perp$ corresponding to eigenvalues $\lambda \in S^1 - \{1\}$. For each such λ , let $\ell(\lambda)$ be the unique element of $(-i, i)$ such that $e^{\ell(\lambda)} = \lambda$. Define A to be the map with eigenspaces N_+ and N_- (for eigenvalues $+i$ and $-i$) as well as eigenspaces V_λ with eigenvalue $\ell(\lambda)$. Then $A \in p^{-1}(f_n)$ and $Ae_n = g$.

- (4) For the stabilization, define $E_n(M) \rightarrow E_n(M \oplus M')$ by $A \mapsto A \oplus (e_n e_{n+1})|_{M'}$. Note that this makes sense because we are assuming that M and M' are Cl_{n+1} -modules. As usual, define $\mathcal{E}x_n(M) \rightarrow \mathcal{E}x_n(M \oplus M')$ by $f \mapsto f \oplus e_n$. The relevant diagrams commute, and we define $p: E_n(\mathcal{U}) \rightarrow \mathcal{E}x_n(\mathcal{U})$ to be the colimit. The space $E_n(\mathcal{U})$ is weakly contractible because it is a filtered colimit of contractible spaces.
- (5) As we discussed in Section 16.7, the space $\mathcal{E}x_n(\mathcal{U})$ is the set of Clifford extension structures f_n such that f_n agrees with e_n on a subspace of o-finite codimension. The topology is a little subtle: the map $f \mapsto -fe_n$ is an injection into $O(\mathcal{U})$ and we give $\mathcal{E}x_n(\mathcal{U})$ the subspace topology.

Similarly, $E_n(\mathcal{U})$ can be described as the set of $A: \mathcal{U} \rightarrow \mathcal{U}$ having the previously-mentioned properties but that also agree with $e_n e_{n+1}$ on a subspace of o-finite codimension. We topologize this via the injective map $A \mapsto -Ae_n e_{n+1}$ into $O(\mathcal{U})$.

For $f_n \in \mathcal{E}x_n(\mathcal{U})$, the fiber $p^{-1}(f_n)$ is the space $\mathcal{E}x_{n+1}(\ker(e_n - f_n))$ of Cl_{n+1} -extension structures that agree with e_{n+1} on a space of o-finite codimension. Note that e_{n+1} does not necessarily preserve $\ker(e_n - f_n)$, but it does preserve an ofc subspace by Exercise 16.6(c). So the phrase “agrees with e_{n+1} on a space of o-finite codimension” makes sense.

- (6) Since $\ker(e_n - f_n)$ has o-finite codimension in $\mathcal{U}$, it is isomorphic to $\mathcal{U}$ as a Clifford module. So $p^{-1}(f_n) \cong \mathcal{E}x_{n+1}(\mathcal{U})$ for all $f_n \in \mathcal{E}x_n(\mathcal{U})$.

This gives a brief outline of the main ideas. The argument that p is a quasifibration is similar to the complex case, and details are in [Behrens2].

21.6. The Harris-Giffen approach. In the previous section we saw that elements of the unitary group $U(n)$ can be described uniquely by specifying their spectral decomposition: this amounts to a collection of subspaces $(V_1, \dots, V_k)$ of $\mathbb{C}^n$ giving an orthogonal direct sum decomposition, together with eigenvalues $\lambda_1, \dots, \lambda_k \in S^1$. The correspondence is essentially unique if we insist that $\lambda_1 < \lambda_2 < \dots < \lambda_k$ under the counterclockwise ordering (starting at 1) on S^1 .

Harris [Harris] realized that this could be used to give a description of $U(n)$ as the geometric realization of a certain naturally-occurring simplicial space that is intimately related to Grassmannians. The basic ideas were later expanded and generalized in [Giffen]. Our discussion below largely follows Giffen’s approach. We should alert the reader that what we are about to describe is essentially the same proof as in the last section, but using somewhat different technology. There is still something useful in seeing this new version of the arguments, though.

Let V be a finite-dimensional inner product space over $\mathbb{C}$. Define $\Gamma_\bullet(V)$ to be the simplicial space given by

$$\Gamma_u(V) = \{(V_0, \dots, V_u) \mid V = V_0 \oplus \dots \oplus V_u \text{ is an orthonormal direct sum}\}.$$

Here $\Gamma_u(V)$ is regarded as a subspace of $\text{Gr}_\bullet(V)^{u+1}$. The face and degeneracy operators are

$$d_j = \text{"merge } V_j \text{ into } V_{j+1}\text{"}, \quad s_j = \text{"insert } 0 \text{ into the } (j+1)\text{st spot}\text{"},$$

where for d_u we interpret V_{u+1} as V_0 . This means that $d_j(V_0, \dots, V_u) = (V_0, \dots, V_{j-1}, V_j \oplus V_{j+1}, \dots, V_u)$ for $j < u$ and $d_u(V_0, \dots, V_u) = (V_u \oplus V_0, V_1, \dots, V_{u-1})$. Think of the tuple $(V_0, \dots, V_u)$ as giving the eigenspaces for a spectral decomposition of an element of $U(V)$, with V_0 always being Eig_1 and the other eigenspaces ordered so that the eigenvalues increase counterclockwise around the circle.

Note for later use that $\Gamma_u(V)$ breaks up into a large disjoint union with components indexed by the dimensions of the V_i 's. If $\underline{k} = (k_0, \dots, k_u)$ denotes a list of positive integers such that $\sum_j k_j = \dim V$ then

$$\Gamma_u(V) = \coprod_{\underline{k}} U(\dim V) / [U(k_0) \times U(k_1) \times \dots \times U(k_u)]$$

where on the right we are quotienting by the evident subgroup of block-diagonal matrices. The components of $\Gamma_u(V)$ are also sometimes described as a flag manifolds, where we identify the tuple $\underline{V}$ with the flag $V_0 \subseteq V_0 \oplus V_1 \subseteq V_0 \oplus V_1 \oplus V_2 \subseteq \dots$. Note that the tuple can be recovered from the flag by taking orthogonal complements with respect to each inclusion.

Recall the simplicial space $\Delta^\bullet$, where our model for Δ^u is the set of tuples $(0 \leq t_1 \leq \dots \leq t_u \leq 1)$. For such a tuple it will be convenient to set $t_0 = 0$ and $t_{u+1} = 1$. The coface and codegeneracy maps are

$$d^j = \text{"repeat } t_j\text{"}, \quad s^j = \text{"omit } t_{j+1}\text{"}.$$

Define maps $\alpha_u: \Gamma_u(V) \times \Delta^u \rightarrow U(V)$ by

$$\alpha((V_0, \dots, V_u), (t_1, \dots, t_u)) \mapsto 1 \cdot \pi_{V_0} + \sum_j e^{2\pi i t_j} \cdot \pi_{V_j}$$

where π_W denotes orthogonal projection onto W .

Exercise 21.7. Check that $\alpha(d_j(\underline{V}), \underline{t}) = \alpha(\underline{V}, d^j(\underline{t}))$ and $\alpha(s_j(\underline{V}), \underline{t}) = \alpha(\underline{V}, s^j(\underline{t}))$, so that α induces a map $|\Gamma_\bullet| \rightarrow U(V)$. Check that this is a homeomorphism.

If the pair $(\underline{V}, \underline{t})$ is regarded as encoding a spectral decomposition for a unitary transformation, the above identities say that having adjacent V 's with the same eigenvalue is the same as merging the two V 's and omitting one of the repeated eigenvalues. Likewise, if one of the V 's is zero then it doesn't matter what the associated eigenvalue is and we can omit it.

There are evident stabilization maps $\Gamma_\bullet(V) \hookrightarrow \Gamma_\bullet(V \oplus W)$ where we send $(V_0, \dots, V_u) \mapsto (V_0 \oplus W, V_1, \dots, V_u)$. Define $\Gamma_\bullet(n) = \Gamma_\bullet(\mathbb{C}^n)$ and $\Gamma_\bullet(\infty) = \text{colim}_n \Gamma_\bullet(\mathbb{C}^n)$. It follows that $|\Gamma_\bullet(\infty)| \cong U$. Observe that $\Gamma_u(\infty)$ is the space of orthogonal decompositions $\mathbb{C}^\infty = V_0 \oplus V_1 \oplus \dots \oplus V_u$ where $V_1, \dots, V_u$ are finite-dimensional (equivalently, V_0 has o-finite codimension).

Notice something important here: in replacing U with $\Gamma_\bullet(\infty)$, we have taken away everything except the geometry of subspaces of $\mathbb{C}^n$. All we have is a homotopical gadget made from that information.

Let us repeat the above ideas but in the setting of Hermitian matrices instead of unitary matrices. The idea remains that of replacing Hermitian matrices with eigenvalues in $[0, 1]$ by the information in their spectral decomposition. To this end, define a simplicial space $E_\bullet(V)$ by

$$E_u(V) = \{(V_0, V_1, \dots, V_u, V_{u+1}) \mid V = \bigoplus_{j=0}^{u+1} V_j \text{ is an orthogonal direct sum}\}.$$

This time think of the tuple as representing the eigenspaces of a Hermitian operator with eigenvalues in $[0, 1]$, ordered as to increasing eigenvalues, with V_0 and V_{u+1} always the eigenspaces for 0 and 1, respectively. The face and degeneracies are given by

$$(21.8) \quad d_j = \text{“merge } V_j \text{ into } V_{j+1}\text{”}, \quad s_j = \text{“add a 0 after } V_j\text{”}.$$

Note that here the last face map does not involve wrapping around back to V_0 , because of the extra V_{u+1} that occurs in degree u .

There is an evident simplicial contracting homotopy for $E_\bullet(V)$ given by

$$(21.9) \quad (V_0, \dots, V_{u+1}) \mapsto (0, V_0, \dots, V_{u+1})$$

(see e.g. [D5, Section 3.10] for background on contracting homotopies). So $|E_\bullet(V)|$ is contractible. The maps $\alpha': E_u(V) \times \Delta^u \rightarrow E(V)$ given by $(\underline{V}, \underline{t}) \mapsto 0 \cdot \pi_{V_0} + \sum_{j=1}^u t_j \pi_{V_j} + 1 \cdot \pi_{V_{u+1}}$ evidently respect the simplicial identities and so induce a map $\alpha': |E_\bullet(V)| \rightarrow E(V)$. This is readily checked to be a homeomorphism.

Observe that we have a canonical map of simplicial spaces $p_V: E_\bullet(V) \rightarrow \Gamma_\bullet(V)$ given by $(V_0, \dots, V_{u+1}) \mapsto (V_0 \oplus V_{u+1}, V_1, \dots, V_u)$. In each simplicial degree this is a coproduct of projection maps $\text{Gr}_{k \leq l}(V) \rightarrow \text{Gr}_l(V)$, which (as we saw in the last section) are Serre fibrations. In degree u , the fiber over an element $(V_0, \dots, V_u)$ is equal to $\text{Gr}_\bullet(V_0)$.

There are two kinds of stabilization maps $\lambda, \rho: E_\bullet(V) \rightarrow E_\bullet(V \oplus W)$. One is $(V_0, \dots, V_{u+1}) \mapsto (V_0 \oplus W, V_1, \dots, V_{u+1})$, the other is $(V_0, \dots, V_{u+1}) \mapsto (V_0, \dots, V_u, V_{u+1} \oplus W)$. These correspond to adding W into Eig_0 and Eig_1 , respectively. They both are compatible (via the p -maps) with the stabilization maps defined for $\Gamma_\bullet$. As we saw in the last section, we want to stabilize with respect to both of them at once. Giffen does this by forming the directed system of $E_\bullet(\mathbb{C}^n)$ spaces where he alternates the λ and ρ maps, and defines $E_\bullet(\infty)$ to be the colimit. It follows that $E_\bullet(\infty) \rightarrow \Gamma_\bullet(\infty)$ is a map of simplicial spaces that is a Serre fibration in each degree. In degree u , given an element $\underline{V} = (V_0, \dots, V_u) \in \Gamma_u(\infty)$ we can write $V_0 = W \oplus \langle e_{n+1}, e_{n+2}, \dots \rangle$ for some n and $W \subseteq \mathbb{C}^n$, and then the fiber over $\underline{V}$ is precisely the space $(\mathbb{Z} \times BU)(W)$ from Exercise ???. In particular, all of the fibers are homeomorphic.

Now we need to invoke a technical result about simplicial spaces. Say that a simplicial space $X_\bullet$ has **free degeneracies** if there exist subspaces $X_u^{nd} \subseteq X_u$ such that for all $k \geq 0$ the natural map

$$\coprod_{[k] \twoheadrightarrow [u]} X_u^{nd} \rightarrow X_k$$

is a homeomorphism, where the coproduct runs over all surjective simplicial operators and the maps are induced by the corresponding degeneracies. This is a common notion, used for example in [DI1, Definition A.4]. Loosely speaking, it says that each X_u decomposes as a disjoint union of a “non-degenerate subspace” X_u^{nd} and the images of all the degeneracies, the latter being as free as possible. If $X_\bullet$ is a simplicial space with free degeneracies then its realization $|X|$ is very manageable: the layers in the skeletal filtration $\text{sk}_\bullet |X|$ sit in pushout diagrams

$$\begin{array}{ccc} X_n \times \partial\Delta^n & \longrightarrow & \text{sk}_{n-1} |X| \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & \text{sk}_n |X|. \end{array}$$

This makes it easy to prove results by inducting up the skeleta. If $X_\bullet$ and $Y_\bullet$ are simplicial spaces with free degeneracies, say that a map $X_\bullet \rightarrow Y_\bullet$ is **nondegenerate** if it maps X_u^{nd} into Y_u^{nd} for all $u \geq 1$.

Proposition 21.10. *Let $f: E_\bullet \rightarrow B_\bullet$ be a nondegenerate map of simplicial spaces that have free degeneracies. Assume that*

- (1) *Each E_n and B_n is cofibrant, and*
- (2) *For every $u \geq 1$, every $b \in B_u$, and every simplicial face operator $d^j: [u-1] \rightarrow [u]$, the induced map $d_j: f^{-1}(b) \rightarrow f^{-1}(d_j b)$ is a weak homotopy equivalence.*

Then $|E| \rightarrow |B|$ is a quasifibration, with fiber having the weak homotopy type of $f^{-1}(b)$ for any $u \geq 0$ and any $b \in B_u$. (Condition (2) is sometimes phrased in terms of all simplicial operators, but that is equivalent by two-out-of-three. Condition (1) can be weakened, but this version is convenient enough).

Proof. This result as stated does not seem to be readily found in the literature, so let us explain where it comes from. The idea is simply to apply Theorem 21.2 to the skeletal filtrations of $|E|$ and $|B|$, and this readily shows that $|E| \rightarrow |B|$ is a quasifibration. The face operators from (2) become the binding maps. $\square$

Remark 21.11. For other results in the literature similar to Proposition 21.10, see [Segal4, Proposition 1.6] (which does not require free degeneracies but uses the fat realization rather than the geometric realization) and [May1, Theorem 12.7] (which requires the B_n spaces to all be connected, as well as other technical conditions). Beware that as soon as one moves away from the free-degeneracies situation the technical difficulties can become quite subtle.

Exercise 21.12. Find simplicial spaces $E_\bullet$ and $B_\bullet$ with free degeneracies, together with a nondegenerate map $E_\bullet \rightarrow B_\bullet$, such that the following conditions all hold:

- (i) $E_0 = \Delta^1$, $E_1^{nd} = pt$, and $E_u^{nd} = \emptyset$ for $u \geq 2$.
- (ii) $B_0 = B_1^{nd} = pt$ and $B_u^{nd} = \emptyset$ for $u \geq 2$.
- (iii) $E_\bullet \rightarrow B_\bullet$ is a levelwise Serre fibration satisfying all of the hypotheses of Proposition 21.10.
- (iv) The geometric realization $|E| \rightarrow |B|$ is not a Serre fibration.

The simplicial spaces $\Gamma_\bullet(V)$ and $E_\bullet(V)$ all have free degeneracies. For $\Gamma_\bullet(V)$, take $\Gamma_u(V)^{nd}$ to be the subspace of tuples $(\underline{V})$ where all V_j except V_0 are required to be nonzero, and likewise for $E_u(V)^{nd}$. Moreover, our maps $E_\bullet(V) \rightarrow \Gamma_\bullet(V)$

are nondegenerate. Now we simply apply Proposition 21.10 to $E_\bullet(\infty) \rightarrow \Gamma_\bullet(\infty)$. For the behavior of the simplicial operators on the fibers, the only cases where these are not the identity are for the face maps that involve taking the direct sum of two subspaces. For these, the map on fibers is a canonical inclusion of the form $(\mathbb{Z} \times BU)(W) \rightarrow (\mathbb{Z} \times BU)(W \oplus W')$, which we know is a homotopy equivalence by Exercise ???. So Proposition 21.10 yields that $|E_\bullet(\infty)| \rightarrow |\Gamma_\bullet(\infty)|$ is a quasifibration with fiber $\mathbb{Z} \times BU$. Since $|E_\bullet(\infty)| \simeq *$ and $|\Gamma_\bullet(\infty)| \cong U$, this proves that $\Omega U \simeq \mathbb{Z} \times BU$. [In [Giffen] the claim is made that $|E_\bullet(\infty)| \rightarrow |\Gamma_\bullet(\infty)|$ is a fibration, but a close look suggests that Giffen uses “fibration” as a synonym for quasifibration.]

So far this argument is so close to the one from the last section that the reader may be wondering why it is worth the extra effort to do things this way. Here is where we get our first payoff: the constructions $\Gamma_\bullet$ and $E_\bullet$ are general enough that they work out of the box not only for complex inner product spaces, but also for real and quaternionic ones. So we actually have three homotopy fiber sequences:

$$\begin{aligned}\mathbb{Z} \times BO &\rightarrow |E_\bullet(\infty, \mathbb{R})| \rightarrow |\Gamma_\bullet(\infty, \mathbb{R})|, \\ \mathbb{Z} \times BU &\rightarrow |E_\bullet(\infty, \mathbb{C})| \rightarrow |\Gamma_\bullet(\infty, \mathbb{C})|, \\ \mathbb{Z} \times BSp &\rightarrow |E_\bullet(\infty, \mathbb{H})| \rightarrow |\Gamma_\bullet(\infty, \mathbb{H})|\end{aligned}$$

where in each case the middle term is weakly contractible. It remains to identify the spaces $|\Gamma_\bullet(\infty, \mathbb{R})|$ and $|\Gamma_\bullet(\infty, \mathbb{H})|$.

The complexification map $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^n$ induces a bijection between subspaces of $\mathbb{R}^n$ and subspaces of $\mathbb{C}^n$ that are closed under complex conjugation. It induces a map of simplicial spaces $\Gamma_\bullet(n, \mathbb{R}) \hookrightarrow \Gamma_\bullet(n, \mathbb{C})$. Taking geometric realizations gives

$$|\Gamma_\bullet(n, \mathbb{R})| \hookrightarrow |\Gamma_\bullet(n, \mathbb{C})| \xrightarrow{\cong} U(n),$$

and the image consists of all unitary matrices having the property that their eigenspaces are closed under complex conjugation. As we saw in Exercise 15.20, this is homeomorphic to the space $U(n)/O(n)$. So we have learned that

$$|\Gamma_\bullet(n, \mathbb{R})| \cong U(n)/O(n)$$

and after stabilizing this yields the loop relation $\Omega(U/O) \simeq \mathbb{Z} \times BO$.

Likewise, the identification $\mathbb{H}^n = \mathbb{C}^{2n}$ gives a tautological bijection between $\mathbb{H}$ -subspaces of $\mathbb{H}^n$ and complex subspaces of $\mathbb{C}^{2n}$ that are closed under the action of j . Call such subspaces of $\mathbb{C}^{2n}$ “quaternionic”. We have an induced map of simplicial spaces $\Gamma_\bullet(n, \mathbb{H}) \hookrightarrow \Gamma_\bullet(n, \mathbb{C})$ and taking geometric realizations gives

$$|\Gamma_\bullet(n, \mathbb{H})| \hookrightarrow |\Gamma_\bullet(n, \mathbb{C})| \xrightarrow{\cong} U(n).$$

The image in $U(n)$ consists of all matrices whose eigenspaces are quaternionic, and we saw in Exercise 15.21 that this space is homeomorphic to $U(2n)/Sp(n)$. So $|\Gamma_\bullet(n, \mathbb{H})| \cong U(2n)/Sp(n)$ and after stabilizing we have the loop relation $\Omega(U/Sp) \simeq \mathbb{Z} \times BSp$.

Bott periodicity consists of $8+2=10$ loop relations. There are the three trivial ones coming from $\Omega(BG) \simeq G$, and we have just derived three more. That leaves four others. With only slight modifications Giffen’s techniques yield those as well. It is easiest to explain this by looking at the most concrete case first and then

pointing out how it generalizes. We start by reviewing how spectral decomposition works for orthogonal matrices:

Exercise 21.13. Let W be a finite-dimensional real inner product space. Show that there is a bijection between orthogonal transformations $T: W \rightarrow W$ and the following information:

- Orthogonal decompositions $W = \text{Eig}_1(T) \oplus \text{Eig}_{-1}(T) \oplus \bigoplus_{\theta \in (0, \pi)} V_{e^{i\theta}}$ (with $V_{e^{i\theta}} \neq 0$ for only finitely many θ), together with
- Orthogonal complex structures on each $V_{e^{i\theta}}$, such that
- $T|_{(V_{e^{i\theta}})}$ is multiplication by $e^{i\theta}$ (interpreted using the given complex structure).

The tuple $(\text{Eig}_1(T), \{V_{e^{i\theta}}, e^{i\theta}\}, \text{Eig}_{-1}(T))$ is called the **real spectral decomposition** of T . The subspaces appearing in the tuple are called the **invariant subspaces** of T . Note that the setup is such that the “eigenvalues” all have argument in $[0, \pi]$.

[Hint: Let $(W_{\mathbb{C}})_{e^{i\theta}}$ be the eigenspace for T acting on $W_{\mathbb{C}}$. Define $V_{e^{i\theta}}$ to be the set of real components of vectors in $(W_{\mathbb{C}})_{e^{i\theta}}$. For v in this space, let Jv be the unique vector such that $v - i(Jv) \in (W_{\mathbb{C}})_{e^{i\theta}}$.]

Define the simplicial space $W_{\bullet}(n)$ by letting $W_u(n)$ be the space of tuples $(V_0, V_1, \dots, V_{u+1})$ such that each $V_j \subseteq \mathbb{R}^n$, $\mathbb{R}^n = \bigoplus_j V_j$ is an orthogonal direct sum, and the spaces $V_1, \dots, V_u$ are each equipped with an orthogonal complex structure. Think of the V_j as invariant subspaces for an orthogonal transformation of $\mathbb{R}^n$, with V_0 and V_{u+1} always contained in the $\lambda = 1$ and $\lambda = -1$ eigenspaces (respectively), and the others sorted in terms of increasing argument of the eigenvalue. The only proviso is that we allow repetition of eigenvalues, so that for example V_j and V_{j+1} might have the same corresponding eigenvalue. The face and degeneracies are as in (21.8).

There is a map $|W_{\bullet}(n)| \rightarrow O(n)$ given by

$$(21.14) \quad (\underline{V}, t) \mapsto 1 \cdot \pi_{V_0} + \sum_j e^{\pi i t_j} |_{V_j} + (-1) \cdot \pi_{V_{u+1}}$$

and this is readily checked to be a homeomorphism.

