A GEOMETRIC INTRODUCTION TO K-THEORY

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Preface

I first learned Serre's definition of intersection multiplicity from Mel Hochster, back when I was an undergraduate. I was immediately intrigued by this surprising connection between homological algebra and geometry. As it has always been for me when learning mathematics, I wanted to know how I could have guessed this definition for myself—what are the underlying principles that *tell us* to go looking in homological algebra for a definition of multiplicity. This question has been in the back of my mind for most of my mathematical life. It took me a long time to accept that the answers to such questions are not often readily available; one has to instead make do with vague hints and partial explanations. I still believe, though, that the answers exist *somewhere*—and that it is the ultimate job of mathematicians to uncover them. So perhaps it is better said this way: those questions often don't have simple answers *yet*.

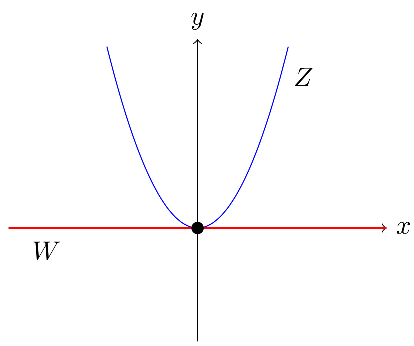
During my first year of graduate school I tried to puzzle out for myself the secrets behind Serre's definition. Thanks to the Gillet-Soulé paper [GS] I was led to K-theory, and similar hints of topology seemed to be operating in work of Roberts [R1, R4]. Coincidentally, MIT had a very active community of graduate students in topology, and I soon joined their ranks. Although there were other factors, it is not far from the truth to say that I became a topologist in order to understand Serre's definition.

In Winter quarter of 2012 I taught a course on this material at the University of Oregon. The graduate students taking the course converted my lectures into LaTeX, and then afterwards I both heavily revised and added to the resulting document. The present notes are the end result of this process. I am very grateful to the attending graduate students for the work they put into typesetting the lectures. These students were: Jeremiah Bartz, Christin Bibby, Safia Chettih, Emilio Gardella, Christopher Hardy, Liz Henning, Justin Hilburn, Zhanwen Huang, Tyler Kloefkorn, Joseph Loubert, Sylvia Naples, Min Ro, Patrick Schultz, Michael Sun, and Deb Vicinsky.

Introduction

1. Algebraic intersection multiplicities

Let Z be the parabola $y = x^2$ in $\mathbb{R}^2$, and let W be the tangent line at the vertex: the line $y = 0$. Then Z and W have an isolated point of intersection at $(0, 0)$:



Since high school you have known how to associate a *multiplicity* with this intersection: it is multiplicity 2, essentially because the polynomial x^2 has a double root at $x = 0$. This multiplicity also has a geometric interpretation, coming from intersection theory. If you perturb the intersection a bit, say by moving either Z or W by some small amount, then you get two points of intersection that are near $(0, 0)$—and these points both converge to $(0, 0)$ as the perturbation gets smaller and smaller.

You might object, rightly so, that I am lying to you. If we perturb $y = 0$ to $y = \epsilon$, with $\epsilon > 0$, then indeed we get two points of intersection: $(\sqrt{\epsilon}, \epsilon)$ and $(-\sqrt{\epsilon}, \epsilon)$. And these do indeed converge to $(0, 0)$ as $\epsilon \to 0$. But if we perturb the line in the other direction, by taking ϵ to be negative, then we get no points of intersection at all! To fix this, it is important to work over the complex numbers rather than the reals: the connection between geometry and algebra works out best (and simplest) in this case. If we work over $\mathbb{C}$, then it is indeed true that almost all small perturbations of our equations yield two solutions close to $(0, 0)$.

Our goal will be to vastly generalize the above phenomenom. Let $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$, and let Z be the algebraic variety defined by the vanishing of the f's. We write

$$Z = V(f_1, \dots, f_k) = \{x \in \mathbb{C}^n \mid f_1(x) = f_2(x) = \cdots = f_k(x) = 0\}.$$

Likewise, let $g_1, \dots, g_l \in \mathbb{C}[x_1, \dots, x_n]$ and let $W = V(g_1, \dots, g_l)$. Assume that P is an isolated point of the intersection $Z \cap W$. Our goal is to determine an algebraic formula, in terms of the f_i's and g_j's, for an intersection multiplicity $i(Z, W; P)$. This multiplicity should have the basic topological property that it coincides with the number of actual intersection points under almost all small deformations of Z and W.

Here are some basic properties, by no means comprehensive, that we would want such a formula to satisfy:

(1) $i(Z, W; P)$ should depend only on local information about Z and W near P.
(2) $i(Z, W; P) \geq 0$ always.

(3) If $\dim Z + \dim W < n$ then $i(Z, W; P) = 0$ (because in this case there is enough room in the ambient space to perturb Z and W so that they don't intersect at all).
(4) If $\dim Z + \dim W = n$ then $i(Z, W; P) > 0$.
(5) If $\dim Z + \dim W = n$ and Z and W meet transversely at P (meaning that $T_P Z \oplus T_P W = \mathbb{C}^n$), then $i(Z, W; P) = 1$.

Note that because of property (1) we can extend the notion of intersection multiplicity to varieties in $\mathbb{C}P^n$, simply by looking locally inside an affine chart for projective space that contains the point P. From now on we will do this without comment. The two statements below are not exactly 'basic properties' along the lines of (1)–(5) above, but they are basic results that any theory of intersection multiplicities should yield as consequences.

(6) Suppose that $X \hookrightarrow \mathbb{C}P^n$ is the vanishing set of a homogeneous polynomial, that is $X = V(f)$. Let L be a projective line in $\mathbb{C}P^n$ that meets X in finitely-many points. Then
$$\sum_{P \in X \cap L} i(X, L; P) = \deg(f).$$
(7) (Bezout's Theorem) Suppose that $X, Y \hookrightarrow \mathbb{C}P^2$ are the vanishing sets of homogeneous polynomials f and g, and that $X \cap Y$ consists of finitely-many points. Then
$$\sum_{P \in X \cap Y} i(X, Y; P) = (\deg f)(\deg g).$$

Note that (6), for the particular case $n = 2$, is a special case of (7).

If you play around with some simple examples, an idea for defining intersection multiplicities comes up naturally. It is

$$i(Z, W; P) = \dim_{\mathbb{C}} \Big[\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k, g_1, \dots, g_l)\Big]_P. \tag{1.1}$$

Here the subscript P indicates localization of the given ring at the maximal ideal $(x_1 - p_1, \dots, x_n - p_n)$ where $P = (p_1, \dots, p_n)$. The localization is necessary because $Z \cap W$ might have points other than P in it, and our definition needs to only depend on what is happening near P.

The best way to get a feeling for the definition in (1.1) is via some easy examples:

Example 1.2. Let $f = y - x^2$ and $g = y$. This is our example of the parabola and the tangent line at its vertex. The point $P = (0, 0)$ is the only intersection point, and our definition tells us to look at the ring
$$\mathbb{C}[x, y]/(y - x^2, y) \cong \mathbb{C}[x]/(x^2).$$
As a vector space over $\mathbb{C}$ this is two-dimensional, with basis 1 and x. So our definition gives $i(Z, W; P) = 2$ as desired. [Note that technically we should localize at the ideal (x, y), which corresponds to localization at (x) in $\mathbb{C}[x]/(x^2)$; however, this ring is already local and so the localization has no effect].

Example 1.3. The above example readily generalizes. If $h(x) \in \mathbb{C}[x]$ then let $f = y - h(x)$ and $g = y$. Factor $h(x) = \prod_i (x - z_i)^{e_i}$ and consider the intersection multiplicity of $V(f)$ and $V(g)$ at the point $(z_1, 0)$. Here we get $\mathbb{C}[x, y]/(f, g) \cong$

$\mathbb{C}[x]/(h(x))$ and after localization at the ideal $(x-z_1)$ the factors $x-z_i$ for $i>1$ become units so that we have

$$\mathbb{C}[x]_{(x-z_1)}/(h(x)) \cong \mathbb{C}[x]_{(x-z_1)}/(x-z_1)^{e_1} \cong \mathbb{C}[u]_{(u)}/(u^{e_1}).$$

The dimension over $\mathbb{C}$ is then e_1, coinciding with the multiplicity of z_1 as a root of $h(x)$.

Example 1.4. Let $f = y^2 - x^3 - 3x$ and $g = y - \frac{3}{2}x - \frac{1}{2}$. Then $Z = V(f)$ is an elliptic curve, and one can check that $W = V(g)$ is the tangent line at the point $P = (1,2)$. Let us recall how this works: the gradient vector to the curve is

$$\nabla f = [-3x^2 - 3, 2y]$$

and this is normal to the curve at (x,y). A tangent vector is then $[2y, 3x^2+3]$ (since this is orthogonal to ∇f), which means the slope of the curve at (x,y) is $(3x^2+3)/2y$. At the point $(1,2)$ we then get slope $\frac{3}{2}$, and $V(g)$ is the line passing through $(1,2)$ with this slope.

The line $V(g)$ intersects the curve at one other point, which we find by simultaneously solving $y^2 = x^3 + 3x$ and $y = \frac{3}{2}x + \frac{1}{2}$. This yields the cubic

$$0 = x^3 + 3x - (\tfrac{3}{2}x + \tfrac{1}{2})^2.$$

Since we know that $x = 1$ is a root, we can factor this out and then solve the resulting quadratic. One finds that the cubic factors as

$$0 = (x-1)^2 \cdot (x - \tfrac{1}{4}).$$

The second point of intersection is found to be $Q = (\frac{1}{4}, \frac{7}{8})$.

Note the appearance of $(x-1)$ with multiplicity two in the above factorization. The fact that we had a tangent line at $x = 1$ guaranteed that the multiplicity would be strictly larger than one. Likewise, the fact that $(x - \frac{1}{4})$ has multiplicity one tells us that $V(g)$ intersects the curve transversely at the second point. These facts suggest that $i(Z,W;P) = 2$ and $i(Z,W;Q) = 1$. Let us consider these in terms of point-counting under small deformations. We can perturb either Z or W, but it is perhaps easiest to perturb the line W: we can write $\tilde{g} = y - Ax - B$ and then consider what happens for all (A,B) near $(\frac{3}{2}, \frac{1}{2})$. We will need to find the intersection of Z and $\widetilde{W} = V(\tilde{g})$, which as before requires us to solve a cubic. Let us again arrange for there to be a known solution which we can factor out. It is possible to have this solution be either $(1,2)$ or $(\frac{1}{4}, \frac{7}{8})$. The calculations turn out to be a little easier for the latter, despite the annoying fractions. So we assume $\frac{7}{8} = \frac{A}{4} + B$, or $\tilde{g} = y - A(x - \frac{1}{4}) - \frac{7}{8}$. Since we want to look at A near $\frac{3}{2}$, it is convenient to write $A = \frac{3}{2} + \epsilon$ where ϵ is near zero.

Finding common solutions of $f = 0$ and $\tilde{g} = 0$ yields a cubic with $(x - \frac{1}{4})$ as a factor, and dividing this out we obtain the quadratic

$$0 = x^2 - x(2 + 3\epsilon + \epsilon^2) + (1 - \epsilon + \tfrac{\epsilon^2}{4}).$$

The discriminant of this quadratic is $D = \epsilon(\epsilon^3 + 6\epsilon^2 + 4\epsilon + 16)$, so the quadratic has a double root when $\epsilon = 0$ (as expected) but simple roots for values of ϵ near but not equal to zero. So for these values of ϵ we get two points of intersection of $V(f)$ and $V(\tilde{g})$ near P, and it is easy to see that they converge to P as ϵ approaches zero.

Let us now see what our provisional definition from (1.1) gives. The quotient ring in our definition is

$$\begin{aligned}\mathbb{C}[x,y]/\big(y^2-x^3-3x,\, y-\tfrac{3}{2}x-\tfrac{1}{2}\big) &\cong \mathbb{C}[x]/\big((\tfrac{3}{2}x+\tfrac{1}{2})^2-x^3-3x\big)\\ &\cong \mathbb{C}[x]/\big((x-1)^2(x-\tfrac{1}{4})\big).\end{aligned}$$

Here we are killing a cubic in $\mathbb{C}[x]$, and so we get a three-dimensional vector space with basis $1, x, x^2$. Note that this is, in some sense, seeing all of the information at P and Q together—this demonstrates the importance of localization. Localization at P corresponds to localizing at $(x-1)$, which turns $(x-\frac{1}{4})$ into a unit. So our localized ring is

$$\mathbb{C}[x]_{(x-1)}/((x-1)^2) \cong \mathbb{C}[t]_{(t)}/(t^2)$$

(where we set $t = x-1$), and this has dimension 2 over $\mathbb{C}$. So $i(Z,W;P) = 2$, as desired.

If we localize at $(x-\frac{1}{4})$ then the $(x-1)^2$ factor becomes a unit, and our localized ring becomes $\mathbb{C}[x]_{(x-\frac{1}{4})}/(x-\frac{1}{4}) \cong \mathbb{C}[t]_{(t)}/(t)$, which is just a copy of $\mathbb{C}$. So $i(Z,W;Q) = 1$.

Note that Example 1.2 through 1.4 involve a key step where the variable y is eliminated, thus bringing the problem down to the multiplicity of a root in a one-variable polynomial. One cannot always do such an elimination—in fact it happens only rarely. So these examples are very special, although they still serve to give some sense of how things are working.

It turns out that our provisional definition from (1.1) is enough to prove Bezout's Theorem for curves in $\mathbb{C}P^2$. But in some sense one is getting lucky here, and it works only because the dimensions of the varieties are so small. When one starts to look at higher-dimensional varieties it doesn't take long to find examples where the definition clearly gives the wrong answers:

Example 1.5. Let $\mathbb{C}^4$ have coordinates u, v, w, y, and let $X, Y \subseteq \mathbb{C}^4$ be given by

$$X = V(u^3-v^2,\, u^2y-vw,\, uw-vy,\, w^2-uy^2), \qquad Y = V(u,y).$$

Note that X is somewhat complicated, but Y is just a plane. If a point (u,v,w,y) is on $X\cap Y$ then $u=y=0$ and therefore the equations for X say that

$$v^2 = 0, \quad vw = 0, \quad \text{and} \quad w^2 = 0$$

as well. So $X\cap Y$ consists of the unique point $(0,0,0,0)$. Our provisional definition of intersection multiplicities would have us look at the ring

$$\mathbb{C}[u,v,w,y]/(u,y,u^3-v^2,u^2y-vw,uw-vy,w^2-uy^2) \cong \mathbb{C}[v,w]/(v^2,vw,w^2)$$

which is three-dimensional over $\mathbb{C}$. If this were the correct answer, then perturbing the plane Y should generically give three points of intersection. However, this is not the case. If we perturb Y to $V(u-\epsilon, y-\delta)$ then the intersection with X is given by the equations

$$u=\epsilon, \quad y=\delta, \quad \epsilon^3=v^2, \quad \epsilon^2\delta = vw, \quad \epsilon w = v\delta, \quad w^2=\epsilon\delta^2.$$

As long as $\epsilon \neq 0$ we have two solutions for v, and then the fourth equation determines w completely (the last two equations 7are redundant). So we only have two points on the intersection, after small perturbations. This is, in fact, the correct answer: $i(Z,W;P) = 2$, and our provisional definition has failed.

Serre discovered the correct formula for the interesection multiplicity [S]. His formula is as follows. If we set $R = \mathbb{C}[x_1, \dots, x_n]$ then

$$i(Z, W; P) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}} \Big[\mathrm{Tor}_j^R \Big(R/(f_1, \dots, f_k), R/(g_1, \dots, g_l) \Big) \Big]_P. \tag{1.6}$$

There are several things to say here. First, although the sum is written to infinity it turns out that the Tor modules vanish for all $j > n$ (we will prove this later). So it is, in fact, a finite sum. Secondly, the condition that P be an isolated point of intersection forces the $\mathbb{C}$-dimension of all the Tor's to be finite. So the formula does make sense. As to why this gives the "correct" numbers, it will take us a while to explain this. But note that the $j = 0$ term is the dimension of

$$\begin{aligned} \mathrm{Tor}_0(R/(f_1, \dots, f_k), R/(g_1, \dots, g_l)) &\cong R/(f_1, \dots, f_k) \otimes_R R/(g_1, \dots, g_l) \\ &\cong R/(f_1, \dots, f_k, g_1, \dots, g_l). \end{aligned}$$

So our provisional definition from (1.1) is just the $j = 0$ term. One should think of the higher terms as "corrections" to this initial term; in a certain sense these corrections get smaller as j increases (this is not obvious).

An algebraist who looks at (1.6) will immediately notice some possible generalizations. The $R/(\underline{f})$ and $R/(\underline{g})$ terms can be replaced by any finitely-generated module M and N, as long as the $\mathrm{Tor}_j(M, N)$ modules are finite-dimensional over $\mathbb{C}$. For this it turns out to be enough that $M \otimes_R N$ be finite-dimensional over $\mathbb{C}$. Also, we can replace $\mathbb{C}[x_1, \dots, x_n]$ with any ring having the property that all finitely-generated modules have finite projective dimension—necessary so that the alternating sum of (1.6) is finite. Such rings are called **regular**. Also, instead of localizing the Tor-modules we can just localize the ring R at the very beginning. And finally, in this generality we need to replace $\dim_{\mathbb{C}}$ with a similar invariant: the notion of *length* (meaning the length of a composition series for our module). This leads to the following setup.

Let R be a regular, local ring (all rings are assumed to be commutative and Noetherian unless otherwise noted). Let M and N be finitely-generated modules over R such that $M \otimes_R N$ has finite length. This implies that all the $\mathrm{Tor}_j(M, N)$ modules also have finite length. Define

$$e(M, N) = \sum_{j=0}^{\infty} (-1)^j \ell\big(\mathrm{Tor}_j(M, N)\big) \tag{1.7}$$

and call this the **intersection multiplicty** of the modules M and N.

Based on geometric intuition, Serre made the following conjectures about the above situation:

(1) $\dim M + \dim N \leq \dim R$ always
(2) $e(M, N) \geq 0$ always
(3) If $\dim M + \dim N < \dim R$ then $e(M, N) = 0$.
(4) If $\dim M + \dim N = \dim R$ then $e(M, N) > 0$.

In [S] Serre proved all of these in the case that R contains a field, the so-called "geometric case" (some non-geometric examples for R include power series rings over the p-adic integers $\mathbb{Z}_p$). Serre also proved (1) in general. Conjecture (3) was proven in the mid 80s by Roberts and Gillet-Soule (independently), using some sophisticated topological ideas that were imported into algebra. Conjecture (2) was proven by Gabber in the mid 90s, using some high-tech algebraic geometry. Conjecture (4) is still open.

1.8. **Where we are headed.** Our main goal in these notes is to describe a particular subset of the mathematics surrounding Serre's definition of multiplicity. It is possible to explore this subject purely in algebraic terms, and that is basically what Serre did in his book [S]. In contrast, our main focus will be topological. Although both commutative algebra and algebraic geometry play a large role in our story, we will always adopt a perspective that concentrates on their relations to topology—and in particular, to K-theory.

Here is a brief summary of some of the main points that we will encounter:

(1) There are certain generalized cohomology theories—called *complex-oriented*—which have a close connection to geometry and intersection theory. Any such cohomology can be used to detect intersection multiplicities.
(2) Topological K-theory is a complex-oriented cohomology theory. Elements of the groups $K^*(X)$ are specified by vector bundles on X, or more generally by bounded chain complexes of vector bundles on X. Fundamental classes for complex submanifolds of X are given by *resolutions*.
(3) When X is an algebraic variety there is another version of K-theory called *algebraic K-theory*, which we might denote $K^*_{\mathrm{alg}}(X)$. The analogs of vector bundles are locally free coherent sheaves, or just finitely-generated projective modules when X is affine. Thus, in the affine case elements of $K^*_{\mathrm{alg}}(X)$ can be specified by bounded chain complexes of finitely-generated projective modules. This is the main connection between homological algebra and K-theory.
(4) Serre's definition of intersection multiplicities essentially comes from the intersection product in K-homology, which is the cup product in K-cohomology translated to homology via Poincaré Duality.

We will spend a large chunk of this book filling in the details behind (1)–(4). But whereas we take our motivation from Serre's definition of multiplicity, that is not the only subject we will cover. Once we have the K-theory apparatus up and running there are lots of neat things to do with it. We have attempted, for the most part, to chose topics that accentuate the relationship between K-theory and geometry in the same way that Serre's definition of multiplicity does.

Part 1. K-theory in algebra

In this first part of the book we investigate the K-theory of modules over a commutative ring R. There are two main varieties: one can study the K-theory of all finitely-generated modules, leading to the group $G(R)$, or one can study the K-theory of finitely-generated *projective* modules, leading to the group $K(R)$. In the following sections we get a taste for these groups and the relations between them.

For the duration of the book all rings are commutative with identity unless otherwise stated. Some of the theory we develop works in greater generality, but we will stay focused on the commutative case.

2. A first look at K-theory

Understanding Serre's alternating-sum-of-Tor's formula for intersection multiplicities will be a gradual process. In particular, there is quite a bit of nontrivial commutative algebra that is needed for the story; we will need to develop this as we go along. We will continue to sweep some of these details under the rug for the moment, but let us at least get a couple of things out in the open. To begin with, we will need the following important result:

Theorem 2.1 (Hilbert Syzygy Theorem)**.** *Let k be a field and let R be $k[x_1,\ldots,x_n]$ (or any localization of this ring). Then every finitely-generated R-module has a free resolution of length at most n.*

We will prove this theorem in Section 18 below. We mention it here because it implies that $\operatorname{Tor}_j(M,N)=0$ for $j>n$. Therefore the sum in Serre's formula is actually finite. More generally, a ring is called **regular** if it is Noetherian and every finitely-generated module has a finite projective resolution. It is a theorem that localizations of regular rings are again regular. Hilbert's Syzygy Theorem simply says that polynomial rings over a field are regular. We will find that regular rings are the 'right' context in which to explore Serre's formula.

We will also need the following simple observation. If P is a prime ideal in any commutative ring R, then

$$[\operatorname{Tor}_R(M,N)]_P=\operatorname{Tor}^{R_P}(M_P,N_P).$$

To see this, let $Q_\bullet\to M\to 0$ be an R-free resolution of M. Since localization is exact, $(Q_\bullet)_P$ is an R_P-free resolution of M_P. Hence

$$\begin{aligned}\operatorname{Tor}_j^{R_P}(M_P,N_P)=H_j\big((Q_\bullet)_P\otimes_{R_P}N_P\big)&=H_j\big(Q_\bullet\otimes_R R_P\otimes_{R_P}N\otimes_R R_P\big)\\&=H_j\big(Q_\bullet\otimes_R N\otimes R_P\big)\\&=H_j\big(Q_\bullet\otimes_R N\big)\otimes R_P\\&=\operatorname{Tor}_j^R(M,N)\otimes_R R_P.\end{aligned}$$

The importance of this observation is that it tells us that each Tor in Serre's formula for $i(Z,W;P)$ may be taken over the ring R_P. So we might as well work over this ring from beginning to end. Moreover, without loss of generality we might as well assume that our point of intersection is the origin, which makes the corresponding maximal ideal $(x_1,\ldots,x_n)$.

Let $R=\mathbb{C}[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$, and let M and N be finitely-generated modules over R. Assume that $\dim_{\mathbb{C}}(M\otimes_R N)<\infty$. It turns out that this implies that

$\dim_{\mathbb{C}} \operatorname{Tor}_j(M,N) < \infty$ for every j, so that we can define

$$e(M,N) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}} \operatorname{Tor}_j(M,N).$$

The above definition generalizes the notion of intersection multiplicity from pairs $(R/I, R/J)$ to pairs of modules (M,N). The reason for making this generalization might not be clear at first, but the following nice property provides some justification:

Lemma 2.2. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, where $R = \mathbb{C}[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$. Then $e(M,N) = e(M',N) + e(M'',N)$, assuming all three multiplicities are defined (that is, under the assumption that $\dim_{\mathbb{C}}(M \otimes N) < \infty$ and similarly with M replaced by M' and M'').*

Proof. Consider the long exact sequence

$$\cdots \to \operatorname{Tor}_j(M',N) \to \operatorname{Tor}_j(M,N) \to \operatorname{Tor}_j(M'',N) \to \cdots$$

This sequence terminates after a finite number of steps, by Hilbert's Syzygy Theorem. By exactness, the alternating sum of the dimensions is zero. This is precisely the desired formula. □

Lemma 2.2 is referred to as the *additivity of intersection multiplicities*. Of course the additivity holds equally well in the second variable, by the same argument.

While exploring ideas in this general area, Grothendieck hit upon the idea of inventing a group that captures **all** the additive invariants of modules. Any invariant such as $e(-,N)$ would then factor through this group. Here is the definition:

Definition 2.3. *Let R be any ring. Let $\mathcal{F}(R)$ be the free abelian group with one generator $[M]$ for every isomorphism class of finitely-generated R-module M. Let $G(R)$ be the quotient of $\mathcal{F}(R)$ by the subgroup generated by all elements $[M]-[M']-[M'']$ for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely-generated R-modules. The group $G(R)$ is called the* ***Grothendieck group*** *of finitely-generated R-modules.*

Remark 2.4. It is important in the definition of $G(R)$ that one use only *finitely-generated* R-modules, otherwise the group would be trivial. To see this, if M is any module then let $M^\infty = M \oplus M \oplus M \cdots$. Note that there is a short exact sequence

$$0 \to M \hookrightarrow M^\infty \to M^\infty \to 0$$

where M is included as the first summand. If we had defined $G(R)$ without the finite-generation condition, we would have $[M^\infty] = [M] + [M^\infty]$ and therefore $[M] = 0$. Since this holds for every module M, the group $G(R)$ would be zero. This is called the "Eilenberg Swindle".

Because of the need to focus on finitely-generated modules, and the fact that arguments will often require us to bring in submodules, results from here on out will often assume that R is Noetherian. The first example of this is part (c) of the next result.

The following proposition records some useful ways of obtaining relations in $G(R)$:

Proposition 2.5. *Let R be any ring.*

(a) *If* $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ *is an exact sequence of finitely-generated* R*-modules, then* $\sum(-1)^i[C_i] = 0$ *in* $G(R)$*.*
(b) *If* $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} = 0$ *is a filtration of* M *by finitely-generated modules, then* $[M] = \sum_i [M_i/M_{i+1}]$ *in* $G(R)$*.*
(c) *Assume that* R *is Noetherian, and let* $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ *be any chain complex of finitely-generataed* R*-modules. Then* $\sum_i(-1)^i[C_i] = \sum_i(-1)^i[H_i(C)]$ *in* $G(R)$*.*

Proof. We prove (a) and (c) at the same time. If $C_\bullet$ is a chain complex, note that one has the short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ where Z_i and B_i are the cycles and boundaries in each dimension. One also has $0 \to B_i \hookrightarrow Z_i \to H_i(C) \to 0$. Assuming everything in sight is finitely-generated, one gets a series of relations in $G(R)$ that immediately yield $\sum(-1)^i[C_i] = \sum(-1)^i[H_i(C)]$. So if R is Noetherian we are done, because everything indeed is finitely-generated; this proves (c). In the general case where R is not necessarily Noetherian, we know that each B_i is finitely-generated because it is the image of C_{i+1}. But if $C_\bullet$ is exact then $B_i = Z_i$ and so the Z_i's are also finitely-generated. We have the relations $[C_i] = [Z_i]+[B_{i-1}] = [Z_i]+[Z_{i-1}]$, and from this it is evident that $\sum(-1)^i[C_i] = 0$. This proves (a).

The proof of (b) is similarly easy; one considers the evident exact sequences $0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0$ and the resulting relations in $G(R)$. $\square$

Here are a series of examples:

(1) Suppose $R = F$, a field. Clearly $G(F)$ is generated by $[F]$, since every finitely-generated F-module has the form F^n. If we observe the existence of the group homomorphism $\dim\colon G(F) \to \mathbb{Z}$, which is clearly surjective because it sends $[F]$ to 1, then it follows that $G(F) \cong \mathbb{Z}$.
(2) More generally, suppose that R is a domain. The **rank** of an R-module M is defined to be the dimension of $M \otimes_R QF(R)$ over $QF(R)$, where $QF(R)$ is the quotient field. The rank clearly gives a homomorphism $G(R) \to \mathbb{Z}$, which is surjective because $[R] \mapsto 1$. So $G(R)$ has $\mathbb{Z}$ as a direct summand.
(3) Next consider $R = \mathbb{Z}$. Then $G(\mathbb{Z})$ is generated by the classes $[\mathbb{Z}]$ and $[\mathbb{Z}/n]$ for $n > 1$, by the classification of finitely-generated abelian groups. The short exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \to 0$ shows that $[\mathbb{Z}/n] = 0$ for all n, hence $G(\mathbb{Z})$ is cyclic. Using (b), it follows that $G(\mathbb{Z}) = \mathbb{Z}$. This computation works just as well for any PID.
(4) So far we have only seen cases where $G(R) \cong \mathbb{Z}$. For a case where this is not true, try $R = F \times F$ where F is a field. You should find that $G(R) \cong \mathbb{Z}^2$ here. More generally, the theory of modules over a product ring $R \times S$ yields that $G(R \times S) \cong G(R) \oplus G(S)$ (this is a nice exercise).
(5) The definition of $G(R)$ can also be made for R non-commutative, using left R-modules (one can of course define another group using right R-modules, but that would be $G(R^{op})$). As an example, let G be a finite group and let $R = \mathbb{C}[G]$ be the group algebra. So R-modules are just representations of G on complex vector spaces. The basic theory of such finite-dimensional representations says that each is a direct sum of irreducibles, in an essentially unique way. Moreover, each short exact sequence is split. A little thought shows that this implies that $G(R)$ is a free abelian group with basis consisting of the isomorphism classes of irreducible representations.

(6) So far all the examples we have computed have $G(R)$ equal to a free abelian group. This is not always the case, although I don't know an example where it is really easy to see this. For a not-so-simple example, let R be the ring of integers in a number field. It turns out that $G(R) \cong \mathbb{Z} \oplus \mathrm{Cl}(R)$, where $\mathrm{Cl}(R)$ is the ideal class group of R. This class group contains some sophisticated number-theoretic information about R. It is known to always be torsion, and it is usually nontrivial. We will work out a simple example when we have more tools under our belt: see Example 4.2.
(7) As another simple example, we look at $R = F[t]/(t^2)$ where F is a field. For any module M over R we have the filtration $M \supseteq tM$, and so $[M] = [M/tM]+[tM]$. But both M/tM and tM are killed by t, hence are direct sums of copies of F (where t acts as zero). This shows that $G(R)$ is generated by $[R/tR]$. We also have the function $\dim_F(-)\colon G(R) \to \mathbb{Z}$. Since this function sends $[R/tR]$ to 1, it must be an isomorphism.
(8) The final example we consider here is a variation of the previous one. Let us look at $R = \mathbb{Z}/p^2$. The R-modules are simply abelian groups killed by p^2. Given any such module A one can consider the sequence $0 \to pA \hookrightarrow A \to A/pA \to 0$, and observe that the first and third terms are $\mathbb{Z}/p$-vector spaces. So $[R/p]$ generates $G(R)$. We claim that $G(R) \cong \mathbb{Z}$, and as in the previous example the easiest way to see this is to write down an additive invariant of R-modules taking its values in $\mathbb{Z}$. All finitely-generated R-modules have a finite composition series, and so we can take the Jordan-Hölder length; this is the same as $\ell(A) = \dim_{\mathbb{Z}/p} A/pA + \dim_{\mathbb{Z}/p} pA$. With some trouble one can check that this is indeed an additive invariant (or refer to the Jordan-Hölder theorem), and of course $\ell(\mathbb{Z}/p) = 1$. This completes the calculation.

Exercise 2.6. Prove that $G(R) \cong \mathbb{Z}$ for $R = F[t]/(t^n)$ or $R = \mathbb{Z}/p^n$.

The above examples help establish some basic intuition. In general, though, it can be very hard to compute $G(R)$. In fact, given two modules M and N it can be hard to decide whether or not $[M] = [N]$ in $G(R)$. The following result (taken from [He, Lemma 2.1]) at least deconstructs the problem into something concrete:

Proposition 2.7. *Let M and N be finitely-generated R-modules. Then the following are equivalent:*
(1) $[M] = [N]$ *in* $G(R)$.
(2) There exist two exact sequences of finitely-generated modules $0 \to A \to X \to B \to 0$ *and* $0 \to A \to Y \to B \to 0$ *and a finitely-generated module C such that* $X \cong M \oplus C$ *and* $Y \cong N \oplus C$.
(3) There exist two exact sequences of finitely-generated modules $0 \to A \to X \to B \to 0$ *and* $0 \to A \to Y \to B \to 0$ *such that* $M \oplus X \cong N \oplus Y$.

Proof. We will prove (3)⇒(2)⇒(1)⇒(3). For (3)⇒(2) use the sequences

$$0 \to M \oplus N \oplus A \to M \oplus N \oplus Y \to B \to 0$$

and

$$0 \to M \oplus N \oplus A \to M \oplus N \oplus X \to B \to 0$$

with $C = M \oplus X \cong N \oplus Y$. (2)⇒(1) is easy since the hypotheses show that $[M] + [C] = [X] = [A] + [B] = [Y] = [N] + [C]$, and therefore $[M] = [N]$. The real content is therefore (1)⇒(3).

Let $\mathcal{R}el \subseteq \mathcal{F}(R)$ be the subgroup generated by all elements $[J]-[J']-[J'']$ for short exact sequences $0 \to J' \to J \to J'' \to 0$. If $[M]-[N] \in \mathcal{R}el$ then there exist two collections of such sequences $0 \to A_i' \to A_i \to A_i'' \to 0$, $1 \le i \le k_1$, and $0 \to B_j' \to B_j \to B_j'' \to 0$, $1 \le j \le k_2$, such that

$$[M]-[N] = \sum_{i=1}^{k_1}\big([A_i]-[A_i']-[A_i'']\big) + \sum_{j=1}^{k_2}\big([B_j']+[B_j'']-[B_j]\big)$$

in $\mathcal{F}(R)$. Rearranging, this gives the identity in $\mathcal{F}(R)$

$$[M] + \sum_{i=1}^{k_1}\big([A_i']+[A_i'']\big) + \sum_{j=1}^{k_2}[B_j] = [N] + \sum_{i=1}^{k_1}[A_i] + \sum_{j=1}^{k_2}\big([B_j']+[B_j'']\big).$$

The only way such sums of basis elements can give the same element of $\mathcal{F}(R)$ is if the collection of summands on the two sides are the same up to permutation. But in that case one can write

$$M \oplus \bigoplus_{i=1}^{k_1}(A_i' \oplus A_i'') \oplus \bigoplus_{j=1}^{k_2} B_j \cong N \oplus \bigoplus_{i=1}^{k_1} A_i \oplus \bigoplus_{j=1}^{k_2}(B_j' \oplus B_j'').$$

Finally, consider the evident short exact sequences

$$0 \to \bigoplus_j B_j' \oplus \bigoplus_i A_i' \to \bigoplus_i A_i \oplus \bigoplus_j (B_j' \oplus B_j'') \to \bigoplus_i A_i'' \oplus \bigoplus_j B_j'' \to 0$$

and

$$0 \to \bigoplus_j B_j' \oplus \bigoplus_i A_i' \to \bigoplus_j B_j \oplus \bigoplus_i (A_i' \oplus A_i'') \to \bigoplus_i A_i'' \oplus \bigoplus_j B_j'' \to 0.$$

Adding N to the middle term of the first sequence is isomorphic to the result of adding M to the middle term of the second. $\square$

We can adapt our definition of intersection multiplicity of two modules to define a product on $G(R)$, at least when R is regular. For finitely-generated modules M and N, define

$$[M] \odot [N] = \sum_j (-1)^j [\mathrm{Tor}_j(M,N)].$$

Regularity of R guarantees that this is a finite sum. The long exact sequence for Tor shows that this definition is additive in the two variables, and hence passes to a pairing $G(R) \otimes G(R) \to G(R)$. It is not *at all* clear that this is associative, although we will prove this shortly (Corollary 2.15).

The above product on $G(R)$ is certainly not the first thing one would think of. It is more natural to try to define a product by having $[M]\cdot[N] = [M \otimes_R N]$, but of course this is not additive in the two variables because of the failure of the tensor product to be exact. The higher Tor's are correcting for this. However, we *can* make this naive definition work if we restrict to a certain class of modules. To that end, let us introduce the following definition:

Definition 2.8. *Let R be any ring. Let $\mathcal{F}_K(R)$ be the free abelian group with one generator $[P]$ for every isomorphism class of finitely-generated, projective R-module M. Let $K(R)$ be the quotient of $\mathcal{F}_K(R)$ by the subgroup generated by all elements $[P]-[P']-[P'']$ for every short exact sequence $0 \to P' \to P \to P'' \to 0$ of* *finitely-generated projectives. The group $K(R)$ is called the* ***Grothendieck group of finitely-generated projective modules****.*

Every short exact sequence of projectives is actually split, so we could also have defined $K(R)$ by imposing the relations $[P \oplus Q] = [P] + [Q]$ for every two finitely-generated projectives P and Q. This makes it a little easier to understand when two modules represent the same class in $K(R)$:

Proposition 2.9. *Let P and Q be finitely-generated projective R-modules. Then $[P] = [Q]$ in $K(R)$ if and only if there exists a finitely-generated projective module W such that $P \oplus W \cong Q \oplus W$. In fact, the same remains true if we require W to be free instead of projective.*

Proof. The first statement is immediate from Proposition 2.7 (which can be reproven verbatim in the present context) using that short exact sequences of projectives always split. For the second claim use that projectives are direct summands of free modules, so that there exists a W' such that $W \oplus W'$ is finitely-generated and free. □

Exercise 2.10. Give a direct proof of Proposition 2.9 along the lines of what we did for Proposition 2.7.

Exercise 2.11. Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$ be an exact sequence of finitely-generated projectives. Prove that $\sum_i (-1)^i [P_i] = 0$ in $K(R)$. [Note that this is almost Proposition 2.5(a) but maybe a tiny bit more thought is required.]

Since projective modules are flat, the product $[P] \cdot [Q] = [P \otimes_R Q]$ is additive and so extends to a product $K(R) \otimes K(R) \to K(R)$. Note that this product is obviously associative, and so makes $K(R)$ into a ring. This is true without any assumptions on R whatsoever (except our standing assumption that R be commutative).

Remark 2.12. Given the motivation of having the tensor product give a ring structure, one might wonder why we used projective modules to define $K(R)$ rather than flat modules. We could have done so, but for finitely-generated modules over commutative, Noetherian rings, being flat and projective are equivalent notions—see [E, Corollary 6.6]. For various reasons it is more common to make the definition using the projective hypothesis.

There is an evident map $\alpha \colon K(R) \to G(R)$ which sends $[P]$ to $[P]$ (note that these two symbols, while they look the same, denote elements of different groups). This brings us to our first important theorem:

Theorem 2.13. *If R is regular then $\alpha \colon K(R) \to G(R)$ is an isomorphism.*

Proof. Surjectivity is easy to see: if M is a finitely-generated module, choose a finite, projective resolution $P_\bullet \to M \to 0$. Then $\sum_j (-1)^j [P_j] = [M]$ in $G(R)$, and this proves that $[M]$ is in the image of α.

Proving injectivity is slightly harder, and it will be most convenient just to define an inverse for α. The above paragraph gives us the definition: for a finitely-generated R-module M, define

$$\beta([M]) = \sum_j (-1)^j [P_j]$$

where $P_\bullet \to M \to 0$ is a finite resolution by finitely-generated projectives. We need to show that this is independent of the choice of P, and that it is additive: these facts will show that β defines a map $G(R) \to K(R)$. It is then obvious that this is a two-sided inverse to α.

Suppose $Q_\bullet \to M \to 0$ is another finite projective resolution of M. Use the Comparison Theorem of homological algebra to produce a map of chain complexes

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Let $T_\bullet$ be the mapping cone of $f\colon P_\bullet \to Q_\bullet$. Recall this means that $T_j = Q_j \oplus P_{j-1}$, with the differential defined by

$$d_T(a,b) = \Big(d_Q(a) + f(b),\ -d_P(b)\Big).$$

There is a short exact sequence of chain complexes

$$0 \to Q \hookrightarrow T \to \Sigma P \to 0$$

where ΣP denotes a copy of P in which everything has been shifted up a dimension (so that $(\Sigma P)_n = P_{n-1}$) and the differential picks up a negative sign ($d_{\Sigma P} = -d_P$). The long exact sequence on homology groups shows readily that T is exact, hence we have $\sum_j (-1)^j[T_j] = 0$ in $K(R)$ by Exercise 2.11. Since $[T_j] = [Q_j] + [P_{j-1}]$ in $K(R)$ this gives that $\sum_j (-1)^j [P_j] = \sum_j (-1)^j [Q_j]$. Hence our definition of β does not depend on the choice of resolution.

A similar argument can be used to show additivity. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, and let $P_\bullet \to M'$ and $Q_\bullet \to M$ be finite projective resolutions. Lift the map $M' \to M$ to a map of complexes $f\colon P_\bullet \to Q_\bullet$, and let $T_\bullet$ be the mapping cone of f. The long exact sequence for homology readily shows that T is a projective resolution of M''. So

$$\beta(M'') = \sum (-1)^j [T_j] = \sum (-1)^j [Q_j] - \sum (-1)^j [P_j] = \beta(M) - \beta(M')$$

and this proves additivity. This completes our proof. $\square$

Using the isomorphism $K(R) \to G(R)$ (when R is regular), we can transplant the ring structure on $K(R)$ to the group $G(R)$. We claim that this gives the product $\odot$ defined via Tor. In the following result, $\beta\colon G(R) \to K(R)$ is the inverse to α defined in the proof of Theorem 2.13.

Proposition 2.14. *Assume that R is regular. Then for any two finitely-generated modules M and N we have*

$$\alpha\Big[\beta([M]) \otimes \beta([N])\Big] = \sum (-1)^j [\mathrm{Tor}_j(M,N)] = [M] \odot [N].$$

Proof. Let $P_\bullet \to M$ and $Q_\bullet \to N$ be finite projective resolutions. Fix j, and consider the complex $P_\bullet \otimes Q_j$. This is a resolution of $M \otimes Q_j$, since Q_j is flat. So

$\sum_i (-1)^i [P_i \otimes Q_j] = [M \otimes Q_j]$ in $G(R)$. Using this for each j, we have that

$$\begin{aligned}\alpha\Big[\beta([M]) \otimes \beta([N])\Big] &= \sum_{i,j} (-1)^{i+j} [P_i \otimes Q_j] \\ &= \sum_j (-1)^j [M \otimes Q_j] \\ &= \sum_j (-1)^j [H_j(M \otimes Q)] \quad \text{using Proposition 2.5(c)} \\ &= \sum_j (-1)^j [\mathrm{Tor}_j(M, N)].\end{aligned}$$

□

Corollary 2.15. *When R is regular the product $\odot$ on $G(R)$ is associative.*

Proof. This follows immediately from the fact that the tensor product gives an associative multiplication on $K(R)$. □

Let us review the above situation. For any ring R, we have the group $K(R)$ which also comes to us with an easily-defined ring structure $\otimes$. We also have the group $G(R)$—but this does not have any evident ring structure. When R is regular, there is an isomorphism $K(R) \to G(R)$ which allows one to transplant the ring structure from $K(R)$ onto $G(R)$: and this leads us directly to our alternating-sum-of-Tors formula.

This situation is very reminiscent of something you have seen in a basic algebraic topology course. When X is a (compact, oriented) manifold, there were early attempts to put a ring structure on $H_*(X)$ coming from the intersection product. This is technically very difficult. In modern times one avoids these technicalities by instead introducing the cohomology groups $H^*(X)$, and here it is easy to define a ring structure: the cup product. When X is a compact, oriented manifold one has the Poincaré Duality isomorphism $H^*(X) \to H_*(X)$ given by capping with the fundamental class, and this lets one transplant the cup product onto $H_*(X)$. This is the modern approach to intersection theory.

The parallels here are intriguing: $K(R)$ is somehow like $H^*(X)$, and $G(R)$ is somehow like $H_*(X)$. The regularity condition is like being a manifold. We will spend the rest of this course exploring these parallels. [The reader might wonder what happened to the assumptions of compactness and orientability. Neither of these is really needed for Poincaré Duality, as long as one does things correctly. For the version of Poincaré Duality for noncompact manifolds one needs to replace ordinary homology with Borel-Moore homology—this is similar to singular homology, but chains are permitted to have infinitely many terms if they stretch out to infinity. For non-orientable manifolds one needs to use twisted coefficients.]

Exercise 2.16. Check that the tensor product makes $G(R)$ into a left module over $K(R)$ in the evident way. The canonical map $\alpha\colon K(R) \to G(R)$ is just multiplication by the class $[R] \in G(R)$.

Exercise 2.17. Let $f\colon R \to S$ be a map of commutative rings. If P is a finitely-generated projective R-module check that $S \otimes_R P$ is a finitely-generated projective S-module. Verify that there is an induced map $f_*\colon K(R) \to K(S)$ sending each $[P]$ to $[S \otimes_R P]$, and that f_* is a ring map.

Exercise 2.18. Let $f\colon R \to S$ be a flat map. Prove that there is an induced map of groups $f_!\colon G(R) \to G(S)$ sending each $[M]$ to $[S \otimes_R M]$.

Exercise 2.19. Let $f\colon R \to S$ be a ring map where S is finitely-generated as an R-module. Prove that there is a map of groups $f^*\colon G(S) \to G(R)$ that sends $[{}_SM]$ to $[{}_RM]$ for every finitely-generated S-module M.

Exercise 2.20. In topology $H_*(X)$ is a module over $H^*(X)$ via the cap product. Given $f\colon X \to Y$ there are maps $f^*\colon H^*(Y) \to H^*(X)$ and $f_*\colon H_*(X) \to H_*(Y)$, and f_* is a map of $H^*(Y)$-modules (where the module structure on the domain is via restriction of scalars along f^*): this last statement is the so-called projection formula $f_*(x \cap f^*y) = f_*x \cap y$. If $f\colon R \to S$ is a map where S is module-finite over R, then we have $f_*\colon K(R) \to K(S)$ and $f^*\colon G(S) \to G(R)$. Prove the analgous statement that f^* is a map of $K(R)$-modules.

2.21. **Some very basic algebraic geometry.** To further develop the analogies between $(K(R), G(R))$ and $(H^*(X), H_*(X))$ we need more of a geometric understanding of the former groups. This starts to require some familiarity with the language of algebraic geometry.

At its most basic level, algebraic geometry attempts to study the geometry of affine n-space $\mathbb{C}^n$ by seeing how it is reflected in the algebra of the ring of polynomial functions $R = \mathbb{C}[x_1, \ldots, x_n]$. Hilbert's Nullstellensatz says that points of $\mathbb{C}^n$ are in bijective correspondence with maximal ideals in R: the bijection sends $q = (q_1, \ldots, q_n)$ to the maximal ideal $m_q = (x_1 - q_1, \ldots, x_n - q_n)$. With a little work one can generalize this bijection. If $S \subseteq \mathbb{C}^n$ is any subset, define $\mathcal{I}(S) = \{f \in R \,|\, f(x) = 0 \text{ for all } x \in S\}$. This is an ideal in R, in fact a radical ideal (meaning that if $f^n \in \mathcal{I}(S)$ then $f \in \mathcal{I}(S)$). In the other direction, if $I \subseteq R$ is any ideal then define $V(I) = \{x \in \mathbb{C}^n \,|\, f(x) = 0 \text{ for all } f \in I\}$. Notice that $V(m_q) = \{q\}$ and $\mathcal{I}(\{q\}) = m_q$.

An **algebraic set** in $\mathbb{C}^n$ is any subset of the form $V(I)$ for some ideal $I \subseteq R$. The algebraic sets form the closed sets for a topology on $\mathbb{C}^n$, called the **Zariski topology**. One form of the Nullstellensatz says that V and $\mathcal{I}$ give a bijection between algebraic sets and radical ideals in R. Under this bijection the prime ideals correspond to **irreducible** algebraic sets—ones that cannot be written as $X \cup Y$ where both X and Y are proper closed subsets. Algebraic sets are also called algebraic **subvarieties**.

The above discussion is summarized in the following table:

Geometry	**Algebra**
$\mathbb{C}^n$ or $\mathbb{A}^n_{\mathbb{C}}$	$\mathbb{C}[x_1, \ldots, x_n] = R$
Points $(q_1, \ldots, q_n)$	Maximal ideals $(x_1 - q_1, \ldots, x_n - q_n)$
Algebraic sets	Radical ideals
Irreducible algebraic sets	Prime ideals

The ring R is best thought of as the set of maps of varieties $\mathbb{A}^n \to \mathbb{A}^1$, with ring operations given by pointwise addition and multiplication. If we restrict to some irreducible subvariety $X = V(P) \subseteq \mathbb{A}^n$ instead, then the ring of functions $X \to \mathbb{A}^1$ is R/P. This ring of functions is commonly called the **coordinate ring** of X. Much of the dictionary between $\mathbb{A}^n$ and R discussed above adapts verbatim to give a dictionary between X and its coordinate ring:

Geometry	**Algebra**
$X = V(P)$	$\mathbb{C}[x_1, \dots, x_n]/P = R/P$
Points in X	Maximal ideals in R/P
Algebraic subsets $V(I) \subseteq X$	Radical ideals in R/P
Irreducible algebraic sets $V(Q) \subseteq X$	Prime ideals in R/P.

Note that ideals in R/P correspond bijectively to ideals in R containing P, and likewise for prime (respectively, radical) ideals.

We need one last observation. Passing from $\mathbb{A}^n$ to $\mathbb{A}^{n+1}$ corresponds algebraically to passing from R to $R[t]$. If $X = V(P) \subseteq \mathbb{A}^n$ is an irreducible algebraic set, then $X \times \mathbb{A}^1 \subseteq \mathbb{A}^{n+1}$ is $V(P[t])$ where $P[t] \subseteq R[t]$. That is, the coordinate ring of X is R/P and the coordinate ring of $X \times \mathbb{A}^1$ is $R[t]/P[t] = (R/P)[t]$. We supplement our earlier tables with the following line:

Geometry	**Algebra**
$X \rightsquigarrow X \times \mathbb{A}^1$	$S \rightsquigarrow S[t]$

So far our story involves pairing rings that are finitely-generated over $\mathbb{C}$ with their corresponding geometric objects. With a leap of faith one can extend this to rings that are finitely-generated over an algebraically closed field, and even to finitely-generated rings over any field. One of the great developments in 20th century algebraic geometry is to go all in and extend the theory even further, to *all* commutative rings. To any commutative ring R we attach the geometric object $\operatorname{Spec} R$. As a set, this is the set of prime ideals in R. We equip it with the Zariski topology, where the closed sets are the ones of the form $V(I) = \{P \in \operatorname{Spec}(R) \mid P \supseteq I\}$. As a topological space this is a very primitive object with only bare-bones geometric information. To find more geometry we have to look to the maps between these gadgets.

The object $\operatorname{Spec} \mathbb{Z}[t]$ will be called the **affine line** and denoted $\mathbb{A}^1_{\mathbb{Z}}$. A basic fact to remember is that everything is going to be set up so that $\operatorname{Hom}(\operatorname{Spec} R, \mathbb{A}^1_{\mathbb{Z}}) = R$. That is, R is the ring of functions from $\operatorname{Spec} R$ to the affine line. A map $\operatorname{Spec} R \to \operatorname{Spec} S$ will then give rise to a map of rings $\operatorname{Hom}(\operatorname{Spec} S, \mathbb{A}^1_{\mathbb{Z}}) \to \operatorname{Hom}(\operatorname{Spec} R, \mathbb{A}^1_{\mathbb{Z}})$, i.e. a map of rings $S \to R$. In fact, let us just define the category of affine schemes $\mathcal{A}ff$ to have objects the $\operatorname{Spec} R$ and where maps from $\operatorname{Spec} R$ to $\operatorname{Spec} S$ are the same as ring maps from S to R. That is, $\mathcal{A}ff = (\mathcal{C}omm\mathcal{R}ing)^{op}$.

One final piece of terminology for now. The phrase "affine scheme" just means one of the objects $\operatorname{Spec} R$. The phrase"affine variety" technically means a $\operatorname{Spec} R$ where R is an integral domain that is finitely-generated over a ground field k. For some reason the word "variety" is the more appealing of the two words, and as a result one sometimes ends up saying "variety" when what one really means is "scheme". In almost all cases the true intent is clear from context, so we won't worry too much about this distinction.

We have defined $G(-)$ and $K(-)$ as functors taking rings as their inputs, but we could also think of them as taking affine schemes as their inputs. We will write $G(R)$ and $G(\operatorname{Spec} R)$ interchangeably, and similarly for the K-groups. It turns out that the geometric perspective and notation are very useful—many properties of these functors take on a familiar "homological" form when written geometrically.

For example, a map $u\colon \operatorname{Spec} R \to \operatorname{Spec} S$ yields a map $u^*\colon K(\operatorname{Spec} S) \to K(\operatorname{Spec} R)$ as one would expect for a cohomology ring.

For the moment we will mostly keep with the algebraic notation, writing $K(R)$ more often than $K(\operatorname{Spec} R)$. But it is good to train oneself to "see" $K(\operatorname{Spec} R)$ even when it is not explicitly written that way.

2.22. **Further properties of $G(R)$.** We return to the study of the groups $G(R)$ and $K(R)$, for the moment concentrating on the former.

Theorem 2.23. *If R is Noetherian, the Grothendieck group $G(R)$ is generated by the set of elements $[R/P]$ where $P \subseteq R$ is prime.*

Before proving this result let us comment on the significance. When X is a topological space, the groups $H_*(X)$ have a geometric presentation in terms of "cycles" and "homologies". The cycles are, of course, generators for the group. The definition of $G(R)$ doesn't look anything like this, but Theorem 2.23 says that the group is indeed generated by classes that have the feeling of "algebraic cycles" on the variety $\operatorname{Spec} R$. One thinks of $G(R)$ as having a generator $[R/P]$ for every irreducible subvariety of R, and then there are some relations amongst these that we don't yet understand. It is worth pointing out that in $H_*(X)$ the cycles are strictly separated by dimension—the dimensions i cycles are confined to the single group $H_i(X)$—whereas in $G(R)$ the cycles of different dimensions are all inhabiting the same group. This is one of the main differences between K-theory and singular homology/cohomology.

To prove Theorem 2.23 we first need a lemma from commutative algebra:

Lemma 2.24. *Let R be a Noetherian ring. For any nonzero finitely-generated R-module M there exists a prime ideal $P \subseteq R$ and an embedding $R/P \hookrightarrow M$. Equivalently, there is some $z \in M$ whose annihilator is prime.*

Proof. Consider the set of ideals

$$\mathcal{S} = \{\operatorname{Ann}(m) \,\big|\, m \in M, m \neq 0\}.$$

Equivalently, $\mathcal{S}$ is the set of ideals I such that R/I embeds into M. Note that $S \neq \emptyset$ because $M \neq 0$. Since R is Noetherian $\mathcal{S}$ has a maximal element $I = \operatorname{Ann}(m)$. We will prove that I is prime. Suppose $ab \in I$ and $b \notin I$. Then $bm \neq 0$ so $\operatorname{Ann}(bm) \in \mathcal{S}$ and $\operatorname{Ann}(bm) \supseteq \operatorname{Ann}(m) = I$. By maximality $\operatorname{Ann}(bm) = I$. But $ab \in I$ so $abm = 0$, hence $a \in \operatorname{Ann}(bm)$ and therefore $a \in I$. This completes the proof that I is prime. $\square$

Proof of Theorem 2.23. Let M be a finitely-generated R-module. We will use repeated applications of Lemma 2.24 to construct a so-called **prime filtration** of M. Pick an embedding $R/P_0 \hookrightarrow M$, and let $M_0 = R/P_0$. Next consider M/M_0. If $M/M_0 = 0$, our filtration is complete. If $M/M_0 \neq 0$, then there exists a prime P_1 and an embedding $R/P_1 \hookrightarrow M/M_0$. Let $\pi\colon M \to M/M_0$ denote the projection and define $M_1 = \pi^{-1}(R/P_1)$. Then $\pi\colon M_1 \to R/P_1$ also has kernel M_0; that is, $M_0 \subseteq M_1$ and $M_1/M_0 \cong R/P_1$. Next consider M/M_1 and repeat. This process yields a filtration of M

$$0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

such that $M_{i+1}/M_i \cong R/P_i$. The filtration must be finite since R is Noetherian. By Proposition 2.5(b) we have that $[M] = \sum [M_{i+1}/M_i] = \sum [R/P_i]$, and so the set $\{[R/P] \,|\, P \text{ is prime in } R\}$ generates $G(R)$. $\square$

Remark 2.25. The prime filtrations constructed in the above proof are very useful, and will appear again and again in our arguments. For future use we note that if an ideal $I \subseteq R$ is such that $IM = 0$, then I also kills any subquotient of M. Consequently, I will be contained in any P_i for which R/P_i appears as a subquotient in a prime filtration of M.

If M is an R-module write $M[t]$ for the $R[t]$-module $M \otimes_R R[t]$. The functor $M \mapsto M[t]$ is exact because $R[t]$ is flat (in fact, free) over R. So we have an induced map $\alpha\colon G(R) \to G(R[t])$ given by $[M] \mapsto [M[t]]$.

Theorem 2.26 (Homotopy invariance)**.** *If R is Noetherian, $\alpha\colon G(R) \to G(R[t])$ is an isomorphism.*

We comment on the name "homotopy invariance" for the above result. If $X = \operatorname{Spec} R$ then $\operatorname{Spec} R[t] = X \times \mathbb{A}^1$, so the result says that $G(-)$ gives the same values on X and $X \times \mathbb{A}^1$. This is reminiscent of a functor on topological spaces giving the same values on X and $X \times I$.

Proof. We will first construct a left inverse $\beta\colon G(R[t]) \to G(R)$. A naive possibility for the map β is $J \mapsto J/tJ = J \otimes_{R[t]} R[t]/(t)$, but this doesn't preserve short exact sequences in general. So we correct this using Tor, and instead define

$$\beta([J]) = [\operatorname{Tor}_0^{R[t]}(J, R[t]/(t))] - [\operatorname{Tor}_1^{R[t]}(J, R[t]/(t))].$$

Before checking that this is well-defined, let us analyze the two Tor-groups. Recall that we can calculate Tor by taking an $R[t]$-resolution of either variable. In this case, it is easier to resolve $R[t]/(t)$:

$$0 \to R[t] \xrightarrow{t} R[t] \to R[t]/(t) \to 0.$$

Tensoring with J yields $0 \to J \xrightarrow{t} J \to 0$, so that $\operatorname{Tor}_0^{R[t]}(J, R[t]/(t)) = J/tJ$ and $\operatorname{Tor}_1^{R[t]}(J, R[t]/(t)) = \operatorname{Ann}_J(t)$. Notice also that $\operatorname{Tor}_i^{R[t]}(J, R[t]/(t)) = 0$ for $i > 1$. We have

$$\beta([J]) = [J/tJ] - [\operatorname{Ann}_J(t)] = \sum_{i=0}^{\infty} (-1)^i [\operatorname{Tor}_i^{R[t]}(J, R[t]/(t))].$$

The fact that β is a well-defined group homomorphism now follows by the usual argument: a short exact sequence of modules induces a long exact sequence of Tor groups, and the alternating sum of these is zero in $G(R)$. It is immediate that $\beta\alpha = Id$: this follows from the fact that for any R-module M one has $M[t]/tM[t] \cong M$ and $\operatorname{Ann}_{M[t]}(t) = 0$. Consequently, α is injective.

The difficult part of the proof is showing that α is surjective. We will use the fact, from Theorem 2.23, that $G(R[t])$ is generated by elements of the form $[R[t]/Q]$ for primes $Q \subseteq R[t]$. It suffices to show that each $[R[t]/Q]$ is in the image of α. Let us write S for $R[t]$, and define

$$T = \{Q \cap R \mid Q \subseteq S \text{ is prime and } [S/Q] \notin \operatorname{im}(\alpha)\}.$$

Our goal is to show that T must be empty.

If $T \neq \emptyset$ then since S is Noetherian it has a maximal element $P = Q \cap R$ for some prime $Q \subseteq S$. Using this P and this Q, we will construct an S-module W which forces $[S/Q]$ to lie in $\operatorname{im}(\alpha)$, thus obtaining a contradiction.

First, some observations:

(1) If $I \subseteq R$ is any ideal then the expansion IS equals $I[t]$, the set of polynomials with coefficients in I. One has $S/IS \cong (R/I)[t]$.
(2) Any S-module M which is killed by $P+u$ for some $u \in R-P$ must lie in $\operatorname{im}(\alpha)$. This is because for each prime Q_i appearing in a prime filtration of M, we have $Q_i \supseteq \operatorname{Ann}_R(M) \supseteq P+u$. In particular, none of these Q_i can be in T since P was chosen to be maximal. So $[S/Q_i] \in \operatorname{im}(\alpha)$ for all these Q_i, and hence $[M] \in \operatorname{im}(\alpha)$ as well.
(3) For any prime $J \subseteq R$ we have $[S/JS] \in \operatorname{im}(\alpha)$, since $S/JS = (R/J)[t] = \alpha([R/J])$.
(4) If $f \in S - JS$ where $J \subseteq R$ is prime, then $[S/(JS+f)] = 0$ in $G(S)$ since $S/(JS+f)$ fits into the short exact sequence
$$0 \longrightarrow S/JS \xrightarrow{f} S/JS \longrightarrow S/(JS+f) \longrightarrow 0.$$
Note that $S/JS \cong (R/J)[t]$, which is a domain—and this is why multiplication by f is injective.

Consider the maps
$$S \twoheadrightarrow S/PS \hookrightarrow (R-P)^{-1}(S/PS).$$
Observe that $(R-P)^{-1}(S/PS) = (R_P/PR_P)[t]$. But R_P/PR_P is a field, so the ring $(R-P)^{-1}(S/PS)$ is a PID. Therefore the image of Q in $(R-P)^{-1}(S/P[t])$ is generated by a single element. Let $f \in Q$ be some lifting of this generator to S.

Consider the S-module $W = Q/(PS+f)$. Since Q and f have the same image in the ring $(R-P)^{-1}(S/PS)$, we have $(R-P)^{-1}W = 0$. Now, W is finitely generated (as an S-module), so there exists some $u \in R-P$ such that $uW = 0$. Since $PW = 0$ by the definition of W, we have that W is killed by $P+u$. By observation (2) above, $[W] \in \operatorname{im}(\alpha)$.

At the same time, W fits into the exact sequence $0 \to W \to S/(PS+f) \to S/Q \to 0$, and we know $[S/(PS+f)] = 0$ in $G(S)$ by observation (4). But this implies that $[W]$ and $[S/Q]$ are additive inverses, and hence $[S/Q]$ lies in $\operatorname{im}(\alpha)$, contradicting our choice of Q. □

Here is an interesting consequence of homotopy invariance:

Corollary 2.27. *Let F be a field. Then $K(F[x_1,\dots,x_n]) \cong \mathbb{Z}$.*

Proof. We have $K(F[x_1,\dots,x_n]) \cong G(F[x_1\dots,x_n])$ by Theorem 2.13, since the ring $F[x_1,\dots,x_n]$ is regular by Hilbert's Syzyzy Theorem. We also have $G(F[x_1,\dots,x_n]) \cong G(F)$ by homotopy invariance, and $G(F) \cong \mathbb{Z}$ via the dimension map. □

In the next section we will see what Corollary 2.27 says about projectives over $F[x_1,\dots,x_n]$. See Proposition 3.6.

2.28. **Regular local rings.** We have seen that regularity is an important condition when dealing with K-theory, and so it is worth giving a crash course on the theory of regular local rings. We start with some examples. All of the following local rings are regular:

(1) $k[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$ where k is a field;
(2) $k[[x_1,\dots,x_n]]$, the ring of formal power series over a field k;

(3) any discrete valuation ring V (equivalently, a local PID), for example $\mathbb{Z}_{(p)}$ or its p-adic completion $\hat{\mathbb{Z}}_p$ or any local ring of the ring of integers in a number field;
(4) $V[[x_1, \dots, x_n]]$ where V is any discrete valuation ring;
(5) $\mathbb{Z}_p[[x_1, \dots, x_n]]/(x_1^2 + x_2^2 + \cdots + x_n^2 - p)$.

We next give some general information that will help sort the above examples into classes. Let (R, m, k) be a Noetherian local ring. This means that R is a Noetherian ring with unique maximal ideal m and $k = R/m$. If $\mathrm{char}(R) = \mathrm{char}(k)$ then R is said to be **equicharacteristic**, and otherwise it is **mixed characteristic**. Regular local rings are always integral domains and so the possibilities for the pair $(\mathrm{char}(R), \mathrm{char}(k))$ are only $(0, 0)$, (p, p), and $(0, p)$ where p is a prime.

Note that $k[[x_1, \dots, x_n]]$ is equicharacteristic, whereas rings like $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p[[x_1, \dots, x_n]]$ are mixed characteristic. The equicharacteristic case has the closer ties to geometry, whereas the mixed characteristic case appears more in number-theoretic situations.

Mixed characteristic local rings (R, m, k) can be further divided into two classes depending on whether or not $p \in m^2$. If $p \in m^2$ one says that R is **ramified**, and otherwise R is **unramified**. Note that $\mathbb{Z}_{(p)}$ and $\hat{\mathbb{Z}}_p$ are unramified, whereas the example from (5) above is ramified.

If one takes an algebraic variety over a field k and looks at the local ring at a nonsingular point, one obtains a regular local ring that is most likely not in the list (1)–(5) above. It is far from true, for example, that the local ring of a dimension n nonsingular variety is isomorphic to $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$. However, it turns out that this is true after completion. The upshot is that there are not very many *complete* regular local rings, and in some sense we know them all. This is part of the Cohen structure theorems for complete local rings. The following theorem brings together several results, but (d) and (e) are the Cohen classification theorems:

Theorem 2.29. *Let (R, m, k) be a local ring and set $n = \dim R$.*
(a) If R is regular then it is a domain.
(b) R is regular if and only if the m-adic completion $\hat{R}$ is regular.
(c) If R is regular and $f \in m - m^2$ then R/f is also regular.
(d) If R is complete and regular and equicharacteristic then $R \cong k[[x_1, \dots, x_n]]$ for some field k.
(e) Suppose R is complete and regular and mixed characteristic. Then there are two possibilities:
- *If R is unramified ($p \notin m^2$) then $R \cong V[[x_1, \dots, x_{n-1}]]$ for some complete discrete valuation ring V with maximal ideal (p).*
- *If R is ramified ($p \in m^2$) then $R \cong V[[x_1, \dots, x_n]]/(p - f)$ for some complete discrete valuation ring V with maximal ideal (p) and some $f \in (p, x_1, \dots, x_n)^2$ such that $p \nmid f$.*

For parts (a)–(c) see [BH, Propositions 2.2.2–2.2.4]. Parts (d) and (e) are from [Co], with [Sam] another good source. The equicharacteristic case in (d) can also be found in [ZS, Section 12 of Chapter VIII].

Theorem 2.29 gives us a very good handle on complete regular local rings in the equicharacteristic and the unramified mixed characteristic cases. But although the theorem says something concrete and useful about the mixed ramified case, things are much more mysterious here because of how open the choice of f is. Many theorems about regular local rings are known in the equicharacteristic and mixed unramified cases but open in the ramified case.

∘ Exercises ∘

Exercise 2.30. Let R be a commutative ring.

(a) Let $Q \subseteq R$ be a prime ideal. Prove that
$$P \mapsto \dim_{QF(R/Q)}(QF(R/Q) \otimes_{R/Q} P/QP)$$
is an additive function on finitely-generated projectives and therefore induces a homomorphism $\text{rank}_Q \colon K(R) \to \mathbb{Z}$.

(b) Let $Q \subseteq R$ be a minimal prime ideal. Then R_Q is Artinian, so every finitely-generated module has finite length. Prove that $M \mapsto \ell_{R_Q}(M_Q)$ is additive and so induces a homomorphism $\ell_Q \colon G(R) \to \mathbb{Z}$.

(c) Prove that in both $K(R)$ and $G(R)$ the class $[R]$ is non-torsion.

Exercise 2.31. If R has the property that every finitely-generated module has a finite free resolution, prove that $K(R) \cong \mathbb{Z}$ and $G(R) \cong \mathbb{Z}$. Give an example of an R for which this property fails. Even better, give an example of a regular ring R for which this property fails.

Exercise 2.32. Let $R = k[x]/(x^n)$ where k is a field and $n \geq 2$. Use the classification of modules over $k[x]$ to prove that every finitely-generated projective R-module is free. Verify that $K(R) \cong G(R) \cong \mathbb{Z}$ but that the canonical map $K(R) \to G(R)$ is isomorphic to multiplication-by-n (and in particular, is not an isomorphism).

Exercise 2.33. Fix a prime p and set $R = \mathbb{Z}[x]/(x^2 - px)$.

(a) Determine all the prime ideals in R and verify that $\operatorname{Spec} R$ looks like two copies of $\operatorname{Spec} \mathbb{Z}$ glued together at the prime (p).

(b) Suppose Q is a prime in R other than (x), $(x-p)$, and (x,p). Show that R/Q has a finite free resolution and use this to prove that $[R/Q] = 0$ in $G(R)$. Likewise, find an exact sequence showing that $[R/(x,p)] = 0$ in $G(R)$. So $G(R)$ is generated by $[R/xR]$ and $[R/(x-p)R]$. Also prove that $[R] = [R/xR] + [R/(x-p)R]$.

(c) Use the maps on G-groups induced by $R \to R/xR$ and $R \to R/(x-p)R$ to prove that $G(R) \cong \mathbb{Z}^2$.

(d) Find a free resolution of R/xR and use this to prove that $\operatorname{Tor}_*(R/xR, R/(x,p))$ is nonzero in all degrees. Conclude that R is not regular.
[Note: We will investigate $K(R)$ in Exercise 3.9].

Exercise 2.34. Let $R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[X]/(X^2+3)$.

(a) Determine all of the prime ideals of R. Prove that $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ is surjective, and that the fiber over (p) is two elements when $3|p-1$ and a singleton otherwise. Prove that every prime other than $(2, 1+X)$ is principal.

(b) Construct a free resolution of $R/(2,1+X)$ and prove that $\operatorname{Tor}_i(R/(2,1+X), R/(2,1+X))$ is nonzero for all i. Deduce that R is not regular. [Hint: The resolution can be made to exhibit a nice periodicity.]

(c) Check that R_P is regular for every prime P except for $(2,1+X)$. Said differently, $(2,1+X)$ is the only singular point of $\operatorname{Spec} R$.

(d) Construct an exact sequence $0 \to R/(2,1+X) \to R/(2) \to R/(2,1+X) \to 0$.

(e) Using the previous parts deduce that $G(R)$ is generated by $[R]$ and $[R/(2,1+X)]$ and is either $\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}/2$, depending on whether $[R/(2,1+X)]$ is zero or nonzero.
(f) ????

Exercise 2.35. Let $S=\mathbb{Z}[w]/(w^2+w+1)$ and let R be as in the previous exercise. It will be convenient to regard $R\subseteq S$ via $X=2w+1$ (if we regard these rings as inside of $\mathbb{C}$ then $X=\sqrt{3}i$ and $w=\frac{-1+\sqrt{3}i}{2}$). Note that w is integral over $\mathbb{Z}$; it turns out that S is the integral closure of R. The ring S is the ring of integers in the number field $\mathbb{Q}(\sqrt{-3})$.

(a) Determine all of the prime ideals in S and prove that $\operatorname{Spec} S\to\operatorname{Spec} R$ is bijective.
(b) Check that the prime in S lying over $(2,1+X)$ is (2), and in particular is principal. Note that $R/(2,1+X)\to S/(2)$ is the field extension $\mathbb{F}_2\hookrightarrow\mathbb{F}_4$.
(c) Verify that S is a PID and conclude $K(S)\cong G(S)\cong\mathbb{Z}$.
[Discussion: What is happening geometrically here is that $\operatorname{Spec} S\to\operatorname{Spec} R$ resolves the singularity in $\operatorname{Spec} R$. The point $(2,1+X)\in\operatorname{Spec} R$, with residue field $\mathbb{F}_2$, is being "blown up" into the point $(2)\in\operatorname{Spec} S$ with residue field $\mathbb{F}_4$. Away from these points the map $\operatorname{Spec} S\to\operatorname{Spec} R$ is an isomorphism.]

3. A closer look at projectives

Recall that a module is projective if and only if it is a direct summand of a free module (there is also a description in terms of lifting criteria, of course). So free modules are projective, and for almost all applications in homological algebra one can get by with using *only* free modules. Consequently, it is common for students not to know many examples of non-free projectives. One of our goals in this section is to remedy this.

It turns out to be very useful to be able to think about projectives geometrically. Projectives over a commutative ring R correspond to vector bundles over $\operatorname{Spec} R$. This is not at all obvious, and we won't fully understand it until Section 10 after we have developed more tools from the theory of vector bundles. But in this section we will start to see these ideas play out as we import some of the geometric language and intuition into our algebraic discussion.

Before diving into some examples we need a couple of small tools. Just as vector bundles are "locally trivial", projectives are locally free. This is a fun piece of local algebra:

Proposition 3.1. *Over a local ring all projectives are free.*

Proof. Let (R,m,k) be a local ring. A set of generators for an R-module M is minimal (in the sense of no proper subset also generating M) if and only if its image in M/mM is an R/m-basis. So all minimal generating sets for a module have the same size. (Of course this is wildly false when R is not local). Every relation amongst the elements of a minimal set of generators must have all coefficients belonging to m, otherwise there would be a unit coefficient and we would contradict minimality.

Pick a generating set for M and map a free module $F_0\twoheadrightarrow M$ by sending the free basis to these generators. Then take the kernel K and repeat, mapping a free module $F_1\twoheadrightarrow K$ by sending the free basis to a minimal generating set. Note that these generating sets might be infinite here. Continuing in this way we build a so-called 'minimal resolution' $F_\bullet \to M \to 0$.

The fact that we picked minimal generating sets at each stage implies that $F_\bullet \otimes R/m$ has vanishing differentials, since all relations amongst the minimal generators have coefficients in m. So $\operatorname{rank}(F_i) = \dim_k \operatorname{Tor}_i(M, k)$.

If M is projective then it is also flat, and hence all of the higher Tor's must vanish. Therefore $F_1 = 0$ and the resolution $0 \to F_0 \to M \to 0$ shows that M is free. $\square$

Let R be a commutative ring, P a finitely-generated projective over R, and $q \subseteq R$ a prime ideal. Then P_q is free and finitely-generated over R_q, so let us define

$$\operatorname{rank}_q(P) = \operatorname{rank}_{R_q}(P_q).$$

Note that rank_q is additive in the sense that $\operatorname{rank}_q(P \oplus Q) = \operatorname{rank}_q(P) \oplus \operatorname{rank}_q(Q)$. If $q \subseteq q'$ then $P_q = (P_{q'})_q$ and so $\operatorname{rank}_q(P) = \operatorname{rank}_{q'}(P)$. Therefore $\operatorname{rank}_?(P)$ is constant on chains of primes. If R is a domain, for example, then all primes can be connected by a chain to (0) and so $\operatorname{rank}_?(P)$ is constant. In this context we will just talk about the rank of P. (An example of a projective with different ranks at different primes is given in (1) below).