Define $E_{\bullet}(n)$ by letting $E_u(n)$ be the space of tuples $(V_0, \dots, V_{u+1})$ as above but where $V_0, \dots, V_u$ are all equipped with complex structures. We have $|E_{\bullet}(n)| \simeq *$ because the formula (21.9) is again a simplicial contracting homotopy. Let $p: E_{\bullet}(n) \rightarrow W_{\bullet}(n)$ be the evident map. This is a levelwise fibration, and its fiber over a given tuple $\underline{V}$ is the space of complex structures on V_0 . Stabilizing with n gives the homotopy fiber sequence $O/U \rightarrow * \rightarrow O$.

Giffen gives three other constructions similar to what we just did that yield the remaining homotopy fiber sequences for the Bott clock, but these involve some ad hoc manipulations that are a little hard to understand. It is better to organize things in terms of Clifford algebras. Here is an outline:

Let M be a Cl_n -module (possibly infinite-dimensional).

- (1) Let $f \in O(M)$ have real spectral decomposition $(V_1, \{V_j, \lambda_j\}, V_{-1})$. Say that this is **regular** if all of the eigenspaces are Cl_{n-1} -submodules, e_n preserves V_1 and V_{-1} , and for each non-real eigenvalue λ_j the complex structure on V_{λ_j} commutes with e_k for $k < n$ and anticommutes with e_n .

- (2) Check that the map $f \mapsto -fe_n$ gives a homeomorphism

$$\mathcal{E}x_n(M) \cong \{g \in O(M) \mid \text{the real spectral decomposition of } g \text{ is regular}\}.$$

The minus sign is there because $-fe_n$ agrees with id on a space of o-finite codimension, and therefore belongs to $O(M)$.

- (3) Let $W_\bullet(M)$ be the simplicial space where $W_u(M)$ is the space of tuples $(V_0, \dots, V_{u+1})$ that are regular real spectral decompositions (loosely interpreted) for some $f \in O(M)$, with V_0 the 1-eigenspace and V_{u+1} the (-1) -eigenspace (note that V_0 will be o-finite codimension). Here we allow repetition of eigenvalues, so that for example V_j and V_{j+1} can have the same eigenvalue, as well as allowing some V_j to be zero. The face and degeneracies are as in (21.8). If $s_{\underline{V}, \underline{t}}$ denotes the orthogonal transformation given by (21.14), prove that there is a homeomorphism $|W_\bullet(M)| \rightarrow \mathcal{E}x_n(M)$ given by $(\underline{V}, \underline{t}) \mapsto s_{\underline{V}, \underline{t}} \circ e_n$.
- (4) For M finite-dimensional, let $E_\bullet(M)$ be the simplicial space where $E_u(M)$ is defined similarly to $W_u(M)$ except that one also assumes that V_0 has an orthogonal complex structure that commutes with the e_k for $k < n$ and anticommutes with e_n . Then $E_\bullet(M)$ has a simplicial contracting homotopy as in (21.9).
- (5) Let $p: E_\bullet(M) \rightarrow W_\bullet(M)$ be the evident map. This is a levelwise fibration, and the fiber over a tuple $(\underline{V})$ is homeomorphic to the space $\mathcal{E}x_{n+1}(V_0)$ (use ???).
- (6) Let $\mathcal{U}$ be a Cl_{n+1} -universe. Stabilize the above constructions using the maps $W_\bullet(M) \rightarrow W_\bullet(M \oplus M')$ given by $\underline{V} \mapsto (V_0 \oplus M', V_1, \dots, V_u)$, and $E_\bullet(M) \rightarrow E_\bullet(M \oplus M')$ given by $((V_0, J_0), \dots) \mapsto ((V_0 \oplus M', J_0 \oplus e_n e_{n+1}), \dots)$. Taking colimits gives quasifibration $|E_\bullet(\mathcal{U})| \rightarrow |W_\bullet(\mathcal{U})|$ with weakly contractible total space, base space $\mathcal{E}x_n(\mathcal{U})$, and fiber $\mathcal{E}x_{n+1}(\mathcal{U})$.

Exercise 21.15. Complete the details of the above argument.

Remark 21.16. The argument in [Harris] is somewhat different from [Giffen]. Harris only proves complex Bott periodicity, and while the key step is still the homeomorphism $|\Gamma_\bullet| \cong U$ the rest of the argument avoids quasifibrations and instead uses group completion techniques. We mostly do not cover these here, but see Section 23 and especially Remark 23.10 for a brief overview.

22. OTHER PROOFS OF BOTT PERIODICITY

In this final section on Bott periodicity our aim is to briefly describe some of the other approaches in the literature. We will cover:

- (i) Atiyah's approach via elliptic operators: [Atiyah6].
- (ii) Wood's approach via Banach algebras: [Wood].
- (iii) The loop-group approach: [GR], [PrSe], and [Mitchell2], [Mitchell3].
- (iv) Latour's approach using spaces of Lagrangians: [Latour].
- (v) Atiyah's approach via KR -theory: [Atiyah4].

This list is not meant to be exhaustive. Some others include [CM], [DL], [Toda], [KT], [BS], [Tian], [AS1], and [LV], and surely there are many more.

Why spend time on this at all? Like many deep results, different proofs seem to demonstrate different aspects of what Bott periodicity is all about. The variety of lines of attack is truly remarkable.

22.1. Atiyah's approach via elliptic operators. In the Atiyah-Bott proof from Section ?? the idea was to take the external tensor product map

$$K(X) \oplus K(X) \cong K(X) \otimes K(S^2) \longrightarrow K(X \times S^2)$$

and produce an explicit inverse. One piece of this inverse is easy: restriction along $X \times pt \hookrightarrow X \times S^2$ gives a map $K(X \times S^2) \rightarrow K(X)$ which can serve as the first component of the splitting. So the work boils down to creating some other interesting map $K(X \times S^2) \rightarrow K(X)$ that will serve as the second component. Atiyah-Bott did this by analyzing clutching functions for bundles on $X \times S^2$, showing they could be linearized after stabilization, and then introducing a novel method for taking a linear S^1 -family of isomorphisms and producing (in a continuous manner) a vector space.

Atiyah later saw this last piece as naturally fitting into the framework he and Singer built around index theory. Diving seriously into this subject would take us too far afield, but let us hint at the general framework. Let M be a compact smooth manifold, let $E \rightarrow M$ and $F \rightarrow M$ be smooth complex vector bundles, and let $d: \Gamma(E) \rightarrow \Gamma(F)$ be an elliptic differential operator. Atiyah shows that d induces a natural map $\text{ind}_d: K(M \times X) \rightarrow K(X)$ for compact spaces X . This is a remarkable realization: elliptic operators give rise to pushforward maps in K -theory. Taking $M = \mathbb{C}P^1$, $E = \underline{1}$, $F = \Omega^{0,1}$ (1-forms of type $(0,1)$, so locally of the form $f(z)d\bar{z}$), and $d = \bar{\partial}$ (so $df = \frac{\partial f}{\partial \bar{z}} d\bar{z}$) gives $\text{ind}_d: K(S^2 \times X) \rightarrow K(X)$. Just a little legwork shows that this gives the desired second component of our splitting. This proves the complex periodicity theorem.

This approach has the feature that real periodicity comes about in essentially the same way, once one has defined the index maps in the KO -setting and produced the correct elliptic operator d . This time d is a Dirac operator on S^8 —in essence, an operator that comes to us from the fundamental Clifford-algebra representations.

Of course the “magic” that we have swept under the rug is the construction of the pushforward maps induced by elliptic operators, and this is not easy (especially in the real case).

22.2. Wood's approach via Banach algebras. This is another cousin of the Atiyah-Bott argument, but it removes the K -groups and deals with the loop spaces directly. A novel component is the recasting of the proof in terms of general Banach algebras, which allows one to reduce most of the work to π_0 statements (see below).

A Banach algebra is an associative algebra over $\mathbb{R}$ or $\mathbb{C}$ that has a norm satisfying $|xy| \leq |x| \cdot |y|$ for all x and y , and that is complete in the norm metric. If B is a Banach algebra and X is a compact space, then the set of continuous maps B^X is a Banach algebra under the sup norm $|f| = \sup_{x \in X} |f(x)|$. Also, if B is a Banach algebra then the set $B(n)$ of $n \times n$ matrices with entries in B is also a Banach algebra under the norm $|M| = \sum_{i,j} |m_{ij}|$.

If B is a Banach algebra let B^* denote the set of units, equipped with the subspace topology. Let $B^\circledast \subseteq B^*$ denote the subspace of elements b such that $b^2 = -1$.

It will be convenient to take as our model for the unit interval I the upper half of S^1 , oriented from $(1,0)$ to $(-1,0)$, with points labelled as (x,y) as usual. For any $b \in B^\circledast$, the path $(x,y) \mapsto xI + yb$ is a path in B^* from I to $-I$. Note that $xI + yb$ is invertible because $(xI + yb)(xI - yb) = (x^2 + y^2)I = I$. So if we fix once

and for all a basepoint $\hat{k} \in B^\otimes$ then we can write down a map

$$B^\otimes \rightarrow \Omega_I B^*, \quad b \mapsto [(x, y) \mapsto (xI + yb)(xI - y\hat{k})].$$

The minus sign in the second term is there so that $\hat{k}$ maps to the constant path. The reader will notice that this is a direct analog of the Bott map. When $B = \mathbb{C}(n)$ then $B^* = GL_n(\mathbb{C})$ and $B^\otimes \cong \{(V, W) \mid V, W \subseteq \mathbb{C}^n, \mathbb{C}^n = V \oplus W\}$ by sending an element $b \in B^\otimes$ to its $+i$ and $-i$ eigenspaces. Since W can be continuously deformed into the orthogonal complement of V , we have $B^\otimes \simeq \text{Gr}_\bullet(\mathbb{C}^n) = \coprod_k \text{Gr}_k(\mathbb{C}^n)$.

In this context stabilization is passing from B to $B(n)$ and letting $n \mapsto \infty$. It will be convenient to change notation and write $B_n = B(n)$, to reduce parentheses. There are the usual inclusions $B_n \hookrightarrow B_{n+1}$, as well as inclusions $(B_n)^\otimes \hookrightarrow (B_{n+1})^\otimes$ given by $b \mapsto b \oplus \hat{k}$. These inclusions commute with the Bott maps so that we can take colimits and get

$$(22.3) \quad \text{colim}_n (B_n)^\otimes \rightarrow \text{colim}_n \Omega_I (B_n)^* \simeq \Omega_I [\text{colim}_n (B_n)^*].$$

Wood shows these stabilized maps are weak homotopy equivalences provided that $\hat{k}$ is path-connected to $-\hat{k}$ in $B^\otimes$. One of the features of this approach is that it is actually enough just to show that the maps are always isomorphisms on π_0 , because for π_k one replaces the Banach algebra B with B^{S^k} and then uses the π_0 -isomorphism there.

The proof of the π_0 -isomorphisms follows the general approach of Atiyah-Bott's argument from Section ??, though the details change somewhat. But one approximates loops using Fourier series and thereby reduces to dealing with polynomial loops, then stabilization tricks let one get down to quadratic loops, and from there it is not hard to massage the loops into ones that are in the image of the Bott map.

Taking $B = \mathbb{C}(2)$ and $\hat{k} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ immediately yields complex Bott periodicity. It is interesting to work out the stabilization map in this case and see that it is what one expects, i.e. the one from Exercise ??(f). One cannot use $B = \mathbb{C}$ because i is not path-connected to $-i$ in $\mathbb{C}^\otimes = \{\pm i\}$, but one can check that $\hat{k}$ and $-\hat{k}$ are connected in $\mathbb{C}(2)^\otimes$.

Exercise 22.4. Prove that $SU(2) \cap \mathbb{C}(2)^\otimes \cong S^2$ and conclude that $\hat{k}$ and $-\hat{k}$ are path-connected in $\mathbb{C}(2)^\otimes$. Using the weak equivalence $\mathbb{C}(n)^\otimes \simeq \text{Gr}_\bullet(\mathbb{C}^n)$ (above), work out the stabilization maps explicitly and show that $\text{colim}_n \mathbb{C}(2n)^\otimes \simeq \mathbb{Z} \times BU$.

This approach extends to give real Bott periodicity with a little extra work. Here one considers pairs (B, σ) of a Banach algebra and a continuous automorphism $\sigma: B \rightarrow B$ such that $\sigma^2 = \text{id}$. For $T \subseteq B$ define T^σ to be the set of elements fixed by σ , and $T^{-\sigma}$ to be the set of elements x such that $\sigma(x) = -x$. Let $\hat{k} \in (B^\otimes)^{-\sigma}$ and suppose that $\hat{k}$ is path-connected to $-\hat{k}$ in this space. In this situation Wood produces a weak homotopy equivalence

$$(22.5) \quad \text{colim}_n (B_n^\otimes)^{-\sigma} \longrightarrow \text{colim}_n [\text{hofib}((B_n^*)^\sigma \hookrightarrow B_n^*)].$$

If G is a Lie group and $H \leq G$ is a closed subgroup then $H \rightarrow G \rightarrow G/H$ is a fibration sequence, and so $\text{hofib}(H \hookrightarrow G) \simeq \Omega(G/H)$. The codomain of (22.5) can therefore be written as $\Omega[\text{colim}_n (B_n^*) / (B_n^*)^\sigma]$.

Wood also provided a nice way of rewriting the domain of (22.5). Let τ be the involution given by conjugation by $\hat{k}$. The map $B_n^* \rightarrow B_n^\otimes$ given by $u \mapsto u\hat{k}u^{-1}$

factors to give an injection $B_n^*/(B_n^*)^\tau \hookrightarrow B_n^\otimes$. This is not always a surjection, but typically it is a surjection onto a collection of path-components. When $\hat{k} \in (B_n^\otimes)^{-\sigma}$ we get $(B_n^*)^\sigma/(B_n^*)^{\sigma,\tau} \rightarrow (B_n^\otimes)^{-\sigma}$ where $T^{\sigma,\tau}$ denotes the elements in T fixed by both σ and τ . So modulo being careful about components (and we will remain vague about this), Wood proves that (22.5) can be rewritten as a weak homotopy equivalence

$$(22.6) \quad \operatorname{colim}_n [(B_n^*)^\sigma / (B_n^*)^{\sigma,\tau}] \longrightarrow \Omega[\operatorname{colim}_n (B_n^*) / (B_n^*)^\sigma]_*$$

where the lower $*$ indicates a possible restriction on path components.

Let $B = \operatorname{Cl}_r$ be a real Clifford algebra, with σ the automorphism that negates all of the e_i . Let $\hat{k} = e_1$. Then one can show $B^\sigma \cong \operatorname{Cl}_{r-1}$ and $B^{\sigma,\tau} \cong \operatorname{Cl}_{r-2}$. As long as $r \geq 2$ we have $\hat{k}$ path-connected to $-\hat{k}$ in $((\operatorname{Cl}_r)^\otimes)^{-\sigma}$, because we can connect them inside the canonical sphere $S^{r-1} = S(\mathbb{R}\langle e_1, \dots, e_r \rangle) \hookrightarrow \operatorname{Cl}_r$. By consulting the table of Clifford algebras one readily works out the units in $\operatorname{Cl}_j(n)$ for any j and n . For example, for $r \leq 3$ we have $\operatorname{Cl}_0(n)^* = GL_n(\mathbb{R})$, $\operatorname{Cl}_1(n)^* = GL_n(\mathbb{C})$, $\operatorname{Cl}_2(n)^* = GL_n(\mathbb{H})$, and $\operatorname{Cl}_3(n)^* = GL_n(\mathbb{H}) \times GL_n(\mathbb{H})$. Applying (22.6) then yields the real Bott periodicity clock but with the compact forms O , U , and Sp replaced by their non-compact analogs $GL(\mathbb{R})$, $GL(\mathbb{C})$, and $GL(\mathbb{H})$.

Wood's paper has a number of unfortunate typos and is not the easiest to read, but the arguments are worth studying. They foreshadow the role of C^* -algebras in the story, as well as the role of $\mathbb{Z}/2$ -equivariance.

22.7. The loop-group approach. Let G be a compact connected Lie group. Let $G_{\mathbb{C}}$ be its complexification, meaning a complex Lie group containing G as a maximal compact subgroup. Note that $G \hookrightarrow G_{\mathbb{C}}$ is a homotopy equivalence by the Iwasawa decomposition. We can assume that $G_{\mathbb{C}}$ comes embedded as a closed subgroup of some $GL_N(\mathbb{C})$. It will be fine if the reader chooses to focus only on $G = U_n$ and $G_{\mathbb{C}} = GL_n(\mathbb{C})$ for the discussion below.

Let LG denote the space of smooth maps $S^1 \rightarrow G$ (not necessarily basepoint-preserving). We can also look at the space $L_{\text{cont}}G$ of continuous loops, but standard smoothing theory yields that $LG \hookrightarrow L_{\text{cont}}G$ is a weak homotopy equivalence. For the rest of this section we will restrict to smooth models. The relation between the free loop space and the space of based loops at the identity is $G \times \Omega G \cong LG$, with $(g, \lambda) \mapsto g \cdot \lambda$.

Equipped with pointwise multiplication, LG and $LG_{\mathbb{C}}$ are groups—called the **loop groups** of G and $G_{\mathbb{C}}$. One can regard these as infinite-dimensional Lie groups, and starting in the 1970s it gradually became understood that many of the nice properties of Lie groups adapt in some way to loop groups.

Inclusion of the constant loops gives $G \subseteq LG$, and the quotient LG/G is the space of (smooth) based loops ΩG . As a homogeneous space for LG this plays the role (up to homotopy) of the flag varieties G/B in the finite-dimensional case. It was realized by Garland and Raghunathan [GR], and independently by Quillen (unpublished), that the Bott cell structure on ΩG that came out of Morse theory could be obtained independently via a Schubert-like cell decomposition in the LG -setting. These ideas can be used to give a new proof of Bott periodicity. The argument for the complex case is only sketched in [GR], but appeared later with complete details in [PrSe].

In Quillen's work on this subject he discovered that similar ideas could be applied to loops on homogeneous spaces, and that this recovered Morse-theoretic work of

Bott-Samelson [BS]. Quillen never published his work, but it was later re-developed and expanded by Mitchell [Mitchell2, Mitchell3]. In this level of generality the ideas led to a new proof of *real* Bott periodicity, worked out in Mitchell's papers.

Our goal here will be to describe the argument for complex Bott periodicity, largely following [PrSe] but simplifying the story down to its core elements. This allows the reader to experience some of the beauty and surprise of the methods. We will not pursue the real version here, but we make a few remarks at the end of the section.

Let H be the complex inner product space with countable orthonormal basis $\{e_k \mid k \in \mathbb{Z}\}$. Recall from Exercise ?? the model $\mathrm{Gr}_{ft}(H)$ for $\mathbb{Z} \times BU$ consisting of finite-type subspaces of H . A quick overview of some details:

- For any $v \in H$ define the **indicator** $\mathrm{indc}(v)$ to be the smallest k for which $v_k \neq 0$.
- For $V \subseteq H$ define $\mathrm{indc}(V) = \{\mathrm{indc}(v) \mid v \in V\} \subseteq \mathbb{Z}$.
- For $m \in \mathbb{Z}$ let $\vec{m} = \mathbb{Z} \cap [m, \infty)$. Let $H_{\geq m} = \mathrm{Span}\{e_k \mid k \in \vec{m}\}$.
- A subset $S \subseteq \mathbb{Z}$ is called finite-type if there exists $m \in \mathbb{Z}$ such that $S \supseteq \vec{m}$ and $S \setminus \vec{m}$ is finite. The *virtual cardinality* is the number $\mathrm{vcard}(S) = \#(S \setminus \vec{0}) - \#(\vec{0} \setminus S)$.
- A subspace $W \subseteq H$ is called finite-type if $\mathrm{indc}(W)$ is finite-type—equivalently, if there exists $m \in \mathbb{Z}$ such that $H_{\geq(-m)} \supseteq W \supseteq H_m$. The *virtual dimension* of W is $\mathrm{v.dim}(W) = \mathrm{vcard}(\mathrm{indc}(W))$. The virtual dimension classifies the connected components of $\mathrm{Gr}_{ft}(H)$.
- Finally, if $S \subseteq \mathbb{Z}$ is finite-type then the **stabilization point** is the smallest m such that $\vec{m} \subseteq \mathbb{Z}$. If $\mathrm{vcard}(S) = d$ then there is a unique increasing sequence $s_{-d}, s_{-d+1}, \dots$ such that $S = \{s_{-d}, s_{-d+1}, \dots\}$, and this has the property that $s_k < k$ for k smaller than the stabilization point, and $s_k = k$ for k greater than or equal to the stabilization point.

Let $T: H \rightarrow H$ be the shift map $e_i \mapsto e_{i+1}$, and let $z = T^n$ be the n -fold shift operator. We will also use z for the corresponding operator on subsets of $\mathbb{Z}$. Say that a subspace $W \subseteq H$ is **z -closed** if $zW \subseteq W$, and likewise for subsets of $\mathbb{Z}$. If a subspace W is z -closed, then $\mathrm{indc}(W)$ is also z -closed.

Note that the z -closed, finite-type subsets of $\mathbb{Z}$ are easy to understand. If S is such a subset, for each $0 \leq i \leq n-1$ let a_i be the smallest element of S that is congruent to i modulo n . The set S is completely determined by the numbers $a_0, \dots, a_{n-1}$, and likewise any collection of a 's representing all n congruence classes corresponds to a unique S .

Let $\mathrm{Gr}^z(H) \subseteq \mathrm{Gr}_{ft}(H)$ be the subspace of z -closed elements. Note the hidden dependence on n . This construction is seemingly coming out of nowhere, but we will see shortly that $\mathrm{Gr}^z(H)$ is a model for $\Omega U(n)$. This is the surprise that comes out of the theory of loop groups.

Also recall from Exercise ?? the Schubert cell structure on $\mathrm{Gr}_{ft}(H)$, having cells C_S indexed by finite-type subsets $S \subseteq \mathbb{Z}$. Again, a brief review:

- Let $S \subseteq \mathbb{Z}$ be finite type, and let $d = \mathrm{vcard}(S)$. Let $C_S \subseteq \mathrm{Gr}_{ft}(H)$ denote the subset of all W whose indicator set equals S . Every element $W \in C_S$

has a canonical basis taking the form of the rows of the matrix

$$(22.8) \quad \begin{bmatrix} 1 & * & * & 0 & * & \cdots \\ 0 & 0 & 0 & 1 & * & \cdots \\ \vdots & & & & & \end{bmatrix}$$

where the columns are indexed by the basis elements e_i for $i \geq s_{-d}$ and the leading ones appear in the columns corresponding to each e_{s_k} . The $*$'s can be any complex number, and this shows that C_S is a complex affine space. The number of $*$'s in the first row is $-d - s_{-d}$, in the second row is $(-d + 1) - s_{-d+1}$, and so forth, so that the total number is $\ell(S) = \sum_{k \geq -d} (k - s_k)$ (note that this is a finite sum).

There is a corresponding cell decomposition on $\text{Gr}^z(H)$ consisting of the open cells $C_S^z = \text{Gr}^z(H) \cap C_S$ where S is z -closed. Subspaces W in C_S^z have a canonical basis taking the form of the rows of (22.8) but where for every $k \in S$ the asterisks are omitted from any row whose leading 1 appears in a column $k + nu$ for $u \geq 1$. The point is that in our canonical basis for W , for any basis vector with indicator $k + nu$ we can use the z^u -shift of the chosen basis vector having indicator k . So the complex dimension of C_S^z is $\sum_{k \notin n+S} (k - s_k)$.

Write $\text{Gr}_{ft}(H)_0$ and $\text{Gr}^z(H)_0$ denote the connected components consisting of subspaces having virtual dimension equal to 0.

Proposition 22.9 ([PrSe, Proposition 8.8.1]). *The inclusion $\text{Gr}^z(H)_0 \hookrightarrow \text{Gr}(H)_0$ is an isomorphism on π_k for $k \leq 2n - 2$.*

Proof. Let $K \subseteq \text{Gr}^z(H)_0$ be the union of all C_S such that $C_S \subseteq \text{Gr}^z(H)_0$. This is a subcomplex of both $\text{Gr}^z(H)_0$ and $\text{Gr}(H)_0$. The result will follow immediately once we show two things:

- (1) All of the cells of $\text{Gr}(H)_0 \setminus K$ have complex dimension at least n .
- (2) All of the cells of $\text{Gr}^z(H)_0 \setminus K$ have complex dimension at least n .

But these are easy calculations. The cell C_S is contained in $\text{Gr}^z(H)_0$ if and only if $S \supseteq \overrightarrow{s_0 + n}$ where s_0 is the minimal element of S (this is not obvious, but it's an exercise). This condition is equivalent to saying that the stabilization point of S is $\leq s_0 + n$. If the condition does not hold then $s_k < k$ for all $k \leq s_0 + n$, and hence $\ell(S) = \sum_{k \geq 0} (k - s_k) \geq (0 - s_0) + \sum_{k=1}^{s_0+n} 1 = n$. This proves (1).

For (2), let S be a z -stable, finite-type subset of $\mathbb{Z}$ having virtual cardinality zero. Let $\{s_k\}_{k \geq 0}$ be the associated sequence. Assume further that $S \not\supseteq \overrightarrow{s_0 + n}$, so that C_S is not contained in $\text{Gr}^z(H)_0$. The complex dimension of $C_S \cap \text{Gr}^z(H)_0$ is equal to $\sum_k (k - s_k)$ where this time the sum ranges over $k \geq 0$ such that $s_k \notin n + S$. Recall that $k - s_k \geq 0$ always, and is > 0 when k is smaller than the stabilization point of S . By assumption, this stabilization point is larger than $s_0 + n$. So for $k \leq s_0 + n$ we have $s_k < k \leq s_0 + n = \min(n + S)$ and therefore $s_k \notin n + S$. Hence when computing the dimension of our cell, the condition $s_k \notin n + S$ does not affect the sum until $k > s_0 + n$. So the (complex) cell dimension is again at least $-s_0 + \sum_{k=1}^{s_0+n} 1 = -s_0 + (s_0 + n) = n$. $\square$

So far we haven't done anything hard, but here is the miracle that comes out of the theory of loop groups:

Theorem 22.10. *There is a weak equivalence $\Omega SU(n) \simeq \text{Gr}^z(H)_0$.*

The composite $\Omega SU(n) \rightarrow \mathrm{Gr}^z(H)_0 \hookrightarrow \mathrm{Gr}(H)_0 = BU$ therefore induces isomorphisms on π_k for $k \leq 2n - 2$, and this is complex Bott periodicity. The Grassmannian model $\mathrm{Gr}^z(H)_0$ for $\Omega SU(n)$ is attributed to Quillen in [Pressley, Section 1] and [Mitchell1, Introduction].

It remains to explain the miracle of Theorem 22.10. There are a number of aspects to this. We switch perspective to $\Omega U(n) \simeq \Omega GL_n(\mathbb{C})$. Define $L_{alg}GL_n(\mathbb{C}) \subseteq LGL_n(\mathbb{C})$ to be the space of functions $f: S^1 \rightarrow GL_n(\mathbb{C})$ such that both f and f^{-1} can be written as finite Laurent series $\sum_{k=-N}^N z^k A_k$ for $A_k \in M_{n \times n}(\mathbb{C})$ (here we use z for the standard complex coordinate on S^1). This can also be interpreted as the $\mathbb{C}[z, z^{-1}]$ -valued points of the algebraic group GL_n , or as $GL_n(\mathbb{C}[z, z^{-1}])$, or as the space of regular maps $\mathbb{G}_m \rightarrow GL_n$. Define $\Omega_{alg}GL_n(\mathbb{C}) \subseteq L_{alg}GL_n(\mathbb{C})$ to be the subspace of functions f such that $f(1) = I_n$.

To understand some of the subtleties of the definition of L_{alg} , here are three examples of elements in $L_{alg}GL_2(\mathbb{C})$:

$$(22.11) \quad \begin{aligned} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & z^2 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1-z & -z \\ z & 1+z \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

We can also multiply the above examples by z^k for any $k \in \mathbb{Z}$. Every element of $L_{alg}GL_n(\mathbb{C})$ can be written as z^k times a polynomial expression in z like the ones above. For comparison, here is a finite Laurent series (in fact, a polynomial) that is *not* in $L_{alg}GL_2(\mathbb{C})$:

$$\begin{bmatrix} z & 0 \\ 0 & z-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, note that even though for $z \in S^1$ the expression takes values in $GL_2(\mathbb{C})$, the *inverse* is not a finite Laurent series in z . It is helpful to note that a matrix with entries in $\mathbb{C}[z, z^{-1}]$ represents an element of $L_{alg}GL_n(\mathbb{C})$ if and only if its determinant is a unit in $\mathbb{C}[z, z^{-1}]$, which means the determinant is z^k for some $k \in \mathbb{Z}$.

These algebraic models are, up to homotopy, the same as the topological models:

Theorem 22.12. *The inclusions $L_{alg}GL_n(\mathbb{C}) \hookrightarrow LGL_n(\mathbb{C})$ and $\Omega_{alg}GL_n(\mathbb{C}) \hookrightarrow \Omega GL_n(\mathbb{C})$ are weak homotopy equivalences.*

This result is by no means obvious. As a reality check, let us look at $n = 1$. Then $L_{alg}GL_1(\mathbb{C})$ consists of the maps uz^k where $u \in \mathbb{C} - 0$ and $k \in \mathbb{Z}$, so this is $\mathbb{Z} \times (\mathbb{C} - 0)$. The space $\Omega_{alg}GL_1(\mathbb{C})$ is just $\{z^k \mid k \in \mathbb{Z}\}$, so this is a copy of $\mathbb{Z}$. Of course $\Omega S^1 \simeq \mathbb{Z}$ and $LS^1 \cong S^1 \times \Omega S^1 \simeq S^1 \times \mathbb{Z}$.

There are versions of Theorem 22.12 for any complex reductive group in place of $GL_n(\mathbb{C})$. Further, if G is a compact Lie group then one defines $L_{alg}G = LG \cap L_{alg}G_{\mathbb{C}}$, and the result is true with G replacing $GL_n(\mathbb{C})$. For sources see [GR, Theorem 1.1], [Pressley, Theorem 5.1], [PrSe, Proposition 8.6.6], [Mitchell1, Theorem 1.1], [Mitchell2, Theorem 1.4], and [Mitchell3, Theorem 4.1]. We will not give the proof of Theorem 22.12 here.

Let $P \subseteq L_{alg}GL_n(\mathbb{C})$ denote the subgroup of maps f that can be extended holomorphically over the disk $|z| \leq 1$. In this case this means that f can be written as a polynomial function, or equivalently that f can be regarded as an invertible matrix with values in $\mathbb{C}[z]$. Note that since $\mathbb{C}[z]^* = \mathbb{C}^*$ this implies that its determinant must be a nonzero scalar. So for example, the map zI_n is not in P , but when $n = 2$ the last two examples of (22.11) do belong to P .

We have the inclusion $GL_n(\mathbb{C}) \hookrightarrow P$ by sending any matrix A to the constant loop, and evaluation at 1 gives a map back. These are readily seen to form a homotopy equivalence. It follows that

$$(22.13) \quad L_{alg}GL_n(\mathbb{C})/P \simeq L_{alg}GL_n(\mathbb{C})/GL_n(\mathbb{C}) \simeq \Omega_{alg}GL_n(\mathbb{C}) \simeq \Omega GL_n(\mathbb{C}).$$

Using the standard action of $GL_n(\mathbb{C})$ on $\mathbb{C}^n$, the group $L_{alg}GL_n(\mathbb{C})$ acts on $L_{alg}\mathbb{C}^n$. By the latter we just mean finite Laurent series with coefficients in $\mathbb{C}^n$, and if $f = \sum_k z^k A_k$ and $v = \sum_m z^m v_m$ then $f.v = \sum_{k,m} z^{k+m} A_k v_m$. Note that $L_{alg}\mathbb{C}^n$ can be identified with $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathbb{C}^n = \mathbb{C}[z, z^{-1}]^n$, but can also be identified with H via $z^j \otimes e_k \mapsto e_{nj+k-1}$. In this last identification the subspace $H_{\geq 0}$ corresponds to $\mathbb{C}[z] \otimes \mathbb{C}^n$. A subspace $W \subseteq H$ that is finite-type and z -stable corresponds to a $\mathbb{C}[z]$ -submodule $W \subseteq \mathbb{C}[z, z^{-1}]^n$ that is finitely-generated and contains $z^N \mathbb{C}[z]^n$ for some $N \geq 0$.

The action of $L_{alg}GL_n(\mathbb{C})$ on H induces an action on $\text{Gr}(H)$. Notice that this action preserves the subspace $\text{Gr}^z(H)$, since $f.(zv) = z(f.v)$ for any f and v .

Proposition 22.14. *The action of $L_{alg}GL_n(\mathbb{C})$ on $\text{Gr}^z(H)$ is transitive, and the stabilizer of $H_{\geq 0}$ is P . So $L_{alg}GL_n(\mathbb{C})/P \cong \text{Gr}^z(H)$. Consequently, $\Omega GL_n(\mathbb{C}) \simeq \text{Gr}^z(H)$.*

Proof. Let us first compute the stabilizer of $H_{\geq 0} = \mathbb{C}[z]^n$. If $f \in P$ then f is a polynomial in z and so clearly $f.H_{\geq 0} \subseteq H_{\geq 0}$. Since $f^{-1} \in P$, we also have $f^{-1}.H_{\geq 0} \subseteq H_{\geq 0}$. Multiplying by f gives $H_{\geq 0} \subseteq f.H_{\geq 0}$, so in fact $H_0 = f.H_{\geq 0}$.

Conversely, suppose $f \in L_{alg}GL_n(\mathbb{C})$ is such that $f.H_{\geq 0} = H_{\geq 0}$. Then clearly f can have no z^k terms for $k < 0$. So f is polynomial in z , and therefore $\det f$ is (up to a nonzero constant) a nonnegative power of z . But multiplying by f^{-1} gives $H_{\geq 0} = f^{-1}.H_{\geq 0}$, so we make the same conclusion for $\det f^{-1}$. This implies that $\det f$ is a nonzero constant, so $f \in P$.

It remains to prove transitivity. Let $W \subseteq H$ be finite-type and z -stable. Up to multiplication by powers of z we can assume $W \subseteq H_{\geq 0} = \mathbb{C}[z]^n$, and then W is a $\mathbb{C}[z]$ -submodule. Since $\mathbb{C}[z]$ is a PID, we know that W must be free. Since $z^N \mathbb{C}[z] \subseteq W$ for $N \gg 0$, W contains a free module of rank n . So in fact W is free of rank n . Choose generators $u_1, \dots, u_n \in \mathbb{C}[z]^n$ for W as a $\mathbb{C}[z]$ -module and let A be the matrix with these as its columns. Then left multiplication by A is a $\mathbb{C}[z]$ -map $L_A: \mathbb{C}[z]^n \rightarrow \mathbb{C}[z]^n$ and the image is W . We claim that $A \in L_{alg}GL_n(\mathbb{C})$ and $A.H_{\geq 0} = W$. The latter is clear, since $A.H_{\geq 0}$ is simply the image of L_A . For the former, since $z^N \mathbb{C}[z]^n \subseteq W$ we can write each vector $z^N e_i$ as a linear combination of the u 's. If B is the matrix giving these linear combinations, then $z^N I_n = B \cdot A$, and therefore $z^{Nn} = \det(B) \cdot \det(A)$. In particular, $\det(A)$ is a power of z . This implies that $A \in L_{alg}GL_n(\mathbb{C})$.

The final statement that $\Omega GL_n(\mathbb{C}) \simeq \text{Gr}^z(H)$ now follows from (22.13). $\square$

Theorem 22.10 is an immediate corollary, and this completes our account of the loop group approach to complex Bott periodicity. While we have largely followed the treatment in [PrSe], we have incorporated some useful perspectives from [Pressley] and [Mitchell1]. To briefly summarize the argument: taking connected components in Proposition 22.14 gives $\Omega SL_n(\mathbb{C}) \simeq \mathrm{Gr}^z(H)_0$, and the cell structure analysis from Proposition 22.9 yields $\pi_k \mathrm{Gr}^z(H)_0 \cong \pi_k BU$ for $k \leq 2n - 2$.

The story for real Bott periodicity follows a related but somewhat different approach, and we do not recount it here because it would involve a long excursion into Lie theory. But we highly recommend the papers [Mitchell2, Mitchell3] as a wealth of intriguing mathematics. The payoff is that the equivalences of real Bott periodicity can be read off of the multiplicity-labelled Dynkin diagrams for the relevant symmetric spaces. At the end of the day this is not so different in spirit from the index calculations that went into the Morse-theoretic arguments, but the repackaging is very elegant and beautiful.

22.15. Karoubi's approach using ϵ -symmetric forms, and Latour's approach using spaces of transversal Lagrangians. We want to simultaneously talk about quadratic, symplectic, Hermitian, and anti-Hermitian forms over the fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The language for this is a bit offputting, but is not so bad once you get used to it. The setting will be a triple $(F, (-)^*, \epsilon)$ where $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $\epsilon \in \{1, -1\}$, and $(-)^*: F \rightarrow F$ is an $\mathbb{R}$ -linear anti-involution—meaning that it is additive and satisfies $(xy)^* = y^*x^*$. We call such a triple a **form type**. In practice the involution will always be either id or the usual conjugation.

In this setting an ϵ -symmetric sesquilinear form is an F -vector space V and a pairing $b: V \times V \rightarrow F$ such that b is additive in both variables and

$$b(\lambda v, w) = b(v, w)\lambda^*, \quad b(v, \lambda w) = \lambda b(v, w), \quad b(w, v) = \epsilon \cdot b(v, w)^*.$$

Such a form is called nondegenerate if the condition $b(v, w) = 0$ for all w implies that $v = 0$.

Let us define a **Karoubian form** to be a nondegenerate ϵ -symmetric sesquilinear form with respect to some form type $(F, (-)^*, \epsilon)$. This is nonstandard language, but will be very convenient. Up to isomorphism every Karoubian form is one of the following:

- $\mathbb{R}^n$ with the standard (p, q) -form $b_{p,q}$ for some $p + q = n$. Here $b_{p,q}(x, y) = \sum_{s=1}^p x_s y_s - \sum_{s=p+1}^{p+q} x_s y_s$ (involution=id, $\epsilon = 1$).
- $\mathbb{R}^{2n}$ with the standard symplectic form ω (involution=id, $\epsilon = -1$).
- $\mathbb{C}^n$ with the sum-of-squares form b_n (the dot product) (involution=id, $\epsilon = 1$).
- $\mathbb{C}^{2n}$ with the standard symplectic form ω (involution=id, $\epsilon = -1$).
- $\mathbb{C}^n$ with the standard Hermitian form $B_{p,q}$ of type (p, q) , for some $p + q = n$. Here $B_{p,q}(v, w) = \sum_{s=1}^p \bar{v}_s w_s - \sum_{s=p+1}^{p+q} \bar{v}_s w_s$ (involution=conjugation, $\epsilon = \pm 1$).
- $\mathbb{H}^n$ with the standard Hermitian form $B_{\mathbb{H}}$ given by $B_{\mathbb{H}}(v, w) = \sum_s w_s \bar{v}_s$ (involution=conjugation, $\epsilon = 1$).
- $\mathbb{H}^n$ with the standard anti-Hermitian form $B_{\mathbb{H}}^-$, given by $B_{\mathbb{H}}^-(v, w) = \sum_s w_s i \bar{v}_s$ (involution=conjugation, $\epsilon = -1$).

Exercise 22.16. Check that forms of type $(\mathbb{C}, \text{conjugation}, 1)$ are the same as those of type $(\mathbb{C}, \text{conjugation}, -1)$, and that the only form of type $(\mathbb{H}, \text{id}, \pm 1)$ is the zero form.

Given a form type and an $n \geq 1$, one can consider the space $\mathcal{K}_n$ of all Karoubian forms on F^n , topologized as a subspace of $\text{Hom}_{\mathbb{R}}(F^n \times F^n, F)$. For reasons that will gradually become clear it is convenient to include the notion of an “empty form” (meaning a vector space with no extra structure), and in this case we define $\mathcal{K}_n$ to be the space of automorphisms. The connection between these two constructions is not transparent, but is demonstrated in Exercise 22.19(f) below.

Proposition 22.17. *The following table shows the homotopy types of all of the $\mathcal{K}_n$ spaces:*

<i>triple</i>	<i>vector space</i>	$\mathcal{K}$
$(\mathbb{R}, \emptyset)$	$\mathbb{R}^n$	$O(n)$
$(\mathbb{R}, \text{id}, 1)$	$\mathbb{R}^n$	$\coprod_{p+q=n} \text{Gr}_p(\mathbb{R}^n)$
$(\mathbb{R}, \text{id}, -1)$	$\mathbb{R}^{2n}$	$O(2n)/U(n)$
$(\mathbb{C}, \emptyset)$	$\mathbb{C}^n$	$U(n)$
$(\mathbb{C}, \text{id}, 1)$	$\mathbb{C}^n$	$U(n)/O(n)$
$(\mathbb{C}, \text{id}, -1)$	$\mathbb{C}^{2n}$	$U(2n)/Sp(n)$
$(\mathbb{C}, \text{conj}, 1)$	$\mathbb{C}^n$	$\coprod_{p+q=n} \text{Gr}_p(\mathbb{C}^n)$
$(\mathbb{H}, \emptyset)$	$\mathbb{H}^n$	$Sp(n)$
$(\mathbb{H}, \text{conj}, 1)$	$\mathbb{H}^n$	$\coprod_{p+q=n} \text{Gr}_p(\mathbb{H}^n)$
$(\mathbb{H}, \text{conj}, -1)$	$\mathbb{H}^n$	$Sp(n)/U(n)$

Proof. To specify a form of type (p, q) on $\mathbb{R}^n$ one must give two subspaces V_+, V_- of dimensions p and $q = n - p$, with $V_+ \oplus V_- = \mathbb{R}^n$, together with an inner product on each of V_+ and V_- . The space of inner products is contractible (??), and the space of pairs (V_+, V_-) is homotopy equivalent to $\text{Gr}_p(\mathbb{R}^n)$. This justifies line two of the table.

For line three, it is known that every symplectic form admits a symplectic basis. So if $J \in M_{2n \times 2n}(\mathbb{R})$ is the matrix for the standard symplectic form, the space of symplectic forms on $\mathbb{R}^{2n}$ is the subspace of matrices PJP^T for $P \in GL_{2n}(\mathbb{R})$, or equivalently $GL_{2n}(\mathbb{R})/\text{stab}(J)$. But we can deformation-retract $GL_{2n}(\mathbb{R})$ down to $O(2n)$, and then $\text{stab}(J)$ deformation-retracts down to $U(n)$.

The other lines of the table are similar, and left to the reader. $\square$

Exercise 22.18. Complete the above proof.

Fix a form type α and let V be a $2n$ -dimensional F -vector space with a Karoubian form b (here we include the possibility of the empty form). Let $\Lambda(V)$ denote the space of n -dimensional isotropic subspaces of V , called **Lagrangian** subspaces. In the empty form case these are just all n -dimensional subspaces. Two Lagrangian subspaces L_1 and L_2 are said to be **transverse** if $N_1 \cap N_2 = 0$, which is equivalent to $N_1 \oplus N_2 = V$. For $L \in \Lambda(V)$, define $\mathcal{U}_L \subseteq \Lambda(V)$ to be the set of Lagrangian subspaces that are transverse to L . This is an open subset, and part (b) of the following exercise shows that it is naturally an $\mathbb{R}$ -affine space.

Exercise 22.19. This exercise, which is just elementary linear algebra, is based on the first section of [Latour].

- (a) Every $M \in \mathcal{U}_L$ determines a projection map $\pi_M: V \rightarrow L$ such that $\pi(x+m) = x$ for $x \in L$ and $m \in M$. Show that for all $x, y \in V$ one has the identity
- (*)
$$b(\pi_M x, y) + b(x, \pi_M y) = b(x, y).$$
- (b) Conversely, suppose given an F -linear map $f: V \rightarrow L$ such that $f|_L = \text{id}_L$ and (*) holds. Show that $\ker f \in \mathcal{U}_L$, and that this yields a homeomorphism $\mathcal{U}_L \cong \{f \in \text{Hom}_F(V, L) \mid f|_L = \text{id}_L, b(fx, y) + b(x, fy) = b(x, y) \text{ for all } x, y \in V\}$.
- Verify that the right-hand-side is an $\mathbb{R}$ -affine space inside of $\text{Hom}_F(V, L)$.
- (c) If $M_1, M_2 \in \mathcal{U}_L$ let f_1, f_2 be the corresponding projections. Confirm that the straight-line homotopy $(1-t)f_1 + tf_2$ determines a path in $\mathcal{U}_L$ from M_1 to M_2 .
- (d) Let $L, L' \in \Lambda(V)$ be transverse. If $N \subseteq V$ is an n -dimensional subspace that is transverse to both L and L' then the projections $\pi_1^{L, L'}: N \rightarrow L$ and $\pi_2^{L, L'}: N \rightarrow L'$ are both isomorphisms, and so N is the graph of a unique F -linear isomorphism $u: L \rightarrow L'$. The requirement that N be isotropic is equivalent to $b(u(x), y) + b(x, u(y)) = 0$ for all $x, y \in L$.
- (e) Given $N \in \mathcal{U}_L \cap \mathcal{U}_{L'}$ (and the corresponding u) define a new form on L by $c(x, y) = b(x, u(y))$. Check that this is nondegenerate and $(-\epsilon)$ -symmetric. Conversely, given such a form c prove that u (and hence N) is uniquely determined.
- (f) Conclude that $\mathcal{U}_L \cap \mathcal{U}_{L'} \cong GL_F(L)$ when we have the empty form, and otherwise one instead has

$$\mathcal{U}_L \cap \mathcal{U}_{L'} \cong \mathcal{K}_{(-\alpha)}(L)$$

where $-\alpha$ is the form-type obtained from α by changing the sign on ϵ .