Note that when m is a maximal ideal we can also write

$$\operatorname{rank}_m(P) = \operatorname{rank}_{R_m}(P_m) = \dim_{R_m/mR_m}(P_m/mP_m) = \dim_{R/m}(P/mP).$$

Recall that maximal ideals are the closed points in $\operatorname{Spec} R$. The R/m-vector space P/mP plays the role of the "fiber" of the projective over our closed point.

Remark 3.2 (The geometry of local rings)**.** In view of the above discussion this might be a good time to clarify some mysteries about local rings. If q is a prime in R then $\operatorname{Spec} R_q$ may be identified with the subset of $\operatorname{Spec} R$ consisting of all primes contained in q. This subset is usually neither open nor closed in $\operatorname{Spec} R$. What we have instead is this:

$$\operatorname{Spec} R_q = \bigcap_{q \in U^{\text{open}} \subseteq \operatorname{Spec} R} U.$$

That is, $\operatorname{Spec} R_q$ is the intersection of all Zariski open neighborhoods of q in $\operatorname{Spec} R$. For Hausdorff topological spaces such an intersection would always just be the point itself, and therefore non-interesting. The analog here is that $\operatorname{Spec} R_q$ has only one *closed* point, but it nevertheless has a significant amount of information lurking in the non-closed points.

For a typical space that appears in algebraic topology—a manifold or CW-complex, for example—while the intersection of open neighborhoods of a point is just the point itself one can nevertheless sense that the open neighborhoods get similar as they get smaller and smaller. In some sense there is some limiting information there that is not captured at just the set level. This is really what is happening in algebraic geometry. Our $\operatorname{Spec} R_q$ behaves like an "infinitesimal" Zariski neighborhood of q. This idea takes some getting used to, but it is important for understanding how information passes back and forth between algebra and geometry.

Now let us turn to some examples of interesting projectives:

(1) Let $R = \mathbb{Z}/6$. Since $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$, both $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective R-modules—and they are clearly not free. More generally, if $R = S \times T$ then we can make S into an R module via $(s,t).u = su$ where $s, u \in S$ and $t \in T$. Then S is projective but not free.

Geometrically, what is happening here is that $\operatorname{Spec} R = \operatorname{Spec} S \amalg \operatorname{Spec} T$. When we make S into an R-module as above we are consructing the vector bundle that is free of rank one on the $\operatorname{Spec} S$ component but zero on the $\operatorname{Spec} T$ component. Check algebraically that $\operatorname{rank}_m(S)$ is either 0 or 1 and depends on the choice of m.

(2) Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1+\sqrt{-5}) = (2, 1-\sqrt{-5})$. For convenience let us write $\mu = \sqrt{-5}$. A standard tool for dealing with this ring is the norm map $N(a+b\mu) = (a+b\mu)(a-b\mu) = a^2+5b^2$. This is multiplicative, i.e. $N(xy) = N(x)N(y)$. Using this one argues that if I were generated by a single element $a+b\mu$ then 2 would be a multiple of a^2+5b^2, and one quickly obtains a contradiction. So I is not principal.

Let K be the kernel of the map $R^2 \to I$ sending e_1 to 2 and e_2 to $1+\mu$. A little work shows that K is spanned by $(1+\mu, -2)$ and $(-3, 1-\mu)$. If one defines $\chi\colon R^2 \to K$ by

$$\chi(e_1) = (3, -1+\mu), \quad \chi(e_2) = (1+\mu, -2),$$

it is readily verified that χ is a splitting for the sequence $0 \to K \to R^2 \to I \to 0$. So $K \oplus I \cong R^2$, and hence both K and I are projective.

The inclusion $I \subseteq R$ becomes an isomorphism after tensoring with $QF(R)$, so I has rank one. If I were free then we would have $I \cong R$. However, this would contradict I being non-principal. So I is a non-free projective.

This example generalizes: if D is a Dedekind domain (such as the ring of integers in an algebraic number field) then every ideal $I \subseteq D$ is projective. Non-principal ideals are never free.

Exercise 3.3. Verify that K is generated by $(1+\mu, -2)$ and $(-3, 1-\mu)$. Note that the second coordinates of elements of K therefore all belong to I. Prove that the composition $K \hookrightarrow R \xrightarrow{\pi_2} I$ is an isomorphism. Deduce that $I \oplus I \cong R \oplus R$.

(3) Let $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$. If $C(S^2)$ denotes the ring of continuous functions $S^2 \to \mathbb{R}$, note that we may regard R as sitting inside of $C(S^2)$: it is the subring of polynomial functions on the 2-sphere. The connections with the topology of the 2-sphere will be important below.

Let $\pi\colon R^3 \to R$ be the map $\pi(f,g,h) = xf+yg+zh$. That is, π is left-multiplication by the matrix $\begin{bmatrix} x & y & z \end{bmatrix}$. Let T be the kernel of π:

$$0 \longrightarrow T \rightarrowtail R^3 \xrightarrow{\pi} R \longrightarrow 0.$$

The map π is split via $\chi\colon R \to R^3$ sending $1 \mapsto (x,y,z)$. We conclude that $T \oplus R \cong R^3$, so T is projective.

We claim that T is not free. Suppose, towards a contradiction, that T is free. Since $T \oplus R \cong R^3$ we have $\operatorname{rank}(T) = 2$, and so $T \cong R^2$. Choose an isomorphism $R^2 \to T$, let e_1 and e_2 be the standard basis for R^2, and let the image of e_1 under our isomorphism be (f,g,h). So f, g, and h are polynomial functions on S^2 and

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} f(p) \\ g(p) \\ h(p) \end{bmatrix} = 0$$

for all $p = (p_1, p_2, p_3) \in S^2$. So $p \mapsto (f(p), g(p), h(p))$ is a tangent vector field on S^2. By the Hairy Ball Theorem we can find a point $q = (q_1, q_2, q_3) \in S^2$ such that $f(q) = g(q) = h(q) = 0$. Let $m = (x - q_1, y - q_2, z - q_3) \subseteq R$ and consider the commutative diagram

$$\begin{tikzcd} R^2 \arrow[r, "\cong"] \arrow[d, two heads] & T \arrow[r, tail] \arrow[d, two heads] & R^3 \arrow[d, two heads] \\ (R/mR)^2 \arrow[r, "\cong"'] & T/mT \arrow[r, tail] & (R/mR)^3. \end{tikzcd}$$

The lower right map is an injection because the upper right map is a split injection. Note that $R/mR \cong \mathbb{R}$ via $F \mapsto F(q)$. Start with e_1 in the upper left corner and compute its image in $(R/mR)^3 \cong \mathbb{R}^3$ under the two outer ways of tracking around the diagram. Along the top route e_1 maps to $(f(q), g(q), h(q))$ which is just $(0, 0, 0)$. On the other hand, along the bottom route e_1 first maps to $(1, 0) \in \mathbb{R}^2$ and then the bottom composite is an injection—so the image in $\mathbb{R}^3$ is nonzero. This is a contradiction, so we conclude that T is not free. (In fact, we have proven more: we have proven that T does not contain R as a direct summand).

Note that T is an algebraic analog of the tangent bundle of S^2. As remarked at the beginning of the section, these parallels between projective modules and vector bundles are very important. We will see much more about them in Section 10.

(4) Let us do one more example where we use topology to produce an example of a non-free projective. This example is based on the Möbius bundle over S^1. Let

$$S = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$$

and let $R \subseteq S$ be the span of the even degree monomials. One should regard S as the ring of polynomial functions on the circle, and R is the ring of polynomial functions $f(x, y)$ satisfying $f(x, y) = f(-x, -y)$. So R is trying to be the ring of polynomial functions on $\mathbb{R}P^1$ (which happens to be homeomorphic to S^1).

Let $P \subseteq S$ be the $\mathbb{R}$-linear span of the homogeneous polynomials with odd total degree. Observe that P is a finitely generated R-module and we have $\pi\colon R^2 \twoheadrightarrow P$ via $\pi(e_1) = x$ and $\pi(e_2) = y$. Define $\chi\colon P \to R^2$ via

$$h \mapsto \chi(h) = \begin{bmatrix} xh \\ yh \end{bmatrix}.$$

One checks that $\pi \circ \chi = id$, so P is projective. We leave it as an exercise for the reader to show that P is not free.

Exercise 3.4. Complete example (4) above by showing that P is not free.

The topological examples (3) and (4), as well as many similar ones, can be found in the lovely paper [Sw]. See also Section 10 below.

Example 3.5. Here is an example of an algebraic problem whose solution involves non-free projectives coming from topology. If A and B are commutative rings such that $A[X] \cong B[X]$, does it follow that $A \cong B$? This is a natural question that once upon a time had people stumped. The first counterexample was due to Hochster [Hoc] and is closely related to our current discussion.

Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ as in (3) above, and let T also be as above. Recall that T is not free but that there is an isomorphism $R^3 \cong T \oplus R$. Apply the symmetric algebra construction $\mathrm{Sym}_R(-)$ to obtain

$$\begin{aligned} R[U, V, W] \cong \mathrm{Sym}_R(R^3) \cong \mathrm{Sym}_R(T \oplus R) &\cong \mathrm{Sym}_R(T) \otimes_R \mathrm{Sym}_R(R) \\ &= \mathrm{Sym}_R(T) \otimes_R R[W] \\ &= \mathrm{Sym}_R(T)[W]. \end{aligned}$$

We take $A = R[U, V]$ and $B = \mathrm{Sym}_R(T)$, and the above gives a ring isomorphism $A[W] \cong B[W]$.

It requires a little work to prove that $A \ncong B$, though it is not too bad. We only give a sktech. One first argues that any isomorphism must be $\mathbb{R}$-linear, then that the isomorphism must map R onto itself. By composing with the inverse to this automorphism of R, one can assume that the isomorphism $A \to B$ is R-linear. But if there is such an R-algebra isomorphism then B is generated as an R-algebra by two elements, and one readily proves that the degree 1 homogeneous components of those two pieces generate T as an R-module. But the arguments from (3) above show that T cannot be generated by two elements, and this is the contradiction. We refer to [Hoc] for more details.

The original question about A and B, together with some variants, are often called the *Zariski Cancellation Problem*. This is an active area of research. For example, one open question is whether or not there exist counterexamples where A is a polynomial ring over a field of characteristic zero (such counterexamples *are* known in positive characteristic, by recent work of Gupta [Gu1]). We mention the survey paper [Gu2] as just one point of entry into this subject.

A projective module P is called **stably free** if there exists a free module F such that $P \oplus F$ is free. The example in (3) gives a projective that is stably-free but not free. It turns out that $K(R)$ can be used to tell us whether such modules exist or not. To see this, recall that if $m \subseteq R$ is a maximal ideal then $\mathrm{rank}_m(-)$ is an additive function on finitely-generated, projective modules. So it induces a map $\mathrm{rank}_m(-)\colon K(R) \to \mathbb{Z}$, which is evidently surjective because $\mathrm{rank}_m(R) = 1$. This shows that $K(R)$ always contains $\mathbb{Z}$ as a direct summand.

Define the **reduced Grothendieck group of** R to be

$$\widetilde{K}(R) = K(R)/\langle [R] \rangle.$$

Here is another way to define this group. Take the set of isomorphism classes of finitely-generated projectives and impose the equivalence relation $(P) \sim (P \oplus R)$ for every P. Such equivalences classes are called **stable projectives**. Define a monoid structure on this set by $(P) + (Q) = (P \oplus Q)$, and note that $(0) = (R)$ is the identity. If P is any projective then there exists a Q such that $P \oplus Q$ is free, and therefore $(P) + (Q) = 0$ in this monoid; hence, we have a group. This is called the **Grothendieck group of stable projectives**. One readily checks that this group is isomorphic to $\widetilde{K}(R)$, with the equivalence class (P) corresponding to the element $[P] \in \widetilde{K}(R)$.

Proposition 3.6. *Let R be a commutative ring. The following are equivalent:*
(1) $K(R) \cong \mathbb{Z}$
(2) $\widetilde{K}(R) = 0$
(3) Every finitely-generated, projective R-module is stably-free.

Proof. Immediate. □

Example 3.7. Recall from Corollary 2.27 that if F is a field then $K(F[x_1, \ldots, x_n]) = \mathbb{Z}$. Thus, every finitely-generated, projective $F[x_1, \ldots, x_n]$-module is stably-free.

In the 1950s, Serre conjectured that every finitely-generated projective over $F[x_1, \ldots, x_n]$ is actually free. As we will see later (Remark 11.6 below), the motivation for this conjecture is inspired by topology and the connection between vector bundles and projective modules. Quillen [Q4] and Suslin [Su] independently proved Serre's conjecture in the 1970s.

Example 3.8. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let I be the ideal $(2, 1 + \sqrt{-5})$. We saw in example (2) from the beginning of this section that I is a rank one projective that is not free. Could I be stably free? If it were, then we would have $I \oplus R^k \cong R^{k+1}$, for some k. Apply the exterior product $\Lambda^{k+1}(-)$ to deduce that

$$R \cong \Lambda^{k+1}(R^{k+1}) \cong \Lambda^{k+1}(I \oplus R^k) \cong \Lambda^1(I) \otimes \Lambda^k(R^k) \cong I \otimes R \cong I$$

(in the third isomorphism we have used the formula for the exterior product of a direct sum, together with the general fact that $\Lambda^j(P) = 0$ for $j > \mathrm{rank}(P)$). However, this is a contradiction as we have already seen that I is not free. Hence, I is not stably free and so $[I]$ determines a nonzero class in $\tilde{K}(R)$. By Exercise 3.3 we know $I \oplus I \cong R^2$ and so $2[I] = 0$ in $\tilde{K}(R)$, therefore we have a nonzero 2-torsion class.

Again, this example generalizes to any Dedekind domain D. If $I \subseteq D$ is a non-principal ideal then I is a rank one projective that is not stably free. So a Dedekind domain has $K(D) \cong \mathbb{Z}$ if and only if D is a PID. As another consequence, we observe that over any commutative ring a rank one projective P cannot be stably free unless it is actually free.

∘ Exercises ∘

Exercise 3.9. Let $R = \mathbb{Z}[x]/(x^2 - px)$ as in Exercise 2.33. Here we will analyze $K(R)$. Recall that geometrically $\operatorname{Spec} R$ looks like two copies of $\operatorname{Spec} \mathbb{Z}$ that are glued together at the point (p). The theme of this exercise is that projectives over R are obtained by taking trivial modules of the same rank on the two copies of $\operatorname{Spec} \mathbb{Z}$ and then gluing them together via an isomorphism on their fibers over (p).

(a) For $u \in (\mathbb{Z}/p)^*$ set $J_u = \{(f, g) \in \mathbb{Z}^2 \mid uf \equiv g \ (p)\}$. Make J_u into an R-module by $X.(f, g) = (0, pg)$. Prove that J_u is generated over R by $(1, u)$ and $(0, p)$ and that the surjection $R^2 \to J_u$ sending e_1 and e_2 to these two generators has an R-linear splitting. So J_u is projective.
(b) Prove that $J_u \cong R$ if and only if $u = \pm 1$, and more generally $J_u \cong J_v \iff u = \pm v$.

(c) Suppose that P is a rank one projective over R. Then P/xP is a rank one projective over the PID R/xR, so we can choose an R-linear isomorphism $\phi_1\colon P/xP \to R/xR$. Likewise, we can choose an R-linear isomorphism $\phi_2\colon P/(x-p)P \to R/(x-p)R$. We then obtain induced isomorphisms $\bar{\phi}_1, \bar{\phi}_2\colon P/(x, x-p)P \to R/(x, x-p)R$, so that the composite $\bar{\phi}_2^{-1}\bar{\phi}_1$ is an R-linear automorphism of $R/(x, x-p)R$ and therefore multiplication by a unit u. Use these ideas to prove that $P \cong J_u$.

(d) The set of isomorphism classes of rank one projectives becomes a group under the tensor product; this is called the Picard group of R and denoted $\operatorname{Pic}(R)$. Putting the previous parts together, prove that $\operatorname{Pic}(R) \cong (\mathbb{Z}/p)^*/\pm 1$.

(e) For $A \in GL_n(\mathbb{Z}/p)$ define $J_A = \{(f, g) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid Af = g(p)\}$. Extend the above ideas to prove that J_A can be generated by $2n$ elements over R and that J_A is projective. Prove that $J_{A\bar{P}} \cong J_A \cong J_{\bar{P}A}$ for any $P \in GL_n(\mathbb{Z})$ (where $\bar{P}$ is the mod p reduction), so that $A \mapsto J_A$ gives a map

$$GL_n(\mathbb{Z})\backslash GL_n(\mathbb{Z}/p)/GL_n(\mathbb{Z}) \longrightarrow \big\{\text{iso. classes of rank } n \text{ projectives over } R\big\}.$$

Verify that this is a bijection. [Discussion: Essentially what is happening here is that we are building vector bundles on $\operatorname{Spec} R$ by taking two trivial bundles on $\operatorname{Spec}\mathbb{Z}$ and then gluing together their fibers over p. The "gluing map" is the matrix in $GL_n(\mathbb{Z}/p)$, and the left and right multiplications by $GL_n(\mathbb{Z})$ correspond to change-of-bases in the two factors.]

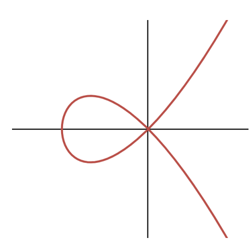
(f) Using the previous part, prove that every finitely-generated projective decomposes as a direct sum of a rank one projective and a free module. Furthermore, prove that $J_u \oplus J_v \cong R \oplus J_{uv}$ and that $R^r \oplus J_u$ is free if and only if $u = \pm 1$. [Hint: Suppose $A, A' \in GL_n(\mathbb{Z}/p)$ and one can obtain A' from A by a row (or column) operation that adds a multiple of one row (or column) to another. Prove that $A \sim A'$ in the orbit space from (e). Then use this technique in the first two proofs. For the third, use the determinant.]

(g) Define a map $\mathbb{Z} \oplus [(\mathbb{Z}/p)^*/\pm 1] \to K(R)$ by $(r, u) \mapsto R^r \oplus J_u$. Prove that this is an isomorphism, so that

$$K(R) \cong \begin{cases} \mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z} \oplus \mathbb{Z}/(\frac{p-1}{2}) & \text{if } p > 2. \end{cases}$$

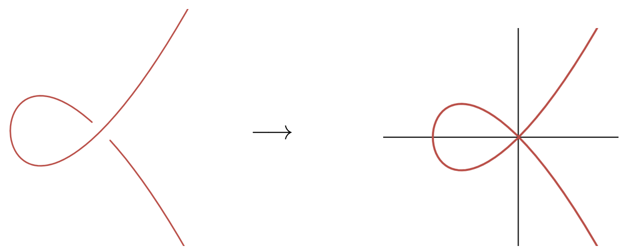
Deduce that the canonical map $K(R) \to G(R)$ is neither injective nor surjective when $p > 3$.

Exercise 3.10. Let k be a field not of characteristic 2 and let $R = k[x, y]/(y^2 - x^2(x+1))$. This is the coordinate ring of a nodal cubic:



In particular, R is not regular. In this exercise we will explore projectives over R.

Set $t = \frac{y}{x}$ and note that $t^2 = x + 1$, hence t is integral over R. It turns out that the integral closure of R is $\tilde{R} = k[t]$, with $R \hookrightarrow \tilde{R}$ sending $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. Geometrically, the map $\operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ looks as follows:



The curve in the left is just the affine line $\operatorname{Spec} k[t]$, but it has been depicted in a way that is compatible with the map to $\operatorname{Spec} R$. The points $t = 1$ and $t = -1$ (shown in the left picture lying above each other) are sent to $(0, 0)$, and away from these points the map is an isomorphism. Note that $R \subseteq k[t]$ may be regarded as the subring of polynomials f such that $f(1) = f(-1)$. In particular note that $(t^2 - 1)k[t] \subseteq R$.

All vector bundles on the affine line $\operatorname{Spec} \tilde{R}$ are trivial (e.g. use the classification of modules over a PID). We can make bundles on $\operatorname{Spec} R$ by taking a trivial rank n bundle on $\operatorname{Spec} \tilde{R}$ and gluing the two fibers at $t = \pm 1$ together via a fixed isomorphism $A \in GL_n(k)$.

Define $P_A = \{g \in k[t]^n \mid g(1) = Ag(-1)\}$ and note that this is naturally an R-module. We claim that P_A is a projective over R and that every projective over R is of this form. Moreover, we can precisely describe the set of isomorphism classes of all rank n projectives, for any n. These are the goals of this exercise.

(a) First consider $n = 1$ and $u \in k^*$, with $J_u = \{g \in k[t] \mid g(1) = ug(-1)\}$. Prove that J_u is generated as an R-module by $t^2 - 1$ and $\alpha_u = \frac{(u+1)+(u-1)t}{2}$ (the 2 in the denominator is not necessary, but leads to nicer-looking formulas in the end). Observe that if $f \in J_u$ and $g \in J_v$ then $fg \in J_{uv}$, or that multiplication gives maps $J_u \otimes J_v \to J_{uv}$. Also note that $J_1 = R$.

(b) Let $\pi\colon R^2 \twoheadrightarrow J_u$ send $e_1 \mapsto t^2 - 1$ and $e_2 \mapsto \alpha_u$. Find elements $P, Q \in J_{u^{-1}}$ such that $f \mapsto (Pf, Qf)$ gives an R-linear splitting for π.

(c) Prove that $J_u \cong R$ if and only if $u = 1$.

(d) Suppose that P is a rank 1 projective over R. Then $P \otimes_R k[t]$ is a rank 1 projective over $k[t]$, hence it is isomorphic to $k[t]$ as a $k[t]$-module. Choose a $k[t]$-linear isomorphism $P \otimes_R k[t] \cong k[t]$. Then the composite

$$P \xrightarrow{p \mapsto p \otimes 1} P \otimes_R k[t] \xrightarrow{\cong} k[t]$$

is an R-linear embedding. Let $\bar{P}$ denote the image. Observe that there exist $p_0, p_1 \in \bar{P}$ such that $1 = p_0 + p_1 t$. Use this to prove that $t^2 - 1, t(t^2 - 1) \in \bar{P}$ and then that $\bar{P} = J_u$ for some $u \in k^*$. [Hint for the last part: Show that $\bar{P}$ is generated by the three elements $t^2 - 1, t(t^2 - 1), a_0 + a_1 t$ for some $a_0, a_1 \in k$. In the cases $a_0 + a_1 = 0$ or $a_0 - a_1 = 0$ prove that $\bar{P}$ could not be projective.]

(e) The Picard group $\mathrm{Pic}(R)$ is the set of isomorphism classes of rank 1 projectives over R, which becomes a group under the tensor product. Verify that the previous parts give an isomorphism $k^* \to \mathrm{Pic}(R)$ sending $u \mapsto J_u$.
(f) Generalize the previous parts to $n > 1$. Show that $J_A = (t^2-1)k[t]^n + R.\langle (I+A)\underline{a}_0 + (A-I)\underline{a}_1 t\rangle$ for some $\underline{a}_0, \underline{a}_1 \in k^n$. In particular, J_A can be generated by $2n$ elements. Produce an R-linear splitting for the projection $R^{2n} \twoheadrightarrow J_A$ to show that J_A is projective. Also, verify that $J_{BAB^{-1}} \cong J_A$ for any $B \in GL_n(k)$.
(g) Prove that the set of isomorphism classes of rank n projectives over R is in bijective correspondence with the quotient set $GL_n(k)/\sim$ where the equivalence relation is conjugation ($A \simeq BAB^{-1}$).

4. A brief tour of localization and dévissage

It would be nice if we could compute the K-groups of more rings. For example, we haven't even computed $K(R)$ for a simple ring like $R = \mathbb{Z}[\sqrt{-5}]$. But so far we don't have many techniques to tackle such a computation. An obvious thing to try is to relate the K-groups of R to those of simpler rings made from R, for example quotient rings R/I and localizations $S^{-1}R$. We will start to explore these ideas in the present section. For the moment it will be easier to do this for G-theory, though, rather than K-theory. Note that $R = \mathbb{Z}[\sqrt{-5}]$ is a regular ring, and so $K(R) \cong G(R)$ by Theorem 2.13; hence, the focus on G-groups still gets us what we want in this case.

Let R be a commutative ring and let $f \in R$. Consider the maps

$$G(R/f) \xrightarrow{d_1} G(R) \xrightarrow{d_0} G(f^{-1}R)$$

where $d_1([M]) = [M]$ and $d_0([W]) = f^{-1}W$. Clearly $d_0 \circ d_1 = 0$. We claim that d_0 is also surjective. To see this, let Z be an $f^{-1}R$-module with generators $z_1, \dots, z_n$. Let $W = R\langle z_1, \dots, z_n\rangle \subseteq Z$ be the R-submodule generated by the z_i's. Then $f^{-1}W \cong Z$, and so d_0 is surjective.

Theorem 4.1. *When R is Noetherian the sequence*

$$G(R/f) \xrightarrow{d_1} G(R) \xrightarrow{d_0} G(f^{-1}R) \longrightarrow 0$$

is exact.

We will delay the proof of this theorem for the moment, as it is somewhat involved. Let us first look at an example.

Example 4.2. Let $R = \mathbb{Z}[\sqrt{-5}]$ and $f = (2)$. Note that R is not a PID but $f^{-1}R$ is. Thus $G(f^{-1}R) \cong \mathbb{Z}$. Now we compute

$$R/f = \mathbb{Z}/2[x]/(x^2+5) = \mathbb{Z}/2[x]/(x^2+1) = \mathbb{Z}/2[x]/((x+1)^2) \cong \mathbb{Z}/2[t]/(t^2).$$

We calculated in example (7) from Section 2 that $G(\mathbb{Z}/2[t]/(t^2)) \cong \mathbb{Z}$ and is generated by the module $\mathbb{Z}/2$ with t acting as zero. Translated into the present situation, we are saying $G(R/f) \cong \mathbb{Z}$ with the group being generated by $R/(2, x+1) = R/(2, 1+\sqrt{-5})$.

We have computed that the exact sequence from Theorem 4.1 has the form

$$\mathbb{Z} \xrightarrow{d_1} G(R) \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

where $d_1(1) = [R/(2, 1+\sqrt{-5})]$ and $d_0([R]) = 1$. Let $I = (2, 1+\sqrt{-5})$ and notice that $G(R)$ is generated by $[R]$ and $[R/I]$.

Now look at the short exact sequence $0 \longrightarrow K \longrightarrow R^2 \xrightarrow{\phi} I \longrightarrow 0$ where $\phi(e_1) = 2$, $\phi(e_2) = 1+\sqrt{-5}$, and $K = ker(\phi) = \{(x,y) \,|\, 2x + (1+\sqrt{-5})y = 0\}$. In example (2) from Section 3 we saw that $K \cong I$. So we have $[I] + [I] = [R^2]$ in $G(R)$, or $2([R] - [I]) = 0$. But $[R] - [I] = [R/I]$, hence $2[R/I] = 0$. It follows that $G(R)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, depending on whether the class $[R/I] = [R] - [I]$ is zero or not.

Now use that R is regular, so that $G(R) \cong K(R)$. Recall that we saw in Example 3.8 that $\tilde{K}(R) \neq 0$, or equivalently $K(R) \neq \mathbb{Z}$. In fact we saw precisely that $[R] - [I]$ is not zero in $K(R)$. We conclude that $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, with generators $[R]$ and $[R/I]$ for each of the two summands.

Remark 4.3. Theorem 4.1 gives another parallel between $G(-)$ and singular homology. If $X = \operatorname{Spec} R$ then $A = \operatorname{Spec} R/f$ is a closed subscheme, and $\operatorname{Spec} f^{-1}R = X - A$ is the open complement. So the sequence in Theorem 4.1 can be written as

$$G(A) \to G(X) \to G(X - A) \to 0.$$

This is somewhat reminiscent of the long exact sequence in singular homology $\cdots \to H_*(A) \to H_*(X) \to H_*(X, A) \to \cdots$ but with some important differences. One obvious difference is that our sequence does not yet extend to the left to give a long exact sequence, but that turns out to be just a lack of knowledge on our part: we will eventually see that there are 'higher G-groups' completing the picture. The other evident difference is the presence of $G(X - A)$ as the 'third term' in the long exact sequence, rather than a relative group $G(X, A)$. There are several things to say about this that would be a distraction to delve into at the moment, but perhaps the most relevant is that $H_*(-)$ is really the wrong analogy to be looking at. If we instead consider Borel-Moore homology, then there are indeed long exact sequences that look like $\cdots \to H_*^{BM}(A) \to H_*^{BM}(X) \to H_*^{BM}(X - A) \to \cdots$

Remark 4.4. It is important in Theorem 4.1 that we are using G-theory rather than K-theory. In K-theory we have maps $K(R) \to K(R/f)$ and $K(R) \to K(f^{-1}R)$, both given by tensoring, but in neither case do we have an evident 'third group' that might form an exact sequence. In essence this is because we need relative K-groups; we will start to encounter these in the next section.

We will now work towards proving Theorem 4.1. The proof is somewhat involved, and the result is actually not going to be used much in the rest of the notes. But the proof is very interesting, as it demonstrates many general issues that arise in the subject of K-theory. So it is worth spending time on this.

The proof comes in two parts. For the first part, let us introduce the multiplicative system $S = \{1, f, f^2, f^3, \dots\}$. Write

$$G(M \,|\, S^{-1}M = 0)$$

for the Grothendieck group of all finitely-generated R-modules M such that $S^{-1}M = 0$. The notation is a little slack, but it is very convenient. There are evident maps

$$G(M \,|\, S^{-1}M = 0) \to G(R) \to G(S^{-1}R) \to 0,$$

and we will prove that this is exact for *any* multiplicative system S. This is called the **localization sequence** for G-theory.

The second step is to notice that if M is an R/f-module then as an R-module it has the property that $S^{-1}M = 0$. So we have a map

$$G(R/f) \to G(M|S^{-1}M = 0). \tag{4.5}$$

If M is an arbitrary finitely-generated R-module, the condition $S^{-1}M = 0$ just says that M is killed by a power of f. So we would have a filtration

$$M \supseteq fM \supseteq f^2M \supseteq \cdots \supseteq f^N M = 0$$

where the factors are all R/f-modules. This shows that the map in (4.5) is surjective, and in fact these ideas allow one to define an inverse. The fact that

$$G(R/f) \cong G(M|S^{-1}M = 0)$$

is an example of a general principle known as **dévissage**. When we come to prove this in a moment we will develop the generalization and get a better understanding of what is going on here.

So those are the two pieces for the proof of Theorem 4.1: a general localization sequence where the third term is something we had not considered before—in essence, a relative G-group—and a dévissage theorem identifying that third term with something more familiar.

4.6. **The localization sequence.** To begin with we will need some basic facts about the localization functor $\gamma\colon \langle\langle R - \mathrm{Mod}\rangle\rangle \to \langle\langle S^{-1}R - \mathrm{Mod}\rangle\rangle$. Note that if M and N are R-modules then the map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ factors through the S-localization to give

$$S^{-1}\,\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N). \tag{4.7}$$

By extension the same is true for the maps $\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N)$, giving us

$$S^{-1}\,\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N). \tag{4.8}$$

Proposition 4.9. *Let R be a commutative ring and let M and N be R-modules. If M is finitely-presented then the map from (4.7) is an isomorphism. If M has a resolution by finitely-generated projective modules then the map from (4.8) is an isomorphism.*

Proof. First observe that (4.7) is readily checked to be an isomorphism when M is free and finitely-generated. If we have a finite presentation $F_1 \to F_0 \to M \to 0$ then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\,\mathrm{Hom}_R(M,N) \arrow[r] \arrow[d] & S^{-1}\,\mathrm{Hom}_R(F_0,N) \arrow[r] \arrow[d] & S^{-1}\,\mathrm{Hom}_R(F_1,N) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_T(S^{-1}M, S^{-1}N) \arrow[r] & \mathrm{Hom}_T(S^{-1}F_0, S^{-1}N) \arrow[r] & \mathrm{Hom}_T(S^{-1}F_1, S^{-1}N)
\end{tikzcd}$$

where we write $T = S^{-1}R$ for typographical reasons and where each vertical map is an instance of (4.7). The top row is exact by the left exactness of $\mathrm{Hom}_R(-,N)$ together with the exactness of S-localization. The bottom row is exact by the exactness of S-localization and then the left exactness of $\mathrm{Hom}_T(-,S^{-1}N)$. The two vertical maps on the right are isomorphisms because the F_i are free and finitely-generated. So the left vertical map is also an isomorphism.

Note that the map from (4.7) is also an isomorphism when M is a finitely-generated projective, either using the fact that projectives are retracts of free modules or by the fact that finitely-generated projectives are automatically finitely-presented.

For the Ext-statement, choose a projective resolution $P_* \to M$ where the P_i are all finitely-generated. Then $S^{-1}P_* \to S^{-1}M$ is a free resolution over $S^{-1}R$, and so

$$\begin{aligned}\operatorname{Ext}^*_{S^{-1}R}(S^{-1}M, S^{-1}N) &= H^*(\operatorname{Hom}_{S^{-1}R}(S^{-1}P, S^{-1}N)) \\ &= H^*(S^{-1}\operatorname{Hom}_R(P, N)) \\ &= S^{-1}H^*(\operatorname{Hom}_R(P, N)) = S^{-1}\operatorname{Ext}^*_R(M, N).\end{aligned}$$

Here the equalities are really canonical isomorphisms, and the second equality is the Hom-isomorphism we have already proven. □

Exercise 4.10. For the verification that (4.7) is an isomorphism when M is finitely-generated and free, think through where the finite-generation hypothesis is needed.

Corollary 4.11. *Let R be Noetherian and let $S \subseteq R$ be a multiplicative system. In each of the parts below the modules are always assumed to be finitely-generated.*

(a) For any $S^{-1}R$-module W there exists an R-module A and an isomorphism $S^{-1}A \cong W$.

(b) For any R-modules A_1 and A_2 and map of $S^{-1}R$-modules $f\colon S^{-1}A_1 \to S^{-1}A_2$, there exists a map of R-modules $g\colon A_1 \to A_2$ and a diagram of $S^{-1}R$-modules

$$\begin{tikzcd} S^{-1}A_1 \arrow[r, "S^{-1}g"] \arrow[d, equal] & S^{-1}A_2 \arrow[d, "\cong"] \\ S^{-1}A_1 \arrow[r, "f"] & S^{-1}A_2. \end{tikzcd}$$

Note that the right vertical map need not be the identity.

(c) For any short exact sequence of $S^{-1}R$-modules

$$0 \to W_1 \to W_2 \to W_3 \to 0,$$

there exists a short exact sequence of R-modules

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

and isomorphisms

$$\begin{tikzcd} 0 \arrow[r] & S^{-1}A_1 \arrow[r] \arrow[d, "\cong"'] & S^{-1}A_2 \arrow[r] \arrow[d, "\cong"'] & S^{-1}A_3 \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & W_1 \arrow[r] & W_2 \arrow[r] & W_3 \arrow[r] & 0 \end{tikzcd}$$

Proof. We saw the technique for (a) already, in the paragraph prior to the statement of Theorem 4.1: pick a set of generators $w_1, \ldots, w_k$ for W as an $S^{-1}R$-module, and let A be the R-linear span of those generators. The inclusion $A \hookrightarrow W$ gives an inclusion $S^{-1}A \hookrightarrow S^{-1}W$ which is surjective, hence an isomorphism.

Parts (b) and (c) are direct consequences of the surjectivity of the Hom- and Ext^1-maps from Proposition 4.9. □

Remark 4.12. The proofs of Corollary 4.11(b) and (c) are very slick, and sometimes it can be good to also think through more concretes approaches. To this end, here is a sketch of an alternative proof.

For (b), choose presentations $F_1 \to F_0 \to A_1 \to 0$ and $G_1 \to G_0 \to A_2 \to 0$ where the F_i and G_i are finitely-generated and free. The map f lifts to a map of complexes $\tilde{f}\colon S^{-1}F \to S^{-1}G$. Represent the maps $\tilde{f}_0$ and $\tilde{f}_1$ by matrices, and choose $t \in S$ large enough to clear the denominators for all entries at once. Since the matrices $t\tilde{f}_i$ have entries in R we may regard them as maps $F_i \to G_i$. Consider the square

$$\begin{tikzcd} F_1 \arrow[r, "t\tilde{f}_1"] \arrow[d] & G_1 \arrow[d] \\ F_0 \arrow[r, "t\tilde{f}_0"] & G_0. \end{tikzcd}$$

This square need not commute, but it commutes after S-localization: so there exists some $u \in S$ such that multiplying the two horizontal maps by u makes the square commutative. Now redefine t to be tu, so that the above square truly does commute. Taking the induced map of cokernels now gives a map $g\colon A_1 \to A_2$. In the diagram

$$\begin{tikzcd} S^{-1}F_1 \arrow[r, "t\tilde{f}_1"] \arrow[d] & S^{-1}G_1 \arrow[r, "t^{-1}"] \arrow[d] & S^{-1}G_1 \arrow[d] \\ S^{-1}F_0 \arrow[r, "t\tilde{f}_0"] \arrow[d] & S^{-1}G_0 \arrow[r, "t^{-1}"] \arrow[d] & S^{-1}G_0 \arrow[d] \\ S^{-1}A_1 \arrow[r, "S^{-1}g"] & S^{-1}A_2 \arrow[r, "t^{-1}", "\cong"'] & S^{-1}A_2 \end{tikzcd}$$

the composition of the first two rows are $\tilde{f}_1$ and $\tilde{f}_0$, so the composition of the bottom row is f. This completes the proof for (b).

The proof of (c) is along similar lines. Choose A_1 and A_3 such that $S^{-1}A_1 \cong W_1$ and $S^{-1}A_3 \cong W_3$. Then choose free presentations $F_1 \to F_0 \to A_1$ and $G_1 \to G_0 \to A_3$. Use the Horseshoe Lemma from homological algebra [W1, 2.2.8] to create a free presentation for W_2 of the form $S^{-1}(F_1 \oplus G_1) \to S^{-1}(F_0 \oplus G_0)$, sitting in the middle of a short exact sequence with the previous two. Play the same game as in (b) with clearing denominators on the maps in this presentation, so that one can lift to a diagram of R-modules. There are a couple of slightly tricky points to think through, but one can in this way create the desired short exact sequence of R-modules (though not necessarily with the originally chosen A_1). We leave the reader to ponder the details here.

With these basic results about localization out of the way, we can now derive some consequences for K-theory.

Corollary 4.13. *When R is Noetherian the following subgroups of $G(R)$ are all equal:*

(1) $\big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle$

(2) $\big\langle [A] - [B] \,\big|\, \textit{there exists a map } f\colon A \to B \textit{ such that } S^{-1}f \textit{ is an isomorphism} \big\rangle$

(3) $\big\langle [J] \,\big|\, S^{-1}J = 0 \big\rangle$.

Proof. Let H_1, H_2, and H_3 be the subgroups listed in (1)–(3). Clearly $H_1 \supseteq H_2 \supseteq H_3$. The opposite subset $H_1 \subseteq H_2$ follows directly from Corollary 4.11(b). To prove $H_2 \subseteq H_3$, let $f\colon A \to B$ be a map of finitely-generated R-modules such that $S^{-1}f$ is an isomorphism. Consider the short exact sequence

$$0 \to \ker f \to A \to B \to \operatorname{coker} f \to 0,$$

and note that our hypothesis implies that $S^{-1}(\ker f) = 0 = S^{-1}(\operatorname{coker} f)$. But $[A] - [B] = [\ker f] - [\operatorname{coker} f]$ in $G(R)$, so we have that $[A] - [B] \in H_3$. $\square$

Proposition 4.14. *Let R be Noetherian and let $S \subseteq R$ be a multiplicative system. The sequence*

$$G(M|S^{-1}M = 0) \xrightarrow{a} G(R) \xrightarrow{b} G(S^{-1}R) \to 0$$

is exact, where a and b are the evident maps.

Proof. Part (a) of Corollary 4.11 gives surjectivity of b. The somewhat tricky thing is to get the exactness in the middle. Let $\mathcal{F}(R)$ denote the free abelian group on isomorphism classes of finitely-generated R-modules, and let $\mathcal{R}el(R) \subseteq \mathcal{F}(R)$ denote the subgroup generated by elements $[M_i'] + [M_i''] - [M_i]$ for short exact sequences $0 \to M_i' \to M_i \to M_i'' \to 0$. Note that $[0] \neq 0$ in $\mathcal{F}(R)$; we could have imposed this as an extra condition, but it is slightly more convenient to not do so. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}el(R) \arrow[r] \arrow[d, "\pi|_{\mathcal{R}el}"] & \mathcal{F}(R) \arrow[r] \arrow[d, "\pi"] & G(R) \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & \mathcal{R}el(S^{-1}R) \arrow[r] & \mathcal{F}(S^{-1}R) \arrow[r] & G(S^{-1}R) \arrow[r] & 0,
\end{tikzcd}$$

which we wish to regard as a short exact sequence of chain complexes (the columns become chain complexes by adding zeros above and below). Corollary 4.11(a) gives surjectivity of π, and Corollary 4.11(c) gives surjectivity of $\pi|_{\mathcal{R}el}$. The long exact sequence in homology then becomes

$$0 \to \ker(\pi|_{\mathcal{R}el}) \to \ker(\pi) \to \ker b \to 0. \tag{4.15}$$

We next analyze the kernel of π.

Assume that $x \in \ker(\pi)$. One can write x in the form

$$x = \Big([M_1] + [M_2] + \cdots + [M_k]\Big) - \Big([J_1] + \cdots + [J_l]\Big)$$

for some modules $M_1, \ldots, M_k, J_1, \ldots, J_l$. We then have

$$0 = \pi(x) = \Big([S^{-1}M_1] + [S^{-1}M_2] + \cdots + [S^{-1}M_k]\Big) - \Big([S^{-1}J_1] + \cdots + [S^{-1}J_l]\Big)$$

in $\mathcal{F}(S^{-1}R)$. How can this happen? It can only be that $k = l$ and that for each module $S^{-1}M_j$ there is some i for which $S^{-1}M_j \cong S^{-1}J_i$. By pairing the terms up two by two we find that

$$x \in \big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle \subseteq \mathcal{F}(R).$$

So $\ker \pi = \big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle$. It then follows from (4.15) and Corollary 4.13 that

$$\ker b = \langle [J] \,|\, S^{-1}J = 0 \rangle \subseteq G(R).$$

This is what we wanted to prove. $\square$

4.16. **Dévissage.** Now we move to the second stage of the proof of Theorem 4.1. We can rephrase what needs to be shown as saying that the map

$$G(M|M \text{ is killed by } f) \to G(M|M \text{ is killed by a power of } f)$$

is an isomorphism. We have seen a baby version of this argument before, namely back in Section 2 when we showed that

$$G(\mathbb{Z}/p) \to G(\mathbb{Z}/p^2) \quad \text{and} \quad G(F) \to G(F[t]/(t^2))$$

are both isomorphisms. These are both maps of the form

$$G(M|M \text{ is killed by } f) \to G(M|M \text{ is killed by } f^2),$$

for the rings $R = \mathbb{Z}$ and $R = F[t]$, respectively. Iterating the same idea we used to prove these—filter by powers of f—allows one to prove the required generalization. But while we're at it, let us generalize even further.

Let $\mathcal{B}$ be an exact category. I will not say exactly what the definition of such a thing is, except that $\mathcal{B}$ is an additive category with a collection of sequences $M' \to M \to M''$ called "exact", and the collection must satisfy a reasonable list of axioms. Any abelian category with its intrinsic notion of short exact sequence is an example. The complete definition is in [Q3]. We are not giving it here in part because the reader can manufacture a suitable definition for themself: just figure out what axioms one needs to make the following proof work.

Theorem 4.17 (Dévissage)**.** *Let $\mathcal{B}$ be an exact category, and let $\mathcal{A} \hookrightarrow \mathcal{B}$ be an exact subcategory such that any object in $\mathcal{B}$ has a finite filtration whose factors are in $\mathcal{A}$. Then $G(\mathcal{A}) \to G(\mathcal{B})$ is an isomorphism.*

Proof. The inclusion $i\colon \mathcal{A} \to \mathcal{B}$ induces a map $\alpha\colon G(\mathcal{A}) \to G(\mathcal{B})$, and we want to define an inverse $\beta\colon G(\mathcal{B}) \to G(\mathcal{A})$. To do so, for $M \in \mathcal{B}$ choose a filtration

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0,$$

whose quotients M_i/M_{i+1} are in $\mathcal{A}$, and define

$$\beta([M]) = \sum [M_i/M_{i+1}].$$

We must check that β is well-defined, because it seems to depend on the choice of filtration. There are two pieces of this. The first and easier one is to check that our formula gives the same class in $G(\mathcal{A})$ if we refine the filtration, meaning that we replace one of the links $M_i \supseteq M_{i+1}$ with a longer chain $M_i \supseteq M_i^1 \supseteq \cdots \supseteq M_i^r = M_{i+1}$. This is trivial using Proposition 2.5(b) (or really, its analog in the present setting).

The second part is to recall something you probably learned in a basic algebra class, namely the Jordan-Hölder Theorem. This says that given any two filtrations of M we can refine each one so that the two refinements have the same quotients up to reindexing. If you accept this, it shows that $\beta([M])$ does not depend on the choice of filtration. It is a simple exercise to prove that β is additive, which we leave to the reader.

At this point we have the map β. It is immediate that $\beta\alpha = \mathrm{id}$ and $\alpha\beta = \mathrm{id}$. □

Remark 4.18. We will not prove the Jordan-Hölder Theorem, as this is something that can be found in basic algebra textbooks, but let us at least recall the main idea for why it is true. Suppose $M \supseteq A \supseteq 0$ and $M \supseteq B \supseteq 0$ are two filtrations for M. Consider the refinement of the first given by

$$M \supseteq A+B \supseteq A \supseteq A \cap B \supseteq 0,$$

having quotients $M/(A+B)$, $(A+B)/A \cong B/(A\cap B)$, $A/(A\cap B) \cong (A+B)/B$, and $A \cap B$, Interchanging the roles of A and B gives a similar filtration refining $M \supseteq B \supseteq 0$, having the same set of filtration quotients.

Once one has the above basic idea, it is not hard to extend to longer filtrations.

Note that it is often true in mathematics that the hard work goes into showing that something is well-defined, and afterwards the rest is easy. This was the case for the Dévissage Theorem, where all the hard work went into constructing the map β.

4.19. **Recap and summary.** We embarked on the above journey in order to prove Theorem 4.1, so let us now come back to that.

Proof of Theorem 4.1. Recall that R is a Noetherian ring and $f \in R$. Let S be the multiplicative system $\{f^i \mid i \geq 0\}$. By Proposition 4.14 we have an exact sequence

$$G(M \mid S^{-1}M = 0) \longrightarrow G(R) \longrightarrow G(S^{-1}R) \longrightarrow 0.$$

The inclusion of R/fR-modules into modules M such that $S^{-1}M = 0$ satisfies the hypotheses of the dévissage theorem (Theorem 4.17), by looking at the filtration $M \supseteq fM \supseteq f^2M \supseteq \cdots$. The conditions that $S^{-1}M = 0$ and M is finitely-generated guarantee that $f^kM = 0$ for some k, so that this filtration is finite. Therefore we have an isomorphism $G(R/fR) \cong G(M \mid S^{-1}M = 0)$, allowing us to write our exact sequence as

$$G(R/fR) \longrightarrow G(R) \longrightarrow G(S^{-1}R) \longrightarrow 0.$$

The composite $G(R/fR) \to G(R)$ is the evident map that regards an R/fR-module as an R-module. □

5. K-theory of complexes and relative K-theory

Recall that there is always a map $K(R) \to G(R)$ sending the K-class of a projective to the G-class of the same projective. We proved in Theorem 2.13 that when R is regular this map is an isomorphism, and we did this by constructing the inverse: it sends a class $[M]$ to $\sum(-1)^i[P_i]$, where $P_\bullet \to M$ is any bounded resolution of M by finitely-generated projectives. If you go back and examine the proof of that result, you might notice that the alternating sums are largely an *annoyance* in the proof—all the key ideas are best expressed without them, and they are only forced into the proof so that we get actual elements of $K(R)$. If you think about this enough, it might eventually occur to you to try to make a definition of $K(R)$ that uses chain complexes instead of modules, thus eliminating the need for these alternating sums. We will show how to do this in the present section.

The importance of using chain complexes extends much further than simply changing language to simplify a proof. We will see that defining K-theory in terms of complexes allows us to write down natural definitions for relative K-groups as well.

Throughout this section let R be a fixed commutative ring. By a *bounded* chain complex over R we mean a chain complex which is nonzero in only finitely-many degrees. We begin by making the following definition.

Definition 5.1.

$$K^{cplx}(R) = \frac{\mathbb{Z}\langle\, [P_\bullet] \,\big|\, P_\bullet \textit{ is a bounded chain complex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[P_\bullet] = [P'_\bullet]$ *if* $P_\bullet$ *and* $P'_\bullet$ *are chain homotopy equivalent,*

(2) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ *if there is a short exact sequence* $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

The second relation is the one that by now we would expect in a K-group, but the first relation is new to us. If one goes back and thinks about the proof of Theorem 2.13, the need for this first relation quickly becomes clear: it guarantees, for instance, that two projective resolutions of a module will represent the same class in the K-group.

Regarding relation (1), let us introduce some common terminology:

Definition 5.2. *A map of chain complexes* $C_\bullet \to D_\bullet$ *is a* ***quasi-isomorphism*** *if the induced maps* $H_i(C_\bullet) \to H_i(D_\bullet)$ *are isomorphisms for all* $i \in \mathbb{Z}$. *Two chain complexes* $C_\bullet$ *and* $D_\bullet$ *are* ***quasi-isomorphic****, written* $C_\bullet \simeq D_\bullet$*, if there is a zig-zag of quasi-isomorphisms*

$$C_\bullet \xrightarrow{\sim} J^1_\bullet \xleftarrow{\sim} J^2_\bullet \xrightarrow{\sim} \cdots \xrightarrow{\sim} J^n_\bullet \xleftarrow{\sim} D_\bullet.$$

The following proposition is basic homological algebra. We omit the proof, but it is very similar to the proof that two projective resolutions of the same module are chain homotopy equivalent.

Proposition 5.3. *If P and Q are bounded below complexes of projectives, then every quasi-isomorphism $P \to Q$ is a chain homotopy equivalence.*

Exercise 5.4. Prove Proposition 5.3, or look up a proof in a book on homological algebra.

Proposition 5.3 lets us replace the words "chain homotopy equivalence" with "quasi-isomorphism" in any statement about bounded, projective complexes. In particular, we do this in relation (1) from the definition of $K^{cplx}(R)$. The advantage of doing this is simply that quasi-isomorphisms are somewhat easier to identify than chain homotopy equivalences.

Here is our main result concerning the K-theory of complexes:

Proposition 5.5. $K(R) \cong K^{cplx}(R)$ *for any commutative ring R.*

Before giving the proof we record two useful results. For any chain complex C, recall that ΣC denotes the chain complex obtained by shifting every module up one degree and adding a sign to all differentials: $(\Sigma C)_n = C_{n-1}$, and $d_{\Sigma C} = -d_C$. Also, if $f\colon A \to B$ is a map of chain complexes then the mapping cone Cf is the chain complex with $(Cf)_n = A_{n-1} \oplus B_n$ and $d_{Cf}(a,b) = (-da + f(b), db)$. Note that ΣA is the mapping cone of $A \to 0$.

Lemma 5.6. *Let P and Q be bounded complexes of finitely-generated projectives.*

(a) $[\Sigma P] = -[P]$ *in* $K^{cplx}(R)$.

(b) If $f\colon P \to Q$ is a map of complexes then $[Cf] = [Q] - [P]$ in $K^{cplx}(R)$.

Proof. Recall that there is a short exact sequence of complexes

$$0 \to Q \hookrightarrow Cf \to \Sigma P \to 0,$$

which shows immediately that $[Cf] = [Q]+[\Sigma P]$ in $K^{cplx}(R)$. Let T be the mapping cone of the identity $P \to P$. Note that T is exact, hence quasi-isomorphic to the zero complex. So $0 = [T] = [P] + [\Sigma P]$, from which we get $[\Sigma P] = -[P]$. It then follows that $[Cf] = [Q] + [\Sigma P] = [Q] - [P]$. □

Exercise 5.7. Prove that if relation (1) in the definition of $K^{cplx}(R)$ is replaced with

(1') $[P_\bullet] = 0$ for every exact complex $P_\bullet$,

then the resulting quotient group is also equal to $K^{cplx}(R)$.

Exercise 5.8. Sometimes the sign on the differential is omitted in the definition of the suspension ΣC. Check that the chain complexes obtained from these two different conventions are naturally isomorphic. (For this reason, occasionally we will be sloppy with the sign on the differential).

We now have enough tools to prove the main result of this section:

Proof of Proposition 5.5. If P is a projective R-module, let $P[n]$ denote the chain complex that has P in degree n and in all other degrees is equal to 0. There is an obvious map $\alpha\colon K(R) \to K^{cplx}(R)$ defined by

$$[P] \mapsto \big[P[0]\big].$$

It is somewhat less obvious, but one can define a map in the other direction $\beta\colon K^{cplx}(R) \to K(R)$ by

$$\beta\Big([P_\bullet]\Big) = \sum (-1)^i [P_i].$$

To see that this is well-defined we need to check that it respects the two defining relations for $K^{cplx}(R)$. Relation (2) is obvious, but for the other relation it is convenient to use Exercise 5.7 to replace (1) with (1'). The fact that β respects (1') is immediate, being a consequence of Exercise 2.11.

It is clear that $\beta \circ \alpha = \mathrm{id}$, so α is injective and β is surjective. To finish the proof, it is easiest to prove that α is surjective; we will do this in several steps. If P is a finitely-generated projective then $P[0]$ is obviously in the image of α, and we know that $P[n] = (-1)^n[P[0]]$ by iterated application of Lemma 5.6(a). So $P[n] \in \operatorname{im}\alpha$ for all $n \in \mathbb{Z}$. Said differently, any complex of projectives of length 0 belongs to the image of α. We next extend this to all bounded complexes by an induction on the length.

Let $P_\bullet$ be a bounded complex of finitely-generated projectives, bounded between degrees k and $n+k$, say. Then $P_k[k]$ is a subcomplex of $P_\bullet$, and the quotient $Q_\bullet$ has length at most $n-1$. We have $[P_\bullet] = [P_k[k]] + [Q_\bullet]$, and both $[P_k[k]]$ and $[Q_\bullet]$ belong to $\operatorname{im}\alpha$ by induction. So $[P_\bullet] \in \operatorname{im}\alpha$, and we are done. □

We will use our identification of $K^{cplx}(R)$ and $K(R)$ implicitly from now on. For example, if P is a bounded complex of projectives we will often write $[P]$ to denote an element of $K(R)$—although of course we mean $\beta([P])$.

Exercise 5.9. Assume that R is Noetherian, and let $G_{\mathrm{fpd}}(R)$ denote the Grothendieck group of finitely-generated modules having finite projective dimension. Prove that $K^{cplx}(R) \cong G_{\mathrm{fpd}}(R)$.

Exercise 5.10. If P is a bounded-below complex of projectives then the functor $(-) \otimes P$ preserves short exact sequences of chain complexes. Using this, show that the tensor product of chain complexes gives a ring structure on $K^{cplx}(R)$ and that the isomorphism $K(R) \cong K^{cplx}(R)$ is an isomorphism of rings.

5.11. **G-theory and chain complexes.** One can prove an analog of Proposition 5.5 in which the 'projective' hypothesis is left out everywhere, showing that $G(R)$ is isomorphic to a Grothendieck group made from bounded chain complexes of arbitrary finitely-generated modules. Here the Grothendieck group of complexes must be defined using relation (1') instead of (1), though, because they are no longer equivalent. Other than this small point, all of the arguments are the same.

What is more interesting, however, is a variant that again uses chain complexes of projectives. Precisely, consider chain complexes $P_\bullet$ such that

(1) Each P_i is a finitely-generated projective,
(2) $P_\bullet$ is bounded-below, in the sense that $P_i = 0$ for all $i \ll 0$.
(3) $P_\bullet$ has bounded homology, in the sense that $H_i(P) \neq 0$ only for finitely many values of i.

Start with the free abelian group on isomorphism classes of such complexes, and define $G^{cplx}(R)$ to be the quotient by the analogs of relations (1) and (2) (or equivalently, (1') and (2)) in the definition of $K^{cplx}(R)$.

Note that one readily obtains maps $\alpha\colon G^{cplx}(R) \to G(R)$ and $\beta\colon G(R) \to G^{cplx}(R)$ by

$$\alpha([P_\bullet]) = \sum_i (-1)^i [H_i(P)] \quad \text{and} \quad \beta([M]) = [Q_\bullet]$$

where $Q_\bullet \to M$ is any resolution by finitely-generated projectives. Though we must be a little careful here, as we are guaranteed that the $H_i(P)$ are finitely-generated only when R is Noetherian. The fact that α respects the short-exact-sequence relation follows from the Snake Lemma.

Proposition 5.12. *When R is Noetherian the maps α and β give inverse isomorphisms $G^{cplx}(R) \cong G(R)$.*

Proof. It is immediate that $\alpha\beta = \mathrm{id}$, so that α is surjective and β is injective. The proof will be completed by showing that β is surjective. Let $P_\bullet$ be a bounded-below, homologically bounded chain complex of finitely-generated projectives. We will prove by induction on the number of nonzero homology groups of $P_\bullet$ that $[P_\bullet] \in \operatorname{im} \beta$. The base is trivial, for if all the homology groups are zero then $P_\bullet \simeq 0$ and so $[P_\bullet] = 0$.

Without loss of generality assume that $P_i = 0$ for $i < 0$. Let n be the smallest integer for which $H_n(P) \neq 0$. If $n > 0$ then $P_1 \to P_0$ is surjective, so there exists a splitting. Using this splitting one sees that P is quasi-isomorphic to a chain complex concentrated in degrees strictly larger than zero. Repeating this argument if necessary, one concludes that P is actually quasi-isomorphic to a chain complex (of f.g. projectives) concentrated in degrees n and higher. So we may assume that P has this property, and then by shifting indices we may assume $n = 0$.

Let $Q_\bullet \to H_0(P)$ be a resolution by finitely-generated projectives (this exists because R is Noetherian). Standard homological algebra gives us a map $f\colon P_\bullet \to Q_\bullet$

inducing an isomorphism on H_0. Let Cf be the mapping cone of f. The long exact homology sequence shows that Cf has one fewer non-vanishing homology group than P, and hence we may assume by induction that $[Cf] \in \operatorname{im}\beta$. But we know from Lemma 5.6 (really, its analog for $G^{cplx}(R)$) that $[Cf] = [Q] - [P]$. Since $[Q] \in \operatorname{im}\beta$ by the definition of β, it follows that $[P] \in \operatorname{im}\beta$ as well. □

When we first learned the definitions of $K(R)$ and $G(R)$, the difference seemed to be about projective versus arbitrary modules. When we look at these groups as $K^{cplx}(R)$ and $G^{cplx}(R)$, however, the difference is about bounded versus bounded-below chain complexes.

5.13. **Relative K-theory.** It may seem like we have introduced an unnecessary level of complexity (no pun intended) by introducing $K^{cplx}(R)$. After all, the proof of Proposition 5.5 shows that for any bounded complex P the class $[P]$ is just the alternating sum $\sum(-1)^i[P_i[0]]$. That is, in $K^{cplx}(R)$ we may decompose any complex into its constituent modules; one really only needs modules, not chain complexes. But we will get some mileage out of these ideas by defining similar K-groups but restricting to complexes *subject to certain conditions*. In these cases we might not be able to 'unravel' the complexes anymore. We give a few examples:

- (i) Let S be a multiplicative system in R. Start with the free abelian group on isomorphism classes of bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that $S^{-1}P_\bullet$ is exact. Define $K(R,S)$ to be the quotient of this free abelian group by the analogs of relations (1) and (2) defining $K^{cplx}(R)$.
- (ii) Let $I \subseteq R$ be an ideal. Start with the free abelian group on isomorphism classes of bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that each $H_k(P)$ is annihilated by I. Define $K(R,I)$ to be the quotient of this free abelian group by the analogs of relations (1) and (2) defining $K^{cplx}(R)$.
- (iii) Fix an $n \geq 0$. Start with the free abelian group on isomorphism classes of bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that each $H_k(P)$ has Krull dimension at most n. (Recall that the dimension of a module M is the dimension of the ring $R/\operatorname{Ann}(M)$). Define $K(R, \leq n)$ to be the quotient of this free abelian group by the usual relations (1) and (2).

Exercise 5.14. In analogy to (iii), define a group $K(R, \geq n)$. Prove that if $n > \dim R$ then $K(R, \geq n) = 0$. If R is a domain and $n \leq \dim R$ then $K(R, \geq n) \cong K(R)$. [Note: By convention the dimension of the zero module is $+\infty$.]

Exercise 5.15. Verify that the tensor product of chain complexes gives a ring structure on $K(R,S)$.

Every map $f\colon P \to Q$ of finitely-generated projectives can be regarded as a chain complex concentrated in degrees 0 and 1. From now on we will often make this identification without comment. If $S^{-1}f$ is an isomorphism then we get a corresponding class in $K(R,S)$. The following lemma about these classes will be very useful:

Lemma 5.16. *Let $\alpha\colon P \to Q$ and $\beta\colon Q \to W$ be maps between finitely-generated projectives, and assume both become isomorphisms after localization at S. Then*

$$[P \xrightarrow{\beta\alpha} W] = [P \xrightarrow{\alpha} Q] + [Q \xrightarrow{\beta} W]$$

in $K(R,S)$.

Proof. Use the following short exact sequence of maps:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "(\alpha,\mathrm{id})"] \arrow[d, "\alpha"'] & Q \oplus P \arrow[r, "(\mathrm{id},-\alpha)"] \arrow[d, "\mathrm{id}_Q \oplus \beta\alpha"] & Q \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & Q \arrow[r, "(\mathrm{id},\beta)"] & Q \oplus W \arrow[r, "(\beta,-\mathrm{id})"] & W \arrow[r] & 0.
\end{tikzcd}$$

This gives that

$$[Q \xrightarrow{\mathrm{id}} Q] + [P \xrightarrow{\beta\alpha} W] = [P \xrightarrow{\alpha} Q] + [Q \xrightarrow{\beta} W],$$

but of course the first term on the left is zero in $K(R,S)$. □

Note that there is an evident map $K(R,S) \to K(R)$ that sends a class $[P]$ in $K(R,S)$ to the similarly-named (but different) class $[P]$ in $K(R)$ (and recall that we identify $K(R)$ and $K^{cplx}(R)$ without comment from now on). In colloquial terms, the map simply 'forgets' that a complex P is S-exact. The composite $K(R,S) \to K(R) \to K(S^{-1}R)$ is clearly zero.

Proposition 5.17. *For any multiplicative system in a commutative ring R the sequence $K(R,S) \to K(R) \to K(S^{-1}R)$ is exact in the middle.*

Proof. Suppose $x \in K(R)$ is in the kernel of the map to $K(S^{-1}R)$. Every element of $K(R)$ may be written as $x = [P] - [Q]$ for some finitely-generated projectives P and Q. Then $[S^{-1}P] = [S^{-1}Q]$ in $K(S^{-1}R)$, so by Proposition 2.9 there exists an n such that $S^{-1}P \oplus (S^{-1}R)^n \cong S^{-1}Q \oplus (S^{-1}R)^n$. Alternatively, write this as $S^{-1}(P \oplus R^n) \cong S^{-1}(Q \oplus R^n)$. By Corollary 4.11(b) there exists a map of R-modules $Q \oplus R^n \to P \oplus R^n$ that becomes an isomorphism after S-localization. Regarding this map as a chain complex concentrated in degrees 0 and 1, it gives an element in $K(R,S)$. The image of this element under $K(R,S) \to K(R)$ is clearly x. □

The reader might have noticed that in the above proof we didn't encounter any kind of complicated chain complex when trying to construct our preimage in $K(R,S)$; in fact, we accomplished everything with chain complexes of length 1. This is a general phenomenon, similar to the fact that elements of $K^{cplx}(R)$ can all be decomposed into modules. For the relative K-groups one can't quite decompose *that* far, but one can always get down to complexes of length 1. To state a theorem along these lines, consider maps $f\colon P \to Q$ where P and Q are finitely-generated R-projectives and $S^{-1}f$ is an isomorphism (it is convenient to regard such maps as chain complexes concentrated in degrees 0 and 1). Let $K(R,S)_{\le 1}$ be the quotient of the free abelian group on such maps by the following relations:

(1) $[f] = 0$ if f is an isomorphism;

(2) $[f] = [f'] + [f'']$ if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "f'"'] & P \arrow[r] \arrow[d, "f"'] & P'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact.

Notice that there is an evident map $K(R,S)_{\leq 1} \to K(R,S)$.

Theorem 5.18. *For any multiplicative system S in a commutative ring R, the map $K(R,S)_{\leq 1} \to K(R,S)$ is an isomorphism.*

The proof of this theorem is a bit difficult, and the techniques are too distant from the topics at hand to merit spending time on them. We give the proof in Appendix F, for the interested reader.

Remark 5.19. Theorem 5.18 naturally suggests the following question: why use chain complexes at all, for relative K-theory? That is to say, if one can access the same groups using only chain complexes of length one, why complicate things by making the definition using complexes of arbitrary length? There are two answers to this question. The first concerns the ring structure: the tensor product of two complexes is again a complex in a natural way, giving a ring structure on $K(R,S)$. In contrast, there is not a particularly natural way of defining a ring structure on $K(R,S)_{\leq 1}$.

The second answer comes from algebraic geometry. Let X be a scheme and let U be an open subset of X. Then the 'correct' way to define a relative K-theory group $K(X,U)$ is to use bounded chain complexes of algebraic vector bundles on X that are exact on U. When $X = \operatorname{Spec} R$ and $U = \operatorname{Spec} S^{-1}R$ then it happens that one can get the same groups using only complexes of length one—as we saw above. But even for $X = \operatorname{Spec} R$ not every open subset is of this form. A general open subset will have the form $U = (\operatorname{Spec} S_1^{-1}R) \cup (\operatorname{Spec} S_2^{-1}R) \cup \cdots \cup (\operatorname{Spec} S_d^{-1}R)$, and to get the same relative K-group here using complexes with a fixed bound on their length the best one can do is to take that bound to be d. See [FH, "Main Theorem"] and [D3, Theorem 1.4] for the proof in this case.

When R is a regular ring all localizations $S^{-1}R$ are also regular. So the groups $K(R)$ and $K(S^{-1}R)$ can be identified with $G(R)$ and $G(S^{-1}R)$, by Theorem 2.13. Comparing the localization sequence in K-theory from Proposition 5.17 to the one in G-theory from Proposition 4.14 suggests an identification of the relative terms. Indeed, observe that the usual Euler characteristic map $\chi(P_\bullet) = \sum (-1)^i [H_i(P)]$ gives a well-defined map $K(R,S) \to G(M \,|\, S^{-1}M = 0)$. We have the following:

Theorem 5.20. *If R is regular then $\chi\colon K(R,S) \to G(M \,|\, S^{-1}M = 0)$ is an isomorphism.*

Proof. The proof repeats the ideas we have already seen in Theorem 2.13, Proposition 5.5, and Proposition 5.12. Define $\beta\colon G(M \,|\, S^{-1}M = 0) \to K(R,S)$ by sending $[M]$ to $[P_\bullet]$ for some finite resolution of M by finitely-generated projectives (which exists because R is regular). The exact same steps as in the proof of Theorem 2.13 show that this is well-defined, and it is clear that $\chi \circ \beta = \mathrm{id}$. So β is injective and χ is surjective. We finish the proof by showing that β is surjective.

Let P be a bounded complex of finitely-generated projectives such that $S^{-1}P$ is exact. If all the homology groups of P are zero then $[P] = 0$ in $K(R,S)$ and so $[P] \in \operatorname{im}(\beta)$. We proceed by induction on the number of non-vanishing homology groups.

By suspending or desuspending we can assume that $P_i = 0$ for $i < 0$. If $H_0(P) = 0$ then we can split off an acyclic complex from the bottom of P and reduce to a complex of smaller length. So we can also assume $H_0(P) \neq 0$. Let Q be a bounded resolution of $H_0(P)$ by finitely-generated projectives. Then there is a map of complexes $f\colon P \to Q$ that is an isomorphism on H_0, so that the mapping cone Cf has fewer nonvanishing homology groups than P. Since $S^{-1}P$ and $S^{-1}Q$ are exact, $S^{-1}(Cf)$ is also exact. We have $[Q] = [Cf] + [P]$ in $K(R,S)$, $[Q]$ is in the image of β by construction, and by induction $[Cf]$ is also in the image of β. So $[P]$ is in the image, and we are done. $\square$

5.21. **Relative K-theory and intersection multiplicities.** We now wish to tie several themes together, and use everything we have learned so far to give a complete, K-theoretic perspective on Serre's definition of intersection multiplicity. This perspective is from the paper [GS].

Let R be a Noetherian ring, and let $Z \subseteq \operatorname{Spec} R$ be any subset. An R-module M is said to be **supported** on Z if $M_P = 0$ for all primes $P \notin Z$. One usually defines $\operatorname{Supp} M$, the **support** of M, to be $\{P \in \operatorname{Spec} R \mid M_P \neq 0\}$. This is known to be a Zariski closed subset of $\operatorname{Spec} R$, and to say that M is supported on Z is just the requirement that $\operatorname{Supp} M \subseteq Z$. When M is finitely-generated, M is supported on $Z = V(I)$ if and only if a power of I annihilates M. Likewise, if S is a multiplicative system then M is supported on $\operatorname{Spec} R - \operatorname{Spec} S^{-1}R$ if and only if $S^{-1}M = 0$.

Let $G(R)_Z$ be the Grothendieck group of all finitely-generated R-modules that are supported on Z.

Similarly, if $C_\bullet$ is a chain complex of R-modules then $\operatorname{Supp} C$ is defined to be $\{P \in \operatorname{Spec} R \mid H_*(C_P) \neq 0\}$. We say that $C_\bullet$ is supported on Z if $\operatorname{Supp} C \subseteq Z$, or if C_Q is exact for every $Q \notin Z$. Note that $C_\bullet$ is supported on Z if and only if all the homology modules $H_*(C)$ are supported on Z.

Similar to our definitions of $K^{cplx}(R)$ and $K(R,S)$, define $K(R)_Z$ to be the Grothendieck-style group of bounded complexes $P_\bullet$ of finitely-generated projective R-modules having the property that $\operatorname{Supp} P_\bullet \subseteq Z$. Note that if $Z = \operatorname{Spec} R - \operatorname{Spec} S^{-1}R$ then $K(R)_Z$ is precisely the group $K(R,S)$ previously defined.

The following statements should be easy exercises for the reader:

(1) The Euler characteristic $\chi(P_\bullet) = \sum_i (-1)^i [H_i(P)]$ defines a group homomorphism $K(R)_Z \to G(R)_Z$.
(2) If R is regular then the map $\chi\colon K(R)_Z \to G(R)_Z$ is an isomorphism.
(3) Tensor product of chain complexes gives pairings
$$\otimes\colon K(R)_Z \otimes K(R)_W \to K(R)_{Z\cap W}$$
for all pairs of closed subsets $Z, W \subseteq \operatorname{Spec} R$.
(4) If M and N are R-modules then $\operatorname{Supp}(M \otimes N) = \operatorname{Supp} M \cap \operatorname{Supp} N$.
(5) Assume that R is regular and transplant the tensor product of chain complexes from (3) to a pairing
$$G(R)_Z \otimes G(R)_W \to G(R)_{Z\cap W}.$$

This sends $[M] \otimes [N]$ to $\sum(-1)^i[\mathrm{Tor}_i(M,N)]$. (Note that this makes sense on the level of supports: If $Z = V(I)$ and $W = V(J)$ then $Z \cap W = V(I+J)$. If M is killed by a power of I and N is killed by a power of J, then $M \otimes N$ and all the $\mathrm{Tor}_i(M,N)$ are killed by some $I^r + J^s$ and therefore by a power of $I+J$).

(6) Let $Z = \{m\}$ where m is a maximal ideal of R. If a module is supported on Z then it has finite length, and the assignment $M \mapsto \ell(M)$ gives an isomorphism $G(R)_Z \xrightarrow{\cong} \mathbb{Z}$.

(7) Let M and N be R-modules such that $\mathrm{Supp}(M \otimes N) = \{m\}$ where m is a maximal ideal of R (geometrically, $\mathrm{Supp}\, M$ and $\mathrm{Supp}\, N$ have an isolated point of intersection). Then Serre's intersection multiplicity $e(M,N)$ is the image of $[M] \otimes [N]$ under the composite

$$G(R)_Z \otimes G(R)_W \longrightarrow G(R)_{Z\cap W} \xrightarrow{\ell} \mathbb{Z},$$

where we have written $Z = \mathrm{Supp}\, M$ and $W = \mathrm{Supp}\, N$ (and the map labelled ℓ is in fact an isomorphism).

Exercise 5.22. Prove (1)–(7) above.

Remark 5.23. We will understand this better after seeing how intersection multiplicities fit into algebraic topology, but it is worth noting that the group $K(R)_Z$ would—from a topological perspective—be better written as $K(X, X-Z)$, where $X = \mathrm{Spec}\, R$. For comparison, relative products in a cohomology theory would give pairings

$$K(X, X-Z) \otimes K(X, X-W) \to K(X, (X-Z) \cup (X-W)) = K(X, X-(Z \cap W)),$$

which is what we saw above in the form $K(R)_Z \otimes K(R)_W \to K(R)_{Z\cap W}$. See Section 18.5 for more on relative topological K-theory.

6. K-theory of exact complexes

We have seen the isomorphism of groups $K(R) \cong K^{cplx}(R)$. If $P_\bullet$ is a bounded, exact complex of projectives then it gives rise to a relation in $K(R)$, and (equivalently) represents the zero object in $K^{cplx}(R)$. Given this, it might seem surprising to learn that there is yet another model for $K(R)$ in which exact complexes can represent nonzero elements—and even more, *all nonzero elements* can be represented this way. The goal of the present section is to explain this model, as well as some variations. This material is adapted from [Gr2].

Note: The contents of this section are only needed once in the remainder of the book, for a certain perspective on Adams operations in Section 35. While the material is intriguing, it can certainly be skipped if desired.

As in the last section, let R be a fixed commutative ring.

Definition 6.1.

$$K^{exact}(R) = \frac{\mathbb{Z}\langle\, [P_\bullet] \,\big|\, P_\bullet \textit{ is a bounded, exact chain complex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ *if there is a short exact sequence* $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$,

(2) $[\Sigma P_\bullet] = -[P_\bullet]$.