Let L_0 and L_1 be two Lagrangian subspaces that meet transversely. Let $\Lambda_{tr}(V; L_0, L_1) = \mathcal{U}_{L_0} \cap \mathcal{U}_{L_1} \subseteq \Lambda(V)$ be the subspace of Lagrangians N that meet both L_0 and L_1 transversely. There is a natural map

$$\beta: \Lambda_{tr}(V; L_0, L_1) \rightarrow \Omega_{L_0, L_1} \Lambda(V)$$

that sends a Lagrangian N to the concatenation of the straight-line path in $\mathcal{U}_{L_0}$ from L_0 to N and the straight-line path in $\mathcal{U}_{L_1}$ from N to L_1 . The following is proven in [Latour]:

Theorem 22.20 (Latour). *After stabilization the map β becomes a weak homotopy equivalence.*

Together with Exercise 22.19(f) this gives the fundamental loop relation

$$(22.21) \quad \Omega[\Lambda_{(F, (-)^*, \epsilon)}] \simeq \mathcal{K}_{(F, (-)^*, -\epsilon)}.$$

To use this to derive the Bott clocks, make a list of the ten triples and the homotopy types of their associated stable $\mathcal{K}$ -spaces and Λ -spaces:

triple	$\mathcal{K}$	Λ
$(\mathbb{R}, \emptyset)$	O	$\mathbb{Z} \times BO$
$(\mathbb{R}, \text{id}, 1)$	$\mathbb{Z} \times BO$	O
$(\mathbb{R}, \text{id}, -1)$	O/U	U/O
$(\mathbb{C}, \emptyset)$	U	$\mathbb{Z} \times BU$
$(\mathbb{C}, \text{id}, 1)$	U/O	O/U
$(\mathbb{C}, \text{id}, -1)$	U/Sp	Sp/U
$(\mathbb{C}, \text{conj}, 1)$	$\mathbb{Z} \times BU$	U
$(\mathbb{H}, \emptyset)$	Sp	$\mathbb{Z} \times BSp$
$(\mathbb{H}, \text{conj}, 1)$	$\mathbb{Z} \times BSp$	Sp
$(\mathbb{H}, \text{conj}, -1)$	Sp/U	U/Sp

For the nontrivial Λ -space calculations see Figure 15.18.1, Exercises 15.19 through 15.23, as well as [Arnold1, Arnold2]. We leave the reader to reflect on the many interesting symmetries in this table.

Now pick any triple—say $(\mathbb{C}, \text{id}, -1)$ —and start with its Λ -space. By (22.21) loops on this is $\mathcal{K}_{(\mathbb{C}, \text{id}, +1)}$, which we then look up in the above table and find is U/O . Observe that this also equals $\Lambda_{(\mathbb{R}, \text{id}, -1)}$, and now continue:

$$\begin{array}{ccccc}
 \Lambda_{(\mathbb{C}, \text{id}, -1)} & \xrightarrow{\Omega} & \mathcal{K}_{(\mathbb{C}, \text{id}, 1)} & & \Lambda_{(\mathbb{R}, \emptyset)} & \xrightarrow{\Omega} & \mathcal{K}_{(\mathbb{R}, \emptyset)} & & \Lambda_{(\mathbb{C}, \text{id}, 1)} & \xrightarrow{\Omega} & \dots \\
 & & \parallel & & \parallel & & \parallel & & \parallel & & \\
 & & \Lambda_{(\mathbb{R}, \text{id}, -1)} & \xrightarrow{\Omega} & \mathcal{K}_{(\mathbb{R}, \text{id}, 1)} & & \Lambda_{(\mathbb{R}, \text{id}, 1)} & \xrightarrow{\Omega} & \mathcal{K}_{(\mathbb{R}, \text{id}, -1)} & &
 \end{array}$$

Exercise 22.22. Complete the cycle that was started above until you produce the full clock for real Bott periodicity. Then do the complex case as well.

22.23. Atiyah's approach via KR -theory. We have seen a few times that it pays to regard $K(X)$ and $KO(X)$, together with the r and c maps, as a single entity. One sophisticated way to do this is to regard $\mathbb{Z} \times BU$ as a $\mathbb{Z}/2$ -space with the conjugation action inherited from that on U . Then the fixed points are precisely $\mathbb{Z} \times BO$, and the $\mathbb{Z}/2$ -equivariant homotopical object $\mathbb{Z} \times BU$ can be thought of as unifying the non-equivariant $\mathbb{Z} \times BU$ and $\mathbb{Z} \times BO$ spaces. This viewpoint leads straight to KR -theory.

Throughout this section let σ denote the generator of $\mathbb{Z}/2$. Whenever involutions occur, we denote them as σ . If T is a set with involution then $T^{\mathbb{Z}/2}$ denotes the fixed set, i.e. those $t \in T$ such that $\sigma(t) = t$.

Atiyah defines a **Real vector space** to be a complex vector space together with a conjugate-linear involution. For any real vector space V the space $\mathbb{C} \otimes_{\mathbb{R}} V$ with $\sigma(\lambda \otimes v) = \bar{\lambda} \otimes v$ is a Real vector space. If W is a Real vector space then its fixed set $W^{\mathbb{Z}/2}$ is a real vector space, and one readily checks that these two functors give an equivalence of categories between Real vector spaces and real vector spaces. Atiyah's "Real" adjective is in some ways unfortunate, but we will stick with it.

Let X be a space with a $\mathbb{Z}/2$ -action. A **Real vector bundle** on X is defined to be a complex vector bundle $E \rightarrow X$ together with a $\mathbb{Z}/2$ -action on the space E , covering the action on X and having the extra property that for every $x \in X$ the map $\sigma: E_x \rightarrow E_{\sigma x}$ is conjugate-linear. When $X = pt$ a Real vector bundle over X is precisely a Real vector space, and when X has trivial action the category of Real vector bundles over X is equivalent to that of real vector bundles over X .

Define $KR(X)$ to be the Grothendieck group of finite-dimensional Real vector bundles on X . Atiyah called this “ K -theory with Reality”. These are best studied in the context of $\mathbb{Z}/2$ -equivariant homotopy theory, which admittedly was not at all developed at the time Atiyah wrote his paper. For $\mathbb{Z}/2$ -spaces X and Y , an equivariant homotopy is an equivariant map $X \times I \rightarrow Y$ where I is equipped with the trivial action. We write $[X, Y]_{\mathbb{Z}/2}$ for equivariant homotopy classes of maps. An equivariant weak homotopy equivalence $X \rightarrow Y$ is an equivariant map such that both $X \rightarrow Y$ and $X^{\mathbb{Z}/2} \rightarrow Y^{\mathbb{Z}/2}$ are non-equivariant weak homotopy equivalences.

A pointed $\mathbb{Z}/2$ -space is a $\mathbb{Z}/2$ -space X with a distinguished $x \in X^{\mathbb{Z}/2}$ (equivalently, an equivariant map $pt \rightarrow X$). If X and Y are pointed $\mathbb{Z}/2$ -spaces define X^Y to be the usual topological function space of (non-equivariant) pointed maps $Y \rightarrow X$, equipped with the induced $\mathbb{Z}/2$ -action where $\sigma.f = \sigma \circ f \circ \sigma$. The fixed points of X^Y are the pointed equivariant maps.

If we give U the action of complex conjugation then there is an induced action on $\mathbb{Z} \times BU$. It is easy to see that $KR(X) = [X, \mathbb{Z} \times BU]_{\mathbb{Z}/2}$. Likewise give $\mathbb{C}P^1$ the conjugation action. Complex Bott periodicity becomes the equivariant equivalence

$$\mathbb{Z} \times BU \simeq (\mathbb{Z} \times BU)^{\mathbb{C}P^1}.$$

Many proofs of (non-equivariant) complex Bott periodicity—for example, the Atiyah-Bott proof of Section ??—are readily adapted to the $\mathbb{Z}/2$ -equivariant setting with almost no change.

So far this is not adding much new into the mix. But Atiyah realized that this equivariant version of complex Bott periodicity yields *real* Bott periodicity as a consequence. The line of reasoning is not trivial, but it can be explained in a page or two. We already saw an idea like this in the Wood paper, where real periodicity came out of complex periodicity by incorporating involutions. We will give an outline of the argument, but the first step is to establish some notation for working with $\mathbb{Z}/2$ -spaces.

Let $\mathbb{R}_{triv}$ and $\mathbb{R}_-$ denote $\mathbb{R}$ with the trivial action and sign action, respectively. For $p \geq q$ set $\mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}_-)^q$. So $\mathbb{R}^{p,q}$ is $\mathbb{R}^p$ but with the $\mathbb{Z}/2$ -action that involves q sign changes.

Let $S(\mathbb{R}^{p,q})$ and $D^{p,q}$ denote the unit sphere and unit disk in $\mathbb{R}^{p,q}$, and let $S^{p,q} = D^{p,q}/S(\mathbb{R}^{p,q})$. Then $S^{p,q}$ can also be regarded as the one-point compactification of $\mathbb{R}^{p,q}$. The space $\mathbb{R}^{2,1}$ can be identified with $\mathbb{C}$ with the conjugation action, and so $S^{2,1} \cong \mathbb{C}P^1$. Observe as well that $S^{n,0}$ is the n -sphere with the trivial action.

The space $S(\mathbb{R}^{p,p})$ is the sphere S^{p-1} with the antipodal action, and since this plays a special role we also write this as S_{anti}^{p-1} . This sphere is somewhat different from the $S^{p,q}$'s because it does not have a fixed point.

The bigraded family of spheres $S^{p,q}$ leads to bigraded homotopy groups $\pi_{p,q}$. Homology and cohomology theories will inherit these bigradings. A $\mathbb{Z}/2$ -equivariant cohomology theory E will have suspension isomorphisms of the form $\tilde{E}^{a,b}(S^{p,q} \wedge X) \cong \tilde{E}^{a-p,b-q}(X)$. Long exact sequences of a pair will look like

$$\cdots \rightarrow E^{p,q}(X, A) \rightarrow E^{p,q}(X) \rightarrow E^{p,q}(A) \rightarrow E^{p+1,q}(X, A) \rightarrow \cdots$$

In these bigradings the first index is called the **topological dimension** and the second index is called the **weight**.

Since $\mathbb{C}P^1 = S^{2,1}$, our equivariant version of complex Bott periodicity takes the form $\Omega^{2,1}(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. This allows us to set up a periodic equivariant spectrum KR , and therefore an associated bigraded cohomology theory. The Bott element is $\beta \in KR^{-2,-1}(pt)$ and multiplication by β yields the periodicity isomorphisms $KR^{p,q}(X) \cong KR^{p-2,q-1}(X)$. The fixed set of $\mathbb{Z} \times BU$ is precisely $\mathbb{Z} \times BO$. So for X a space with trivial $\mathbb{Z}/2$ -action and $n > 0$ one has $KR^{-n,0}(X) = [S^{n,0} \wedge X_+, \mathbb{Z} \times BU]_{*,\mathbb{Z}/2} = [S^n \wedge X_+, \mathbb{Z} \times BO]_* = KO^{-n}(X)$. The second equality uses that a map from a space with trivial action into $\mathbb{Z} \times BU$ must factor through the fixed set.

If X is a space with trivial action then $\mathbb{Z}/2 \times X$ consists of two copies of X that are switched by the involution. Then $KR^{0,0}(\mathbb{Z}/2 \times X) = [\mathbb{Z}/2 \times X, \mathbb{Z} \times BU]_{\mathbb{Z}/2} = [X, \mathbb{Z} \times BU] = K^0(X)$, and more generally $KR^{n,0}(\mathbb{Z}/2 \times X) = K^n(X)$ for all $n \in \mathbb{Z}$. So we start to see how both K and KO are together built into KR .

Remark 22.24. We now must pause to warn the reader that the bigraded indexing we have introduced above is *different* than the one used in [Atiyah4]. We use the modern “motivic indexing”, explained for example in [D3]. There is nothing inherently wrong with Atiyah’s indexing, but the motivic indexing leads to statements that are easier to parse—one can always drop the weight and get a statement that makes sense in the nonequivariant world. For example, in our system KR is $(2, 1)$ -periodic, generalizing the usual 2-periodicity of KU . In Atiyah’s indexing KR is $(1, 1)$ -periodic.

In addition to this indexing difference, the paper [Atiyah4] made some non-optimal choices like defining $KR^{p,q}$ to be a group that probably should have been denoted $KR^{-p,-q}$ in the language of that paper. It is quite understandable that this first excursion into equivariant topology would have used conventions that were less than ideal, but it can make the paper hard going for a modern reader.

We also mention that the reader can consult [D3] for an introduction to the framework of $\mathbb{Z}/2$ -equivariant homotopy theory, which here has been presented very quickly.

Fact 1: Suppose $p \in \{1, 2, 4\}$. For any bigraded $\mathbb{Z}/2$ -equivariant cohomology theory E there are canonical isomorphisms $E^{n,-p}(X \times S_{anti}^{p-1}) \cong E^{n,0}(X \times S_{anti}^{p-1})$.

Proof. The point is that for $p \in \{1, 2, 4\}$ the space $\mathbb{R}^{p,p}$ has the structure of a (nonequivariant) division algebra ($\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$). One therefore gets equivariant homeomorphisms $X \times S_{anti}^{p-1} \times \mathbb{R}^{p,0} \cong X \times S_{anti}^{p-1} \times \mathbb{R}^{p,p}$ by sending $(x, s, u) \mapsto (x, s, su)$. These restrict to homeomorphisms where one replaces $\mathbb{R}^{p,0}$ and $\mathbb{R}^{p,p}$ with $S(\mathbb{R}^{p,0})$ and $S(\mathbb{R}^{p,p})$. Applying $E^{n+p,0}(-)$ gives

$$\begin{aligned} E^{n,0}(X \times S_{anti}^{p-1}) &\cong E^{n+p,0}(X \times S_{anti}^{p-1} \times D^{p,0}, X \times S_{anti}^{p-1} \times S(\mathbb{R}^{p,0})) \\ &\cong E^{n+p,0}(X \times S_{anti}^{p-1} \times D^{p,p}, X \times S_{anti}^{p-1} \times S(\mathbb{R}^{p,p})) \\ &\cong E^{n,-p}(X \times S_{anti}^{p-1}) \end{aligned}$$

where the first and last steps are suspension isomorphisms. $\square$

Fact 2: For $p \in \{1, 2, 4\}$ there are canonical isomorphisms $KR^{a,0}(X \times S_{anti}^{p-1}) \cong KR^{a+2p,0}(X \times S_{anti}^{p-1})$. (Note that for $p = 1$, $S_{anti}^0 = \mathbb{Z}/2$ and so this is the 2-fold periodicity of complex K -theory.)

Proof. This comes from combining Fact 1 with (2,1)-periodicity:

$$KR^{a,0}(X \times S_{anti}^{p-1}) \cong KR^{a,-p}(X \times S_{anti}^{p-1}) \cong KR^{a+2p,0}(X \times S_{anti}^{p-1}).$$

□

Fact 3: There are canonical short exact sequences

$$0 \rightarrow KR^{n,0}(X) \xrightarrow{\pi^*} KR^{n,0}(X \times S_{anti}^3) \rightarrow KR^{5+n,0}(X) \rightarrow 0.$$

Proof. Start with the long exact sequence for the pair $(D^{4,4}, S_{anti}^3) = (D, S)$, which takes the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & KR^{n,0}(X \times D, X \times S) & \longrightarrow & KR^{n,0}(X \times D) & \xrightarrow{(\text{id} \times i)^*} & KR^{n,0}(X \times S) \longrightarrow \cdots \\ & & \uparrow \cong & & \uparrow \pi^* \cong & & \\ & & KR^{n-4,-4}(X) & \xrightarrow{\cdot \rho^4} & KR^{n,0}(X) & & \end{array}$$

We need to explain the ρ^4 map. Let $\rho: S^{0,0} \hookrightarrow S^{1,1}$ denote the inclusion of the points $\{0, \infty\}$. This determines an element in the stable homotopy groups of spheres $\pi_{-1,-1}(S)$ and therefore an element in $KR^{1,1}(pt)$ (which we will also call ρ). The map $S^{0,0} \hookrightarrow S^{4,4}$ is the fourth smash power of ρ , and this is what induces the map $KR^{n-4,-4}(X) \rightarrow KR^{n,0}(X)$ in the diagram.

Now use Bott periodicity $KR^{1,1}(pt) \cong KR^{-1,0}(pt) = KO^{-1}(pt) = \mathbb{Z}/2$. We know that the third power of the generator in $KO^{-1}(pt)$ vanishes (see note below), and it follows that $\rho^3 = 0$ (ρ maps to either the generator or zero in $KO^{-1}(pt)$, and we don't actually care which one). So the above long exact sequence breaks up into short exact sequences

$$0 \rightarrow KR^{n,0}(X) \xrightarrow{\pi^*} KR^{n,0}(X \times S_{anti}^3) \rightarrow KR^{n+1,0}(X \times D^{4,4}, X \times S_{anti}^3) \rightarrow 0.$$

The group on the right is $KR^{n-3,-4}(X)$ using the suspension isomorphism, and then using (2,1)-periodicity this is also $KR^{n+5,0}(X)$.

[Note that we are only using a small bit about $KO^*(pt)$ in this argument. The fact that the third power of the generator of $KO^{-1}(pt)$ vanishes can be proven geometrically as in Corollary 20.33, or one could compute $KO^{-3}(pt) = \widetilde{KO}^0(S^3) = 0$ by hand and then get it for free.] □

Fact 4: Multiplication by the usual Bott generator $\lambda_8 \in KO^{-8}(pt) = KR^{-8,0}(pt)$ induces isomorphisms $KR^{a,0}(X) \cong KR^{a-8,0}(X)$ for all X and all a, b .

Proof. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & KR^{n,0}(X) & \longrightarrow & KR^{n,0}(X \times S_{anti}^3) & \longrightarrow & KR^{5+n,0}(X) \longrightarrow 0 \\ & & \downarrow \cdot \lambda_8 & & \downarrow \cdot \lambda_8 & & \downarrow \cdot \lambda_8 \\ 0 & \longrightarrow & KR^{n-8,0}(X) & \longrightarrow & KR^{n-8,0}(X \times S_{anti}^3) & \longrightarrow & KR^{n-3,0}(X) \longrightarrow 0 \end{array}$$

where the two rows are from Fact 3. The horizontal maps are combinations of pullback maps and (2,1)-periodicity maps (which are multiplication by β), and so the squares commute—possibly up to signs, but one is free to alter the signs on the

vertical maps without changing the rest of the argument. The middle vertical map is an isomorphism, as this turns out to equal the composite of two of the $(4,0)$ -periodicity isomorphisms from Fact 2 (this must be checked carefully, of course). It follows from a diagram chase that the left vertical map is an injection. But as this holds for all n , the right vertical map is also an injection. Another diagram chase now shows that the right vertical map is surjective as well. But as this holds for all n , we are done. $\square$

Conclusion: Recall that for X a space with trivial $\mathbb{Z}/2$ -action, $KR^{-n,0}(X) = KO^{-n}(X)$ for $n \geq 0$. Therefore Fact 5 gives Bott periodicity for the negative KO -groups. Taking $X = pt$ and using $KO^{-n}(pt) = \pi_n(\mathbb{Z} \times BO)$ gives real Bott periodicity in π_*O .

Of course we have only tried to give the general flavor of Atiyah's arguments and have therefore skipped over some key details. The main one is the claim that the middle $\cdot\lambda_8$ map in the proof of Fact 4 is the composite of the two $(4,0)$ -periodicity isomorphisms. This involves some manipulations with Clifford algebras along the lines of things we did in previous sections. The KR -theory proof is not exactly 'simple', but packaging some of the argument in terms of general facts about $\mathbb{Z}/2$ -equivariant topology does lead to a proof that is more streamlined than some others.

We close our summary of this KR -approach by quoting what Frank Adams wrote in his Math Review of Atiyah's paper: "*The reviewer is conscious that the paper contains points of interest not mentioned above [in the review]; he pleads that this is a paper of 19 pages which cannot be summarised adequately in less than 20, and urges topologists to read it.*"

22.25. Even more approaches. As we wrap up our discussion of Bott periodicity, it is worth mentioning that there were some early attempts to take all of the geometry away and try to prove the Bott loop relations by entirely homotopy-theoretic means. See for example Cartan's chapters in [CM], or Dyer-Lashof [DL], or [Toda]. Typically these involve doing hard work to compute the cohomology of the spaces involved (via spectral sequences or other means) and then using—for example—that a map of H -spaces $X \rightarrow Y$ is a weak equivalence if and only if it induces an isomorphism on homology. The work of Toda is slightly different, in that he tried to uniquely characterize the space $\mathbb{Z} \times BU$ via a short list of special properties. By showing that ΩU also had those properties, he proved that $\Omega U \simeq \mathbb{Z} \times BU$. A more refined approach along these lines is in [KT].

There are many novel ways of getting at Bott periodicity via geometry. The papers [BS] and [Tian] find it in moduli spaces of instantons. [LV] gives a proof entirely in the world of algebraic geometry. Atiyah-Singer [AS1] gave a proof using Fredholm operators on infinite Clifford modules, and there are many proofs in the setting of C^* -algebras. As the reader might suspect at this point, any survey of this subject is necessarily incomplete!

Part 4. What is K -theory?

Of course there is no single answer to this question, and also no easy answer. It is a bit like asking “What is physics?” or “What is a cow?” What constitutes a satisfying answer all depends on one’s perspective. A reporter once asked William Faulkner what he was trying to say in one of his novels, and the author’s response was something like “If I could tell you in two minutes I wouldn’t have had to write the whole book!” I feel much the same way about our journey through K -theory over the past $\infty+$ pages. All of the material we have covered has been trying to unveil different aspects of what K -theory “is”, but the reader might understandably be feeling that we have, in the end, raised more questions than answers. There are too many places where the question of “what is really going on here?” seems to call out for a deeper explanation.

In this final section I want to approach the question “What is K -theory?” directly and try to sketch a certain kind of answer. But this will involve very different techniques than in the rest of the book. Here I want to mostly leave geometry behind and focus instead on the world of homotopy theory. Unfortunately, that opens up a whole Pandora’s box of evils that could easily take us another $\infty+$ pages to unravel. And at some point we all need to retire for the night and get some sleep.

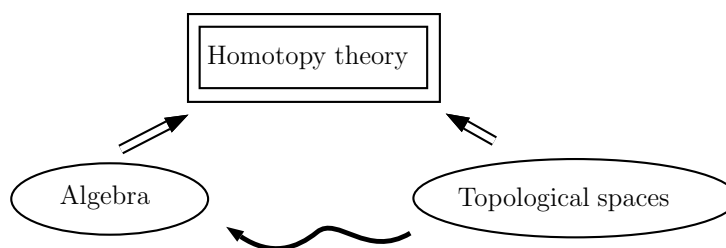
So here is what is going to happen. Sit back, get comfortable, take your feet off the ground. Peter Pan and Tinkerbell are going to take us sailing into the heavens, on a whirlwind tour, and we are going to see some strange lands. Not everything we see is going to make sense to us tonight, and that’s okay. We will take in what we can, not ask too many questions, and then come back home and go to sleep before the parents catch on. Deal?

23. THE SECOND STAR TO THE RIGHT

Warning: This section is more about painting a picture than being perfectly rigorous, and so we are going to be cavalier about including the occasional “mathematical white lie”. The reader should proceed beyond this point at their own risk.

23.1. What is homotopy theory? Since homotopy theory is going to be the main backdrop for this journey, we had better get some things straight about it up front.

In an introductory algebraic topology course one gets the idea right away that there are deep and interesting processes for passing from topological information to algebraic information. As researchers delved into this subject and found themselves creating ever more sophisticated machinery to accomplish this transfer, it gradually became apparent that it was best to think of this technology as operating inside a certain realm that had previously been hidden. What came into view was a world lying in between topology and algebra, sharing characteristics with both: this is the world of homotopy theory. More precisely, I should call this the “homotopy theory of topological spaces”; but this is a mouthful and it gives the erroneous impression that the theory is mainly about topology. Here let me just use “homotopy theory” as an abbreviation. The following crude diagram captures the situation somewhat:



When students first encounter homotopy theory it is usually from the perspective that the whole point is to study topological spaces, and as a result topology feels inherently inseparable from what homotopy theory *is*. But that viewpoint is an anchor that keeps one from ascending into the heavens, and it is important to somehow let it go. It is best to view homotopy theory as its own thing, a wonderland that was waiting to be discovered. Topological spaces were our first way into it, but are by no means the only way in.

It is true that the homotopical world can at times feel very geometric. Objects like spheres and manifolds, Lie groups and their homogeneous spaces, cycles and intersections—the list goes on and on—all play a big role. But at other times the homotopical world feels like some kind of highly-sophisticated algebra: it has localizations, completions, bar constructions, resolutions, etc. Every topological space yields a corresponding object in the homotopical world, and in fact every object comes from a topological space—but the realm of spaces is often not the best way to think about what is happening. For example, it turns out that every chain complex of abelian groups also specifies an object in the homotopical world; yet it is far from evident what topological space that should correspond to.

This strange blending of worlds can make it hard for homotopy theorists to talk to mathematicians in other fields. For a homotopy theorist, topological spaces and chain complexes live in the same playpen and get manipulated according to the same rules. This idea can seem very disorienting to algebraists and geometers! The intrepid homotopy theorists have developed all kinds of techniques for working in—and thinking about—the homotopical world, and it can take years to master. We can't cover all of that in one night, but fasten your seatbelts and we will do a quick flyover.

I need to start assigning some names to things, so let me call this mystery world HS —for the Homotopy theory of Spaces. As the name suggests, the objects in this world are typically called “spaces”, but this is hopelessly confusing when one is first getting the lay of the land. Inevitably one starts to confuse the objects with topological spaces, which they are not. Let me instead call the objects “ h -sets”, short for “homotopical sets”. This is not a standard name, but will do for the time being.

If you have been brought up on 20th century mathematics you would undoubtedly expect me to describe HS to you as a category: that is, I should give you precise definitions of the objects and the morphisms. Somewhat surprisingly, this has never quite worked. One can indeed describe certain categories that serve as gateways into HS , but in each case one has to specify extra rules that govern when various constructions in the category match HS and when they don't. Categories with such rules go under the name *model categories*.

The inability to describe HS in terms of ordinary category theory is one of the first oddities one must adapt to. In modern language one might say that HS has to be described as an ∞ -category. We won't go into the technology of either model categories or ∞ -categories, and instead will just keep most of that structure hidden in the background. But it is important to know that it is there.

After going on at length about the difficulties in describing HS , we still need to describe it somehow. To get a sense of the problem, we will make an attempt to give an “algebraic” description of the objects. Remember that these are supposed to model homotopy types, and we know something about those. It seems clear enough that an object X should come with a set of points X_0 , and that we should be able to talk about paths—or maybe “edges” is a better term—connecting two points. That seems simple enough, but now things get bumpy. We should be able to concatenate paths (when they line up appropriately) to make new paths, and we should be able to talk about deformations—or homotopies—between paths. Then we are going to need homotopies between homotopies, and so on ad infinitum. Here it gradually becomes clear that there are probably lots and lots of ways to encode all of this stuff: one could use cubes, or simplices, or any other form of higher-dimensional “blobs”, and then there is a combinatorial challenge of how to make this stuff all fit together. None of the choices seem particularly canonical, and they all seem fairly complicated. Blech.

Of the many possible approaches there are two that are most commonly used:

- (a) *Simplicial sets*. Here we use simplices as our model for “higher homotopies”. The definitions of the objects are not too bad, and one gets a completely combinatorial world in which to work. There are issues, though. One cannot compose edges in a simplicial set, and so one has to find ways around that and related issues. Also, there is certainly nothing canonical about simplices—one could have used cubes just as well, and some problems seem to call out for cubes or even more complicated “parameter spaces”. As a result, certain constructions end up seeming unnecessarily complicated when done in the simplicial world.
- (b) *Topological spaces*. This is a kind of Gordian-knot style solution. Observe that topological spaces already have all of the structure we are trying to model, the only problem is that they have too *much* structure. If we only care about the homotopies and higher homotopies inside a space then we don't care about the open sets or other homeomorphism-type information, and we have to find a way to systematically forget that data.

We are not going to introduce the language of model categories here, but the point of that technology is that it offers a systematic approach to “doing homotopy theory” that applies in both of the above worlds (as well as many others). We have two model categories $sSet$ and Top as well as adjoint functors $sSet \rightleftarrows Top$ allowing us to pass back and forth between them.

Via these models every topological space yields an object in HS , but we also obtain other sources for such objects. Every small category $\mathcal{C}$ has a nerve $N\mathcal{C} \in sSet$ —with the n -simplices being strings of n -composable maps—and the associated object of HS is denoted $B\mathcal{C}$ and called the *classifying space* of $\mathcal{C}$ (perhaps we should say “classifying h -set”, but we are going to start reverting to more standard terminology). Likewise, there is the Dold-Kan equivalence of categories $Ch_{\geq 0}(\mathbb{Z}) \cong sAb$ and via this every non-negatively-graded chain complex of abelian groups yields an associated simplicial set—and hence an object of HS .

23.2. Homotopical monoids. Now that we have the notion of an h -set (or at least, the vague idea that there is a world of such things), perhaps an algebraic-minded person would next ask: what is an h -monoid? For historical reasons these usually go under the name **A_∞ -spaces**: such a thing is an h -set X together with a distinguished point $1 \in X$ and a map $\mu: X \times X \rightarrow X$ that is “associative and unital up to infinite homotopies”. As one would expect, it takes some serious work to make sense of this. We sail past this point without much comment.

Let us note that an A_∞ -space is more than just the triple $(X, \mu, 1)$: it is also the infinite collection of “higher homotopies” alluded to in our vague definition. So there can exist multiple A_∞ -structures with the same underlying triple $(X, \mu, 1)$. The convention is to be somewhat sloppy with language and to say things like “ X is an A_∞ -space” in place of “ X , with a bunch of hidden data that you might guess but we are not exactly going to spell out, is an A_∞ -space”. Such is life.

Although we will not need them, let us mention that an A_n -space is like an A_∞ -space but where one only has higher homotopies up through level n , with level 2 just consisting of the product $X \times X \rightarrow X$ itself.

Here are several examples of A_∞ -spaces:

- (1) Any monoid in sets, or more generally any topological space with a continuous multiplication that is associative and unital on the nose.
- (2) Any loop space ΩX .
- (3) Any (homologically graded) differential graded algebra concentrated in non-negative degrees (i.e. a monoid in $\text{Ch}_{\geq 0}(\mathbb{Z})$).
- (4) If $\mathcal{C}$ is a small monoidal category then the monoidal structure induces a product on $B\mathcal{C}$ making this into an A_∞ -space.
- (5) For any space X we have the monoid $\text{Vect}(X) = \coprod_n \text{Vect}_n(X)$ of complex vector bundles under direct sum. The sets $\text{Vect}^n(X)$ are represented by $BU(n)$ (or $BGL_n(\mathbb{C})$ if you prefer), and direct sum is represented by the maps of spaces $BU(k) \times BU(n) \rightarrow BU(k+n)$ induced by the block sum homomorphism maps $U(k) \times U(n) \rightarrow U(k+n)$, i.e.

$$(A, B) \mapsto \begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

These maps give a multiplication on the space

$$\coprod_n BU(n)$$

that makes it into an A_∞ -space

- (6) In analogy to (5), if R is any discrete ring then $\coprod_n BGL_n(R)$ has the structure of an A_∞ -space. We can also consider

$$\coprod_P B \text{Aut}(P)$$

where the coproduct is over the finitely-generated R -projectives. Here one has to be careful as to how to get an indexing *set* for the coproduct. Dodging this point, this is another example of an A_∞ -space. Another approach to the same (up to homotopy) space is to take the nerve of the category of finitely-generated R -projectives with isomorphisms, with the monoidal structure of direct sum (here again one should technically restrict to a small skeletal subcategory).

- (7) We can replace $U(n)$ or $GL_n(R)$ with the symmetric groups Σ_n , again using the map $B\Sigma_k \times B\Sigma_n \rightarrow B\Sigma_{k+n}$ induced by block sum of permutations. The space $\coprod_n B\Sigma_n$ is again A_∞ .
- (8) If X is a pointed topological space then the James construction $J(X)$ is the free monoid generated by the points of X with the basepoint as unit, suitably topologized. As $J(X)$ is a topological monoid, it is an A_∞ -space.
- (9) If X is an A_∞ -space then any mapping space $\text{Map}(W, X)$ will inherit an A_∞ -structure. In particular, applying this to example (5) shows that if W is path-connected then

$$\coprod_n \text{Map}(W, BU(n))$$

is A_∞ .

Every monoid M can be regarded as a category with one object (and endomorphism monoid M), and as such it has a classifying space BM . If we model BM as the geometric realization of the nerve, there is an evident map $M \rightarrow \Omega(BM)$ and this is compatible with multiplication (up to homotopy). This generalizes: every A_∞ -space X has a classifying space BX , which comes with a map of A_∞ -spaces $X \rightarrow \Omega BX$. (We have of course not explained what a map of A_∞ -spaces is, but so it goes). In fact in most incarnations the B construction comes with a canonical quasifibration $EX \rightarrow BX$ with EX contractible and X mapping into (but not equal to) the fibers, analogous to the $G \rightarrow EG \rightarrow BG$ story for groups G . This will become important later on.

23.3. Homotopical groups and abelian groups. Now that we know about h -monoids, how about h -groups? We might as well keep this party going. Here the answer turns out to be pretty easy, though: an h -group is just an h -monoid X with the property that $\pi_0(X)$ is a group. One can imagine other, more sophisticated, definitions, but they all turn out to be equivalent to this more simplistic one.

Note that every loop space ΩX will be an h -group. It turns out these are all of them!

Theorem 23.4. *Every h -group X is equivalent (as an h -group) to ΩY for some space Y . In fact the map $X \rightarrow \Omega BX$ is an equivalence.*

What about h -analogs of commutative monoids and abelian groups? The analogs of commutative monoids are called E_∞ -spaces: these are A_∞ -spaces together with an infinite collection of higher homotopies governing commutativity. An E_n -space is the associated concept where one only has homotopies up through level n . For example, E_1 -spaces are just A_∞ -spaces. We define E_n -groups to be E_n -spaces where π_0 is a group.

Theorem 23.5. *The E_n -groups are precisely the n -fold loop spaces $\Omega^n X$.*

Again, several examples:

- (1) Every topological commutative monoid is an E_∞ -space. For example, if X is a pointed space then the infinite symmetric product $\text{Sp}^\infty(X)$ is E_∞ .
- (2) If $\mathcal{C}$ is a symmetric monoidal category then $B\mathcal{C}$ is an E_∞ -space.
- (3) The spaces $\coprod_n B\Sigma_n$, $\coprod_n BGL_n(R)$, $\coprod_P B\text{Aut}(P)$, and $\coprod_n BU(n)$ are all E_∞ -spaces. The first three are the classifying spaces for the categories of finite sets with isomorphisms, finitely-generated free R -modules with isomorphisms, and finitely-generated projective R -modules with isomorphisms. The last is

the classifying space for the *topological* category of finitely-generated $\mathbb{C}$ -vector spaces with isomorphisms.

There is a subtlety in Theorem 23.5 that requires some explanation. An n -fold loop space is a space $X = X_0$ together with a sequence of deloopings: spaces $X_1, \dots, X_n$ and weak equivalences $X_i \simeq \Omega X_{i+1}$. But observe that the space X_0 can have different choices for the deloopings: X_1 can be replaced with any of its components, X_2 can be replaced with its universal cover, etc. The information in the original E_n -group really only corresponds to the *connective* deloopings: the choices where each X_i is $(i-1)$ -connected.

The notion of E_∞ -space is a reasonable candidate for what “ h -abelian group” should mean, but it is not the only candidate. Another perspective is that the “ h -abelian groups” are spectra. The difference here is not huge, but is worth commenting on; it is related to the concerns of the preceding paragraph. An E_∞ -space X is an infinite-loop space, and therefore is $\Omega^\infty(E)$ for some spectrum E . However, the spectrum E is not unique. For example, a spectrum and its connective cover always have the same Ω^∞ .

What is true is that there is (essentially) an equivalence between E_∞ -spaces and *connective* spectra. It is common to be somewhat sloppy about the distinction between these two, and we will follow that approach in what follows. So the two different perspectives on “ h -abelian groups” amount to just the difference between connective and non-connective spectra. We will not need to care about this distinction for our present purposes.

23.6. Homotopical group completion. In usual (non-homotopical) algebra, group completion is the left adjoint to the inclusion i of Groups into Monoids:

$$(\text{Monoids}) \xrightleftharpoons[i]{\quad} (\text{Groups}).$$

If M is a monoid then the group completion is often denoted M^+ . It can be constructed as the quotient of the free group on the underlying pointed set of M by the normal subgroup generated by elements $[x][y][xy]^{-1}$ for all $x, y \in M$. Sometimes the group completion is also referred to as the Grothendieck group of the monoid M . Of course $\mathbb{Z}$ is the group completion of $\mathbb{N}$, and when X is compact $K^0(X)$ is the group completion of the monoid of isomorphism classes of vector bundles on X .

In the homotopical setting we can ask for a homotopical adjoint to the inclusion of h -groups into h -monoids. This is “homotopical group completion”. If one starts with an E_∞ -monoid then homotopical group completion will produce an E_∞ -group, i.e. an infinite loop space. Homotopical group completion is usually denoted by $M \mapsto M^+$ just as in the non-homotopical setting, but we will write $M \mapsto M^{+h}$ to avoid confusion.

If X is an A_∞ -space then there is always an A_∞ map $X \rightarrow \Omega(BX)$, and the target is an h -group. It is reasonable to expect this target to be the group completion, and it is:

Theorem 23.7. *If X is an A_∞ -space then $X^{+h} \simeq \Omega(BX)$.*

If M is a discrete monoid then the simplicial model of BM together with the Van Kampen theorem immediately gives that $\pi_1(BM) = M^+$, so that we conclude $\pi_0(M^{+h}) = \pi_0(\Omega BM) = \pi_1(BM) = M^+$. It is not hard to believe that this holds more generally: if M is an h -monoid then $\pi_0(M^{+h}) \cong \pi_0(M)^+$. But the interesting

question is what $(-)^{+h}$ does to the higher portions of the homotopy type. This is one answer to the question of “what is K -theory?”—at some fundamental level it is the study of homotopical group completion. We constructed the group $K^0(X)$ as the ordinary group completion of the monoid of isomorphism classes of vector bundles on X , but what we should have really been doing is taking homotopical group completion of an A_∞ -space of of vector bundles on X . We will say more about this in a moment.

One would almost certainly guess that if M is a discrete monoid then M^{+h} is homotopy equivalent to M^+ . But this turns out to be far from the case! In Exercise 23.23 below we give an example of a discrete monoid M for which $BM \simeq S^2$, so that $M^{+h} \simeq \Omega S^2$; in particular, this M^{+h} has non-vanishing higher homotopy groups. In fact McDuff proved that *every* h -group can be realized as the homotopical group completion of a discrete monoid [McDuff2]. The moral is that even discrete monoids can be quite complicated. The naive guess about M^{+h} is indeed true, however, when M is sufficiently nice: see (1) and (2) in the list below for some cases of this. The exercises at the end of this section give an introduction to the study of M^{+h} when M is discrete.

Here are several examples of homotopical group completion:

- (1) If M is a commutative discrete monoid then $M^{+h} \simeq M^+$ (see Exercise 23.21).
- (2) If M is a free discrete monoid then $M^{+h} \simeq M^+$.
- (3) $(\coprod_n BU(n))^{+h} \simeq \mathbb{Z} \times BU$.
- (4) $(\coprod_n BGL_n(R))^{+h} \simeq \mathbb{Z} \times K'(R)$, where $K'(R)$ is a connected space whose homotopy groups are the higher K -groups of R .
- (5) $(\coprod_P B\text{Aut}(P))^{+h} \simeq K_0(R) \times K'(R) \simeq K(R)$, where $K(R)$ is the Quillen K -theory space of R .
- (6) $(\coprod_n B\Sigma_n)^{+h} \simeq Q(S^0) = \Omega^\infty \Sigma^\infty(S^0)$, the 0th space in the Ω -spectrum for the sphere spectrum. Note that $\pi_i(Q(S^0)) = \pi_i(S)$, the i th stable homotopy group of spheres.
- (7) If $\mathcal{C}$ is a symmetric monoidal category then $(B\mathcal{C})^{+h}$ is the Quillen-Segal algebraic K -theory space of $\mathcal{C}$ (the 0th space of the algebraic K -theory spectrum).

We have seen that even if M is discrete then M^{+h} can nevertheless be quite complicated, in the sense of having nonvanishing higher homotopy. In examples (4)–(6) above we find examples of A_∞ -spaces M having $\pi_i(M) = 0$ for $i > 1$ but where M^{+h} is again quite complex. The algebraic K -groups of a ring are in general unknown, and there are open questions even in the case $R = \mathbb{Z}$. In example (6) we find the stable homotopy groups of spheres in M^{+h} , and of course those are also notoriously complex. Generally speaking, there is no known process for starting with the homotopy groups of M and then cooking up the homotopy groups of M^{+h} .

We will next try to explain some of the common themes one might notice in examples (3)–(6).

For an h -group X , all of the components must have the same homotopy type: any $x \in X$ has an inverse y in $\pi_0(X)$, and then multiplication-by- x and multiplication-by- y give homotopy inverses between the components X_e and X_x . Here X_e is the component of the identity. So for an h -group we will always have $X \simeq \pi_0(X) \times X_e$. Thus, one thing group completion must do is “equalize” the different components in an h -monoid. In many of our examples we have an h -monoid X with $\pi_0(X) = \mathbb{N}$, so let us focus on that situation for a moment. Choose a point $p_1 \in X_1$, noting

that here X_1 is *not* the component of the identity element (which is $0 \in \mathbb{N}$). Form the direct limit system

$$X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_1} \dots$$

Let X_∞ denote the (homotopy) colimit. The composites $X_i \rightarrow X^{+h} \xrightarrow{p_1^{-i}} X^{+h}$ are compatible with the maps in the direct limit system, and so yield an induced map from X_∞ to the 0-component of X^{+h} . If we are lucky and X_∞ is actually an h -group then this will be the end of the story: we will get $X^{+h} \simeq \pi_0(X)^+ \times X_\infty$. This happens in example (3), for instance.

In example (4) an issue arises, which is that the X_∞ space is $BGL(R)$ and therefore has nonabelian π_1 . But any A_∞ -monoid will have an abelian π_1 , so in this example X_∞ cannot be the 0-component of the group completion. The same problem occurs in example (6), where π_1 is the infinite symmetric group Σ_∞ . In a moment we will see how to fix up these examples: the short answer is one can recover the group completion by just altering X_∞ in order to get rid of the bad parts of π_1 . But to explain this, we need a brief detour on homology.

Quillen discovered a general formula for the homology of certain “well-behaved” group completions. The idea is essentially as follows. To construct X^{+h} from X one can examine how $\pi_0(X)^+$ is made from $\pi_0(X)$ and then try to mimic that at the homotopical level. The difficulty is that if M is a general monoid then the passage from M to M^+ might do all kinds of strange things. But if M satisfies some mild hypothesis then M^+ can be obtained from M by a colimit procedure: for example, the colimit (in the category of sets) of the diagram

$$(23.8) \quad \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \dots$$

can be identified with $\mathbb{Z}$ in an evident manner. This was secretly what was lying behind the $\pi_0(X) = \mathbb{N}$ examples we looked at above; note how we replaced the colimit with a homotopical one in order to get X_∞ .

If M is any monoid, let EM be the translation category for M : the object set is M and for every $x, y \in M$ there is a map $\rho_x: y \rightarrow yx$ with the properties that $\rho_1 = \text{id}$ and $\rho_a \rho_b = \rho_{ba}$. Let $\underline{M}$ be the diagram $EM \rightarrow \text{Set}$ that sends every object to M and each map ρ_x to right-multiplication-by- x . Let M_∞ be the colimit in Set of $\underline{M}$. Note that this has a very simple description: an element in this colimit can be described as a pair (x, y) where y is the object of EM and $x \in \underline{M}(y)$. The relations in the colimit just say that $(x, y) \sim (xz, yz)$ for every $x, y, z \in M$.

There is a map from the diagram $\underline{M}$ to M^+ sending the pair (x, y) to xy^{-1} , and this induces a map of sets $M_\infty \rightarrow M^+$. When M is commutative this is an isomorphism: this is the usual construction of the group completion via pairs of elements. The map is even an isomorphism somewhat more generally, whenever the monoid M admits a certain kind of “calculus of fractions”. We will leave the reader to dream about the details there.

We now come to the so-called “Group Completion Theorem”, due independently to Quillen and to Barratt-Priddy. If X is an A_∞ -space and T is a ring then $H_*(X; T)$ becomes a ring under the Pontryagin product, given by

$$H_p(X; T) \otimes H_q(X; T) \longrightarrow H_{p+q}(X \times X; T) \xrightarrow{\mu_*} H_{p+q}(X; T).$$

The degree 0 subring is precisely the monoid ring $T[\pi_0(X)]$, where $\pi_0(X)$ has the monoid structure inherited from the product on X . The following result is from [BPr] and [Quillen7]:

Theorem 23.9 (Quillen, Barratt-Priddy). *Let T be a ring and let X be an A_∞ -space. Suppose that $\pi_0(X)$ is commutative. Then there is an isomorphism*

$$H_*(X^{+h}; T) \cong H_*(X; T)[\pi_0(X)^{-1}].$$

(This also holds under somewhat less restrictive hypothesis, where one only assumes that $\pi_0(X)$ admits a certain kind of calculus of fractions).

Now let us return to the example of the A_∞ -space $X = \coprod_n BGL_n(R)$. We have seen that the map $X \rightarrow X^{+h}$ induces a map $BGL(R) \rightarrow (X^{+h})_0$ (the path-component of the identity element $0 \in \pi_0(X) = \mathbb{N}$), but that $BGL(R)$ cannot be $(X^{+h})_0$ because of π_1 issues. However, the group completion theorem shows that $H_*(BGL(R); T) \rightarrow H_*((X^{+h})_0; T)$ will be an isomorphism for every ring T : this is because $H_*(X; T)[\pi_0(X)^{-1}]$ is precisely $\mathbb{Z}[t, t^{-1}] \otimes H_*(BGL(R))$. This tells us that $(X^{+h})_0$ is pretty close to being $BGL(R)$, but it is just that the π_1 is getting in the way. So it is reasonable to envision that if we could just carefully kill off some of the π_1 in $BGL(R)$ then we might get X^{+h} .

Quillen invented his “plus construction” to do exactly this. This construction is related to group completion, though not exactly equal to it, and so the terminology “plus construction” is a bit annoying. If W is a space with a normal subgroup $N \leq \pi_1(W)$ that is perfect (meaning $N = [N, N]$) then Quillen defines a new space W^{+Q} (usually just denoted W^+ , sadly) by attaching 2-cells to W to kill off N . This space has the property that $\pi_1(W) \rightarrow \pi_1(W^{+Q})$ has kernel N and $W \rightarrow W^{+Q}$ induces isomorphisms on homology with any coefficients. [For more details on the plus-construction see [Quillen2, Section 3], [Quillen4, Section 12], and [Loday]].