If P is a projective module let CP denote the mapping cone of the identity map $P \to P$. Specifically, CP is a chain complex concentrated in dimensions 0 and 1 where the only nonzero differential is the identity map on P. Observe that there is a group homomorphism $K(R) \to K^{exct}(R)$ that sends $[P]$ to $[CP]$.

Proposition 6.2. *The map $K(R) \to K^{exct}(R)$ is an isomorphism. The inverse is denoted $\chi'\colon K^{exct}(R) \to K(R)$ and called the **derived** (or **secondary**) **Euler characteristic**. If $P_\bullet$ is an exact bounded complex of finitely-generated projectives then*

$$\chi'(P_\bullet) = \sum_j (-1)^{j+1} j[P_j] = \sum_j (-1)^{j-1}[\operatorname{im} d_j]$$

where $d_j\colon P_j \to P_{j-1}$.

Technically speaking the second formula given for χ' doesn't make sense unless we know that each $\operatorname{im} d_j$ is a finitely-generated projective module. This is a simple exercise, but let us record it in a lemma.

Lemma 6.3. *Let $P_\bullet$ be a bounded, exact complex of projectives. Then each $\operatorname{im} d_j$ is projective, and is finitely-generated if P_j is.*

Proof. Without loss of generality we can assume that $P_\bullet$ has the form $0 \to P_n \to \cdots \to P_0 \to 0$. So $d_1\colon P_1 \to P_0$ is surjective, hence $\operatorname{im} d_1 = P_0$ and there is nothing to prove here. Exactness gives us short exact sequences $0 \to \operatorname{im} d_{j+1} \to P_j \to \operatorname{im} d_j \to 0$, for each j. We can assume by induction that $\operatorname{im} d_j$ is projective, hence the sequence is split-exact and therefore $\operatorname{im} d_{j+1}$ is also projective.

Since $\operatorname{im} d_j$ is a quotient of P_j, it is finitely-generated if P_j is. □

The above proof of course gives more than was explicity stated: by choosing splittings one level at a time one can see that $P_\bullet$ decomposes as a direct sum of exact complexes of length 1. This decomposition is non-canonical, however, depending on the choices of splitting. For variety we will see a weaker, but more canonical, version of this decomposition in the next proof.

Proof of Proposition 6.2. Let α denote the map $K(R) \to K^{exct}(R)$ sending $[P] \mapsto [CP]$. It is easy to see that α is surjective, because if $0 \to P_n \to \cdots \to P_0 \to 0$ is an exact complex then there is an evident short exact sequence

$$0 \to \Sigma^{n-1}(CP_n) \to P_\bullet \to Q_\bullet \to 0$$

where $Q_\bullet$ is exact and has length at most $n-1$. Since $[P_\bullet] = [\Sigma^{n-1}(CP_n)] + [Q_\bullet] = (-1)^{n-1}[CP_n] + [Q_\bullet]$, an immediate induction shows that $K^{exct}(R)$ is generated by the classes $[CP]$ as P ranges over all finitely-generated projectives.

Note that $P_n \cong \operatorname{im} d_n$, and $Q_{n-1} = \operatorname{coker}(P_n \to P_{n-1}) \cong \operatorname{im} d_{n-1}$. The induction mentioned in the preceding paragraph shows that $[P_\bullet] = \sum_j (-1)^{j-1}[C(\operatorname{im} d_j)]$. From this it is clear that if an inverse to α exists it must send $[P_\bullet]$ to $\sum_j (-1)^{j-1}[\operatorname{im} d_j]$. It is only left to check that this formula does indeed define a map $K^{exct}(R) \to K(R)$.

Let $P_\bullet$ be any bounded, exact complex of finitely-generated projectives, and assume that the smallest degree containing a nonzero module is degree n. Write

$I_j = \operatorname{im} d_j$. Since $\cdots \to P_{j+1} \to P_j \to I_j \to 0$ is exact, we have that $[I_j] = \sum_{k\geq j}(-1)^{k-j}[P_k]$ in $K(R)$. So in $K(R)$ we have

$$\begin{aligned}\sum_{j\geq n}(-1)^{j-1}[I_j] = \sum_{j\geq n}\sum_{k\geq j}(-1)^{j-1}(-1)^{k-j}[P_k] &= \sum_k (-1)^{k-1}\sum_{n\leq j\leq k}[P_k] \\ &= \sum_k (-1)^{k-1}(k-n+1)[P_k] \\ &= \sum_k (-1)^{k-1}k[P_k] + (n-1)\chi(P_\bullet) \\ &= \sum_k (-1)^{k-1}k[P_k].\end{aligned}$$

In the last equality we have used that $\chi(P_\bullet) = 0$ since $P_\bullet$ is exact.

Define $\chi'(P_\bullet) = \sum_k (-1)^{k-1}k[P_k]$. One easily checks that this satisfies relations (1) and (2) in the definition of $K^{exct}(R)$, and hence defines a map $\chi' \colon K^{exct}(R) \to K(R)$. It is trivial to check that $\chi' \circ \alpha = \mathrm{id}$. Therefore α is injective, and since we already proved surjectivity it is an isomorphism and χ' is its inverse. $\square$

6.4. **Derived Euler characteristics.** Now that we have encountered the derived Euler characteristic it seems worthwhile to take a moment and place it into a broader context. Consider the definition

$$\chi_t(P_\bullet) = \sum t^j[P_j] \in K(R)[t, t^{-1}].$$

This function is additive, and in fact one can see that it is the universal additive invariant for bounded complexes of finitely-generated projectives. The usual Euler characteristic is $\chi(P_\bullet) = \chi_t(P_\bullet)|_{t=-1}$. Of course we do not have $\chi_t(\Sigma P_\bullet) = -\chi_t(P_\bullet)$, this only becomes true after the substitution $t = -1$; what we have instead is the identity

$$\chi_t(\Sigma P_\bullet) = t \cdot \chi_t(P_\bullet). \tag{6.5}$$

If we differentiate χ_t with respect to t then we obtain $\chi_t'(P_\bullet) = \sum jt^{j-1}[P_j]$. Clearly this is also an additive invariant of complexes. The invariant we called χ' is just $\chi_t'(P_\bullet)|_{t=-1}$. Differentiating (6.5) yields the formula

$$\chi_t'(\Sigma P_\bullet) = \chi_t(P_\bullet) + t \cdot \chi_t'(P_\bullet), \tag{6.6}$$

and consequently $\chi'(\Sigma P_\bullet) = \chi(P_\bullet) - \chi'(P_\bullet)$. This is not the kind of behavior we are used to, but notice that if we restrict to complexes $P_\bullet$ with $\chi(P_\bullet) = 0$ then we get the nicer behavior $\chi'(\Sigma P_\bullet) = -\chi'(P_\bullet)$.

One can, of course, iterate this procedure. Let $\chi_t^{(n)}(P_\bullet)$ denote the nth derivative of $\chi_t(P_\bullet)$, and white $\chi^{(n)}(P_\bullet) = \chi_t^{(n)}(P_\bullet)|_{t=-1}$. Call this the ***n*th derived Euler characteristic**. It is an additive function, and if one restricts to complexes such that $0 = \chi^{(n-1)}(P_\bullet)$ then it satifies $\chi^{(n)}(\Sigma P_\bullet) = -\chi^{(n)}(P_\bullet)$.

6.7. **Doubly-exact complexes.** A bicomplex $C_{\bullet,\bullet}$ will be called **bounded** if the modules $C_{i,j}$ are nonzero for only finitely many values of (i,j). The bicomplex will be called **doubly-exact** if every row and every column is exact. By abuse of terminology an ordinary chain complex $D_\bullet$ will be called doubly-exact if it is isomorphic to the total complex of a bounded, doubly-exact bicomplex. Doubly exact complexes all represent zero in $K^{exct}(R)$:

Proposition 6.8. *If $P_\bullet$ is a bounded, doubly-exact complex of finitely-generated projectives then $[P_\bullet] = 0$ in $K^{exct}(R)$.*

Proof. Let $M_{\bullet,\bullet}$ be a doubly-exact bicomplex of finitely-generated projectives. For some n and k, $M_{i,j}$ is zero outside of the rectangle $0 \leq i \leq n$ and $0 \leq j \leq k$. Write $M_{i,*}$ for the ordinary complex whose jth term is $M_{i,j}$, and write $M_{\leq i,*}$ for the sub-bicomplex of $M_{\bullet,\bullet}$ consisting of all $M_{a,j}$ for $a \leq i$. Observe that there are short exact sequences

$$0 \to \mathrm{Tot}(M_{\leq(i-1),*}) \hookrightarrow \mathrm{Tot}(M_{\leq i,*}) \twoheadrightarrow \Sigma^i M_{i,*} \to 0,$$

for all i. Induction shows that each $\mathrm{Tot}(M_{\leq i,*})$ is exact, and therefore in $K^{exct}(R)$ we have

$$[\mathrm{Tot}\, M_{\bullet,\bullet}] = \sum_i [\Sigma^i M_{i,*}] = \sum_i (-1)^i [M_{i,*}]$$

(using the analog of Proposition 2.5(b)). But $M_{\bullet,\bullet}$ may be regarded as an exact sequence of chain complexes

$$0 \to M_{n,*} \to M_{n-1,*} \to \cdots \to M_{1,*} \to M_{0,*} \to 0.$$

The image of each map in this sequence is a chain complex of finitely-generated projectives (using Lemma 6.3), and we have the short exact sequences of chain complexes

$$0 \longrightarrow \mathrm{im}(M_{i+1,*}) \longrightarrow M_{i,*} \longrightarrow \mathrm{im}(M_{i,*}) \longrightarrow 0.$$

By a straightforward induction, each of these image complexes is exact. Each of these short exact sequences gives a relation in $K^{exct}(R)$, and taking their alternating sum shows that $\sum_i (-1)^i [M_{i,*}] = 0$. We have therefore shown that $[\mathrm{Tot}\, M_{\bullet,\bullet}] = 0$ in $K^{exct}(R)$. $\square$

The reader will notice the beginnings of a pattern here. Exact complexes $P_\bullet$ represent zero in $K^{cplx}(R)$, but then we produced a new model for this same group where the exact complexes are the generators. In this new group $K^{exct}(R)$ the doubly-exact complexes represent zero. It is natural, then, to wonder if there is yet another model for this group where the doubly-exact complexes are the generators. Indeed, this works out in what is now a completely straightforward manner, and can be repeated ad infinitum.

Let us use the term *multicomplex* for the evident generalization of bicomplexes to n dimensions. We will denote a multicomplex by $M_\star$, where the symbol $\star$ stands for an n-tuple of integers. Say that the multicomplex is ***n*-exact** if every linear 'row' (obtained by fixing $n-1$ of the indices) is exact.

Definition 6.9.

$$K^{n-exct}(R) = \frac{\mathbb{Z}\langle\, [M_\star] \,\big|\, M_\star \textit{ is a bounded, n-exact multicomplex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[M_\star] = [M'_\star] + [M''_\star]$ *if there is an exact sequence* $0 \to M'_\star \to M_\star \to M''_\star \to 0$,

(2) $[\Sigma M_\star] = -[M_\star]$, *where* Σ *stands for any of the* n *suspension operators on* n*-multicomplexes.*

Given an $(n+1)$-multicomplex $M_\star$ there are $\binom{n+1}{2}$ ways to totalize it to get an n-multicomplex—one needs to choose two of the $n+1$ directions to combine. One can follow the proof of Proposition 6.8 to show that if $M_\star$ is $(n+1)$-exact then each of these totalizations represents zero in $K^{n-exct}(R)$.

If $M_\star$ is an n-multicomplex then let $CM_\star$ denote the cone on the identity map $M_\star \to M_\star$. This is an $(n+1)$-multicomplex, defined in the evident manner. This cone construction induces a group homomorphism $K^{n-exct}(R) \to K^{(n+1)-exct}(R)$.

Proposition 6.10. *The map $K^{n-exct}(R) \to K^{(n+1)-exct}(R)$ is an isomorphism, with inverse given by*

$$\chi'(M_\star) = \sum (-1)^{j+1} j [M_{j,\star}]$$

where the symbols $M_{j,\star}$ represent the various slices of $M_\star$ in any fixed direction.

Proof. Follow the proof of Proposition 6.2 almost verbatim, but where each P_i represents an n-exact multicomplex rather than an R-module. □

We have the sequence of isomorphisms

$$K(R) \to K^{exct}(R) \to K^{2-exct}(R) \to \cdots$$

The composite map $K(R) \to K^{n-exct}(R)$ sends $[P]$ to the n-dimensional cube consisting of P's and identity maps. The composite of the χ' maps in the other direction yields the map $K^{n-exct}(R) \to K(R)$ given by

$$M_\star \mapsto \sum_{j_1,\ldots,j_n} (-1)^{j_1+\cdots+j_n+n} j_1 \cdots j_n [M_{j_1,\ldots,j_n}].$$

If one considers the formal Laurent polynomial

$$\chi_{t_1,\ldots,t_n}(M) = \sum_{j_1,\ldots,j_n} t_1^{j_1} \cdots t_n^{j_n} [M_{j_1,\ldots,j_n}]$$

then this is the nth order partial derivative $\partial_{t_1} \cdots \partial_{t_n} \chi_{t_1,\ldots,t_n}(M)$ evaluated at $t_1 = t_2 = \cdots = t_n = -1$.

Remark 6.11. Grayson [Gr2] suggests a perspective where exact complexes are analogous to the formal infinitesimals from nonstandard analysis. Doubly-exact complexes are analogues of products of infinitesimals, and so forth.

7. A taste of K_1

Note: The material in this section will not be needed for most of what follows. We include it for general interest, and because the material fits naturally here. But this section can safely be skipped.

Given a commutative ring R and a multiplicative system $S \subseteq R$, we have seen the exact sequences

$$G(M \mid S^{-1}M = 0) \to G(R) \to G(S^{-1}R) \to 0$$

and

$$K(R, S) \to K(R) \to K(S^{-1}R).$$

It is natural to wonder if these extend to long exact sequences, and the answer is that they do: in the first case there is an extension to the left, and in the latter case there is an extension in both directions. These extensions are not easy to produce, however—they are the subject of 'higher algebraic K-theory', a field that involves some very deep and difficult mathematics. Our aim here is not to start a long journey into that subject, but rather to just give some indications of the very beginnings.

Remark 7.1. From now on the groups $K(R)$ and $G(R)$ will be written $K_0(R)$ and $G_0(R)$.

7.2. **The basic theory of $K_1(R)$.** Let us adopt the perspective that $K_0(R)$ is, in essence, constructed with the goal of generalizing the familiar notion of dimension for vector spaces. The key property of dimension is additivity for short exact sequences, so consequently one forms the universal group with that property. The group $K_1(R)$ is obtained similarly but with the goal of generalizing the *determinant*.

Determinants are invariants of self-maps—maps with the same domain and target—and we need some language for dealing with such things. Given two self-maps $f\colon A \to A$ and $g\colon B \to B$, we define a map from f to g to be a map $u\colon A \to B$ giving a commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A \arrow[r, "u"] & B. \end{tikzcd}$$

Likewise, an exact sequence of self-maps is a diagram

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r, "u_0"] \arrow[d, "f'"] & P \arrow[r, "u_1"] \arrow[d, "f"] & P'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & P' \arrow[r, "u_0"] & P \arrow[r, "u_1"] & P'' \arrow[r] & 0 \end{tikzcd} \tag{7.3}$$

in which the rows are short exact sequences of modules.

Definition 7.4. *Form the free abelian group generated by isomorphism classes of maps $[P \xrightarrow{\alpha} P]$ where P is a finitely-generated projective and α is an isomorphism. Let $K_1(R)$ be the quotient of this group by the following relations:*

(a) $[P \xrightarrow{\alpha} P] = [P' \xrightarrow{\alpha'} P'] + [P'' \xrightarrow{\alpha''} P'']$ whenever there is a short exact sequence as in (7.3);

(b) $[P \xrightarrow{\alpha\beta} P] = [P \xrightarrow{\alpha} P] + [P \xrightarrow{\beta} P]$ for all self-maps $\alpha, \beta\colon P \to P$.

As a consequence of relation (b) one has that $[P \xrightarrow{\mathrm{id}} P] = [P \xrightarrow{\mathrm{id}} P] + [P \xrightarrow{\mathrm{id}} P]$, and so $[P \xrightarrow{\mathrm{id}} P] = 0$ for any finitely-generated projective P. Note also that if $\alpha\colon P \to P$ and $\beta\colon Q \to Q$ are automorphsms then

$$[P \oplus Q \xrightarrow{\alpha\oplus\beta} P \oplus Q] = [P \xrightarrow{\alpha} P] + [Q \xrightarrow{\beta} Q], \tag{7.5}$$

as a consequence of relation (a).

The use of projective modules in the definition of $K_1(R)$ turns out to be unnecessarily complicated—one can get the same group by only using automorphisms of free modules. Even more, the use of short exact sequences in relation (a) is unnecessarily complicated; one can get the same group by only imposing the weaker relation from (7.5). We will prove both of these claims in just a moment.

Observe that there is a map of groups $GL_n(R) \to K_1(R)$ that sends a matrix A to the class $[R^n \xrightarrow{A} R^n]$ (left-multiplication-by-A). Relation (b) guarantees that this is indeed a group homomorphism. If we let $j\colon GL_n(R) \hookrightarrow GL_{n+1}(R)$ be the usual inclusion, obtained by adding an additional row and column and a 1 along the diagonal, then it is clear that $[R^{n+1} \xrightarrow{j(A)} R^{n+1}] = [R^n \xrightarrow{A} R^n]$. This follows from (7.5) and the fact that $[R \xrightarrow{\mathrm{id}} R] = 0$. Let $GL(R)$ denote the colimit

$$GL(R) = \operatorname{colim}[GL_1(R) \to GL_2(R) \to GL_3(R) \to \cdots],$$

and call this the **infinite general linear group** of R. We have obtained a map $GL(R) \to K_1(R)$, and of course this will factor through the abelianization to give

$$GL(R)_{ab} = GL(R)/[GL(R), GL(R)] \to K_1(R).$$

Exercise 7.6. We abelianized after taking the colimit, but could have just as well done it the other way around. Verify that if $G_1 \to G_2 \to \cdots$ is any sequence of group homomorphisms then $[\operatorname{colim}_n G_n]_{ab} = \operatorname{colim}_n[(G_n)_{ab}]$.

The most fundamental result in the theory of K_1 is the following:

Theorem 7.7. *The map $GL(R)_{ab} \to K_1(R)$ is an isomorphism.*

It will be convenient to prove this at the same time that we give other descriptions for $K_1(R)$. In particular, we make the following definitions:

(1) $K_1^{fr}(R)$ is the group defined similarly to $K_1(R)$ but changing all occurrences of 'projective' to 'free'.
(2) $K_1^{sp}(R)$ is the group defined similarly to $K_1(R)$ but replacing relation (a) by the direct sum relation of (7.5). The "sp" stands for "split".
(3) $K_1^{sp,fr}(R)$ is the group defined by making both the changes indicated in (1) and (2).

One obtains a large diagram as follows:

(7.8)
$$\begin{tikzcd}
 & \operatorname{colim}_P \operatorname{Aut}(P)_{ab} \arrow[r, two heads] & K_1^{sp}(R) \arrow[r, two heads] & K_1(R) \\
GL(R)_{ab} \arrow[r, equal] & \operatorname{colim}_n GL_n(R)_{ab} \arrow[r, two heads] \arrow[u] & K_1^{sp,fr}(R) \arrow[r, two heads] \arrow[u] & K_1^{fr}(R). \arrow[u]
\end{tikzcd}$$

The maps labelled as surjections are obviously so. Let us explain the colimit over projectives P. Let $\mathcal{M}$ denote the monoid of isomorphism classes of finitely-generated projectives, with the operation of $\oplus$. The *translation category* $T(\mathcal{M})$ of this monoid has object set equal to $\mathcal{M}$, and the maps from A to B are the elements $C \in \mathcal{M}$ such that $A + C = B$; composition is given by the multiplication in $\mathcal{M}$. This is the indexing category for our colimit. Given an isomorphism $f\colon P \to Q$, there is an induced map of groups $\operatorname{Aut}(P) \to \operatorname{Aut}(Q)$ sending α to $f\alpha f^{-1}$. Changing f gives a different induced map, but it gives the same induced map on $\operatorname{Aut}(P)_{ab} \to \operatorname{Aut}(Q)_{ab}$: this is a consequence of the formula

$$f\alpha f^{-1} = (fg^{-1})(g\alpha g^{-1})(gf^{-1}).$$

We can therefore construct a functor $T(\mathcal{M}) \to \mathcal{A}b$ sending each $[P]$ to $\operatorname{Aut}(P)_{ab}$. If $[Q]$ is a map from $[P]$ to $[J]$ then we choose an isomorphism $f\colon P \oplus Q \to J$ and have T send the map $[Q]$ to the composite $\operatorname{Aut}(P)_{ab} \hookrightarrow \operatorname{Aut}(P \oplus Q)_{ab} \to \operatorname{Aut}(J)_{ab}$. The first map is direct sum with id_Q and the second map is independent of the choice of f, so this is well-defined. The upper left term in our diagram is the colimit of the functor T. The map from this colimit to $K_1^{sp}(R)$ is induced by the one sending an element $\alpha \in \mathrm{Aut}(P)$ to the class $[P \xrightarrow{\alpha} P]$.

Theorem 7.7 will follow as an immediate consequence of the following stronger result:

Theorem 7.9. *All of the maps in (7.8) are isomorphisms.*

We are almost ready to prove this theorem, but we will need one key result first. Let $E(R) \subseteq GL(R)$ be the subgroup generated by the elementary matrices—matrices that have ones along the diagonal and a single nonzero, off-diagonal entry. We likewise define $E_n(R) \subseteq GL_n(R)$, and observe that $E(R) = \mathrm{colim}_n\, E_n(R)$. We will implicitly identify matrices in $E_n(R)$ with their image in $E(R)$; note that this involves adding trailing ones down the diagonal of the matrix.

Note that right multiplication by an elementary matrix amounts to performing a column operation where a multiple of one column is added to another; similarly, left multiplication amounts to performing the analogous row operation. One very useful way to recognize a matrix as belonging to $E(R)$ is to observe that it can be obtained from the identity matrix by using these types of row and column operations. It will be convenient to call a column or row operation of this type **allowable**.

Lemma 7.10.

(a) For any $X \in M_n(R)$ the matrix $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ and its transpose belong to $E(R)$.

(b) If $A \in GL_n(R)$ then $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \in E(R)$.

(c) Let A be a matrix obtained from the identity by switching two columns and multiplying one of the switched columns by -1. Then $A \in E(R)$, and similarly for the transpose of A.

Proof. For part (a) just note that $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ can be obtained from the identity matrix by a sequence of allowable column operations. Same for the transpose.

For (b) consider the following chain of matrices:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} I & A \\ A^{-2} - A^{-1} & A^{-1} \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ A^{-2} - A^{-1} & A^{-1} \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.$$

Passage from each matrix to the next can be done by allowable row and column operations. Alternatively, each matrix can be obtained from its predecessor by left multiplication by a matrix of the type considered in (a): use the matrices $\left[\begin{smallmatrix} I & A \\ 0 & I \end{smallmatrix}\right]$, $\begin{bmatrix} I & 0 \\ A^{-2}-A^{-1} & I \end{bmatrix}\cdot$, $\begin{bmatrix} I & -A^2 \\ 0 & I \end{bmatrix}\cdot$, and $\cdot\begin{bmatrix} I & 0 \\ A^{-2}-A^{-3} & I \end{bmatrix}$.

Finally, for (c) we argue directly in terms of column operations. If v and w are two columns consider the following chain

$$v, w \quad \mapsto \quad v, w - v \quad \mapsto \quad w, w - v \quad \mapsto \quad w, -v.$$

Each link involves adding a multiple of one column to another, and is therefore allowable; therefore the composite operation is allowable. The argument is similar for $v, w \mapsto -w, v$, or one could use the fact that $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in E(R)$ by part (b). $\square$

The following is the key lemma that we will need in our proof of Theorem 7.9:

Lemma 7.11 (Whitehead Lemma)**.** $E(R) = [GL(R), GL(R)]$

Proof. For the $\subseteq$ direction we consider three columns u, v, w, and the following chain of operations (where $r, s \in R$):

$$\begin{tikzcd} u,v,w \arrow[r, "1"] & u,v+ru,w \arrow[r, "2"] & u,v+ru,w+sv+sru \arrow[r, "3"] & u,v,w+sv+sru \arrow[d, "4"] \\ & & & u,v,w+sru. \end{tikzcd}$$

It should be clear what column operation is being used in each step. Note that the third and fourth operations are the inverses of the first and second, so the composite it a commutator. This shows that any column operation of the type "add a multiple of one column to another" is a commutator, and therefore $E(R) \subseteq [GL(R), GL(R)]$. (We have actually shown $E_n(R) \subseteq [GL_n(R), GL_n(R)]$ for $n \geq 3$).

For the other subset direction, let $A, B \in GL_n(R)$. Consider the following identity:

$$\begin{bmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{bmatrix}.$$

The first matrix is identified with the commutator of A and B inside $GL(R)$, and all of the other matrices are in $E(R)$ by Lemma 7.10(b). So $[A, B] \in E(R)$ as well. $\square$

Corollary 7.12. *For any $A \in GL_n(R)$, $B \in GL_k(R)$, and $X \in M_{n\times k}(R)$, $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ in $GL(R)_{ab}$. If $n = k$ then this matrix also equals $\left[\begin{smallmatrix} AB & 0 \\ 0 & I \end{smallmatrix}\right]$ in $GL(R)_{ab}$.*

Proof. For the first claim simply observe that

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} I & A^{-1}X \\ 0 & I \end{bmatrix}.$$

The second matrix in the product is in $E(R)$ by Lemma 7.10(a), and hence in $[GL(R), GL(R)]$ by the Whitehead Lemma.

For the second claim notice that $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} B & 0 \\ 0 & B^{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} AB & 0 \\ 0 & I \end{smallmatrix}\right]$ and use Lemma 7.10(b) together with the Whitehead Lemma. $\square$

We are now ready to prove that all of our descriptions of $K_1(R)$ give the same group:

Proof of Theorem 7.9. Let $\alpha \colon P \to P$ be an automorphism of a finitely-generated projective, and let Q be a free complement to P: that is, $P \oplus Q \cong R^n$ for some n. Then

$$[P \xrightarrow{\alpha} P] = [P \oplus Q \xrightarrow{\alpha \oplus \mathrm{id}_Q} P \oplus Q]$$

in $K_1^{sp}(R)$, which shows that $K_1^{sp,fr}(R) \to K_1^{sp}(R)$ is surjective. The same proof works for all of the vertical maps in diagram (7.8).

The fact that $\mathrm{colim}_n\, GL_n(R)_{ab} \to \mathrm{colim}_P \mathrm{Aut}(P)_{ab}$ is an isomorphism is very easy: it is just because the subcategory of $T(\mathcal{M})$ consisting of the free modules is final in $T(\mathcal{M})$ (see [ML, IX.3] for the notion of final functors).

Define a map $K_1^{fr}(R) \to GL(R)_{ab}$ by sending $[R^n \xrightarrow{A} R^n]$ to the matrix A. To see that this is well-defined we need to verify that it respects relations (a) and (b) from Definition 7.4. Relation (b) is self-evident. For (a), suppose that

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d, "\alpha'"] & F \arrow[r] \arrow[d, "\alpha"] & F'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\ 0 \arrow[r] & F' \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence of automorphisms between free modules. Then there is a basis for F with respect to which the matrix for α has the form $\left[\begin{smallmatrix} \alpha' & * \\ 0 & \alpha'' \end{smallmatrix}\right]$. Corollary 7.12 verifies that this matrix equals $\left[\begin{smallmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{smallmatrix}\right]$ in $GL(R)_{ab}$.

Now that we have the map $K_1^{fr}(R) \to GL(R)_{ab}$, it is trivial to check that this is a two-sided inverse for the map from (7.8). It follows that all the maps in the bottom row of that diagram are isomorphisms.

The proof for the maps along the top row proceeds in a similar manner. Define a map $K_1(R) \to \operatorname{colim}_P \operatorname{Aut}(P)_{ab}$ by sending $[P \xrightarrow{\alpha} P]$ to the element $\alpha \in \operatorname{Aut}(P)_{ab}$. One has to check that this respects relations (a) and (b) in the definition of $K_1(R)$, and relation (b) is again trivial. Suppose that

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "\alpha'"] & P \arrow[r] \arrow[d, "\alpha"] & P'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\ 0 \arrow[r] & P' \arrow[r] & P \arrow[r] & P'' \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence of automorphisms between finitely-generated projectives. Choose free complements Q' for P', and Q'' for P''. Consider the new short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & P' \oplus Q' \arrow[r] \arrow[d, "\alpha' \oplus \mathrm{id}_{Q'}"] & P \oplus Q' \oplus Q'' \arrow[r] \arrow[d, "\alpha \oplus \mathrm{id}_{Q'} \oplus \mathrm{id}_{Q''}"] & P'' \oplus Q'' \arrow[r] \arrow[d, "\alpha'' \oplus \mathrm{id}_{Q''}"] & 0 \\ 0 \arrow[r] & P' \oplus Q' \arrow[r] & P \oplus Q' \oplus Q'' \arrow[r] & P'' \oplus Q'' \arrow[r] & 0. \end{tikzcd}$$

All of the modules in this diagram are free (recall that $P \cong P' \oplus P''$), and so this diagram gives a relation in $K_1^{fr}(R)$. Using the map $K_1^{fr}(R) \to GL(R)_{ab}$ already constructed, we find that

$$\alpha \oplus \mathrm{id}_{Q'} \oplus \mathrm{id}_{Q''} = \big(\alpha' \oplus \mathrm{id}_{Q'}\big) + \big(\alpha'' \oplus \mathrm{id}_{Q''}\big)$$

in $GL(R)_{ab}$ and hence also in $\operatorname{colim}_P \operatorname{Aut}(P)_{ab}$. But this says precisely that $\alpha = \alpha' + \alpha''$ as elements in $\operatorname{colim}_P \operatorname{Aut}(P)_{ab}$, and this is what we needed to check. We have now constructed our map $K_1(R) \to \operatorname{colim}_P \operatorname{Aut}(P)_{ab}$, and it readily follows that it is an inverse for the map in the other direction from (7.8). So all the maps in the top horizontal row of (7.8) are isomorphisms.

We have shown that all horizontal maps in (7.8) are isomorphisms, and that the left vertical map is an isomorphism. So all the maps are isomorphisms. $\square$

We now work towards computing a few examples of $K_1(R)$ in some easy cases. The defining relations we used to construct $K_1(R)$ captured familiar properties of the determinant, so it should not be surprising that the determinant plays a large role here.

Observe that $\det\colon GL(R) \to R^*$ factors through the abelianization and therefore yields an induced map $\det\colon K_1(R) \to R^*$. This map is split, since we can send any $r \in R^*$ to the class of the automorphism $R \xrightarrow{r} R$ (this is a group homomorphism using relation (b) of Definition 7.4). So we always have $K_1(R) \cong R^* \oplus (???)$. The mystery factor is usually called $SK_1(R)$.

We will not calculate K_1 for many rings, but in the easiest examples $SK_1(R)$ always vanishes. We explain this next.

Lemma 7.13. *If A is a diagonal matrix of determinant 1 then A lies in $E(R)$.*

Proof. This can be proven directly by using row and column operations, but the following argument is a bit easier to write. We use that $GL(R)/E(R) \cong K_1(R)$. Let $d_1, \ldots, d_n$ be the diagonal entries of A. Working in $K_1(R)$ we write

$$[R^n \xrightarrow{A} R^n] = [R \xrightarrow{d_1} R] + \cdots + [R \xrightarrow{d_n} R] = [R \xrightarrow{d_1 \cdots d_n} R] = [R \xrightarrow{1} R] = 0$$

where the first equality is by relation (a) in Definition 7.4 and the second equality is by relation (b). $\square$

Proposition 7.14. *If F is a field then $K_1(F) = F^*$.*

Proof. One must show that if $A \in GL(F)$ satisfies $\det(A) = 1$ then $A \in [GL(F), GL(F)] = E(F)$. Lemma 7.13 verifies this in the case where A is diagonal. The proof proceeds by using row and column operations to reduce to this case.

We will use two types of column and row operations: adding a multiple of one column/row to another, and switching two columns (or rows) together with a sign change of one of them. Both types of operation are allowable, the latter by Lemma 7.10(c). Pick any nonzero entry in the matrix, move it into the $(1, 1)$ position, and then use it to clear out the rest of its row and column. Proceeding inductively, this transforms A into a diagonal matrix. That is, there exist matrices $E_1, E_2 \in E(F)$ such that $E_1 A E_2$ is diagonal. But by the preceding paragraph we then have $E_1 A E_2 \in E(F)$, and so $A \in E(F)$. $\square$

Essentially the same proof as above also shows the following:

Proposition 7.15. *Let R be a Euclidean domain. Then $\det\colon K_1(R) \to R^*$ is an isomorphism. In particular, $K_1(\mathbb{Z}) = \{1, -1\} \cong \mathbb{Z}/2$ and when F is a field $K_1(F[t]) \cong F^*$. The conclusion also holds when R is a local ring.*

Proof. We must again show that if $A \in GL_n(R)$ has $\det(A) = 1$ then $A \in E(R)$. For any fixed row of A, the ideal generated by the elements in that row contains $\det(A)$ and is therefore the unit ideal. Pick an element x of smallest degree in this row and then use column operations (and the Euclidean division property) to arrange all other elements in this row to be either zero or have degree smaller than x. By repeating this process, eventually the row will contain a unit and all other entries will be zero. Do a signed transposition to switch this unit into position $(1, 1)$, and then do row operations to clear out all other terms in the first column. Repeat this process for the submatrix obtained by deleting the first row and column, and so forth. Eventually the matrix will be reduced to a diagonal matrix, necessarily of determinant 1. Such a diagonal matrix lies in $E(R)$ by Lemma 7.13, so this proves A also lies in $E(R)$.

Essentially the same proof works for local rings, by finding units in the matrix and then using them to clear out their row and column. $\square$

It is somewhat of a challenge to come up with an easy ring R for which one can prove by elementary means that $SK_1(R) \neq 0$, or equivalently that $\det\colon K_1(R) \to$

R^* is not an isomorphism. We have just seen that there are no examples amongst fields or Euclidean domains or local rings, and Milnor [Mi2, Corollary 16.3] proves that one also cannot find examples amongst rings of integers in number fields. There do exist examples where R is a PID, but they are a bit exotic—see Example 7.17 below. In some sense the "easiest" example of such a ring (though not a PID) involves some topology. We outline this in the next example, but skim over the details.

Example 7.16. Let R be a commutative ring.

(a) If $a, b \in R$ are such that $(a, b)R = R$, choose $c, d \in R$ such that $ad - bc = 1$. The matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ lies in $SL_2(R)$ and therefore gives us an element in $K_1(R)$ via the inclusions $SL_2(R) \hookrightarrow SL(R) \hookrightarrow GL(R)$. It turns out this element does not depend on the choice of c and d, for if $aq - bp = 1$ one computes that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ p & q \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} q & -b \\ -p & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ?? & 1 \end{bmatrix} \in E(R).$$

Write $[a, b]$ for the element in $K_1(R)$ represented in this way, and note that in fact we have $[a, b] \in SK_1(R)$. This is called the **Mennicke symbol** represented by a and b.

(b) With some trouble one can prove that for any $\lambda \in R$ one has

$$[a, b] = [a + \lambda b, b] = [a, b + \lambda a] \qquad \text{and} \qquad [a, b] = [b, a]$$

(see [Mi2, Lemma 13.2]). We will not need these facts, we just list them to spark the reader's interest.

(c) Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, which we regard as the ring of polynomial functions on the circle S^1. We will argue that the Mennicke symbol $[x, y]$ is nonzero, so that $SK_1(R) \neq 0$.

Note that we get maps $GL_n(R) \to [S^1, GL_n(\mathbb{R})]$ (unpointed homotopy classes) in the evident way: if $A \in GL_n(R)$ then we get a map $S^1 \to GL_n(\mathbb{R})$ by sending the point $(a, b) \in S^1$ to the matrix obtained by plugging in $x = a$ and $y = b$ into A. This is a map of groups when we give $[S^1, GL_n(\mathbb{R})]$ the operation coming from pointwise multiplication. Elementary matrices all go to the identity under this map: if $E = I + N$ where N has a single nonzero entry off the diagonal, then $t \mapsto I + tN$ gives a homotopy (of course elements of $E(R)$ might be products of such matrices, but then one does the homotopy in each factor simultaneously). In this way one gets a map $K_1(R) \to [S^1, GL(\mathbb{R})]$, and likewise a map $SK_1(R) \to [S^1, SL(\mathbb{R})]$. Since $SL(\mathbb{R})$ is a path-connected topological group, unpointed homotopy classes of maps from S^1 agree with the pointed version; that is to say, $[S^1, SL(\mathbb{R})] \cong \pi_1(SL(\mathbb{R}))) \cong \pi_1(SO)$ (the latter because $SO_n \hookrightarrow SL_n(\mathbb{R})$ is a deformation retract for all n). It is known that this homotopy group is $\mathbb{Z}/2$ and is generated by the image of the standard generator in $\pi_1 SO(2)$ (see Section 12.7 below for this fact).

The Mennicke symbol $[x, y]$ denotes the matrix $\left[\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right] \in SL_2(R)$, and therefore gives the map $S^1 \to SO(2)$ sending $(a, b) \in S^1$ to $\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$. Said differently, this sends $e^{i\theta} \in S^1$ to the 2×2 matrix for (clockwise) rotation by θ, and so is a generator for $\pi_1 SO_2$. In particular, this proves that $[x, y] \neq 0$.

(d) In fact Bass [B1, XIII.2, p. 714] proves that $SK_1(R) \cong \mathbb{Z}/2$, but we will not do this much. Let us at least prove that $SK_1(R)$ is 2-torsion, as this is easy. Let $T = \mathbb{C} \otimes_{\mathbb{R}} R = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ and note that as an R-module T is just R^2.

So there is an induced map $\pi\colon K_1(T) \to K_1(R)$ sending $[T^n \xrightarrow{f} T^n] \in K_1(T)$ to $[T^n \xrightarrow{f} T^n] \in K_1(R)$ where in the latter case we regard T^n as an R-module. Consider the diagram

$$\begin{tikzcd} K_1(R) \arrow[r, "i"] & K_1(T) \arrow[r, "\pi"] & K_1(R) \\ SK_1(R) \arrow[u, hook] \arrow[r, "i"] & SK_1(T) \arrow[u, hook] & \end{tikzcd}$$

where i is induced by the inclusion $R \hookrightarrow T$. Then $\pi \circ i$ sends $[R^n \xrightarrow{g} R^n]$ to $[R^n \oplus R^n \xrightarrow{g\oplus g} R^n \oplus R^n]$ and is therefore multiplication by 2. But if we set $z = x + iy$ and $w = x - iy$ then $T \cong \mathbb{C}[z, w]/(zw - 1) = \mathbb{C}[z, z^{-1}]$. This is a localization of the Euclidean domain $\mathbb{C}[z]$, and essentially the same proof as for Proposition 7.15 shows that $SK_1(T) = 0$ (see also Exercise 7.20 below). Hence, $SK_1(R)$ is 2-torsion.

Example 7.17. Here we describe an example due to Grayson [Gr1] of a PID R with $SK_1(R) \neq 0$. Let $T = \mathbb{Z}[x]$ and let S be the multiplicative system generated by x and all polynomials $x^m - 1$ where $m \geq 1$. Let $R = S^{-1}T$. Observe that $\dim T = 2$ and that T is a UFD. The latter implies that all height one primes are principal, and the former gives that the height two primes are maximal. But quotienting out by a maximal ideal will yield a finite field, which means that x will map to either zero or a root of unity. Consequently, every maximal ideal of T must intersect S. Since the primes in R are all extended from the primes in T that do not intersect S, we conclude that all of the primes in R are principal. So R is a PID.

The computation of $SK_1(R)$ is a combination of results from [Gr1], [Le], and [Sc]. This computation is far too complex for us to include here, but the upshot is that

$$SK_1(R) \cong \bigoplus_{n\geq 2} \mathbb{Z}/n\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \bigoplus_{i\geq 1} (\mathbb{Z}/p^i\mathbb{Z})^\infty.$$

So this group is quite big. See [Gr1] for details.

Other examples of PIDs with $SK_1 \neq 0$ had previously been given by Bass [B2] and Ischebeck [I].

◦ Exercises ◦

Exercise 7.18. Let R be a commutative ring. Prove that if D is an invertible diagonal matrix and N is strictly lower triangular then $[D + N] = [D]$ in $K_1(R)$.

Exercise 7.19. Prove that the following are equivalent:

(1) $SK_1(R) = 0$,
(2) $K_1(R)$ is generated by the classes $[R \xrightarrow{r} R]$ for $r \in R^*$,
(3) Every invertible matrix over R can be transformed via allowable column operations into a diagonal matrix,
(4) Every invertible matrix over R can be transformed via allowable column operations into a diagonal matrix with exactly one entry that is not a 1.

Exercise 7.20. Let R be a Euclidean domain.

(a) Prove that every matrix can be transformed to a lower triangular matrix using allowable column operations. Use this together with the previous two exercises to give a (slightly) different proof that $SK_1(R) = 0$.
(b) For every $f \in R - \{0\}$ prove that $SK_1(f^{-1}R) = 0$.

Exercise 7.21. Prove that $GL_n(R \times S) = GL_n(R) \times GL_n(S)$ and that $K_1(R \times S) = K_1(R) \oplus K_1(S)$.

Exercise 7.22. Calculate $K_1(\mathbb{Z}/360)$.

Exercise 7.23. Attempt to prove that if R is a PID then $SK_1(R) = 0$ and get a sense of what goes wrong.

7.24. **Longer localization sequences.** We next work on extending the localization sequence from Proposition 5.17 to the left using K_1 terms. See Theorem 7.27 below.

Proposition 7.25. *Let $S \subseteq R$ be a multiplicative system.*

(a) The group $K_1(S^{-1}R)$ is generated by classes $[S^{-1}R^n \xrightarrow{S^{-1}\alpha} S^{-1}R^n]$ where $\alpha\colon R^n \to R^n$ is such that $S^{-1}\alpha$ is an isomorphism.
(b) There is a unique map $\partial\colon K_1(S^{-1}R) \to K_0(R,S)$ having the property that if $\alpha\colon R^n \to R^n$ is such that $S^{-1}\alpha$ is an isomorphism, then ∂ sends $[S^{-1}R^n \xrightarrow{S^{-1}\alpha} S^{-1}R^n]$ to the class of the chain complex $0 \to R^n \xrightarrow{\alpha} R^n \to 0$ (concentrated in degrees 0 and 1).

Proof. First let $\beta\colon (S^{-1}R)^n \to (S^{-1}R)^n$ be an automorphism. Let A be the matrix for β with respect to the standard basis, and let $u \in S$ be an element such that uA has entries in R (e.g., take u to be the product of all the denominators of the entries in A). Then uA represents a map $\beta'\colon R^n \to R^n$, and we have the commutative diagram

$$\begin{tikzcd}
(S^{-1}R)^n \arrow[r, "\beta"] & (S^{-1}R)^n \arrow[r, "uI_n"] & (S^{-1}R)^n \\
R^n \arrow[u] \arrow[rr, "\beta'"] & & R^n \arrow[u]
\end{tikzcd}$$

where the vertical maps are localization. This diagram gives $uI_n \circ \beta = S^{-1}\beta'$, and so $[uI_n] + [\beta] = [S^{-1}\beta']$ in $K_1(S^{-1}R)$. Note that $[uI_n] = n[uI_1]$, and uI_1 is itself the localization of the multiplication-by-u map on R; so we can write

$$[\beta] = [S^{-1}\beta'] - n[S^{-1}u]. \tag{7.26}$$

This shows that $K_1(S^{-1}R)$ is generated by classes $[S^{-1}\alpha]$ for $\alpha\colon R^n \to R^n$, and we have thereby proven (a) and the uniqueness part of (b).

For the existence part of (b), we will define a map $\partial\colon K_1^{sp,fr}(S^{-1}R) \to K_0(R,S)$ and then appeal to Theorem 7.9. Given an automorphism $\beta\colon (S^{-1}R)^n \to (S^{-1}R)^n$, choose a $u \in S$ such that the standard matrix representing $u\beta$ has entries in R. Consider the assignment

$$\beta \mapsto F(\beta, u) = [R^n \xrightarrow{u\beta} R^n] - n[R \xrightarrow{u} R] \in K_0(R,S).$$

Note that this expression doesn't come out of thin air: the expected homomorphism ∂, if it exists, must have this form by (7.26). It remains to show that the above formula does indeed define a homomorphism.

We first show that $F(\beta,u)$ does not depend on the choice of u. It suffices to prove that $F(\beta,tu)=F(\beta,u)$ for any $t\in S$; for if u' is another choice for u then we would have $F(\beta,u)=F(\beta,u'u)=F(\beta,u')$. But now we just compute that

$$\begin{aligned}
F(\beta,tu) &= [R^n \xrightarrow{tu\beta} R^n] - n[R \xrightarrow{tu} R]\\
&= [R^n \xrightarrow{t} R^n] + [R^n \xrightarrow{u\beta} R^n] - n\Big[[R \xrightarrow{t} R] + [R \xrightarrow{u} R]\Big]\\
&= [R^n \xrightarrow{u\beta} R^n] - n[R \xrightarrow{u} R]
\end{aligned}$$

(the second equality is by Lemma 5.16, applied twice).

Let us now write $F(\beta)$ instead of $F(\beta,u)$. The last things that must be checked are that $F(\beta\oplus\beta')=F(\beta)+F(\beta')$ and $F(\beta\gamma)=F(\beta)F(\gamma)$, but these are both immediate (the latter using Lemma 5.16). So we have established the existence of $\partial\colon K_1(S^{-1}R)\to K_0(R,S)$ having the desired properties. $\square$

Theorem 7.27 (Localization sequence for K-theory)**.** *Let R be a commutative ring and $S\subseteq R$ a multiplicative system. The following sequence is exact:*

$$K_1(R)\longrightarrow K_1(S^{-1}R)\xrightarrow{\partial} K_0(R,S)\longrightarrow K_0(R)\longrightarrow K_0(S^{-1}R).$$

Proof. We will not prove exactness at $K_1(S^{-1}R)$, as this is a bit difficult and would take us too far afield. Exactness at $K_0(R)$ was already proven in Proposition 5.17, so it only remains to verify exactness at $K_0(R,S)$.

Let $x\in K_0(R,S)$. We know by Theorem 5.18 that x can be written in the form $x=[P_1\to P_0]-[Q_1\to Q_0]$ for finitely-generated projectives P_0,P_1,Q_0,Q_1 over R and maps $\alpha\colon P_1\to P_0$, $\beta\colon Q_1\to Q_0$ that become isomorphisms after S-localization. Consider the isomorphism $S^{-1}Q_0\to S^{-1}Q_1$ that is the inverse to $S^{-1}\beta$. By Corollary 4.11(b) there is a map $\gamma\colon Q_0\to Q_1$ whose localization is isomorphic to this map. Notice that

$$\begin{aligned}
x &= [P_1 \xrightarrow{\alpha} P_0] + [Q_0 \xrightarrow{\gamma} Q_1] - \big([Q_0 \xrightarrow{\gamma} Q_1] + [Q_1 \xrightarrow{\beta} Q_0]\big)\\
&= [P_1\oplus Q_0 \xrightarrow{\alpha\oplus\gamma} P_0\oplus Q_1] - [Q_0\oplus Q_1 \xrightarrow{\gamma\oplus\beta} Q_1\oplus Q_0].
\end{aligned}$$

So by replacing our original P's and Q's we can assume that $Q_0=Q_1$.

Let G be a projective such that $Q_0\oplus G$ is free, and observe that

$$x = x-[G\xrightarrow{\mathrm{id}} G] = [P_1\to P_0]-[Q_0\oplus G\to Q_0\oplus G].$$

So again, by replacing our chosen $Q_0=Q_1$ we can actually assume that $Q_0=Q_1$ is free. That is, $x=[P_1\xrightarrow{\alpha} P_0]-[R^n\xrightarrow{\beta} R^n]$.

Now assume that x maps to zero in $K_0(R)$. This just says that $[P_0]=[P_1]$ in $K_0(R)$, and so there exists a free module G such that $P_0\oplus G\cong P_1\oplus G$ (Proposition 2.9). Since $x=x+[G\xrightarrow{id} G]$ we see that we can write x as

$$x=[R^k\xrightarrow{\alpha} R^k]-[R^n\xrightarrow{\beta} R^n]$$

where α and β become isomorphisms after S-localization. It is now immediate that x is in the image of ∂; to be completely specific,

$$x=\partial\Big([S^{-1}R^k\xrightarrow{S^{-1}\alpha} S^{-1}R^k]-[S^{-1}R^n\xrightarrow{S^{-1}\beta} S^{-1}R^n]\Big).$$

$\square$

Example 7.28. This example will be a "reality check". We won't learn anything new, but we will see that the localization sequence is doing something sensible. Let R be a discrete valuation ring (a regular local ring whose maximal ideal is principal), and let F be the quotient field. Let x be a generator for the maximal ideal, and let $S = \{1, x, x^2, \ldots\}$. Note that $S^{-1}R = F$. The localization sequence for K-theory takes on the form

$$R^* \to F^* \xrightarrow{\partial} K_0(R, S) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$$

where we are using $K_1(R) \cong R^*$, $K_1(F) \cong F^*$, $K_0(R) \cong \mathbb{Z}$ (because R is a PID), and the map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ sends $[R]$ to $[F]$ and is therefore an isomorphism. So the localization sequence distills into a single isomorphism

$$F^*/R^* \xrightarrow{\partial} K_0(R, S).$$

The group F^*/R^* is readily checked to be $\mathbb{Z}$, where the isomorphism $\mathbb{Z} \cong F^*/R^*$ sends n to $[x^n]$. On the other hand, we also know by Theorem 5.20 that $K_0(R, S) \cong G(M \,|\, S^{-1}M = 0)$. A finitely-generated module M satisfies $S^{-1}M = 0$ if and only if M is killed by a power of x, or equivalently if M has finite length over R. The length map $\ell\colon G(M \,|\, S^{-1}M = 0) \to \mathbb{Z}$ is easily checked to be an isomorphism.

Finally, let us analyze the map ∂. Given an element $a \in F^*$, we write $a = r/x^n$ for some $r \in R^*$ and $n \geq 0$. The description of ∂ given in the proof of Proposition 7.25 shows that

$$\partial(a) = [R \xrightarrow{r} R] - [R \xrightarrow{x^n} R] = [R \xrightarrow{r} R] - n[R \xrightarrow{x} R].$$

The isomorphism $K_0(R, S) \to G(M \,|\, S^{-1}M = 0)$ sends a complex $P_\bullet$ to the alternating sum of its homology modules, so under this isomorphism we would write

$$\partial(a) = [R/rR] - n[R/xR].$$

Note that $\ell(R/xR) = 1$. We can write $r = ux^k$ for some unit $u \in R$ and $k \geq 0$, in which case $R/rR \cong R/x^kR$ and so $\ell(R/rR) = k$. It follows that the composite

$$F^* \xrightarrow{\partial} K_0(R, S) \xrightarrow{\cong} G(M \,|\, S^{-1}M = 0) \xrightarrow{\cong} \mathbb{Z}$$

sends $\frac{ux^k}{x^n}$ to $k - n$ and so is just the usual x-adic valuation on F^*.

Exercise 7.29. Let $R = \mathbb{Z}$ and $S = \{2^i \,|\, i \geq 0\}$. Compute all the groups and maps in the K-theory localization sequence, and also compare with the isomorphism $K(R, S) \cong G(M \,|\, S^{-1}M = 0)$.

The following example generalizes the previous one, but is a bit more interesting.

Example 7.30. Let D be a Dedekind domain—a regular ring of dimension one. In such a ring all nonzero primes are maximal ideals. Let $S = D - \{0\}$ and let $F = S^{-1}D$ be the quotient field. Our localization sequence looks like

$$K_1(D) \to F^* \to K_0(D, S) \to K_0(D) \to \mathbb{Z}.$$

Although we have not calculated $K_1(D)$, the commutative diagram

$$\begin{tikzcd} K_1(D) \arrow[r] \arrow[d, "\det"'] & K_1(F) \arrow[d, "\det"', "\cong"] \\ D^* \arrow[r, tail] & F^* \end{tikzcd}$$

shows that the image of $K_1(D)$ in F^* is just D^*. The map $K_0(D) \to \mathbb{Z}$ is just the usual rank map, so its kernel is $\widetilde{K}_0(D)$. So we get a short exact sequence

$$0 \to F^*/D^* \xrightarrow{\partial} K_0(D,S) \to \widetilde{K}_0(D) \to 0.$$

We know $K_0(D,S) \cong G(M \mid S^{-1}M = 0)$ by Theorem 5.20. The condition $S^{-1}M = 0$ just says that M is a torsion module. Consider the evident map

$$j\colon \bigoplus_{P\neq 0} G_0(D/P) \to G(M \mid S^{-1}M = 0)$$

where the direct sum is over all nonzero prime ideals and where the map just forgets that a module is defined over D/P and instead regards it as a D-module. This map is clearly surjective: a torsion D-module M will have a prime filtration in which the primes appearing are all maximal, and $[M]$ will be the sum of the corresponding $[D/P]$'s by the usual argument (see Theorem 2.23 and its proof).

Note that each D/P is a field, and so $G_0(D/P) \cong \mathbb{Z}$. If M is a torsion D-module then M_P is a torsion D_P-module. Since D_P is a discrete valuation ring, this means that M_P has finite length. Define

$$\chi\colon G(M \mid S^{-1}M = 0) \to \bigoplus_{P\neq 0} G_0(D/P)$$

by sending $[M]$ to the tuple of integers $\ell_{D_P}(M_P)$, as P runs over all maximal ideals of D (the only ones that give nonzero lengths are the ones containing $\operatorname{Ann} M$, and there are only finitely-many of these since they are precisely the minimal primes of $\operatorname{Ann} M$). It is easy to check that $\chi \circ j = \mathrm{id}$. Since j was already known to be surjective this means they are inverse isomorphisms. So we can rewrite our short exact sequence as

$$0 \to F^*/D^* \xrightarrow{\partial} \bigoplus_{P\neq 0} \mathbb{Z} \to \widetilde{K}_0(D) \to 0.$$

It will be convenient to write e_P for the basis element of the free abelian group in the middle corresponding to the maximal ideal P. Note that these basis elements correspond to the closed points of $\operatorname{Spec} D$, and so we are looking at a group of 0-cycles.

It remains to analyze the map ∂. By Proposition 7.25, if $r \in D - \{0\}$ then $\partial(r) = [D \xrightarrow{r} D] \in K_0(D,S)$. Under the isomorphisms described above this corresponds to the tuple of integers $\ell_{D_P}(D_P/rD_P)$. This is usually called the **divisor class** of r, and written

$$\operatorname{div}(r) = \sum \ell_{D_P}(D_P/rD_P)e_P.$$

It should be thought of as listing all the zeros of the “function” r, together with their orders of vanishing (see below for an example). For a general element $x \in F^*$ we would just write $x = r/s$ for $r, s \in D - \{0\}$, and then $\partial(x) = \operatorname{div}(r) - \operatorname{div}(s)$; this gives the zeros and poles of x, with multiplicities.

The quotient of $\bigoplus_{P\neq 0} \mathbb{Z}$ by the classes $\operatorname{div}(x)$ for $x \in F^*$ is called the **divisor class group** of D; it is isomorphic to the ideal class group from algebraic number theory. Our short exact sequence shows that $\widetilde{K}_0(D)$ is also isomorphic to this group.

To demonstrate the geometric intuition behind $\operatorname{div}(r)$, consider the case $D = F[t]$ where F is algebraically closed. If $r = p(t)$ is nonzero then the maximal ideals containing r are the ones $(t - a_i)$ where a_i is a root of $p(t)$. If we write

$r = u\prod(t-a_j)^{m_j}$ where $u \in F^*$ and we localize at $P = (t-a_i)$, then r becomes a unit multiple of $(t-a_i)^{m_i}$ and the number $\ell_{D_P}(D_P/rD_P)$ is precisely m_i. So

$$\operatorname{div}(r) = \sum_i m_i \cdot e_{(t-a_i)},$$

as expected. Note that the divisor class group is not very interesting in this case: clearly div is surjective, and so the group is zero. We already knew this for another reason, because $\widetilde{K}_0(D) = 0$ whenever D is a PID.

Example 7.31. As one more example, let us return to the ring $R = \mathbb{Z}[\sqrt{-5}]$ and $S = \{2^i \mid i \geq 0\}$. Some of the computations here are a little difficult, but one can determine that $K_1(R) \to R^*$ is an isomorphism because [Mi2, Corollary 16.3] tells us this holds for all rings of integers in number fields. One can also compute that $R^* = \{1, -1\}$ by using the fact that the norm map $N(a+b\sqrt{-5}) = a^2 + 5b^2$ is multiplicative. The localization sequence takes the following form:

$$\begin{tikzcd}
K_1(R) \arrow[r] \arrow[d, "\cong"'] & K_1(S^{-1}R) \arrow[r, "\partial"] \arrow[d, "\cong"'] & K_0(R,S) \arrow[r] \arrow[d, "\cong"'] & K_0(R) \arrow[r] \arrow[d, "\cong"'] & K_0(S^{-1}R) \arrow[d, "\cong"'] \\
\mathbb{Z}/2 \arrow[r, "c"] & SK_1(S^{-1}R) \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "\partial"] & \mathbb{Z} \arrow[r, "b"] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "a"] & \mathbb{Z}
\end{tikzcd}$$

Let us recall how these isomorphisms work, identify generators for the groups, and see what the maps do. Since $S^{-1}R$ is a PID one has that $K_0(S^{-1}R)$ is $\mathbb{Z}$, generated by $[S^{-1}R]$. We have seen in Example 4.2 that $K_0(R)$ is generated by $[R]$ and $[R]-[I]$ where $I = (2, 1+\sqrt{-5})$ (recall that I is projective), with the former generating the $\mathbb{Z}$ summand and the latter the $\mathbb{Z}/2$. The map a is S-localization and therefore sends $[R]$ to $[S^{-1}R]$ and $[R]-[I]$ to zero.

Since R is regular we have isomorphisms $K_0(R,S) \cong G(M \mid S^{-1}M = 0) \cong G_0(R/(2))$ where the first is the Euler characteristic and the second is devissage (Theorem 4.17). Note that $R/(2) \cong \mathbb{Z}/2[x]/(x^2+1) \cong \mathbb{Z}/2[x]/((x+1)^2) \cong \mathbb{Z}/2[u]/(u^2)$ and so $G_0(R/(2))$ is $\mathbb{Z}$ generated by $\mathbb{Z}/2[u]/(u) = R/(2, 1+\sqrt{-5}) = R/I$. Tracing this back into $K_0(R,S)$, we see that the group is generated by the chain complex $[0 \to I \hookrightarrow R \to 0]$. The map b sends this to $[R]-[I]$, which is the generator of the $\mathbb{Z}/2$. So it must be that $2[I \hookrightarrow R]$ is in the image of ∂, but let us check this.

Recall that $K_1(S^{-1}R) = SK_1(S^{-1}R) \oplus (S^{-1}R)^*$. The second summand is clearly $\mathbb{Z} \oplus \mathbb{Z}/2$ with generators 2 and -1. Under ∂ these map to $[R \xrightarrow{2} R]$ and $[R \xrightarrow{-1} R]$, the latter of which is zero (being acyclic). Under the isomorphism $K_0(R,S) \cong G_0(R/(2))$ the complex $[R \xrightarrow{2} R]$ maps to $R/(2)$, and this is twice the generator (look at the dimension over $\mathbb{Z}/2$). So $\partial(2) = 2[I \hookrightarrow R]$, as expected. The long exact sequence now shows that $SK_1(S^{-1}R)$ must be zero.

Remark 7.32. The localization sequence of Theorem 7.27 can be extended further to the left, by definining K-groups $K_n(R)$ and $K_n(R,S)$ for all $n \geq 1$. This is the subject of higher algebraic K-theory, a deep field with intricate connections to many area of mathematics. It does not really do justice to the subject for us to give just a few references, but some places to get started learning about it are [B1], [Mi2], [Q3], [Ro], and [W2].

◦ Exercises ◦

Part 2. K-theory in topology

Let's take as our starting point that we understand finite-dimensional linear algebra extremely well. There aren't that many isomorphism types of objects (one for each dimension), and we have a pretty good understanding of the maps between them. Our next goal in these notes is to explore the idea of doing linear algebra *continuously* over a fixed parameter space X. What this means is that rather than have only one vector space we will have a continuously varying family of vector spaces, parameterized by the points of X.

The way to talk about such "continuously varying families" is to bundle the objects together into a single topological space E together with a map $p\colon E \to X$, so that the members of our family appear as the fibers of p. A map from the family $p\colon E \to X$ to the family $p'\colon E' \to X$ will then be a continuous map $F\colon E \to E'$, commuting with the maps down to X, such that F is a linear transformation on each fiber. It turns out that much of linear algebra carries over easily to this enhanced setting. But there are more isomorphism types of objects here, because the topology of X allows for some twisting in the vector space structure of the fibers. The surprise is that studying these 'twisted vector spaces' over a base space X quickly leads to interesting homotopy invariants of X! From a topological viewpoint, K-theory is a cohomology theory for topological spaces that arises out of this study of fiberwise linear algebra.

8. Vector bundles

The point of this section and the next one is to establish all of the foundational results we will require for working with vector bundles. Unfortunately, going through all of this carefully ends up being somewhat tedious. The reader might do well to skim these two sections for the basic ideas but not get bogged down in details, referring back for those only as needed.

A (real) vector space is a set V together with operations $+\colon V \times V \to V$ and $\cdot\colon \mathbb{R} \times V \to V$ satisfying a familiar (but long) list of properties. If X is a topological space, a *family of vector spaces* over X will be a continuous map $p\colon E \to X$ together with extra data making each fiber $p^{-1}(x)$ into a vector space, with the operations varying in a continuous manner. The easiest way to say this is as follows:

Definition 8.1. *A* ***family of (real) vector spaces*** *is a map $p\colon E \to X$ together with operations $+\colon E \times_X E \to E$ and $\bullet\colon \mathbb{R} \times E \to E$ making the two diagrams*

$$\begin{tikzcd} E \times_X E \arrow[rr, "+"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R} \times E \arrow[rr, "\bullet"] \arrow[dr, "p\circ\pi_2"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commute, together with a map $\zeta\colon X \to E$ called the "zero section", such that the operations make each fiber $p^{-1}(x)$ into a real vector space with zero element $\zeta(x)$.

One could write down the above definition completely category-theoretically, in terms of maps and commutative diagrams. Essentially one is defining a "vector space object" in the category of topological spaces over X. In the case where X is a point, E is simply called a **topological vector space**.

The space X is called the **base** of the family. If $x \in X$ we will write E_x for the fiber $p^{-1}(x)$ regarded with its vector space structure. The dimension of E_x is called the **rank** of the family at x, and denoted $\mathrm{rank}_x(E)$. The rank of E is defined to be

$$\mathrm{rank}(E) = \sup\{\mathrm{rank}_x(E) \,|\, x \in X\},$$

where we include the possibility that $\mathrm{rank}(E)$ is infinite (though we will need this case only rarely).

A **section** of $p\colon E \to X$ is a map $s\colon X \to E$ such that $ps = \mathrm{id}_X$. The condition is equivalent to saying $s(x) \in E_x$ for each $x \in X$. So a section is a continuous choice of element in each fiber. This explains the term "zero-section" for ζ. By abuse of terminology the image of ζ is also sometimes referred to as the zero-section.

Remark 8.2. The additive inverse map $E \to E$ is continuous, as it can be expressed as the composite $E = \{-1\} \times E \hookrightarrow \mathbb{R} \times E \xrightarrow{\cdot} E$. Similarly, the zero-section can be recoved set-theoretically from the scalar multiplication as the image of

$$\{0\} \times E \hookrightarrow \mathbb{R} \times E \xrightarrow{\cdot} E,$$

and this determines a set-theoretic section $X \to E$. However, continuity of this section is not automatic from the other axioms; this is why it is included as part of Definition 8.1.

Definition 8.3. *Given two familes of vector spaces $p\colon E \to X$ and $p'\colon E' \to X$, a* ***map of families*** *is a continuous map $f\colon E \to E'$ such that $p'f = p$ and such that f restricts to a linear map on each fiber. Write $\mathsf{FamVS}(X)$ for the category of families of vector spaces over X.*

A ***subfamily*** *of E is a topological subspace $J \hookrightarrow E$ that contains the zero section and is closed under the operations of addition and scalar multiplication.*

Given a map of families $f\colon E \to E'$, the usual image of f (denoted $\mathrm{im}\, f$) is a subfamily of E'. Let us define the **kernel** of f, denoted $\ker f$, to be the subspace of E consisting of all elements mapped to zero in E'. Equivalently, $\ker f$ is defined to be the pullback

$$\begin{tikzcd}
\ker f \arrow[r] \arrow[d] & E \arrow[d, "f"] \\
X \arrow[r, "\zeta'"] & E'.
\end{tikzcd}$$

Exercise 8.4. Review the categorical notions of kernels and cokernels from Appendix G. Verify that the family $X \xrightarrow{\mathrm{id}} X$ is the zero object of the category $\mathsf{FamVS}(X)$ and that the kernel of a map, as defined above, is a kernel in the categorical sense.

Note that kernels in $\mathsf{FamVS}(X)$ are also fiberwise kernels, in the sense that $(\ker f)_x = \ker(f_x)$ for all $x \in X$. Be warned that this property will turn out not to hold for cokernels, though. REF???

We next explore a few examples of these basic concepts:

Example 8.5.

(a) The simplest example of a family of vector spaces is $E = X \times \mathbb{R}^n$, with the projection map $X \times \mathbb{R}^n \to X$ (here $\mathbb{R}^n$ is equipped with the standard topology). This is called the **trivial family of rank** n, and it is often denoted simply by $\underline{n}_X$. It is also denoted by just $\underline{n}$, with the space X understood. Note that $\underline{0}_X$ is the family $X \xrightarrow{\text{id}} X$.

(b) Let $E = \{(x, v) \,|\, x \in \mathbb{R}^2, v \in \mathbb{R}.\langle x\rangle\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, and let $p\colon E \to \mathbb{R}^2$ be projection onto the first coordinate. Define $(x, v) + (x, v') = (x, v + v')$ and $r.(x, v) = (x, rv)$. This makes $E \to \mathbb{R}^2$ into a family of vector spaces, in fact a subfamily of the trivial family $\underline{2}$. Note that the fiber $p^{-1}(x)$ is one-dimensional for $x \neq 0$, but 0-dimensional when $x = 0$.

(c) Let $X = \mathbb{R}$ and $E = X \times \mathbb{R}$ be the trivial family of rank 1. Consider the map of familes $E \to E$ given by $(t, v) \mapsto (t, tv)$. This is multiplication-by-t on the fiber over t. The kernel of this map is the subfamily $K \hookrightarrow E$ given by $K = (X \times \{0\}) \cup (\{0\} \times \mathbb{R})$. Note that most of the fibers are 0-dimensional, but the fiber over 0 is one-dimensional.

(d) Let $X = \mathbb{R}$. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$. Let $E \subseteq X \times \mathbb{R}^2$ be the union of $\{(x, re_1) \,|\, x \in \mathbb{Q}, r \in \mathbb{R}\}$ and $\{(x, re_2) \,|\, x \in X \backslash \mathbb{Q}, r \in \mathbb{R}\}$. Recall from (a) that $X \times \mathbb{R}^2 \to X$ is a family of vector spaces, and note that E becomes a sub-family of vector spaces under the same operations.

Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the linear transformation such that $\pi(e_1) = \pi(e_2) = 1$. Define a map $E \to \underline{1}_X$ by $(x, v) \mapsto (x, \pi(v))$. This is a map of families that is an isomorphism on each fiber, but is not an isomorphism of families.

(e) Let $\mathbb{R}^n_{ind}$ denote the vector space $\mathbb{R}^n$ but with the indiscrete topology, so that the only open sets are $\emptyset$ and $\mathbb{R}^n$. Check that $\mathbb{R}^n_{ind}$ is a topological vector space, i.e. that $\mathbb{R}^n_{ind} \to *$ is a family of vector spaces.

Part (d) of the previous example shows that the concept of "family of vector spaces" admits some unpleasant pathology. Even the examples in (b) and (c) show that families can have jumps in the fibers, which doesn't give the feeling of a "continuously varying" family. Part (e) gives the warning that the topology on the fibers might not be what one expects. Very shortly we will start imposing some conditions that eliminate these kind of phenomena.

If $p\colon E \to X$ is a family of vector spaces and $A \hookrightarrow X$ is a subspace, then $p^{-1}(A) \to A$ is also a family of vector spaces. We will usually write this restriction as $E|_A$.

More generally, suppose that $p\colon E \to X$ is a family of vector spaces and $f\colon Y \to X$ is a map. One may form the pullback $Y \times_X E$, more commonly denoted f^*E in this context:

$$\begin{tikzcd} f^*E \arrow[r, equal] & Y \times_X E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ & Y \arrow[r, "f"] & X. \end{tikzcd}$$

A point in f^*E is a pair (y, e) such that $f(y) = p(e)$, and one defines addition and scalar multiplication on f^*E by $(y, e) + (y, e') = (y, e + e')$ and $r \cdot (y, e) = (y, re)$. This makes $f^*E \to Y$ into a family of vector spaces, called the **pullback family**. If $y \in Y$ then there is an evident map of vector spaces $(f^*E)_y \to E_{f(y)}$ which one readily checks is an isomorphism. Note that if f is a subspace inclusion then $f^*E = E|_Y$.

Exercise 8.6. Check carefully that f^*E is a family of vector spaces.

We will need to develop some tools for dealing with maps in $\mathsf{FamVS}(X)$. The following proposition is easy but will be used very often:

Proposition 8.7. *Giving a map of trivial families $X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ is equivalent to giving a continuous map $X \to \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n) \cong \mathbb{R}^{kn}$, where the target has the standard Euclidean topology.*

Proof. Let $F\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ be a map of families. Define $F_i\colon X \to \mathbb{R}^n$ by $F_i(x) = \pi_2 F(x, e_i)$. These maps are continuous, and so they induce a continuous map $\phi\colon X \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k factors) by $\phi(x) = (F_1(x), \ldots, F_k(x))$.

In the other direction, given $\phi\colon X \to \mathbb{R}^{kn} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k factors) define F_2 to be the composite

$$X \times \mathbb{R}^k \xrightarrow{\phi \times \mathrm{id}} \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^k \xrightarrow{LC} \mathbb{R}^n$$

where LC is the "linear combination map" sending $(v_1, \ldots, v_k, \underline{r}) \mapsto r_1 v_1 + \cdots + r_k v_k$. Both maps are continuous, so their composite is as well. Then define $F\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ by $F(x, \underline{r}) = (x, F_2(x, \underline{r}))$.

It is now routine to check that the two assignments given above are inverses to each other. $\square$

Here is one more example showing the oddities of the category $\mathsf{FamVS}(X)$:

Exercise 8.8. Let $X = [-1, 1]$ and let $f\colon \underline{1}_X \to \underline{1}_X$ be the map that is multiplication-by-t on the fiber over t. Prove that the zero bundle $\underline{0}_X$ is the cokernel of f (see Appendix G for the definition). In particular, note that in this case the cokernel is not the fiberwise cokernel.

Sometimes we will have to deal with families of vector spaces where the fibers are infinite-dimensional. Mostly we will only need trivial families, but even here there are some subtleties. We describe these next.

If S is any set, write $\mathbb{R}\langle S\rangle$ for the vector space with basis S (occasionally we will abbreviate this to $\mathbb{R}S$). There is a natural isomorphism of vector spaces

$$\operatorname*{colim}_{F_{finite} \subseteq S} \mathbb{R}\langle F\rangle \to \mathbb{R}\langle S\rangle$$

where the colimit is over the finite subsets of S. If we give each $\mathbb{R}\langle F\rangle$ the standard Euclidean topology then the colimit inherits an induced topology, giving a topology on $\mathbb{R}\langle S\rangle$. This is also called the "finite" topology in some of the literature on topological vector spaces, but we will call it the **colimit topology**. When S is infinite this topology is *different* than the subspace topology induced from the evident embedding $\mathbb{R}\langle S\rangle \subseteq \prod_S \mathbb{R}$, where the product is given the product topology. In fact, the colimit topology is the subspace topology induced by the box topology. See Appendix B.5 for a complete discussion.

A theorem of Kakutani-Klee [KK] says that when $\mathbb{R}\langle S\rangle$ is given the colimit topology, addition is continuous only when S is countable. We will only ever work with $\mathbb{R}\langle S\rangle$ in this case, and from now on we only ever consider families of vector spaces where the rank is countable.

Note that there are at least three distinct topologies that make $\mathbb{R}^\infty$ into a topological vector space: these are the colimit, product, and metric topologies, described in detail in Appendix B.5. So when talking about trivial families with fiber $\mathbb{R}^\infty$ one needs to be careful about specifying which topology is being used. The product and metric topologies have certain algebraic deficiencies—e.g., linear functionals are not always continuous (see ???)—and so we will only use $\mathbb{R}^\infty$ with the colimit topology in our applications.

The background category for families of vector spaces is the overcategory $\mathcal{T}op \downarrow X$ of spaces over X. The terminal object of this category is X itself (or more precisely, the identity map $X \to X$). A "vector" in a family of vector spaces E should be a map in $\mathcal{T}op \downarrow X$ from X to E, which is the same thing as a section of $E \to X$. Given a collection of such "vectors" $s_\alpha \colon X \to E$, for α in an indexing set A, say that these are **linearly independent** (resp. **spanning**) if for every $x \in X$ the vectors $\{s_\alpha(x)\}_{\alpha\in A}$ are linearly independent (resp. spanning) in E_x. Say that the collection is a **weak basis** for E if it is both linearly independent and spanning, i.e. for every $x \in X$ the collection $\{s_\alpha(x)\}_{\alpha\in A}$ is a basis for the vector space E_x. When A is finite a weak basis gives a map of families $X \times \mathbb{R}\langle A\rangle \to E$ which is a bijection (see Exercise 8.40 for the case when A is infinite). However, it need not be an isomorphism! The inverse map need not be continuous: as an example consider a basis $b_1, \dots, b_n$ in $\mathbb{R}^n_{ind}$ and the associated map $\mathbb{R}^n \to \mathbb{R}^n_{ind}$ sending $e_i \mapsto b_i$.

Define a **strong basis** for a family E to be a collection of sections s_α with the property that the induced map $X \times \mathbb{R}\langle A\rangle \to E$ is an isomorphism. So a family is trivial if and only if it has a strong basis.