Returning to our space $BGL(R)$, the normal subgroup $[GL(R), GL(R)] \subseteq GL(R) = \pi_1(BGL(R))$ turns out to be perfect—this is not just by chance, see Exercise 23.17 for an explanation. So we can form $BGL(R)^{+Q}$, and the map $BGL(R) \rightarrow X^{+h}$ must factor through it:

$$\begin{array}{ccc} BGL(R) & \xrightarrow{\quad\quad\quad} & X^{+h} \\ & \searrow & \nearrow \\ & BGL(R)^{+Q} & \end{array}$$

The two solid-arrow maps are homology isomorphisms (with any coefficients), so the same is true of the dotted arrow map. But for the dotted arrow map the domain and codomain are both simple spaces (π_1 is abelian and the action on the higher homotopy groups is trivial). A homology isomorphism between simple spaces is a weak homotopy equivalence, so we find that $BGL(R)^{+Q} \simeq X_0^{+h}$. That is, the Quillen plus-construction for $BGL(R)$ gives a model for the 0-component of the homotopical group completion of X .

The same exact story plays out for $\coprod_n B\Sigma_n$ to show that $(B\Sigma_\infty)^{+Q} \simeq (\coprod_n B\Sigma_n)_0^{+h}$.

Finally, let us look briefly at example (5), which was $X = \coprod_P B\text{Aut}(P)$. Here $\pi_0(X)$ is not $\mathbb{N}$ but rather the monoid of isomorphism classes of finitely-generated R -projectives under direct sum. This is still commutative and so we can form the corresponding directed system and construct our X_∞ as the homotopy colimit. But the free R -modules are cofinal in this system, since every projective is a direct summand of a free. So we find that X_∞ is weakly equivalent to the corresponding homotopy colimit taken over the free R -modules, which is the same space obtained

in example (4). The same homology-based arguments as before now show that $(X^{+h})_0 \simeq BGL(R)^{+Q}$.

Remark 23.10. One can imagine a generalization of A_∞ -monoids where the multiplication is not defined on the entire space but only on a “big enough” subspace that captures the entire homotopy type. That is, instead of a map $\mu: X \times X \rightarrow X$ we have an inclusion $X_{[2]} \subseteq X \times X$ that is a weak homotopy equivalence and a map $\mu_2: X_{[2]} \rightarrow X$. Think of the space $X_{[2]}$ as giving “multipliable pairs”. Homotopy associativity is now harder to state because we need triples where all possible pairs are multipliable. It turns out to be better to approach this by having a specified inclusion $X_{[3]} \subseteq X \times X \times X$ of “multipliable triples” that is again a weak homotopy equivalence, a map $\mu_3: X_{[3]} \rightarrow X$, and so forth for tuples of longer length. One can develop such a theory of “partial A_∞ -monoids” together with classifying spaces and group completions, and prove that any such partial object is equivalent to an actual A_∞ -monoid—so the theories are the same.

Segal’s paper [Segal4] developed this story for the commutative case. He called his objects “ Γ -spaces” and proved (essentially) that the homotopy theory of Γ -spaces is equivalent to that of E_∞ -spaces (to be more precise, Segal did this for grouplike Γ -spaces and connective spectra). It should be explained that Segal’s work actually predates the A_∞ -analogs, I believe.

Here is an important example that we have secretly already seen: take $X = \coprod_n \text{Gr}_n(\mathbb{C}^\infty)$. Given subspaces $U, V \subseteq \mathbb{C}^\infty$ of dimensions n and k , there is no canonical way to produce a sum inside $\mathbb{C}^\infty$ of dimension $n+k$. But if the subspaces are orthogonal then we can do it. So define $X_{[k]} = \{(V_1, \dots, V_k) \mid V_i \perp V_j\} \subseteq X^k$, which is readily seen to be a weak equivalence. The addition maps $X_{[k]} \rightarrow X$ are just $\underline{V} \mapsto \sum_j V_j$. This gives a Γ -space structure, and the classifying space BX turns out to be exactly the space $|\Gamma_\bullet(\infty)|$ that we encountered back in Section 21.6. There we saw the homeomorphism $|\Gamma_\bullet(\infty)| \cong U$, which gives $\Omega BX \cong \Omega U$. But $\Omega BX \simeq X^{+h}$ and we have already said that $X^{+h} \simeq \mathbb{Z} \times BU$, so this gives the Bott loop relation $\Omega U \simeq \mathbb{Z} \times BU$. In some sense this is the most conceptual proof of complex Bott periodicity. To summarize, spectral theory shows that U is the classifying space of the E_∞ -monoid $\coprod_n BU_n$, group-completion arguments show that $(\coprod_n BU_n)^{+h} \simeq \mathbb{Z} \times BU$, and therefore $\Omega U \simeq \mathbb{Z} \times BU$. This is the argument from [Harris]. Similar arguments (as in Section 21.6) give $B(\coprod_n BO_n) \simeq U/O$ and $B(\coprod_n BSp_n) \simeq U/Sp$, hence $\mathbb{Z} \times BO \simeq \Omega(U/O)$ and $\mathbb{Z} \times BSp \simeq \Omega(U/Sp)$.

We could stay on this subject of group completion for a long time, but let us just mention one more theme. Suppose that $\mathcal{C}$ is a symmetric monoidal category, so that $B\mathcal{C}$ is an A_∞ -space. Can we do something to $\mathcal{C}$ —back in the combinatorial world of categories—that will mirror the passage from $B\mathcal{C}$ to $(B\mathcal{C})^{+h}$? For example, can we construct a category $\mathcal{C}^{+h}$ having the property that $B(\mathcal{C}^{+h}) \simeq (B\mathcal{C})^{+h}$? This was another question considered by Quillen, and the answer works out very nicely. This is known as Quillen’s $S^{-1}S$ -construction, though in our case it would technically be $\mathcal{C}^{-1}\mathcal{C}$. We will give a brief overview, but see [Grayson1] for details.

Let $(\mathcal{C}, \oplus)$ be a monoidal category where every map is an isomorphism. Define $\mathcal{C}^{-1}\mathcal{C}$ to be the category with object set $\text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{C})$, and where a morphism $(X_1, X_2) \rightarrow (Y_1, Y_2)$ is an equivalence class of triples $[c, f: c \oplus X_1 \rightarrow Y_1, g: c \oplus X_2 \rightarrow Y_2]$ where the maps f and g are isomorphisms. The equivalence relation

has $[c, f, g] \sim [c', f', g']$ if there is an isomorphism $c \rightarrow c'$ making the two evident diagrams commute. If $(\mathcal{C}, \oplus)$ is symmetric monoidal then one can define an action of $\mathcal{C}$ on $\mathcal{C}^{-1}\mathcal{C}$ by $c \odot (X_1, X_2) = (X_1, c \oplus X_2)$, and this is readily checked to be an invertible action (the inverse is given by $c \odot (X_1, X_2) = (c \oplus X_1, X_2)$).

Continuing to assume that $\mathcal{C}$ is symmetric monoidal, we get an induced monoidal structure on $\mathcal{C}^{-1}\mathcal{C}$ where $(X_1, X_2) \oplus (Y_1, Y_2) = (X_1 \oplus Y_1, X_2 \oplus Y_2)$. The symmetry isomorphism is needed for the induced behavior on maps. It is easy enough to check that $\pi_0(\mathcal{C}^{-1}\mathcal{C}) \cong (\pi_0\mathcal{C})^+$. In fact we have the analogous statement at the homotopical level:

Theorem 23.11 (Quillen). *Suppose $(\mathcal{C}, \oplus)$ is a symmetric monoidal category where every map is an isomorphism. Then $B(\mathcal{C}^{-1}\mathcal{C}) \simeq (B\mathcal{C})^{+h}$.*

Although Theorem 23.11 is stated for discrete categories, one can play similar games in the generalized setting of topological categories. Here is one example:

Theorem 23.12. *Let X be a compact Hausdorff space, and let $\mathcal{V}_X$ denote the topological category of complex vector bundles on X with the maps being the isomorphisms. Then*

- (a) *If $\mathcal{V} = \mathcal{V}_{pt}$ then $B\mathcal{V} \simeq \coprod_n BU(n)$ and $B(\mathcal{V}^{-1}\mathcal{V}) \simeq (\coprod_n BU(n))^{+h} \simeq \mathbb{Z} \times BU$.*
- (b) *$B\mathcal{V}_X \simeq \coprod_n \text{Map}(X, BU(n))$ and $B(\mathcal{V}_X^{-1}\mathcal{V}_X) \simeq \text{Map}(X, \mathbb{Z} \times BU)$. In particular, $\pi_p(B\mathcal{V}_X) \cong K^{-p}(X)$.*

23.13. Exact sequences. As we know from our study of K^0 , sometimes the Grothendieck group of interest arises not as the group completion of a monoid but rather as the universal additive group in which exact sequences are forced to split. To incorporate this kind of framework into our picture we should look for constructions having the form

$$(\text{homotopical gadgets involving a notion of exact sequence}) \longrightarrow (E_\infty\text{-spaces}),$$

where historically the domain has been made explicit with progressively more sophisticated notions. Going very quickly through some history here:

- (a) Quillen defined his “ Q -construction”, which takes a category $\mathcal{C}$ with a notion of exact sequence and outputs a spectrum $K(\mathcal{C})$.
- (b) Waldhausen developed machinery that takes a so-called “category with a cofibrations” $\mathcal{D}$ (a category with a well-behaved notion of cofiber sequence) and outputs a spectrum $K_{Wald}(\mathcal{D})$. This allows one to talk about the K -theory spectrum for certain categories of chain complexes, with their homotopical notion of cofiber sequence.
- (c) After the introduction of ∞ -categories, it was realized early on that the Waldhausen machinery could be adapted to take as its input any stable ∞ -category. Blumberg-Gepner-Tabuada [BGT] proved that in this setting one can describe Waldhausen K -theory as the universal construction satisfying a short list of familiar axioms (Morita invariance, additivity, and compatibility with colimits, for those who know what all those words mean).

When $\mathcal{C}$ is a symmetric monoidal category equipped with the notion of split-exact sequence, Quillen’s Q -construction agrees (up to homotopy) with the output of the $\mathcal{C}^{-1}\mathcal{C}$ -construction we saw earlier. Likewise, Waldhausen K -theory agrees with Quillen K -theory in the cases where one would hope for that. For example, if X is a scheme then the Waldhausen K -theory of perfect complexes over X agrees with

the Quillen Q -construction applied to the category of locally-free coherent sheaves on X , and if X is affine then this in turn agrees with the $S^{-1}S$ -construction applied to the category of finitely-generated projective R -modules where $R = \mathcal{O}(X)$, which in turn agrees with $K_0(R) \times BGL(R)^{+Q} \simeq (\coprod_P B\text{Aut}(P))^{+h}$.

23.14. This is all very interesting, but what is the punchline?

In all of our discussion of K -theory throughout this book, the groups $K^i(X)$ for $i \neq 0$ have always been a bit nebulous. All too often we end up getting our hands on them by using Bott periodicity to shift them to a K^0 , which often feels unsatisfying. Even the relative groups $K^0(X, A)$ are a bit intimidating; here we saw different geometric models (e.g. Atiyah's difference bundles, or chain complexes of bundles, or the $\mathcal{L}_\infty$ model from ???) but a preferred model never seemed to rise to the top.

Keeping the above comments in mind, let us now return to the beginnings—but from a somewhat different perspective—and see if we can come to terms with some of this. If X is a compact Hausdorff topological space let $\underline{\text{Vect}}(X)$ be the *space* of vector bundles on X , regarded as an A_∞ -space via direct sum. The homotopy type is

$$\underline{\text{Vect}}(X) \simeq \coprod_{n \geq 0} \text{Map}(X, BU(n)).$$

K -theory (and really we should say *connective* K -theory) is the study of the homotopical group completion $\underline{\text{Vect}}(X)^{+h}$, but as we have discussed this is *a priori* a somewhat nebulous object. There is no known general procedure for understanding the homotopy groups of M^{+h} in terms of algebraic information about the original A_∞ -space M .

One of the themes of K -theory is finding models for $\underline{\text{Vect}}(X)^{+h}$ in terms of the geometry of X , and perhaps it is not surprising to discover that there are many such models. This is a general aspect of homotopy theory: an abstract homotopy type, when it manifests out in the real world—in essence, as a “representation” of the homotopy type—often occurs in many different forms. Each form has the homotopy type embedded in it, but has extra features that fill out the object, like flesh and organs attached to a skeleton. The same skeleton can support quite different-looking golems.

Some of these are ????. A recent line of research involves describing a model in terms of certain kind of topological field theories on X : see [ST], [HST], [Markert], and [Ulrickson] as just a few places to get started in that area.

NEED A PUNCHLINE HERE

23.15. Addendum. In our flyover of the neverland of A_∞ - and E_∞ -spaces, one thing we failed to do is give many references. Unfortunately, it is not so easy to find a sunlit path through this particular field of ideas. The original study of A_∞ spaces is from [Stasheff1], but further development became a hot topic throughout the 1960s and 1970s with competing approaches and much cross-fertilization of ideas. A partial list of references is

- (i) Boardman and Vogt [BV1, BV2]. The “E” of E_∞ is due to them and comes from their notion of “homotopy Everything” spaces. (For what it's worth, in my experience almost anything by Vogt is filled with beautiful ideas that reward serious study.)

- (ii) The extensive work of Peter May and his collaborators, starting with [May1] but continuing on with many later tomes. The use of operads was popularized heavily here, though the concept itself appeared in Boardman and Vogt's work under a different name.
- (iii) Segal introduced Γ -spaces in [Segal4], which as we have explained are a version of E_∞ -spaces.

See also [Stasheff3] for an interesting survey article from that period, as well as [Stasheff2], [May2], and many other works of May. For homotopical group-completion see [BPr], [Quillen7], [May3, Section 15], and [MS] as places to get started.

In the 1990s, after additional categorical foundations for homotopy theory had been developed in the form of model categories and their localizations, people returned to many of these ideas with fresh eyes. The papers are again too numerous to mention, but see for example [Badzioch1], [Badzioch2], [BCV], [Stelzer], as well as the many papers on model categories of operads and their modules. Finally, let us mention that Lurie later developed an approach to A_∞ - and E_∞ -structures in the setting of ∞ -categories, which can be found in [Lurie].

23.16. Exercises.

Exercise 23.17. Recall that a group G is called **perfect** if $G = [G, G]$, or equivalently if the abelianization G_{ab} is zero. Suppose that X and Y are path-connected spaces, $\pi_1(Y)$ is abelian, and $f: X \rightarrow Y$ is a map having the property that for every local coefficient system $\mathcal{F}$ on Y the map $f_*: H_*(X; f^*\mathcal{F}) \rightarrow H_*(Y; \mathcal{F})$ is an isomorphism. Let $P = \ker(f_*: \pi_1(X) \rightarrow \pi_1(Y))$. Prove that P is perfect, for example via the following steps:

- (a) Using homology with (untwisted) $\mathbb{Z}$ coefficients and the Hurewicz isomorphism, prove that $P = [\pi_1(X), \pi_1(X)]$.
- (b) Let $G = \pi_1(X)$ and $A = \pi_1(Y) = G/P$. Let $\tilde{X}$ and $\tilde{Y}$ be the universal covering spaces of X and Y . Let $\mathcal{F}$ be the local coefficient system on Y corresponding to the $\pi_1(Y)$ -module $\mathbb{Z}[A]$. Prove that
 - (i) $H_1(Y; \mathcal{F}) = H_1(\tilde{Y}) = 0$.
 - (ii) $H_1(X; f^*\mathcal{F}) = H_1(\tilde{X}/P; \mathbb{Z})$.
 - (iii) $P_{ab} = 0$.

[Note that this exercise applies to the examples we saw where W was an A_∞ -monoid with $\pi_0(X) = \mathbb{N}$, $Y = W^{+h}$, and $X = W_\infty$ was the stabilization of W . By the group-completion theorem $X \rightarrow Y$ is an isomorphism on homology with respect to any coefficient system, and therefore the difference between X and Y is controlled by $\ker(\pi_1(X) \rightarrow \pi_1(Y))$, which this exercise says must be perfect. Quillen's plus construction gives $Y \simeq X^{+Q}$.]

The exercises below concern discrete monoids M together with their group completions M^+ and homotopical group completions M^{+h} .

Exercise 23.18. Let $M = \{1, h\}$ where $h^2 = h$. Check that M is a monoid and $M^+ = 1$. For each $n \in \mathbb{Z}_+$ find a monoid generated by one element, that is not a group, but whose group completion is $\mathbb{Z}/n$. In particular, these are examples where $M \rightarrow M^+$ is not injective.

Exercise 23.19. If M is a finite monoid prove that $M \rightarrow M^+$ is always surjective.

Exercise 23.20. Let M be a monoid. Say that $x \in M$ is a **right absorbing element** if $mx = x$ for all $m \in M$. Define left absorbing elements analogously, and let **absorbing element** mean any element that is either right or left absorbing.

- (a) If M has an absorbing element, prove that $M^+ = 1$ and that $BM \simeq *$ (give a contracting homotopy for the simplicial set $B.M$). So $M^{+h} \simeq M^+$ here.
- (b) Let $M \triangleright g$ denote the monoid whose elements are words involving the formal variable g and elements of M , subject to the relations that $g^2 = g$ and $mg = g$ for all $m \in M$. Check that this really is a monoid, which by (a) satisfies $(M \triangleright g)^{+h} \simeq (M \triangleright g)^+ = 1$. The monoid $M \triangleright g$ acts like a cone on M .

Exercise 23.21. Let M be a discrete commutative monoid.

- (a) Explain why Theorem 23.9 shows that $M^{+h} \simeq M^+$.
- (b) One can also give a more elementary proof of the result in (a), though still requiring a bit of homotopical machinery. Let $\mathcal{C}_M$ be the category with one object and endomorphism monoid M , and likewise for $\mathcal{C}_{M^+}$. Let $F: \mathcal{C}_M \rightarrow \mathcal{C}_{M^+}$ be the evident functor. Quillen's Theorem A [Quillen5] says that F induces a weak equivalence on nerves provided that the nerve of the undercategory $(* \downarrow F)$ is contractible (one can also use overcategories here). The undercategory has objects $x \in M^+$, and morphisms from x to y are elements $m \in M$ such that $y = x + m$. Prove that $(* \downarrow F)$ is a filtered category, and hence has contractible nerve.

Exercise 23.22. Using the simplicial model for $B.M$ one sees immediately that $H_*(BM) = \text{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z})$. And more generally, if J is an M -module on which all of the generators act as isomorphisms then J is a $\pi_1(BM)$ -module and $H_*(BM; J) \cong \text{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, J)$.

- (a) Let $M = \langle f \rangle$, the free monoid on one generator f . Let $\alpha: S^1 \rightarrow BM$ be the evident map sending the 1-simplex of S^1 to f . Prove that α induces an isomorphism on π_1 and on $H_*(-; J)$ for every local coefficient system, and is therefore a weak equivalence.
- (b) Generalize part (a) to show that if M is any free discrete monoid then BM is weakly equivalent to a wedge of circles. Deduce that $M^{+h} \simeq M^+$.

Exercise 23.23. Following [F] we will produce a 5-element monoid M such that $BM \simeq S^2$. Then $M^{+h} \simeq \Omega BM \simeq \Omega S^2$, which has nonvanishing higher homotopy groups. So this will be an example where $M^{+h} \not\simeq M^+$.

Let $M = \{1, x_{ij}\}_{1 \leq i, j \leq 2}$ with the product given by $x_{ij}x_{mn} = x_{in}$.

- (a) Verify that M is associative and that $M^+ = 1$.
- (b) Let $P_1 = \{x_{11}, x_{12}\}$ and $P_2 = \{x_{21}, x_{22}\}$. As subsets of M these are closed under right multiplication by M . Prove that the map of right $\mathbb{Z}[M]$ -modules $\mathbb{Z}[M] \rightarrow \mathbb{Z}\langle P_1 \rangle$ sending $1 \mapsto x_{11}$ is split surjective, proving that $\mathbb{Z}\langle P_1 \rangle$ is projective as a right $\mathbb{Z}[M]$ -module. Repeat for P_2 .
- (c) Show that $\mathbb{Z}$ has a projective resolution as right $\mathbb{Z}[M]$ -module taking the form

$$0 \longrightarrow \mathbb{Z}\langle P_1 \rangle \oplus \mathbb{Z}\langle P_2 \rangle \longrightarrow \mathbb{Z}[M] \longrightarrow \mathbb{Z}\langle P_1 \rangle \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Use this to prove that $\text{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z}) \cong H_*(S^2)$.

- (d) $\pi_1(BM) \cong M^+ = 1$, so by the Hurewicz theorem the map $\pi_2(BM) \rightarrow H_2(BM)$ is an isomorphism. Verify that if $f: S^2 \rightarrow BM$ is a generator for π_2 then f is a homotopy equivalence.

Exercise 23.24. Here we will give another example of a monoid M such that $BM \simeq S^2$. This monoid will not be finite, but the construction will suggest some generalizations.

Let $L = \langle f \rangle$ be the free monoid on one generator. Let M consists of words in the symbols f , h_1 , and h_2 subject to the relations that $fh_1 = h_1$, $fh_2 = h_2$, $h_1^2 = h_1$, and $h_2^2 = h_2$. Note that M is the pushout $(L \triangleright h_1) *_L (L \triangleright h_2)$ in the category of monoids, so one can think of it as being the monoid L with two cones attached. We claim that $BM \simeq S^2$.

- (a) As a plausibility check, try to compute $H_*(BM)$ algebraically as $\text{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z})$. If we set

$$R = \mathbb{Z}[M] = \mathbb{Z}[f, h_1, h_2] / (fh_1 = h_1, fh_2 = h_2, h_1^2 = h_1, h_2^2 = h_2)$$

then try to show that $\mathbb{Z} = R/(f - 1, h_1 - 1, h_2 - 1)$ has a free resolution that starts out as

$$\cdots \rightarrow R^4 \rightarrow R^3 \rightarrow R \rightarrow \mathbb{Z} \rightarrow 0$$

and find a nonzero class in $\text{Tor}_2^R(\mathbb{Z}, \mathbb{Z})$.

- (b) Here is a better way to compute $H_*(BM) = \text{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z})$. Set $N_i = L \triangleright h_i \subseteq M$. Before diving in, note that we have the inclusion

$$j: BN_1 \cup_{BL} BN_2 \hookrightarrow BM$$

and that the domain of j , being the union of two contractible “cones” over BL , has the homotopy type of $\Sigma(BL) \simeq S^2$. The idea of our computation will be to break the Tor into pieces analogous to the decomposition of S^2 into two hemispheres. Start by proving that M is the disjoint union of left cosets sL where s ranges over the set consisting of 1 and all words in M ending in either h_1 or h_2 . Consequently, $\mathbb{Z}[M]$ is free as a right $\mathbb{Z}[L]$ -module, on the corresponding set of generators. Do something similar to analyze $\mathbb{Z}[M]$ as a right $\mathbb{Z}[N_i]$ module for $i = 1, 2$, in particular showing that it is free.

- (c) Use the generators from the previous part to prove that there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}[M] \otimes_{\mathbb{Z}[L]} \mathbb{Z} \longrightarrow (\mathbb{Z}[M] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}) \oplus (\mathbb{Z}[M] \otimes_{\mathbb{Z}[N_2]} \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Then use the long exact sequence for $\text{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z}, -)$ to prove that $\text{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z}) \cong \text{Tor}_{i-1}^{\mathbb{Z}[L]}(\mathbb{Z}, \mathbb{Z})$ for $i \geq 1$ (use the change-of-ring isomorphisms $\text{Tor}_i^S(M, S \otimes_R N) \cong \text{Tor}_i^R(M, N)$ that hold when $R \rightarrow S$ is flat). Deduce that $H_*(BM) \cong H_*(S^2)$.

- (d) Prove that j is an isomorphism on π_1 and on $H_*(-; \mathbb{Z})$, and so obtain that j is a weak homotopy equivalence.

Exercise 23.24 is a particular case of the analysis done in [McDuff2]. In fact it is not a huge leap to go from the above ideas to seeing how to construct monoids whose classifying spaces achieve any desired homotopy type.

Exercise 23.25. In Exercise 23.23 we saw a 5-element monoid where $M^{+h} \not\cong M^+$. Are there smaller examples of this? If M is a discrete monoid we know that we do have $M^{+h} \simeq M^+$ if M has an absorbing element, or if M is commutative, or (trivially) if M is a group.

Parts (a) and (b) below can be done by hand, but for later parts one should probably use a computer algebra system such as GAP.

- (a) Check that up to isomorphism there are exactly two monoids having two elements. Verify that both have $M^{+h} \simeq M^+$.
- (b) Prove that up to isomorphism there are exactly seven monoids with three elements, and construct them all. Actually, for our purposes it is useful to work up to both isomorphism and anti-isomorphism, in which case there are only six classes. Of these six, check that four have absorbing elements and the remaining two are commutative (one has $M^+ = \mathbb{Z}/3$ and one has $M^+ = \mathbb{Z}/2$).
- (c) There are 35 monoids with four elements, or 27 up to isomorphism and anti-isomorphism. Verify that 19 of these have absorbing elements, two are groups, and all of the remaining six are commutative (five have $M^+ = \mathbb{Z}/2$ and one has $M^+ = \mathbb{Z}/3$).
- (d) For monoids with five elements, there are now 228 of these (see [OEIS, A058129]), or only 156 if we work up to anti-isomorphism. So the level of complexity has dramatically increased. 126 of them have absorbing elements, and of the remaining 30 only 5 are noncommutative. If you are brave, try to determine how many of these five have $M^{+h} \simeq M^+$. (Of these five, four have $M^+ = \mathbb{Z}/2$ and one has $M^+ = 1$. The latter is the monoid from Exercise 23.23).

REFERENCES

- [AM] A. Abbondandolo and P. Majer, *Infinite dimensional Grassmannians*, J. Operator Theory **61** (2009), no. 1, 19–62.
- [Adams1] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104.
- [Adams2] J. F. Adams, *Vector fields on spheres*, Annals of Math. **75**, no. 3 (1962), 603–632.
- [Adams3] J. F. Adams, *On the groups $J(X)$ I–IV*, Topology **2** (1963), 181–195; Topology **3** (1965), 137–171; Topology **3** (1965), 193–222; Topology **5** (1966), 21–71.
- [Adams4] J. F. Adams, *Stable homotopy theory*, 3rd edition, Lecture Notes in Math., No. 3, Springer-Verlag, Berlin-New York, 1969.
- [Adams5] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995 (reprint of the 1974 original).
- [Adams6] J. F. Adams, *Geometric dimension of bundles over $\mathbb{R}P^n$* , Proceedings of the International Conference on Prospects in Mathematics (Taniguchi Internat. Sympos., Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1973), pp. 1–17. Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1974.
- [AA] J. F. Adams and M. Atiyah, *K-theory and the Hopf invariant*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 31–38.
- [AGJ] J. Adem, S. Gitler, and I. M. James, *On axial maps of a certain type*, Bol. Soc. Mat. Mexicana (2) **17** (1972), 59–62.
- [AP] M. A. Aguilar and C. Prieto, *Quasifibrations and Bott periodicity*, II Iberoamerican Conference on Topology and its Applications (Morelia, 1997), Topology Appl. **98** (1999), no. 1–3, 3–17.
- [Arnold1] V. I. Arnold, *On a characteristic class entering into conditions of quantization*, Funkcional. Anal. i Priložen. **1** (1967), 1–14.
- [Arnold2] V. I. Arnold, *The complex Lagrangian Grassmannian*, Funkcional. Anal. i Priložen. **34** (2000), no. 3, 63–65; translation in Funct. Anal. Appl. **34** (2000), no. 3, 208–210.
- [Astey] L. Astey, *Geometric dimension of bundles over real projective spaces*, Quart J. Math. Oxford Ser. (2) **31** (1980), no. 122, 139–155.
- [Atiyah1] M. F. Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 23–64.
- [Atiyah2] M. F. Atiyah, *Thom complexes*, Proc. Lond. Math. Soc. **11** (1961), 291–310.
- [Atiyah3] M. F. Atiyah, *Immersion and embeddings of manifolds*, Topology **1** (1962), 125–132.

- [Atiyah4] M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386.
- [Atiyah5] M. F. Atiyah, *K-theory*, W.A. Benjamin, Inc., New York-Amsterdam, 1967.
- [Atiyah6] M. F. Atiyah, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 113–140.
- [AB] M. F. Atiyah and R. Bott, *On the periodicity theorem for complex bundles*, Acta Math. **112** (1964), 229–247.
- [ABS] M. F. Atiyah, R. Bott, A. Shapiro, *Clifford modules*, Topology **3** (1964), 3–38.
- [AH1] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, pp. 7–38. Amer. Math. Soc., Providence, RI, 1961.
- [AH2] M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1961), 25–45.
- [AH3] M. F. Atiyah and F. Hirzebruch, *The Riemann-Roch theorem for analytic embeddings*, Topology **1** (1962), 151–166.
- [ASe] M. F. Atiyah and G. Segal, *Equivariant K-theory and completion*, J. Differential Geom. **3** (1969), 1–18.
- [AS1] M. F. Atiyah and I. M. Singer, *The index of elliptic operators I*, Ann. of Math. (2) **87** (1968), 484–530.
- [Badzioch1] B. Badzioch, *Algebraic theories in homotopy theory*, Ann. of Math. (2) **155** (2002), no. 3, 895–913.
- [Badzioch2] B. Badzioch, *From Γ -spaces to algebraic theories*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 1779–1799.
- [BCV] B. Badzioch, K. Chung, and A. A. Voronov, *The canonical delooping machine*, J. Pure Appl. Algebra **208** (2007), no. 2, 531–540.
- [Baez] J. C. Baez, *The tenfold way*, Notices Amer. Math. Soc. **67** (2020), no. 10, 1599–1601.
- [BCC] E. Ballico, F. Catanese, C. Ciliberto (eds.), *Classification of irregular varieties. Proceedings of the conference held in Trento, December 17–21, 1990*, Trento Examples (pp. 134–139), Lecture Notes in Math. **1515**, Springer-Verlag, Berlin, 1992.
- [BP] M. G. Barratt and G. F. Paechter, *A note on $\pi_r(V_{n,m})$* , Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 119–121.
- [BPr] M. G. Barratt and S. Priddy, *On the homology of non-connected monoids and their associated groups*, Comment. Math. Helv. **47** (1972), 1–14.
- [Bass1] H. Bass, *Algebraic K-theory*, W.A. Benjamin, Inc., New York, NY, 1968.
- [Bass2] H. Bass, *Some problems in “classical” algebraic K-theory*, Algebraic K-theory II: “Classical” algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Institute, Seattle, Washington, 1972), pp. 3–73, Lecture Notes in Math. **342**, Springer, Berlin, 1973.
- [BFM1] P. Baum, W. Fulton, and R. MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math., No. 45 (1975), 101–145.
- [BFM2] P. Baum, W. Fulton, and R. MacPherson, *Riemann-Roch and topological K-theory for singular varieties*, Acta Math. **143** (1979), no. 3–4, 155–192.
- [Beck] J. Beck, *On H-spaces and infinite loop spaces*, in *Category Theory, Homology Theory, and their Applications III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pp. 139–153, Lecture Notes in Math. **99**, Springer-Verlag, Berlin-New York, 1969.
- [BF] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88.
- [Behrens1] M. Behrens, *A new proof of the Bott periodicity theorem*, Topology Appl. **119** (2002), no. 2, 167–183.
- [Behrens2] M. Behrens, *Addendum to “A new proof of the Bott periodicity theorem”*, Topology Appl. **143** (2004), no. 1–3, 281–290.
- [BDM] M. Bendersky, D. Davis, and M. Mahowald, *Stable geometric dimension of vector bundles over even-dimensional real projective spaces*, Trans. Amer. Math. Soc. **358** (2006), no. 4, 1585–1603.
- [BO] O. Benoist and J. C. Ottem, *Failure of the integral Hodge conjecture for threefolds of Kodaira dimension zero*, Comment. Math. Helv. **95** (2020), no. 1, 27–35.

- [BGH] A. J. Blumberg, T. Gerhardt, M. A. Hill (editors), *Stable categories and structured ring spectra*, Math. Sci. Res. Inst. Publ., **69**, Cambridge University Press, Cambridge, 2022.
- [BGT] A. J. Blumberg, D. Gepner, G. Tabuada, *A universal characterization of higher algebraic K -theory*, *Geom. Topol.* **17** (2013), no. 2, 733–838.
- [BV1] J. M. Boardman and R. M. Vogt, *Homotopy everything H -spaces*, *Bull. Amer. Math. Soc.* **74** (1968), 1117–1122.
- [BV2] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, *Lecture Notes in Math.* **347**, Springer-Verlag, Berlin-New York, 1973.
- [BS] A. Borel and J. P. Serre, *Le théorème de Riemann-Roch*, *Bull. Soc. Math. France* **86** (1958), 97–136.
- [Borsuk] K. Borsuk, *Theory of retracts*, *Monogr. Mat.*, Tom. 44, PWN – Polish Scientific Publishers, Warsaw, 1966.
- [Bott1] R. Bott, *The stable homotopy of the classical groups*, *Annals of Math.* **70** (1959), no. 2, 313–337.
- [Bott2] R. Bott, *Quelques remarques sur les théorèmes de périodicité*, *Bull. Soc. Math. France* **87** (1959), 293–310.
- [BS] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, *Amer. J. Math.* **80** (1958), 964–1029.
- [Bredon1] G. E. Bredon, *Sheaf theory*, *Grad. Texts in Math.* **170**, Springer-Verlag, New York, 1997.
- [Bredon2] G. E. Bredon, *Topology and geometry*, *Grad. Texts in Math.* **139**, Springer-Verlag, New York, 1997.
- [BG] W. Bruns and J. Gubeladze, *Polytopes, rings, and K -theory*, *Springer Monogr. Math.*, Springer, Dordrecht, 2009.
- [BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, second edition, *Cambridge Studies in Advanced Math.* **39**, Cambridge University Press, Cambridge, UK, 1998.
- [BS] J. Bryan and M. Sanders, *Instantons on S^4 and $\mathbb{CP}^2$, rank stabilization, and Bott periodicity*, *Topology* **39** (2000), no. 2, 331–352.
- [Buchsbaum] D. Buchsbaum, *Exact categories and duality*, *Trans. Amer. Math. Soc.* **80** (1955), no. 1, 1–34.
- [BH] S. Buoncrisiano and D. Hacon, *On the formal groups approach to cobordism*, *Proc. Amer. Math. Soc.* **97** (1986), no. 3, 559–564.
- [Cairns] S. S. Cairns, *Triangulation of the manifold of class one*, *Bull. Amer. Math. Soc.* **41** (8), 549–552.
- [CM] H. Cartan and J. C. Moore, *Périodicité des groupes d'homotopie stables des groupes classiques, d'après Bott*, *Séminaire Henri Cartan-John C. Moore*, E.N.S. 12e année, Fasc. 1 & 2, Paris, 1959/1960.
- [CR] L. M. Chaves and A. Rigas, *Complex reflections and polynomial generators for homotopy groups*, *J. Lie Theory* **6** (1996), no. 1, 19–22.
- [CN] B. Chen and T. Nagano, *A Riemannian geometric invariant and its application to a problem of Borel and Serre*, *Trans. Amer. Math. Soc.* **308** (1988), no. 1, 273–297.
- [Christensen] O. Christensen, *An introduction to frames and Riesz bases*, second edition. *Appl. Numer. Harmon. Anal.*, Birkhäuser/Springer, 2016.
- [CohenI] I. Cohen, *On the structure and ideal theory of complete local rings*, *Trans. Amer. Math. Soc.* **59** (1946), 54–106.
- [CohenR] R. Cohen, *The immersion conjecture for differentiable manifolds*, *Ann. of Math.* (2) **122** (1985), no. 2, 237–328.
- [Conner] P. E. Conner, *Differentiable periodic maps*, second edition, *Lecture Notes in Math.*, Springer, Berlin, 1979.
- [CF] P. E. Conner and E. E. Floyd, *The relation of cobordism to K -theories*, *Lecture Notes in Math.* **28**, Springer-Verlag, Berlin-New York, 1966.
- [Conway] J. B. Conway, *A course in functional analysis*, second edition, *Grad. Texts in Math.* **96**, Springer-Verlag, New York, 1990.

- [Davis1] D. Davis, *The mathematical work of Sam Gitler 1960–2003*, Recent developments in algebraic topology 1–9, Contemporary Math. **407**, Amer. Math. Soc., Providence, RI, 2006.
- [Davis2] D. Davis, *Table of immersions and embeddings of real projective spaces*, <http://www.lehigh.edu/~dmd1/immtable>.
- [DGM] D. Davis, S. Gitler, and M. Mahowald, *The stable geometric dimension of vector bundles over real projective spaces*, Trans. Amer. Math. Soc. **268** (1981), no. 1, 39–61.
- [Deligne] P. Deligne, *Notes on spinors*. Quantum fields and strings: a course for mathematicians, Vol. 1–2 (Princeton, NJ, 1996/1997), 99–135. American Mathematical Society, Providence, RI, 1999.
- [DM] P. Deligne and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)* Quantum fields and strings: a course for mathematicians, Vol. 1–2 (Princeton, NJ, 1996/1997), 41–97. American Mathematical Society, Providence, RI, 1999.
- [Dold] A. Dold, *K-theory of non-additive functors of finite degree*, Math. Ann. **196** (1972), 177–197.
- [DT] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische produkte*, Ann. of Math. (2) **67** (1958), 239–281.
- [D1] D. Dugger, *Multiplicative structures on homotopy spectral sequences I*, 2003 preprint, available at arXiv:math/0305173v1.
- [D2] D. Dugger, *Multiplicative structures on homotopy spectral sequences II*, 2003 preprint, available at arXiv:math/0305187v1.
- [D3] D. Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*, K-theory **35** (2005), no. 3-4, 213–256.
- [D4] D. Dugger, *Grothendieck groups of complexes with null homotopies*, J. K-theory **13** (2014), no. 3, 17–531.
- [D4.5] D. Dugger, *Stable categories and spectra via model categories*, in *Stable categories and structured ring spectra*, 75–150. Math. Sci. Res. Inst. Publ., 69 Cambridge University Press, Cambridge, 2022
- [D5] D. Dugger, *A primer on homotopy colimits*, draft, available at <http://pages.uoregon.edu/ddugger>.
- [D6] D. Dugger, *Another book on K-theory*, draft, available at <http://pages.uoregon.edu/ddugger>.
- [DI1] D. Dugger and D. C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$ -realizations*, Math. Z. **246** (2004), 667–689.
- [DI2] D. Dugger and D. C. Isaksen, *Weak equivalences for simplicial presheaves*, in *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, pp. 97–113, Contemp. Math. **346**, American Mathematical Society, Providence, RI 2004.
- [DI3] D. Dugger and D. C. Isaksen, *Motivic Hopf elements and relations*, New York J. Math. **19** (2013), 823–871.
- [Dugundji] J. Dugundji, *Absolute neighborhood retracts and local connectedness in arbitrary metric spaces*, Compositio Math. **13** (1958), 229–246.
- [DZ] W. G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, in *Algebraic topology, Barcelona, 1986*, 106–119. Lecture Notes in Math. **1298**, Springer-Verlag, Berlin, 1987.
- [DS] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of Algebraic Topology, 73–126, North-Holland, Amsterdam, 1995.
- [DL] E. Dyer and R. Lashof, *A topological proof of the Bott periodicity theorems*, Ann. Mat. Pura Appl. (4) **54** (1961), 231–254.
- [Eagon] J. A. Eagon, *The Grothendieck group of finitely-generated modules*, Proc. Amer. Math. Soc. **19** (1968), 1341–1345.
- [EN] J. A. Eagon and D. G. Northcott, *Ideals defined by matrices and a certain complex associated to them*, Proc. Roy. Soc. London Ser. A **269** (1962), 188–204.
- [ERW] J. Ebert and O. Randal-Williams, *Semisimplicial spaces*, Algebr. Geom. Topol. **19** (2019), no. 4, 2099–2150.

- [Eckmann1] B. Eckmann, *Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen*, Comment. Math. Helv. **15** (1942/43), 358–366.
- [Eckmann2] B. Eckmann, *Hurwitz-Radon matrices and peridocity modulo 8*, Enseign. Math. (2) **35** (1989), no. 1–2, 77–91.
- [Eckmann3] B. Eckmann, *Hurwitz-Radon matrices revisited: from effective solutions of the Hurwitz matrix equations to Bott periodicity*, CRM Proc. Lecture Notes **6**, American Mathematical Society, Providence, RI, 1994, 23–35.
- [Eisenbud] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Springer-Verlag, New York, 1995.
- [EH] D. Eisenbud and J. Harris, *The geometry of schemes*, Graduate Texts in Math. **197**, Springer-Verlag, New York, 2000.
- [EM] A. D. Elmendorf and M. A. Mandell, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. **205** (2006), no. 1, 163–228.
- [ES] J. Eschenburg and B. Hanke, *Bott periodicity, submanifolds, and vector bundles*, in *Hermitian-Grassmannian submanifolds*, 295–309. Springer Proc. Math. Stat. **203**, Springer, Singapore, 2017.
- [Farber] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), no. 2, 211–221.
- [FTY] M. Farber, S. Tabachnikov, S. Yuzvinsky, *Topological robotics: motion planning in projective spaces*, Int. Math. Res. Not. **34** (2003), 1853–1870.
- [F] Z. Fiedorowicz, *A counterexample to a group completion conjecture of J C Moore*, Algebr. Geom. Topol. **2** (2002), 33–35.
- [FR1] J. Fogarty and D. S. Rim, *Serre sequences and Chern classes*, Bull. Amer. Math. Soc. **74** (1968), 919–921.
- [FR2] J. Fogarty and D. S. Rim, *Serre sequences and Chern classes*, J. Algebra **10** (1968), 436–447.
- [Fomenko] A. T. Fomenko, *Variational principles in topology: multi-dimensional minimal surface theory*, translated from the Russian by O. Efimov, Math. Appl. (Soviet Ser.) **42**, Kluwer Academic Publishers Group, Dordrecht, 1990.
- [FH] H.-B. Foxby and E. B. Halvorsen, *Grothendieck groups for categories of complexes*, J. K-theory **3** (2009), no. 1, 165–203.
- [FLS] V. Franjou, J. Lannes, and L. Schwartz, *Autour de la cohomologie de Mac Lane des corps finis*, Invent. Math. **115** (1994), no. 3, 513–538.
- [FM] E. M. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties*, Mem. Amer. Math. Soc. **110** (1994), no. 529, x+110pp.
- [FP] R. Fritsch and R. A. Piccinini, *Cellular structures in topology*, Cambridge Stud. Adv. Math. **19**, Cambridge University Press, Cambridge, 1990.
- [Fulton] W. Fulton, *Intersection theory*, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 2, Springer-Verlag, Berlin-Heidelberg, 1998.
- [Gabriel] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
- [GR] H. Garland and M. S. Raghunathan, *A Bruhat decomposition for the loop space of a compact group: a new approach to results of Bott*, Proc. Nat. Acad. Sci. USA **72** (1975), 4716–4717.
- [Geoghegan] R. Geoghegan, *Topological methods in group theory*, Grad. Texts. in Math. **243**, Springer, New York, 2008.
- [Gersten] S. M. Gersten, *The localization theorem for projective modules*, Comm. Algebra **2** (1974), 317–350.
- [GS] H. Gillet and C. Soulé, *Intersection theory using Adams operations*, Invent. Math. **90** (1987), no. 2, 243–277.
- [Giffen] C. H. Giffen, *Bott periodicity and the Q-construction*, in Algebraic K-theory (Poznań, 1995), 107–124. Contemp. Math. **199**, American Mathematical Society, Providence, RI, 1996.
- [Ginsburg] M. Ginsburg, *Some immersions of projective space in Euclidean space*, Topology **2** (1963), 69–71. T
- [GM1] S. Gitler and M. Mahowald, *The geometric dimension of real stable vector bundles*, Bol. Soc. Mat. Mexicana (2) **11** (1966), 85–107.