Remark 8.9. If $s_1, \dots, s_n$ is a weak basis for E, then let $\phi_i \colon E \to X \times \mathbb{R}$ be the ith coordinate function: it is the map that sends a vector $v \in E_x$ to the pair (x, t) where t is the ith coordinate of v with respect to the basis $s_1(x), \dots, s_n(x)$. The condition that $s_1, \dots, s_n$ be a strong basis is equivalent to the coordinate functions $\phi_1, \dots, \phi_n$ all being continuous. (Note that the case of an infinite basis is more subtle, though, essentially because of the difference between the colimit and product topologies on $\mathbb{R}^\infty$—see ???).

Most families of vector spaces will have neither a weak nor a strong basis, as the topology of X gets in the way. The families in Example 8.5(b,c,d) do not have any nonvanishing sections at all! However, demanding that we have strong bases *locally* turns out to be a reasonable thing to require. To this end, if $p \colon E \to X$ is a family of vector spaces and $U \subseteq X$ recall that $E|_U$ denotes the family $p^{-1}(U) \to U$ (equipped with the subspace topology from E). If $x \in X$ say that a **local weak (resp. strong) basis** at x is a neighborhood U of x together with a weak (resp. strong) basis for $E|_U$.

Definition 8.10. *A **vector bundle** is a family of finite rank vector spaces $p \colon E \to X$ that has a local strong basis at every point of X. In other words, for each $x \in X$ there is a neighborhood $x \in U \subseteq X$, an $n \in \mathbb{Z}_{\geq 0}$, and an isomorphism of families of vector spaces*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & U \times \mathbb{R}^n \arrow[dl] \\ & U. & \end{tikzcd}$$

The isomorphism in the above diagram is called a "local trivialization". Usually one simply says that a vector bundle is a family of finite rank vector spaces that is locally trivial.

A map of vector bundles is just a map of the underlying families, and $\mathsf{Vect}(X)$ *will denote the category of vector bundles over* X*.*

Remark 8.11. Note that the n appearing in Definition 8.10 is $\operatorname{rank}_x(E)$, and it depends on the point x. It is easy to prove, though, that this number is constant on each connected component of X. Vector bundles of constant rank 1 are called **line bundles**, and bundles of constant rank 2 are called **plane bundles**.

Remark 8.12. Definition 8.10 defines *real* vector bundles, but one could do the same with $\mathbb{C}$ or $\mathbb{H}$ replacing $\mathbb{R}$ to define complex and quaternionic bundles.

Remark 8.13. Some authors allow vector bundles to have countably-infinite rank. There is nothing wrong with that, though one has to specify the topology one uses for $\mathbb{R}^\infty$. But if we take this approach then we end up having to add "finite rank" hypotheses in an annoying number of places. In our approach, we will on occasion abuse terminology and use the phrase "vector bundle of infinite rank" for this concept, even though such an object is not a vector bundle according to our definition.

Exercise 8.14. Let $f\colon E_1 \to E_2$ be a map of vector bundles on X that is an isomorphism on each fiber. Prove that f is an isomorphism in $\mathsf{Vect}(X)$.

Of the families of vector spaces we considered in Example 8.5, only the trivial family from (a) is a vector bundle. Before discussing more interesting examples, though, we need some general remarks. Establishing that a given family of vector spaces is a vector bundle requires producing weak local bases *and* proving that the associated coordinate functions are continuous—oftentimes one forgets the latter part, but it is important. This second part can be somewhat annoying, though, so it is useful to know that in many situations one can avoid it. In order to discuss these situations, we start with the following definition:

Definition 8.15. *A family of vector spaces* $E \to X$ *is said to be* ***tame*** *if for every point* $x \in X$ *and every local weak basis* $\{s_\alpha\}$ *defined on an open neigborhood* U *of* x*, there is an open set* $x \in V \subseteq U$ *such that the set* $\{s_\alpha|_V\}$ *is a local strong basis. That is to say, every local weak basis near a point can be restricted to a local strong basis.*

Proposition 8.16. *Let* $E \to X$ *be a tame family of vector spaces. Then* E *is a vector bundle if and only if for every* $x \in X$ *there exists a local weak basis on a neighborhood of* x*.*

Proof. Immediate. □

Of course every vector bundle is tame. The families from Example 8.5(b,c,d) are also tame, essentially because these families do not have *any* local weak bases at the "exotic" points. The family $\mathbb{R}^n_{ind}$ from Example 8.5(e) is not tame. The concept of tameness is not particularly natural or important, but it is useful to us because of the following result which guarantees tameness in many common situations:

Proposition 8.17. *Let* $E \to X$ *be a family of vector spaces of finite rank. Then* E *is tame if either of the following conditions is satisfied:*

(a) E *is a subfamily of a trivial family (perhaps of countably infinite rank).*
(b) X *is locally compact and* E *is Hausdorff.*

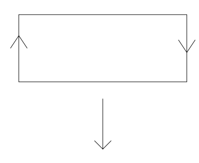
Recall that a CW-complex is locally compact if it is locally finite, in the sense that every point lies in the closures of only finitely-many open cells. So Proposition 8.17(b) applies in a large percentage of the cases one naturally encounters. But as just one example note that the CW-complex S^∞ is *not* locally compact; so Proposition 8.17(b) is not a universal panacea.

The proof of Proposition 8.17 is technical and a distraction from our goals at the moment; it can be found in Appendix B. However, from now on we will use Proposition 8.17 often and—except for the first few times—mostly without comment.

Example 8.18.

(a) Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector space isomorphism. Let $E' = [0,1] \times \mathbb{R}^n$ and let E be the quotient of E' by the relation $(0, v) \sim (1, \phi(v))$. Identifying S^1 with the quotient of $[0, 1]$ by $0 \sim 1$, we obtain a map $E \to S^1$ that is clearly a family of vector spaces. We claim this is a vector bundle. If $x \in (0, 1)$ then it is evident that E is locally trivial at x, so the only point of concern is $x = 0 = 1 \in S^1$. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$, and let $s_i\colon [0, \frac{1}{4}) \to E'$ be the constant section whose value is e_i. Likewise, let $s'_i\colon (\frac{3}{4}, 1] \to E'$ be the constant section whose value is $\phi(e_i)$. Projecting into E we obtain $s_i(0) = s'_i(1)$, and so the sections s_i and s'_i patch together to give a section $S_i\colon U \to E$, where $U = [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$. The sections $S_1, \dots, S_n$ are independent and therefore give a local trivialization of E over U.

When $n = 1$ and $\phi(x) = -x$ the resulting bundle is the *Möbius bundle* M, depicted below:



We further discuss the case of general n and ϕ in Example 11.3.

(b) Let $X = \mathbb{R}P^n$, regarded as the space of lines in $\mathbb{R}^{n+1}$. Let $L \subseteq X \times \mathbb{R}^{n+1}$ be the set

$$L = \{(l, v) \,|\, l \in \mathbb{R}P^n, x \in l\}.$$

Then L is a subfamily of the trivial family, and we claim that it is a line bundle over X. To see this, for any $l \in X$ we must produce a local trivialization. By Propositions 8.17(a) and 8.16 it suffices to just produce a local weak basis at every point. By symmetry it suffices to do this at the point $l = \langle e_0 \rangle$. Let $U \subseteq \mathbb{R}P^n$ be the set of lines whose orthogonal projection to $\langle e_0 \rangle$ is nonzero. Then $\mathbb{R}^n \cong U$ via the homeomorphism $\underline{x} \mapsto \langle e_0 + \underline{x} \rangle$, where here we regard $\mathbb{R}^n$ as having basis $e_1, \dots, e_n$. Define $s\colon U \to L$ by sending l to $(l, e_0 + u)$ where $e_0 + u$ is the unique point on l with $u \in \mathbb{R}^n$. Via the homeomorphism $\mathbb{R}^n \cong U$ we verify at once that s is continuous. This section is clearly nonzero everywhere, so it gives the desired weak basis of $L|_U$. Thus, L is a vector bundle.

To be clear, it is not hard to prove that s induces a local trivialization of L without referencing Propositions 8.16 and 8.17. But the point is that those results allow us to avoid having to think about the extra steps that would be involved for that.

The bundle L is called the **tautological line bundle** over $\mathbb{R}P^n$. Do not confuse this with the *canonical* line bundle over $\mathbb{R}P^n$ that we will define shortly (they are duals of each other). Note that when $n=1$ the bundle L is isomorphic to the Möbius bundle on S^1. (Exercise: Check this!)

(c) One may generalize the previous example as follows. Let V be a vector space and fix an integer $k>0$. Consider the Grassmannian $\mathrm{Gr}_k(V)$ of k-planes in V. Let
$$\eta = \{(W,x)\,|\,W\in \mathrm{Gr}_k(V), x\in W\}.$$
Projection to the first coordinate $\pi\colon \eta\to \mathrm{Gr}_k(V)$ makes η into a rank k vector bundle, called the **tautological bundle** over $\mathrm{Gr}_k(V)$. To see that it is indeed a bundle, let $W\in \mathrm{Gr}_k(V)$ be an arbitrary k-plane. By choosing an appropriate basis for V we can just assume $W=\langle e_1,\dots,e_k\rangle$. Equip V with the standard dot product with respect to the e-basis, and let $U\subseteq \mathrm{Gr}_k(V)$ be the collection of all k-planes whose orthogonal projection onto W is surjective (equivalently, an isomorphism). One readily checks that this is an open set of W. For each $J\in U$ let $s_1(J),\dots,s_k(J)$ be the unique vectors in J that orthogonally project onto $e_1,\dots,e_k$. One checks that these are continuous sections of $\eta|_U$, and of course they are clearly independent and hence give a local trivialization.

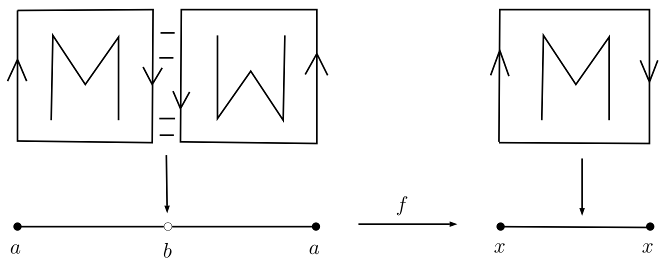
(d) Let M be a smooth manifold, and let $TM\to M$ be its tangent bundle. So the fiber over each $x\in M$ is the tangent space at x. Let $x\in M$ and let U be a local coordinate patch about x. Let $x_1,\dots,x_n$ be local coordinates in U, and let $\partial_1,\dots,\partial_n$ be the associated vector fields (giving the tangent vectors to the coordinate curves in this system). Then $\partial_1,\dots,\partial_n$ are independent sections of TM, and hence give a local trivialization.

Note that if $f\colon E\to F$ is a map of vector bundles over X then neither $\ker f$ nor $\operatorname{coker} f$ will necessarily be a vector bundle. For an example, let $X=[-1,1]$ and let $E=\underline{1}$. Define $f\colon E\to E$ by letting it be multiplication-by-t on the fiber over $t\in X$. We will give a thorough discussion of kernels and cokernels in Section 9.

8.19. **Pullback bundles.** If $E\to X$ is a vector bundle and $f\colon Y\to X$ then it is easy to check that f^*E is also a vector bundle. Given composable maps $Z\xrightarrow{g} Y\xrightarrow{f} X$, there is an evident natural isomorphism $(fg)^*E\cong g^*(f^*E)$. For each topological space X and each integer $k\geq 0$, let $\mathrm{Vect}_k(X)$ denote the set of isomorphism classes of vector bundles of rank k on X. The pullback construction then makes $\mathrm{Vect}_k(-)$ into a contravariant functor from $\mathcal{T}op$ into $\mathcal{S}et$.

Example 8.20. Pullback bundles can be slightly non-intuitive. Let $M\to S^1$ be the Möbius bundle, and let $f\colon S^1\to S^1$ be the map $z\mapsto z^2$. We claim that $f^*M\cong \underline{1}$. This is easiest to see if one uses the following model for M:
$$M=\Big\{\big(e^{i\theta}, re^{i\frac{\theta}{2}}\big)\,\Big|\,\theta\in[0,2\pi], r\in\mathbb{R}\Big\}.$$
The bundle map is projection onto the first coordinate $\pi\colon M\to S^1$. Then $f^*M=\{(e^{i\theta},re^{i\theta})\,|\,\theta\in[0,2\pi], r\in\mathbb{R}\}$. This is clearly isomorphic to $\underline{1}_{S^1}$, via the map $S^1\times\mathbb{R}\to f^*M$ given by $(e^{i\theta},r)\mapsto(e^{i\theta},re^{i\theta})$.

We can also demonstrate the isomorphism $f^*M\cong\underline{1}$ by the following picture:



Here f is the evident map that wraps the circle around itself twice, so that $f^{-1}(x) = \{a, b\}$. We see that f^*M can be thought of as two copies of M that are cut open and then sewn together as shown, thereby producing a cylinder.

8.21. **Constructing new vector bundles out of old ones.** Let $p\colon E \to X$ and $p'\colon E' \to X$ be two vector bundles. We may form a new bundle $E \oplus E'$, whose underlying topological space is just the pullback $E \times_X E'$. So a point in $E \oplus E'$ is a pair (e, e') where $p(e) = p'(e')$. The rules for vector addition and scalar multiplication are the evident ones. Note that the fiber of $E \oplus E'$ over a point x is simply $E_x \oplus E'_x$. The local trivializations of E and E' combine to give local trivializations of $E \oplus E'$, showing that this is indeed a vector bundle.

More generally, any canonical construction that one can apply to vector spaces may be extended to apply to vector bundles. So one can talk about the bundle $E \otimes E'$, the dual bundle E^*, the hom-bundle $\underline{\mathrm{Hom}}(E, E')$, the exterior product bundle $\bigwedge^i E$, and so on. We will only carefully define $E \otimes E'$, and leave the other definitions to the reader.

Set-theoretically define

$$E \otimes E' = \{(x, v) \mid x \in X, v \in E_x \otimes E'_x\}.$$

This is clear enough, and it is clear how to define addition and scalar multiplication in the fibers. The only thing that takes thought is how to define the topology on $E \otimes E'$, and to check that the operations are continuous. But it is enough to define the topology *locally*, and to check continuity locally. If $x \in X$, let U be a neighborhood of x over with both E and E' are trivializable. Choose isomorphisms $\phi\colon U \times \mathbb{R}^k \to E|_U$ and $\phi'\colon U \times \mathbb{R}^l \to E'|_U$. Then one gets a bijection of sets $U \times (\mathbb{R}^k \otimes \mathbb{R}^l) \to (E \otimes E')|_U$ which is a linear isomorphism on each fiber: one sends $(u, v \otimes w)$ to $(u, \phi(u, v) \otimes \phi'(u, w))$ and then extends linearly. Finally, one uses this bijection to transplant the topology from $U \times (\mathbb{R}^k \otimes \mathbb{R}^l)$ to $(E \otimes E')|_U$. We leave the reader to fill in all the details here.

A brief summary of this technique is “define the new bundle set-theoretically and then use the local trivializations to induce the topology”. Technically one should check that different choices of local trivialization yield the same topology, but this is usually routine and left implicit.

Remark 8.22 (External sums and products)**.** Let $E \to X$ and $F \to Y$ be two vector bundles, but this time over possibly different base spaces. One may construct an external direct sum $E \hat{\oplus} F \to X \times Y$ whose fiber over (x, y) is $E_x \oplus F_y$. The underlying topological space of $E \hat{\oplus} F$ is just $E \times F$, and it has the evident operations. Note that $E \hat{\oplus} F$ can also be constructed as $\pi_1^*(E) \oplus \pi_2^*(F)$, where π_1 and π_2 are the projections from $X \times Y$ onto the two factors.

In the case $X = Y$ we can construct the (internal) direct sum from the external one: namely $E \oplus F = \Delta^*(E \hat{\oplus} F)$ where $\Delta\colon X \to X \times X$ is the diagonal map. Thus, the internal and external direct sums determine each other.

One can tell a similar story about external tensor products, or external hom-bundles.

Exercise 8.23. Given two vector bundles $E_1 \to X$ and $E_2 \to X$ construct the bundle $\underline{\mathrm{Hom}}(E_1, E_2)$ whose fiber over $x \in X$ is $\mathrm{Hom}((E_1)_x, (E_2)_x)$. Prove an adjointness formula

$$\mathsf{Vect}(X)(E_1 \otimes E_2, E_3) \cong \mathsf{Vect}(X)(E_1, \underline{\mathrm{Hom}}(E_2, E_3)).$$

Even better, establish an isomorphism of bundles

$$\underline{\mathrm{Hom}}(E_1 \otimes E_2, E_3) \cong \underline{\mathrm{Hom}}(E_1, \underline{\mathrm{Hom}}(E_2, E_3)).$$

8.24. **Constructing vector bundles by patching.** Let X be a space and let A and B be subspaces such that $A \cup B = X$. Recall that if $f_A\colon A \to Y$ and $f_B\colon B \to Y$ are continuous maps that agree on $A \cap B$ then we may patch these together to get a continuous map $f\colon X \to Y$ provided that either (i) A and B are both closed, or (ii) A and B are both open. This is a basic fact about topological spaces. The analogous facts for vector bundles are very similar in the case of an open cover, but a little more subtle for closed covers.

Proposition 8.25. *Let $E \to X$ be a family of vector spaces.*

(a) If $\{U_\alpha\}$ is an open cover of X and each $E|_{U_\alpha}$ is a vector bundle, then E is a vector bundle.

(b) Let $\{A, B\}$ be a cover of X by closed subspaces. Suppose that either

  (i) B is regular and has a countable basis, or

  (ii) For every $x \in A \cap B$ and every neighborhood $x \in U \subseteq X$ there exists a neighborhood $x \in V \subseteq U$ such that $V \cap A \cap B \hookrightarrow V \cap B$ has a retraction.

  Then if $E|_A$ and $E|_B$ are both vector bundles, so is E.

Proof. Part (a) is trivial, so we focus on (b). The main issue is producing local trivializations around points $x \in A \cap B$: one can do so in the "A-part" of a neighborhood and in the "B-part" of the neighborhood, but then some care is required in doing the two simultaneously in a compatible way.

Let $x \in X$, with the goal of producing a local trivialization around x. There are three cases: $x \in X - A$, $x \in X - B$, and $x \in A \cap B$. If $x \in X - B$ then we have $x \in (X - B)^{open} \subseteq A$. Since $E|_A$ is a vector bundle there is a subset $x \in U \subseteq X - B$ such that U is open in A and $E|_U$ is trivializable. But then U is open in $X - B$ and hence also open in X, so $E \to X$ has a local trivialization at x. A similar argument works if $x \in X - A$.

We have left to analyze the case $x \in A \cap B$. The fact that $E|_A$ is a vector bundle implies that there exists an open set $x \in U_1 \subseteq X$ such that E is trivializable over $U_1 \cap A$. Similarly, there exists an open set $x \in U_2 \subseteq X$ such that E is trivializable over $U_2 \cap B$. Let $U = U_1 \cap U_2$. Then $E|_{U \cap A}$ and $E|_{U \cap B}$ is a closed cover of $E|_U$, and so $E|_U$ is the pushout of the diagram

$$E|_{U \cap A} \longleftarrow E|_{U \cap A \cap B} \longrightarrow E|_{U \cap B}.$$

Choose a trivializaion $E|_{U\cap A} \cong (U \cap A) \times \mathbb{R}^n$, which induces a trivializaion of $E|_{U\cap A\cap B}$. So we have the following diagram

$$\begin{tikzcd} E|_{U\cap A} & E|_{U\cap A\cap B} \arrow[l] \arrow[r] & E|_{U\cap B} \\ (U\cap A)\times\mathbb{R}^n \arrow[u, "f_A"', "\cong"] & (U\cap A\cap B)\times\mathbb{R}^n \arrow[l] \arrow[r] \arrow[u, "f_{A\cap B}"', "\cong"] & (U\cap B)\times\mathbb{R}^n. \arrow[u, dotted, "f_B"'] \end{tikzcd}$$

and the question is whether we can find an isomorphism f_B that makes the diagram commute.

Let j denote the various horizontal maps in the above diagram, which are all subspace inclusions. Choose an isomorphism $h\colon E|_{U\cap B} \to (U \cap B) \times \mathbb{R}^n$. Then the composite $hjf_{A\cap B}$ may be represented by a map $U \cap A \cap B \to GL_n(\mathbb{R})$ (via adjointness, essentially). The construction of f_B is the question of extending this over $U \cap B$.

Under hypothesis (ii), by passing to a smaller neighborhood $x \in U' \subseteq U$ we can find a retraction $U'\cap B \to U'\cap A\cap B$, in which case the required extension is evident. Under hypothesis (i) we know that all subspaces of B are normal [Mu, Theorems 31.2 and 32.1], and hence we can apply the Tietze extension theorem: the composite map $U\cap A\cap B \to GL_n(\mathbb{R}) \hookrightarrow M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ extends to $U\cap B$. By continuity we can land in the open set $GL_n(\mathbb{R})$ after passing to a smaller neighborhood $x \in U' \subseteq U$. Thus, we have produced the localization trivialization of E at x. $\square$

Remark 8.26. A good example of Proposition 8.25(b) is the covering of a sphere S^n by its upper and lower hemispheres, intersecting in the equator. In this situation *both* hypotheses (i) and (ii) happen to be satisfied.

Corollary 8.27 (Patching vector bundles)**.** *Let $\{A, B\}$ be a cover of the space X. Suppose given vector bundles $E_A \to A$ and $E_B \to B$, together with a vector bundle isomorphism $\phi\colon E_A|_{A\cap B} \to E_B|_{A\cap B}$. Then there exists a vector bundle $E \to X$ such that $E|_A$ is isomorphic to E_A and $E|_B$ is isomorphic to E_B provided that one of the following conditions holds:*

(i) A and B are both open, or

(ii) A and B is a closed cover satisfying either of the hypotheses in part (b) of Proposition 8.25.

Proof. Define E to be the pushout of the following diagram:

$$\begin{tikzcd} E_A|_{A\cap B} \arrow[r, "\phi"] \arrow[d] & E_B|_{A\cap B} \arrow[r, tail] & E_B \arrow[d, dotted] \\ E_A \arrow[rr, dotted] & & E. \end{tikzcd}$$

The composite maps $E_A \to A \to X$ and $E_B \to B \to X$ yield a map $E \to X$, and one readily checks that this inherits the structure of a family of vector spaces (see Exercise 8.39). It is also evident that $E|_A \cong E_A$ and $E|_B \cong E_B$. It only remains to verify that E is a vector bundle, and this is a direct application of Proposition 8.25. $\square$

Corollary 8.27(a) admits a generalization to arbitrary open coverings. Suppose $\{U_\alpha\}$ is an open cover of X, and assume given a collection of vector bundles $E_\alpha \to$

U_α. For each α and β further assume given an isomorphism

$$\phi_{\beta,\alpha}\colon E_\alpha|_{U_\alpha\cap U_\beta} \xrightarrow{\cong} E_\beta|_{U_\alpha\cap U_\beta}.$$

Let E be the quotient of $\amalg_\alpha E_\alpha$ by the equivalence relation generated by saying $(\alpha, v_\alpha) \sim (\beta, \phi_{\beta,\alpha}(v_\alpha))$ for every α, β, and $v_\alpha \in E_\alpha|_{U_\alpha\cap U_\beta}$. Here we are writing (α, v_α) for the element v_α in $\amalg_\gamma E_\gamma$ that lies in the summand indexed by α.

It is easy to see that in this generality E is a family of vector spaces. It is not necessarily the case, however, that $E|_{U_\alpha} \cong E_\alpha$. If this were true for all α then of course E would be a vector bundle and we would be done. Here is the trouble, though. Suppose $\alpha_0, \alpha_1, \dots, \alpha_n$ are a sequence of indices such that $\alpha_0 = \alpha_n = \alpha$. If $v \in E_\alpha$ then we identify v with $\phi_{\alpha_1,\alpha_0}(v)$, which is in turn identified with $\phi_{\alpha_2,\alpha_1}(\phi_{\alpha_1,\alpha_0}(v))$, and so forth—so that v ends up being identified with

$$\Big(\phi_{\alpha_n,\alpha_{n-1}} \circ \phi_{\alpha_{n-1},\alpha_{n-2}} \circ \cdots \circ \phi_{\alpha_1,\alpha_0}\Big)(v). \tag{8.28}$$

Note that, like v, this expression is an element of E_α. So identifications are possibly being made within individual summands of $\amalg_\alpha E_\alpha$, rather than just between different summands. The fibers of $E|_{U_\alpha}$ are quotients of those in E_α, but they might not be identical. To prohibit this from happening we impose some extra conditions: for any indices α, β, γ we require that

(i) $\phi_{\alpha,\alpha} = \mathrm{id}$,
(ii) The two isomorphisms $\phi_{\gamma,\alpha}$ and $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}$ agree on their common domain of definition, which is $E_\alpha|_{U_\alpha\cap U_\beta\cap U_\gamma}$.

We leave it to the reader to check that these conditions force any expression as in (8.28), with $\alpha_0 = \alpha_n$, to just be equal to v (in particular, note that they force $\phi_{\alpha,\beta} = \phi_{\beta,\alpha}^{-1}$). So the fibers of E coincide with the fibers of the E_α's, we get isomorphisms $E|_{U_\alpha} \cong E_\alpha$, and hence E is a vector bundle.

Condition (ii) above is usually called the **cocycle condition**. To see why, consider the case where all of the E_α's are trivial bundles of rank n. Then the data in the $\phi_{\alpha,\beta}$ maps is really just the data of a map $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. These $g_{\alpha,\beta}$ maps are called **transition functions**. Condition (ii) is the requirement that the transition functions assemble to give a Čech 1-cocycle with values in the group $GL_n(\mathbb{R})$. Condition (i) is just a normalization condition, so that we are dealing with 'normalized' Čech 1-cocycles. Elements of the (continuous) Čech cohomology group $\check{H}^1_{cts}(U_\bullet; GL_n(\mathbb{R}))$ can be seen to be in bijective correspondence with isomorphism classes of vector bundles on X that are trivializable over the U_α's; if we take the direct limit over all open coverings then we obtain a bijection between isomorphism classes of vector bundles on X and elements of the Čech cohomology group $\check{H}^1_{cts}(X; GL_n(\mathbb{R}))$. But we are getting ahead of ourselves here; see Section 13 for related discussion.

8.29. **Dual bundles.** Let $E \to X$ be a vector bundle of rank n. Using the method of Section 8.21 we can define the dual bundle E^*, which set-theoretically is $\{(x, v) \,|\, x \in X, v \in E_x^*\}$. One can examine this construction in terms of patching trivial bundles. Choose an open cover $\{U_\alpha\}$ of X with respect to which E is trivializable; a choice of trivialization over each U_α then yields a collection of gluing maps $\phi_{\alpha,\beta}$. We think of E as being built from the trivial bundles $E_\alpha = U_\alpha \times \mathbb{R}^n$ via these gluing maps. Then the dual bundle E^* is built from the trivial bundles $U_\alpha \times (\mathbb{R}^n)^*$ via the duals of the gluing maps: that is, $(\phi^{E^*})_{\beta,\alpha} = \big(\phi^E_{\alpha,\beta}\big)^*$.

We will see in a moment (Corollary 8.34) that for real vector bundles over paracompact Hausdorff spaces one always has $E \cong E^*$, although the isomorphism is not canonical. This is not true for complex bundles, however (see Example 8.37).

Let $L \to \mathbb{C}P^n$ be the tautological complex line bundle over $\mathbb{C}P^n$. Its (complex) dual L^* is called the **canonical line bundle** over $\mathbb{C}P^n$. Whereas from a topological standpoint neither L nor L^* holds a preferential position over the other, in algebraic geometry there is an important difference between the two. The difference comes from the fact that L^* has certain "naturally defined" sections, whereas L does not. For a point $z = [z_0 : \cdots : z_n] \in \mathbb{C}P^n$, L_z is the complex line in $\mathbb{C}^{n+1}$ spanned by $(z_0, \ldots, z_n)$. Given only $z \in \mathbb{C}P^n$ there is no evident way of writing down a point on L_z, without making some kind of arbitrary choice; said differently, the bundle L does not have any easily-described sections. In contrast, it is much easier to write down a *functional* on L_z. For example, let ϕ_i be the unique functional on L_z that sends the point $(z_0, \ldots, z_n)$ to z_i. Notice that this description depends only on $z \in \mathbb{C}P^n$, not the point $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ that represents it; that is, the functional sending $(\lambda z_0, \ldots, \lambda z_n)$ to λz_i is the same as ϕ_i. In this way we obtain an entire $\mathbb{C}^{n+1}$'s worth of sections for L^*, by taking linear combinations of the ϕ_i's.

To be clear, it is important to realize that L has plenty of sections—it is just that one cannot describe them by simple formulas. The slogan to remember is that the bundle L^* has *algebraic* sections, whereas L does not. In algebraic geometry the bundle L^* is usually denoted $\mathcal{O}(1)$, whereas L is denoted $\mathcal{O}(-1)$. More generally, $\mathcal{O}(n)$ denotes $(L^*)^{\otimes n}$ when $n \geq 0$ (so that $\mathcal{O}(0)$ is the trivial line bundle), and denotes $L^{\otimes(-n)}$ when $n < 0$.

8.30. **Inner products on bundles.** It is nearly possible to develop everything we need from bundle theory without using inner products, and in the rest of the text we do try to minimize our use of them. But for some results the use of inner products provides significant simplifications of proofs, and so it is good to know about them.

Definition 8.31. *Let $E \to X$ be a real vector bundle. An **inner product** on E is a map of vector bundles $E \otimes E \to \underline{1}_X$ that induces a positive-definite, symmetric, bilinear form on each fiber E_x. A vector bundle with an inner product is usually called an **orthogonal** vector bundle.*

There is a similar notion for Hermitian inner products on complex vector bundles, but here we cannot phrase things in terms of the tensor product because of conjugate-linearity in one variable. So perhaps the simplest thing is just to say that if $E \to X$ is a complex bundle then a Hermitian inner product is a map $E \times_X E \to \underline{1}$ (over X) which induces a Hermitian inner product on each fiber of E.

The next result is the first of several places where we will need to use partitions of unity, so let us take a moment to review this concept.

Definition 8.32. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a space X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection of continuous functions $\phi_\alpha \colon X \to [0,1]$ such that*

(1) [Support condition] For each α, $\overline{\operatorname{Supp}\phi_\alpha} \subseteq U_\alpha$ where $\operatorname{Supp}\phi_\alpha = \phi_\alpha^{-1}(\mathbb{R} - 0)$.
(2) [Local finiteness] For each $x \in X$ there is a neighborhood $x \in V$ such that V has nonempty intersection with $\overline{\operatorname{Supp}\phi_\alpha}$ for only finitely many indices α.
(3) [Partition of 1] For each $x \in X$, $\sum_\alpha \phi_\alpha(x) = 1$ (note that this is a finite sum by (2)).

The basic theory of partitions of unity is discussed in [Mu, Section 41], though note that Munkres's terminology is slightly different from ours: he defines the support of ϕ_α to be the **closure** of $\phi_\alpha^{-1}(\mathbb{R} - 0)$. The main result we will need is that on a paracompact Hausdorff space any open cover has a partition of unity subordinate to it [Mu, Theorem 41.7].

Proposition 8.33. *Assume that X is paracompact and Hausdorff. Then any real bundle on X admits an inner product, and any complex bundle on X admits a Hermitian inner product.*

Proof. The idea is to produce the necessary inner products locally, and then use a partition of unity to average the results into a global inner product. We may as well assume X is connected, and then E has a constant rank n.

Let $E \to X$ be a real vector bundle, and let $\{U_\alpha\}$ be an open cover over which the bundle is trivial. Choose bundle isomorphisms $f_\alpha\colon E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$, for each α. Equip $\mathbb{R}^n$ with the standard Euclidean inner product, and let $\langle -, - \rangle_\alpha$ be the inner product on $E|_{U_\alpha}$ obtained by transplanting the Euclidean product across the isomorphisms f_α.

Let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. For $x \in X$ and $v, w \in E_x$ define

$$\langle v, w \rangle = \sum_\alpha \phi_\alpha(x) \cdot \langle v, w \rangle_\alpha.$$

It is clear that this is continuous in v and w, bilinear, symmetric, and positive-definite—these follow from the corresponding properties of the forms $\langle -, - \rangle_\alpha$ (note that continuity uses the local finiteness property of the partition). So this completes the construction.

The proof for Hermitian inner products on a complex bundle is basically identical. □

Corollary 8.34. *Let $E \to X$ be a real vector bundle on a paracompact Hausdorff space X. Then E is isomorphic to its dual E^*.*

Proof. Start by equipping E with an inner product $E \otimes E \to \underline{1}$, and note that the fiberwise forms are nondegenerate (since they are positive-definite). The adjoint of the above bundle map is a map $E \to E^*$, and nondegeneracy of the fiberwise forms shows that this is a fiberwise isomorphism. So it is an isomorphism of bundles. □

Exercise 8.35. Here is an illuminating problem to think through. Every complex vector space may be equipped with a nondegenerate, symmetric bilinear form. Check that the proof of Proposition 8.33 does *not* generalize to show that every complex vector bundle may be equipped with a symmetric bilinear form that is nondegenerate on the fibers—in particular, find the point where the proof breaks down. Note that if the proof did generalize, one could show just as in Corollary 8.34 that every complex bundle was isomorphic to its own dual. This is false, as we will see in Example 8.37 below. The complex version of Corollary 8.34 says that if $E \to X$ is a complex bundle over a paracompact Hausdorff space then E is isomorphic to the conjugate of E^* (the bundle obtained from E^* by changing the complex structure so that $z \in \mathbb{C}$ acts as $\bar{z}$).

Consider a trivial bundle $X \times \mathbb{R}^n \to X$ and equip it with the standard inner product. This bundle may be considered as trivial in two different ways: the vector bundle structure is trivial, and the inner product structure is also trivial. It is not clear *a priori* that the former property implies the latter, but in fact it does:

Proposition 8.36. *Let X be a space and let $n \in \mathbb{Z}_+$. Every inner product on $\underline{n}_X$ is isomorphic (in the category of vector bundles with inner product) to the 'constant' inner product provided by the standard Euclidean metric.*

Proof. Consider $\mathbb{R}^n$ with its standard basis $e_1, \dots, e_n$. Inner products on $\mathbb{R}^n$ are in bijective correspondence with symmetric, positive-definite matrices $A \in M_{n\times n}(\mathbb{R})$, by sending an inner product $\langle -, - \rangle$ to the matrix $a_{ij} = \langle e_i, e_j \rangle$. Let $M^{sym,+}$ denote the space of such matrices. To give an inner product on the trivial bundle $\underline{n}_X$ is therefore equivalent to giving a map $X \to M^{sym,+}$.

Given an isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ we may transplant an inner product from the target onto the domain; this gives rise to an action of $GL_n(\mathbb{R})$ on the space of inner products. If $P \in GL_n(\mathbb{R})$ and $A \in M^{sym,+}$ then the action is $P.A = PAP^T$. The fact that every inner product on $\mathbb{R}^n$ has an orthonormal basis shows that $M^{sym,+}$ equals the orbit of the identity matrix I_n under this action. The stabilizer of the identity is of course the orthogonal group O_n, and so we obtain the homeomorphism $GL_n(\mathbb{R})/O_n \cong M^{sym,+}$.

Since O_n is a compact Lie group the quotient $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})/O_n$ is a fiber bundle and hence a fibration (cf. [Pa, Section 4.1]). So we have the fibration sequence $O_n \hookrightarrow GL_n(\mathbb{R}) \to M^{sym,+}$, where the projection map sends a matrix P to $PI_nP^T = PP^T$. The inclusion $O_n \hookrightarrow GL_n(\mathbb{R})$ is a homotopy equivalence by Gram-Schmidt, and so $M^{sym,+}$ is weakly contractible. But actually it is not hard to show directly that there is a homeomorphism $M^{sym,+} \cong \mathbb{R}^{\binom{n+1}{2}}$. For this, recall from linear algebra that a real symmetric matrix A is positive definite if and only if the minors $A[1\cdots k|1\cdots k]$ are positive, for every $1 \le k \le n$. When $n = 2$, for example, we can use this to write down a homeomorphism

$$(0,\infty)\times\mathbb{R}\times(0,\infty) \xrightarrow{\cong} M_2^{sym,+}, \qquad (a,b,c) \mapsto \begin{bmatrix} a & b \\ b & \frac{b^2}{a}+c \end{bmatrix}$$

So $M_2^{sym,+} \cong \mathbb{R}^3$. For the general case one works inductively to show $M_n^{\mathrm{sym},+} \cong M_{n-1}^{sym,+} \times \mathbb{R}^{n-1} \times (0,\infty)$. The idea is that after specifying the upper $(n-1)\times(n-1)$ submatrix and the top $n-1$ entries of the last column, we rearrange the cofactor expansion of the determinant about the last column to obtain a lower bound for the final diagonal entry.

Since $M_n^{\mathrm{sym},+}$ is a Euclidean space, it can be given the structure of a CW-complex. So the commutative square

$$\begin{tikzcd} \{I_n\} \arrow[r, tail] \arrow[d, tail, "\simeq"'] & GL_n(\mathbb{R}) \arrow[d, two heads, "{[P\mapsto PP^T]}"] \\ M_n^{sym,+} \arrow[r, "\mathrm{id}"] \arrow[ur, dotted, "r"] & M_n^{sym,+} \end{tikzcd}$$

has a lift as indicated.

As we have discussed, our given inner product on $\underline{n}_X$ is represented by a map $X \to M_n^{sym,+}$. Compose with r to obtain $X \to GL_n(\mathbb{R})$. This map specifies a bundle isomorphism $\underline{n}_X \to \underline{n}_X$. If we equip the domain with our given inner product and the codomain with the standard inner product, this map preserves the inner products and therefore proves the proposition. $\square$

Suppose that $E \to X$ is a rank n real vector bundle with an inner product. Choose a trivializing open cover $\{U_\alpha\}$, and for each α fix an inner-product-preserving trivialization $f_\alpha \colon E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ where the codomain has the standard inner product (this is possible by Proposition 8.36). The transition functions $g_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ therefore factor through O_n, as they must preserve the inner product. This process is usually referred to as *reduction of the structure group*.

We may use these ideas to give another proof of Corollary 8.34, one that is perhaps more down-to-earth. Let $E \to X$ be a real vector bundle on a compact space, and choose a trivializing cover $\{U_\alpha\}$ with respect to which there exist local trivializations where the transition functions are maps $g_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to O_n$. So we obtain E by gluing together the spaces $E_\alpha = U_\alpha \times \mathbb{R}^n$ via the maps $g_{\alpha,\beta}$. But then we obtain E^* by gluing together the spaces $U_\alpha \times (\mathbb{R}^n)^*$ via the maps

$$h_{\beta,\alpha} = g^*_{\alpha,\beta}.$$

Recall that in terms of matrices the dual is represented by the transpose. Since each $g_{\alpha,\beta}(x)$ is in O_n we can write $h_{\beta,\alpha}(x) = g_{\alpha,\beta}(x)^{-1}$, or

$$h_{\alpha,\beta}(x) = h_{\beta,\alpha}(x)^{-1} = g_{\alpha,\beta}(x).$$

In other words, the transition functions for E and E^* are exactly the same, and that is why the bundles are isomorphic.

We close this section with the promised example of a complex bundle that is not isomorphic to its dual:

Example 8.37. Let D_+ and D_- denote the upper and lower hemispheres of S^2. Let S^1 be the equator, which we identify with the unit complex numbers. Given a map $f \colon S^1 \to GL_n(\mathbb{C})$ we may construct a complex bundle on S^2 by taking two trivial bundles $\underline{n}_{D_+}$ and $\underline{n}_{D_-}$ and gluing them together using the map f: precisely, for $z \in S^1$ an element $v \in \left(\underline{n}_{D_+}\right)_z$ is glued to $f(z) \cdot v \in \left(\underline{n}_{D_-}\right)_z$. Here we are using Corollary 8.27(b). Let $E(f)$ denote the resulting bundle.

Observe that giving an isomorphism $E(f) \to E(g)$ is equivalent to giving two maps $A \colon D_+ \to GL_n(\mathbb{C})$ and $B \colon D_- \to GL_n(\mathbb{C})$ such that $g(z) \cdot A(z) = B(z) \cdot f(z)$ for all $z \in S^1$. Let us rewrite this as $A(z) = g(z)^{-1}B(z)f(z)$. Now, the map $B|_{S^1} \colon S^1 \to GL_n(\mathbb{C})$ is null-homotopic because it extends over D_-; so it is (unbased) homotopic to the constant map at I_n (this uses connectivity of $GL_n(\mathbb{C})$). Therefore the map $z \mapsto g(z)^{-1}B(z)f(z)$ is homotopic to $g(z)^{-1}f(z)$. But $A|_{S^1}$ is also (unbased) homotopic to the constant map at I_n, because it extends over D_+. So we have proven that if $E(f) \cong E(g)$ then $z \mapsto g(z)^{-1}f(z)$ is unbased homotopic to a constant map.

Next, observe that the dual of $E(f)$ is $E(f')$, where $f'(z) = \left[f(z)^T\right]^{-1}$. So if $E(f)$ is isomorphic to its dual then the map $z \mapsto f(z)^T \cdot f(z)$ is null-homotopic.

Consider the case where $n = 1$ and $f(z) = z$. Since $z \mapsto z^2$ is not null-homotopic, we see that $E(f)$ is not isomorphic to its dual.

Exercise 8.38. Check that $E(f)$ from the end of the above example is isomorphic to the tautological line bundle $L \to \mathbb{C}P^1$.

The above example starts to give an indication that the classification of vector bundles on spheres reduces to a problem in homotopy theory. We will explore this in detail in Section 12, and in Section 13 we will see that this is true not only for spheres but for vector bundles on any paracompact Hausdorff space X.

◦ Exercises ◦

Exercise 8.39. Let $E_A \to A$, $E_X \to X$, and $E_Y \to Y$ be families of vector spaces and assume given maps of families

$$E_X \longleftarrow E_A \longrightarrow E_Y$$

that lie over subset inclusions $X \supseteq A \subseteq Y$. Assume either that

(i) A is open in X and Y, and the maps $E_A \to E_X$ and $E_A \to E_Y$ are open injections, or
(ii) A is closed in X and Y, and the maps $E_A \to E_X$ and $E_A \to E_Y$ are closed injections.

Prove that the pushout $E = E_X \amalg_{E_A} E_Y$ is a family of vector spaces over $\Omega = X \amalg_A Y$.
[Hint: The difficult part is producing the addition $E \times_\Omega E \to E$. Show that the domain is covered by $E_X \times_X E_X$ and $E_Y \times_Y E_Y$ and that these are open (resp. closed).]

Exercise 8.40. Let $E \to X$ be a family of vector spaces and let $s_1, s_2, \ldots$ be a weak basis for E. If X is locally compact and Hausdorff prove that there is an associated map of families $X \times \mathbb{R}^\infty \to E$ that is a bijection, where $\mathbb{R}^\infty$ is given the colimit topology. [Hint: First show that the natural map $\operatorname{colim}_n(X \times \mathbb{R}^n) \to X \times \operatorname{colim}_n \mathbb{R}^n$ is a homeomorphism. Consult [Mu, Theorem 46.11].]

Exercise 8.41. Given a vector bundle $E \to X$, explain how to construct an associated projective bundle $\mathbb{P}(E) \to X$ that is a fiber bundle whose fiber over a point $x \in X$ is the projective space $\mathbb{P}(E_x)$. Verify that your construction admits a natural map of fiber bundles $E \to \mathbb{P}(E \oplus \underline{1})$ that sends a point $e \in E_x$ to the line spanned by $(e, 1)$ in $E_x \oplus \mathbb{R}$.

Exercise 8.42. Let $E \to X$ be a rank n vector bundle and $Y \subseteq X$ a closed set. Suppose given a trivialization $u\colon E|_Y \xrightarrow{\cong} \underline{n}_Y$. Define E' to be the pushout of the top row in this diagram:

$$\begin{tikzcd} \mathbb{R}^n \arrow[d] & E|_Y \arrow[l, "\pi_2 \circ u"'] \arrow[r, tail] \arrow[d] & E \arrow[d] \\ * & Y \arrow[l] \arrow[r, tail] & X. \end{tikzcd}$$

In words, E' is made by gluing all of the fibers of $E|_Y$ together according to the isomorphism u. Prove that $E' \to X/Y$ is a vector bundle IF there exists an open subset $Y \subseteq U$ and an extension of u to an isomorphism $E|_U \cong \underline{n}_U$.

Exercise 8.43. Let $E \to X$ be a vector bundle and let $Y \subseteq X$ be a closed subset.

(a) Prove that every section $s\colon Y \to E$ extends to a section defined on all of X in each of the following two cases:
  (i) $Y \hookrightarrow X$ is a cofibration, or
  (ii) X is locally compact and paracompact and Hausdorff.

[Note: This demonstrates a common theme in bundle theory, which is that certain results work either in the presence of sufficient cofibration hypotheses or in the presence of technical point-set topology hypotheses. For (ii), use the Tietze extension theorem locally and then patch the results together using a partition of unity.]

(b) Again under hypotheses (i) or (ii), prove that a nonvanishing section $s\colon Y \to E$ extends to a nonvanishing section on some neighborhood of Y.

(c) Suppose that $F \to X$ is another bundle and $u\colon E|_Y \to F|_Y$ is an isomorphism. Under hypotheses (i) or (ii) prove that u extends to an isomorphism $E|_U \to F|_U$ for some neighborhood U of Y.

9. Some results from fiberwise linear algebra

Recall that our basic goal is to learn to do linear algebra "over a base space". The fundamental objects in this setting are the vector bundles, and the maps are the bundle maps. This section contains a miscellany of foundational results that are frequently useful. This material can be safely skipped the first time through and referred back to as needed.

Lemma 9.1. *Let X be any space, and let $f\colon \underline{n} \twoheadrightarrow \underline{k}$ be a surjective map of bundles. Then f has a splitting.*

Note that the result is not immediately obvious. Of course one can choose a splitting in each fiber, but what guarantees that these can be chosen in a continuous manner?

Proof. Let $W = \{A \in M_{k\times n} \,|\, \operatorname{rank} A = k\}$, which is the space of surjective maps $\mathbb{R}^n \to \mathbb{R}^k$ (our matrices act on the left). Let Z be the space

$$Z = \{(A, B) \,|\, A \in M_{k\times n},\ B \in M_{n\times k},\ AB = I\},$$

which is the space of surjective maps with a chosen splitting. We claim that the projection map $p_1\colon Z \to W$ is a fiber bundle with fiber $\mathbb{R}^{k(n-k)}$, but defer the proof for just a moment. The fact that the fiber is contractible then shows that p_1 is weak homotopy equivalence.

Consider the diagram

$$\begin{tikzcd} & Z \arrow[d, "p_1"', "\sim"] \\ W \arrow[r, equal] & W. \end{tikzcd}$$

The space W is an open set of $M_{k\times n} \cong \mathbb{R}^{kn}$; indeed, it is the union of the $\binom{n}{k}$ open sets defined by one of the $k \times k$ minors being nonzero. Any open set of Euclidean space may be given a CW-structure (ref???), so the standard lifting theorems now show that there is a lifting $r\colon W \to Z$ in the above diagram.

Our surjective bundle map $f\colon \underline{n} \to \underline{k}$ is determined by a map $X \to W$. Composing with $W \to Z$, and then projecting to the second coordinate of Z, gives the desired splitting for f.

It remains to prove the claim about p_1 being a fiber bundle. Let $A \in W$. Since $\operatorname{rank}(A) = k$ there is a $k \times k$ minor of A that is nonzero; without loss of generality let us assume that it is the minor made up of the first k columns of the matrix. Let $U \subseteq W$ be the subspace consisting of all matrices where this same minor is nonzero, which is an open neighborhood of A in W. Writing matrices in block form, U consists of matrices $[X|Y]$ where $\det(X) \neq 0$. Then $p_1^{-1}(U)$ consists of pairs

$$\alpha_{X,Y,J,K} = \left(\begin{bmatrix} X & Y \end{bmatrix}, \begin{bmatrix} J \\ K \end{bmatrix} \right)$$

having the property that $\det(X) \neq 0$ and $XJ + YK = I_k$. We obtain an isomorphism $U \times M_{n-k,k}(\mathbb{R}) \cong p_1^{-1}(U)$ by sending $([X|Y], K)$ to $\alpha_{X,Y,J,K}$ with $J = X^{-1}(I_k - YK)$. □

Note the significance of the map $W \to Z$ that is produced in the above proof. This assigns to every surjection $\mathbb{R}^n \to \mathbb{R}^k$ a splitting, and it does so in a continuous manner. Of course there is no claim that there is a nice formula for how to do this, and in fact there almost certainly is not—but the proof shows that there does exist *some* way of doing so.

The previous result implies that every surjection of vector bundles is locally split. The following is a global version of this:

Proposition 9.2. *Let X be a paracompact Hausdorff space. Then any surjection of bundles $E \twoheadrightarrow F$ has a splitting.*

Proof. Briefly, we choose local splittings and then use a partition of unity to patch them together.

Choose an open cover $\{U_\alpha\}$ such that both E and F are trivializable over each U_α. Lemma 9.1 shows that there are splittings $\chi_\alpha \colon F|_{U_\alpha} \to E|_{U_\alpha}$. Now choose a partition of unity $\{\phi_\alpha\}$ subordinate to our open cover (cf. Definition 8.32). Set $\chi = \sum_\alpha \phi_\alpha \chi_\alpha$. This sum makes sense and is continuous because the partition of unity is locally finite, and one readily checks that it is a splitting for f. □

Given a map of vector bundles $f \colon E \to F$ over a space X, we would like to construct kernels, images, and cokernels—bundle operations that reflect the usual constructions of linear algebra. But we have already seen a good example that shows the subtleties here: take $X = [-1, 1]$ and let $f \colon \underline{1} \to \underline{1}$ be multiplication by t on the fiber over $t \in X$. Taking fiberwise kernels or images does not give a vector bundle. To avoid these issues we will require that f have constant rank on the fibers (actually locally constant is enough, as this is equivalent to constant rank on each connected component) . Under this assumption the construction of kernel and image bundles is faily straightforward, as these can be realized as subobjects of bundles we already have. The construction of cokernels is a little more challenging, but also works.

We start by looking at kernels and images. If $f \colon E \to F$ is a map of bundles and $x \in X$, write $\operatorname{rank}_x(f)$ for the rank of $f_x \colon E_x \to F_x$. If these fiberwise ranks are independent of x then we will also write $\operatorname{rank}(f)$ for the common value.

Proposition 9.3. *Let X be any space, and let $f \colon E \to F$ be a map of vector bundles over X. If f has constant rank then* $\ker f$ *and* $\operatorname{im} f$ *are vector bundles.*

Proof. Let $x \in X$, let $n = \operatorname{rank}_x(E)$, let $k = \operatorname{rank}_x(F)$, and let $r = \operatorname{rank}(f)$. It will suffice to produce a neighborhood U of x together with $n - r$ independent sections of $\ker f$ over U and r independent sections of $\operatorname{im} f$ over U. In particular, this makes it clear that we might as well assume that E and F are both trivial bundles; in this case f is specified by a map $X \to W_r$ where $W_r = \{A \in M_{k\times n} \mid \operatorname{rank}(A) = r\}$.

Let Z_r be the space

$$Z_r = \{(A, v_1, \dots, v_{n-r}) \mid A \in W_r \text{ and } v_1, \dots, v_{n-r} \text{ span the kernel of } A\}.$$

One can check that the projection $Z_r \to W_r$ is a fiber bundle with fiber $GL_{n-r}(\mathbb{R})$, but this is stronger than what we actually need. We only need that the map is locally split: any point in W_r has a neighborhood over which there exists a section. Given a map $X \to W_r$, it will then follow that every point in x has a neighborhood over which there exists a lifting into Z_r, and this will give the $n-r$ independent local sections of $\ker f$.

So let A be a point in W_r. Since $\operatorname{rank}(A) = r$, some $r \times r$ minor of A is nonzero. Without loss of generality we might as well assume it is the upper left $r \times r$ minor. Since $\operatorname{rank}(A) = r$, then for $j > r$ the jth column of A is a linear combination of the first r columns in a unique way; said differently, there is a unique vector of the form

$$v_j = e_j - s_1 e_1 - s_2 e_2 - \cdots - s_r e_r$$

that is in the kernel of A. Here the s_i's are certain rational expressions in the matrix entries of A that can be determined using Cramer's Rule. These formulas define sections on the neighborhood U of A consisting of all $k \times n$ matrices of rank r whose upper left $r \times r$ minor is nonzero. This finishes the proof of our claim.

We have established that $\ker f$ is a vector bundle. The proof for $\operatorname{im} f$ is entirely similar but a little easier. Let Y_r be the space

$$Y_r = \{(A, v_1, \dots, v_r) \mid A \in W_r \text{ and } v_1, \dots, v_r \text{ span the image of } A\}.$$

Again, it suffices to show that $Y_r \to W_r$ is locally split. For $A \in W_r$ there is some non-vanishing $r \times r$ minor, and the subset $U \subseteq W_r$ consisting of all matrices with that minor nonzero is a neighborhood of A. We get a section $U \to Y_r$ be sending each $B \in U$ to the pair consisting of B and the r columns of B that are chosen by that minor. $\square$

Next we turn to the construction of cokernels. Let us begin with a precise definition:

Definition 9.4. *Let $f\colon E \to F$ be a map of vector bundles over a space X. A **fiberwise cokernel** for f is a vector bundle Q together with a map of bundles $F \to Q$ with the property that for every $x \in X$ the map $F_x \to Q_x$ makes Q_x a cokernel for f_x.*

The following result says that fiberwise cokernels are also cokernels in the categorical sense (see Appendix G).

Proposition 9.5. *Suppose that Q is a fiberwise cokernel for the bundle map $f\colon E \to F$. Then for any map of bundles $g\colon F \to G$ such that $gf = 0$, there is a unique map of bundles $Q \to G$ making the diagram*

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & G \\ & Q \arrow[ur] & \end{tikzcd}$$

commute.

Proof. Each fiber Q_x is a cokernel for f_x, so there is a unique map of vector spaces $Q_x \to G_x$ making the evident triangle commute. This defines a map of sets $Q \to G$, but we need to check continuity. However, continuity is a local condition and so we can assume that all of the bundles are trivial. In this case the map $p\colon F \to Q$ has a splitting χ by Lemma 9.1. Then $p(\mathrm{id} - \chi p) = 0$, therefore $\mathrm{id} - \chi p$ factors through the kernel of p, which is the image of f. Since $gf = 0$ it follows that $g(\mathrm{id} - \chi p) = 0$ (note that this identity can be checked on fibers). So $g = g\chi p$, and therefore $g\chi$ is the set map $Q \to G$ constructed in the first line. As both g and χ are continuous, we are done. $\square$

Exercise 9.6. Explain why categorical cokernels in the category of vector bundles over X are not always fiberwise cokernels.

Our aim is to prove that fiberwise cokernels for bundle maps $f\colon E \to F$ exist when f has locally constant rank. Let us start with the case where E is a subbundle of F. Define an equivalence relation on F by saying that $v_1 \sim v_2$ if v_1 and v_2 are in the same fiber and $v_1 - v_2 \in E$. Let Q be the quotient space of F under this relation, and let $\pi\colon F \to Q$ be the quotient map. Note that the bundle projection $F \to X$ respects the equivalence relation and so induces $Q \to X$.

Proposition 9.7. *Under the above setup we have*

(a) For the standard inclusion $E = X \times \mathbb{R}^k \hookrightarrow X \times \mathbb{R}^n = F$, then $Q \cong X \times \mathbb{R}^{n-k}$ with the map $F \to Q$ induced by projection onto the last $n-k$ coordinates $\mathbb{R}^n \to \mathbb{R}^{n-k}$.
(b) For any open set $U \subseteq X$, the map $F|_U \to Q|_U$ is a quotient map.
(c) $\pi\colon F \to Q$ is an open map;
(d) The map $F \times_X F \to Q \times_X Q$ is a quotient map;
(e) There is a unique structure of addition and scalar multiplication that makes Q into a family of vector spaces and $F \to Q$ a map of families.
(f) Q is a vector bundle.

Proof. The projection onto the final $n-k$ coordinates $X \times \mathbb{R}^n \to X \times \mathbb{R}^{n-k}$ respects the equivalence relation and so induces a map $Q \to X \times \mathbb{R}^{n-k}$. But we also have the inclusion $\mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^n$ and so can consider the composition $X \times \mathbb{R}^{n-k} \hookrightarrow X \times \mathbb{R}^n \to Q$. One readily verifies that the two maps are inverses, and so $Q \cong X \times \mathbb{R}^{n-k}$.

Part (b) is trivial: for any quotient map $f\colon A \to B$, when $W \subseteq B$ is open then $f^{-1}(W) \to W$ is a quotient map. Apply this to $W = Q|_U$.

By (b) it is enough to prove (c) locally on X, so we immediately reduced to the case where E and F are trivial. The inclusion $f\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ is represented by a map $X \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ where LE denotes the space of linear embeddings. Consider the restriction map $\mathrm{LE}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$. We claim this is locally split. Given a linear embedding $f\colon \mathbb{R}^k \to \mathbb{R}^n$, the associated $n \times k$ matrix has a non-vanishing $k \times k$ minor—without loss of generality assume it is the initial $k \times k$ minor. Then on the open subset of $\mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ where this minor is nonzero, the assignment

$$\begin{bmatrix} A \\ B \end{bmatrix} \mapsto \begin{bmatrix} A & 0 \\ B & I_{n-k} \end{bmatrix}$$

gives a continuous splitting.

Since $\mathrm{LE}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ is locally split, by passing to smaller open sets in X we can assume that $X \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ lifts to $\mathrm{LE}(\mathbb{R}^n, \mathbb{R}^n)$ (via composition with a splitting). This says that there is a bundle isomorphism $X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ making the following diagram commute:

$$\begin{tikzcd} X \times \mathbb{R}^k \arrow[r, "f"] \arrow[d, equal] & X \times \mathbb{R}^n \arrow[r] \arrow[d, "\cong"'] & Q \arrow[d, dotted] \\ X \times \mathbb{R}^k \arrow[r, "i"] & X \times \mathbb{R}^n \arrow[r] & X \times \mathbb{R}^{n-k}. \end{tikzcd}$$

Here i is the standard inclusion. The diagram shows that we can get an induced isomorphism $Q \to X \times \mathbb{R}^{n-k}$. So verifying that $X \times \mathbb{R}^n \to Q$ is open is equivalent to the same statement for $X \times \mathbb{R}^n \to X \times \mathbb{R}^{n-k}$, which is true because the standard projection $\mathbb{R}^n \to \mathbb{R}^{n-k}$ is open.

Now we turn to (d). Let π be the map $F \times_X F \to Q \times_X Q$. Suppose $U \subseteq Q \times_X Q$ is such that $\pi^{-1}(U)$ is open in $F \times_X F$. Let $(q_1, q_2) \in U$, and choose preimages v_1 for q_1 and v_2 for q_2. Since $\pi^{-1}(U)$ is open in $F \times_X F$, there exist open neighborhoods $v_1 \in V_1 \subseteq F$ and $v_2 \in V_2 \subseteq F$ such that $V_1 \times_X V_2 \subseteq \pi^{-1}(U)$. But then $(q_1, q_2) \in \pi(V_1 \times_X V_2) \subseteq U$. But observe that $\pi(V_1 \times_X V_2) = (\pi V_1) \times_X (\pi V_2)$, and by (c) both πV_1 and πV_2 are open in Q. This proves that U contains a neighborhood of (q_1, q_2).

For (e) we refer to the diagram

$$\begin{tikzcd} F \times_X F \arrow[r, "+"] \arrow[d] & F \arrow[d] \\ Q \times_X Q \arrow[r, dotted] & Q \end{tikzcd}$$

and observe that the composition $F \times_X F \xrightarrow{+} F \longrightarrow Q$ respects the equivalence relation on F and so induces the map $Q \times_X Q \to Q$ using (b). A similar argument works for scalar multiplication, using that $\mathbb{R} \times F \to \mathbb{R} \times Q$ is a quotient map [Mu, Section 46, Exercise 9]. One readily checks that this structure makes Q into a family of vector spaces with the correct properties.

Finally we prove (f). Since we are verifying a local condition, it is enough to do so under the assumption that E and F are trivial. Just as in the proof of (c), we can reduce to the case where $E \hookrightarrow F$ is the standard inclusion $X \times \mathbb{R}^k \hookrightarrow X \times \mathbb{R}^n$, and here we know by (a) that Q is $X \times \mathbb{R}^{n-k}$. $\square$

Corollary 9.8. *Let $f \colon E \to F$ be a map of vector bundles of constant rank. Then a fiberwise cokernel for f exists.*

Proof. By Proposition 9.3 we know that $\operatorname{im} f$ is a vector bundle. Now apply Proposition 9.7 to the inclusion $\operatorname{im} f \hookrightarrow F$. $\square$

The result below is an easy variation on Proposition 9.2; it will be used often, so it is useful to have it stated explicitly.

Corollary 9.9. *Let X be a paracompact Hausdorff space. Then any injection of bundles $E \hookrightarrow F$ has a splitting.*

Proof. We can assume X is connected. Let Q be the cokernel, which is a vector bundle by Corollary 9.8. By Proposition 9.2 the map $F \twoheadrightarrow Q$ has a splitting, which then induces an isomorphism $F \cong E \oplus Q$. The composition $F \xrightarrow{\cong} E \oplus Q \xrightarrow{\pi_1} E$ gives the required splitting of $E \hookrightarrow F$. $\square$

The next result is of a somewhat different nature:

Proposition 9.10. *Suppose that X is compact and Hausdorff. Then every bundle is a subbundle of some finite-rank trivial bundle.*

Proof. Let $\pi\colon E \to X$ be a vector bundle on X. Choose a finite cover $U_1, \dots, U_s$ over which E is trivializable, which exists because of compactness. For each i choose a trivialization $f_i\colon E|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{R}^n$. Write $F_i = \pi_2 \circ f_i$.

Let $\{\phi_i\}$ be a partition of unity subordinate to the open cover (Definition 8.32). Define a map

$$\beta\colon E \longrightarrow X \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$$

(where there are s copies of $\mathbb{R}^n$) by the formula

$$\beta(v) = \Big(\pi v,\, \phi_1(\pi v)F_1(v),\, \dots,\, \phi_s(\pi v)F_s(v)\Big). \tag{*}$$

We have written πv instead of $\pi(v)$ here, to avoid being overwhelmed by parentheses. Note that if v is not in $E|_{U_i}$ then $F_i(v)$ is undefined, but in this case $\phi_i(\pi v)$ equals 0 and so the formula still makes sense. More rigorously, we can interpret the formula $\phi_i(\pi v)F_i(v)$ as specifying continuous functions on the two closed sets $\pi^{-1}(\overline{\operatorname{Supp}\phi_i})$ (which is contained in $E|_{U_i}$) and on $E\backslash\pi^{-1}(\operatorname{Supp}\phi_i)$ (in this case the zero function), which agree on the overlap—so these patch to give a continuous function on all of X. It is routine to check that formula (*) gives an embedding of bundles. ☐

We can generalize Proposition 9.10 to paracompact Hausdorff spaces at the expense of passing to trivial bundles of countably-infinite rank. The proof is essentially the same once we have established the following lemma and its corollary:

Lemma 9.11. *Let X be a paracompact Hausdorff space. Let $\mathcal{P}$ be a property of open subsets of X such that (1) if U has $\mathcal{P}$ and $V \subseteq U$ then V also has $\mathcal{P}$, (2) if $\{U_\alpha\}$ are disjoint open subsets that all have $\mathcal{P}$ then so does $\bigcup_\alpha U_\alpha$, and (3) X can be covered by open sets with property $\mathcal{P}$. Then X can be covered by countably many open subsets with property $\mathcal{P}$.*

Proof. We take the argument from [MS, Lemma 5.9]. First choose an open cover $\{V_\alpha\}$ by sets with property $\mathcal{P}$, indexed by some set I. Then choose a partition of unity $\{\phi_\alpha\}$ subordinate to this cover. For each finite subset $S \subseteq I$, define

$$W(S) = \{x \in X \mid \phi_\alpha(x) > \phi_\beta(x) \text{ for all } \alpha \in S \text{ and } \beta \in I - S\}.$$

Said in words, $W(S)$ is the collection of points for which the ϕ's from S are all larger than the ϕ's from outside of S. This is an open set. One sees readily that if $W(S) \cap W(S') \neq \emptyset$ then either $S \subseteq S'$ or $S' \subseteq S$. So if S and S' have the same size then $W(S)$ and $W(S')$ are disjoint.

Let $W_m = \bigcup_{\#S=m} W(S)$. Note that this is a disjoint union. If $x \in X$ then let S_x be the (necessarily finite) collection of all $\alpha \in I$ for which $\phi_\alpha(x)$ is maximal. Then $x \in W(S_x)$, which shows that $\bigcup_m W_m = X$. Finally, for each $S \subseteq I$ note that if $\alpha \in S$ then $W(S) \subseteq \operatorname{Supp}\phi_\alpha \subseteq V_\alpha$. So each $W(S)$ has property $\mathcal{P}$, and therefore each W_m has property $\mathcal{P}$. ☐

Corollary 9.12. *Let $E \to X$ be a vector bundle, where X is paracompact and Hausdorff. Then there is a countable open cover $\{U_i\}$ for X such that each $E|_{U_i}$ is trivializable.*

Proof. Apply the lemma where a subset U has $\mathcal{P}$ if E is trivializable over U. $\square$

Proposition 9.13. *Let X be paracompact and Hausdorff, and let $E \to X$ be a vector bundle. Then there is an embedding of bundles $E \hookrightarrow X \times \mathbb{R}^\infty$, where $\mathbb{R}^\infty$ denotes the vector space of countably-infinite dimension topologized as the colimit of its finite-dimensional subspaces (see Appendix B.5).*

Proof. One chooses a countable trivializing cover as in Corollary 9.12, and then the proof is essentially identical to that for Proposition 9.10 except for one subtlety. The construction from that proof gives a collection of continuous maps $E \to \mathbb{R}^n$ which can be assembled to give an embedding

$$g\colon E \to X \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \cdots \cong X \times \prod_{i=1}^{\infty} \mathbb{R}.$$

The image of each point is a tuple where only finitely many entries are nonzero, so g factors through $X \times \mathbb{R}^\infty$. But the topology on $\mathbb{R}^\infty$ here is the *product topology*—or more precisely, the subspace topology induced by the product topology on $\prod_{i=1}^{\infty} \mathbb{R}$—and this is different from the colimit topology (see Appendix B.5 for discussion of this). So an extra argument is required to see that the map is still continuous when $\mathbb{R}^\infty$ has the colimit topology.

Since continuity is a local condition, it is sufficient to prove that every $v \in E$ has a neighborhood on which the function is continuous. But by the local finiteness of the partition of unity, there is a neighborhood $\pi(v) \in W$ with the property that W only intersects finitely-many of the $\overline{\operatorname{Supp} \phi_i}$ spaces—$i = 1$ through $i = N$, say. So the coordinates of $g(v')$ past the Nth all vanish, for all $v' \in \pi^{-1}(W)$. Then g maps $\pi^{-1}(W)$ into $X \times \mathbb{R}^N$, and is continuous as such a map. Therefore g is also continuous as a map $g\colon \pi^{-1}(W) \to X \times \mathbb{R}^\infty_{\text{colim}}$. $\square$

Finally, we close this section with a few useful results related to ranks and exactness:

Proposition 9.14. *Let $\alpha\colon E \to F$ be a map of vector bundles over X. Then for any $n \in \mathbb{Z}_{\geq 0}$, the set $\mathcal{R}_n = \{x \in X \mid \operatorname{rank}(\alpha_x) \geq n\}$ is an open subset of X.*

Proof. Let $k = \operatorname{rank}(E)$ and $l = \operatorname{rank}(F)$. Let $x \in \mathcal{R}_n$. We can choose a neighborhood V of x over which both E and F are trivial. The map α is then specified by a continuous function $\alpha\colon V \to \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l) = M_{l\times k}(\mathbb{R})$. Since $\operatorname{rank}(\alpha_x) \geq n$, some $n \times n$ minor of $\alpha(x)$ is nonzero. If $U \subseteq M_{l\times k}(\mathbb{R})$ is the subspace of all matrices for which the corresponding minor is nonzero, this is an open subset of $M_{l\times k}(\mathbb{R})$. Then $\phi^{-1}(U)$ is a neighborhood of x that is completely contained in $\mathcal{R}_n$. $\square$

Remark 9.15. Let $\mathbb{Z}_{+\infty}$ denote the set $\mathbb{Z}$ equipped with the topology where the open sets are the intervals (n, ∞) for $n \in \mathbb{Z}$. Note that since $(n, \infty) = [n+1, \infty)$, the open sets can also be described as half-open intervals. The following two conditions on a function $f\colon X \to \mathbb{Z}$ are readily seen to be equivalent:

(a) f is continuous when $\mathbb{Z}$ is given the topology $\mathbb{Z}_{+\infty}$,
(b) for any convergent net $x\colon I \to X$, there exists $\alpha \in I$ such that $f(\lim_I x) \leq f(x_i)$ for all $i \geq \alpha$.

(Recall that a net is simply a function whose domain is a directed set, and the notion of convergent net is the evident one; a standard result of point-set topology is that a function is continuous if and only if it sends convergent nets to convergent nets [Mu, Section 29, Supplementary Exercise 7]). When these properties are satisfied the function f is said to be **lower semi-continuous**. Briefly, in a lower semi-continous function the value on the limit of a convergent sequence can jump down—but not up—from the limiting values in the sequence. Proposition 9.14 says that the map $x \mapsto \operatorname{rank}(\alpha_x)$ is lower semi-continuous. We leave the reader to think about the inverted notion of upper semi-continuity, and the fact that the map $x \mapsto \operatorname{nullity}(\alpha_x)$ is upper semi-continuous.

Proposition 9.16. *Let $E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ be an exact sequence of vector bundles. Then $\operatorname{im}\alpha$ (which equals $\ker\beta$) is a vector bundle.*

Proof. Using Proposition 9.3 it suffices to prove that α has constant rank on each connected component of X. Without loss of generality we can assume that X is connected. Since the question is local on X, we can assume that E, F, and G are all trivial bundles. Let $n = \operatorname{rank}(F)$.

Pick an $x \in X$ and let $p = \operatorname{rank}(\alpha_x)$. Then $\operatorname{rank}(\beta_x) = n - p$ by exactness. Let $U = \{z \in X \mid \operatorname{rank}(\alpha_z) \geq p\}$ and $V = \{z \in X \mid \operatorname{rank}(\beta_z) \geq n - p + 1\}$. Note that exactness implies that $U = X - V$. But both U and V are open by Proposition 9.14, which means they are also both closed. By connectedness, U is either empty or the whole of X. Since $x \in U$, we must have $U = X$.

A similar argument proves that $\{z \in X \mid \operatorname{rank}(\alpha_z) \leq p\} = X$. So for every $z \in X$ we have $p \leq \operatorname{rank}(\alpha_z) \leq p$; that is, the rank of α is constant on X. □

If $E_\bullet$ is a chain complex of vector bundles on X and $x \in X$, write $(E_x)_\bullet$ for the chain complex of vector spaces formed by the fibers over x. Define the **support** of $E_\bullet$, denoted $\operatorname{Supp} E_\bullet$, to be the subspace $\{x \in X \,|\, (E_x)_\bullet \text{ is not exact}\} \subseteq X$. We will occasionally write $\operatorname{Supp}_i E_\bullet$ for $\{x \in X \,|\, H_i((E_x)_\bullet) \neq 0\}$. Note that $\operatorname{Supp} E_\bullet = \bigcup_i \operatorname{Supp}_i E_\bullet$.

Proposition 9.17. *Let $E_\bullet$ be a chain complex of vector bundles on X. Then for any $i \in \mathbb{Z}$, the subspace $\operatorname{Supp}_i E_\bullet$ is closed in X. If $E_\bullet$ is a bounded chain complex then $\operatorname{Supp} E_\bullet$ is closed.*

Proof. We will prove that $X - \operatorname{Supp}_i E_\bullet$ is open, so assume x belongs to this set. Write the maps in the chain complex as

$$E_{i+1} \xrightarrow{\alpha} E_i \xrightarrow{\beta} E_{i-1}.$$

Let $n = \operatorname{rank}_x(E_i)$, $a = \operatorname{rank}_x(\alpha)$, and $b = \operatorname{rank}_x(\beta)$. Since the complex is exact at x in the ith spot we have $a + b = n$. By Proposition 9.14 applied twice, there is an open neighborhood U of x such that $\operatorname{rank}_y(\alpha) \geq a$ and $\operatorname{rank}_y(\beta) \geq b$ for all $y \in U$. Then we can write

$$a \leq \operatorname{rank}_y(\alpha) \leq n - \operatorname{rank}_y(\beta) \leq n - b$$

where the middle inequality follows from the fact that $(E_y)_\bullet$ is a chain complex. Since $a = n - b$ all the inequalities are in fact equalities, and so we have exactness at y for all $y \in U$. That is, $U \subseteq X - \operatorname{Supp}_i E_\bullet$.

The final statement follows from the fact that $\operatorname{Supp} E_\bullet$ is a finite union of the $\operatorname{Supp}_i E_\bullet$ spaces. □

∘ Exercises ∘

10. Swan's Theorem

In this section we explore our first connection between topology and algebra. We will see that vector bundles are closely related to projective modules.

When X is a space let $C(X)$ denote the ring of continuous functions from X to $\mathbb{R}$, where the addition and multiplication are pointwise. Recall that if $E \to X$ is a family of vector spaces, then $\Gamma(E)$ denotes the vector space of sections. In addition to being a vector space, it is easy to see that this is actually a module over $C(X)$: if $f \in C(X)$ and $s \in \Gamma(E)$ then fs is the section $x \mapsto f(x)s(x)$. The assignment $E \mapsto \Gamma(E)$ gives a functor from the category $\mathsf{FamVS}(X)$ to $C(X)$-modules.

Proposition 10.1. *Suppose that $E' \xrightarrow{i} E \xrightarrow{p} E''$ are maps in $\mathsf{FamVS}(X)$ where i is the kernel of p. Then $0 \to \Gamma(E') \to \Gamma(E) \to \Gamma(E'')$ is exact.*

Proof. We have the pullback diagram

$$\begin{tikzcd} E' \arrow[r, "i"] \arrow[d] & E \arrow[d, "p"] \\ 0_X \arrow[r, "\zeta"] & E''. \end{tikzcd}$$

This is a pullback both in $\mathsf{FamVS}(X)$ and in $\mathcal{T}op$, since the limits in these categories are the same. The conclusions about $\Gamma(-)$ follow immediately from categorical diagram chasing. $\square$

Remark 10.2. Note in particular that Proposition 10.1 applies when $0 \to E' \to E \to E'' \to 0$ is a fiberwise exact sequence of vector bundles over X, by ????.