- [GM2] S. Gitler and M. Mahowald, *The immersion of manifolds*, Bull. Amer. Math. Soc. **73** (1967), 696–700.
- [Gray] B. Gray, *Homotopy Theory: An Introduction to Algebraic Topology*, Pure Appl. Math. **64**, Academic Press, New York-London, 1975.
- [Grayson1] D. Grayson, *Higher algebraic K-theory. II (after Daniel Quillen)*, Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976), pp. 217–240. Lecture Notes in Math., Vol. 551, Springer-Verlag, Berlin-New York, 1976.
- [Grayson2] D. Grayson, *SK_1 of an interesting principal ideal domain*, J. Pure Appl. Algebra **20** (1981), 157–163.
- [Grayson3] D. Grayson, *Adams operations on higher K-theory*, K-theory **6** (1992), no. 2, 97–111.
- [GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.
- [Grothen1] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. (2) **9** (1957), 119–221.
- [Grothen2] A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154.
- [GP] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.
- [Gupta1] N. Gupta, *On Zariski's cancellation problem in positive characteristic*, Adv. Math. **264** (2014), 296–307.
- [Gupta2] N. Gupta, *A survey on Zariski cancellation problem*, Indian J. Pure Appl. Math. **46**(2015), no. 6, 865–877.
- [HH] A. Haeffliger and M. W. Hirsch, *Immersions in the stable range*, Annals Math. **75** (1962), no. 2, 231–241.
- [Hanner] O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. **1** (1951), 389–408.
- [Harris] B. Harris, *Bott periodicity via simplicial spaces*, J. Algebra **62** (1980), no. 2, 450–454.
- [Hartshorne] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York-Heidelberg, 1977.
- [Hatcher1] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
- [Hatcher2] A. Hatcher, *Vector bundles and K-theory*, preprint. Available on the webpage <http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html>
- [Hazewinkel] M. Hazewinkel, *Formal groups and applications*, Academic Press, New York, 1978.
- [HS] D. Heard and V. Stojanoska, *K-theory, reality, and duality*, J. K-Theory **14** (2014), no. 3, 526–555.
- [Helgason] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Grad. Stud. Math. **34**, American Mathematical Society, Providence, RI, 2001.
- [Heller] A. Heller, *Some exact sequences in algebraic K-theory*, Topology **3** (1965), no. 4, 389–408.
- [Hironaka1] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; Ann. of Math. (2) **79** (1964), 205–326.
- [Hironaka2] H. Hironaka, *Triangulations of algebraic sets*, Algebraic Geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 165–185. Amer. Math. Soc., Providence, RI, 1975.
- [Hirsch1] M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), no. 2, 242–276.
- [Hirsch2] M. W. Hirsch, *Differentiable topology*, Graduate Texts in Mathematics **33**, Springer-Verlag, New York, 1976.
- [Hirschhorn] P.S. Hirschhorn, *Model categories and their localizations*, Math. Surveys Monogr. **99**, American Mathematical Society, Providence, RI, 2003.
- [Hochster1] M. Hochster, *Topics in the homological theory of modules over commutative rings*. Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974. CBMS Regional Conference Series in Mathematics, No. 24. Published for the Conference Board of the Mathematical

- Sciences, Washington, DC, by American Mathematical Society, Providence, RI, 1975.
- [Hochster2] M. Hochster, *Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors*, in *Conference on Commutative Algebra-1975 (Queen's Univ., Kingston, Ont., 1975)*, pp. 106–195, Queen's University, Kingston, ON, 1975.
- [Hochster3] M. Hochster, *Non-uniqueness of coefficient rings in a polynomial ring*, Proc. Amer. Math. Soc. **34** (1972), no. 1, 81–82.
- [Hochster4] M. Hochster, *Euler characteristics over unramified regular local rings*, Illinois J. Math. **28** (1984), no. 2, 281–285.
- [Hochster5] M. Hochster, *Lecture notes from a course on intersection multiplicities, 1990*, unpublished lecture notes.
- [Hochster6] M. Hochster, *Nonnegativity of intersection multiplicities in ramified regular local rings following Gabber/de Jong/Berthelot*, unpublished preprint, 1997.
- [HST] H. Hohnhold, S. Stolz, P. Teichner, *From minimal geodesics to supersymmetric field theories*. A celebration of the mathematical legacy of Raoul Bott, 207–274. CRM Proc. Lecture Notes., 50. Amer. Math. Soc., Providence, RI, 2010.
- [Hopf] H. Hopf, *Ein topologischer Beitrag zur reellen Algebra*, Comment. Math. Helv. **13** (1940–41), 219–239.
- [Hovey] M. Hovey, *Model categories*, Math. Surveys Monogr. **63**, American Mathematical Society, Providence, RI, 1999.
- [Hu] S. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965.
- [Hurwitz] A. Hurwitz, *Über die Komposition der quadratischen Formen*, Math. Ann. **88** (1922), no. 1–2, 1–25.
- [Husemoller] D. Husemoller, *Fibre bundles*, 3rd edition, Grad. Texts in Math. **20**, Springer-Verlag, New York, 1994.
- [Ischebeck] F. Ischebeck, *Hauptidealringe mit nichttrivialer SK_1 -Gruppe*, Arch. Math. (Basel) **35** (1980), no. 1–2, 138–139.
- [Iversen] B. Iversen, *Local Chern classes*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 155–169.
- [IM] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math., Vol. 25 (1965), 5–48.
- [JO] S. Jackowski and B. Oliver, *Vector bundles over classifying spaces of compact Lie groups*, Acta Math. **176** (1996), 109–143.
- [James1] I. M. James, *The space of bundle maps*, Topology **2** (1963), 45–59.
- [James2] I. M. James, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc. **69** (1963), 231–238.
- [Jänich] K. Jänich, *Vektorraumbündel und der Raum der Fredholm-Operatoren*, Math. Ann. **161** (1965), 129–142.
- [Joachim] M. Joachim, *A symmetric ring spectrum representing KO -theory*, Topology **40** (2001), no. 2, 299–308.
- [Joyce1] D. Joyce, *An introduction to d -manifolds and derived differential geometry*, in *Moduli spaces*, pp. 230–281, London Math. Soc. Lecture Note Ser., **411**, Cambridge University Press, Cambridge, 2014.
- [Joyce2] D. Joyce, *Algebraic geometry over C^∞ -rings*, Mem. Amer. Math. Soc. **260** (2019), no. 1256.
- [Karoubi] M. Karoubi, *K -theory: an introduction*, Classics Math., Springer-Verlag, Berlin, 2008.
- [KK] S. Kakutani and V. Klee, *The finite topology of a linear space*, Arch. Math. (Basel) **14** (1963), 55–58.
- [KT] A. Kono and K. Tokunaga, *A topological proof of Bott periodicity theorem and a characterization of BU* , J. Math. Kyoto Univ. **34** (1994), no. 4, 873–880.
- [Kosinski] A. A. Kosinski, *Differential manifolds*, Pure Appl. Math. **138**, Academic Press, Inc., Boston, MA, 1993.
- [Kuiper] N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology **3** (1965), 19–30.

- [Lam1] K. Y. Lam, *Construction of nonsingular bilinear maps*, Topology **6** (1967), 423–426.
- [Lam2] K. Y. Lam, *Topological methods for studying the composition of quadratic forms*, Quadratic and Hermitian Forms, (Hamilton, Ont., 1983), pp 173–192, Canadian Mathematical Society Conference Proceedings **4**, Amer. Math. Soc., 1984.
- [LR1] K. Y. Lam and D. Randall, *Geometric dimension of bundles on real projective spaces*, Homotopy theory and its applications (Cocoyoc, 1993), 137–160. Contemporary Math. **188**, Amer. Math. Soc., Providence, RI, 1995.
- [LR2] K. Y. Lam and D. Randall, *Vector bundles of low geometric dimension over real projective spaces*, Math. Proc. Cambridge Philos. Soc. **139** (2005), no. 2, 229–241.
- [LS] T. Y. Lam and T. Smith, *On the Clifford-Littlewood-Eckmann groups: a new look at periodicity mod 8*, Rocky Mount. J. Math. **19** (1989), no. 3, 749–783.
- [Lamotke] K. Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Topology **20** (1981), no. 1, 15–51.
- [LV] H. Larson and R. Vakil, *Complex Bott periodicity in algebraic geometry*, 2024 preprint, arXiv:2411.09122.
- [Latour] F. Latour, *Transversales lagrangiennes, périodicité de Bott et formes génératrices par une immersion lagrangienne dans un cotangent*, Ann. Scient. Éc. Norm. Sup. (4) **24** (1991), no. 1, 3–55.
- [Lee] J. M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics **218**, Springer-Verlag New York, 2003.
- [Lenstra] H. W. Lenstra Jr., *Grothendieck groups of abelian group rings*, J. Pure Appl. Algebra **20** (1981), 173–193.
- [Lichtenbaum] S. Lichtenbaum, *On the vanishing of Tor in regular local rings*, Illinois J. Math. **10** (1966), 220–226.
- [Loday] J. L. Loday, *Introduction to algebraic K-theory and cyclic homology*, Higher algebraic K-theory: an overview, pp. 31–54, Lecture Notes in Math. **1491**, Springer-Verlag, Berlin-Heidelberg, 1992.
- [Lundell] A. T. Lundell, *A Bott map for non-stable homotopy of the unitary group*, Topology **8** (1969), 209–217.
- [LW] A. T. Lundell and S. Weingram, *The topology of CW-complexes*, Univ. Ser. Higher Math., van Nostrand Reinhold Co., New York, 1969.
- [Lurie] J. Lurie, *Higher algebra*, 2017 preprint, <http://www.math.ias.edu/~lurie>.
- [Mac Lane] S. Mac Lane, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics **5**, Springer-Verlag New York, 1998.
- [MarSe] S. Mardešić and J. Segal, *Shape Theory. The Inverse System Approach*, North-Holland Publishing Company, Amsterdam, 1982.
- [Markert] E. K. Markert, *Field theory configuration spaces for connective ko-theory*, Algebr. Geom. Topol. **10** (2010), no. 2, 1187–1219.
- [May1] J. P. May, *Geometry of iterated loop spaces*, Lecture Notes in Math. **271**, Springer-Verlag, Berlin-New York, 1972.
- [May2] J. P. May, *E_∞ -spaces, group completions, and permutative categories*, in *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pp. 61–93. London Math. Soc. Lecture Note Ser., No. 11, Cambridge University Press, London-New York, 1974.
- [May3] J. P. May, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc., **1** (1975), no. 155. American Mathematical Society, Providence, RI, 1975.
- [May4] J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, with contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. Lecture Notes in Math., Vol. 577, Springer-Verlag, Berlin-New York, 1977.
- [May5] J. P. May, *Multiplicative infinite loop space theory*, J. Pure Appl. Algebra **26** (1982), no. 1, 1–69.
- [McCleary] J. McCleary, *A user's guide to spectral sequences*, Second edition, Cambridge Studies in Advanced Mathematics **58**, Cambridge University Press, Cambridge, 2001.
- [McDuff1] D. McDuff, *Configuration spaces, K-theory and operator algebras* (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 88–95. Lecture Notes in Math. **575**, Springer, Berlin, 1977.

- [McDuff2] D. McDuff, *On the classifying spaces of discrete monoids*, Topology **18** (1979), no. 4, 313–320.
- [MS] D. McDuff and G. Segal, *Homology fibrations and the “group-completion” theorem*, Invent. Math. **31** (1975/76), no. 3, 79–284.
- [Michael] E. Michael, *Continuous selections I*, Ann. of Math. (2) **63** (1956), 361–382.
- [Milgram] J. Milgram, *Immersing projective spaces*, Ann. of Math. (2) **85** (1967), 473–482.
- [Miller] H. Miller, *Course Notes: Homotopy theory of the vector field problem*, available at <http://www-math.mit.edu/~hrm/papers/vf1.pdf>.
- [MiT] L. M. Milne-Thomson, *The calculus of finite differences*, MacMillan and Co., Limited, London, 1933.
- [Milnor1] J. Milnor, *Construction of universal bundles I*, Ann. of Math. (2) **63** (1956), 272–284.
- [Milnor2] J. Milnor, *Construction of universal bundles II*, Ann. of Math. (2) **63** (1956), 430–436.
- [Milnor3] J. Milnor, *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. **90** (1959), no. 2, 272–280.
- [Milnor4] J. Milnor, *On the cobordism ring Ω^* and a complex analog, Part 1*, Amer. J. Math. **82** (1960), no. 3, 505–521.
- [Milnor5] J. Milnor, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville, VA, 1965.
- [Milnor6] J. Milnor, *Morse Theory*, Annals of Mathematics Studies **51**, Princeton University Press, Princeton, NJ, 1969.
- [Milnor7] J. Milnor, *Algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton University Press, Princeton, NJ, 1971.
- [MS] J. Milnor and J. Stasheff, *Characteristic classes*, Princeton University Press, 1974.
- [Mitchell1] S. A. Mitchell, *A filtration of the loops on $SU(N)$ by Schubert varieties*, Math. Z. **193** (1986), 347–362.
- [Mitchell2] S. A. Mitchell, *The Bott filtration of a loop group*, in Algebraic topology, Barcelona, 1986, 215–226. Lecture Notes in Math. **1298**, Springer-Verlag, Berlin, 1987.
- [Mitchell3] S. A. Mitchell, *Quillen’s theorem on buildings and the loops on a symmetric space*, Enseign. Math. (2) **34** (1988), no. 1-2, 123–166.
- [Mitchell4] S. A. Mitchell, *On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic viewpoint*, in Algebraic topology and its applications, 163–240. Math. Sci. Res. Inst. Publ., **27**, Springer-Verlag, New York, 1994.
- [MT] R.E. Mosher and M.C. Tangora, *Cohomology operations and applications in homotopy theory*, Dover Books on Mathematics, 2008.
- [Mumford] D. Mumford, *The red book of schemes and varieties, second expanded edition*, Lect. Notes in Math. **1358**, Springer-Verlag, Berlin, 1999.
- [Munkres] J. Munkres, *Topology*, second edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.
- [Novikov] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951; English translation in USSR-Izv. 1 (1967).
- [OEIS] On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc. (2026), published electronically at <https://oeis.org>.
- [Palais] R. S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. **73** (1961), no. 2, 295–323.
- [PS1] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass, and A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119.
- [PS2] C. Peskine and L. Szpiro, *Szygies et multiplicites*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 1421–1424.
- [Pedersen] G. K. Pedersen, *Analysis now*, Grad. Texts in Math. **118**, Springer-Verlag, New York, 1989.
- [Peterson] F. P. Peterson, *Some non-embedding problems*, Bol. Soc. Mat. Mexicana (2) **2** (1957), 9–15.

- [PW] G. Piepmeyer and M. E. Walker, *A new proof of the new intersection theorem*, J. Algebra **322** (2009), no. 9, 3366–3372.
- [Pressley] A. N. Pressley, *Decompositions of the space of loops on a Lie group*, Topology **19** (1980), 65–79.
- [PrSe] A. Pressley and G. Segal, *Loop groups*, Oxford Math. Monogr., Oxford Sci. Publ., The Clarendon Press, Oxford University Press, New York, 1986.
- [PR] T. Püttmann and A. Rigas, *Presentations of the first homotopy groups of the unitary groups*, Comment. Math. Helv. **78** (2003), no. 3, 648–662.
- [Quast] P. Quast, *Centrioles in symmetric spaces*, Nagoya Math. J. **211** (2013), 51–77.
- [Quillen1] D. Quillen, *On the formal group laws of unoriented and complex bordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.
- [Quillen2] D. Quillen, *Cohomology of groups*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 47–51, Gauthier-Villars Éditeur, Paris, 1971.
- [Quillen3] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56.
- [Quillen4] D. Quillen, *On the cohomology and K-theory of general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586.
- [Quillen5] D. Quillen, *Higher algebraic K-theory: I*, in Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147, Lecture Notes in Mathematics **341**, Springer-Verlag, Berlin-New York, 1973.
- [Quillen6] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.
- [Quillen7] D. Quillen *On the group completion of a simplicial monoid*, appearing as an appendix in [FM].
- [Ray1] N. Ray, *Extensions of umbral calculus: penumbral coalgebras and generalised Bernoulli numbers*, Adv. in Math. **61** (1986), no. 1, 49–100.
- [Ray2] N. Ray, *Symbolic calculus: a 19th century approach to MU and BP*, Homotopy theory (Durham, 1985), 195–238. London Math. Soc. Lecture Note Ser. **177**, Cambridge Univ. Press, Cambridge, 1987.
- [Roberts1] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 103–106.
- [Roberts2] P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. **13** (1985), no. 2, 127–130.
- [Roberts3] P. Roberts, *Le théorème d’intersection*, C.R. Acad. Sci. Paris Ser. I Math. **304** (1987), no. 7, 177–180.
- [Roberts4] P. Roberts, *Intersection theorems*, in Commutative Algebra (Berkeley, CA, 1987), 417–436. Math. Sci. Res. Inst. Publ., **15**, Springer-Verlag, New York, 1989.
- [Roberts5] P. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics **133**, Cambridge University Press, Cambridge, 1998.
- [Roberts6] P. Roberts, *Recent developments on Serre’s multiplicity conjectures: Gabber’s proof of the nonnegativity conjecture*, Enseign. Math. (2) **44** (1998), no. 3-4, 305–324.
- [Rosenberg] J. Rosenberg, *Algebraic K-theory and its Applications*, Graduate Texts in Mathematics **147**, Springer-Verlag, New York, 1994.
- [Rudyak] Y. B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer Monogr. Math., Springer-Verlag, Berlin, 1998.
- [Sakai] K. Sakai, *Geometric aspects of general topology*, second edition, Springer Asia Pac. Math. Ser., **2**, Springer, Singapore, 2026.
- [Samuel] P. Samuel, *Algèbre locale*, Mémor. Sci. Math., no. 123, Gauthier-Villars, Paris, 1953, 76pp.
- [Sanderson1] B.J. Sanderson, *A non-immersion theorem for real projective spaces*, Topology **2** (1963), 209–211.
- [Sanderson2] B.J. Sanderson, *Immersion and embeddings of projective spaces*, Proc. London. Math. Soc. (3) **14** (1964), 137–153.
- [Schoof] R. Schoof, *Cohomology of class groups of cyclotomic fields: an application to Morse-Smale diffeomorphisms*, J. Pure. Appl. Algebra **53** (1988), no. 1-2, 125–137.

- [Segal1] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math., No. 34 (1968), 105–112.
- [Segal2] G. Segal, *The representation ring of a compact Lie group*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 113–128.
- [Segal3] G. Segal, *Fredholm complexes*, Quart. J. Math. Oxford Ser. (2) **21** (1970), 385–402.
- [Segal4] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
- [SW] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. **61** (1985), 5–65.
- [Serre1] J. P. Serre, *Sur la topologie des variétés algébriques en caractéristique p* , Symposium Internacional de Topologia Algebraica (Mexico, 1958), pp. 24–53, Universidad Nacional Autónoma de México and UNESCO, México, 1958.
- [Serre2] J. P. Serre, *Algèbre locale. Multiplicités*, Lecture Notes in Math. **11**, Springer-Verlag, Berlin-New York, 1965.
- [Shafarevich1] I. R. Shafarevich, *Basic algebraic geometry 1, Varieties in projective space, third edition*, Springer-Verlag Berlin-Heidelberg 1977, 1994, 2013.
- [Shafarevich2] I. R. Shafarevich, *Basic algebraic geometry 2, Schemes and complex manifolds, third edition*, Springer-Verlag Berlin Heidelberg 1977, 1994, 2013.
- [Shapiro] D. B. Shapiro, *Compositions of quadratic forms*, De Gruyter Expositions in Mathematics **33**, Walter De Gruyter & Co., Berlin, 2000.
- [SV] C. Soulé and C. Voisin, *Torsion cohomology classes and algebraic cycles on complex projective manifolds*, Adv. Math. **198** (2005), 107–127.
- [Spivak] D. I. Spivak, *Derived smooth manifolds*, Duke Math. J. **153** (2010), no. 1, 55–128.
- [Stanley] R. P. Stanley, *Enumerative combinatorics, Volume 1*, Cambridge Studies in Advanced Math. **49**, Cambridge University Press, 1997, 2012.
- [Stasheff1] J. D. Stasheff, *Homotopy associativity of H -spaces I and II*, Trans. Amer. Math. Soc. **108** (1963), 275–292 and 293–312.
- [Stasheff2] J. D. Stasheff, *H -spaces from a homotopy point of view*, Lecture Notes in Math. **161**, Springer-Verlag, Berlin-New York, 1970.
- [Stasheff3] J. D. Stasheff, *Infinite loop spaces—a historical survey*, in *H -spaces (Actes Réunion Neuchâtel, 1970)*, pp. 43–53. Lecture Notes in Math. **196**, Springer-Verlag, Berlin-New York, 1971.
- [Steenrod1] N. Steenrod, *The topology of fibre bundles*, Princeton Math. Ser., vol. 14, Princeton University Press, Princeton, NJ, 1951.
- [Steenrod2] N. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.
- [Stelzer] M. Stelzer, *A model categorical approach to group-completion of E_n -algebras*, J. Homotopy Relat. Struct. **7** (2012), no. 2, 207–221.
- [Stiefel] E. Stiefel, *Über Richtungsfelder in den projektiven Räumen und einen Satz aus der reellen Algebra*, Comment. Math. Helv. **13** (1940/41), 201–218.
- [ST] S. Stolz and P. Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical foundations of quantum field theory and perturbative string theory, 279–340. Proc. Symp. Pure. Math., **83**. Amer. Math. Soc., Providence, RI, 2011.
- [Stong] R. E. Stong, *Notes on cobordism theory*, Mathematical Notes, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1968.
- [Suslin] A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk. SSSR **229** (1976), no. 5, 1063–1066.
- [Swan] R. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105**, no. 2 (1962), 264–277.
- [Switzer] R. M. Switzer, *Algebraic topology—homotopy and homology*, Classics in Mathematics, Springer-Verlag, Berlin, 2002 (reprint of the 1975 original).
- [Thomason] R. W. Thomason, *Algebraic K -theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 437–552.
- [TT] R. W. Thomason and T. Trobaugh, *Higher algebraic K -theory of schemes and derived categories*, Progr. Math. **88**, Birkhäuser Boston, Inc., Boston, MA, 1990.
- [Tian] Y. Tian, *The Atiyah-Jones conjecture for classical groups and Bott periodicity*, J. Differential Geom. **44** (1996), no. 1, 178–199.

- [Toda] H. Toda, *A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups*, Mem. Coll. Sci. Univ. Kyoto Ser. A **32** (1959), 103–119.
- [Totaro1] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. **10** (1997), no. 2, 467–493.
- [Totaro2] B. Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22.
- [Totaro3] B. Totaro, *On the integral Hodge and Tate conjectures over a number field*, Forum Math. Sigma **1** (2013), Paper No. e4, 13pp.
- [Totaro4] B. Totaro, *The integral Hodge conjecture for 3-folds of Kodaira dimension zero*, J. Inst. Math. Jussieu **20** (2021), no. 5, 1697–1717.
- [Ulrickson] P. Ulrickson, *Supersymmetric Euclidean field theories and K-theory*, J. Geom. Phys. **161** (2021), Paper No. 104015, 13pp.
- [Vakil] R. Vakil, *The rising sea—foundations of algebraic geometry*, Princeton University Press, Princeton, NJ, 2025.
- [van de Ven] A. van de Ven, *A property of algebraic varieties in complex projective spaces*, in *Colloque de Géométrie Différentielle Globale, tenu à Bruxelles du 19 au 22 décembre 1958*, pp. 151–152, Centre Belge de Recherches Mathématiques. Librairie Universitaire, Louvain; Gauthier-Villars, Paris, 1959.
- [Voisin] C. Voisin, *On integral Hodge classes on uniruled or Calabi-Yau threefolds*, in *Moduli spaces and arithmetic geometry*, 43–73, Adv. Stud. Pure Math. **45**, Mathematical Society of Japan, Tokyo, 2006.
- [Wall] C. T. C. Wall, *Graded Brauer groups*, J. Reine Angew. Math. **213** (1963/64), 187–199.
- [Weibel1] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, 1994.
- [Weibel2] C. Weibel, *The K-book: An Introduction to Algebraic K-theory*, Graduate Studies in Mathematics **145**, American Mathematical Society, Providence, RI, 2013.
- [Weyman] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Math., **149**, Cambridge University Press, Cambridge, 2003.
- [Whitehead1] J. H. C. Whitehead, *On C^1 -complexes*, Ann. of Math. (2) **41** (1940), 809–824.
- [Whitehead2] J. H. C. Whitehead, *On the groups $\pi_r(V_{n,m})$ and sphere bundles*, Proc. London Math. Soc. (2) **48** (1944), 243–291.
- [Whitney1] H. Whitney, *On the topology of differentiable manifolds*, in *Lectures in Topology*, pp. 101–141, University of Michigan Press, Ann Arbor, MI, 1941.
- [Whitney2] H. Whitney, *The self-intersections of a smooth n -manifold in $2n$ -space*, Ann. of Math. (2) **45** (1944), 220–246.
- [Whitney3] H. Whitney, *The singularities of a smooth n -manifold in $(2n - 1)$ -space*, Ann. of Math. (2) **45** (1944), 247–293.
- [Wood] R. Wood, *Banach algebras and Bott periodicity*, Topology **4** (1965/66), 371–389.
- [Yuzvinsky] S. Yuzvinsky, *Orthogonal pairings of Euclidean spaces*, Michigan Math. J. **28** (1981), 131–145.
- [ZS] O. Zariski and P. Samuel, *Commutative algebra, Volume II*, Grad. Texts in Math., Vol. 29, Springer-Verlag, New York-Heidelberg, 1975, x+414pp.
- [EGA2] A. Grothendieck, *Éléments de géométrie algébrique II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 5–222.
- [EGA3] A. Grothendieck, *Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents*, parts 1 and 2, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167pp and Inst. Hautes Études Sci. Publ. Math. (1963), no. 17, 91pp.
- [SGA1] A. Grothendieck and M. Raynaud, *Revêtements étale et groupe fondamentale*, Springer Lect. Notes in Math. **224** (1971).
- [SGA6] P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Springer Lect. Notes in Math. **225** (1971).
- [Stacks] The Stacks Project Authors, *Stacks Project*, <https://stacks.math.columbia.edu>, accessed July 2026.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403