If $E \to X$ is a vector bundle then of course the modules of the form $\Gamma(E)$ are not just arbitrary $C(X)$-modules; there is something special about them. It is easiest to say what this is under some assumptions on X:

Proposition 10.3. *If X is compact and Hausdorff, and E is a vector bundle over X, then $\Gamma(E)$ is a finitely-generated, projective module over $C(X)$.*

Proof. By Proposition 9.10 we can embed E into a trivial bundle $\underline{N}$. This embedding has constant rank on each connected component, so by Proposition 9.3 the quotient Q is also a vector bundle. So we have the exact sequence $0 \to E \rightarrowtail \underline{N} \twoheadrightarrow Q \to 0$ of vector bundles on X. Now apply $\Gamma(-)$, which yields the exact sequence

$$0 \to \Gamma(E) \to \Gamma(\underline{N}) \to \Gamma(Q)$$

of $C(X)$-modules. This much is for free. But by Proposition 9.2 the map $\underline{N} \to Q$ has a splitting, and this splitting shows that $\Gamma(\underline{N}) \to \Gamma(Q)$ is split-surjective. So

$$\Gamma(E) \oplus \Gamma(Q) \cong \Gamma(\underline{N}) = C(X)^n.$$

That is, $\Gamma(E)$ is a direct summand of a free module; hence it is projective. $\square$

For the rest of this section we will assume that our base spaces are compact and Hausdorff. Let $\mathsf{Vect}(X)$ denote the category of vector bundles over X, and let $C(X)$-Mod denote the category of modules over the ring $C(X)$. Let $C(X)$-proj denote the full subcategory of finitely-generated, projective modules. Then Γ is a functor $\mathsf{Vect}(X) \to C(X)$-$\mathsf{proj}$. It is proven in [Sw] that this is actually an equivalence:

Theorem 10.4 (Swan's Theorem). *Let X be a compact, Hausdorff space. Then*
$$\Gamma\colon \mathsf{Vect}(X) \longrightarrow C(X)\text{-}\mathsf{proj}$$
is an equivalence of categories.

Remark 10.5. Theorem 10.4 is also known as the "Serre-Swan theorem", as Serre had earlier proven the analogous result in algebraic geometry. The content of Swan's paper was to demonstrate that the basic framework provided by Serre could also be made to work in the purely topological context. For more on the algebraic geometry side of things, see Section 19 below.

To prove Swan's theorem we need to verify two things:

- Every finitely-generated projective over $C(X)$ is isomorphic to $\Gamma(E)$ for some vector bundle E.
- For every two vector bundles E and F, the induced map
$$\Gamma\colon \operatorname{Hom}_{\mathsf{Vect}(X)}(E,F) \to \operatorname{Hom}_{C(X)}(\Gamma E, \Gamma F)$$
is a bijection.

That is to say, we need to prove that Γ is surjective on isomorphism classes, and is fully faithful. Here is the first part:

Proposition 10.6. *If X is paracompact Hausdorff and P is a finitely-generated projective module over $C(X)$, then $P \cong \Gamma(E)$ for some vector bundle $E \to X$.*

Proof. Choose a surjection $p\colon C(X)^n \twoheadrightarrow P$. Since P is projective, there is a splitting χ. Then $e = \chi p$ satisfies $e^2 = e$, and P is isomorphic to $\operatorname{im}(e)$.

Since $e\colon C(X)^n \to C(X)^n$ we can represent e by an $n \times n$ matrix whose elememts are in $C(X)$. Denote the entries of this matrix as e_{ij}. Note that for any $x \in X$ we can evaluate all these functions at x to get an element $e(x) \in M_{n\times n}(\mathbb{R})$. In this way we can regard e as a continuous map $X \to M_{n\times n}(\mathbb{R})$.

Define a map of vector bundles $\alpha\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ by the formula $\alpha(x,v) = e(x)\cdot v$. Then the sequence
$$\underline{n} \xrightarrow{\alpha} \underline{n} \xrightarrow{1-\alpha} \underline{n}$$
is exact in the middle. Let $E = \operatorname{im}(\alpha)$, which by Proposition 9.16 is a vector bundle on X; the proof of that lemma also shows that α has locally constant rank. We claim that $\Gamma(E) \cong P$. To see this, consider the following diagram of vector bundles:
$$\begin{tikzcd}
\ker\alpha \arrow[r, tail] & \underline{n} \arrow[rr, "\alpha"] \arrow[dr, two heads] & & \underline{n} \\
& & \operatorname{im}\alpha. \arrow[ur, hook] &
\end{tikzcd}$$
The map $\underline{n} \to \operatorname{im}\alpha$ is split by Proposition 9.2, because X is paracompact. Applying Γ to the above diagram gives
$$\begin{tikzcd}
\Gamma(\ker\alpha) \arrow[r, tail] & C(X)^n \arrow[rr, "e"] \arrow[dr, two heads] & & C(X)^n \\
& & \Gamma(\operatorname{im}\alpha). \arrow[ur, hook] &
\end{tikzcd}$$
The sequence $0 \to \Gamma(\ker\alpha) \to C(X)^n \to \Gamma(\operatorname{im}\alpha) \to 0$ is exact because it was split-exact before applying Γ, and the identification $\Gamma(\ker\alpha) = \ker(\Gamma\alpha)$ shows that

$\Gamma(\ker\alpha)$ is the kernel of e. It now follows that $\Gamma(\operatorname{im}\alpha)$ is isomorphic to the image of e, which is P. □

Our next goal is to prove that Γ is fully faithful. To do this, it is useful to relate the fibers E_x of our bundle to an algebraic construction based on the module $\Gamma(E)$. For each $x \in X$ consider the evaluation map $\mathrm{ev}_x \colon C(X) \to \mathbb{R}$, and let m_x be the kernel. The ideal $m_x \subseteq C(X)$ is maximal, since the quotient is a field.

Note that we have the evaluation map $\mathrm{ev}_x \colon \Gamma(E) \to E_x$. This map clearly sends the submodule $m_x\Gamma(E)$ to zero so that we get the induced map $\Gamma(E)/m_x\Gamma(E) \to E_x$. In reasonable cases this map is an isomorphism:

Lemma 10.7. *Assume that X is paracompact Hausdorff. Then for any vector bundle $E \to X$ and any $x \in X$, the map $\mathrm{ev}_x \colon \Gamma(E)/m_x\Gamma(E) \xrightarrow{\cong} E_x$ is an isomorphism.*

Proof. We first record the following important fact, which we label (*): if s is a section of E defined on some neighborhood U of x, then there exists a section s' defined on all of X such that s and s' agree on some (potentially smaller) neighborhood of x. To see this, first choose a neighborhood V of x such that $\overline{V} \subseteq U$ (this exists because X is normal). By Urysohn's Lemma there is a continuous function $f \colon X \to \mathbb{R}$ such that $f|\overline{V} = 1$ and $f|_{X-U} = 0$. The assignment $z \mapsto f(z) \cdot s(z)$ is readily checked to be a continuous section of E that agrees with s on V.

To prove surjectivity of ev_x, let $v \in E_x$. Since E is locally trivial, one can find a section s defined locally about x such that $s(x) = v$. By principle (*) there is a section s' defined on all of X that agrees with s near x; in particular, $s'(x) = v$.

For injectivity we must work a little harder. Suppose that $s \in \Gamma(E)$ and $s(x) = 0$. We must prove that $s \in m_x\Gamma(E)$. Choose independent sections $e_1, \dots, e_n$ defined on a neighborhood U of x. Fact (*) says that by replacing U by a smaller neighborhood of x we can assume that the sections are defined on all of X (but only independent on U).

Using that $e_1(y), \dots, e_n(y)$ is a basis for E_y when $y \in U$, we can write $s(y) = a_1(y)e_1(y) + \cdots + a_n(y)e_n(y)$ for uniquely defined numbers $a_1(y), \dots, a_n(y) \in \mathbb{R}$. The functions a_i are continuous (under the local trivialization given by the e_i's the a_i are just compositions of s with projection operators). Regarding the a_i's as local sections of the trivial bundle $X \times \mathbb{R}$, (*) shows we may assume the a_i's are defined on all of X (again, we might need to replace U with a smaller neighborhood here). Since $s(x) = 0$ note that $0 = a_1(x) = a_2(x) = \cdots = a_n(x)$.

Let $t = s - a_1e_1 - \cdots - a_ne_n \in \Gamma E$. Note that t vanishes throughout the neighborhood U of x. Again using the Urysohn Lemma, choose a continuous function $b \colon X \to \mathbb{R}$ such that $b(x) = 0$ and $b|_{X-U} = 1$. So $t = bt$. Then $s = t + \sum a_ie_i = bt + \sum a_ie_i \in m_x\Gamma(E)$, since b and all the a_i are in m_x. □

Remark 10.8. Lemma 10.7 should be thought of as giving an algebraic construction of the geometric fiber E_x. We already encountered this idea back in Section 3 and will develop it in more detail in Section 19.

Proposition 10.9. *Assume that X is paracompact Hausdorff. Then for any vector bundles E and F over X, the map $\Gamma \colon \mathrm{Hom}_{\mathrm{Vect}(X)}(E,F) \to \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$ is a bijection.*

Proof. First of all, it is easy to check this when E and F are both trivial. A map of vector bundles $X \times \mathbb{R}^k \to X \times \mathbb{R}^l$ is uniquely specified by a map $X \to M_{l\times k}(\mathbb{R})$, and likewise a map of $C(X)$-modules $C(X)^k \to C(X)^l$ is specified by an $l \times k$ matrix with entries in $C(X)$. One observes that continuous maps $X \to M_{l\times k}(\mathbb{R})$ bijectively correspond with $l \times k$ matrices with entries in $C(X)$.

For the general case, consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Vect}(X)}(E,F) \arrow[r, "\Gamma"] \arrow[d, hook] & \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F) \arrow[d] \\
\prod_{x\in X} \mathrm{Hom}(E_x, F_x) \arrow[r, "\cong"] & \prod_{x\in X} \mathrm{Hom}(\Gamma E/m_x\Gamma E, \Gamma F/m_x\Gamma_F).
\end{tikzcd}$$

The bottom horizontal map is an isomorphism by Lemma 10.7. The left vertical arrow sends a bundle map $\alpha\colon E \to F$ to the collection of its restrictions to each fiber; surely this map is an injection. It follows at once that Γ is also an injection.

Next we show that the right vertical map is injective. Suppose $\beta \in \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$ is sent to zero. If $\beta \neq 0$ then there is an $s \in \Gamma E$ such that $\beta(s) \neq 0$. Then $(\beta s)(x) \neq 0$ for some $x \in X$. The square

$$\begin{tikzcd}
\Gamma E \arrow[r, "\beta"] \arrow[d, "\mathrm{ev}_x"'] & \Gamma F \arrow[d, "\mathrm{ev}_x"] \\
\Gamma E/m_x\Gamma E \arrow[r, "\bar{\beta}"] & \Gamma F/m_x\Gamma F
\end{tikzcd}$$

immediately shows that $\bar{\beta}(s(x))$ cannot be zero, which is a contradiction. So indeed the right vertical map is injective.

It remains to show that the top horizontal map (labelled Γ) in our diagram is a surjection, so let $\beta \in \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$. We can apply the right vertical arrow to β, and then find a unique preimage in $\prod_x \mathrm{Hom}(E_x, F_x)$ using that the bottom map is an isomorphism. This gives us a map of sets $\alpha\colon E \to F$, by defining it on each of the fibers. We need to prove that α is continuous. However, this is a local question: so it suffices to do so in the case that E and F are trivial, and this case has already been verified. So we have produced a bundle map $\alpha\colon E \to F$ whose restriction to each fiber agrees with the map β. Then $\Gamma\alpha$ and β are sent to the same object under the right vertical map, therefore they must be equal. $\square$

Note that we have now completed the proof of Swan's Theorem, via Propositions 10.6 and 10.9.

10.10. **Variants of Swan's Theorem.** While Swan's theorem is very pretty, it is unfortunate that the rings $C(X)$ are quite large and unwieldy from an algebraic perspective. For example, these are typically non-Noetherian: choose an infinite descending sequence of sets $X \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and look at the ideals of functions vanishing on each A_i. In many cases, however, a projective coming from Swan's Theorem can be seen to be extended from a projective over a smaller, more manageable ring. As one example, let us consider the ring R of *polynomial* functions on the 2-sphere: $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1) \subseteq C(S^2)$. Let T be the module appearing as the kernel in the following short exact sequence:

$$0 \longrightarrow T \longrightarrow R^3 \xrightarrow{[x\ y\ z]} R \longrightarrow 0.$$

Note that the map $R^3 \to R$ is split by the map $1 \mapsto (x, y, z)$, so T is projective. By applying the functor $(-) \otimes_R C(S^2)$ one sees that $T \otimes_R C(S^2)$ is the $C(S^2)$-module corresponding to the tangent bundle of S^2 under Swan's Theorem.

It is reasonable to regard the finitely-generated projective modules over R as the "algebraic" vector bundles over S^2, and this is exactly what happens in the context of algebraic geometry. Given an affine scheme $X = \operatorname{Spec} S$, the category of algebraic vector bundles over X is defined in such a way that it is equivalent (in some sense tautologically) to the category of finitely-generated S-projectives.

As a slight generalization of the previous example, let R be a finitely-generated $\mathbb{C}$-algebra that is an integral domain and let $X = \operatorname{Spec} R$. We can consider the topological space $X_{\mathbb{C}}$ of $\mathbb{C}$-valued points in X with its analytic topology, and we get an inclusion $R \hookrightarrow C(X_{\mathbb{C}})$. From this we get a functor $P \mapsto P_{\mathbb{C}} = P \otimes_R C(X_{\mathbb{C}})$ from the category of finitely-generated projective R-modules into the category of finitely-generated projective $C(X_{\mathbb{C}})$-modules, the latter of which is equivalent to the category of complex vector bundles on $X_{\mathbb{C}}$ by Swan's Theorem. For example, when $R = \mathbb{C}[x_1, \ldots, x_n]$ then every finitely-generated R-projective gives us an associated complex vector bundle on $\mathbb{C}^n$. This technique of passing from algebra into topology will be used often.

Exercise 10.11. Keeping the above discussion in mind, revisit the examples of Section 3 and think about how they fit into this perspective.

Exercise 10.12. Check that Swan's Theorem holds for any paracompact Hausdorff space having the property that every vector bundle is trivializable on a finite open cover. Using the results of the next section, check that this property holds for all paracompact Hausdorff spaces that are homotopy equivalent to a finite CW-complex. [By ??? this includes all algebraic varieties over $\mathbb{C}$.]

11. Homotopy invariance of vector bundles

For a fixed n, let $\operatorname{Vect}_n(X)$ denote the set of isomorphism classes of vector bundles on X. It turns out that when X is a finite complex this set is always countable, and often finite. It actually gives a homotopy invariant of the space X. In this section we prove the homotopy invariance property, and in the next section we will start to compute $\operatorname{Vect}_n(X)$ for some simple spaces X.

Write i_0 and i_1 for the two inclusions $X \hookrightarrow X \times I$ coming from the boundary points of the interval. The key to homotopy invariance is the following result.

Proposition 11.1. *Let X be paracompact Hausdorff, and let $E \to X \times I$ be a vector bundle. Then there is an isomorphism $i_0^*(E) \cong i_1^*(E)$.*

Before proving this let us give the evident corollaries:

Corollary 11.2. *Let X be paracompact Hausdorff.*

(a) If $f, g\colon X \to Y$ are homotopic then f^* and g^* give the same map $\operatorname{Vect}_n(Y) \to \operatorname{Vect}_n(X)$, for any $n \geq 0$.
(b) If Y is also paracompact Hausdorff and $f\colon X \to Y$ is a homotopy equivalence then $f^*\colon \operatorname{Vect}_n(Y) \to \operatorname{Vect}_n(X)$ is a bijection, for all $n \geq 0$.
(c) If X is contractible then all vector bundles on X are trivializable.

Proof. For (a), let $H\colon X \times I \to Y$ be a homotopy and consider the diagram

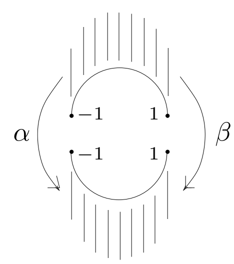
$$\mathrm{Vect}_n(X) \underset{i_1^*}{\overset{i_0^*}{\leftleftarrows}} \mathrm{Vect}_n(X \times I) \xleftarrow{H^*} \mathrm{Vect}_n(Y).$$

One of the compositions is f^*, the other is g^*, and Proposition 11.1 says that the two compositions are the same.

Parts (b) and (c) are simple consequences of (a). □

Example 11.3. To give an idea of how we will apply these results, let us think about vector bundles on S^1. Divide S^1 into an upper hemisphere D_+ and a lower hemisphere D_-, intersecting in two points. Each of D_+ and D_- are contractible, so any vector bundle will be trivializable when restricted to these subspaces.

Given two elements $\alpha, \beta \in GL_n(\mathbb{R})$, let $E_n(\alpha, \beta)$ be the vector bundle on S^1 obtained by taking $\underline{n}_{D_+}$ and $\underline{n}_{D_-}$ and gluing them together via α and β at the two points on the equator. The considerations of the previous paragraph tell us that every vector bundle on S^1 is of this form (up to isomorphism). The following picture depicts the construction of $E_n(\alpha, \beta)$:



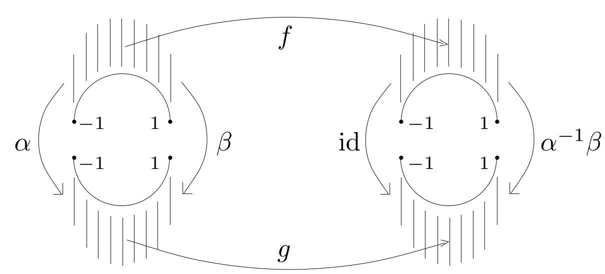
Note that $E_n(\mathrm{id}, \mathrm{id}) = \underline{n}$, and $E_1(\mathrm{id}, -1) = M$ (the Möbius bundle). It is easy to check the following:

(1) $E_n(\alpha, \beta) \cong E_n(\mathrm{id}, \alpha^{-1}\beta)$

(2) $E_n(\mathrm{id}, \beta) \cong E_n(\mathrm{id}, \beta')$ if and only if β and β' are in the same path component of $GL_n(\mathbb{R})$ (or equivalently, if $\det(\beta)$ and $\det(\beta')$ have the same sign).

In (2) we have used the fact that $\pi_0(GL_n(\mathbb{R})) = \mathbb{Z}/2$, with the isomorphism being given by the sign of the determinant.

Let us explain the above facts. The isomorphism in (1) can be depicted as



Here f and g are maps $D_+ \to GL_n(\mathbb{R})$ and $D_- \to GL_n(\mathbb{R})$ giving the isomorphisms on each fiber; compatibility with the gluing requires that we have $g(-1)\alpha = f(-1)$ and $\alpha^{-1}\beta f(1) = g(1)\beta$. This can be achieved by letting $f(t) = I_n$ and $g(t) = \alpha^{-1}$, for all t.

The proof of (2) is a little more subtle. To give an isomorphism $E(\mathrm{id}, \beta) \cong E(\mathrm{id}, \beta')$ we must again specify maps f and g as above, but this time satisfying $g(-1) = f(-1)$ and $\beta' f(1) = g(1)\beta$. If we paste D_+ and D_- together at -1 and identify the resulting interval with $[0,1]$, then we are just asking for a map $h\colon [0,1] \to GL_n(\mathbb{R})$ such that $\beta' h(1) = h(0)\beta$.

If β and β' are in the same path component then choose a path $h\colon I \to GL_n(\mathbb{R})$ such that $h(0) = \beta'$ and $h(1) = \beta$. Since we then have $\beta' h(1) = h(0)\beta$, this yields the desired isomorphism. Conversely, if we have a map h satisfying $\beta' h(1) = h(0)\beta$ then we can rearrange this as $\beta' = h(0)\beta h(1)^{-1}$. The term on the right is path-connected to $h(0)\beta h(0)^{-1}$, using the homotopy $t \mapsto h(0)\beta h(t)^{-1}$. But $h(0)\beta h(0)^{-1}$ has the same determinant as β, so these are also in the same path-component. Hence, β' and β are themselves path-connected and this proves (2).

To summarize, from (1) and (2) it follows that isomorphism types for rank n bundles over S^1 are in bijective correspondence with the path components of $GL_n(\mathbb{R})$. We know that for $n > 0$ there are two such path components, given by the sign of the determinant. They can be represented by the identity matrix and the diagonal matrix J whose diagonal entries are $-1, 1, 1, \ldots, 1$. The corresponding bundles $E_n(\mathrm{id}, \beta)$ are $\underline{n}$ and $M \oplus \underline{(n-1)}$.

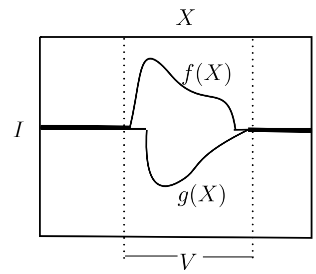
Most of the basics of this discussion generalize readily from S^1 to S^k. We discuss this in Proposition 12.3.

The methods of the above example apply in much greater generality, and with little change allow one to get control over vector bundles on any suspension. We will return to this topic in Section 12.

At this point let us now give the proof of Proposition 11.1, starting with an overview of the basic approach. For a bundle on $E \to X \times I$ one can look at the slices $E|_{X\times\{t\}}$ and try to track potential "twisting" that develops as t increases. We need to differentiate between the twisting that was already present in $E|_{X\times\{0\}}$ and twisting that is getting added (or subtracted) as the time variable progresses. To this end, it is useful if we can arrange things so that the new twists only occur in sectors of $E_{X\times\{0\}}$ where the bundle was already trivial. The heart of the proof is reducing to this situation, which will then be handled by the following technical lemma:

Lemma 11.4. *Let X be a Hausdorff space and suppose given maps $f, g\colon X \to X \times I$ that are sections of the projection map $X \times I \to X$. Let $A = \{x \in X \mid f(x) = g(x)\}$. Suppose there exists an open subset V of X such that $\overline{(X - A)} \subseteq V$. Finally, assume that $E \to X \times I$ is a vector bundle that is trivializable on $V \times I$. Then $f^*E \cong g^*E$.*

The statement of the lemma is a mouthful, so let us explain a bit. The following picture shows the two images $f(X)$ and $g(X)$ inside $X \times I$, with the intersection $f(A) = g(A)$ drawn in bold. The region where the two sections disagree is contained inside of $V \times I$, where we assume the bundle E is trivializable. The proof will construct a fiberwise isomorphism $E|_{f(X)} \to E|_{g(X)}$ that equals the identity on the portion of the bundles over the bold region.



Proof of Lemma 11.4. Let $X_f = f(X)$ and $X_g = g(X)$. The maps $f\colon X \to X_f$ and $X \to X_g$ are homeomorphisms, and we have isomorphisms of bundles $f^*E \cong E|_{X_f}$ and $g^*E \cong E|_{X_g}$. Write $B\colon X_f \to X_g$ for the homeomorphism $g \circ f^{-1}$.

Note that our hypotheses say that the points where X_f and X_g differ are concentrated in the region $V \times I$, where E is trivializable.

The subset A is closed because X is Hausdorff. Let $C = f(A) = g(A)$, $D_f = f\left(\overline{X - A}\right)$, and $D_g = g\left(\overline{X - A}\right)$. We have $X_f = C \cup D_f$ and $X_g = C \cup D_g$, and the subsets appearing in the unions are all closed. Observe that B maps D_f homeomorphically to D_g. Also note that $D_f \cap D_g \subseteq C$.

For $S \subseteq X \times I$ write $E_S = E|_S$ for brevity. Then $E_{X_f} = E_C \cup E_{D_f}$ and $E_{X_g} = E_C \cup E_{D_g}$ are again decompositions into closed sets. We will define a map of bundles $\phi\colon E_{X_f} \to E_{X_g}$ that covers B:

$$\begin{tikzcd} E_{X_f} \arrow[r, "\phi"] \arrow[d] & E_{X_g} \arrow[d] \\ X_f \arrow[r, "B"] & X_g. \end{tikzcd}$$

Note that $D_f \subseteq V \times I$ and $D_g \subseteq V \times I$. Choose a trivialization $\alpha\colon E_{V\times I} \xrightarrow{\cong} \underline{n}$. This gives the diagram

$$\begin{tikzcd} E_{D_f} \arrow[r, dotted, "\phi_1"] \arrow[d, "\alpha"', "\cong"] & E_{D_g} \arrow[d, "\cong"', "\alpha"] \\ D_f \times \mathbb{R}^n \arrow[r, "B\times \mathrm{id}", "\cong"'] & D_g \times \mathbb{R}^n \end{tikzcd}$$

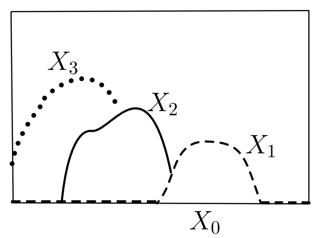
and we define ϕ_1 to be the indicated fill-in. This is clearly an isomorphism of bundles. Moreover, for $x \in D_f \cap D_g$ one has $B(x) = x$ and therefore ϕ_1 is the identity on the fiber over x.

We now have a continuous map $\phi_1\colon E_{D_f} \to E_{D_g} \hookrightarrow E_{X_g}$, and we define $\phi_0\colon E_C \hookrightarrow E_C \cup E_{D_g} = E_{X_g}$ to be the inclusion. We have just seen that ϕ_0 and ϕ_1 agree on the overlap, and so they patch together to define a continuous map $\phi\colon E_{X_f} \to E_{X_g}$. This is an isomorphism on each fiber by construction, and hence an isomorphism of vector bundles. □

Proof of Proposition 11.1. This proof is taken from [Ha2]. Pick an $x \in X$. Using the compactness of I and the definition of vector bundle, we may find a neighborhood $U \subseteq X$ of x and values $0 = a_0 < a_1 < \dots < a_{n-1} < a_n = 1$ such that E is trivial over each $U \times [a_i, a_{i+1}]$. Patching these together gives a trivialization of the vector bundle over $U \times I$.

Now assume for a moment that X is compact. Then we can cover X by open sets $U_1, \dots, U_n$ such that E is trivial over each $U_i \times I$. Choose a partition of unity $\phi_1, \dots, \phi_n$ subordinate to this cover, and set $\beta_0 = 0$, $\beta_i = \phi_1 + \ldots + \phi_i$. Define X_i to be the graph of β_i in $X \times I$. If $f_i \colon X \to X \times I$ is $f_i(x) = (x, \beta_i(x))$, then $X_i = f_i(X)$.

Observe that $\beta_n = 1$ and thus $X_n = X \times \{1\}$, $X_0 = X \times \{0\}$. Also, β_{i-1} and β_i agree except on the support of ϕ_i, whose closure is inside of U_i. Because of this, the pair of maps f_{i-1}, f_i satisfies the hypotheses of Lemma 11.4, for each i. The following picture gives an example of the first four sections X_i; each section agrees with the previous one except for a new "bubble" that appears in a region over the set U_i.



Define $B_i = f_{i-1} \circ f_i^{-1}$. These are homeomorphisms $X_i \to X_{i-1}$ that in terms of the picture can be described as"push down until you hit the next graph".

Lemma 11.4 yields fiberwise isomorphisms ϕ_i making the diagram

$$\begin{array}{ccc} E|_{X_i} & \xrightarrow{\phi_i} & E|_{X_{i-1}} \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{B_i} & X_{i-1} \end{array}$$

commute. Here we have used that X_i and X_{i-1} coincide except over the open set U_i, and that $E|_{U_i \times I}$ is trivial.

Via the identifications $X \cong X_i$, each $E|_{X_i}$ is a vector bundle on X and we have isomorphisms

$$i_1^*(E) = E|_{X_n} \xrightarrow{\cong} E|_{X_{n-1}} \xrightarrow{\cong} E|_{X_{n-2}} \xrightarrow{\cong} \cdots \xrightarrow{\cong} E|_{X_1} \xrightarrow{\cong} E|_{X_0} = i_0^*(E).$$

This gives us what we wanted.

The paracompact case is similar, except for a few details. Let $\mathcal{P}$ be the property that an open subset $U \subseteq X$ is such that E is trivializable on $U \times I$. By Lemma 9.11 there is a countable cover $\{U_i\}$ of X where each U_i has this property. Choose a partition of unity ϕ_i and work as previously to produce fiberwise bundle isomorphisms

$$\cdots \to E_{X_n} \longrightarrow E_{X_{n-1}} \longrightarrow \cdots \longrightarrow E_{X_1} \longrightarrow E_{X_0}.$$

For any particular $x \in X$ the maps on the fiber over x are eventually identities as one moves to the left, and after a finite number of steps one has $(E_{X_n})_x = (E_{X \times \{1\}})_x$ (equality, not just isomorphism). So we have a fiberwise isomorphism of sets $E|_{X \times \{1\}} \to E|_{X_0}$, and it only remains to check continuity. But continuity is a local condition, and by local finiteness of the partition of unity one knows that each $x \in X$ has a neigborhood where the above sequence stabilizes after a finite number of steps. Continuity is then immediate. $\square$

Remark 11.5. The isomorphism $i_0^*(E) \cong i_1^*(E)$ is not canonical, as is clear from the proof of the theorem.

Remark 11.6. We have seen that all bundles on contractible spaces are trivial, and that there is a close connection between vector bundles and projective modules. Recall that when k is a field then $k[x_1, \dots, x_n]$ is the algebraic analog of affine space $\mathbb{A}^n$, and that projectives over this ring correspond to algebraic vector bundles. The analogy with topology is what led Serre to conjecture that all finitely-generated projectives over $k[x_1, \dots, x_n]$ are free, as we discussed in Example 3.7.

We have proven that if E is a vector bundle on $X \times I$ then $i_0^*(E) \cong i_1^*(E)$. It is natural to wonder if this result has a converse, but stating such a thing is somewhat tricky. Here is one possibility: if F and F' are isomorphic vector bundles on X, is there a vector bundle E on $X \times I$ such that $i_0^*(E) \cong F$ and $i_1^*(E) \cong F'$? Unfortunately, this has a trivial answer: yes, just take $E = \pi^*(F)$ where $\pi\colon X \times I \to X$ is the projection. So this phrasing of the question was not very informative.

Here is another possibility: if F and F' are isomorphic vector bundles on X, is there a vector bundle E on $X \times I$ such that $i_0^*(E) = F$ and $i_1^*(E) = F'$? Note the presence of equalities here, as opposed to isomorphisms. This question does not have an obvious answer, but it is also the kind of question that one really doesn't want to be asking: saying that two abstract gadgets are *equal*, rather than just isomorphic, is going to force us down a path that requires us to keep track of too much data.

So we find ourselves in somewhat of a muddle. Perhaps there is an interesting question here, but we don't quite know how to ask it. One approach is to restrict to a class of bundles where "equality" is something we can better control. For example, one can restrict to bundles on X that sit inside of $X \times \mathbb{R}^\infty$. Here, finally, we have an interesting question: if F and F' are two such bundles, which are abstractly isomorphic, is there a bundle E inside of $(X \times I) \times \mathbb{R}^\infty$ that restricts to F and F' at times 0 and 1? The answer is yes, and we will discuss this further in Section 13.

11.7. **Isomorphisms and homotopy invariance.** Let E and F be two vector bundles on a space X, and let $u, v\colon E \to F$ be two bundle maps. Let $\pi\colon X \times I \to X$ be the projection. Define a **homotopy** from u to v to be a bundle map $\pi^*E \to \pi^*F$ that restricts to u on $X \times 0$ and to v on $X \times 1$. Intuitively, this is the same as deforming u to v through bundle maps (we will make this precise in a moment).

For arbitrary bundle maps the set of homotopy classes isn't very interesting, since via scalar multiplication we find that every bundle map is homotopic to the zero map. It is more interesting to put restrictions on the maps, and to require that the homotopies respect these restrictions. For example, we can require u and v to be isomorphisms and then that the homotopy $\pi^*E \to \pi^*F$ also be an isomorphism. When we talk about homotopies between bundle isomorphisms this is always what we will mean.

The following result is a bit silly but will be useful later on. We will also outline a massive generalization in the exercises.

Proposition 11.8. *Let X be a paracompact Hausdorff space, and let E and F be bundles on $X \times I$. Any bundle isomorphism $u_0\colon E|_{X\times 0} \to F|_{X\times 0}$ may be extended to a bundle isomorphism $u\colon E \to F$, and given any two such extensions their restrictions to $X \times 1$ are homotopic (through isomorphisms).*

Proof. Note that the second statement is trivial, because by definition an extension is a homotopy from $u|_{X\times 1}$ to $u|_{X\times 0}$. Given two such extensions u and u', we simply glue them together at $u|_{X\times 0}$ to obtain a homotopy from $u|_{X\times 1}$ to $u'|_{X\times 1}$.

For the first statement, by Corollary 11.2(b) the map $\pi^*\colon \mathrm{Vect}_n(X) \to \mathrm{Vect}_n(X \times I)$ is a bijection. So there exist bundles E' and F' on X together with isomorphisms $E \cong \pi^* E'$ and $F \cong \pi^* F'$, and therefore it is enough to prove the result for $\pi^* E'$ and $\pi^* F'$. In this case, the first statement simply claims that any bundle map $E' \to F'$ extends to a bundle map $\pi^* E' \to \pi^* F'$, which is obvious. $\square$

◦ Exercises ◦

Exercise 11.9. Let E and F be rank n bundles on X, and recall the bundle $\underline{\mathrm{Hom}}(E, F) \to X$ whose fiber over a point $x \in X$ is $\mathrm{Hom}(E_x, F_x)$. Define $\underline{\mathrm{Iso}}(E, F) \subseteq \underline{\mathrm{Hom}}(E, F)$ to be the subspace whose fibers over x are $\mathrm{Iso}(E_x, F_x)$. Finally, define $\mathrm{Iso}_X(E, F)$ to be the space of sections of $\underline{\mathrm{Iso}}(E, F) \to X$, equipped with the compact-open topology. Said differently, $\mathrm{Iso}_X(E, F)$ is defined by the pullback diagram

$$\begin{tikzcd} \mathrm{Iso}(E,F) \arrow[r] \arrow[d] & \underline{\mathrm{Iso}}(E,F)^X \arrow[d] \\ * \arrow[r, "\mathrm{id}"] & X^X \end{tikzcd}$$

where the function spaces have the usual compact-open topology.

For the following exercises assume that X is locally compact and Hausdorff, so that one has the natural adjunction isomorphism $\mathcal{T}op(Z, Y^X) \cong \mathcal{T}op(Z \times X, Y)$.

(a) If $E = \underline{n} = F$ verify that $\mathrm{Iso}_X(E, F) \cong GL_n(\mathbb{R})^X$.
(b) If Z is a space verify that $\mathrm{Iso}_X(E, F)^Z \cong \mathrm{Iso}_{Z\times X}(\pi^* E, \pi^* F)$ where $\pi\colon Z \times X \to X$ is the projection.
(c) Let $u, v\colon E \to F$ be bundle isomorphisms. Prove that u and v are homotopic in the sense defined in Section 11.7 if and only if there is a path $I \to \mathrm{Iso}_X(E, F)$ from u to v.
(d) Given a map $g\colon Y \to X$ explain how to get an induced map of spaces $\mathrm{Iso}_X(E, F) \to \mathrm{Iso}_Y(g^* E, g^* F)$ that has the evident behavior on points.
(e) Let J and M be bundles on $X \times I$, and assume that X is paracompact Hausdorff. Prove that the restriction maps
$$\mathrm{res}_i\colon \mathrm{Iso}_{X\times I}(J, M) \to \mathrm{Iso}_{X\times\{i\}}(J|_{X\times\{i\}}, M|_{X\times\{i\}})$$
(induced by the inclusions $X \times \{i\} \hookrightarrow X \times I$ as in (d)) are acyclic fibrations, for $i \in \{0, 1\}$. [Hint: It suffices to prove this when $J = \pi^* E$ and $M = \pi^* F$ for some bundles E and F on X. Relate the restriction maps to the evaluation maps $\mathrm{ev}_i\colon W^I \to W$ for a certain space W.]
(f) It follows from (e) that the restriction maps are surjective and induce bijections on path components. Relate this statement to Proposition 11.8.

12. Vector bundles on spheres

In this section we explore the set of isomorphism classes $\mathrm{Vect}_n(S^k)$ for various values of k and n. Our goal is to compute as much as we can by elementary means and see how far this gets us.

In the end there are two important points to keep in mind. First, for a fixed k these sets stabilize for $n \gg 0$. Secondly, Bott was able to compute these stable values completely and found an 8-fold periodicity (with respect to k) in the case of real vector bundles, and a 2-fold periodicity in the case of complex bundles. Bott's periodicity theorems are of paramount importance in modern algebraic topology and will be discussed at length later in the book.

12.1. **The clutching construction.** Let X be a pointed space, and let C_+ and C_- be the positive and negative cones in ΣX. Fix $n \geq 0$. For a map $f\colon X \to GL_n(\mathbb{R})$, let $E_n(f)$ be the vector bundle obtained by gluing $\underline{n}|_{C_+}$ and $\underline{n}|_{C_-}$ via the map f (we use Corollary 8.27(b) here). Precisely, if $x \in X$ and v belongs to the fiber of $\underline{n}_{C_+}$ over x then we glue v to $f(x)\cdot v$ in the fiber of $\underline{n}_{C_-}$ over x. This procedure for constructing vector bundles on ΣX is called *clutching*, and when X is paracompact Hausdroff every bundle on ΣX arises in this way (use Corollary 11.2(c) to see that the bundle is trivial on the two cones). By changing basis in one of the trivial bundles we can always require $f(*) = I_n$; that is, we can require f to be a based map.

Proposition 12.2. *Let X be a paracompact Hausdorff space with a chosen basepoint. If $f, f'\colon X \to GL_n(\mathbb{R})$ are homotopic relative to the basepoint, then $E_n(f) \cong E_n(f')$. Therefore we have a well-defined map $E_n\colon [X, GL_n(\mathbb{R})]_* \to \mathrm{Vect}_n(\Sigma X)$, and in addition this map is surjective.*

Proof. Given a homotopy H between f and f', we use H to make a bundle E on $(\Sigma X) \times I$ by gluing together trivial bundles on $C_+ \times I$ and $C_- \times I$ via H. Then $E|_{X\times\{0\}} \cong E_n(f)$ and $E|_{X\times\{1\}} \cong E_n(f')$ by construction, so $E_n(f) \cong E_n(f')$ by Proposition 11.1.

To see that the map E_n is surjective, just note that a given vector bundle $\mathcal{E}$ on ΣX can be trivialized on C_+X and C_-X by Corollary 11.2(c). Choose such isomorphisms $\phi_+\colon \mathcal{E}|_{C_+X} \to (C_+X) \times \mathbb{R}^n$ and $\phi_-\colon \mathcal{E}|_{C_-X} \to (C_-X) \times \mathbb{R}^n$, then let F be the composite isomorphism

$$X \times \mathbb{R}^n \xrightarrow{(\phi_+)^{-1}} \mathcal{E}|_X \xrightarrow{\phi_-} X \times \mathbb{R}^n.$$

By Proposition 8.7 this has the form $F(x, v) = (x, f_x(v))$ for a unique map $f\colon X \to GL_n(\mathbb{R})$.

The maps ϕ_+ and ϕ_- can be regarded as vector bundle maps $\mathcal{E}|_{C_+X} \to E_n(f)|_{C_+X}$ and $\mathcal{E}|_{C_-X} \to E_n(g)|_{C_-X}$, and by our construction of f they agree on $\mathcal{E}|_X$. So they induce a vector bundle map $\phi\colon \mathcal{E} \to E_n(f)$ that is a fiberwise isomorphism, and therefore an isomorphism of vector bundles by Exercise 8.14. $\square$

It is natural to guess that $E_n\colon [X, GL_n(\mathbb{R})]_* \to \mathrm{Vect}_n(\Sigma X)$ is an isomorphism, but this is not quite true. It does turn out to be true when $\mathbb{R}$ is replaced by $\mathbb{C}$ and we are dealing with complex vector bundles, but over the reals there is an issue. To see this, observe that $GL_n(\mathbb{R})$ acts on $[X, GL_n(\mathbb{R})]_*$ by pointwise conjugation: if $A \in GL_n(\mathbb{R})$ and $f\colon X \to GL_n(\mathbb{R})$ then define $A.f$ by $(A.f)(x) = A\cdot f(x)\cdot A^{-1}$. If A and B in $GL_n(\mathbb{R})$ are connected by a path then $A.f$ and $B.f$ are based homotopic, and so the action factors through $\pi_0 GL_n(\mathbb{R}) = \mathbb{Z}/2$. For the analogous story over the complex numbers note that the action factors through $\pi_0 GL_n(\mathbb{C}) = *$, and so the action is trivial.

Notice that $E_n(f) \cong E_n(A.f)$ for all A and f; the isomorphism from the former to the latter just consists of left multiplication by A on the trivial bundles $C_+X \times \mathbb{R}^n$ and $C_-X \times \mathbb{R}^n$, which are readily checked to be compatible with the gluing maps. So our E_n map factors as

$$[X, GL_n(\mathbb{R})]_*/(\mathbb{Z}/2) \twoheadrightarrow \mathrm{Vect}_n(\Sigma X)$$

where in the domain we factor out by the group action of $\mathbb{Z}/2$.

Proposition 12.3. *The above maps $E_n^{\mathbb{R}} \colon [X, GL_n(\mathbb{R})]_*/(\mathbb{Z}/2) \to \mathrm{Vect}_n(\Sigma X)$ and $E_n^{\mathbb{C}} \colon [X, GL_n(\mathbb{C})]_* \to \mathrm{Vect}_n^{\mathbb{C}}(\Sigma X)$ are bijections. Moreover, when n is odd the action of $\mathbb{Z}/2$ on $[X, GL_n(\mathbb{R})]_*$ is trivial.*

Proof. Let $f, g \colon X \to GL_n(\mathbb{R})$ be based maps with $E_n(f) \cong E_n(g)$. A choice of isomorphism α amounts to giving maps $\alpha_+ \colon C_+X \to GL_n(\mathbb{R})$ and $\alpha_- \colon C_-X \to GL_n(\mathbb{R})$ such that

$$(*) \qquad g \cdot (\alpha_+|_X) = (\alpha_-|_X) \cdot f.$$

Evaluating at the basepoint gives $\alpha_+(*) = \alpha_-(*)$. Let A denote this element of $GL_n(\mathbb{R})$.

Since $\alpha_+|_X$ has an extension to C_+X there is a homotopy relative to the basepoint between $\alpha_+|_X$ and the constant map with value A. The same holds for $\alpha_-|X$. Then (*) gives that $g \cdot A \simeq A \cdot f$, or $g \simeq A \cdot f \cdot A^{-1}$. This verifies that E_n is an injection when we mod out by the $\mathbb{Z}/2$ conjugation action on the domain.

Finally, we need to prove that the $\mathbb{Z}/2$ action is trivial when n is odd. But in this case the two components of $GL_n(\mathbb{R})$ are represented by I and $-I$, and conjugation by both of these elements is trivial. $\square$

Example 12.4. To see the importance of the $\mathbb{Z}/2$-action in the above result, consider the case where $X = S^1$ and $n = 2$. Here we are dealing with

$$E_2 \colon [S^1, GL_2(\mathbb{R})]_* \to \mathrm{Vect}_2(S^2).$$

We can replace $GL_2(\mathbb{R})$ by its homotopy-equivalent subgroup $O(2)$, and then by $SO(2)$ since any pointed map $S^1 \to O(2)$ must land entirely inside $SO(2)$. Let us also identify $S^1 \cong SO(2)$ via $e^{i\theta} \mapsto \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$. We of course know that $[S^1, S^1]_* \cong \mathbb{Z}$ via the degree map. So $[S^1, SO(2)]_* \cong \mathbb{Z}$ with $n \in \mathbb{Z}$ corresponding to the map

$$f_n \colon S^1 \to SO(2), \quad e^{i\theta} \mapsto \left[\begin{smallmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{smallmatrix}\right].$$

The nontrivial path component of $GL_2(\mathbb{R})$ is represented by $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Conjugating f_n by A gives

$$e^{i\theta} \mapsto \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{smallmatrix}\right]$$

and this is precisely f_{-n}. So we see that the $\mathbb{Z}/2$-action is nontrivial here, and the map E_2 is not injective before we quotient out by this action.

Exercise 12.5. Think through the isomorphism $E(f_1) \cong E(f_{-1})$ (or more generally, $E(f_n) \cong E(f_{-n})$) for bundles on S^2 until it feels second nature to you.

Our next goal is to apply Proposition 12.3 when X is a sphere S^{k-1}, in order to obtain a classification of bundles on S^k. For rank 2 bundles on S^2 this was the content of Example 12.4, but we want to see how much further we can push those ideas. The proposition gives a bijection $\mathrm{Vect}_n(S^k) \simeq [\pi_{k-1} GL_n(\mathbb{R})]/(\mathbb{Z}/2)$. It will be convenient to replace $GL_n(\mathbb{R})$ with its subgroup O_n. Recall that $O_n \hookrightarrow GL_n(\mathbb{R})$

is a deformation retract, as a consequence of the Gram-Schmidt process. When $k \geq 2$ any based map $S^{k-1} \to O_n$ must actually factor through the connected component of the identity, which is SO_n. So we have

$$\mathrm{Vect}_n(S^k) \cong [\pi_{k-1}GL_n(\mathbb{R})]/(\mathbb{Z}/2) \cong [\pi_{k-1}O_n]/(\mathbb{Z}/2) \cong [\pi_{k-1}SO_n]/(\mathbb{Z}/2)$$

(where the last isomorphism needs $k \geq 2$).

12.6. **Vector bundles on S^1.** For $k = 1$ and $n > 0$ we need to look at $\pi_0 GL_n(\mathbb{R}) = \mathbb{Z}/2$. The conjugation action is trivial here (it is literally the conjugation on the group $\mathbb{Z}/2$, which is trivial since the group is abelian). So we find that $\mathrm{Vect}_n(S^1) \cong \mathbb{Z}/2$, and we have previously seen in Example 11.3 that the two isomorphism classes are represented by $\underline{n}$ and $M \oplus \underline{(n-1)}$ where M is the Möbius bundle.

12.7. **Vector bundles on S^2.** Here we have $\mathrm{Vect}_n(S^2) \cong [\pi_1 SO_n]/(\mathbb{Z}/2)$. When $n = 1$ we have $SO_1 = \mathbb{Z}/2$ and $\pi_1 SO_1 = *$, so all line bundles on S^2 are trivial. We analyzed $n = 2$ in Example 12.4 and found a bijection $\mathrm{Vect}_2(S^2) \cong \mathbb{Z}_{\geq 0}$. We claim that for $n > 2$ one has $\pi_1 SO_n \cong \mathbb{Z}/2$ and the action of $\mathbb{Z}/2$ is trivial, so that we have the following:

Proposition 12.8. $\mathrm{Vect}_n(S^2) \cong \begin{cases} * & \text{\textit{if} } n = 1, \\ \mathbb{Z}_{\geq 0} & \text{\textit{if} } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{\textit{if} } n \geq 3. \end{cases}$

Proof. For $n = 3$ recall that $SO_3 \cong \mathbb{R}P^3$, so that $\pi_1(SO_3) \cong \mathbb{Z}/2$. To see the homeomorphism use the model $\mathbb{R}P^3 \cong D^3/\!\sim$ where the equivalence relation has $x \sim -x$ for $x \in \partial D^3$. Map $D^3 \to SO_3$ by sending a vector v to the rotation of $\mathbb{R}^3$ with axis $\langle v \rangle$, through $|v| \cdot \pi$ radians, in the direction given by a right-hand-rule with the thumb pointed along v. Note that this makes sense even for $v = 0$, since the corresponding rotation is through 0 radians. For $x \in \partial D^3$ this map sends x and $-x$ to the same rotation, and so induces a map $\mathbb{R}P^3 \to SO_3$. This is clearly a continuous bijection, and therefore a homeomorphism since the spaces are compact and Hausdorff.

For $n \geq 4$ one can use the long exact sequence associated to the fibration $SO_{n-1} \hookrightarrow SO_n \twoheadrightarrow S^{n-1}$ to deduce that $SO_{n-1} \hookrightarrow SO_n$ induces an isomorphism on π_1.

To deduce that the conjugation action is trivial, we recall that the nonzero element of $\pi_0 O_n$ is represented by any orthogonal matrix of determinant -1. A convenient choice is the diagonal reflection matrix $A_n = \mathrm{diag}(-1, 1, 1, \ldots, 1)$. We know that conjugation by A_n acts trivially on $\pi_1 SO_n$ when $n > 1$ is odd (see the last sentence of Proposition 12.3), and the same statement for even numbers $n \geq 4$ follows from lifting the generator of $\pi_1 SO_n$ back to $\pi_1 SO_{n-1}$. $\square$

Definition 12.9. *Let $\mathcal{O}(n) \in \mathrm{Vect}_2(S^2)$ be the vector bundle E_{f_n} where $f_n \colon S^1 \to SO_2$ is a map of degree n. Note that $\mathcal{O}(0) \cong \underline{2}$.*

The bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}_{\geq 0}$, give a complete list of the rank 2 bundles on S^2. To get to higher ranks we consider the operation of adding on a trivial line bundle, and note that we have commutative diagrams

$$\begin{tikzcd}
\mathrm{Vect}_2(S^2) \arrow[r, "\oplus 1"] & \mathrm{Vect}_3(S^2) \\
\pi_1(SO_2)/(\mathbb{Z}/2) \arrow[u, "\cong"] \arrow[r, "(i_2)_*"] & \pi_1(SO_3) \arrow[u, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Vect}_{n-1}(S^2) \arrow[r, "\oplus 1"] & \mathrm{Vect}_n(S^2) \\
\pi_1(SO_{n-1}) \arrow[u, "\cong"] \arrow[r, "(i_{n-1})_*"] & \pi_1(SO_n) \arrow[u, "\cong"]
\end{tikzcd}$$

where in the second diagram $n \geq 4$, the bottom maps are induced by the inclusions $i_{n-1}\colon SO_{n-1} \hookrightarrow SO_n$, and we are using the fact that the conjugation action on $\pi_1(SO_n)$ is trivial when $n \geq 3$. We saw in the proof of Proposition 12.8 that $(i_{n-1})_*$ is an isomorphism for $n \geq 4$ and surjective for $n = 3$, and so the same is true for the top maps in the two squares. That is, bundles of rank at least 3 on S^2 all come from bundles of rank 2 by adding on trivial bundles.

The map $\pi_1(SO_2) \to \pi_1(SO_3)$ is readily seen to be the projection $\mathbb{Z} \to \mathbb{Z}/2$ (use the fibration sequence $SO_2 \hookrightarrow SO_3 \to S^2$). This shows that $\mathcal{O}(j) \oplus 1$ is trivial when j is even, and is isomorphic to the nontrivial bundle $\mathcal{O}(1) \oplus 1$ when j is odd.

Putting all of this information together, the following table shows all the vector bundles on S^2:

n	1	2	3	4	5	6
$\mathrm{Vect}_n(S^2)$	$\underline{1}$	$\mathcal{O}(n)$, $n \in \mathbb{Z}_{\geq 0}$	$\underline{3}$, $\mathcal{O}(1) \oplus \underline{1}$	$\underline{4}$, $\mathcal{O}(1) \oplus \underline{2}$	$\underline{5}$, $\mathcal{O}(1) \oplus \underline{3}$	$\cdots$

The operation $(-) \oplus \underline{1}$ moves us from one column of the table to the next, and is completely clear except from column 2 to column 3; as we saw above, there it is given by $\mathcal{O}(j) \oplus \underline{1} \cong \underline{3}$ if j is even, and $\mathcal{O}(j) \oplus \underline{1} \cong \mathcal{O}(1) \oplus \underline{1}$ if j is odd.

To complete our study of these bundles there is one final question that we should answer, namely what happens when one adds two rank 2 bundles (all other sums can be figured out once one knows how to do these):

Theorem 12.10. $\mathcal{O}(j) \oplus \mathcal{O}(k) \cong \begin{cases} \underline{4} & \textit{if } j+k \textit{ is even,} \\ \mathcal{O}(1) \oplus \underline{2} & \textit{if } j+k \textit{ is odd.} \end{cases}$

Proof. Let $f_j\colon S^1 \to SO_2$ and $f_k\colon S^1 \to SO_2$ be the clutching functions for $\mathcal{O}(j)$ and $\mathcal{O}(k)$, respectively. The clutching function for the bundle $\mathcal{O}(j) \oplus \mathcal{O}(k)$ is the map $f_j \oplus f_k\colon S^1 \to SO_4$, where $\oplus$ is the (pointwise) block diagonal sum $SO_2 \times SO_2 \to SO_4$, given by

$$(A, B) \mapsto \begin{bmatrix} A & O \\ 0 & B \end{bmatrix}.$$

We can factor $f_j \oplus f_k = (f_j \oplus f_0) \cdot (f_0 \oplus f_k)$ where $\cdot$ is pointwise multiplication and f_0 is the constant map at the identity. It is a standard fact in topology that the group structure on $[S^1, SO_4]_*$ given by pointwise multiplication agrees with the group structure given by concatenation of loops (this is true with SO_4 replaced by any topological group). Note that the homotopy classes of $f_0 \oplus f_k$ and $f_k \oplus f_0$ are the same, since these clutching functions give rise to isomorphic bundles. So we have

$$[f_j \oplus f_k] = [f_j \oplus f_0] + [f_k \oplus f_0]$$

where this is a statement abouts sums of homotopy classes in $\pi_1(SO_4)$.

But $\pi_1(SO_4) = \mathbb{Z}/2$. The function $f_j \oplus f_0$ is the nontrivial element of $\pi_1 SO_4$ precisely when j is odd, and similarly for $f_k \oplus f_0$. It follows that the sum of these elements is trivial/non-trivial when $j+k$ is even/odd. $\square$

The core argument used in the above proof actually works verbatim in other dimensions, so we record the result below for later use:

Proposition 12.11. *For any n and k, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Vect}_n(S^k) \times \mathrm{Vect}_n(S^k) \arrow[rr, "\oplus"] & & \mathrm{Vect}_{2n}(S^k) \\
(\pi_{k-1}SO_n)/(\mathbb{Z}/2) \times (\pi_{k-1}SO_n)/(\mathbb{Z}/2) \arrow[u, "\cong"] & & (\pi_{k-1}SO_{2n})/(\mathbb{Z}/2) \arrow[u, "\cong"'] \\
\pi_{k-1}SO_n \times \pi_{k-1}SO_n \arrow[r, "\mu"] \arrow[u, two heads] & \pi_{k-1}SO_n \arrow[r, "i_*"] & \pi_{k-1}SO_{2n}. \arrow[u, two heads]
\end{tikzcd}$$

Here μ is the group operation on $\pi_{k-1}SO_n$.

Exercise 12.12. Think through the proof of Proposition 12.11.

12.13. **Vector bundles on $\boldsymbol{S^3}$.** Now we have to calculate $\pi_2 SO_n$. This is trivial for $n \leq 2$ (easy), and for $n = 3$ it also trivial: use $SO_3 \cong \mathbb{R}P^3$ and the fibration sequence $\mathbb{Z}/2 \hookrightarrow S^3 \twoheadrightarrow \mathbb{R}P^3$. Finally, the fibration sequences $SO_{n-1} \hookrightarrow SO_n \twoheadrightarrow S^{n-1}$ now show that $\pi_2 SO_n = 0$ for all n. We have proven

Proposition 12.14. $\mathrm{Vect}_n(S^3) \cong \pi_2(SO_n) \cong 0$. *That is, every vector bundle on S^3 is trivializable.*

12.15. **Vector bundles on $\boldsymbol{S^4}$.** Once again, the first step is to calculate $\pi_3 SO_n$. Eventually one expects to get stuck here, but so far we have been fortunate. The group is trivial for $n \leq 2$, and for $n = 3$ it is $\mathbb{Z}$ using $SO_3 \cong \mathbb{R}P^3$ and $\mathbb{Z}/2 \to S^3 \twoheadrightarrow \mathbb{R}P^3$. Next look at the long exact homotopy sequence for the fibration $SO_3 \hookrightarrow SO_4 \to S^3$:

$$\cdots \to \mathbb{Z}/2 = \pi_4(S^3) \to \mathbb{Z} \to \pi_3 SO_4 \to \mathbb{Z} \to \pi_2(SO_3) = 0.$$

It follows that $\pi_3 SO_4 \cong \mathbb{Z}^2$. Next do the same thing for $SO_4 \hookrightarrow SO_5 \to S^4$:

$$\cdots \to \mathbb{Z} = \pi_4 S^4 \to \pi_3 SO_4 \to \pi_3 SO_5 \to 0.$$

Unfortunately we cannot go further without calculating the map $\pi_4 S^4 \to \pi_3 SO_4$, which is $\mathbb{Z} \to \mathbb{Z}^2$. So now we are indeed stuck, unless we can resolve this issue. Note, however, that the fibrations $SO_{n-1} \hookrightarrow SO_n \to S^{n-1}$ show that $\pi_3 SO_5 = \pi_3 SO_n$ for $n \geq 5$, so once we've figured this one out we know everything. It will take us a moment, but we will show that the map $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ is a split inclusion. So we get that

Proposition 12.16. $\pi_3(SO_n) \cong \begin{cases} 1 & n \leq 2 \\ \mathbb{Z} & n = 3 \\ \mathbb{Z}^2 & n = 4 \\ \mathbb{Z} & n \geq 5. \end{cases}$

Moreover, the maps $\pi_3 SO_n \to \pi_3 SO_{n+1}$ are isomorphisms for $n \geq 5$, and a surjection when $n = 4$.

The way we "get lucky" here is that we can think of S^3 as the unit quaternions, and then use quaternionic arithmetic to get our hands on both $\pi_3 SO_3$ and $\pi_3 SO_4$. This gets a bit gnarly and takes a couple of pages, but is worth sketching. For each $q \in S^3$ the conjugation map $\gamma_q \colon \mathbb{H} \to \mathbb{H}$ given by $x \mapsto qx\bar{q}$ is orthogonal and fixes 1. So it induces an orthogonal transformation of $\langle 1\rangle^{\perp} = \langle i,j,k\rangle$, which we may identify with an element of SO_3. The resulting map $C_1\colon S^3 \to SO_3$ given by $q \mapsto \gamma_q|_{\langle i,j,k\rangle}$ has the property that $C_1(q) = C_1(-q)$ for all $q \in S^3$, and the induced map $\mathbb{R}P^3 \to SO_3$ is readily seen to be a homeomorphism. Since elements of $\pi_3 SO_3$ are classified by their degree (use $SO_3 \cong \mathbb{R}P^3$ and $\mathbb{Z}/2 \to S^3 \to \mathbb{R}P^3$), C_1 is a generator. Write C_n for $q \mapsto \gamma_{q^n} = (\gamma_q)^n$, which is the n-fold multiple of C_1 in $\pi_3 SO_3$.

Let f be the image of C_1 in $\pi_3 SO_4$. Here we think of SO_4 as acting on the quaternions $\mathbb{H}$ by taking $1, i, j, k$ as the standard basis, so that the inclusion $SO_3 \hookrightarrow SO_4$ is $X \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 0 & X \end{smallmatrix}\right]$. Then $f(q)$ is still conjugation by q, since that operation fixes 1. Things are about to become slightly awkward in that we will want to use both additive and multiplicative notation for the group operation in $\pi_3 SO_4$, the latter mostly when we are using matrix multiplication. We will write f or nf when using additive notation, and C_1 or C_n when using multiplicative notation.

The element f (or C_1) gives us the generator of $\pi_3 SO_4$ that comes from $\pi_3 SO_3$. Using the long exact sequence for $SO_3 \to SO_4 \xrightarrow{p_1} S^3$, where p_1 is projection onto the first column, for the second generator we can take any map $g\colon S^3 \to SO_4$ having the property that the composite

$$S^3 \xrightarrow{g} SO_4 \xrightarrow{p_1} S^3$$

has degree 1. Consider the maps $L_n\colon S^3 \to SO_4$ sending $q \in S^3$ to left multiplication by q^n (so $L_n = (L_1)^n$). The fact that $[L_1(q)](1) = q$ shows $p_1 \circ L_1 = \mathrm{id}$, and so L_1 is a choice for the second generator for $\pi_3 SO_4$. The elements of $\pi_3 SO_4$ can therefore be written as $C_k L_n = L_n C_k$ for $k, n \in \mathbb{Z}$. When working additively we will write L_1 as g, so that f and g form our additive basis for $\pi_3 SO_4 \cong \mathbb{Z}^2$.

The next step is to think about the conjugation action on $\pi_3 SO_{n-1}$ that comes into our bijection $\mathrm{Vect}_n(S^4) \cong (\pi_3 SO_{n-1})/(\mathbb{Z}/2)$. We know the action on $\pi_3 SO_n$ is trivial when n is odd, and then the isomorphisms $\pi_3 SO_n \cong \pi_3 SO_{n+1}$ for $n \geq 5$ show that the action is trivial in that range. So the only place it has to be analyzed is for $n = 4$.

For conjugation in SO_4 we use the matrix $B = \mathrm{diag}(1, -1, -1, -1)$, which as a transformation of $\mathbb{H}$ is $B(x) = \bar{x}$ (we could use $A = \mathrm{diag}(-1, 1, 1, 1)$ instead to match our previous work, but that introduces an unwelcome and unhelpful minus sign into the formulas on $\mathbb{H}$). Then we readily compute that BC_1B^{-1} sends $q \in S^3$ to the map

$$x \mapsto \overline{(q\bar{x}\bar{q})} = qx\bar{q} = C_1(x),$$

or $BC_1B^{-1} = C_1$. Likewise, BL_1B^{-1} sends $q \in S^3$ to the map

$$x \mapsto \overline{(q\bar{x})} = x\bar{q} = q(q^{-1}x)\bar{q} = [C_1(q)L_{-1}(q)](x).$$

This is, $BL_1B^{-1} = C_1 L_{-1}$. So in additive notation the conjugation action on $\pi_3 SO_4 = \mathbb{Z}\langle f, g\rangle$ is $f \mapsto f$, $g \mapsto f - g$, or $(a, b) \mapsto (a + b, -b)$. Each orbit has a unique element with non-negative second coordinate, therefore the quotient $(\pi_3 SO_4)/(\mathbb{Z}/2)$ can be identified with $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

What is left to do is to understand the connecting homomorphism for the long exact homotopy sequence for the fibration $SO_4 \hookrightarrow SO_5 \to S^4$

$$\mathbb{Z} \xrightarrow{\partial} \pi_3 SO_4 \xrightarrow{i_*} \pi_3 SO_5 \to 0.$$

But this is the same as understanding the kernel of i_*, which by the long exact sequence can have rank at most 1. We identify $\pi_3 SO_4$ with $\mathbb{Z}^2$ via the basis f, g, and recall that the conjugation action is $(a,b) \mapsto (a+b, -b)$. Since the conjugation action on $\pi_3 SO_5$ is trivial, we know that $(0,1)$ and $(1,-1)$ will both map to the same element. So $(1,-2)$ is in the kernel of i_*. But the only subgroup of $\mathbb{Z}^2$ that contains $(1,-2)$ and has rank at most 1 is $\langle(1,-2)\rangle$, and so this must be $\ker(i_*)$. Therefore the image of ∂ is this subgroup, and this completes our analysis.

The following table summarizes some of what we now know about vector bundles on S^4:

n	1	2	3	4	5	6	7
$\mathrm{Vect}_n(S^4)$	$*$	$*$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_{\geq 0}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$

All of the bundles of rank 5 and higher come from adding trivial bundles to rank 4 bundles. Let us give names to some of the bundles and thereby give a more precise description. Let $\mathcal{F}_n$ be the rank 3 bundle whose clutching function is nf (or C_n). Let $\mathcal{G}_{n,k}$ be the rank 4 bundle whose clutching function is $nf + kg$, where $n, k \in \mathbb{Z}$ and $k \geq 0$. Note that $\mathcal{G}_{n,0} = \mathcal{F}_n \oplus \underline{1}$. We have shown that $\mathcal{G}_{a,b} \oplus \underline{1} \cong \mathcal{G}_{a+1,b-2} \oplus \underline{1}$, and so for a complete list of rank 5 bundles we can use $\mathcal{G}_{n,0} \oplus \underline{1} = \mathcal{F}_n \oplus \underline{2}$ and $\mathcal{G}_{n,1} \oplus \underline{1}$, for $n \in \mathbb{Z}$. In our basis the map $\pi_3 SO_4 \to \pi_3 SO_5$ is the map $\mathbb{Z}^2 \to \mathbb{Z}$ given by $(a,b) = 2a + b$, and so in our bijection $\mathrm{Vect}_5(S^4) \cong \mathbb{Z}$ we have $\mathcal{F}_n \oplus \underline{2} \mapsto 2n$ and $\mathcal{G}_{n,1} \oplus \underline{1} \mapsto 2n+1$.

The following table compactly summarizes all of the vector bundles on S^4:

n	1	2	3	4	5	6
$\mathrm{Vect}_n(S^4)$	$*$	$*$	$\mathcal{F}_n$ $(n\in\mathbb{Z})$	$\mathcal{G}_{n,k}$ $(n\in\mathbb{Z}, k\in\mathbb{Z}_{\geq 0})$	$\mathcal{F}_n \oplus \underline{2}$, $\mathcal{G}_{n,1} \oplus \underline{1}$ $(n\in\mathbb{Z})$	$\cdots$

The direct sum relations are all deduced from $\mathcal{F}_n \oplus \mathcal{F}_{n'} \cong \underline{3} \oplus \mathcal{F}_{n+n'}$ and $\mathcal{G}_{a,b} \oplus \mathcal{G}_{a',b'} \cong \underline{4} \oplus \mathcal{G}_{a+a',b+b'}$. These follow directly from Proposition 12.11.

Exercise 12.17 (Challenge)**.** Show that the tangent bundle TS^4 is isomorphic to $\mathcal{G}_{-1,2}$ (or equivalently, $\mathcal{G}_{1,-2}$). One way is to follow the steps below:

(1) First check that $TS^4 \oplus \underline{1}$ is trivial, and deduce that $TS^4 \cong \mathcal{G}_{-a,2a}$ for some $a \geq 0$.

(2) Regard S^4 as the sphere inside $\mathbb{R}\langle e_N\rangle \oplus \mathbb{H}$, where e_N is the "north pole". The equator of S^4 is then the unit quaternions, and the tangent space of S^4 at e_N is identified with $\mathbb{H}$ by projection. Orient S^4 by having $1, i, j, k$ be an oriented basis of $T_{e_N} S^4$, and then verify that $-1, i, j, k$ is an oriented basis of $T_{-e_N} S^4$.

(3) Let $\langle x, q\rangle = \mathrm{Re}(x\bar{q})$ be the standard inner product on $\mathbb{H}$. For each $q \in S^3$ consider the transformation $R_{\theta,q}$ of $\mathbb{R} \oplus \mathbb{H}$ that rotates the plane $\mathrm{Span}\{e_N, q\}$ by θ radians and is the identity on $\mathrm{Span}\{e_N, q\}^\perp$. We rotate according to the orientation that when $\theta = \frac{\pi}{2}$ the vector e_N rotates to q, and q to $-e_N$. Prove that for $x \in \mathbb{H} = T_{e_N} S^4$ we have
$$R_{\theta,q}(x) = -(\sin\theta)\langle x, q\rangle e_N + (\cos\theta - 1)\langle x, q\rangle q + x.$$
Conclude that $R_{\pi,q}(x) = x - 2\langle x, q\rangle q$, which is the reflection of x in the hyperplane $\langle q\rangle^\perp$ of $\mathbb{H}$.

(4) Verify that to get the clutching function for TS^4 we can use $q \mapsto R_{\pi,q}$ but with one caveat. The way we have set things up, this maps S^3 into the determinant -1 component of O_4. To instead get into SO_4 we need to compose this with a fixed orthogonal transformation of determinant -1; choose the transformation

$A(x) = -\bar{x}$. In this way we get the clutching function $h\colon S^3 \to SO_4$ given by

$$q \mapsto [x \mapsto -(\overline{x - 2\langle x, q\rangle q})].$$

(5) We know that $h = (-a, 2a)$ for some a in our basis for $\pi_3 SO_4 = \mathbb{Z}^2$, so to determine a it suffices to compute the image of h under the map $(p_1)_*\colon \pi_3 SO_4 \to \pi_3 S^3$. This image is $p_1 \circ h$, which is

$$q \mapsto R_{\pi,q}(1) = 2\langle 1, q\rangle \bar{q} - 1.$$

Verify that this map has degree -2 as a map $S^3 \to S^3$, for example by using the local degree theorem.

(6) The previous part completes the problem, but for good measure let us compute just a little more. The element $2g+h$ will have $(p_1)_*(2g+h) = 0$, and so $2g+h$ will be a multiple of f. We already know what multiple this must be, but let us find it by brute force. The element $2g + h$ is represented by

$$q \mapsto [x \mapsto -q^2\overline{(x - 2\langle x, q\rangle q)} = -q^2\bar{x} + 2\langle x, q\rangle q].$$

Verify that restricting to $x = 1$ gives the map $q \mapsto 1$, i.e. the trivial map $S^3 \to S^3$. If we instead restrict to $\langle x, 1\rangle = 0$ (i.e. $x \in \langle i, j, k\rangle$) show that the above formula reduces to

$$q \mapsto [x \mapsto qx\bar{q}] = C_1(q).$$

This confirms that $2g+h$ is represented by the image of the element $C_1 \in \pi_3 SO_3$ under $i_*\colon \pi_3 SO_3 \to \pi_3 SO_4$, i.e. that $2g + h = f$. So $h = -f + 2g$.

12.18. **Vector bundles on S^k.** Although we cannot readily do the calculations for $k > 4$, at this point one sees the general pattern. One must calculate $\pi_{k-1} SO_n$ for each n, and these groups vary for a while but eventually stabilize. In fact, $\pi_i SO_n \cong \pi_i SO_{n+1}$ for $n > i + 1$. The calculation of these stable groups was an important problem back in the 1950s, that was eventually solved by Bott. (There is again the conjugation action that must be dealt with, but because of the stabilization isomorphisms this action is trivial when n is large enough.)

Let us phrase things as follows. Consider the inclusions

$$O_1 \hookrightarrow O_2 \hookrightarrow O_3 \hookrightarrow \cdots$$

that send a matrix A to $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. The colimit of this sequence is denoted O and called the **stable orthogonal group**. The homotopy groups of O are the stable values that we encountered above. We computed the first few: $\pi_0 O = \mathbb{Z}/2$, $\pi_1 O = \mathbb{Z}/2$, $\pi_2 O = 0$, and $\pi_3 O = \mathbb{Z}$. Bott's calculation showed the following:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\pi_i O$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The pattern is 8-fold periodic: $\pi_{i+8} O \cong \pi_i O$ for all $i \geq 0$. One is supposed to remember the pattern of groups to the tune of "Twinkle, Twinkle, Little Star":

zee - two - zee - two - ze - ro - zee ze - ro - ze - ro - ze - ro - zee.

We will eventually have to understand Bott's computations at a deeper level; in particular, we will need to get our hands on explicit generators. See Part 6 for an extensive treatment. But for now we will just accept that the values are as given above.

Exercise 12.19. Using the methods we have demonstrated for S^n, $n \leq 4$, attempt to understand all vector bundles on S^5. Look up whatever homotopy groups of spheres you need to do the calculations. Establishing a complete classification is challenging and perhaps requires more tools than we have developed, but you should be able to at least determine the following:

- All bundles of rank 1 and 2 are trivial.
- There is exactly one nontrivial bundle of rank 3. Describe the clutching function as concretely as you can.
- There are at most four different isomorphism types of rank 4 bundles.
- All bundles of rank 5 or more are obtained by adding trivial bundles to rank 4 bundles, and therefore there are at most four isomorphism classes in each rank.
- [Accepting the Bott results] All bundles of rank 6 or higher are trivial.

12.20. **Complex vector bundles on spheres.** One can repeat the above analysis for complex vector bundles on a sphere. One finds that

$$\mathrm{Vect}_n^{\mathbb{C}}(S^k) \cong \pi_{k-1}(GL_n(\mathbb{C})) \cong \pi_{k-1}(U_n),$$

where $U_n \hookrightarrow GL_n(\mathbb{C})$ is the unitary group. Analogously to the real case, one has fiber bundles $U_{n-1} \hookrightarrow U_n \twoheadrightarrow S^{2n-1}$ coming from the fact that when U_n acts on $\mathbb{C}^n$ the orbit of e_1 is S^{2n-1} and the stabilizer is U_{n-1}. Using that $U_1 \cong S^1$ one can again compute $\mathrm{Vect}_n^{\mathbb{C}}(S^k)$ for small values of k. Here is what you get:

n	1	2	3	4	5	6	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^1)$	0	0	0	0	0	0	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^3)$	0	0	0	0	0	0	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^4)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^5)$	0	$\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	$\cdots$

The stable value in the last row turns out to be 0, although one cannot figure this out without computing a connecting homomorphism in the long exact homotopy sequence.

Exercise 12.21. Verify all of the computations in the above table.

The fiber bundles $U_n \hookrightarrow U_{n+1} \twoheadrightarrow S^{2n+1}$ again imply that $\pi_i U_n$ stabilizes as n grows. In fact, $\pi_i U_n \cong \pi_i U_{n+1}$ for $n > \frac{i}{2}$. We can write the stable value as $\pi_i U$ where U is the **infinite unitary group** defined as the colimit of

$$U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \cdots$$

Bott computed the homotopy groups of U to be 2-fold periodic, with

$$\pi_i U = \begin{cases} \mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Again, for now we will just accept this result; but eventually we will have to understand the computation in more detail, and in particular we will need to get our hands on specific generators.

13. Classifying spaces

Consider the functor sending a paracompact Hausdorff space X to the set $\mathrm{Vect}_n(X)$. This is contravariant and homotopy invariant, so it has the feel of a functor $[-, Y]$ for some space Y—that is, a functor that is representable in the homotopy category. In this section we prove that the functor $\mathrm{Vect}_n(-)$ is indeed representable. A representing space Y is called a **classifying space** for rank n vector bundles. There are different methods for constructing such a space, some more homotopical and some more geometric.

Before embarking on the construction, let us give an overview of some of the key ideas. Suppose given a space VB_n and a rank n vector bundle $\gamma_n \to \mathrm{VB}_n$. To every map $f\colon X \to \mathrm{VB}_n$ we can associate the pullback bundle $f^*\gamma_n$, and by Proposition 11.1(a) this gives a natural transformation $[X, \mathrm{VB}_n] \to \mathrm{Vect}_n(X)$ for paracompact Hausdorff spaces X. It will then be a matter of proving surjectivity and injectivity of this natural map, and our construction of VB_n will give surjectivity almost automatically: that is, for any vector bundle $E \to X$ there is a map $f\colon X \to \mathrm{VB}_n$ and an isomorphism $E \cong f^*\gamma_n$. The bundle $\gamma_n \to \mathrm{VB}_n$ is called a "universal bundle", since every other rank n bundle is pulled back from it. Injectivity of our natural transformation is the slightly subtle part.

For injectivity we will produce an analogous classifying space for isomorphisms between bundles of rank n. This will be a space VBI_n together with two canonical maps $\pi_1, \pi_2\colon \mathrm{VBI}_n \to \mathrm{VB}_n$ and an isomorphism $\phi\colon \pi_1^*\gamma_n \to \pi_2^*\gamma_n$. For any two maps $f, g\colon X \to \mathrm{VB}_n$ and an isomorphism $f^*\gamma_n \cong g^*\gamma_n$ there will exist a map $u\colon X \to \mathrm{VBI}_n$ such that $\pi_1 \circ u \simeq f$ and $\pi_2 \circ u \simeq g$. This is the analog of the surjectivity part of the previous paragraph, and is again fairly easy.

Finally, we will construct a map $\mathrm{VBI}_n \to (\mathrm{VB}_n)^I$ sitting in a homotopy-commutative diagram

$$\begin{tikzcd}
 & & \mathrm{VB}_n \\
\mathrm{VBI}_n \arrow[r] \arrow[urr, bend left, "\pi_1"] \arrow[drr, bend right, "\pi_2"'] & (\mathrm{VB}_n)^I \arrow[ur, "ev_0"] \arrow[dr, "ev_1"'] & \\
 & & \mathrm{VB}_n .
\end{tikzcd}$$

As an exercise, check that injectivity of $[X, \mathrm{VB}_n] \to \mathrm{Vect}_n(X)$ follows formally from these considerations. So the work ahead of us comes down to constructing VB_n and VBI_n with the properties outlined above.

One may view a vector bundle as a family of vector spaces indexed by the base space. In general, it is often useful to view a map $E \to B$ as a family of ____ if each fiber is a ____. Taking our cue from the subject of moduli spaces, one could naively hope that families of some mathematical object over B are in bijection with maps from B to some space, called the moduli space corresponding to that mathematical object. With this naive idea, we would have that families over $*$ are in bijective correspondence with points of our moduli space. But for rank n vector bundles this would mean that the moduli space has only one point, since there is only one real vector space of rank n. So this can't work.

What turns out to work instead is a homotopical version of the theory, where rank n vector bundles over B are in bijective correspondence with *homotopy classes* *of maps* from B to a 'homotopical moduli space'. Our goal in this section is to construct this space—called the *classifying space* of rank n vector bundles—and prove the bijective correspondence.

There are several different approaches to the construction of classifying spaces, but we will only focus on the most geometric of these. Instead of just one rank n vector space we look at all of the rank n subspaces of some fixed infinite-dimensional vector space. This gets somewhat technical because of the ins-and-outs of dealing with $\mathbb{R}^\infty$.

13.1. **Grassmannians.** We start by developing the basic theory of Grassmannians for topological vector spaces. We include the case where the vector space is infinite-dimensional, though one is free to think mostly about the finite-dimensional case. To this end, let W be a topological vector space. Let $VI_n(W) \subseteq W^n$ be the subset of tuples $(w_1, \dots, w_n)$ that are linearly independent, equipped with the subspace topology (the 'V' is the usual notation for Stiefel manifolds, and here 'I' stands for 'independent'). Define $\mathrm{Gr}_n(W)$ to be the quotient space of $VI_n(W)$ with respect to the relation where $(w_1, \dots, w_n) \sim (w_1', \dots, w_n')$ if the vectors span the same subspace. As usual, we identify points of $\mathrm{Gr}_n(W)$ with n-dimensional subspaces of W.

From another perspective, change-of-basis gives a continuous group action of $GL_n(\mathbb{R})$ on $VI_n(W)$, and $VI_n(W) \to \mathrm{Gr}_n(W)$ is just quotienting by this action. Let us construct the action carefully. If $\underline{w} = (w_1, \dots, w_n)$ is a tuple of vectors in W and $P \in GL_n(\mathbb{R})$, define $[\underline{w} \cdot P]_j = \sum_i w_i P_{ij}$. Note that if $W = \mathbb{R}^k$ and we regard each w_i as a column vector then $\underline{w}$ is a $k \times n$ matrix and this definition is just matrix multiplication. Our definition clearly gives a map of sets $VI_n(W) \times GL_n(\mathbb{R}) \to VI_n(W)$. To see continuity start with the "linear combination map" $LC\colon W^n \times \mathbb{R}^n \to W$, $(\underline{w}, \underline{r}) \mapsto \sum_i r_i w_i$. This is continuous because W is a topological vector space. Next consider the diagram

$$\begin{tikzcd}
VI_n(W) \times GL_n(\mathbb{R}) \arrow[r, dotted] \arrow[d, tail] & VI_n(W) \arrow[dd, tail] \\
W^n \times GL_n(\mathbb{R}) \arrow[d, tail] & \\
W^n \times (\mathbb{R}^n)^n \arrow[r] & W^n.
\end{tikzcd}$$

The map $GL_n(\mathbb{R}) \hookrightarrow (\mathbb{R}^n)^n$ sends a matrix to the n-tuple of its columns, and the bottom horizontal map is a product of n copies of LC and so is continuous. The down-bottom composite factors through the subspace $VI_n(W)$, thereby giving the upper horizontal map.

Let W' be another topological vector space. If $f\colon W \to W'$ is a continuous linear embedding then the map $W^n \to (W')^n$ restricts to $VI_n(W) \to VI_n(W')$ and therefore induces a map $\mathrm{Gr}_n(W) \to \mathrm{Gr}_n(W')$. We will generally denote all of these maps by f as well. The following result just says that homotopies among the maps $W \to W'$ descend to homotopies among the induced maps of Grassmannians.

Lemma 13.2. *Suppose that $H\colon W\times I\to W'$ is a homotopy such that each H_t is a linear embedding. Then there is a homotopy $\mathrm{Gr}_n(W)\times I\to\mathrm{Gr}_n(W')$ that at each time t is the map induced by H_t.*

Proof. Let $\tilde{H}$ be the composite

$$W^n\times I\xrightarrow{\mathrm{id}\times\Delta} W^n\times I^n\xrightarrow{H^n}(W')^n$$

which sends $(w_1,\dots,w_n,t)\mapsto(H_t(w_1),\dots,H_t(w_n))$. This is continuous by construction. Since each H_t is a linear embedding the composite restricts to give

$$\tilde{H}\colon VI_n(W)\times I\longrightarrow VI_n(W').$$

In the diagram

$$\begin{tikzcd} VI_n(W)\times I \arrow[r] \arrow[d, two heads] & VI_n(W') \arrow[r] & \mathrm{Gr}_n(W') \\ \mathrm{Gr}_n(W)\times I \arrow[urr, dotted] & & \end{tikzcd}$$

the projection $VI_n(W)\times I\to\mathrm{Gr}_n(W)\times I$ is a quotient map by [Mu, Chapter 29, Exercise 11], and one readily checks that the quotient relations are respected by the horizontal map. So the indicated extension exists. $\square$

Corollary 13.3. *If $k<N\leq\infty$ then all linear embeddings $\mathbb{R}^k\hookrightarrow\mathbb{R}^N$ induce homotopic maps on Grassmannians.*

Proof. First assume $N<\infty$. Since a linear map $\mathbb{R}^k\to\mathbb{R}^N$ is completely determined by the image of the standard basis, the space of all such embeddings is homeomorphic to $VI_k(\mathbb{R}^N)$. The usual Stiefel manifold $V_k(\mathbb{R}^N)$ of orthornormal frames sits inside $VI_k(\mathbb{R}^N)$ as a deformation retract, by Gram-Schmidt. But the space $V_k(\mathbb{R}^N)$ is connected, by the usual inductive argument using the fibrations $V_{k-1}(\mathbb{R}^{N-1})\to V_k(\mathbb{R}^N)\to V_1(\mathbb{R}^N)$ (this is where we need $k<N$). So any two points are connected by a path, i.e. any two linear embeddings are connected by a homotopy through linear embeddings.

For the case $N=\infty$ just use that any linear map $\mathbb{R}^k\to\mathbb{R}^\infty$ factors through a finite $\mathbb{R}^s$, and then appeal to what has already been proven. $\square$

Here is a useful property of the VI-spaces:

Proposition 13.4. *There is a natural bijection between $\mathcal{T}op(X,VI_n(W))$ and the set of fiberwise injections of families of vector spaces $\underline{n}_X\hookrightarrow X\times W$.*

Proof. Start with the linear combination map $LC\colon VI_n(W)\times\mathbb{R}^n\to W$. Given a map $f\colon X\to VI_n(W)$, let ϕ_f be the composite

$$X\times\mathbb{R}^n\xrightarrow{f\times\mathrm{id}} VI_n(W)\times\mathbb{R}^n\xrightarrow{LC} W.$$

This is evidentally continuous. We then get $(\pi_1,\phi_f)\colon X\times\mathbb{R}^n\to X\times W$ and one readily checks that this is a fiberwise injective map of families.

In the other direction, suppose given a fiberwise injective map of families $j\colon X\times\mathbb{R}^n\hookrightarrow X\times W$. For each i let α_i be the composite

$$X\times\{e_i\}\hookrightarrow X\times\mathbb{R}^n\xrightarrow{j} X\times W\xrightarrow{\pi_2} W,$$

which is clearly continuous. These induce $\alpha\colon X \to W^n$, and the image lies in the subspace $VI_n(W)$. So we get a continuous map $X \to VI_n(W)$. This construction is readily checked to be a two-sided inverse for the construction of the previous paragraph. $\square$

To prove more about $VI_n(W)$ and $\mathrm{Gr}_n(W)$ we will have to assume that W is a reasonable topological vector space in the sense of Definition B.10. This ensures that finite-dimensionsal subspaces of W are homeomorphic to $\mathbb{R}^n$, and that when $W = F + W'$ is a direct sum decomposition with F finite-dimensionsal then the corresponding projections from W to F and W' are continuous. The main non-trivial example to keep in mind is $W = \mathbb{R}^\infty$ (with the colimit topology), though the finite-dimensional spaces $\mathbb{R}^n$ are also examples. Note that $\mathbb{R}^\infty$ in the metric or product topologies are not reasonable topological vector spaces in this sense.

Proposition 13.5. *If W is a reasonable topological vector space then the quotient map $p\colon VI_n(W) \to \mathrm{Gr}_n(W)$ is a principal $GL_n(\mathbb{R})$-bundle.*

Proof. Let $F \subseteq W$ be a subspace of dimension n. Choose a complement W' for F and write $\pi_F\colon W \to F$ and $\pi_{W'}\colon W \to W'$ for the associated projections. These are continuous because W is reasonable (Proposition B.11). Set $U = \{J \in \mathrm{Gr}_n(W) \,|\, \pi_F(J) = F\}$. We will prove that U is a neighborhood of F in $\mathrm{Gr}_n(W)$ that trivializes p.

Observe that

$$p^{-1}(U) = \{(w_1, \ldots, w_n) \in W^n \,|\, \pi_F(w_1), \ldots, \pi_F(w_n) \text{ is a basis for } F\}.$$

The set $p^{-1}(U)$ is open in $VI_n(W)$ because it sits in the pullback diagram

$$\begin{tikzcd}
VI_n(W) \arrow[r, tail] & W^n \arrow[r, two heads, "\pi_F"] & F^n \\
p^{-1}(U) \arrow[u, tail] \arrow[rr] & & VI_n(F) \arrow[u, tail]
\end{tikzcd}$$

and $VI_n(F) \hookrightarrow F^n$ is open. The last statement is because $F \cong \mathbb{R}^n$ (this uses that W is reasonable) and for n vectors in $\mathbb{R}^n$ independence is determined by the nonvanishing of the determinant. Also, note that we are using continuity of π_F.

Since $p^{-1}(U)$ is open in $VI_n(W)$, we have that U is open in $\mathrm{Gr}_n(W)$.

We have the continuous map

$$\alpha\colon p^{-1}(U) \to (W')^n \times F^n, \qquad \underline{w} \mapsto (\pi_{W'}(w_1), \ldots, \pi_{W'}(w_n), \pi_F(w_1), \ldots, \pi_F(w_n)).$$

This is continuous because both π_F and $\pi_{W'}$ are continuous, by Proposition B.11. Note that the image of α lands in the subspace $(W')^n \times VI_n(F)$. We also have the map β given by the composite

$$(W')^n \times VI_n(F) \hookrightarrow (W')^n \times F^n = (W' \times F)^n \hookrightarrow (W \times W)^n \xrightarrow{+} W^n$$

and the image lands in $p^{-1}(U)$. One readily checks that α and β are inverses, so we have the homeomorphism $p^{-1}(U) \cong (W')^n \times VI_n(F)$.

Choose a basis $b_1, \ldots, b_n$ for F. Let f be the composite

$$(W')^n = (W')^n \times \{(b_1, \ldots, b_n)\} \hookrightarrow (W')^n \times VI_n(F) \xrightarrow{\beta} p^{-1}(U) \xrightarrow{p} U.$$

This is readily seen to be a bijection, but we need to prove it is a homeomorphism. For this consider the composite

$$\begin{tikzcd} p^{-1}(U) \arrow[d, "\alpha"] & & & \\ (W')^n \times VI_n(F) \arrow[r, "a"] & (W')^n \times GL_n(\mathbb{R}) \arrow[r, "b"] & (W')^n \times (\mathbb{R}^n)^n \arrow[r, "c"] & (W')^n. \end{tikzcd}$$

The map a comes from the identification $F \cong \mathbb{R}^n$ provided by the chosen basis. The map b sends a matrix in $GL_n(\mathbb{R})$ to its tuple of rows. The map c consists of several instances of the linear combination map $(W')^n \times \mathbb{R}^n \to W'$ given by $((\underline{w}), (\underline{r})) \mapsto \sum r_i w_i$. All of these maps are continuous. The composite takes a tuple of the form $(v_1 + w_1', \ldots, v_n + w_n')$ where $(v_1, \ldots, v_n) \in VI_n(F)$ and each $w_i' \in W'$ and sends it to the W'-projections of the unique basis for the same span that has the form $(b_1 + w_1'', \ldots, b_n + w_n'')$ with $w_i'' \in W'$ for all i. This map respects the quotient relation for $p^{-1}(U) \to U$ and so descends to give a map $U \to (W')^n$. This is the inverse to the map f, thereby proving that f is a homeomorphism.

Putting everything together, at this point we have produced the homeomorphism in the commutative triangle

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"'] \arrow[dr] & & U \times GL_n(\mathbb{R}) \arrow[dl] \\ & U. & \end{tikzcd}$$

It is routine to check that the horizontal map respects the right $GL_n(\mathbb{R})$ action. This completes the proof that p is a principal $GL_n(\mathbb{R})$-bundle. □

In the course of the above proof we also established the following result, which we record for future use:

Proposition 13.6. *Every point in* $\mathrm{Gr}_n(W)$ *has a neighborhood that is homeomorphic to* $(W')^n$ *for some subspace* $W' \subseteq W$ *such that* $\dim(W/W') = n$.

Define $\gamma_n(W)$ to be the vector bundle associated to the principal $GL_n(\mathbb{R})$-bundle $VI_n(W) \to \mathrm{Gr}_n(W)$. That is,

$$\gamma_n(W) = VI_n(W) \times_{GL_n(\mathbb{R})} \mathbb{R}^n.$$

There is a continuous map $\phi\colon \gamma_n(W) \to W$ defined by $((w_1, \ldots, w_n), (r_1, \ldots, r_n)) \mapsto r_1 w_1 + \cdots + r_n w_n$. From this we can construct $\gamma_n(W) \to \mathrm{Gr}_n(W) \times W$ which is the projection in the first coordinate and ϕ in the second. As a map of sets this is readily checked to be an injection.

Proposition 13.7. *The canonical map* $\gamma_n(W) \to \mathrm{Gr}_n(W) \times W$ *is a homeomorphism onto its image (in the subspace topology).*

Proof. Let f denote the canonical map in question, and let Z denote the image of f with the subspace topology. We have the triangle

$$\begin{tikzcd} \gamma_n(W) \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "\pi_1"] \\ & \mathrm{Gr}_n(W). & \end{tikzcd}$$

The map $Z \to \mathrm{Gr}_n(W)$ is a rank n subfamily of the trivial family of vector spaces $\mathrm{Gr}_n(W) \times W$, and f is an isomorphism on each fiber. Our approach will be to prove that $Z \to \mathrm{Gr}_n(W)$ is a vector bundle. It will then follow that f is an isomorphism of vector bundles and hence a homeomorphism.

For each n-dimensional subspace $F \subseteq W$ with chosen complement W' let $U_{F,W'} \subseteq \mathrm{Gr}_n(W)$ be the subspace of n-planes J for which $\pi_F(J) = F$. In a previous proof we showed that these are open subsets, and clearly they cover $\mathrm{Gr}_n(W)$. (In fact we only need one choice of W' for each F, but that fact will not be used). We also showed that if $b_1, \dots, b_n$ is a basis for F then the map $\phi_{\underline{b}}\colon (W')^n \to U_{F,W'}$ given by $(w_1', \dots, w_n') \mapsto \mathrm{Span}(b_1 + w_1', \dots, b_n + w_n')$ is a homeomorphism.

Given a pair (F, W'), choose a basis $\underline{b}$ for F. For convenience write $Z_{F,W'} = \pi_1^{-1}(U_{F,W'}) = Z \cap (U_{F,W'} \times W)$. Consider the diagram

$$\begin{tikzcd}
(W')^n \times \mathbb{R}^n \arrow[r, "LC"] \arrow[d, "\phi_{\underline{b}} \times \mathrm{id}"', "\cong"] & Z_{F,W'} \arrow[r, tail] & U_{F,W'} \times W \arrow[d, "\mathrm{id} \times \pi_F"] \\
U_{F,W'} \times \mathbb{R}^n \arrow[rr, "\mathrm{id} \times LC_{\underline{b}}", "\cong"'] & & U_{F,W'} \times F.
\end{tikzcd}$$

Here $LC_{\underline{b}}$ is the map $\underline{r} \mapsto \sum_i r_i b_i$, and LC is the map $(\underline{w}', \underline{r}) \mapsto (\phi_{\underline{b}}(\underline{w}'), \sum_i r_i(b_i + w_i'))$. All of the maps are obviously continuous and the diagram is readily checked to commute. Also, LC is clearly a set-theoretic bijection. It then follows immediately that LC is a homeomorphism, as the inverse is given by going around the other side of the diagram (using the inverses of the maps we already know are homeomorphisms).

We have produced a homeomorphism $\pi_1^{-1}(U_{F,W'}) = Z_{F,W'} \cong U_{F,W'} \times \mathbb{R}^n$, namely the map $(\phi_{\underline{b}} \times \mathrm{id}) \circ LC^{-1}$ in the above diagram. This is the desired local trivialization, showing that $Z \to \mathrm{Gr}_n(W)$ is indeed a vector bundle. $\square$

The standard inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2} \hookrightarrow \cdots$ give an induced system of maps

$$\mathrm{Gr}_n(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{n+1}) \hookrightarrow \cdots \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$$

and similarly for the VI_n and γ_n constructions.

Proposition 13.8. *The canonical maps* $\mathrm{colim}_s\, VI_n(\mathbb{R}^s) \to VI_n(\mathbb{R}^\infty)$, $\mathrm{colim}_s\, \mathrm{Gr}_n(\mathbb{R}^s) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, *and* $\mathrm{colim}_s\, \gamma_n(\mathbb{R}^s) \to \gamma_n(\mathbb{R}^\infty)$ *are all homeomorphisms.*

Proof. Consider the following three directed systems and their abutments:

$$\begin{tikzcd}
(\mathbb{R}^n)^n \arrow[r] & (\mathbb{R}^{n+1})^n \arrow[r] & (\mathbb{R}^{n+2})^n \arrow[r] & \cdots \arrow[r] & (\mathbb{R}^\infty)^n \\
VI_n(\mathbb{R}^n) \arrow[r] \arrow[u, tail] \arrow[d, two heads, "p_n"] & VI_n(\mathbb{R}^{n+1}) \arrow[r] \arrow[u, tail] \arrow[d, two heads, "p_{n+1}"] & VI_n(\mathbb{R}^{n+2}) \arrow[r] \arrow[u, tail] \arrow[d, two heads, "p_{n+2}"] & \cdots \arrow[r] & VI_n(\mathbb{R}^\infty) \arrow[u, tail] \arrow[d, two heads, "p"] \\
\mathrm{Gr}_n(\mathbb{R}^n) \arrow[r] & \mathrm{Gr}_n(\mathbb{R}^{n+1}) \arrow[r] & \mathrm{Gr}_n(\mathbb{R}^{n+2}) \arrow[r] & \cdots \arrow[r] & \mathrm{Gr}_n(\mathbb{R}^\infty).
\end{tikzcd}$$

It is trivial that in each case the map from the colimit to the rightmost entry is a continuous bijection, but a little work is required to check that these are homeomorphisms.

By Corollary B.18 we have that $(\mathbb{R}^\infty)^n$ is the colimit of the top directed system. The intersection of $VI_n(\mathbb{R}^\infty)$ with $(\mathbb{R}^k)^n$ is $VI_n(\mathbb{R}^k)$, and this is open in $(\mathbb{R}^k)^n$ because it is the set of $n\times k$ matrices with at least one non-vanishing $n\times n$ minor. So $VI_n(\mathbb{R}^\infty)$ is open in $(\mathbb{R}^\infty)^n$ and is the colimit of the middle system. Finally, suppose $U \subseteq \mathrm{Gr}_n(\mathbb{R}^\infty)$ is such that its intersection with each $\mathrm{Gr}_n(\mathbb{R}^s)$ is open in $\mathrm{Gr}_n(\mathbb{R}^s)$. Then $p_s^{-1}(U \cap \mathrm{Gr}_n(\mathbb{R}^s))$ is open in $VI_n(\mathbb{R}^s)$, but this set is also $p^{-1}(U) \cap VI_n(\mathbb{R}^s)$. Therefore $p^{-1}(U)$ is open in $VI_n(\mathbb{R}^\infty)$, and hence U is open. This confirms that $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is the colimit of the bottom system.

Finally, we turn to the map $\operatorname{colim}_s \gamma_n(\mathbb{R}^s) \to \gamma_n(\mathbb{R}^\infty)$. For this consider the two directed systems

$$\begin{tikzcd}
VI_n(\mathbb{R}^n)\times\mathbb{R}^n \arrow[r, tail] \arrow[d, two heads] & VI_n(\mathbb{R}^{n+1})\times\mathbb{R}^n \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & VI_n(\mathbb{R}^\infty)\times\mathbb{R}^n \arrow[d, two heads] \\
\gamma_n(\mathbb{R}^n) \arrow[r] & \gamma_n(\mathbb{R}^{n+1}) \arrow[r] & \cdots \arrow[r] & \gamma_n(\mathbb{R}^\infty)
\end{tikzcd}$$

Since $\mathbb{R}^n$ is locally compact and Hausdorff the colimit of the top row is the space on the right; this uses Exercise A.1 together with what we have already proven about VI_n. It then follows formally that the induced map $VI_n(\mathbb{R}^\infty)\times\mathbb{R}^n \to \operatorname{colim}_s \gamma_n(\mathbb{R}^s)$ is quotient map (see Exercise A.2). The quotient relations are the same as the ones that define $\gamma_n(\mathbb{R}^\infty)$, so we deduce that $\operatorname{colim}_s \gamma_n(\mathbb{R}^s) \to \gamma_n(\mathbb{R}^\infty)$ is a homeomorphism. □

The bundle $\gamma_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ allows us to construct the natural transformation $\Theta_X \colon [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n(X)$ sending $f \mapsto f^*\gamma_n$ (for X paracompact Hausdorff).

Lemma 13.9. *For X paracompact Hausdorff the map Θ_X is a surjection.*

Proof. Let $\pi\colon E \to X$ be a rank n vector bundle. By Proposition 9.13 we know that there is a fiberwise embedding of bundles $j\colon E \hookrightarrow X\times\mathbb{R}^\infty$. Define the function $f\colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by $f(x) = \pi_2(j(E_x))$. We need to check that f is continuous. This can be done locally on X, and so we can assume E is trivial. In that case Proposition 13.4 says that the map j is determined by a continuous map $X \to VI_n(\mathbb{R}^\infty)$. The composite of this map with the projection to $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is precisely f, and this verifies continuity.

Consider the map $(f\pi, \pi_2 j)\colon E \to \mathrm{Gr}_n(\mathbb{R}^\infty)\times\mathbb{R}^\infty$. This is clearly continuous, and its image lands in the subspace $\gamma_n(\mathbb{R}^\infty)$ (this uses ???). The diagram

$$\begin{tikzcd}
E \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& f^*\gamma_n(\mathbb{R}^\infty) \arrow[r] \arrow[d] & \gamma_n(\mathbb{R}^\infty) \arrow[d] \\
& X \arrow[r, "f"] & \mathrm{Gr}_n(\mathbb{R}^\infty)
\end{tikzcd}$$

gives us an induced map $E \to f^*\gamma_n$ which is readily seen to be an isomorphism on each fiber and hence an isomorphism of bundles. So the map f is a preimage for E under Θ_X. □

Remark 13.10. Note in the above proof that if X were actually compact then the bundle embedding could be chosen to be $E \hookrightarrow X\times\mathbb{R}^s$ for some $s < \infty$, and then f maps to the finite Grassmannian $\mathrm{Gr}_n(\mathbb{R}^s)$.

Proposition 13.11. *Let $j^{ev}, j^{odd} \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be given by $j^{ev}(x_1, x_2, ...) = (0, x_1, 0, x_2, ...)$ and $j^{odd}(x_1, x_2, ...) = (x_1, 0, x_2, 0, ...)$. Then $j^{ev} \simeq \mathrm{id}$ and $j^{odd} \simeq \mathrm{id}$, via homotopies H having the property that each H_t is a linear embedding. Consequently, the maps $j^{ev}, j^{odd} \colon \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ are homotopic.*

Proof. We prove the claim for j^{ev}; the proof for j^{odd} is analogous. Define a homotopy $H \colon \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ by $H(x,t) = tj^{ev}(x) + (1-t)x$. This is continuous using that $\mathbb{R}^\infty \times I$ is the colimit of the $\mathbb{R}^k \times I$ (Exercise A.1). This is clearly a homotopy between j^{ev} and id. It remains to be shown that this is a homotopy through linear embeddings. Let $t \in (0,1)$ and suppose that $H(x,t) = 0$. We need to show that $x = 0$. Our assumption yields $0 = ((1-t)x_1, tx_1 + (1-t)x_2, (1-t)x_3, tx_2 + (1-t)x_4, ...)$. Therefore $(1-t)x_i = 0$ for all odd i; but since $t \neq 1$, this means that $x_i = 0$ for all odd i. Likewise, observe that $tx_n + (1-t)x_{2n} = 0$ for all $n \in \mathbb{N}$. So $x_n = 0$ implies $x_{2n} = 0$. Since we have $x_i = 0$ for all odd i and every natural number n can be written in the form $n = 2^e \cdot (\text{odd})$, it follows that $x = 0$.

The last statement in the proposition follows from Lemma 13.2 $\square$

Corollary 13.12. *Any two linear embeddings $f, g \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ are homotopic through linear embeddings, and therefore induced homotopic maps on $\mathrm{Gr}_n(\mathbb{R}^\infty)$.*

Proof. By Proposition 13.11 the map f is homotopic to $j_{ev} \circ f$, and the map g is homotopic to $j_{odd} \circ g$. So we may assume that $\mathrm{im}(f) \subseteq \mathbb{R}^\infty_{ev}$ and $\mathrm{im}(g) \subseteq \mathbb{R}^\infty_{odd}$. Now consider the map $H \colon \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ given by $H(x,t) = tf(x) + (1-t)g(x)$. $\square$

Theorem 13.13. *The map $\phi \colon [X, \mathrm{Gr}_n(\mathbb{R}^\infty] \to \mathrm{Vect}_n(X)$ is a bijection when X is paracompact and Hausdorff.*

Proof. Surjectivity was established in Lemma 13.9, so only injectivity remains to be shown. Assume $f, g \colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ are such that $f^*(\gamma_n) \cong g^*(\gamma_n)$ as vector bundles over X. We will show that f is homotopic to g. By Proposition 13.11, we may replace f by $j^{ev} \circ f$ and g by $j^{odd} \circ g$. In doing so, we are effectively assuming that $f(x) \subseteq \mathbb{R}^\infty_{ev}$ and $g(x) \subseteq \mathbb{R}^\infty_{odd}$, for each $x \in X$.

Let $\alpha \colon f^*\gamma_n \to g^*\gamma_n$ be a choice of isomorphism. Define a map $H \colon X \times I \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by the formula

$$H(x,t) = \{tv + (1-t)\alpha(v) \,|\, v \in f(x)\}.$$

It is easy to see that $H(x,t)$ is indeed an n-dimensional subspace of $\mathbb{R}^\infty$, using that $f(x) \subseteq \mathbb{R}^\infty_{ev}$ and $g(x) \subseteq \mathbb{R}^\infty_{odd}$. Also $H(x,0) = g$ and $H(x,1) = f$ by definition. It only remains to check continuity of H. This is a local issue, so we can restrict to an open set $U \subseteq X$ on which $f^*\gamma_n$ is trivializable.

Choose an isomorphism $A \colon \underline{n}_U \to f^*\gamma_n|_U$ and let $B = \alpha \circ A$. The composite $U \times \mathbb{R}^n \xrightarrow{A} (f^*\gamma_n)|_U \hookrightarrow U \times \mathbb{R}^\infty$ is a fiberwise embedding and so corresponds to a map $\tilde{f} \colon U \to VI_n(\mathbb{R}^\infty)$ by Proposition 13.4, and the image of $\tilde{f}$ actually lies in $VI_n(\mathbb{R}^\infty_{ev})$. Likewise, the composite $U \times \mathbb{R}^n \xrightarrow{B} (g^*\gamma_n)|_U \hookrightarrow U \times \mathbb{R}^\infty$ corresponds to a map $\tilde{g} \colon U \to VI_n(\mathbb{R}^\infty)$ that factors through $VI_n(\mathbb{R}^\infty_{odd})$.

The assignment $(u,t) \mapsto t\tilde{f}(u) + (1-t)\tilde{g}(u)$ gives a continuous map $J\colon U \times I \to VI_n(\mathbb{R}^\infty)$. To check continuity just refer to the diagram

$$\begin{tikzcd} VI_n(\mathbb{R}^\infty_{ev}) \times VI_n(\mathbb{R}^\infty_{odd}) \times I \arrow[r, dotted] \arrow[d, tail] & VI_n(\mathbb{R}^\infty) \arrow[d, tail] \\ (\mathbb{R}^\infty)^n \times (\mathbb{R}^\infty)^n \times I \arrow[r] & (\mathbb{R}^\infty)^n \end{tikzcd}$$

where the bottom map is $(\underline{a}, \underline{b}, t) \mapsto t\underline{a} + (1-t)\underline{b}$ and is cleary continuous. The image of the top left corner in $(\mathbb{R}^\infty)^n$ lands in $VI_n(\mathbb{R}^\infty)$ by algebra, and so we get the induced dotted arrow. The map J is obtained just by precomposing the dotted arrow with $(\tilde{f}, \tilde{g})$.

The projection of J down to $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is exactly $H|_{U\times I}$. This verifies continuity of H and completes the proof. $\square$

13.14. **Representing operations on vector bundles.** If $\mathcal{C}$ is a category then a map $f\colon W \to Z$ in $\mathcal{C}$ induces a map of representable functors $f_*\colon \mathcal{C}(-,W) \to \mathcal{C}(-,Z)$. The Yoneda lemma says this in fact gives a bijection between $\mathcal{C}(W,Z)$ and the set of natural transformations $\mathcal{C}(-,W) \to \mathcal{C}(-,Z)$. In particular, this implies that natural transformations $\mathrm{Vect}_n(X) \to \mathrm{Vect}_r(X)$ correspond to homotopy classes of maps maps $\mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_r(\mathbb{R}^\infty)$.

A simple example of this is the assignment $E \mapsto E \oplus 1$, regarded as a natural transformation $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$. The corresponding map $P_1\colon\ \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$ (really, just a homotopy class) can be described as the composite

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_{n+1}(\mathbb{R} \oplus \mathbb{R}^\infty) \xrightarrow{\cong} \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$$

where the first map sends an n-plane W to $\langle e_0 \rangle \oplus W$, where e_0 is the standard basis for the added copy of $\mathbb{R}$. The second map is the one induced by any linear homeomorphism $\mathbb{R} \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ (they all induced homotopic maps on the Grassmannisn by Corollary 13.12). For specificity let us choose the linear homeomorphism that sends $e_i \mapsto e_{i+1}$ for all i. To check that P_1 really does induce the "$\oplus \underline{1}$" natural transformation, just verify that $P_1^*\gamma_{n+1} \cong \gamma_n \oplus 1$; this is routine.

As another example consider the direct sum operation $\mathrm{Vect}_n(X) \times \mathrm{Vect}_k(X) \to \mathrm{Vect}_{n+k}(X)$, sending $(E, E') \mapsto E \oplus E'$. This will be induced by a homotopy class of maps $\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+k}(\mathbb{R}^\infty)$. Such a map can be constructed as

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty) \xrightarrow{\oplus} \mathrm{Gr}_{n+k}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \xrightarrow{\cong} \mathrm{Gr}_{n+k}(\mathbb{R}^\infty)$$

where the first map sends a pair of subspaces (J_1, J_2) to $J_1 \oplus J_2$ and the second is induced by any linear homeomorphism $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ (they all induce homotopic maps by Corollary 13.12. For example, choose any bijection $\alpha\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ and then use the linear map that sends $e_i \oplus e_j \mapsto e_{\alpha(i,j)}$. Again, to justify that P does represent the direct sum map one verifies that $P^*\gamma_{n+k} \cong \gamma_n \hat{\oplus} \gamma_k$, which is an easy exercise.

Continuity of the $\oplus$ map requires some comment. If the product $\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty)$ is given the compactly-generated topology then it may be identified with $\mathrm{colim}_{s,t}[\mathrm{Gr}_n(\mathbb{R}^s) \times \mathrm{Gr}_k(\mathbb{R}^t)]$ and then continuity of the map is immediate. However, one can also verify continuity when the domain has the product topology. Start with the composite

$$\begin{tikzcd} VI_n(\mathbb{R}^\infty)\times VI_k(\mathbb{R}^\infty) \arrow[r, tail] & (\mathbb{R}^\infty)^n\times(\mathbb{R}^\infty)^k \arrow[r, equal] & (\mathbb{R}^\infty\times\{0\})^n\times(\{0\}\times\mathbb{R}^\infty)^k \arrow[d, tail] \\ & & (\mathbb{R}^\infty\oplus\mathbb{R}^\infty)^{n+k} \end{tikzcd}$$

and observe that the image lands inside $VI_{n+k}(\mathbb{R}^\infty\oplus\mathbb{R}^\infty)$. Next consider the diagram

$$\begin{tikzcd} VI_n(\mathbb{R}^\infty)\times VI_k(\mathbb{R}^\infty) \arrow[r] \arrow[d] & VI_{n+k}(\mathbb{R}^\infty) \arrow[r] & \mathrm{Gr}_{n+k}(\mathbb{R}^\infty) \\ \mathrm{Gr}_n(\mathbb{R}^\infty)\times\mathrm{Gr}_k(\mathbb{R}^\infty). \arrow[urr, dotted] & & \end{tikzcd}$$

The maps $VI_n(\mathbb{R}^\infty)\to\mathrm{Gr}_n(\mathbb{R}^\infty)$ and $VI_k(\mathbb{R}^\infty)\to\mathrm{Gr}_k(\mathbb{R}^\infty)$ are fiber bundles and therefore are open quotient maps. The product of two open quotient maps is again an open quotient map (exercise!). The quotient relations for $VI_n(\mathbb{R}^\infty)\times VI_k(\mathbb{R}^\infty)\to\mathrm{Gr}_n(\mathbb{R}^\infty)\times\mathrm{Gr}_k(\mathbb{R}^\infty)$ are satisfied by the composite (*), so this yields the dotted arrow in the diagram. This arrow is readily verified to by $\oplus$.

For a third method of verifying the continuity of $\oplus$, see Exercise13.17.

There are many examples similar to the previous two where one constructs explicit maps on classifying spaces representing some algebraic construction. Let us do just one more. Given any vector space V we know how to construct the exterior product Λ^kV and this is functorial. An n-plane $J\subseteq\mathbb{R}^\infty$ therefore gives an $\binom{n}{k}$-plane $\Lambda^kJ\subseteq\Lambda^k\mathbb{R}^\infty$. Choosing a linear homeomorphism $\Lambda^k\mathbb{R}^\infty\cong\mathbb{R}^\infty$ therefore gives the map L obtained as the composite

$$\mathrm{Gr}_n(\mathbb{R}^\infty)\xrightarrow{\Lambda^k}\mathrm{Gr}_{\binom{n}{k}}(\Lambda^k\mathbb{R}^\infty)\xrightarrow{\cong}\mathrm{Gr}_{\binom{n}{k}}(\mathbb{R}^\infty).$$

Given a rank n bundle $E\to X$ we could define its kth exterior product to be the bundle represented by $L\circ f$, where f is any representing map for E.

Exercise 13.15. Verify continuity of the Λ^k map in the above composite. [Hint: It suffices to show that the map preserves convergent sequences. Use the fiber bundle $VI_n(\mathbb{R}^\infty)\to\mathrm{Gr}_n(\mathbb{R}^\infty)$.]

Exercise 13.16. Think about some of the pros and cons of constructing Λ^kE in the above manner, compared to the construction that chooses a local trivialization of the bundle and applies Λ^k to each patch.

Exercise 13.17. Prove that $\mathrm{Gr}_n(\mathbb{R}^\infty)\times\mathrm{Gr}_k(\mathbb{R}^\infty)$ is sequentially determined, and then prove that the map $\oplus\colon\ \mathrm{Gr}_n(\mathbb{R}^\infty)\times\mathrm{Gr}_k(\mathbb{R}^\infty)\to\mathrm{Gr}_{n+k}(\mathbb{R}^\infty\oplus\mathbb{R}^\infty)$ is continuous by proving that it preserves convergent sequences. [Hint: Consult Proposition 13.6 and Exercise A.7(g).]

13.18. **Stabilization of vector bundles.** Here is a simple application of classifying spaces that we will occasionally find useful. Fix a space X. If $E\to X$ is a vector bundle of rank n, then of course $E\oplus\underline{1}$ is a vector bundle of rank $n+1$. We get a sequence of maps

$$\mathrm{Vect}_0(X)\xrightarrow{\oplus 1}\mathrm{Vect}_1(X)\xrightarrow{\oplus 1}\mathrm{Vect}_2(X)\xrightarrow{\oplus 1}\cdots$$

Are these maps injective? Surjective? Are there more and more isomorphism classes of vector bundles as one goes up in rank, or is it the case that all "large" rank vector bundles actually come from smaller ones via addition of a trivial bundle? A homotopical analysis of classifying spaces yields some partial answers here. We handle both the case of real and complex bundles:

Proposition 13.19. *Let X be a finite-dimensional CW-complex. For real vector bundles,* $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ *is a bijection for* $n \geq \dim X + 1$ *and a surjection for* $n = \dim X$. *For complex bundles,* $\mathrm{Vect}_n^{\mathbb{C}}(X) \to \mathrm{Vect}_{n+1}^{\mathbb{C}}(X)$ *is a bijection for* $n \geq \frac{1}{2}\dim X$ *and a surjection for* $n \geq \frac{1}{2}(\dim X - 1)$.

Proof. The map $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ is represented by a map of spaces $f\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$. One model for this map is the one that sends a subspace $V \subseteq \mathbb{R}^\infty$ to $\mathbb{R}\oplus V \subseteq \mathbb{R}\oplus\mathbb{R}^\infty$ and then uses a fixed isomorphism $\mathbb{R}\oplus\mathbb{R}^\infty \cong \mathbb{R}^\infty$ to obtain a point in $\mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$. To establish the proposition we must analyze when $[X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \xrightarrow{f_*} [X, \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)]$ is injective/surjective.

Now, the inclusion $\mathrm{Gr}_n(\mathbb{R}^\infty) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$ is n-connected. This can be argued in different ways, but one way is to examine the Schubert cell decompositions of each space and observe that they are identical until one reaches dimension $n+1$. This connectivity result implies that $[B, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to [B, \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)]$ is bijective for CW-complexes with $\dim B \leq n-1$, and surjective for CW-complexes with $\dim B = n$. We simply apply this to $B = X$.

For the complex case, $\mathrm{Gr}_n(\mathbb{C}^\infty) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{C}^\infty)$ is now $(2n+1)$-connected. So we get the analogous bijection for CW-complexes B of dimension at most $2n$, and the surjection when $\dim B = 2n+1$. □

◦ Exercises ◦

Exercise 13.20. Consider the natural transformation $\mathrm{Vect}_1(X) \times \mathrm{Vect}_1(X) \to \mathrm{Vect}_1(X)$ given by $(L_1, L_2) \mapsto L_1 \otimes L_2$. This is represented by a map $m\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$. Prove that the induced map m^* on $H^*(-;\mathbb{Z}/2)$ sends the generator $x \in H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$ to $x\otimes 1 + 1\otimes x$. [Note that since $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$ this characterization completely determines the homotopy class of m.]

Exercise 13.21. If $L \to X$ is a real line bundle then $L\otimes L$ is trivial. Give three proofs of this fact, using the following strategies:

(i) Equip L with an inner product and then construct a nonzero section of $L\otimes L$ by patching local sections together.

(ii) Prove that if L is any line bundle then the evaluation homomorphism $L\otimes L^* \to \underline{1}$ is an isomorphism, and then use that $L \cong L^*$.

(iii) Write down a model for the map $f\colon \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ that represents the natural transformation $\mathrm{Vect}_1(X) \to \mathrm{Vect}_1(X)$ given by $L \to L\otimes L$, and prove that f is homotopic to the identity.

14. Topological K-theory

For a compact and Hausdorff space X, let $KO(X)$ denote the Grothendieck group of (finite rank) real vector bundles over X. Swan's Theorem gives that

$KO(X) \cong K_{\text{alg}}(C(X))$, where the latter denotes the Grothendieck group of finitely-generated projectives. We can repeat this definition for both complex and quaternionic bundles, to define groups $KU(X)$ and $KSp(X)$, respectively. The group $KU(X)$ is most commonly just written $K(X)$ for brevity. In this section we start to develop the general theory of these groups, mostly concentrating on $KO(X)$ because the story is very analogous in the three cases.

The main point is that $KO(X)$ is the 0th group in a generalized cohomology theory (and likewise for $KU(X)$ and $KSp(X)$). We will sketch the construction of this theory, though some key elements will be postponed until later.

Until we explicitly mention otherwise, all spaces in this section are assumed to be compact and Hausdorff.

14.1. **Initial observations on *KO*.** Observe that $KO(-)$ is a contravariant functor: if $f\colon X \to Y$ then $f^*\colon KO(Y) \to KO(X)$ sends $[E]$ to $[f^*E]$. In particular, the squash map $p\colon X \to *$ yields a split-inclusion $p^*\colon KO(*) \to KO(X)$, where the splitting is induced by any choice of basepoint in X. One has $KO(*) \cong \mathbb{Z}$, so $\mathbb{Z}$ is a direct summand of $KO(X)$. To analyze the complement we can take two different approaches:

Definition 14.2. *For $x \in X$ let $\widetilde{KO}(X,x) = \ker[KO(X) \xrightarrow{i^*} KO(x)]$ where $i\colon \{x\} \hookrightarrow X$. Further, define $KO^{st}(X) = KO(X)/p^*KO(*)$.*

The group $\widetilde{KO}(X,x)$ is called the **reduced KO-group** of the pointed space X. We call $KO^{st}(X)$ the **Grothendieck group of stable vector bundles on X**. The reason for the latter terminology will be clear momentarily. These two groups are isomorphic; algebraically, this is coming from the split-exact sequence

$$0 \longrightarrow KO(*) \xrightarrow{p^*} KO(X) \longrightarrow KO^{st} \longrightarrow 0.$$

If $i\colon \{x\} \hookrightarrow X$ is the inclusion then i^* is a splitting for the first map in the sequence. One gets an isomorphism between $KO^{st}(X)$ and $\ker i^*$ in the evident way, by sending a class $[E]$ to $[E] - p^*i^*[E]$. This isomorphism is used so frequently that it is worth recording more visibly:

$$KO^{st}(X) \cong \widetilde{KO}(X,x) \text{ via } [E] \mapsto [E] - [\underline{\operatorname{rank}_x(E)}]. \tag{14.3}$$

Remark 14.4. Both $KO^{st}(X)$ and $\widetilde{KO}(X,x)$ appear often in algebraic topology, and topologists are somewhat cavalier about mixing them up. We give here one example where this can cause confusion.

Tensor product of bundles makes $KO(X)$ into a ring, via the formula $[E]\cdot[F] = [E \otimes F]$ and extending linearly. Then $\widetilde{KO}(X,x)$ is an ideal of this ring. Therefore $KO^{st}(X)$ may be given a product via the above isomorphism, but this product is *not* $[E]\cdot[F] = [E\otimes F]$. Indeed, it is clear that this definition would not be invariant under $E \mapsto E \oplus 1$. The product on $KO^{st}(X)$ is instead $[E]\cdot[F] = [E\otimes F] - (\operatorname{rank} E)[F] - (\operatorname{rank} F)[E] + (\operatorname{rank} E)(\operatorname{rank} F)$.

We offer the following alternative description of $KO^{st}(X)$. Let $\operatorname{Vect}(X)$ be the set of isomorphism classes of vector bundles on X, and impose the equivalence relation $E \simeq E \oplus \underline{1}$ for every vector bundle E. The set of equivalence classes is obviously a monoid under direct sum, but it is actually more than a monoid: it is a group. To see this, recall that if E is any vector bundle over X then there exists an embedding $E \hookrightarrow \underline{N}$ for sufficiently large N (Proposition 9.10). If Q is the quotient then we have the exact sequence $0 \to E \to \underline{N} \to Q \to 0$, which is split by Proposition 9.2. So $E \oplus Q \cong \underline{N}$. Yet $\underline{N} = 0$ under our equivalence relation, and so E has an additive inverse. It is easy to see that $KO^{st}(X)$ is precisely this set of equivalence classes.

Finally, here is a third description of $KO^{st}(X)$. Consider the chain of maps

$$\mathrm{Vect}_0(X) \xrightarrow{\oplus 1} \mathrm{Vect}_1(X) \xrightarrow{\oplus 1} \mathrm{Vect}_2(X) \xrightarrow{\oplus 1} \cdots$$

When X is path-connected the colimit is the set of equivalence classes described in the preceding paragraph, and therefore coincides with $KO^{st}(X)$. Note that if X were not path connected then we would only be getting the monoid of vector bundles of constant rank on X. Recall that $\mathrm{Vect}_n(X) = [X, \mathrm{Gr}_n(\mathbb{R}^\infty)]$, and one easily sees that the $\oplus 1$ map is represented by the map of spaces

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_{n+1}(\mathbb{R} \oplus \mathbb{R}^\infty) = \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$$

that sends a subspace $U \subseteq \mathbb{R}^\infty$ to $\mathbb{R} \oplus U \subseteq \mathbb{R} \oplus \mathbb{R}^\infty$. Let $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ denote the colimit of these maps

$$\mathrm{Gr}_1(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \mathrm{Gr}_2(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \mathrm{Gr}_3(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \cdots$$

(we really want the homotopy colimit, if you know what that is, but in this case the colimit has the same homotopy type and is good enough). You might recall that $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is also called BO_n, and likewise $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ is also called BO.

Then for compact, path-connected Hausdorff spaces X we have a bijection

$$\operatorname*{colim}_n\, [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \longrightarrow [X, \mathrm{Gr}_\infty(\mathbb{R}^\infty)].$$

So we have learned that $KO^{st}(X) \simeq [X, BO]$.

If X has a basepoint then we can consider $[X, BO]_*$ instead of $[X, BO]$. There is the evident map $[X, BO]_* \to [X, BO]$. Typically there would be no reason for this to be a bijection, but BO is a path-connected H-space: and in that setting the map *is* a bijection. So in fact we can write

$$KO^{st}(X) \simeq [X, BO]_* \qquad (X \text{ path-connected}).$$

Applying this in particular to $X = S^k$ we have that for $k \geq 1$

$$\widetilde{KO}(S^k) \cong KO^{st}(S^k) \cong [S^k, BO] \cong [S^k, BO]_* = \pi_k(BO) = \pi_{k-1}(O).$$

The calculations of Bott therefore give us the values of $\widetilde{KO}(S^k)$. For $k = 0$ observe that $KO(S^0) = KO(* \sqcup *) \cong \mathbb{Z} \oplus \mathbb{Z}$, so we have $\widetilde{KO}(S^0) \cong \mathbb{Z}$. This lets us fill out the table:

TABLE 14.4. Reduced KO-theory of spheres

k	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\widetilde{KO}(S^k)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\cdots$

Now let X be an arbitrary CW-complex, not necessarily compact or connected. We define

$$KO(X) = [X_+, \mathbb{Z} \times BO]_* = [X, \mathbb{Z} \times BO],$$

where X_+ denotes X with a disjoint basepoint added. For a pointed space X we define $\widetilde{KO}(X) = [X, \mathbb{Z} \times BO]_*$.

As we have seen before, Bott Periodicity shows that the homotopy groups of $\mathbb{Z} \times BO$ are 8-fold periodic. This is a consequence of the following stronger statement:

Theorem 14.5 (Bott Periodicity, Strong version)**.** *There is a weak equivalence of spaces $\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO)$.*

Using Bott Periodicity we can then calculate that for every pointed space X one has

$$\widetilde{KO}(\Sigma^8 X) = [\Sigma^8 X, \mathbb{Z} \times BO]_* = [X, \Omega^8(\mathbb{Z} \times BO)]_* = [X, \mathbb{Z} \times BO]_* = \widetilde{KO}(X).$$

Remark 14.6. In the complex case, Bott Periodicity gives the weak equivalence $\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU)$. Consequently one obtains $\widetilde{K}(\Sigma^2 X) \cong \widetilde{K}(X)$ for all pointed spaces X.

14.7. **K-theory as a cohomology theory.** When X is compact and Hausdorff we have seen that $KO(X) \cong [X_+, \mathbb{Z} \times BO]_*$, where X_+ is X with a disjoint basepoint added. The point of this isomorphism is that it immediately gives us several tools for computing $KO(X)$ that we didn't have before. These are tools that work for homotopy classes of maps in reasonable generality, so let us discuss them in that broader context.

Let X and Z be pointed spaces. Then $[X, Z]_*$ is just a pointed set, but if we suspend the space in the domain then we get a bit more structure: $[\Sigma X, Z]_*$ is a group, where ΣX is the reduced suspension of X. One way to see this is to collapse the equatorial copy of X in ΣX, to get $\Sigma X \vee \Sigma X$; write this collapse map as

$$\nabla \colon \Sigma X \to \Sigma X \vee \Sigma X.$$

The operation on $[\Sigma X, Z]_*$ is defined by precomposing the wedge of two homotopy classes with ∇. With some trouble one checks that ΣX is a cogroup object in the homotopy category of pointed spaces, which yields that $[\Sigma X, Z]_*$ is a group.

Here is another way to think about this, which relates it to something we already know. Let $F(X, Z)$ be the set of functions from X to Z, equipped with the compact-open topology. We can write

$$[\Sigma X, Z]_* = [S^1, F(X, Z)]_* = \pi_1(F(X, Z))$$

where the basepoint of $F(X, Z)$ is the map sending all of X to the basepoint of Z. Now just use that $\pi_1(F(X, Z))$ is a group.

When $k \geq 2$ then we have $[\Sigma^k X, Z]_* = \pi_k(F(X, Z))$ by a similar argument, and so $[\Sigma^k X, Z]_*$ is an abelian group. Alternatively, one proves that now $\Sigma^k X$ is a *cocommutative* cogroup object in the homotopy category.

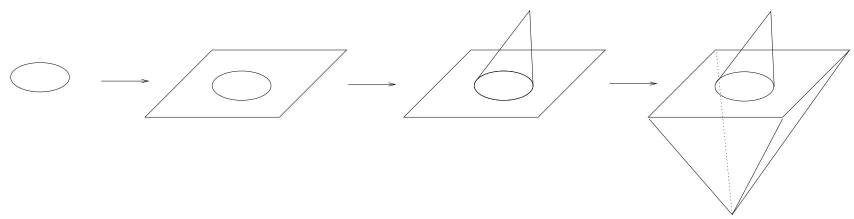
Similar results are obtained by putting conditions on Z rather than X. If Z is a loop space, say $Z \simeq \Omega Z_1$, then $[X, Z]_* \cong [X, \Omega Z_1]_* \cong [\Sigma X, Z_1]_*$, and this is a group by the above arguments. Similarly, if Z is a k-fold loop space for $k \geq 2$, say $Z \simeq \Omega^k Z_1$, then $[X, Z]_* \cong [\Sigma^k, Z_1]_*$ and this is an abelian group.

Homotopy classes of maps into a fixed space Z always give rise to exact sequences:

Proposition 14.8. *Let X, Y be pointed spaces, and let $f \colon X \to Y$ be a pointed map. Consider the mapping cone Cf and the natural map $p \colon Y \to Cf$. For any pointed space Z, the sequence of pointed sets $[X, Z]_* \leftarrow [Y, Z]_* \leftarrow [Cf, Z]_*$ is exact in the middle (meaning that anything in $[Y, Z]_*$ which is sent to the basepoint is in the image of the previous map).*

Proof. Let $h\colon Y \to Z$ and suppose $h \circ f$ is homotopic to the constant map. Choose a pointed homotopy $H\colon X \times I \to Z$ so that $H(X \times \{1\}) = *$. Then H induces a map $CX \to Z$. Let $g\colon Cf \to Z$ be given by H on CX and h on Y. Then clearly $g \circ p = h$. □

Given $f\colon X \to Y$ we form the mapping cone Cf, which comes to us with an inclusion $j_0\colon Y \hookrightarrow Cf$. Next form the mapping cone on j_0, which comes with an inclusion $j_1\colon Cf \hookrightarrow Cj_0$. Keep doing this forever to get the sequence of spaces $X \to Y \to Cf \to Cj_0 \to Cj_1 \to \cdots$ depicted below:



Note that $Cj_0 \simeq \Sigma X$ and $Cj_1 \simeq \Sigma Y$ (this is clear from the pictures). Up to sign the map $Cj_0 \to Cj_1$ is just Σf, so that the sequence of spaces becomes periodic:

$$X \to Y \to Cf \to \Sigma X \to \Sigma Y \to \Sigma(Cf) \to \Sigma^2 X \to \dots$$

This is called the **Puppe sequence**. Note that the composition of two subsequent maps is null-homotopic, and that every three successive terms form a cofiber sequence.

Now let Z be a fixed space and apply $[-, Z]_*$ to the Puppe sequence. We obtain the sequence of pointed sets

$$[X, Z]_* \leftarrow [Y, Z]_* \leftarrow [Cf, Z]_* \leftarrow [\Sigma X, Z]_* \leftarrow [\Sigma Y, Z]_* \leftarrow [\Sigma(Cf), Z]_* \leftarrow \dots$$

By Proposition 14.8 this sequence is exact at every spot where this makes sense (everywhere except at $[X, Z]_*$). At the left end this is just an exact sequence of pointed sets, but as one moves to the right at some point it becomes an exact sequence of groups (namely, at $[\Sigma Y, Z]_*$). As one moves further to the right, it becomes an exact sequence of *abelian* groups by the time one gets to $[\Sigma^2 Y, Z]_*$.

If $Z \simeq \Omega Z_1$ then we can extend the above sequence a little further to the left, by noticing that the sequences for $[-, Z_1]_*$ and $[-, Z]_*$ mesh together:

$$\begin{tikzcd}
{[X,Z_1]_*} & {[Y,Z_1]_*} \arrow[l] & {[Cf,Z_1]_*} \arrow[l] & {[\Sigma X,Z_1]_*} \arrow[l] \arrow[d, equal] & {[\Sigma Y,Z_1]_*} \arrow[l] \arrow[d, equal] & {[\Sigma(Cf),Z_1]} \arrow[l] \arrow[d, equal] & \arrow[l] \\
 & & & {[X,\Omega Z_1]_*} \arrow[d, equal] & {[Y,\Omega Z_1]_*} \arrow[l] \arrow[d, equal] & {[Cf,\Omega Z_1]} \arrow[l] \arrow[d, equal] & \arrow[l] \\
 & & & {[X,Z]_*} & {[Y,Z]_*} \arrow[l] & {[Cf,Z]} \arrow[l] & \arrow[l]
\end{tikzcd}$$

Note that the leftmost cycle of the original sequence, which we had thought consisted just of pointed sets, in fact consisted of groups! If in turn we have $Z_1 \simeq \Omega Z_2$ then we can play this game again and extend the sequence one more cycle to the left, and so forth. If we are *really* lucky then we can do this forever:

Definition 14.9. *An* ***infinite loop space*** *is a space Z_0 together with spaces $Z_1, Z_2, Z_3, \dots$ and weak homotopy equivalences $Z_n \simeq \Omega Z_{n+1}$ for all $n \geq 0$.*

Note that if Z is an infinite loop space then we really do get a long exact sequence—infinite in both directions—consisting entirely of abelian groups, having the form

$$\cdots \leftarrow [Cf, Z_{i+1}]_* \leftarrow [X, Z_i]_* \leftarrow [Y, Z_i]_* \leftarrow [Cf, Z_i]_* \leftarrow [X, Z_{i-1}]_* \leftarrow \cdots$$

where it is convenient to use the indexing convention $Z_{-n} = \Omega^n Z$ for $n > 0$.

This situation is very reminiscent of a long exact sequence in cohomology, so let us adopt the following notation: write

$$E^i_Z(X) = [X_+, Z_i]_* = \begin{cases} [X_+, Z_i]_* & i \geq 0, \\ [\Sigma^{-i}(X_+), Z_0]_* & i < 0. \end{cases}$$

For an inclusion of subspaces $j\colon A \hookrightarrow X$ write

$$E^i_Z(X, A) = [Cj, Z_i]_* = \begin{cases} [Cj, Z_i]_* & i \geq 0, \\ [\Sigma^i(Cj), Z_0]_* & i < 0. \end{cases}$$

It is not hard to check that this *is* a generalized cohomology theory. So we get a generalized cohomology theory whenever we have an infinite loop space. (You may know that it works the other way around, too: every generalized cohomology comes from an infinite loop space. But we won't need that fact here.)

For us the importance of all of this is that by Bott's theorem we have

$$\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO) \simeq \Omega^{16}(\mathbb{Z} \times BO) \simeq \dots.$$

Thus, $\mathbb{Z} \times BO$ is an infinite loop space and the above machinery applies. We obtain a cohomology theory KO^*. Moreover, periodicity gives us that $KO^{i+8}(X, A) \cong KO^i(X, A)$, for any i.

This all works in the complex case as well. There we have $\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU)$, so $\mathbb{Z} \times BU$ is again an infinite loop space. We get a cohomology theory K^* that is 2-fold periodic.

14.10. **Afterward.** The point of this section was to construct the cohomology theories KO and K, having the properties that when X is compact and Hausdorff the groups $KO^0(X)$ and $K^0(X)$ coincide with the Grothendieck groups of real and complex vector bundles over X. We have now accomplished this! We will spend the rest of these notes exploring what one can do with such cohomology theories, i.e., what they are good for. We have already said that one thing they are good for is calculation; we close this section with an example demonstrating the benefits and limitations here.

Let us try to compute $KO(\mathbb{R}P^2)$. Recall the ubiquitous decomposition $KO(\mathbb{R}P^2) = \mathbb{Z} \oplus \widetilde{KO}(\mathbb{R}P^2) = \mathbb{Z} \oplus KO^{st}(\mathbb{R}P^2)$. Next use the fact that $\mathbb{R}P^2$ can be built by attaching a 2-cell to $\mathbb{R}P^1 = S^1$, where the attaching map wraps S^1 around itself twice. That is, $\mathbb{R}P^2$ is the mapping cone for $S^1 \xrightarrow{2} S^1$. The Puppe sequence for this map looks like

$$S^1 \xrightarrow{2} S^1 \longrightarrow \mathbb{R}P^2 \longrightarrow S^2 \xrightarrow{2} S^2 \longrightarrow \cdots$$

hence we have an exact sequence

$$\leftarrow \cdots \leftarrow \widetilde{KO}(S^1) \leftarrow \widetilde{KO}(S^1) \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \widetilde{KO}(S^2) \leftarrow \widetilde{KO}(S^2) \leftarrow \cdots$$

Note that this is just the long exact sequence for the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$ in $\widetilde{KO}$-cohomology, where we are using the identification $\widetilde{KO}(S^2) = \widetilde{KO}^0(S^2) = \widetilde{KO}^{-1}(S^1)$.

We know $\widetilde{KO}(S^k)$ for all $k \geq 0$, so the above sequence becomes

$$\mathbb{Z}/2 \leftarrow \mathbb{Z}/2 \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \mathbb{Z}/2 \leftarrow \mathbb{Z}/2$$

where both maps $\mathbb{Z}/2 \leftarrow \mathbb{Z}/2$ are multiplication by 2, i.e. the 0 map. Hence we have a short exact sequence

$$0 \leftarrow \mathbb{Z}/2 \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \mathbb{Z}/2 \leftarrow 0, \tag{14.11}$$

and so $\widetilde{KO}(\mathbb{R}P^2)$ is either $(\mathbb{Z}/2)^2$ or $\mathbb{Z}/4$. It remains to decide which one.

The short exact sequence in (14.11) is really

$$0 \longleftarrow \widetilde{KO}(S^1) \xleftarrow{i^*} \widetilde{KO}(\mathbb{R}P^2) \xleftarrow{p^*} \widetilde{KO}(S^2) \longleftarrow 0.$$

We have previously seen that the generator of $\widetilde{KO}^0(S^1) = KO^{st}(S^1)$ corresponds to the Mobius bundle $[M]$, and the generator of $\widetilde{KO}(S^2) = KO^{st}(S^2)$ is $[\mathcal{O}(1)]$, the rank 2 bundle whose clutching map is the isomomorphism $S^1 \to SO(2)$. The image of $[\mathcal{O}(1)]$ in $\widetilde{KO}^0(\mathbb{R}P^2)$ is $p^*\mathcal{O}(1)$, where $p\colon \mathbb{R}P^2 \to S^2$ is the projection.

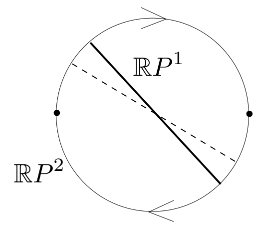
We happen to know one bundle on $\mathbb{R}P^2$, the tautological line bundle γ. When we restrict γ to $\mathbb{R}P^1$ we get M, and so $[\gamma]$ is a preimage for $[M]$ under i^*. We need to decide if $2[\gamma] = 0$ in $KO^{st}(\mathbb{R}P^2)$; if it is, then $\widetilde{KO}(\mathbb{R}P^2) \cong (\mathbb{Z}/2)^2$ and if it is not then $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$. So the question becomes: is $\gamma \oplus \gamma$ stably trivial?

The answer turns out to be that $\gamma \oplus \gamma$ is *not* stably trivial; this is an elementary exercise using characteristic classes (Stiefel-Whitney classes), but we have not discussed such techniques yet—see Section 26.9 below for complete details. For now we will just accept this fact, and conclude that $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$. Note that this calculation demonstrates an important principle to keep in mind: often the machinery of cohomology theories get you a long way, but not quite to the end, and one has to do some geometry to complete the calculation.

There is a better way to think about this calculation, and we can't resist pointing it out even though it won't make complete sense yet. But it ties in to intersection theory, which is our overarching theme in these notes. In our discussion above we used $KO^{st}(\mathbb{R}P^2)$ as our model for $\widetilde{KO}(\mathbb{R}P^2)$, but let us change perspective and use the model that is the kernel of $KO(\mathbb{R}P^2) \to KO(*)$, for some chosen basepoint. Recall that $[E]$ in $KO^{st}(\mathbb{R}P^2)$ corresponds to $[E] - \mathrm{rank}(E)$ in $\widetilde{KO}(\mathbb{R}P^2)$; so the class we wrote as $[\gamma]$ is $[\gamma] - 1$ in the shifted perspective, and we need to decide if $2([\gamma] - 1) = 0$ in $KO(\mathbb{R}P^2)$. The element $1 - [\gamma]$ should be thought of as corresponding to a chain complex of vector bundles

$$0 \to \gamma \to 1 \to 0,$$

and thinking of it this way one finds that it plays the role of the K-theoretic fundamental class of the submanifold $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$. Then $(1 - [\gamma])^2$ represents the self-intersection product of $\mathbb{R}P^1$ inside $\mathbb{R}P^2$, which we know is a point by the standard geometric argument (shown in the picture below, depicting an $\mathbb{R}P^1$ and a small perturbation of it)):



In particular, the self-intersection is not empty. This translates to the statement that $(1-[\gamma])^2 \neq 0$. But

$$(1-[\gamma])^2 = 1 - 2[\gamma] + [\gamma^2] = 1 - 2[\gamma] + 1 = 2(1-[\gamma])$$

where we have used that $\gamma \otimes \gamma \cong \underline{1}$ (this is true for any real line bundle, over any base); so this explains why $2(1-[\gamma]) \neq 0$. Again, we understand this argument doesn't make much sense yet. We will come back to it in Section ?????. For the moment just get the idea that it is the intersection theory of submanifolds in $\mathbb{R}P^2$ that is ultimately forcing $\widetilde{KO}(\mathbb{R}P^2)$ to be $\mathbb{Z}/4$ rather than $(\mathbb{Z}/2)^2$.

Remark 14.12. It seems worth pointing out that in fact for every n one has $\widetilde{KO}(\mathbb{R}P^n) \cong \mathbb{Z}/2^k$ for a certain value k depending on n. We will return to this calculation (and complete it) in Section 37.

Exercise 14.13. It is a good idea for the reader to try their hand at similar calculations, to see how the machinery is working. Try calculating some of the groups below, at least for small values of n:

- $K(\mathbb{C}P^n)$ (reasonably easy)
- $KO(\mathbb{C}P^n)$ (a little harder)
- $K(\mathbb{R}P^n)$ (even harder)
- $KO(\mathbb{R}P^n)$ (hardest).

Don't worry if you can't completely determine some of the groups; just see how far the machinery takes you.

14.14. ***K*-theory for non-compact spaces.** The reader will perhaps have noticed that for arbitrary spaces X we now have two competing definitions:

$$\begin{aligned} KO_{Grt}(X) &= \text{Grothendieck group of vector bundles over } X \\ KO^0(X) &= [X, \mathbb{Z} \times BO]. \end{aligned}$$

In older literature the second group is sometimes called **representable *KO*-theory**. When X is compact and Hausdorff we have seen that these groups are isomorphic. What about more generally?

For X paracompact and Hausdorff we get a natural map $\Phi\colon KO_{Grt}(X) \to KO^0(X)$. To explain this we might as well assume that X is path-connected as well. Recall from Proposition 9.2 that all short exact sequences of vector bundles are split. From this it follows that $KO_{Grt}(X)$ is the group completion of the monoid $\mathrm{Vect}(X)$ of isomorphism classes of vector bundles. To a rank n vector bundle $E \to X$ we assign the pair $(n, j \circ f)$ where f is a classifying map $X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ and $j\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \hookrightarrow \mathrm{Gr}_\infty(\mathbb{R}^\infty)$ is the standard inclusion. The map f involves a choice, but it is well-defined up to homotopy. So this assignment gives a map of monoids $\mathrm{Vect}(X) \to [X, \mathbb{Z} \times BO]$, and since the latter is a group it induces $KO_{Grt}(X) \to KO^0(X)$.

We know that Φ is an isomorphism when X is compact. We also know that both the domain and range are homotopy invariant, and so we can generalize this slightly.

Definition 14.15. *A space is* ***homotopically compact*** *if it is weakly equivalent to a finite cell complex. A pair (X, A) is* ***homotopically compact*** *is there is a finite CW-complex Y and subcomplex B for which there is a zig-zag of weak equivalences between (X, A) and (Y, B).*

Proposition 14.16. *If X is a homotopically compact CW-complex then Φ is a bijection.*

Proof. Since X is a CW-complex, homotopically compact implies that X is homotopy equivalent (not just weakly equivalent) to a finite cell complex X'. Then the map Φ_X is isomorphic to $\Phi_{X'}$, and the latter is a bijection since X' is compact Hausdorff. $\square$

The map Φ is in general neither injective nor surjective. Here is an example where the latter fails:

Example 14.17. We claim that the map $K_{Grt}(\mathbb{R}P^\infty) \to K^0(\mathbb{R}P^\infty)$ is not surjective. Let $L \to \mathbb{R}P^\infty$ be the tautological bundle, and let $J = cL$ be its complexification. With some trouble one can completely analyze $\mathrm{Vect}_n^{\mathbb{C}}(\mathbb{R}P^\infty) = [\mathbb{R}P^\infty, BU(n)]$, and one finds that it consists of the bundles $rJ \oplus \underline{(n-r)}$ for $0 \le r \le n$. Conseqeuntly, $K_{Grt}(\mathbb{R}P^\infty) = \mathbb{Z} \oplus \mathbb{Z}$, with the summands generated by $\underline{1}$ and $[J]$. In contrast, $K^0(\mathbb{R}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}_2^\wedge$ where the second summand denotes the 2-adic integers. The completion is a phenomenon that often arises when dealing with homotopy classes of maps out of infinite complexes. Using the standard skeletal filtration of $\mathbb{R}P^\infty$ by finite projective spaces, one has an associated Milnor exact sequence of the form

$$0 \longrightarrow \lim_n{}^1 K^{-1}(\mathbb{R}P^n) \longrightarrow K^0(\mathbb{R}P^\infty) \longrightarrow \lim_n K^0(\mathbb{R}P^n) \longrightarrow 0.$$

In this case the $\lim^1$ term turns out to vanish, the $K^0(\mathbb{R}P^n)$ groups are all of the form $\mathbb{Z} \oplus \mathbb{Z}/2^{??}$ with the exponents increasing with n, and so one obtains $\mathbb{Z} \oplus \mathbb{Z}_2^\wedge$ for the inverse limit.

Elements of this example can be generalized to BG, for any finite group G. There is a map $\mathrm{Rep}_{\mathbb{C}}(G) \to K_{Grt}(BG)$ induced by sending a representation V to the bundle $EG \times_G V \to BG$. This map is not always an isomorphism, but it is so when G is a p-group (ref?).

Let $I(G)$ be the kernel of the dimension map $\dim\colon \mathrm{Rep}_{\mathbb{C}}(G) \to \mathbb{Z}$, which is typically called the **augmentation ideal**. The Atiyah-Segal completion theorem [AS] says that $K^0(BG) = \mathrm{Rep}_{\mathbb{C}}(G)_I^\wedge$. We get the diagram

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{C}}(G) \arrow[r] \arrow[d] & \mathrm{Rep}_{\mathbb{C}}(G)_I^\wedge \arrow[d, "\cong"] \\
K_{Grt}(BG) \arrow[r, "\Phi"] & K^0(BG).
\end{tikzcd}$$

When G is a p-group the left vertical map is an isomorphism, the top horizontal map is injective, and so the bottom horizontal map is injective as well [JO, Corollary 1.9].

Let us explore what the above says when $G = \mathbb{Z}/2$. Here there are two irreducible representations, namely the trivial and sign representations on $\mathbb{C}$. If we denote the latter as x then $\mathrm{Rep}_{\mathbb{C}}(G) = \mathbb{Z}[x]/(x^2-1)$. The augmentation ideal is $I = (x-1)$. If we set $R = \mathrm{Rep}_{\mathbb{C}}(G)$ then $\mathrm{Rep}_{\mathbb{C}}(G)^{\wedge}_I$ is the inverse limit of

$$\cdots \twoheadrightarrow R/I^3 \twoheadrightarrow R/I^2 \twoheadrightarrow R/I.$$

One quickly finds that $R/I = \mathbb{Z}$ (generated by 1), $R/I^2 = \mathbb{Z} \oplus \mathbb{Z}/2$ where the first summand is generated by 1 and the second by $x-1$, and $R/I^3 = \mathbb{Z} \oplus \mathbb{Z}/4$ with the same generators. To find the order of $x-1$ in R/I^n we use the division algorithm in $\mathbb{Z}[x]$ to write

$$(x-1)^n = (x^2-1)f(x) + (\text{linear polynomial})$$

and observe that since both $(x-1)^n$ and (x^2-1) vanish for $x = 1$, so must the linear polynomial. Hence we have

$$(x-1)^n = (x^2-1)f(x) + k(x-1)$$

for some k, or equivalently

$$(x-1)^{n-1} = (x+1)f(x) + k.$$

We can obtain k by plugging in $x = -1$, and so $k = (-2)^{n-1}$. Thus one finds that $R/I^n = \mathbb{Z} \oplus \mathbb{Z}/2^{n-1}$, with the first summand generated by 1 and the second by $x-1$. The completion $R^{\wedge}_I$ is therefore $\mathbb{Z} \oplus \mathbb{Z}^{\wedge}_2$, as desired.

15. Vector fields on spheres

It is a classical problem to determine how many independent vector fields one can construct on a given sphere S^n. This problem was heavily studied throughout the 1940s and 1950s, and then finally solved by Adams in 1962 using K-theory. It is one of the great successes of generalized cohomology theories. In this section we discuss some background to the vector field problem. We will not tackle the solution until Section 38, when we have more tools at our disposal.

15.1. **The vector field problem.** Given a nonzero vector $u = (x, y)$ in $\mathbb{R}^2$, there is a formula for producing a (nonzero) vector that is orthogonal to u: namely, $(-y, x)$. However, there is no analog of this that works in $\mathbb{R}^3$. That is, there is no single formula that takes a vector in $\mathbb{R}^3$ and produces a (nonzero) orthogonal vector. If such a formula existed then it would give a nonvanishing vector field on S^2, and we know that such a thing does not exist by elementary topology.

Let us next consider what happens in $\mathbb{R}^4$. Given $u = (x_1, x_2, x_3, x_4)$, we can produce an orthogonal vector via the formula $(-x_2, x_1, -x_4, x_3)$. But of course this is not the only way to accomplish this: we can vary what pairs of coordinates we choose to flip. In fact, if we consider

$$v_1 = \begin{bmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -x_4 \\ -x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

then we find that v_1, v_2, and v_3 are not only orthogonal to u but they are orthogonal to each other as well. In particular, at each point of S^3 we have given an orthogonal basis for the tangent space.

We aim to study this problem for any $\mathbb{R}^n$. What is the maximum k for which there exist formulas for starting with $u \in \mathbb{R}^n - \{0\}$ and producing k orthogonal (nonzero) vectors, with u as the first of the set? The following gives a different phrasing for essentially the same question:

Question 15.2. *On S^n, how many vectors fields $v_1, v_2, \ldots, v_r$ can we find so that $v_1(x), v_2(x), \ldots, v_r(x)$ are linearly independent for each $x \in S^n$?*

In colloquiual usage we will sometimes drop the phrase "linearly independent" and leave it to be understood. So for example, if we talk about constructing two vector fields on S^5 it is implicit that we mean independent vector fields, as otherwise the problem would be trivial!

Note that by the Gram-Schmidt process we can replace "linearly independent" by "orthonormal." If n is even, the answer is zero because there does not exist even a single nonvanishing vector field on an even sphere. To start to see what happens when n is odd, we look at a couple of more examples.

Let $u \in S^5$ have the standard coordinates. We notice that the vector $v_1 = (-x_2, x_1, -x_4, x_3, -x_6, x_5)$ is orthogonal to u. However, a little legwork shows that no other pattern of switching coordinates will produce a vector that is orthogonal to both u and v_1. Of course this does not mean that there isn't some more elaborate formula that would do the job, but it shows the limits of what we can do using our naive constructions.

For $u \in S^7$ we can divide the coordinates into the top four and the bottom four. Take the construction that worked for S^3 and repeat it simultaneously in the top and bottom coordinates—this yields a set of three orthonormal vector fields on S^7, given by the formulas

$$\begin{aligned} &(-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7), \\ &(-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6), \\ &(-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5). \end{aligned} \tag{15.2}$$

This idea generalizes at once to prove the following:

Proposition 15.3. *If there exist r (independent) vector fields on S^{n-1}, then there also exist r vector fields on S^{kn-1} for all k.*

For example, since there is one vector field on S^1 we also know that there is at least one vector field on S^{2k-1} for every k. Likewise, since there are three vector fields on S^3 we know that there are at least three vector fields on S^{4k-1} for every k.

We have constructed three vector fields on S^7, but one can actually make seven of them. This can be done via trial-and-error attempts at extending the patterns in (15.2), but there is a slicker way to accomplish this as well. Recall that S^3 is a Lie group, being the unit quaternions inside of $\mathbb{H}$. We can choose an orthonormal frame at the origin and then use the group structure to push this around to any point, thereby obtaining three independent vector fields; in other words, for any point $x \in S^3$ use the derivative of right-multiplication-by-x to transport our vectors in T_1S^3 to T_xS^3. The space S^7 is not quite a Lie group, but it still has a multiplication coming from being the set of unit octonions. The multiplication is not associative, but this is of no matter—the same argument works to construct 7 vector fields on S^7. Note that this immediately gives us 7 vectors fields on S^{15}, S^{23}, etc.

Based on the data so far, one would naturally guess that if $n = 2^r$ then there are $n-1$ vector fields on S^{n-1}. However, this guess turns out to fail already when $n = 16$ (and thereafter). To give a sense of how the numbers grow, we give a chart showing the maximum number of vector fields that exist on low-dimensional spheres:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$n-1$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
vf on S^{n-1}	1	3	1	7	1	3	1	8	1	3	1	7	1	3	1	9

Notice that we have explained how to construct the requisite number of vector fields until we get to S^{15}—there we have shown how to make seven of them, but the claim is that one more can be made. Once we know how to make eight on S^{15} we automatically know how to make eight on S^{31}, but the claim is again that one more can be made beyond this. By the end of this section we will know how to construct all of these.

Okay. Now that we have a basic sense of the problem let us explain the numerology behind the answer.

Definition 15.4. *If $n = m \cdot 2^{a+4b}$ where m is odd, then the **Hurwitz-Radon number** for n is $\rho(n) = 2^a + 8b - 1$.*

Theorem 15.5 (Hurwitz-Radon)**.** *There exist at least $\rho(n)$ independent vector fields on S^{n-1}.*

Consider $n = 32 = 2^5 = 2^{1+4\cdot 1}$. In the terms of Definition 15.4 we have $a = b = 1$, so that $\rho(32) = 2^1 + 8(1) - 1 = 9$. That is, there are at least 9 vector fields on S^{31}. If $n = 1024 = 2^{10} = 2^{2+4\cdot 2}$ then $\rho(n) = 2^2 + 8\cdot 2 - 1 = 19$; one can make 19 independent vector fields on S^{1023}. One should of course notice that these numbers are not going up very quickly.

We will prove the Hurwitz-Radon theorem by a slick, modern method using Clifford algebras. But it is worth pointing out that the theorem can be proven through very naive methods, too (it was proven in the 1920s). All of the Hurwitz-Radon vector fields follow the general patterns that we have seen, of switching pairs of coordinates and changing signs—one only has to find a way to organize the bookkeeping behind these patterns.

15.6. **Sums-of-squares formulas.** Hurwitz and Radon were not actually thinking about vector fields on spheres. They were instead considering an algebraic question about the existence of certain kinds of "composition formulas" for quadratic forms. For example, the following identity is easily checked:

$$(*) \qquad (x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2.$$

Hurwitz and Radon were looking for more formulas such as this one, for larger numbers of variables:

Definition 15.7. *A **sum-of-squares formulas** of type $[r, s, n]$ is an identity*

$$(x_1^2 + x_2^2 + \ldots + x_r^2)(y_1^2 + y_2^2 + \ldots + y_s^2) = z_1^2 + z_2^2 + \ldots + z_n^2$$

in the polynomial ring $\mathbb{R}[x_1, \dots, x_r, y_1, \dots, y_s]$*, where each* z_i *is a bilinear expression in* x*'s and* y*'s.*

We will often just refer to an "$[r,s,n]$-formula", for brevity. For what values of r, s, and n does such a formula exist? This is currently an open question. There are three formulas that are easily produced, coming from the normed algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The multiplication is a bilinear pairing, and the identity $|xy|^2 = |x|^2|y|^2$ is the required sums-of-squares formula. These algebras give formulas of type $[2,2,2]$, $[4,4,4]$, and $[8,8,8]$. (Check that the $[2,2,2]$ formula that comes from $\mathbb{C}$ is exactly formula (*) above). In a theorem from 1898 Hurwitz proved that these are the only normed algebras over the reals, and in doing so ruled out the existence of $[n,n,n]$-formulas for $n \notin \{1,2,4,8\}$. The question remained (and remains) about other types of formulas. See [Sh] for a detailed history of this problem.

Exercise 15.8. Use the multiplication table for $\mathbb{H}$ to write down the corresponding $[4,4,4]$ formula.

Perhaps surprisingly, most of what is known about the non-existence of sums-of-squares formulas comes from topology. To phrase the question differently, we are looking for a function $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ such that $|\phi(x,y)|^2 = |x|^2 \cdot |y|^2$ for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. The bilinear expressions $z_1, \dots, z_n$ are just the coordinates of $\phi(x,y)$.

Write $z = \phi(x,y) = \sum x_j A_j y$, where the A_j's are $n \times s$ matrices of real numbers. The sum of squares formula says that $z^T z = (x^T x) \cdot (y^T y)$. But $z^T z = \sum_{i,j} (y^T A_j^T x_j)(x_i A_i y)$, hence

$$y^T \left(\sum_{i,j} x_i x_j A_j^T A_i \right) y = z^T z = (x^T x)(y^T y) = y^T \Big((x^T x) I \Big) y$$

for all y. The first and last expressions are quadratic forms in y, and they are equal only if $\sum_{i,j} x_i x_j A_j^T A_i = (x^T x) I = \sum x_i^2 I$. This must hold for all x, so equating coefficients of the monomials in x we find that

- $A_i^T A_i = I$ (that is, $A_i \in O_n$) for every i, and
- $A_j^T A_i + A_i^T A_j = 0$ for every $i \neq j$.

The case $s = n$ turns out to be significantly simpler to address than the general case. If $s = n$ we may set $B_i = A_1^{-1} A_i = A_1^T A_i$. Then $B_1 = Id$ and the conditions to satisfy for $i \geq 2$ become

- $B_i^T = B_i^{-1}$
- $B_i^2 = -I_n$
- $B_i B_j = -B_j B_i$ for all $i \neq j$.

Note that the first two conditions imply $B_i^T = -B_i$, and using this the third condition can be rewritten as $B_j^T B_i + B_i^T B_j = 0$. So by replacing the A's with the B's we have proven the following:

Corollary 15.9. *If an* $[r,n,n]$*-formula exists, then one exists where* $A_1 = I$ *and* $A_i^T = -A_i$ *for* $i \geq 2$*.*

In the setting of the corollary, the necessary conditions on the matrices $A_2, A_3, \dots, A_r$ become that $A_i^2 = -I$ and $A_i A_j = -A_j A_i$.

Corollary 15.10. *If an $[r,n,n]$-formula exists, then there exist $r-1$ independent vector fields on S^{n-1}.*

Proof. Assume the A_i are chosen as in Corollary 15.9. If $y \in S^{n-1}$ then $\phi(e_i, y) \in S^{n-1}$ for $i = 1, 2, \ldots, r$: this follows from the identity $|\phi(e_i, y)|^2 = |e_i|^2 \cdot |y|^2$. We also have that $\phi(e_1, y) = y$ since $A_1 = I$. We claim that $\phi(e_i, y) \perp \phi(e_j, y)$ if $i \neq j$. To see this, note that by the norm formula

$$|\phi(e_i + e_j, y)|^2 = |e_i + e_j|^2 \cdot |y|^2 = 2|y|^2.$$

On the other hand,

$$\begin{aligned} |\phi(e_i + e_j, y)|^2 &= |\phi(e_i, y) + \phi(e_j, y)|^2 \\ &= |\phi(e_i, y)|^2 + |\phi(e_j, y)|^2 + 2\phi(e_i, y) \cdot \phi(e_j, y) \\ &= 2|y|^2 + 2\phi(e_i, y) \cdot \phi(e_j, y). \end{aligned}$$

We conclude $\phi(e_i, y) \cdot \phi(e_j, y) = 0$. Therefore we have established that $\phi(e_2, -)$, $\phi(e_3, -), \ldots, \phi(e_r, -)$ are orthonormal vector fields on S^{n-1}. □

15.11. **Clifford algebras.** We have seen that we get $r-1$ independent vector fields on S^{n-1} if we have a sums-of-squares formula of type $[r, n, n]$. Having such a formula amounts to producing matrices $A_2, A_3, \ldots, A_r \in O_n$ such that $A_i^2 = -I$ and $A_iA_j + A_jA_i = 0$ for $i \neq j$. If we disregard the condition that the matrices be orthogonal, we can encode the latter two conditions by saying that we have a representation of a certain algebra:

Definition 15.12. *The* ***Clifford algebra*** Cl_k *is defined to be the quotient of the tensor algebra* $\mathbb{R}\langle e_1, \ldots, e_k\rangle$ *by the relations* $e_i^2 = -1$ *and* $e_ie_j + e_je_i = 0$ *for all* $i \neq j$.

The first few Clifford algebras are familiar: $\mathrm{Cl}_0 = \mathbb{R}$, $\mathrm{Cl}_1 = \mathbb{C}$, and $\mathrm{Cl}_2 = \mathbb{H}$. After this things become less familiar: for example, it turns out that $\mathrm{Cl}_3 = \mathbb{H} \times \mathbb{H}$ (we will see why in just a moment). It is somewhat of a miracle that it is possible to write down a precise description of *all* of the Clifford algebras, and *all* of their modules. Before doing this, let us be clear about *why* we are doing it:

Theorem 15.13. *An $[r,n,n]$-formula exists if and only if there exists a Cl_{r-1}-module structure on $\mathbb{R}^n$. Consequently, if there is a Cl_{r-1}-module structure on $\mathbb{R}^n$ then there are $r-1$ independent vector fields on S^{n-1}.*

Before giving the proof, we need one simple fact. The collection of monomials $e_{i_1} \cdots e_{i_r}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ give a vector space basis for Cl_k, which has size 2^k (note that we include the empty monomial, corresponding to 1, in the basis). This is an easy exercise. This is the "standard basis" for Cl_k.

Proof of Theorem 15.13. The forward direction is trivial: Given an $[r,n,n]$-formula, Corollary 15.9 gives us such a formula with $A_1 = I$, $A_i^2 = -I$, and $A_iA_j = -A_jA_i$ for $i, j \geq 2$. Then define a Cl_{r-1}-module structure on $\mathbb{R}^n$ by letting e_i act as multiplication by A_{i+1}, for $1 \leq i \leq r-1$.

Conversely, assume that Cl_{r-1} acts on $\mathbb{R}^n$. We can almost reverse the procedure of the previous paragraph, except that there is no guarantee that the e_i's act orthogonally on $\mathbb{R}^n$—and we need $A_i \in O_n$ to get an $[r,n,n]$-formula.

Equip $\mathbb{R}^n$ with a positive-definite inner product, denoted $x, y \mapsto x \cdot y$. This inner product probably has no compatibility with the Clifford-module structure. So define a new inner product on $\mathbb{R}^n$ by

$$\langle v, w\rangle = \sum_{1\leq i_1<i_2<\cdots<i_j\leq r-1} (e_I v)\cdot(e_I w),$$

where $e_I = e_{i_1}e_{i_2}\cdots e_{i_j}$ and the sum runs over all 2^{r-1} elements of the standard basis for Cl_{r-1}. Basically we are averaging out the dot product. Our inner product $\langle v, w\rangle$ is a symmetric bilinear form, and it is positive definite because the dot product is positive definite. It also has the property that it is invariant under the Clifford algebra action: $\langle e_i v, e_i w\rangle = \langle v, w\rangle$ for all i.

Now let $v_1, \ldots, v_n$ be an orthonormal basis for $\mathbb{R}^n$ with respect to our new inner product. Let A_i be the matrix for e_i with respect to this basis. Then the A_i's are orthogonal matrices, and the relations $A_i^2 = -I$ and $A_iA_j + A_jA_i = 0$ are automatic because they are satisfied in Cl_{r-1}. In this way we obtain the desired $[r, n, n]$-formula. □

Remark 15.14. Most modern treatments of vector fields on spheres go straight to Clifford algebras and their modules, without ever talking about sums-of-squares formulas. But the sums-of-squares material is an interesting part of this whole story, both for historical reasons and for its own sake.

From now on we can focus on the following question: For what values of n do we have a Cl_{r-1}-module structure on $\mathbb{R}^n$? This is one of the most intriguing parts of the story, because on the face of things it doesn't seem like we have accomplished anything by shifting our perspective onto Clifford algebras. We have, after all, just rephrased the basic question. But a miracle now occurs, in that we can analyze all the Clifford algebras by a simple technique.

15.15. **Clifford algebras over general rings.** To make it clear that our analysis doesn't use anything special about $\mathbb{R}$, let us change our ground right to any ring R not of characteristic two. Define $\mathrm{Cl}_n(R)$ to be the algebra generated over R by symbols $e_1, \ldots, e_n$ that commute with elements of R and satisfy the relations $e_i^2 = -1$ and $e_ie_j = -e_je_i$ for $i \neq j$. It turns out that these algebras show an interesting pattern that depends on the residue of n mod 8, reflecting how the elements e_i interact.

Here $\mathrm{Cl}_n(R)$ is a free R-module of rank 2^n with a basis consisting of the monomials $e_{i_1}\cdots e_{i_k}$ with $i_1 < \cdots < i_k$. Also recall that if S is a ring then $S[e]/(e^2-1) \cong S \times S$ via the isomorphism $f(e) \mapsto (f(1), f(-1))$. Our derivation will proceed by finding elements in the Clifford algebras that square to 1 and commute with certain others.

In this discussion it will be convenient to abbreviate $\mathrm{Cl}_n(R)$ as just R_n. Also, if S is a ring then we will abbreviate the matrix algebra $M_{n\times n}(S)$ as just $S(n)$.

Here are a series of observations:

(1) $(e_1\cdots e_n)^2 = (-1)^{\binom{n+1}{2}}$. So the product squares to 1 when n is 0 or 3 mod 4, and to -1 when n is 1 or 2 mod 4.
(2) $e_1\cdots e_n$ commutes with all of e_1, e_2, …, e_{n-1} precisely when n is odd.

(3) When $n \equiv 3 \bmod 4$ then $e_1 \cdots e_n$ squares to 1 and commutes with all elements of R_{n-1} and so we conclude that $R_n \cong R_{n-1} \times R_{n-1}$. Let us restate this as

$$R_{4k+3} \cong R_{4k+2} \times R_{4k+2}.$$

(4) In R_{4k+r} consider the elements $\gamma_i = e_1 \cdots e_{4k} e_{4k+i}$ for $i \leq 1 \leq r$. Then $\gamma_i^2 = -1$, γ_i anticommutes with γ_j for $i \neq j$, and each γ_i commutes with elements of $\mathrm{Cl}_{4k}(R)$. This proves that

$$R_{4k+r} \cong R_{4k} \otimes_R R_r.$$

So once we know R_i for $i \leq 4$ we can derive all of the others.

(5) $R_4 \cong R_2(2)$. We can derive an isomorphism as follows. If we think of the ring $R_3 = R_2 \times R_2$ as sitting inside $R_2(2)$ as the diagonal matrices, then what we are missing are the anti-diagonal matrices. The obvious anti-diagonal matrix that squares to -1 is $f = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Note that f anticommutes with elements $\mathrm{diag}(a, -a)$ for $a \in R_2$. So define $\phi\colon R_4 \to R_2(2)$ by the formulas $e_1 \mapsto \mathrm{diag}(e_1, -e_1)$, $e_2 \mapsto \mathrm{diag}(e_2, -e_2)$, $e_3 \mapsto \mathrm{diag}(e_1e_2, -e_1e_2)$, and $e_4 \mapsto f$. The images satisfy the defining relations for the Clifford algebra, so ϕ is an algebra map. One readily checks that ϕ is surjective (this uses that $\mathrm{char}(R) \neq 2$), then since the domain and codomain are free modules of the same rank it is an isomorphism.

At this point we can make the following partial table of Clifford algebras over R, and could continue on ad infinitum:

n	$\mathrm{Cl}_n(R)$
0	R
1	R_1
2	R_2
3	$R_2 \times R_2$
4	$R_2(2)$
5	$(R_2 \otimes_R R_1)(2)$
6	$(R_2 \otimes_R R_2)(2)$
7	$(R_2 \otimes_R R_2)(2) \times (R_2 \otimes_R R_2)(2)$
8	$(R_2 \otimes_R R_2)(4)$

In fact we can simplify this table a bit, because we can identify the algebras $R_2 \otimes_R R_1$ and $R_2 \otimes_R R_2$. However, this requires a few more tools.

(6) $\mathrm{Cl}_n(R)$ is a $\mathbb{Z}/2$-graded algebra where the even part is the R-linear span of monomials in the e's having an even number of factors, and similarly for the odd part. The algebra $\mathrm{Cl}_n(R)$ also has a canonical automorphism α given by $e_i \mapsto -e_i$ for all i; $\alpha^2 = 1$ and the above even and odd parts of $\mathrm{Cl}_n(R)$ are the $+1$ and -1 eigenspaces of α.

(7) There is also an anti-automorphism of $\mathrm{Cl}_n(R)$ that sends $e_{i_1} \cdots e_{i_k} \mapsto e_{i_k} \cdots e_{i_1}$. This is called the **transpose** and written $x \mapsto x^t$. Observe that $(ab)^t = b^t a^t$ for all a and b. Note that the transpose map on R_1 is just the identity.

(8) Using the transpose we can construct algebra maps $\theta_n\colon \mathrm{Cl}_n(R) \otimes_R \mathrm{Cl}_n(R) \to \mathrm{End}_R(\mathrm{Cl}_n(R))$ via $a \otimes b \mapsto [x \mapsto axb^t]$ (note that it would not be an algebra map without the transpose). Note here that $\mathrm{End}_R(\mathrm{Cl}_n(R))$ denotes the algebra of R-module endomorphisms, so it is isomorphic to $R(2^n)$. The domain and codomain of θ_n are both free R-modules of rank 2^{2n}. The θ maps are not always

isomorphisms, e.g. when $n = 1$ one has $\theta(e_1 \otimes 1) = \theta(1 \otimes e_1)$. But sometimes they are, as we are about to see.

Proposition 15.16. *θ_2 is an isomorphism, so $R_2 \otimes_R R_2 \cong R(4)$. If R is commutative then the restriction of θ_2 to $R_2 \otimes_R R_1 \subseteq R_2 \otimes_R R_2$ is an isomorphism $R_2 \otimes_R R_1 \cong R_1(2)$.*

Proof. We prove the second claim first. It concerns the map $\alpha\colon R_2 \otimes_R R_1 \to \mathrm{End}_R(R_2)$ given by

$$a \otimes b \mapsto [x \mapsto axb^t].$$

Since R is commutative, R_1 is also commutative. It follows that the map $x \mapsto axb^t$ is R_1-linear when we give R_2 the action of R_1 via right multiplication. So in fact α is a map $R_2 \otimes_R R_1 \to \mathrm{End}_{R_1}(R_2)$. Both the domain and codomain are free R_1-modules of rank 4, and α is R_1-linear, so it suffices to prove that α is surjective. Recalling that we treat R_2 as a right R_1-module, and choosing the R_1-basis $1, e_2$, we identify $\mathrm{End}_{R_1}(R_2)$ with 2×2 matrices over R_1. Then we compute by hand that the elements $1 \otimes 1$, $e_1 \otimes 1$, $e_2 \otimes 1$, and $e_1e_2 \otimes 1$ map to the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} e_1 & 0 \\ 0 & -e_1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -e_1 \\ -e_1 & 0 \end{bmatrix}.$$

For example, $e_1 \otimes 1$ becomes the map $1 \mapsto e_1$, $e_2 \mapsto e_1e_2 = -e_2e_1$, giving us the second matrix. As these four matrices are clearly a spanning set for $R_1(2)$, this verifies the surjectivity of α.

To prove that θ_2 is an isomorphism we can use a similar strategy, but here we would need 4×4 matrices over R and it would be nice not to have to write down 16 of them. We can simplify things somewhat by using the above observation that $R_2 \otimes_R R_1$ maps into the subspace of R_1-linear endomorphisms, as well as the parallel observation that $R_2 \otimes_R (e_2R_1)$ maps into the subspace of R_1-antilinear endomorphisms (those such that $f(xe_1) = -f(x)e_1$). Since $\mathrm{End}_R(R_2)$ is the direct sum of R_1-linear and R_1-antilinear subspaces, we are reduced to checking that $R_2 \otimes_R (e_2R_1) \to \mathrm{End}_{R_1-anti}(R)$ is an isomorphism. Again representing endomorphisms via matrices in the usual way, the images of the basis elements $1 \otimes e_2$, $e_1 \otimes e_2$, $e_2 \otimes e_2$, and $e_1e_2 \otimes e_2$ are

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -e_1 \\ -e_1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -e_1 & 0 \\ 0 & e_1 \end{bmatrix}.$$

Again, these are a basis for $M_{2\times 2}(R_1)$ and the proof is complete. □

We can use Proposition 15.16 to simplify our table of Clifford algebras. For convenience we show both the table for general R and also the special case $R = \mathbb{R}$:

Note that with the appearance of $R(16)$ in spot 8 the rest of the table becomes periodic, using the isomorphisms $R_{4k+r} \cong R_{4k} \otimes_R R_r$. So $R_9 \cong R_1(16)$, $R_{10} \cong R_2(16)$, and so on: we just add "__(16)" to each of the terms in the above table.

So we have calculated all of the Clifford algebras!

Exercise 15.18. For the ground ring $R = \mathbb{C}$ prove that $R_1 \cong \mathbb{C}\times\mathbb{C}$ and $R_2 \cong \mathbb{C}(2)$, and observe that the resulting table of Clifford algebras actually has its 8-fold quasi-periodicity absorbed into a 2-fold quasi-periodicity.

TABLE 15.17. Clifford algebras

n	$\mathrm{Cl}_n(R)$
0	R
1	R_1
2	R_2
3	$R_2 \times R_2$
4	$R_2(2)$
5	$R_1(4)$
6	$R(8)$
7	$R(8) \times R(8)$
8	$R(16)$

r	$\mathrm{Cl}_r(\mathbb{R})$
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \times \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \times \mathbb{R}(8)$
8	$\mathbb{R}(16)$

15.19. **Modules over Clifford algebras.** Now that we know all the Clifford algebras, it is actually an easy process to determine all of their finitely-generated modules. We need three facts:

- If A is a division algebra then all finitely-generated modules over A are free;
- By Morita theory, the finitely-generated modules over $A(n)$ are in bijective correspondence with the finitely-generated modules over A. The bijection sends an A-module M to the $A(n)$-module M^n.
- If R and S are algebras then modules over $R \times S$ can all be written as $M \times N$ where M is an R-module and N is an S-module.

In the following table we list each Clifford algebra Cl_r and the dimension of its smallest nonzero module.

TABLE 15.20. Dimensions of Clifford modules

r	Cl_r^+	Smallest dim. of a module over Cl_r
0	$\mathbb{R}$	1
1	$\mathbb{C}$	2
2	$\mathbb{H}$	4
3	$\mathbb{H} \times \mathbb{H}$	4
4	$\mathbb{H}(2)$	8
5	$\mathbb{C}(4)$	8
6	$\mathbb{R}(8)$	8
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	8
8	$\mathbb{R}(16)$	16
9	$\mathbb{C}(16)$	32
10	$\mathbb{H}(16)$	64

Note that the third column has a quasi-periodicity, where row $k+8$ is obtained from row k by multiplying by 16.

After all of this, we are ready to prove the Hurwitz-Radon theorem about constructing vector fields on spheres. Recall that if Cl_{r-1} acts on $\mathbb{R}^n$ then there are $r-1$ independent vector fields on S^{n-1}. Going down the rows of the above table, we make the following deductions:

$$
\begin{aligned}
&\mathrm{Cl}_1 \text{ acts on } \mathbb{R}^2, \text{ therefore we have 1 vector field on } S^1\\
&\mathrm{Cl}_2 \text{ acts on } \mathbb{R}^4, \text{ therefore we have 2 vector fields on } S^3\\
&\mathrm{Cl}_3 \text{ acts on } \mathbb{R}^4, \text{ therefore we have 3 vector fields on } S^3\\
&\mathrm{Cl}_4 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 4 vector fields on } S^7\\
&\mathrm{Cl}_5 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 5 vector fields on } S^7\\
&\mathrm{Cl}_6 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 6 vector fields on } S^7\\
&\mathrm{Cl}_7 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 7 vector fields on } S^7\\
&\mathrm{Cl}_8 \text{ acts on } \mathbb{R}^{16}, \text{ therefore we have 8 vector fields on } S^{15}.
\end{aligned}
$$

It is not hard to deduce the general pattern here. The key is knowing where the jumps in dimension occur, and then just doing bookkeeping. To this end, note that the smallest dimension of a nonzero module over Cl_r is $2^{\sigma(r)}$ where

$$\sigma(r) = \#\{s : 0 < s \leq r \text{ and } s \equiv 0, 1, 2, \text{ or } 4 \mod (8)\}$$

(the numbers 0,1,2,4 mod 8 are the rows where the jumps occur in the third column of Table 15.19). Our analysis has shown that we can construct r independent vector fields on $S^{2^{\sigma(r)}-1}$.

Proof of Theorem 15.5 (Hurwitz-Radon Theorem). First note that we know much more about Clifford modules than is indicated in Table 15.20. For each Clifford algebra Cl_r we know the complete list of all isomorphism classes of finitely-generated modules, and their dimensions are all *multiples* of the dimension listed in the table. This is important.

Given an $n \geq 1$, our job is to determine the largest r for which Cl_r acts on $\mathbb{R}^n$. We will then know that there are r vector fields on S^{n-1}. If we write $n = 2^u \cdot (\text{odd})$ it is clear from Table 15.20 and the previous paragraph that the only way Cl_r could act on $\mathbb{R}^n$ is if it actually acts on $\mathbb{R}^{2^u}$. Moreover, the quasi-periodicity in the table shows that if we add 4 to u then the largest r goes up by 8. It follows at once that if $u = a + 4b$ then the formula for the largest r is $8b+$??? where the missing expression just needs to be something that works for the values $a = 0, 1, 2, 3$. One readily finds that $r = 8b + 2^a - 1$ does the job.

So we know that there are $8b + 2^a - 1$ vector fields on S^{n-1}, where n has the form $(\text{odd}) \cdot 2^{a+4b}$. $\square$

Remark 15.21 (First connection with KO^*)**.** Return to Table 15.20 and look at the column with the smallest dimensions of the modules. As one reads down the column, consider where the jumps in dimensions occur: we have "jump-jump-nothing-jump-nothing-nothing-nothing-jump," which then repeats. This is strangely reminiscent of the periodic sequence

$$\mathbb{Z}_2 \; \mathbb{Z}_2 \; 0 \; \mathbb{Z} \; 0 \; 0 \; 0 \; \mathbb{Z} \ldots$$

At first blush this feels like quite an amazing and unexpected connection! We will eventually see, following [ABS], that there is a very direct connection between the groups KO^* and the module theory of the Clifford algebras. For now we leave it as an intriguing coincidence, but see Section ??? for more discussion.

15.22. **Adams's Theorem.** So far we have done all this work just to construct collections of independent vector fields on spheres. The Hurwitz-Radon lower bound is classical, and was probably well-known in the 1940's. The natural question is, can one do any better? Is there a different construction that would yield *more* vector fields than we have managed to produce, or is the bound provided by the Hurwitz-Radon construction the best possible? People were actively working on this problem throughout the 1950's. Adams finally proved in 1962 [Ad2] that the Hurwitz-Radon bound was maximal, and he did this by using K-theory:

Theorem 15.23 (Adams)**.** *There do not exist $\rho(n)+1$ independent vector fields on S^{n-1}.*

This is a difficult theorem, and it will be a long while before we are able to prove it. We are introducing it here largely to whet the reader's appetite. Note that it is far from being immediately clear how a cohomology theory would help one prove the result. There are several reductions one must make in the problem, but the first one we can explain without much effort:

Proposition 15.24. *If there are $r-1$ vector fields on S^{n-1} then the projection $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1} \to \mathbb{R}P^{un-1}/\mathbb{R}P^{un-2} \cong S^{un-1}$ has a section in the homotopy category, for every $u > \frac{2r-2}{n}$.*

The existence of a section in the homotopy category is something that can perhaps be contradicted by applying a suitable cohomology theory $E^*(-)$. See Exercise 15.25 below for a simple example.

We close this section by sketching the proof of Proposition 15.24. Recall that the Stiefel manifold of k-frames in $\mathbb{R}^n$ is the space

$$V_k(\mathbb{R}^n) = \{(u_1,\dots,u_k) \,|\, u_i \in \mathbb{R}^n \text{ and } u_1,\dots,u_k \text{ are orthonormal}\}.$$

Consider the map $p_1\colon V_k(\mathbb{R}^n) \to S^{n-1}$ which sends $(u_1,\dots,u_n) \mapsto u_1$. There exist r vector fields on S^{n-1} if and only if there is a section of $p_1\colon V_{r+1}(\mathbb{R}^n) \to S^{n-1}$.

We need a fact from basic topology, namely that there is a cell structure on $V_k(\mathbb{R}^n)$ where the cells look like

$$e^{i_1} \times \cdots \times e^{i_s}$$

with $n-k \le i_1 < i_2 < \cdots < i_s \le n-1$ and s is arbitrary. We will not prove this here: see Hatcher [Ha, Section 3.D] or Mosher-Tangora [MT, Chapter 5].

The cell structure looks like

$$\big[e^{n-k} \cup e^{n-k+1} \cup \cdots \cup e^{n-1}\big] \cup \big[(e^{n-k+1} \times e^{n-k}) \cup (e^{n-k+2} \times e^{n-k}) \cup \cdots\big] \cup \cdots$$

If $n-1 < (n-k+1)+(n-k)$ (these are the dimensions of the last cell in the first group and the first cell in the second group) then the $(n-1)$-skeleton just consists of the cells e^{n-k} through e^{n-1}. This looks like the top part of the cell structure for $\mathbb{R}P^{n-1}$, and indeed it is. To begin to see this, start with the map $\rho\colon \mathbb{R}P^{n-1} \to O(n)$ that sends a line $\ell \subseteq \mathbb{R}^n$ to the reflection in the hyperplane $\ell^\perp$. Let $f\colon \mathbb{R}P^{n-1} \to V_k(\mathbb{R}^n)$ be the composite

$$\mathbb{R}P^{n-1} \xrightarrow{\rho} O(n) \xrightarrow{p_{\le k}} V_k(\mathbb{R}^n)$$

where $p_{\le k}$ sends a matrix $A \in O(n)$ to the tuple of its first k columns. The subspace $\mathbb{R}P^{n-k-1} \subseteq \mathbb{R}P^{n-1}$ consisting of points $[0:\cdots:0:x_{k+1}:\cdots:x_n]$ is all sent to the

standard frame $(e_1, \ldots, e_k)$ under f, so we obtain the induced map
$$\tilde{f}\colon \mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \longrightarrow V_k(\mathbb{R}^n).$$
If r is a reflection in a hyperplane H then $r(v)-v$ lies in $H^{\perp}$ for every vector v. So one can recover $H^{\perp}$ from any vector v such that $r(v) \neq v$. Using this, it is easy to see that $\tilde{f}$ is an injection. The cell structure on $V_k(\mathbb{R}^n)$ (which we have kept in a black box) is defined in such a way that the image of $\tilde{f}$ is indeed the $(n-1)$-skeleton when $n+2 > 2k$.

We will need one more fact about this situation. The composite map $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \xrightarrow{\tilde{f}} V_k(\mathbb{R}^n) \xrightarrow{p_1} S^{n-1}$ sends the subspace $\mathbb{R}P^{n-2} = \{[0 : x_2 : \cdots : x_n]\}$ to e_1 and so factors through $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}$. The induced map $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \to S^{n-1}$ is a homeomorphism, using the same considerations described above for proving that $\tilde{f}$ is injective. So we have a commutative diagram of the form

$$(*)\qquad \begin{tikzcd} \mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \arrow[r, tail, "\tilde{f}"] \arrow[d, "\pi"'] & V_k(\mathbb{R}^n) \arrow[d, "p_1"] \\ \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \arrow[r, "\cong"] & S^{n-1}. \end{tikzcd}$$

Proof of Proposition 15.24. If there exist $r-1$ vector fields on S^{n-1} there also exist $r-1$ vector fields on S^{un-1} for any u (see Proposition 15.3). Then $p_1\colon V_r(\mathbb{R}^{un}) \to S^{un-1}$ has a section s. By the cellular approximation theorem the map s is homotopic to a cellular map s'. So s' factors through the $(un-1)$-skeleton of $V_r(\mathbb{R}^{un})$, which by the above remarks is $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1}$ provided u is large enough so that $un+2 > 2r$. From diagram (*) we deduce that the composition
$$S^{un-1} \xrightarrow{s'} \mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1} \xrightarrow{\pi} \mathbb{R}P^{un-1}/\mathbb{R}P^{un-2} \cong S^{un-1}.$$
is homotopic to $\pm id$. If it happened to be $-\mathrm{id}$, alter s' by precomposing with a degree -1 map to fix this. $\square$

Exercise 15.25. Use singular cohomology to prove that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-3} \to S^{n-1}$ does not have a section when n is odd. Deduce that an even sphere does not have a non-vanishing vector field (which you already knew).

Exercise 15.26 (For those who know Steenrod operations)**.** Use Steenrod operations to prove that there do not exist two independent vector fields on S^5, by proving that the projection $\pi\colon \mathbb{R}P^5/\mathbb{R}P^2 \to S^5$ does not have a section. [See Section 38.4 for further discussion of this method.]

◦ Exercises ◦

The following exercises take us through another approach to computing the Clifford algebras over $\mathbb{R}$. For this we will need slight generalization of Clifford algebras, one that will also be very useful later on (???).

Given a real vector space V and a quadratic form $q\colon V \to \mathbb{R}$, define
$$\mathrm{Cl}(V,q) = T_{\mathbb{R}}(V)/\langle v \otimes v = q(v)\cdot 1 \mid v \in V\rangle.$$

For $\mathbb{R}^k$ with $q(x_1, \dots, x_k) = -(x_1^2 + \cdots + x_k^2)$ this recovers the algebra Cl_k. For $q(x_1, \dots, x_k) = x_1^2 + \cdots + x_k^2$ this gives a new algebra we will call Cl_k^-. It will be convenient to temporarily rename Cl_k as Cl_k^+. Of course there are other quadratic forms on $\mathbb{R}^k$, but these will be the only two we need for our present purposes.

Exercise 15.27. Prove that there are isomorphisms of algebras $\mathrm{Cl}_k^\pm \cong \mathrm{Cl}_2^\pm \otimes_\mathbb{R} \mathrm{Cl}_{k-2}^\mp$. [Hint: Make some suitable guesses for where to send each algebra generator e_i and then just prove that it works.]

Exercise 15.28. We have already remarked that $\mathrm{Cl}_0^+ \cong \mathbb{R}$, $\mathrm{Cl}_1^+ \cong \mathbb{C}$, and $\mathrm{Cl}_2^+ \cong \mathbb{H}$. Prove that $\mathrm{Cl}_0^- \cong \mathbb{R}$, $\mathrm{Cl}_1^- \cong \mathbb{R} \times \mathbb{R}$, and $\mathrm{Cl}_2^- \cong \mathbb{R}(2)$. [To get the last isomorphism, note that Cl_2^- is generated by e_1 and e_2 subject to the relations $e_1^2 = 1$, $e_2^2 = 1$, and $e_1 e_2 = -e_2 e_1$. The conditions $e_i^2 = 1$ might make you think of reflections, and we can try to realize the skew-commutativity relation by using two reflections through carefully chosen lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$. Get an algebra homomorphism $\mathrm{Cl}_2^- \to \mathbb{R}(2)$ by sending e_i to the matrix for reflection in ℓ_i, and then prove that this map is an isomorphism.]

Exercise 15.29. Use the previous two exercises to prove that $\mathrm{Cl}_3^+ \cong \mathbb{H} \times \mathbb{H}$, $\mathrm{Cl}_4^+ \cong \mathbb{H}(2)$, $\mathrm{Cl}_3^- \cong \mathbb{C}(2)$, and $\mathrm{Cl}_4^- \cong \mathbb{H}(2)$. Then continue with the same method to show

$$\mathrm{Cl}_5^+ \cong (\mathbb{H} \otimes_\mathbb{R} \mathbb{C})(2), \ \mathrm{Cl}_6^+ \cong (\mathbb{H} \otimes_\mathbb{R} \mathbb{H})(2), \ \mathrm{Cl}_5^- \cong \mathbb{H}(2) \times \mathbb{H}(2), \ \mathrm{Cl}_6^- \cong \mathbb{H}(4).$$

Miller [M] describes this process as being like lacing up a shoe.

Exercise 15.30. The algebras $\mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ and $\mathbb{H} \otimes_\mathbb{R} \mathbb{H}$ can be written in a simpler form: in fact we have $\mathbb{H} \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C}(2)$ and $\mathbb{H} \otimes_\mathbb{R} \mathbb{H} \cong \mathbb{R}(4)$. Find where we already proved these in this section and write down explicit isomorphisms.

Exercise 15.31. Now continue "lacing up the shoe" and deduce that

$$\mathrm{Cl}_7^+ \cong \mathbb{R}(8) \times \mathbb{R}(8), \ \mathrm{Cl}_8^+ \cong \mathbb{R}(16), \ \mathrm{Cl}_7^- \cong \mathbb{C}(8), \ \mathrm{Cl}_8^- \cong \mathbb{R}(16).$$

Note that coincidence of $\mathrm{Cl}_8^+ \cong \mathrm{Cl}_8^- \cong \mathbb{R}(16)$, and convince yourself that the interlacing process now yields the 8-fold quasi-periodicity.

For future reference we give the entire table of our generalized Clifford algebras:

TABLE 15.32. The Clifford algebras $\mathrm{Cl}_r^\pm$

r	Cl_r^+	Cl_r^-
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \times \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \times \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \times \mathbb{H}(2)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	$\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$

Part 3. K-theory and geometry I

At this point we have seen that there exist cohomology theories $K^*(-)$ and $KO^*(-)$. We have not *proven* their existence, but we have seen that their existence falls out as a consequence of the Bott periodicity theorems $\Omega^2(\mathbb{Z}\times BU)\simeq\mathbb{Z}\times BU$ and $\Omega^8(\mathbb{Z}\times BO)\simeq\mathbb{Z}\times BO$. If the only cohomology theory you have even seen is singular cohomology, this will seem like an amazing thing: suddenly you know three times as many cohomology theories as you used to. But a deeper study reveals that cohomology theories are actually quite common—to be a little poetic about it, that they are as plentiful as grains of sand on the beach. What is rare, however, is to have cohomology theories with a close connection to geometry: and both K and KO belong to this (vaguely-defined) class. In the following sections we will begin to explore what this means.

To some extent we have a "geometric" understanding of $K^0(-)$ and $KO^0(-)$ in terms of Grothendieck groups of vector bundles, at least for compact Hausdorff spaces. We also know that any $K^n(-)$ (or $KO^n(-)$) group can be shifted to a $K^0(-)$ group using the suspension isomorphism and Bott periodicity. One often hears a slogan like "The connection between K-theory and geometry is via vector bundles". This slogan, however, doesn't really say very much; our goal will be to develop a more detailed story along these lines.

One way to encode geometry into a cohomology theory is via Thom classes for vector bundles. Such classes give rise to fundamental classes for submanifolds and a robust connection wth intersection theory. In the next section we begin our story by recalling how all of this works for singular cohomology.

16. Thom classes, Thom isomorphism, and Thom spaces

The theory of Thom classes begins with the cohomological approach to orientations. Recall that

$$H^*(\mathbb{R}^n,\mathbb{R}^n-0)\cong H^*(D^n,S^{n-1})\cong\tilde{H}^*(S^n)\cong\begin{cases}\mathbb{Z} & \text{if } *=n,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, an orientation on $\mathbb{R}^n$ determines a generator for $H^n(\mathbb{R}^n,\mathbb{R}^n-0)\cong\mathbb{Z}$. (For a review of how this correspondence works, see the proof of Lemma 17.2 in the next section).

Now consider a vector bundle $p\colon E\to B$ of rank n. Let $\zeta\colon B\to E$ be the zero section, and write $E-0$ as shorthand for $E-\mathrm{im}(\zeta)$. We know that $H^n(E_x,E_x-0)\cong\mathbb{Z}$, and an orientation of the fiber gives a generator. We wish to consider the problem of giving compatible orientations for all the fibers at once; this can be addressed through the cohomology of the pair $(E,E-0)$.

For a neighborhood V of x, let $E_V=E|_V=p^{-1}(V)$. If E_V is trivial, then there is an isomorphism $E_V\cong V\times\mathbb{R}^n$, and $(E_V-0)\cong V\times(\mathbb{R}^n-0)$. Hence, $H^*(E_V,E_V-0)\cong H^*(V\times\mathbb{R}^n,V\times(\mathbb{R}^n-0))$. If V is contractible (which we will temporarily assume), this gives that

$$H^*(E_V,E_V-0)\cong H^*(\mathbb{R}^n,\mathbb{R}^n-0)\cong\begin{cases}\mathbb{Z} & \text{if } *=n,\\ 0 & \text{otherwise.}\end{cases}$$

Pick a generator $\mathcal{U}_V\in H^n(E_V,E_V-0)\cong\mathbb{Z}$. For all $x\in V$ the inclusion $j_x\colon(E_x,E_x-0)\hookrightarrow(E_V,E_V-0)$ gives a map $j_x^*\colon H^*(E_V,E_V-0)\to H^*(E_x,E_x-0)$.

Since we are assuming that V is contractible, j_x^* is an isomorphism. So $\mathcal{U}_V$ gives rise to generators in $H^n(E_x, E_x - 0)$ for all $x \in V$. We think of $\mathcal{U}_V$ as orienting all of the fibers simultaneously.

Even when V is not contractible the conclusions of the last paragraph still hold. One has that $H^*(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0)) \cong H^*(V) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$ by the Künneth Theorem, and so

$$H^i(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0)) \cong H^{i-n}(V) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \begin{cases} H^{i-n}(V) & \text{if } i \geq n, \\ 0 & \text{if } i < n. \end{cases}$$

Let $\mathcal{U}_V \in H^n(E_V, E_V - 0)$ be an element that corresponds to $1 \in H^0(V)$ under the above isomorphism. Then one checks that $j_x^*(\mathcal{U}_V)$ is a generator for $H^n(E_x, E_x - 0)$ for every $x \in V$.

Next suppose that we have two open sets $V, W \subseteq B$ where E is trivializable over each one, together with classes $\mathcal{U}_V \in H^n(E_V, E_V - 0)$ and $\mathcal{U}_W \in H^n(E_W, E_W - 0)$ that restrict to generators (orientations) on the fibers E_x for every $x \in V$ and every $x \in W$, respectively. We would like to require that these orientations match: so we require that the images of $\mathcal{U}_V$ and $\mathcal{U}_W$ in $H^n(E_{V\cap W}, E_{V\cap W} - 0)$ coincide. Consider the (relative) Mayer-Vietoris sequence:

$$\begin{tikzcd}
 & & H^{n-1}(E_{V\cap W}, E_{V\cap W} - 0) \arrow[d] \\
H^n(E_{V\cap W}, E_{V\cap W} - 0) & \begin{matrix} H^n(E_V, E_V - 0) \\ \bigoplus \\ H^n(E_W, E_W - 0) \end{matrix} \arrow[l] & H^n(E_{V\cup W}, E_{V\cup W} - 0) \arrow[l]
\end{tikzcd}$$

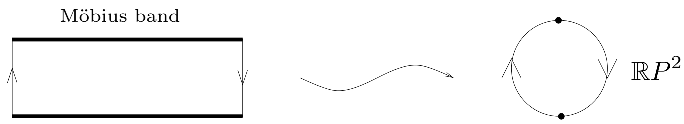
Under our requirement of compatibility between $\mathcal{U}_V$ and $\mathcal{U}_W$, the class $\mathcal{U}_V \oplus \mathcal{U}_W$ maps to zero; so it is the image of a class $\mathcal{U}_{V\cup W}$. Since $H^{n-1}(E_{V\cap W}, E_{V\cap W} - 0) = 0$ (see the computation in the previous paragraph), the class $\mathcal{U}_{V\cup W}$ is unique. Note that the Mayer-Vietories sequence also shows that $H^*(E_{V\cup W}, E_{V\cup W} - 0) = 0$ for $* < n$, which leaves us poised to inductively continue this argument. In other words, the argument shows that we may patch more and more $\mathcal{U}$-classes together, provided that they agree on the regions of overlap. This is the kind of behavior one would expect for orientation classes.

The above discussion suggests the following definition:

Definition 16.1. *Given a (constant) rank n bundle $E \to B$, a* ***Thom class*** *for E is an element $\mathcal{U}_E \in H^n(E, E - 0)$ such that for all $x \in B$, $j_x^*(\mathcal{U}_E)$ is a generator in $H^n(E_x, E_x - 0)$. (Here $j_x \colon E_x \hookrightarrow E$ is the inclusion of the fiber).*

There is no guarantee that a bundle has a Thom class. Indeed, consider the following example:

Example 16.2. Let $M \to S^1$ be the Möbius bundle. Take two contractible open subsets V and W of S^1, where $V \cup W = S^1$. We can choose a Thom class for $M|_V$, and one for $M|_W$, but the orientations won't line up correctly to give us a Thom class for M. In fact, notice that by homotopy invariance $H^*(M, M - 0)$ is the cohomology of the Möbius band relative to its boundary. But collapsing the boundary of the band gives an $\mathbb{R}P^2$



and we know $H^1(\mathbb{R}P^2) = 0$. So a Thom class cannot exist in this case.

If a bundle $E \to B$ has a Thom class then the bundle is called **orientable**. Said differently, an **orientation** on a vector bundle $E \to B$ is simply a choice of Thom class in $H^n(E, E-0; \mathbb{Z})$. One can readily prove that this notion of orientability agrees with other notions one may have encountered, and we leave this to the reader.

One can also talk about Thom classes with respect to the cohomology theories $H^*(-; R)$ for any ring R. Typically one only needs $R = \mathbb{Z}$ and $R = \mathbb{Z}/2$, however. In the latter case, note that any n-dimensional real vector space V has a *canonical* orientation in $H^n(V, V-0; \mathbb{Z}/2)$. It follows that local Thom classes always patch together to give global Thom classes, and so every vector bundle has a Thom class in $H^*(-; \mathbb{Z}/2)$.

Finally, note that we can repeat all that we have done for complex vector spaces and complex vector bundles. However, a complex vector space V of dimension n has a *canonical* orientation on its underlying real vector space, and therefore a canonical generator in $H^{2n}(V, V-0)$. Just as in the last paragraph, this implies that local Thom classes always patch together to give global Thom classes; so every complex vector bundle has a Thom class.

The following theorem summarizes what we have just learned:

Theorem 16.3. *Suppose that B is connected.*

(a) Every complex bundle $E \to B$ of rank n has a Thom class in $H^{2n}(E, E-0)$.
(b) Every real bundle $E \to B$ of rank n has a Thom class in $H^n(E, E-0; \mathbb{Z}/2)$.

The Mayer-Vietoris argument preceding Definition 16.1 shows that if $p\colon E \to B$ is a rank n orientable real vector bundle then $H^*(E, E-0)$ vanishes for $* < n$ and equals $\mathbb{Z}$ for $* = n$. A careful look at the argument reveals that it also gives a complete determination of the cohomology groups for $* > n$. We describe this next.

For any $z \in H^*(B)$, we may first apply p^* to obtain an element $p^*(z) \in H^*(E)$. We may then multiply by the Thom class $\mathcal{U}_E$ to obtain an element $p^*(z) \cup \mathcal{U}_E \in H^{*+n}(E, E-0)$. This gives a map $H^*(B) \to H^*(E, E-0)$ that increases degrees by n.

Theorem 16.4 (Thom Isomorphism Theorem)**.** *Suppose that $p\colon E \to B$ has a Thom class $\mathcal{U}_E \in H^*(E, E-0)$. Then the map $H^*(B) \to H^*(E, E-0)$ given by*

$$z \mapsto p^*(z) \cup \mathcal{U}_E$$

is an isomorphism of graded abelian groups that increases degrees by n.

Proof. If the bundle is trivial, then $E = B \times \mathbb{R}^n$, and $E - 0 = B \times (\mathbb{R}^n - 0)$. Here one just uses the suspension and Künneth isomorphisms to get

$$H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0)) \cong H^{*-n}(B).$$

One readily checks that the map from the statement of the theorem gives the isomorphism.

For the case of a general bundle one uses Mayer-Vietoris and the Five Lemma to reduce to the case of trivial bundles. The argument is easy, but one can also look it up in [MS]. □

Exercise 16.5. Write out the details for the above proof.

16.6. **Thom spaces.** The relative groups $H^*(E, E-0)$ coincide with the reduced cohomology groups of the mapping cone of the inclusion $E-0 \hookrightarrow E$. This mapping cone is sometimes called the **Thom space** of the bundle $E \to B$, although that name is more commonly applied to more geometric models that we will introduce next (the various models are all homotopy equivalent). For the most common model we require that the bundle have an inner product (see Section 8.30).

Definition 16.7. *Suppose that $E \to B$ is a bundle with an inner product. Define the* ***disk bundle*** *of E as $D(E) = \{v \in E \mid \langle v, v\rangle \le 1\}$, and the* ***sphere bundle*** *of E as $S(E) = \{v \in E \mid \langle v, v\rangle = 1\}$.*

If E has rank n over each component of B, note that $D(E) \to B$ and $S(E) \to B$ are fiber bundles with fibers D^n and S^{n-1}, respectively. Note also that we have the following diagram:

$$\begin{tikzcd} E-0 \arrow[r, tail] & E \\ S(E) \arrow[u, "\simeq"] \arrow[r, tail] & D(E) \arrow[u, "\simeq"'] \end{tikzcd}$$

This diagram shows that $E-0 \hookrightarrow E$ and $S(E) \hookrightarrow D(E)$ have weakly equivalent mapping cones. Unlike $E-0 \hookrightarrow E$, however, the map $S(E) \hookrightarrow D(E)$ is a cofibration (under the mild condition that B is a CW-complex, say): so the mapping cone is weakly equivalent to the quotient $D(E)/S(E)$. This quotient is what is most commonly meant by the term 'Thom space':

Definition 16.8. *For a bundle $E \to B$ with inner product, the* ***Thom space*** *of E is* $\operatorname{Th} E = D(E)/S(E)$.

Remark 16.9. The notation B^E is also commonly used in the literature to denote the Thom space.

Note that if B is compact then $\operatorname{Th} E$ is homeomorphic to the one-point compactification of the space E. To see this it is useful to first compactify all the fibers separately, which amounts to forming the pushout of $B \leftarrow S(E) \to D(E)$. The inclusion from B into the pushout P is the 'section at infinity', and the quotient P/B is readily seen to be the one-point compactification of E. But clearly the quotients P/B and $D(E)/S(E)$ are homeomorphic.

Example 16.10. We will show that $\operatorname{Th}(nL \to \mathbb{R}P^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$, where L is the tautological line bundle. First we define a homeomorphism of spaces over $\mathbb{R}P^k$:

$$\begin{tikzcd} \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & nL^* \arrow[dl] \\ & \mathbb{R}P^k. & \end{tikzcd}$$

Consider $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{n+k}$ as embedded via the last n coordinates. Take a point $\ell = [x_0 : \cdots : x_k : y_1 : \cdots : y_n] \in \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1}$, and note that at least one x_i is nonzero. The map $\pi\colon \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \to \mathbb{R}P^k$ is defined to send ℓ to $[x_0 : \cdots : x_k]$.

Regard ℓ as a line in $\mathbb{R}^{n+k+1}$, and $\pi(\ell)$ as a line in $\mathbb{R}^{k+1}$. The formula $(x_0, \ldots, x_k) \mapsto y_1$ specifies a unique functional $\pi(\ell) \to \mathbb{R}$ (obtained by extending linearly). Likewise, we obtain n functionals on $\pi(\ell)$ via the formulas

$$(x_0, \ldots, x_k) \mapsto y_1, \qquad \ldots \qquad (x_0, \ldots, x_k) \mapsto y_n.$$

Note also that these functionals are independent of the choice of the homogeneous coordinates for ℓ: multiplying all the x_i's and y_j's by λ gives rise to the same functionals. We have therefore described a continuous map $\mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \to nL^*$, and this is readily checked to be a homeomorphism.

Since the Thom space is the one-point compactification, we get that

$$\mathrm{Th}(nL^* \to \mathbb{R}P^k) \cong \widehat{(nL^*)} \cong (\mathbb{R}P^{n+k} - \mathbb{R}P^{n-1})^\wedge \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}.$$

We know by Corollary 8.34 that any real vector bundle over a paracompact space is isomorphic to its dual. So $nL^* \cong nL$, and we have shown that $\mathrm{Th}(nL \to \mathbb{R}P^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$.

Remark 16.11. Note the case $n = 1$ in the above example: $\mathrm{Th}(L \to \mathbb{R}P^k) \cong \mathbb{R}P^{k+1}$.

Remark 16.12. A similar analysis to above shows that $\mathrm{Th}(nL^* \to \mathbb{C}P^k) \cong \mathbb{C}P^{n+k}/\mathbb{C}P^{n-1}$, but note that unlike the real case the dual is important here.

There is another approach to Thom spaces that does not require a metric for the bundle. If $E \to B$ is any vector bundle, let $\mathbb{P}(E) \to B$ be the corresponding fiber bundle of projective spaces where the fiber of $\mathbb{P}(E) \to B$ over a point b is $\mathbb{P}(E_b)$ (see Exercise 8.41). Another definition of Thom space is then

$$\mathrm{Th}\, E = \mathbb{P}(E \oplus \underline{1})/\mathbb{P}(E).$$

Note that this definition does not require a metric.

To see that our definitions are equivalent, note that if V is a vector space then there is a canonical inclusion $V \hookrightarrow \mathbb{P}(V \oplus \mathbb{R})$ given by $v \mapsto \langle v \oplus 1 \rangle$. A little thought shows that we get a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathbb{P}(V \oplus \mathbb{R}) \arrow[d] \\ \hat{V} \arrow[r, "\cong"] & \mathbb{P}(V \oplus \mathbb{R})/\mathbb{P}(V) \end{tikzcd}$$

where the bottom map is a homeomorphism. Extending this to the bundle setting, it is clear that the pushout of $B \longleftarrow \mathbb{P}(E) \to \mathbb{P}(E \oplus \underline{1})$ is the fiberwise one-point compactification of E. Then $\mathbb{P}(E \oplus \underline{1})/\mathbb{P}(E)$ is obtained by taking this fiberwise one-point compactification and collapsing the section at infinity: this clearly agrees with the other descriptions we have given of the Thom space.

It is sometimes useful to be able to connect the pairs $(\mathbb{P}(E \oplus \underline{1}), \mathbb{P}(E))$ and $(E, E - 0)$ in a way that doesn't make use of any metric. To do so, observe that every vector space V gives rise to a commutative diagram

$$\begin{tikzcd} V-0 \arrow[r, tail] \arrow[d, tail] & \mathbb{P}(V\oplus\mathbb{R})-* \arrow[d, tail] & \mathbb{P}(V) \arrow[l, "\sim"'] \arrow[d, tail] \\ V \arrow[r, tail] & \mathbb{P}(V\oplus\mathbb{R}) \arrow[r, equal] & \mathbb{P}(V\oplus\mathbb{R}). \end{tikzcd}$$

Here $*\in\mathbb{P}(V\oplus\mathbb{R})$ is the line formed by the distinguished copy of $\mathbb{R}$, and $V\to\mathbb{P}(V\oplus\mathbb{R})$ is the map $v\mapsto\langle v\oplus 1\rangle$. All the other maps are the evident inclusions. To see that the indicated map is a homotopy equivalence, use the map in the other direction that projects a line in $V\oplus\mathbb{R}$ down to V; this is readily checkd to be a deformation retraction of $\mathbb{P}(V\oplus\mathbb{R})-*$ down to $\mathbb{P}(V)$. The left square is an open covering diagram (with open sets along the antidiagonal and their intersection in the upper left corner), and therefore a homotopy pushout. Taking homotopy cofibers of the three columns therefore yields a zig-zag of weak equivalences between the homotopy cofiber of $V-0\hookrightarrow V$ and the homotopy cofiber of $\mathbb{P}(V)\hookrightarrow\mathbb{P}(V\oplus\mathbb{R})$. The latter is weakly equivalent to its cofiber, because $\mathbb{P}(V)\hookrightarrow\mathbb{P}(V\oplus\mathbb{R})$ is a cofibration.

Now consider a fiberwise version of the above diagram. If $E\to B$ is a real bundle then we have maps

$$\begin{tikzcd} E-0 \arrow[r, tail] \arrow[d, tail] & \mathbb{P}(E\oplus\underline{1})-B \arrow[d, tail] & \mathbb{P}(E) \arrow[l, "\sim"'] \arrow[d, tail] \\ E \arrow[r, tail] & \mathbb{P}(E\oplus\underline{1}) \arrow[r, equal] & \mathbb{P}(E\oplus\underline{1}). \end{tikzcd}$$

The only difference worth noting is that $B\hookrightarrow\mathbb{P}(E\oplus\underline{1})$ is the evident section that in each fiber selects out the distinguished line determined by the trivial bundle $\underline{1}$. The left square is again a homotopy pushout square, and so taking homotopy cofibers of the columns gives a zig-zag of weak equivalences between the homotopy cofibers of $E-0\hookrightarrow E$ and $\mathbb{P}(E)\hookrightarrow\mathbb{P}(E\oplus\underline{1})$.

16.13. **Thom spaces for virtual bundles.** Thom spaces behave in a very simple way in relation to adding on trivial bundles:

Proposition 16.14. *For any real bundle $E\to X$ one has $\mathrm{Th}(E\oplus\underline{n})\cong\Sigma^n\,\mathrm{Th}(E)$. For a complex bundle $E\to X$ one has $\mathrm{Th}(E\oplus\underline{n})\cong\Sigma^{2n}\,\mathrm{Th}(E)$.*

Proof. We only prove the statement for real bundles, as the case of complex bundles works the same (and is even a consequence of the real case). Also, we will give the proof assuming the bundle has a metric, although the result is true in more generality. Note the isomorphisms

$$D(E\oplus\underline{n})\cong D(E)\times D^n,\qquad S(E\oplus\underline{n})\cong (S(E)\times D^n)\amalg_{S(E)\times S^{n-1}}(D(E)\times S^{n-1}).$$

From this one readily sees that

$$D(E\oplus\underline{n})/S(E\oplus\underline{n})\cong[D(E)/S(E)]\wedge[D^n/S^{n-1}]\cong\mathrm{Th}(E)\wedge S^n.$$

□

Proposition 16.14 allows one to make sense of Thom spaces for virtual bundles, provided that we use *spectra*. This material will only be needed briefly in the rest of the book, but we include it here because these Thom spectra play a large role in modern algebraic topology.

Assume that X is compact and let $E \to X$ be a bundle. Then by Proposition 9.10 E embeds in some trivial bundle $\underline{N}$. Let Q denote the quotient, so that we have $E \oplus Q \cong \underline{N}$. Assuming that $\mathrm{Th}(-E)$ had some meaning then we would expect

$$\mathrm{Th}(Q) = \mathrm{Th}(\underline{N} - E) = \Sigma^N \mathrm{Th}(-E).$$

This suggests the definition

$$\mathrm{Th}(-E) = \Sigma^{-N} \mathrm{Th}(Q),$$

where the negative suspension must of course be interpreted as taking place in a suitable category of spectra.

Our definition seems to depend on the choice of embedding $E \hookrightarrow \underline{N}$. To see that this dependence is an illusion, let $E \hookrightarrow \underline{N'}$ be another embedding and let Q' be the quotient. Then $\underline{N'} \oplus Q \cong Q' \oplus E \oplus Q \cong Q' \oplus \underline{N}$. On Thom spaces this gives $\Sigma^{N'} \mathrm{Th}(Q) \cong \Sigma^N \mathrm{Th}(Q')$, or $\Sigma^{-N} \mathrm{Th}(Q) \simeq \Sigma^{-N'} \mathrm{Th}(Q')$.

The above discussion can be extended to cover any element $\alpha \in KO(X)$. Write $\alpha = E - F$ for vector bundles E and F, and choose an embedding $F \hookrightarrow \underline{N}$. Let Q denote the quotient $\underline{N}/F$. Note that $\alpha + \underline{N} = (E - F) + (F + Q) = E + Q$. If $\mathrm{Th}(\alpha)$ makes sense then we would expect $\Sigma^N(\mathrm{Th}\,\alpha) \simeq \mathrm{Th}(\alpha + \underline{N}) \simeq \mathrm{Th}(E + Q)$, and so this suggests the definition

$$\mathrm{Th}(\alpha) = \Sigma^{-N} \mathrm{Th}(E \oplus Q).$$

Again, one readily checks that up to homotopy this does not depend on the choice of E, F, N, or the embedding $F \hookrightarrow \underline{N}$.

16.15. **An application to stunted projective spaces.** To demonstrate the usefulness of Thom spaces we give an application to periodicities amongst stunted projective spaces. This material will be needed later, in the solution of the vector fields on spheres problem presented in Section 38.

Consider the space $\mathbb{R}P^{a+b}/\mathbb{R}P^a$. This has a cell structure with exactly b cells (not including the zero cell), in dimensions $a+1$ through $a+b$. The space $\mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}$ has a similar cell structure, although here the cells are in dimensions $a+1+r$ through $a+b+r$. The natural question arises: fixing a and b, what values of r (if any) satisfy

$$\Sigma^r[\mathbb{R}P^{a+b}/\mathbb{R}P^a] \simeq \mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}?$$

One can use singular cohomology and Steenrod operations to produce some necessary conditions here. For example, integral singular homology easily yields that if if $b \geq 2$ then r must be even. Use of Steenrod operations produces more stringent conditions (we leave this for the reader to think about).

We will use Thom spaces to provide some *sufficient* conditions for a stable homotopy equivalence between stunted projective spaces. We begin with a simple lemma:

Lemma 16.16. *The element $\lambda = [L] - 1 \in \widetilde{KO}(\mathbb{R}P^n)$ satisfies $\lambda^2 = -2\lambda$ and $\lambda^{n+1} = 0$. Consequently, $\lambda^k = (-2)^{k-1}\lambda$ for all k, and $2^n\lambda = 0$.*

Proof. The square of any real line bundle is trivializable (Exercise 13.21), so $L^2 \cong \underline{1}$. This immediately yields $\lambda^2 = -2\lambda$. The second statement follows from the fact that $\mathbb{R}P^n$ may be covered by $n+1$ contractible sets $U_0, \ldots, U_n$. (With respect to homogeneous coordinates $[x_0 : \cdots : x_n]$ on $\mathbb{R}P^n$, one may take U_i to be the open set $x_i \neq 0$). The element $\lambda \in KO(\mathbb{R}P^n, *)$ lifts to a class $\lambda_i \in KO(\mathbb{R}P^n, U_i)$, and therefore λ^{n+1} is the image of $\lambda_0\lambda_1 \cdots \lambda_n$ under the natural map

$$KO(\mathbb{R}P^n, U_0 \cup \cdots \cup U_n) \to KO(\mathbb{R}P^n).$$

But since $\cup_i U_i = \mathbb{R}P^n$, the domain of the above map is zero; hence $\lambda^{n+1} = 0$.

Finally, since $\lambda^2 = -2\lambda$ it follows that $\lambda^e = (-2)^{e-1}\lambda$ for all e. In particular, $(-2)^n\lambda = \lambda^{n+1} = 0$. $\square$

Proposition 16.17. *Let r be any positive integer such that $r([L]-1) = 0$ in $\widetilde{KO}(\mathbb{R}P^{b-1})$. Then there is a stable homotopy equivalence*

$$\Sigma^r[\mathbb{R}P^{a+b}/\mathbb{R}P^a] \simeq \mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}.$$

Proof. The assumption that $r([L]-1) = 0$ implies that $rL \oplus \underline{s} \cong \underline{r} \oplus \underline{s}$ for some $s \geq 0$. We have

$$\begin{aligned}
\mathbb{R}P^{a+b}/\mathbb{R}P^a \cong \mathrm{Th}\begin{pmatrix} (a+1)L \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} &\simeq \Sigma^{-r-s}\,\mathrm{Th}\begin{pmatrix} (a+1)L\oplus\underline{r+s} \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} \\
&\simeq \Sigma^{-r-s}\,\mathrm{Th}\begin{pmatrix} (a+1+r)L\oplus\underline{s} \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} \\
&\simeq \Sigma^{-r}\,\mathrm{Th}\begin{pmatrix} (a+1+r)L \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} \\
&\simeq \Sigma^{-r}\big[\mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}\big].
\end{aligned}$$

The first and last steps use the identification of stunted projective spaces with a corresponding Thom space—see Example 16.10 for this. $\square$

Combining Lemma 16.16 and Proposition 16.17 we see that stunted projective spaces with b cells have a periodicity of 2^{b-1}:

$$\Sigma^{2^{b-1}}\big[\mathbb{R}P^{a+b}/\mathbb{R}P^a\big] \simeq \mathbb{R}P^{a+b+2^{b-1}}/\mathbb{R}P^{a+2^{b-1}}$$

(here $\simeq$ means stable homotopy equivalence). However, this is not the best result along these lines: we will get a better result by finding the exact order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{b-1})$. This was determined by Adams; see Theorem 37.14

17. Thom classes and intersection theory

In this section we will see how Thom classes give rise to fundamental classes for submanifolds, and we will develop the connection between products of such classes and intersection theory.

Let $E \to B$ be a real vector bundle of rank n. In general, E may not have a Thom class; and if it does have a Thom class, it actually has *two* Thom classes (since $H^n(E, E-0) \cong \mathbb{Z}$ by the Thom Isomorphism Theorem). The situation is familiar, as it matches the usual behavior of orientations. It is, of course, possible—and necessary!—to do geometry in a way that includes keeping track of orientations and computing signs according to whether orientations match up or not. But it is easier if we are in a situation where we don't have to keep track of quite so much, and there are two situations with that property: we can work always with mod 2 coefficients, or we can work in the setting of complex geometry. In either case we have canonical Thom classes all the time. In this section, and for most of the rest of these notes, we choose to work in the setting of complex bundles and complex geometry. But it is important to note that almost everything works verbatim for real bundles if we use $\mathbb{Z}/2$ coefficients, and that many things can be made to work for oriented real bundles if one is diligent enough about keeping track of signs.

For a rank n complex bundle $E \to B$ we have a canonical Thom class $\mathcal{U}_E \in H^{2n}(E, E-0)$. The following result gives two useful properties:

Proposition 17.1.

(a) (Naturality) Suppose $E \to B$ is a rank n complex vector bundle, and $f\colon A \to B$. Consider the pullback

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

Then $\bar{f}^*\colon H^{2n}(E, E-0) \to H^{2n}(f^*E, f^*E-0)$ sends $\mathcal{U}_E$ to $\mathcal{U}_{f^*E}$; that is, $\bar{f}^*(\mathcal{U}_E) = \mathcal{U}_{f^*E}$

(b) (Multiplicativity) Suppose that $E \to B$ is a rank n complex vector bundle, and $E' \to B$ is a rank k complex vector bundle, with Thom classes $\mathcal{U}_E \in H^{2n}(E, E-0)$ and $\mathcal{U}_{E'} \in H^{2k}(E', E'-0)$. Then $\mathcal{U}_E \times \mathcal{U}_{E'} = \mathcal{U}_{E\oplus E'}$ in $H^{2n+2k}(E\times E', (E\times E')-0)$.

Proof. Recall that the Thom class of a rank n complex bundle $E \to B$ is the unique class in $H^{2n}(E, E-0)$ that restricts to the canonical generator in $H^{2n}(E_x, E_x-0)$ for every fiber E_x. Part (a) follows readily from this characterization. Using the same reasoning, part (b) is reduced to the case where B is a point; this is checked in the lemma below. $\square$

Lemma 17.2. *Let V and W be two real vector spaces, of dimensions n and k, respectively. Assume given orientations on V and W, and let $V \oplus W$ have the product orientation. Let $U_V \in H^n(V, V-0)$, $U_W \in H^k(W, W-0)$, and $U_{V\oplus W} \in H^{n+k}(V\oplus W, (V\oplus W)-0)$ be the corresponding orientation classes. Then $U_{V\oplus W} = U_V \times U_W$.*

Proof. Let $v_1, \dots, v_n$ be an oriented basis for V, and let $\sigma_V\colon \Delta^n \to V$ be the affine simplex whose ordered list of vertices is $0, v_1, \dots, v_n$. Let σ_V^t denote any translate of σ that contains the origin of V in the interior. Then $[\sigma_V^t]$ is a generator for $H_n(V, V-0)$, and any relative cocycle in $C^*_{sing}(V, V-0)$ that evaluates to 1 on σ_V^t is a generator (in fact, the same generator) for $H^n(V, V-0)$. This is how an orientation of V determines a generator of $H^n(V, V-0)$.

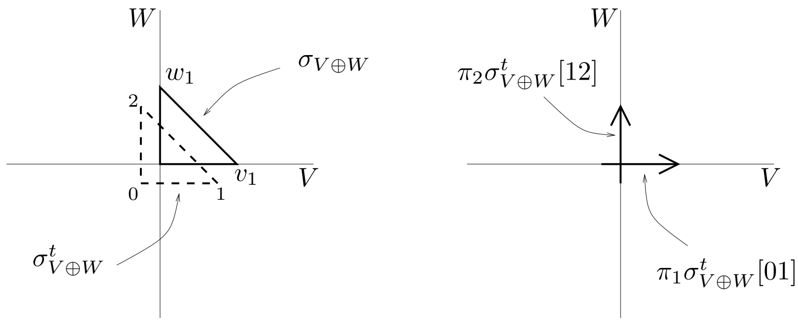
Now let $w_1, \dots, w_k$ be an oriented basis for W. Let $\sigma_{V\oplus W}\colon \Delta^{n+k} \to V \oplus W$ be the affine simplex whose ordered list of vertices is $0, v_1, \dots, v_n, w_1, \dots, w_k$ (note that omitting 0 gives an oriented basis for $V \oplus W$). Again, let $\sigma_{V\oplus W}^t$ denote a translate of $\sigma_{V\oplus W}$ that contains the origin in its interior.

Recall that $U_V \times U_W = (\pi_1)^*(U_V) \cup (\pi_2^*)(U_W)$, where $\pi_1\colon V \times W \to V$ and $\pi_2\colon V \times W \to W$ are the two projections. The definition of the cup product gives

$$\begin{aligned} (U_V \times U_W)(\sigma_{V\oplus W}^t) &= (\pi_1^* U_V)(\sigma_{V\oplus W}^t[01\cdots n]) \cdot (\pi_2^* U_W)(\sigma_{V\oplus W}^t[n\cdots(n+k)]) \\ &= U_V(\pi_1 \circ \sigma_{V\oplus W}^t[01\cdots n]) \cdot U_W(\pi_2 \circ \sigma_{V\oplus W}^t[n\cdots(n+k)]). \end{aligned}$$

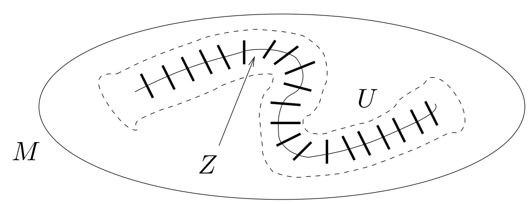
It is clear that $\pi_1 \circ \sigma^t_{V\oplus W}[01\cdots n]$ gives a simplex in the same homology class as σ^t_V, and so U_V evaluates to 1 on this simplex. Similarly, $\pi_2 \circ \sigma^t_{V\oplus W}[n\cdots(n+k)]$ gives a simplex in the same homology class as σ^t_W, and so U_W evaluates to 1 here. Since $1 \cdot 1 = 1$, we see that $U_V \times U_W$ satisfies the defining property of $U_{V\oplus W}$.

Given that a picture is worth a thousand words, here is a picture showing what is happening in the smallest nontrivial case:



□

17.3. **Fundamental classes.** Next we use the Thom isomorphism to define fundamental classes for submanifolds. Let M be a complex manifold, and let Z be a regularly embedded submanifold of complex codimension c. By "regularly embedded" we mean that there exists a neighborhood U of Z and a homeomorphism $\phi\colon U \to N$ between U and the normal bundle $N = N_{M/Z}$, with the property that ϕ carries Z to the zero section of N. The neighborhood U is called a **tubular neighborhood** of Z. Keep in mind the following rough picture:



In the above situation we have that $H^*(U, U-Z) \cong H^*(N, N-0)$. Notice that $N \to Z$ is a complex bundle of rank c, with Thom class $\mathcal{U}_N \in H^{2c}(N, N-0)$, and so by the Thom Isomorphism we get $H^{i-2c}(Z) \cong H^i(N, N-0)$. Also, by excision one has $H^*(M, M-Z) \cong H^*(U, U-Z)$. So we have isomorphisms

$$H^{i-2c}(Z) \xrightarrow{Thom} H^i(N, N-0) \cong H^i(U, U-Z) \xleftarrow{\cong} H^i(M, M-Z).$$

Now consider the long exact sequence for the pair $(M, M-Z)$, but use the above isomorphisms to rewrite the relative groups $H^*(M, M-Z)$ and $H^{*-2c}(Z)$:

$$\begin{tikzcd}
\cdots & H^*(M-Z) \arrow[l] & H^*(M) \arrow[l] & H^*(M, M-Z) \arrow[l] \arrow[d, "\cong"] & \cdots \arrow[l] \\
& & & H^*(U, U-Z) \arrow[d, "\cong"] & \\
& & & H^*(N, N-0) & \\
& & & H^{*-2c}(Z) \arrow[u, "\cong"] \arrow[uuul, bend left, "j_!"] &
\end{tikzcd}$$

If $j\colon Z \hookrightarrow M$ is the inclusion, then the indicated composition in the above diagram is denoted $j_!$ and called a **pushforward map** or **Gysin map**. We can rewrite the long exact sequence to get the **Gysin sequence**, also called a **localization sequence** by algebraic geometers:

$$\cdots \longleftarrow H^i(M-Z) \longleftarrow H^i(M) \xleftarrow{j_!} H^{i-2c}(Z) \longleftarrow H^{i-1}(M-Z) \longleftarrow \cdots$$

Definition 17.4. *Let Z be a regularly embedded, codimension c submanifold of the complex manifold M. Let $j_!$ be the Gysin map described above. We define the* ***fundamental class*** *of Z to be $[Z]_M = j_!(1) \in H^{2c}(M)$, where $1 \in H^0(Z)$ is the unit. We also define the* ***relative fundamental class*** *$[Z]_{rel} \in H^{2c}(M, M-Z)$ to be the image of 1 under the chain of isomorphisms from $H^0(Z)$ to $H^{2c}(M, M-Z)$. Note that $j^*([Z]_{rel}) = [Z]$, where j^* denotes the induced map in cohomology associated to the inclusion $(M, \emptyset) \hookrightarrow (M, M-Z)$.*

On an intuitive level one should think of $[Z]$ as being the Poincaré dual of the usual fundamental class of Z in $H_*(M)$. The point, however, is that we don't need to think through the hairiness of the Poincaré duality isomorphism; this has been replaced with the machinery of vector bundles and Thom classes.

One must of course prove a collection of basic results showing that the classes $[Z]$ really do behave as one would expect fundamental classes to behave, and have the expected ties with geometry. We will do a little of this, just enough to give the reader the idea that it is not hard. Before tackling this let us do the most trivial example:

Exercise 17.5.

(a) Check that the relative fundamental class of the origin in $\mathbb{C}^d$ is the canonical generator: i.e., $[0]_{rel} \in H^{2d}(\mathbb{C}^d, \mathbb{C}^d - 0)$ is the canonical generator provided by the complex orientation on $\mathbb{C}^d$.

(b) Let M be a d-dimensional complex manifold. If $a, b \in M$ are path-connected, verify that $[a] = [b]$. Hint: Reduce to the case where a and b belong to a common chart U of M, with $U \cong \mathbb{C}^d$. Let I be a line joining a and b inside of U, and consider the diagram

$$\begin{tikzcd}
H^*(M, M-a) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H^*(M, M-I) \arrow[d, "\cong"'] & H^*(M, M-b) \arrow[l, "\cong"'] \arrow[d, "\cong"'] \\
H^*(U, U-a) \arrow[r, "\cong"] & H^*(U, U-I) & H^*(U, U-b). \arrow[l, "\cong"']
\end{tikzcd}$$

Using an argument similar to that in the proof of Lemma 17.2, show that $[a]_{rel,U}$ and $[b]_{rel,U}$ map to the same element in $H^*(U, U-I)$.

(c) Suppose M is compact and connected. Verify that if $a \in M$ then $[a] \in H^{2d}(M)$ is a generator. (Use that the map $H^{2d}(M) \to H^{2d}(M, M-a)$ is an isomorphism in this case).

The following theorem connects our fundamental classes to intersection theory. It is far from the most general statement along these lines, but it will suffice for our applications later in the text. The diligent reader will find that the proof readily generalizes to tackle more complicated situations, for example where the intersection is not discrete.

Theorem 17.6. *Let M be a connected complex manifold. Suppose that Z and W are regularly embedded submanifolds of M that intersect transversely in d points. Assume also that W is connected (this is mostly for convenience in the statement of (b)). Then*

(a) $[Z]_M \cup [W]_M = d[*]_M$

(b) $j^*([Z]_M) = d[*]_W$, *where* $j\colon W \hookrightarrow M$.

Proof. We begin by proving (a). Suppose that $\dim Z = k$ and $\dim W = \ell$, so that $\dim M = k + \ell$. Let $Z \cap W = \{p_1, \dots, p_d\}$, and for each i let U_i be a Euclidean neighborhood of p_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Consider the following diagram:

$$\begin{tikzcd}
H^k(M, M-Z) \otimes H^l(M, M-W) \arrow[r] \arrow[d, "\cup_{rel}"] & H^k(M) \otimes H^l(M) \arrow[d, "\cup"] \\
H^{k+l}(M, M-(Z\cap W)) \arrow[r] \arrow[d, equal] & H^{k+l}(M) \\
H^{k+l}(M, M-\{p_1, \dots, p_d\}) & \\
\oplus_r H^{k+l}(M, M-\{p_r\}) \arrow[u, "\cong"] \arrow[uur] &
\end{tikzcd}$$

Since $[Z]$ and $[W]$ lift to relative classes $[Z]_{rel}$ and $[W]_{rel}$, it will suffice to show that if we take $[Z]_{rel} \cup [W]_{rel}$ and take its projection to the rth factor $H^{k+l}(M, M-\{p_r\})$ of the summand then we get $[p_r]_{rel}$. From this it will follow from the diagram that $[Z] \cup [W] = [p_1] + \dots + [p_d]$ in $H^{k+l}(M)$. Since we have already seen in Exercise 17.5 that $[p_i] = [p_j]$ for any i and j, this will complete the proof of (a).

Next, fix an index r and consider the second diagram

$$\begin{tikzcd}
H^k(U_r, U_r - Z) \otimes H^l(U_r, U_r - W) \arrow[d, "\cup_{rel}"] & H^k(M, M-Z) \otimes H^l(M, M-W) \arrow[l] \arrow[d, "\cup_{rel}"] \\
H^{k+l}(U_r, U_r - \{p_r\}) & H^{k+l}(M, M-\{p_1, \dots, p_d\}) \arrow[l] \\
& H^{k+l}(M, M-\{p_r\}). \arrow[u] \arrow[ul, "\cong"]
\end{tikzcd}$$

Thanks to this diagram, it is enough to replace M by U_r, Z by $Z \cap U_r$, and W by $W \cap U_r$, and to prove that $[Z]_{rel} \cup [W]_{rel} = [p_r]_{rel}$.

But now M is just $\mathbb{C}^{k+l}$. By choosing our neighborhood small enough, we can find local coordinates so that Z is just $\mathbb{C}^k \times \{0\}$ and W is just $\{0\} \times \mathbb{C}^l$, intersecting transversely at the origin (we will write $\mathbb{C}^k \times 0$ and $0 \times \mathbb{C}^l$ for brevity). We need to compute $[\mathbb{C}^k \times 0]_{rel} \cup [0 \times \mathbb{C}^l]_{rel} \in H^{k+l}(\mathbb{C}^{k+l}, \mathbb{C}^{k+l} - 0)$. By writing $\mathbb{C}^{k+l} = \mathbb{C}^k \times \mathbb{C}^l$ one sees that $[\mathbb{C}^k \times 0]_{rel}$ coincides with the Thom class for the bundle $\underline{l} \to \mathbb{C}^k$. Likewise, $[0 \times \mathbb{C}^l]_{rel}$ coincides with the Thom class for the bundle $\underline{k} \to \mathbb{C}^l$. These are trivial bundles, so they are pulled back from $\mathbb{C}^l \to *$ and $\mathbb{C}^k \to *$ along the projection maps $\mathbb{C}^k \to *$ and $\mathbb{C}^l \to *$, respectively. In particular, by Proposition 17.1(a) we can write

$$[\mathbb{C}^k \times 0]_{rel} \cup [0 \times \mathbb{C}^l]_{rel} = \pi_1^*(\mathcal{U}_1) \cup \pi_2^*(\mathcal{U}_2)$$

where $\mathcal{U}_1 \in H^{2l}(\mathbb{C}^l, \mathbb{C}^l - 0)$ and $\mathcal{U}_2 \in H^{2k}(\mathbb{C}^k, \mathbb{C}^k - 0)$ are the canonical classes and $\pi_1 \colon \mathbb{C}^{k+l} \to \mathbb{C}^l$, $\pi_2 \colon \mathbb{C}^{k+l} \to \mathbb{C}^k$ are the projection maps. But $\pi_1^*(\mathcal{U}_1) \cup \pi_2^*(\mathcal{U}_2)$ is the external cross product $\mathcal{U}_1 \times \mathcal{U}_2$, and so Lemma 17.2 says that this is the same as the canonical generator in $H^{2k+2l}(\mathbb{C}^{k+l}, \mathbb{C}^{k+l} - 0)$. This canonical generator is $[0]_{rel}$, by Exercise 17.5(a). We have therefore shown that $[\mathbb{C}^k \times 0]_{rel} \cup [0 \times \mathbb{C}^l]_{rel} = [0]_{rel}$, and this completes the proof of (a).

The proof of (b) is very similar. One considers the diagram

$$\begin{tikzcd}
 & H^k(W) & H^k(M) \arrow[l, "j^*"'] \\
\oplus_r H^k(W, W - p_r) \arrow[r, "\cong"] & H^k(W, W - \{p_1, \dots, p_d\}) \arrow[u] \arrow[d] & H^k(M, M - Z) \arrow[l] \arrow[u] \arrow[d] \\
 & H^k(W \cap U_r, (W \cap U_r) - p_r) & H^k(U_r, U_r - p_r) \arrow[l]
\end{tikzcd}$$

where r is an arbitrary choice of index. The top square implies that it suffices to show that the projection of $j^*([Z]_{rel})$ to $H^k(W, W - p_r)$ equals $[p_r]_{rel}$, for any choice of r. The bottom square then allows us to replace M by U_r and Z and W by $Z \cap U_r$ and $W \cap U_r$. That is, we are again reduced to the case where $M = \mathbb{C}^{k+l}$, $Z = \mathbb{C}^k \times 0$, and $W = 0 \times \mathbb{C}^l$. Here we are considering the map

$$H^{2l}(\mathbb{C}^l, \mathbb{C}^l - 0) \xleftarrow{j^*} H^{2l}\big(\mathbb{C}^k \times \mathbb{C}^l, (\mathbb{C}^k \times \mathbb{C}^l) - (\mathbb{C}^k \times 0)\big)$$

and must show that the image of $[\mathbb{C}^k \times 0]_{rel}$ is the canonical generator in the target. But if we identity $\mathbb{C}^k \times \mathbb{C}^l$ with the bundle $\underline{l} \to \mathbb{C}^k$ then $[\mathbb{C}^k \times 0]_{rel}$ is just the Thom class $\mathcal{U}$, and the map j^* is restriction to the fiber over $0 \in \mathbb{C}^k$; so it becomes the canonical generator by definition of the Thom class. $\square$

Exercise 17.7. If W were not assumed to be connected in Theorem 17.6, what would need to change in the statement of part (b)?

It is important to notice that for the most part the above proof used nothing special about singular cohomology—we only used the basic properties of Thom classes, together with generic properties that hold in *any* cohomology theory. In the proof of Lemma 17.2 we apparently used particular details about the definition of the cup product, but in fact what we needed could have been written in a way that doesn't reference the peculiar definition of the cup product at all. Indeed, we have the identifications $H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^k) = H^*(\mathbb{C}^k \times \mathbb{C}^{n-k}, \mathbb{C}^k \times (\mathbb{C}^{n-k} - 0)) = H^*(\mathbb{C}^{n-k}, \mathbb{C}^{n-k} - 0) = H^*(D^{2n-2k}, \partial D^{2n-2k}) \cong \tilde{H}^*(S^{2(n-k)})$ (for the second identification we use the map induced by projection $\mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$, and for the third identification we use the induced map of any orientation-preserving embedding of the disk into $\mathbb{C}^{n-k}$). Similarly, we have a canonical identification $H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^{n-k}) = \tilde{H}^*(S^{2k})$. Considering the commutative diagram

$$\begin{array}{ccc} H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^k) \otimes H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^{n-k}) & \xrightarrow{\mu} & H^*(\mathbb{C}^n, \mathbb{C}^n - 0) \\ \| & & \| \\ \tilde{H}^*(S^{2(n-k)}) \otimes \tilde{H}^*(S^{2k}) & \xrightarrow{\mu} & \tilde{H}^*(S^{2n}) \end{array}$$

where μ denotes our product, the property needed for the proof of Lemma 17.2 boils down to the requirement that

$$\sigma^{2(n-k)}(1) \otimes \sigma^{2k}(1) \xrightarrow{\mu} \sigma^{2n}(1).$$

In other words, the computation comes down to the fact that the product behaves well with respect to the suspension isomorphism.

Example 17.8. To demonstrate Theorem 17.6 we will be content with the usual first example. Let $Z \hookrightarrow \mathbb{C}P^n$ be a codimension c complex submanifold. Then $[Z] \in H^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}$. A generator for this group is $[\mathbb{C}P^{n-c}]$, so $[Z] = d[\mathbb{C}P^{n-c}]$ for a unique integer d. This integer is called the **degree** of the submanifold Z. A generic, c-dimensional, linear subspace of $\mathbb{C}P^n$ will intersect Z transversely in finitely many points, say e of them. Theorem 17.6 gives that $[Z] \cup [\mathbb{C}P^c] = e[*]$, but we also have $d[\mathbb{C}P^{n-c}] \cup [\mathbb{C}P^c] = d[*]$ since $[\mathbb{C}P^{n-c}] \cup [\mathbb{C}P^c] = [*]$ (again by Theorem 17.6). So $d = e$, and this gives the geometric description of the degree: the number of intersection points with a generic linear subpace of complementary dimension.

The following result is the evident generalization of Theorem 17.6.

Theorem 17.9. *Let M be a connected complex manifold. Suppose that Z and W are regularly embedded submanifolds of M that intersect transversely. Then*
(a) $[Z]_M \cup [W]_M = [Z \cap W]$;
(b) $j^*([Z]_M) = [Z \cap W]$, where $j\colon W \hookrightarrow M$.

Outline of proof. We omit the details here, since the proof is largely similar to that of Theorem 17.6. For (a) use the relative fundamental classes $[Z]_{rel}$ and $[W]_{rel}$, and show that $[Z]_{rel} \cup [W]_{rel} = [Z \cap W]_{rel}$ in $H^*(M, M - (Z \cap W))$. For this, restrict to a tubular neighborhood and then show that both classes restrict to the canonical generators on the fibers of the normal bundle. For $[Z \cap W]_{rel}$ this is the definition, and for $[Z]_{rel} \cup [W]_{rel}$ this is a computation with the cup product. The proof of (b) is similar. □

17.10. **Topological intersection multiplicities.** We can now use our machinery to give a topological definition of intersection multiplicity. Suppose that Z and W are complex submanifolds of the complex manifold M, and that Z and W have an isolated point of intersection at p. Let k denote the complex codimension of Z in M, and l denote the complex codimension of W in M. Let d be the complex dimension of M.

Let U be a Euclidean neighborhood of p that contains no other points of $Z \cap W$. Consider the classes $[Z]_{rel} \in H^{2k}(M, M - Z)$ and $[W]_{rel} \in H^{2l}(M, M - W)$. Then the cup product $[Z]_{rel} \cup [W]_{rel}$ lies in the group $H^{2(k+l)}(M, M-(Z\cap W))$. This group is zero if $k+l \neq d$, and in this case we define the intersection multiplicity $i(Z,W;p)$ to be zero. In the case where $k+l=d$ we have that

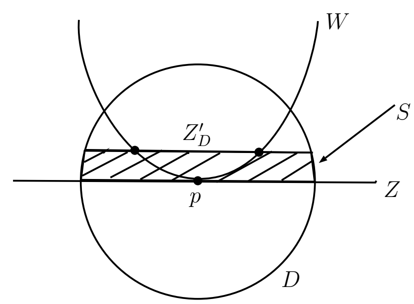
$$H^{2(k+l)}(M, M-(Z\cap W)) = H^{2d}(M, M-\{p\}) = \mathbb{Z},$$

which has the preferred generator $[p]_{rel}$. Here we define $i(Z,W;p)$ to be the unique integer such that

$$[Z]_{rel} \cup [W]_{rel} = i(Z,W;p)[p].$$

The same proof as for Theorem 17.6 shows that when Z and W intersect transversely at p we have $i(Z,W;p)=1$.

The final subject we turn to is the invariance of the intersection multiplicity under small deformations. There are different ways one might approach this; we just give one version. Assume given a Euclidean neighborhood E of p in which the normal bundle of Z is trivializable, and a closed disk $D=D^{2d}\subseteq E$ where p is in the interior. Let $Z_D = Z\cap D$ and $W_D = W\cap D$. Next suppose we have a nonzero smooth section s of the normal bundle of Z_D, and consider the associated homotopy $h\colon Z_D\times I \to D$ given by $h(z,t)=ts(z)$. Let S be the image of h and let Z'_D be the image of s. Assume that $S\cap W\subseteq \mathrm{int}(D)$ and that $Z'_D\cap W$ has only finitely-many points $q_1,\dots,q_r$. The following picture shows an example of this setup:



The space Z'_D need not be a complex submanifold of M, but we can still define the intersection multiplicities $i(Z'_D,W;q_j)$. For these we only need a Thom class for the normal bundle of Z'_D in M, which itself is determined by a consistent choice of local orientations along the fibers. For each $x\in Z_D$ we can use the path h_x ($t\mapsto h(x,t)$) to transport the normal orientation for Z_D to Z'_D.

Proposition 17.11. *In the above setting, we have*

$$i(Z,W;p) = \sum_{i=1}^{r} i(Z'_D,W;q_i).$$

In particular, if Z'_D is a complex submanifold of M and meets W transversely at each point q_i then $i(Z,W;p)=r$.

Proof. We give a sketch and leave some of the details to the reader. Consider the following diagram:

$$\begin{tikzcd}
H^{2k}(D,D-Z_D)\otimes H^{2l}(D,D-W) \arrow[r, "\cup"] \arrow[d] & H^{2d}(D,D-p) \arrow[r] \arrow[d] & H^{2d}(D,\partial D) \arrow[d, equal] \\
H^{2k}(D,D-S)\otimes H^{2l}(D,D-W) \arrow[r, "\cup"] & H^{2d}(D,D-(S\cap W)) \arrow[r] & H^{2d}(D,\partial D) \\
H^{2k}(D,D-Z'_D)\otimes H^{2l}(D,D-W) \arrow[r, "\cup"] \arrow[u] & H^{2d}(D,D-\{q_1,\dots,q_r\}) \arrow[r] \arrow[u] & H^{2d}(D,\partial D) \arrow[u, equal]
\end{tikzcd}$$

Note that in the middle line we are using that $S\cap W$ does not meed ∂D. In the top left corner we have $[Z]_{rel}\otimes[W]_{rel}$, and in the bottom left corner we have $[Z']_{rel}\otimes[W]_{rel}$. The diagram shows that these map to the same element in $H^{2d}(D,\partial D)=\mathbb{Z}$, provided that $[Z]_{rel}$ and $[Z']_{rel}$ map to the same element in $H^{2k}(D,D-S)$. Assuming this for the moment, along the top row the class maps to $i(Z,W;p)$ and along the bottom row the class maps to $\sum_j i(Z,W;q_j)$ (using the same argument as in the proof of Theorem 17.6).

So it only remains to analyze the two maps

$$H^{2k}(D,D-Z_D)\longrightarrow H^{2k}(D,D-S)\longleftarrow H^{2k}(D,D-Z'_D)$$

and see that $[Z]_{rel}$ and $[Z']_{rel}$ map to the same element in the middle. Note that in our setup the inclusions $D-S\hookrightarrow D-Z_D$ and $D-S\hookrightarrow D-Z'_D$ are homotopy equivalences, so the above two maps are isomorphisms. Pick any point $z\in Z_D$ and let $N_z\subseteq D$ be the fiber of the normal bundle to Z. Consider the diagram

$$\begin{tikzcd}
H^{2k}(D,D-Z_D) \arrow[r, "\cong"] \arrow[d, "\cong"] & H^{2k}(D,D-S) \arrow[d] & H^{2k}(D,D-Z'_D) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
H^{2k}(N_z,N_z-0) \arrow[r, "\cong"] & H^{2k}(N_z,N_z-(N_z\cap S)) & H^{2k}(N_z,N_z-(N_z\cap Z'_D)) \arrow[l, "\cong"']
\end{tikzcd}$$

where the indicated maps are all isomorphisms (and therefore every map is an isomorphism). The elements $[Z]_{rel}$ and $[Z']_{rel}$ map to the orientation classes in the left and right groups from the bottom row, and these map to the same class in the middle precisely because we oriented the normal directions to Z' in the way that was determined by the given orientation on the normal spaces of Z. $\square$

18. Thom classes in K-theory and Koszul complexes

In the last section we saw how Thom classes for complex vector bundles give rise to cohomological fundamental classes for submanifolds, and we saw that these fundamental classes have the expected connections to geometry. The discussion was carried out in the case of singular cohomology, but very little specific information about this cohomology theory was actually used. In fact, once we showed that Thom classes existed everything else followed formally. So let us now generalize a bit:

Definition 18.1. *A **multiplicative generalized cohomology theory** is a cohomology theory $\mathcal{E}$ equipped with product maps*

$$\mu\colon \mathcal{E}^p(X,A)\otimes\mathcal{E}^q(Y,B)\longrightarrow\mathcal{E}^{p+q}(X\times Y, X\times B\cup A\times Y)$$

satisfying the following requirements:

1. *Naturality in both* (X, A) *and* (Y, B)*;*
2. *There exists a two-sided unit* $1 \in \mathcal{E}^0(pt, \emptyset) = \mathcal{E}^0(pt)$*;*
3. *The pairings are associative;*
4. *The pairings are compatible with the connecting homomorphisms, in the sense that when* (X, A) *and* (Y, B) *are CW-pairs the following two diagrams commute:*

$$\begin{tikzcd}
\mathcal{E}^p(A) \otimes \mathcal{E}^q(Y,B) \arrow[r, "\mu"] \arrow[d, "\delta\otimes\mathrm{id}"'] & \mathcal{E}^{p+q}(A\times Y, A\times B) \\
\mathcal{E}^{p+1}(X,A) \otimes \mathcal{E}^q(Y,B) \arrow[dr, "\mu"'] & \mathcal{E}^{p+q}(X\times B\cup A\times Y, X\times B) \arrow[u, "\cong"'] \arrow[d, "\delta"] \\
 & \mathcal{E}^{p+q+1}(X\times Y, X\times B\cup A\times Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E}^p(X,A) \otimes \mathcal{E}^q(B) \arrow[r, "\mu"] \arrow[d, "(-1)^p\,\mathrm{id}\otimes\delta"'] & \mathcal{E}^{p+q}(X\times B, A\times B) \\
\mathcal{E}^{p}(X,A) \otimes \mathcal{E}^{q+1}(Y,B) \arrow[dr, "\mu"'] & \mathcal{E}^{p+q}(X\times B\cup A\times Y, A\times Y) \arrow[u, "\cong"'] \arrow[d, "\delta"] \\
 & \mathcal{E}^{p+q+1}(X\times Y, X\times B\cup A\times Y)
\end{tikzcd}$$

(the vertical δ *maps are the connecting homomorphisms for the evident triples).*

Let $\mathcal{E}$ be a multiplicative generalized cohomology theory.

Definition 18.2. *Let* $E \to B$ *be a rank* n *complex vector bundle. A* ***Thom class*** *for* E *is an element* $\mathcal{U}_E \in \mathcal{E}^{2n}(E, E-0)$ *such that for all* $x \in B$ *one has* $i^*(\mathcal{U}_E)$ *mapping to* 1 *under the string of isomorphisms*

$$\mathcal{E}^{2n}(E_x, E_x\backslash 0) \cong \mathcal{E}^{2n}(\mathbb{C}^n, \mathbb{C}^n{-}0) \cong \mathcal{E}^{2n}(D^{2n}, \partial D^{2n}) \cong \tilde{\mathcal{E}}^{2n}(S^{2n}) \cong \tilde{\mathcal{E}}^0(S^0) = \mathcal{E}^0(pt).$$

In the above definition note that the first isomorphism depends on an identification $E_x \cong \mathbb{C}^n$, which is equivalent to a choice of basis in E_x. However, any two such bases give homotopic maps of pairs $(\mathbb{C}^n, \mathbb{C}^n - 0) \to (E_x, E_x - 0)$ (since $GL_n(\mathbb{C})$ is path-connected) and therefore induce the same map on $\mathcal{E}^*$.

Definition 18.3. *A* ***complex orientation*** *for* $\mathcal{E}$ *is a choice, for every rank* n *complex bundle* $E \to B$*, of a Thom class* $\mathcal{U}_E \in \mathcal{E}^{2n}(E, E-0)$ *such that*

1. *(Naturality)* $\mathcal{U}_{f^*E} = f^*(\mathcal{U}_E)$ *for every map* $f\colon A \to B$*;*
2. *(Multiplicativity)* $\mathcal{U}_{E\oplus E'} = \mathcal{U}_E \cdot \mathcal{U}_{E'}$

A given cohomology theory may or may not admit a complex orientation—most likely, it will not. The complex-orientable cohomology theories are a very special class. Note that once a complex orientation is provided one gets the Thom isomorphism, Gysin sequences, and fundamental classes for complex submanifolds just as before—as well as the same connections to intersection theory.

Our goal in this section is the following:

Theorem 18.4. *Complex K-theory is a multiplicative cohomology theory that admits a complex orientation.*

Actually, it will take us many more sections to complete the details of our discussion of this result. But in this section we set down the basic ideas.

We will spend a long time exploring the geometric consequences of Theorem 18.4 (essentially all of Part 4 of this book), but let us go ahead and give one example right away. Let $Z \hookrightarrow \mathbb{C}P^n$ be a complex submanifold of codimension c. The above theorem implies that we have a fundamental class $[Z] \in K^{2c}(\mathbb{C}P^n)$, just as we did in the case of singular cohomology. Whereas $H^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}$ and only results in one integral invariant, we will find that $K^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$. This is a much larger group, and so there is suddenly the potential for detecting more information: the K-theoretic fundamental class $[Z]$ is an $(n+1)$-tuple of integers rather than just a single integer. Of course it might end up that all of these new invariants are just zero, or some algebraic function of the invariant we already had—we will have to do some computations to find out. But this demonstrates the general situation: K-theory has an inherent ability to detect more information than singular cohomology did.

To prove Theorem 18.4 we need to give a construction, for every rank n complex vector bundle $E \to B$, of a Thom class in $K^{2n}(E, E-0)$. By Bott periodicity this group is the same as $K^0(E, E-0)$. Our first step will be to develop some tools for producing elements in relative K-groups.

18.5. **Relative K-theory.** Let $A \hookrightarrow X$ be an inclusion of topological spaces. When we talked about algebraic K-theory back in Part 1, we defined the relative K-group $K^0(X, A)$ using quasi-isomorphism classes of chain complexes that were exact on A (Section 5.13). We will make a similar construction in the topological case, with some important differences.

Definition 18.6. *Let $\mathcal{F}(X, A)$ be the free abelian group on isomorphism classes of bounded chain complexes of vector bundles $E_\bullet$ on X that are exact on A (meaning that for every $x \in X$ the complex of vector spaces $(E_x)_\bullet$ is exact). Define $\mathcal{K}(X, A)$ to be the quotient of $\mathcal{F}(X, A)$ by the following relations:*

(1) $[E_\bullet \oplus F_\bullet] = [E_\bullet] + [F_\bullet]$;
(2) $[E_\bullet] = 0$ *whenever $E_\bullet$ is exact on all of X;*
(3) If $\mathcal{E}_\bullet$ is a boundex complex of vector bundles on $X \times I$ that is exact on $A \times I$ then $[\mathcal{E}|_{X\times 0}] = [\mathcal{E}|_{X\times 1}]$.

Note that pullback of vector bundles makes $\mathcal{K}(X, A)$ into a contravariant functor.

Relations (1) and (2) are familiar from Part 1, although the reader might be surprised that (1) only deals with direct sums and not short exact sequences. We will say more about this in a moment. Let us first make some remarks on relation (3), since in our work on algebraic Grothendieck groups we did not encounter relations of this type.

If $d\colon E \to F$ is a map of vector bundles over X there is a clear, intuitive notion of a *deformation* of d. One way to make this rigorous is to consider the subspace $\mathrm{VB}(E, F) \subseteq \mathcal{T}op(E, F)$ consisting of the vector bundle maps; then a deformation of d is just a continuous map $I \to \mathrm{VB}(E, F)$ that sends 0 to d. If $\pi\colon X \times I \to X$ is the projection, a little thought shows that the above notion of deformation is the same as a map of vector bundles $\pi^*E \to \pi^*F$ over $X \times I$ that restricts to d on $X \times \{0\}$.

Likewise, a deformation of a chain complex $E_\bullet$ over X can be thought of in two ways. One way involves a collection of deformations for all the maps of $E_\bullet$, having the property that at any given time t the maps in the deformation assemble into a chain complex. The other way is simply as a chain complex structure on the set of vector bundles $\{\pi^* E_i\}$ which when restricted to $X \times \{0\}$ is isomorphic to $E_\bullet$. These notions are equivalent.

The point here is that if $(E_\bullet, d)$ is a given chain complex and d' is a deformation of the differential on d, then relation (3) implies that $[(E_\bullet, d)] = [(E_\bullet, d')]$. (To be precise, there is an exactness condition required for the deformation, namely that at every time t the differential d_t is exact on A). Moreover, if X is paracompact Hausdorff then by Corollary 11.2(b) every bundle on $X \times I$ is isomorphic to the pullback of a bundle from X; it follows that *every* relation from (3) can be recast in this form. That is to say, for paracompact Hausdorff spaces it is equivalent to replace (3) by

(3') $[(E_\bullet, d)] = [(E_\bullet, d')]$ for any bounded chain complex $(E_\bullet, d)$ that is exact on A and any deformation d' of d (where each d_t is also required to be exact on A).

The following important lemma will help give a feeling for the idea of deforming a chain complex:

Lemma 18.7. *Let $E_\bullet$ be a bounded complex of vector bundles on X that is exact on A. Then $[E_\bullet] = -[\Sigma E_\bullet]$ in $\mathcal{K}(X, A)$, where $\Sigma E_\bullet$ is the shifted complex having E_i in degree $i+1$ and $d_{\Sigma E} = -d_E$.*

Proof. First note that if $V_\bullet$ and $W_\bullet$ are exact complexes of vector spaces and $f\colon V_\bullet \to W_\bullet$ is any map, then the mapping cone Cf is still exact. This follows by the long exact sequence on homology. Consequently, if $E_\bullet \to F_\bullet$ is a map between complexes of vector bundles on X, each of which is exact on A, then the mapping cone is also exact on A.

Let C denote the mapping cone of the identity map $E_\bullet \xrightarrow{\text{id}} E_\bullet$. We depict this complex as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & E_3 \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & E_0 \\
 & \oplus & \oplus & \oplus & \\
\cdots \arrow[r] \arrow[uur] & E_2 \arrow[r] \arrow[uur, "\text{id}"] & E_1 \arrow[r] \arrow[uur, "\text{id}"] & E_0 \arrow[uur, "\text{id}"] &
\end{tikzcd}$$

The arrows depict the various components of the differentials in the mapping cone; recall that $d(a, b) = (da + \text{id}(b), -db)$ for $(a, b) \in E_n \oplus E_{n-1}$, where we have written $\text{id}(b)$ instead of b just to indicate the role of the original chain map.

Consider the deformation of C obtained by putting a t in front of all the diagonal arrows and letting $t \mapsto 0$. That is, $C(t)$ is the mapping cone for $t(\text{id})\colon E_\bullet \to E_\bullet$. Then $C(t)$ is exact on A for every t, and when $t = 0$ we have $C(0) = E_\bullet \oplus \Sigma E_\bullet$. So $[C] = [C(1)] = [C(0)] = [E_\bullet] + [\Sigma E_\bullet]$ in $\mathcal{K}(X, A)$.

But C is exact on all of X, being the mapping cone of an identity map. So $[C] = 0$ in $\mathcal{K}(X, A)$, and hence $[E_\bullet] = -[\Sigma E_\bullet]$. $\square$

Exercise 18.8. Let $E_\bullet$ and $F_\bullet$ be complexes of vector bundles on X that are exact on A and let $f\colon E_\bullet \to F_\bullet$ be any map, with Cf denoting the mapping cone. Use the ideas in the above proof to show that $[Cf] = [F_\bullet] - [E_\bullet]$ in $\mathcal{K}(X, A)$.

Our next task is to analyze exact complexes, and see that just as in homological algebra they split up into basic pieces.

Definition 18.9. *An* ***elementary complex*** *is one of the form*

$$[0 \to \cdots \to 0 \to E \xrightarrow{\text{id}} E \to 0 \to \cdots \to 0]$$

where E is a vector bundle on X and the E's occur in some dimensions i and $i+1$. Denote this complex as $D_i(E)$.

Proposition 18.10. *Let X be a paracompact Hausdorff space. If $E_\bullet$ is a bounded complex of vector bundles on X that is exact, then $E_\bullet$ is a direct sum of elementary complexes.*

Proof. The proof is really the same as in homological algebra. Assume without loss of generality that $E_i = 0$ for $i < 0$. Then $E_1 \to E_0$ is a surjection, so the kernel K_1 is a vector bundle by Proposition 9.3. By Proposition 9.2 the sequence $0 \to K_1 \to E_1 \to E_0 \to 0$ is split-exact, and a choice of splitting allows us to write $E_1 = K_1 \oplus Q_1$ where the composite $Q_1 \hookrightarrow E_1 \to E_0$ is an isomorphism. Noting that $E_2 \to E_1$ has image contained in K_1, the complex $E_\bullet$ splits as the direct sum of $D_0(E_0)$ and a complex that is zero in dimensions smaller than 1. Now continue inductively, replacing $E_\bullet$ with this smaller factor, until the nonzero degrees of $E_\bullet$ have been exhausted. □

Remark 18.11. Observe now that relation (2) of Definition 18.6 could be replaced with the relation that $[D_i(E)] = 0$ for any vector bundle E on X and any $i \in \mathbb{Z}$. This fact is sometimes useful.

The next result explains why we were able to forego short exact sequences in relation (1) from Definition 18.6.

Proposition 18.12. *Let X be paracompact and Hausdorff, and let $A \subseteq X$. Assume given a short exact sequence $0 \to E'_\bullet \to E_\bullet \to E''_\bullet \to 0$ of complexes of vector bundles on X, where each complex is exact on A. Then $[E_\bullet] = [E'_\bullet] + [E''_\bullet]$ in $\mathcal{K}(X, A)$.*

Proof. Let $C_\bullet$ be the mapping cone of $E'_\bullet \hookrightarrow E_\bullet$, and recall that there is a natural map $C_\bullet \twoheadrightarrow E''_\bullet$. Let $K_\bullet$ be the kernel, which is a chain complex of vector bundles by Proposition 9.3. Elementary homological algebra (applied in each fiber) shows that $K_\bullet$ is exact on X. By Lemma 18.13 below the inclusion $K_\bullet \hookrightarrow C_\bullet$ is split, and so $C_\bullet \cong K_\bullet \oplus E''_\bullet$. So $[C_\bullet] = [K_\bullet] + [E''_\bullet] = [E''_\bullet]$ in $\mathcal{K}(X, A)$. Yet Exercise 18.8 gives $[C_\bullet] = [E_\bullet] - [E'_\bullet]$. □

Lemma 18.13. *Let X be a paracompact Hausdorff space. Let $j \colon K_\bullet \hookrightarrow C_\bullet$ be an inclusion between bounded complexes of vector bundles on X, and assume that $K_\bullet$ is exact. Then the map j admits a splitting $\chi \colon C_\bullet \to K_\bullet$.*

Proof. Without loss of generality assume that $K_i = 0 = C_i$ for $i < 0$. By Proposition 18.10 we can write $K = \oplus_{i=0}^{N} D_i(A_i)$ for some vector bundles $A_0, \ldots, A_N$ on

X. The inclusion j looks as follows:

$$\begin{tikzcd}
\vdots \arrow[d, "\mathrm{id}"] & & & & & \vdots \arrow[d] \\
A_2 & \oplus & A_1 \arrow[rrr] \arrow[d, "\mathrm{id}"] & & & C_2 \arrow[d] \\
& & A_1 & \oplus & A_0 \arrow[r] \arrow[d, "\mathrm{id}"] & C_1 \arrow[d] \\
& & & & A_0 \arrow[r] & C_0
\end{tikzcd}$$

Starting at the bottom, choose a splitting χ_0 for the inclusion $A_0 \hookrightarrow C_0$, using Corollary 9.9. Likewise, choose a splitting α_1 for the inclusion $A_1 \hookrightarrow C_1/A_0$ (note that C_1/A_0 is a vector bundle by Proposition 9.3). Define $\chi_1 \colon C_1 \to A_1 \oplus A_0$ to be the sum of $C_1 \to C_1/A_0 \xrightarrow{\alpha_1} A_1$ and $C_1 \to C_0 \xrightarrow{\chi_0} A_0$. It is readily checked that χ_1 is a splitting for j_1 and that $d\chi_1 = \chi_0 d$. Continue inductively to define χ at all levels. $\square$

The groups $\mathcal{K}(X, A)$ are readily seen to be homotopy invariant constructions, essentially because this was built into the definition:

Proposition 18.14. *For any map of pairs $f \colon (X, A) \to (Y, B)$ that is part of a homotopy equivalence (of pairs), the induced map $f^* \colon \mathcal{K}(Y, B) \to \mathcal{K}(X, A)$ is an isomorphism.*

Proof. If $j_0, j_1 \colon (X, A) \hookrightarrow (X \times I, A \times I)$ are the evident inclusions then it is clear that $j_0^* = j_1^*$, by relation (3) in Definition 18.6. It then follows by category theory that homotopic maps $(X, A) \to (Y, B)$ induce the same map upon applying $\mathcal{K}(-, -)$. Consequently, if $f \colon (X, A) \to (Y, B)$ is part of a relative homotopy equivalence then it induces an isomorphism on $\mathcal{K}$-groups. $\square$

Before finishing with our basic exploration of the group $\mathcal{K}(X, A)$, let us note the following simple result:

Proposition 18.15. *For any paracompact Hausdorff space X there is an isomorphism $\mathcal{K}(X, \emptyset) \to K^0_{Grt}(X)$ given by the formula $[E_\bullet] \to \sum_i (-1)^i [E_i]$, where $K^0_{Grt}(X)$ is the Grothendieck group of vector bundles on X.*

Proof. It is immediate that the indicated formula gives a group homomorphism $\chi \colon \mathcal{K}(X, \emptyset) \to K^0_{Grt}(X)$; the only nontrivial part is verifying relation (3), but here one uses that if F is a vector bundle on $X \times I$ then $F|_{X \times 0} \cong F|_{X \times 1}$ by Proposition 9.2.

There is also the evident map $j \colon K^0_{Grt}(X) \to \mathcal{K}(X, \emptyset)$ sending a vector bundle E to the chain complex $E[0]$ consisting of E in degree 0 and zeros in all other degrees. Certainly $\chi \circ j = \mathrm{id}$.

If $E_\bullet$ is any chain complex of vector bundles on X then we may deform $E_\bullet$ to the complex with zero differentials, by putting a t in front of all the d maps and letting $t \mapsto 0$. So $[E_\bullet] = [(E_\bullet, d = 0)] = \sum_i [\Sigma^i E_i]$ in $\mathcal{K}(X, \emptyset)$. But by Lemma 18.7 we know $[\Sigma^i E_i] = (-1)^i [E_i]$. This proves that $j \circ \chi = \mathrm{id}$. $\square$

Let $E_\bullet$ and $F_\bullet$ be bounded chain complexes of vector bundles on X. Let $E_\bullet \otimes F_\bullet$ denote the usual tensor product of chain complexes, giving another complex of vector bundles on X. In contrast to this, there is also an *external* tensor product. If $G_\bullet$ is a complex of vector bundles on a space Y, define

$$E_\bullet \hat{\otimes} G_\bullet = \pi_1^*(E_\bullet) \otimes \pi_2^*(G_\bullet)$$

where $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are the two projections. Note that if $\Delta \colon X \to X \times X$ is the diagonal map then $E_\bullet \otimes F_\bullet \cong \Delta^*(E_\bullet \hat{\otimes} F_\bullet)$.

The internal and external tensor products induce pairings on the $\mathcal{K}$-groups defined above, taking the form

$$\otimes \colon \mathcal{K}(X, A) \otimes \mathcal{K}(X, B) \to \mathcal{K}(X, A \cup B)$$

and

$$\hat{\otimes} \colon \mathcal{K}(X, A) \otimes \mathcal{K}(Y, B) \to \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)).$$

The main point is that if $V_\bullet$ and $W_\bullet$ are bounded exact sequences of vector spaces and $V_\bullet$ is exact, then $V_\bullet \otimes W_\bullet$ is exact. It follows that if $E_\bullet$ is exact on A and $F_\bullet$ is exact on B, then $E_\bullet \otimes F_\bullet$ is exact on $A \cup B$, with a similar analysis for the external case. Note again that the internal and external tensor products are connected by the formula

$$[E_\bullet] \otimes [F_\bullet] = \Delta^*\big([E_\bullet] \hat{\otimes} [F_\bullet]\big).$$

The following theorem is essentially due to Atiyah, Bott and Shapiro [ABS].

Theorem 18.16. *There is a natural transformation of functors $\chi \colon \mathcal{K}(X, A) \to K^0(X, A)$ such that when $A = \emptyset$ one has $\chi(E_\bullet) = \sum_i (-1)^i [E_i]$, and all such natural transformations agree on pairs (X, A) where both X and A are homotopically compact. Moreover, χ is a natural isomorphism on pairs that are homotopy equivalent to a finite CW-pair. Also, χ is compatible with (external and internal) products in the sense that $\chi(E_\bullet \otimes F_\bullet) = \chi(E_\bullet) \cdot \chi(F_\bullet)$ for pairs (X, A) and (Y, B) that are either finite CW-complexes or are homotopically compact with A being open in X and B being open in Y.*

The proof of Theorem 18.16 involves some technicalities that would be a distraction at this particular moment, so we postpone the proof until Section 22 below. See, in particular, Section 22.43 for the final proof.

In light of the above theorem, it is unclear how well-behaved the groups $\mathcal{K}(X, A)$ are for pairs (X, A) that are not homotopy equivalent to a finite CW-pair. This is unfortunate, because we have already seen that we need to work with groups like $K^0(E, E - 0)$ (E a vector bundle) and $K^0(X, X - Z)$ (Z a closed subvariety of X). The above result does not allow us to replace these with the analogous $\mathcal{K}$ groups. Still, we do have a map from the latter to the former, and that is often enough for us. Essentially, the $\mathcal{K}$ construction is good for producing elements and relations in the K^0 groups, even when it is not good for computing them explicitly.

From now on we assume that a specific natural transformation χ has been chosen, but the indeterminancy in this choice will not effect any arguments we ever need to make.

18.17. **Koszul complexes.** Now that we know how to produce classes in relative K-theory, we will put this knowledge to good use.

Let V be a complex vector space of dimension n. For any $v \in V$ consider the chain complex

$$0 \longrightarrow \Lambda^0 V \xrightarrow{v \wedge -} \Lambda^1 V \xrightarrow{v \wedge -} \cdots \xrightarrow{v \wedge -} \Lambda^{n-1} V \xrightarrow{v \wedge -} \Lambda^n V \longrightarrow 0.$$

Denote this chain complex by $J_{V,v}$. It is easy to see that this is exact when $v \neq 0$: indeed, pick a basis $e_1, \dots, e_n$ for V where $e_1 = v$, then use the usual induced basis for the exterior products. It is clear that if $e_1 \wedge \omega = 0$ then all the basis elements appearing in ω have an e_1 in them.

Exercise 18.18. Check that $J_{V,v} \otimes J_{W,w} \cong J_{V\oplus W, v\oplus w}$, and the isomorphism is canonical.

For various reasons we will need to consider the dual of $J_{V,v}$. Recall the existence of a natural isomorphism $\Lambda^k(V^*) \to (\Lambda^k V)^*$: for any $\phi_1, \dots, \phi_k \in V^*$ it sends $\phi_1 \wedge \cdots \wedge \phi_k$ to the functional on $\Lambda^k V$ given by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{\sigma \in \Sigma_k} (-1)^\sigma \phi_1(v_{\sigma(1)})\phi_2(v_{\sigma(2)}) \cdots \phi_k(v_{\sigma(k)}).$$

If $e_1, \dots, e_n$ is a basis for V, let $e_1^*, \dots, e_n^*$ denote the dual basis for V^*. As a basis for $\Lambda^k V$ use the standard basis of wedge products $e_{i_1\cdots i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $i_1 < \cdots < i_k$, and write $e^*_{i_1\cdots i_k}$ for the corresponding elements of the dual basis for $(\Lambda^k V)^*$. One readily checks that our map sends $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ to the functional $e^*_{i_1\cdots i_k}$, which shows that we have an isomorphism.

Using these isomorphisms, the dual complex of $J_{V,v}$ has the form

$$0 \to \Lambda^n V^* \xrightarrow{d_v} \Lambda^{n-1} V^* \xrightarrow{d_v} \cdots \xrightarrow{d_v} \Lambda^1 V^* \xrightarrow{d_v} \Lambda^0 V^* \to 0$$

We denote this by $J^*_{V,v}$, and it is called a **Koszul complex**. Here is a description of the differential:

Proposition 18.19. *Let $e_1, \cdots, e_n$ be a basis for V and write $v = \sum v_i e_i$. Let $e_1^*, \dots, e_n^*$ be the dual basis for V^*. Then the differential in $J^*_{V,v}$ is given by*

$$d_v(e^*_{i_0} \wedge \cdots \wedge e^*_{i_k}) = \sum_{j=0}^{k} (-1)^j v_{i_j} e^*_{i_0} \wedge \cdots \wedge \widehat{e^*_{i_j}} \wedge \cdots \wedge e^*_{i_k},$$

where the hat indicates that that term is omitted from the wedge.

Proof. Left to the reader. $\square$

Example 18.20. Prove that $J_{V,v}$ and $J^*_{V,v}$ are isomorphic as chain complexes. A good exercise is to try to do this without any help, starting with a specific map $\Lambda^0 V \to \Lambda^n V^*$ and figuring out what the other maps would have to be to get a map of complexes. This essentially leads one to discover the so-called "Hodge star-operator". But for good measure we also give an outline of how to do this.

Fix a basis $e_1, \dots, e_n$ of V, and let $\omega = e_1 \wedge \cdots \wedge e_n \in \Lambda^n V$. For $I = (i_1, \dots, i_k)$, let $\hat{I} = (j_1, \dots, j_{n-k})$ denote any tuple of indices for which $e_I \wedge e_{\hat{I}} = \omega$. Note that $e_{\hat{I}} \in \Lambda^{n-k}(V)$ only depends on $e_I \in \Lambda^k(V)$ and not the choice of ordering in either tuple. Define $S\colon \Lambda^k(V) \to \Lambda^{n-k} V^*$ by

$$S(e_I) = (-1)^{\binom{k}{2}} \cdot e^*_{\hat{I}}.$$

A few examples are: $S(1) = \omega$, $S(e_1) = e^*_{2\cdots n}$, $S(e_2) = -e^*_{1\cdots n}$, and $S(e_{12}) = -e^*_{3\cdots n}$. Prove that S gives the desired isomorphism of complexes. (Note that the isomorphism between $J_{V,v}$ and $J^*_{V,v}$ is not canonical, though, as it depends on the chosen basis of V.)

Recall that K-theory is largely about 'doing linear algebra fiberwise over a base space'. Anything canonical that we can do for vector spaces can be done for vector bundles as well. So let $E \to B$ be a rank n complex vector bundle, and let $s\colon B \to E$ be a section. We get a chain complex of vector bundles

$$0 \longrightarrow \Lambda^0 E \xrightarrow{s\wedge -} \Lambda^1 E \xrightarrow{s\wedge -} \cdots \xrightarrow{s\wedge -} \Lambda^{n-1} E \xrightarrow{s\wedge -} \Lambda^n E \longrightarrow 0$$

which we will denote $J_{E,s}$. For $x \in B$ this chain complex is exact over x provided that $s(x) \neq 0$. Thus it determines an element in $K^0(B, B - s^{-1}(0))$. We can just as well consider the dual complex, which also determines a (likely different) element $[J^*_{E,s}] \in K^0(B, B - s^{-1}(0))$.

Now let V be a complex vector space of dimension n. Consider the vector bundle $\pi_1\colon V \times V \to V$, with section given by the diagonal map $\Delta\colon V \to V \times V$. Our Koszul complex $J^*_{V\times V,\Delta}$ is exact on $V - 0$, and so defines an element

$$\beta(V) = [J^*_{V\times V,\Delta}] \in K^0(V, V - 0).$$

Example 18.21. One readily checks that $\beta(\mathbb{C})$ is the complex

$$\begin{tikzcd} \underline{1} \arrow[rr, "\cdot z"] \arrow[dr] & & 1 \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

where the fiber over $z \in \mathbb{C}$ is the chain complex $0 \to \mathbb{C} \xrightarrow{z} \mathbb{C} \to 0$ (multiplication by z). The Koszul complex $\beta(\mathbb{C}^2)$ has the form

$$\begin{tikzcd} \underline{1} \arrow[r, "A"] \arrow[dr] & \underline{2} \arrow[r, "B"] \arrow[d] & \underline{1} \arrow[dl] \\ & \mathbb{C}^2 & \end{tikzcd},$$

where over a point $(z, w) \in \mathbb{C}^2$ we have

$$A = \begin{bmatrix} -w \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} z & w \end{bmatrix}.$$

Finally we look at $\beta(\mathbb{C}^3)$, which has the form

$$\underline{1} \xrightarrow{A} \underline{3} \xrightarrow{B} \underline{3} \xrightarrow{C} \underline{1}$$

where the fiber over $(z, w, u) \in \mathbb{C}^3$ has

$$A = \begin{bmatrix} u \\ -w \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -w & -u & 0 \\ z & 0 & -u \\ 0 & z & w \end{bmatrix}, \quad C = \begin{bmatrix} z & w & u \end{bmatrix}.$$

Let us return to our element $\beta(V) \in K^0(V, V - 0)$. If we pick a basis for V then we get isomorphisms

$$\begin{aligned} K^0(V, V-0) \cong K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^0(D^{2n}, \partial D^{2n}) &\cong \widetilde{K}^0(S^{2n}) \\ &\cong \widetilde{K}^{-2n}(S^0) = K^{-2n}(pt). \end{aligned}$$

Moreover, one checks that any two choices of basis for V give rise to the same isomorphism (essentially because a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ is orientation-preserving). So we may regard $\beta(V)$ as giving us an element of $K^{-2n}(pt)$. Using Exercise 18.18 we have $\beta(V \oplus W) = \beta(V) \cdot \beta(W)$.

When we first learned about K-theory as a cohomology theory, we set ourselves the goal of having explicit generators for $K^*(pt)$. We can now at least state the basic result:

Theorem 18.22.
(a) $K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^{-2n}(pt) \cong \mathbb{Z}$ and $\beta(\mathbb{C}^n) = (\beta(\mathbb{C}))^n$ is a generator.
(b) $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$, where $\beta = \beta(\mathbb{C}) \in K^{-2}(pt)$.

The element $\beta = \beta(\mathbb{C}) \in K^{-2}(pt)$ is often called the **Bott element**, although sometimes this name is applied to $\beta^{-1} \in K^2(pt)$ instead. This theorem is best regarded as part of Bott periodicity. And just as for the periodicity theorem, we again postpone the proof in favor of moving forward and seeing how to use it. The proof can be found in ????.

Let $p\colon E \to B$ be a rank n complex vector bundle. Consider the pullback p^*E, which is $\pi_1\colon E \times_B E \to E$. This bundle has an evident section given by the diagonal map $\Delta\colon E \to E \times_B E$, and we may consider the Koszul complex with respect to this section. Since Δ is nonzero away from the zero-section of E, this gives us an element in $K^0(E, E - 0)$: we define

$$\mathcal{U}_E = [J^*_{p^*E,\Delta}] \in K^0(E, E - 0). \tag{18.23}$$

Note that if $x \in B$ and $j_x\colon E_x \hookrightarrow E$ is the inclusion of the fiber, it is completely obvious that $j_x^*(\mathcal{U}_E) = \beta(E_x) \in K^0(E_x, E_x - 0)$.

The element $\mathcal{U}_E$ is not quite our desired Thom class, since the Thom class is supposed to live in $K^{2n}(E, E-0)$ rather than $K^0(E, E-0)$. Of course these groups are the same because of Bott periodicity. To be completely precise, we should define our Thom class to be $\mathcal{U}_E = \beta^{-n} \cdot [J^*_{p^*E,\Delta}]$. However, it is common practice to leave off the factors of β and just do constructions in K^0. We will often follow this practice, but sometimes we will put the factors of β back into the equations in order to emphasize a point. Hopefully this won't be too confusing.

We have now completed our outline of the proof that K-theory admits Thom classes, modulo Theorems 18.16 and 18.22 whose proofs will come in Sections 22 and ???, respectively.

18.24. **Koszul complexes in algebra.** Now that we have seen Koszul complexes in geometry it seems worthwhile to also see how they appear in algebra. They turn out to be very important tools in homological algebra.

Let R be a commutative ring, and let $x_1, \ldots, x_n \in R$. Define the Koszul complex $K(x_1, \ldots, x_n; R)$ to be the complex

$$0 \longrightarrow \Lambda^n R^n \xrightarrow{d} \Lambda^{n-1} R^n \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^2 R^n \xrightarrow{d} \Lambda^1 R^n \xrightarrow{d} \Lambda^0 R^n \longrightarrow 0,$$

where the differential d is given by

$$d(e_{i_0} \wedge \cdots \wedge e_{i_k}) = \sum_{j=0}^{k} (-1)^j x_{i_j} (e_{i_0} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}).$$

Note that d is the unique derivation such that $d(e_i) = x_i$. Define the **Koszul homology groups** as $H_*(x_1, \dots, x_n; R) = H_*(K(x_1, \dots, x_n; R))$. We will often abbreviate the sequence $x_1, \dots, x_n$ to just $\underline{x}$, and write $K(\underline{x}; R)$ and so forth. It is easy to see that $H_0(\underline{x}; R) = R/(x_1, \dots, x_n)$.

In some cases the Koszul complex $K(\underline{x}; R)$ is actually a resolution of $R/(x_1, \dots, x_n)$, and this is perhaps the main reason it is useful. To explain when this occurs we need a new definition. The sequence $x_1, \dots, x_n$ is said to be a **regular sequence** if x_i is a non-zero-divisor in $R/(x_1, \dots, x_{i-1})$ for every $1 \le i \le n$ (in particular, x_1 is a non-zero-divisor in R). For example, in the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$ the indeterminates $z_1, \dots, z_n$ are a regular sequence.

Theorem 18.25. *Let $x_1, \dots, x_n \in R$.*

(a) If $x_1, \dots, x_n$ is a regular sequence, then $H_i(\underline{x}; R) = 0$ for all $i \ge 1$.

(b) Suppose R is local Noetherian and $x_1, \dots, x_n \in m$, where m is the maximal ideal. Then $x_1, \dots, x_n$ is a regular sequence if and only if $H_i(\underline{x}; R) = 0$ for all $i \ge 1$.

Proof. The subalgebra of $\Lambda^* R^n$ generated by $e_1, \dots, e_{n-1}$ is a subcomplex of $K(x_1, \dots, x_n; R)$, and is isomorphic to $K(x_1, \dots, x_{n-1}; R)$. The quotient complex has a free basis consisting of wedge products that contain e_n; and in fact the process of 'wedging with e_n' gives an isomorphism between $K(x_1, \dots, x_{n-1}; R)$ and this quotient complex that shifts degrees by one. We can summarize this by saying that there is a short exact sequence of chain complexes

(18.26)

$$0 \to K(x_1, \dots, x_{n-1}; R) \hookrightarrow K(x_1, \dots, x_n; R) \twoheadrightarrow \Sigma K(x_1, \dots, x_{n-1}; R) \to 0.$$

Denote the sequence $x_1, \dots, x_n$ by $\underline{x}$ and $x_1, \dots, x_{n-1}$ by $\underline{x'}$.

Our short exact sequence induces a long exact sequence in homology groups:

$$\cdots \to H_i(\underline{x'}; R) \to H_i(\underline{x}; R) \to H_{i-1}(\underline{x'}; R) \xrightarrow{d} H_{i-1}(\underline{x'}; R) \to H_{i-1}(\underline{x}; R) \to \cdots$$

and one easily checks that the connecting homomorphism is multiplication by $\pm x_n$ (we leave this as Exercise 18.27).

Our proof of part (a) now proceeds by induction on the length of the sequence n. When $n = 1$ the Koszul complex is $0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$, so $H_1(\underline{x}; R) = \operatorname{Ann}_R x_1 = 0$ since x_1 is a non-zero-divisor.

Now assume that we know part (a) for all regular sequences of length $n-1$. By the induction hypothesis and the above long exact sequence, it is easy to see that $H_i(\underline{x}; R) = 0$ for $i \ge 2$. So we only need to worry about $H_1(\underline{x}; R)$, for which we have

$$H_1(\underline{x'}; R) \to H_1(\underline{x}; R) \to H_0(\underline{x'}; R) \xrightarrow{\pm x_n} H_0(\underline{x'}; R) \to H_0(\underline{x}; R) \to 0$$

By induction $H_1(\underline{x'}; R) = 0$, and we know $H_0(\underline{x'}; R) = R/(x_1, \dots, x_{n-1})$. Since x_n is a non-zero-divisor in this ring, the kernel of the map labelled $\pm x_n$ is zero—hence $H_1(\underline{x}; R) = 0$ as well. This completes the proof of (a).

For (b), the point is that the above argument is almost reversible. For $n = 1$ the other direction works without any assumptions on R, because $H_1(x; R) = \operatorname{Ann}(x)$. So assume by induction that the result holds for sequences of length $n - 1$. It follows from the long exact sequence we saw in part (a) that there are short exact sequences

$$0 \to H_i(\underline{x'}; R)/x_n H_i(\underline{x'}; R) \to H_i(\underline{x}; R) \to \operatorname{Ann}_{H_{i-1}(\underline{x'}; R)}(x_n) \to 0.$$

The assumption that $H_i(\underline{x}; R) = 0$ implies that $x_n H_i(\underline{x'}; R) = H_i(\underline{x'}; R)$. But $x_n \in m$, so by Nakayama's Lemma this yields $H_i(\underline{x'}; R) = 0$. This holds for all $i \geq 1$, so induction gives that $\underline{x'}$ is a regular sequence. The assumption $H_i(\underline{x}; R) = 0$ also yields (using the other half of the above short exact sequence) that x_n is a non-zero-divisor on $H_{i-1}(\underline{x'}; R)$; so for $i = 1$ this says that x_n is a non-zero-divisor on $R/(x_1, \ldots, x_{n-1})$. Hence, $\underline{x}$ is a regular sequence. $\square$

Exercise 18.27. In the above proof, work through the definition of the connecting homomorphism for the short exact sequence (18.26) to check that it is multiplication by $\pm x_n$ (the sign can be determined, we just don't care about it).

We can use our knowledge of Koszul complexes to prove the Hilbert Syzygy Theorem:

Theorem 18.28 (Hilbert Syzygy Theorem)**.** *Let L be a field. Then every finite-generated module over $L[x_1, \ldots, x_n]$ has a finite projective resolution.*

Proof. We first prove the result in the graded case. Let $R = L[x_1, \ldots, x_n]$, and grade R by setting $\deg(x_i) = 1$. Assume that M is a finitely-generated, graded R-module. We construct the so-called "minimal resolution" of M: Start by picking a minimal set of homogeneous generators $w_1, \ldots, w_k$ for M. Define $F_0 = R^k$, graded so that the ith generator has degree equal to $\deg(w_i)$. Let $d_0 \colon F_0 \to M$ be the map sending e_i to w_i, and let K_0 be the kernel. Then d_0 preserves degrees, so K_0 is again a graded module. Repeat this process to construct $F_1 \twoheadrightarrow K_0$, let K_1 be the kernel, repeat to get $F_2 \twoheadrightarrow K_1$, and so forth. This constructs a free resolution $F_\bullet \to M$ of the form

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to M \to 0$$

Each differential has entries in the ideal $(x_1, \ldots, x_n)$: this follows from the fact that at each stage we chose a *minimal* set of generators.

Next, form the complex $F_\bullet \otimes_R R/(x_1, \ldots, x_n)$ and take homology. Tensoring with $R/(x_1, \ldots, x_n)$ kills all the entries of the matrices and changes every R to an L; so we have

$$L^{b_i} \cong H_i(F_\bullet \otimes_R R/(x_1, \ldots, x_n)) = \operatorname{Tor}_i(M, R/(x_1, \ldots, x_n)).$$

Now we use the fact that we can also compute Tor by resolving $R/(x_1, \ldots, x_n)$ and tensoring with M. Yet by Theorem 18.25(a) $R/(x_1, \ldots, x_n)$ is resolved by the Koszul complex, which has length n: so this immediately yields that $\operatorname{Tor}_i(M, R/(x_1, \ldots, x_n)) = 0$ for $i > n$. It follows that $b_i = 0$ for $i > n$, which says that $F_\bullet$ was actually a finite resolution.

Now we prove the general case, for modules that are not necessarily graded. Choose a presentation of the module

$$R^{b_1} \xrightarrow{A} R^{b_0} \twoheadrightarrow M$$

where A is a matrix with entries in R. Now introduce a new variable x_0 and homogenize A to $\tilde{A}$: that is, multiply factors of x_0 onto the monomials appearing in the entries of A so that all the entries have the same degree. Put $S = L[x_0, \ldots, x_n] = R[x_0]$, and let $\widetilde{M}$ be the cokernel of $\tilde{A}$:

$$S^{b_1} \xrightarrow{\tilde{A}} S^{b_0} \longrightarrow \widetilde{M} \longrightarrow 0\,.$$

Note that $\widetilde{M}$ is a graded module over S, and $\widetilde{M} \otimes_S S/(1 - x_0) \cong M$.

What we have already proven in the graded case guarantees a finite S−free resolution $\widetilde{F}_\bullet \to \widetilde{M} \to 0$. Let $F_\bullet = \widetilde{F}_\bullet \otimes_S (S/(1-x_0))$. This is an R-free chain complex, and $H_0(F_\bullet) \cong M$. Note that $H_i(F) = \mathrm{Tor}_i^S(\widetilde{M}, S/(1-x_0))$, and the Tor-module can again also be computed by resolving $S/(1-x_0)$. We use the resolution $0 \to S \xrightarrow{1-x_0} S \to 0$ and immediately find that $H_i(F) = 0$ if $i \geq 2$. We also have that $H_1(F) \cong \mathrm{Ann}_{\widetilde{M}}(1-x_0)$, but such an annihilator is zero for any finitely-generated, graded module. So $F_\bullet \to M$ is a finite free resolution over R. $\square$

Remark 18.29. In the above proof, the deduction of the general case from the graded case was taken from [E, Corollary 19.8].

19. Interlude: More algebraic geometry

In preparation for some arguments in the next section we need to develop a little more algebraic geometry: we need the concepts of schemes and coherent sheaves, as well as some experience with specific examples. The good news is that we only require the basic ideas here. An extremely brief summary is:

(i) The category of affine schemes is the opposite category of commutative rings. Affine algebraic geometry is just commutative algebra.

(ii) Schemes are affine schemes that have been pasted together along open inclusions, analogously to the way one passes from open subsets of Euclidean space to manifolds.

(iii) Here is an SAT-style analogy:

$$[R : R\text{-modules}] \quad :: \quad [X : \text{quasi-coherent } \mathcal{O}_X\text{-modules}].$$

That is, the category of quasi-coherent $\mathcal{O}_X$-modules is the algebraic geometers' version of the category of R-modules. When $X = \operatorname{Spec} R$ the categories are equivalent. *Coherent* $\mathcal{O}_X$-modules are the analog of *finitely-presented* R-modules, but we will always deal with Noetherian situations and so one might as well read *finitely-generated* here. If you know and love the theory of modules over a ring, the theory of quasi-coherent $\mathcal{O}_X$-modules will soon be your good friend.

This section will give an introduction to these topics, and along the way will introduce blowup varieties and a few other geometric concepts.

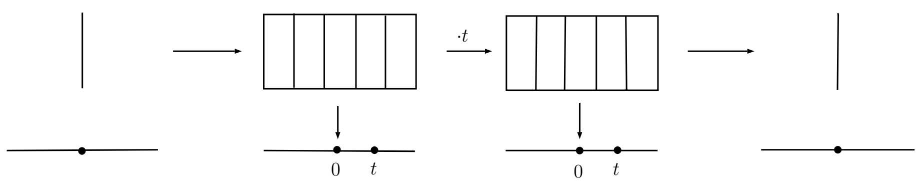
19.1. **Getting started.** Let us start by recalling what we have learned so far about the correspondence between geometry and algebra:

Geometry	**Algebra**
$\mathbb{C}^n$ or $\mathbb{A}^n_{\mathbb{C}}$	$\mathbb{C}[x_1, \ldots, x_n] = R$
Points $(q_1, \ldots, q_n)$ in $\mathbb{A}^n$	Maximal ideals $(x_1 - q_1, \ldots, x_n - q_n)$
Algebraic sets	Radical ideals
Irreducible algebraic sets	Prime ideals
subvarieties $X = V(P) \subseteq \mathbb{A}^n$	$\mathbb{C}[x_1, \ldots, x_n]/P = R/P$
(Closed) Points in X	Maximal ideals in R/P
Algebraic subsets $V(I) \subseteq X$	Radical ideals in R/P
Irreducible algebraic sets $V(Q) \subseteq X$	Prime ideals in R/P
algebraic vector bundles E on X	f.g. projective R/P-modules $\mathcal{E}$
fiber E_x at the closed point $x \in X$	$\mathcal{E}/m_x\mathcal{E}$ where m_x is the maximal ideal corresponding to x

One of the questions we aim to answer is this: if finitely-generated R/P-projectives on the Algebra side correspond to algebraic vector bundles on the Geometry side, what do more general finitely-generated R/P-modules correspond to? The answer will be that they represent "coherent sheaves", but our job will be to figure out what these are.

Here is a related issue that will drive our discussion. Once we have the idea that a chain complex of projectives corresponds to a chain complex of algebraic vector bundles, we immediately notice an issue: exactness in the module setting does not match our usual fiberwise notion of exactness in the geometric setting.

Recall that the most basic example is the map of trivial bundles $f\colon \underline{1} \to \underline{1}$ on $\mathbb{A}^1$ that corresponds to the map $\mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x]$. So on the fiber over $t \in \mathbb{C}$ this map is multiplication by t. On fibers these are isomorphisms for every $t \neq 0$, but at $t = 0$ we have both a kernel and a cokernel. We might draw the following picture:



Notice that on the module side $\mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x]$ we have a cokernel, namely $Q = \mathbb{C}[x]/(x)$, but zero kernel. In other words, regarding these maps as chain complexes concentrated in degrees 0 and 1, the complex on the module side is exact in degree 1 but the complex on the geometric side is not. The module Q satisfies $Q/mQ = 0$ for every maximal ideal $m \subseteq \mathbb{C}[x]$ other than $m = (x)$, and $Q/(x)Q = \mathbb{C}$. So the "fibers" of Q are matching the cokernel in our picture, but something is not lining up when we look at kernels.

The fix for this—that is to say, the way to get the geometry to better match the algebra—is that we need to be looking at *stalks* rather than fibers. So let us pause for a moment to say what these are.

If $p\colon E \to X$ is an algebraic vector bundle and $x \in X$, then the stalk $E_{(x)}$ is the vector space of germs of sections of p: that is, it consists of pairs (U, s) where U is a Zariski neighborhood of x and $s\colon U \to X$ is a section of p, subject to the identification $(U, s) = (V, s')$ if $s|_{U\cap V} = s'|_{U\cap V}$. Note that the stalk contains much more information about sections than the fiber does: the fiber just records the values of sections, but the stalk remembers all of the local information (for example, the values of all of the derivatives). For our example of $E = \underline{1}$ on $\mathbb{A}^1$, the stalk $E_{(t)}$ may be identified with the localization $\mathbb{C}[x]_{(x-t)}$.

Returning to our map of bundles $f\colon E \to E$ on $\mathbb{A}^1$, on the stalk over t we have

$$\mathbb{C}[x]_{(x-t)} \xrightarrow{\cdot x} \mathbb{C}[x]_{(x-t)}.$$

When $t \neq 0$ the element x is a unit in $\mathbb{C}[x]_{(x-t)}$, therefore this map is an isomorphism—kernel and cokernel are both zero. When $t = 0$ we have the map $\mathbb{C}[x]_{(x)} \xrightarrow{x} \mathbb{C}[x]_{(x)}$, which has cokernel $\mathbb{C}$ but which is an *injective* map.

The lesson is that if we are to create a geometric version of the category of modules, which in some way expands on our existing picture of algebraic vector bundles, we need to be paying attention to stalks and not just fibers. This provides some motivation for the notions of sheaves and quasi-coherent $\mathcal{O}_X$-modules that we survey below.

19.2. **Crash course on schemes.** We start with the category of affine schemes over $\mathbb{C}$, which is defined to be the opposite of the category of $\mathbb{C}$-algebras. For R a $\mathbb{C}$-algebra we define $\operatorname{Spec} R$ to be the associated affine scheme. Maps from $\operatorname{Spec} R$ to $\operatorname{Spec} T$ are the same as maps of $\mathbb{C}$-algebras from T to R. We define the underlying topological space of $\operatorname{Spec} R$ to be the set of prime ideals in R, equipped with the Zariski topology.

It will be most convenient for us to build up the category of schemes by working backwards. So let us assume given a category of objects called "$\mathbb{C}$-schemes" which will be our preferred setting for doing algebraic geometry. It should contain the affine schemes over $\mathbb{C}$ as a full subcategory, but it should also contain any Zariski open subset of an affine variety. The category should also contain objects, like complex projective space, that are constructed by gluing affine varieties together in certain allowable ways. We will have the property that for any $\mathbb{C}$-scheme X and any point $p \in X$ there is a neighborhood of p that is affine. Denote this imagined category by $\mathcal{S}ch/\mathbb{C}$. Since we will be working entirely over $\mathbb{C}$ throughout this section, we shorten $\mathbb{C}$-scheme to just "scheme" and write $\mathcal{S}ch$ instead of $\mathcal{S}ch/\mathbb{C}$.

Let $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. We have maps $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ called addition and multiplication, corresponding to the maps of rings $\mathbb{C}[t] \to \mathbb{C}[t_1, t_2]$ given by $t \mapsto t_1 + t_2$ and $t \mapsto t_1 t_2$. For any scheme X we can consider $\mathcal{S}ch(X, \mathbb{A}^1)$, which is a ring via the operations induced by the above two maps. We denote this ring as $\mathcal{O}(X)$ and call it the "ring of algebraic functions" on X. For any open covering $\{U_i\}$ of X we would expect to have an isomorphism

$$\mathcal{O}(X) \xrightarrow{\cong} \lim \Big[\prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U_i \cap U_j) \Big] \tag{19.3}$$

where the top map sends a tuple $(f_i)_i$ to $(f_i|_{U_i \cap U_j})_{ij}$, and the bottom map sends the same tuple to $(f_j|_{U_i \cap U_j})_{ij}$. The isomorphism captures the notion that functions patch well: giving an algebraic function defined on all of X is the same as giving functions defined on each U_i which agree on the pairwise intersections.

Observe that when $X = \operatorname{Spec} R$ we have

$$\mathcal{O}(X) = \mathcal{S}ch(X, \mathbb{A}^1) = \mathbb{C}\text{−Alg}(\mathbb{C}[t], R) = R.$$

That is, the ring of algebraic functions on $\operatorname{Spec} R$ is just R.

The isomorphism of (19.3) is called a **descent condition**. It is useful to consider other assignments satisfying these conditions, so define a **presheaf** F on X to be an assignment $U \mapsto F(U)$ that maps each open set $U \subseteq X$ to a set (or abelian group, or ring) $F(U)$, together with restriction maps $F(V) \to F(U)$ whenever $U \subseteq V$. Say that a presheaf F is a **sheaf** if it satisfies the descent condition saying that for every open covering $\{U_i\}$ of X the natural map

$$F(X) \xrightarrow{\cong} \lim \Big[\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \Big] \tag{19.4}$$

is an isomorphism. The functor $\mathcal{O}$, when restricted to open subsets of X, is usually written $\mathcal{O}_X$. It is a sheaf of rings on X. More generally, for any scheme Z the functor $U \mapsto \mathcal{S}ch(U, Z)$ will give a sheaf on X.

For any presheaf F and any point $x \in X$, define the **stalk** of F at x to be

$$F_{(x)} = \operatorname*{colim}_{x \in U^{open} \subseteq X} F(U),$$

where the colimit is taken over the inverse system of Zariski neighborhoods of x in X. The stalk represents the truly local information in F near x. The stalk $(\mathcal{O}_X)_{(x)}$ is usually written as $\mathcal{O}_{X,x}$ and called the ring of **germs of algebraic functions** at x.

Starting with a presheaf F, there is a universal way to impose the descent condition, resulting in a universal map $F \to \tilde{F}$ where $\tilde{F}$ is a sheaf. This is called the **sheafification** of F. We will not explain the construction, but it is not hard (consult any introductory text on sheaf theory, e.g. [Br]). In addition to the universality, the main property is that $F_{(x)} \to \tilde{F}_{(x)}$ is an isomorphism for every point x. So F and $\tilde{F}$ contain the same local information.

One can mimic the definition of topological vector bundle and define an *algebraic vector bundle* to be a map of schemes $p\colon E \to X$ with certain extra structures, that is locally trivial in the evident way. We omit the details only because it is cumbersome to write them all out. For U an open subset of X write $\Gamma_E(U)$ for the subset of $\mathcal{S}ch(U, E)$ consisting of sections of p. Then Γ_E will be a sheaf on X. Even more, each $\Gamma_E(U)$ will be an abelian group under pointwise addition from the fibers of E, and an $\mathcal{O}_X(U)$-module under pointwise multiplication. Let us generalize this scenario and define a **sheaf of $\mathcal{O}_X$-modules** to be a sheaf of abelian groups F on X together with maps $\mathcal{O}_X(U) \otimes F(U) \to F(U)$ that make $F(U)$ into an $\mathcal{O}_X(U)$-module and are compatible with the restriction maps in $\mathcal{O}_X$ and F. Often one just says "$\mathcal{O}_X$-module" as shorthand for "sheaf of $\mathcal{O}_X$-modules".

Observe that Γ_E isn't just any $\mathcal{O}_X$-module, but it has the additional property of being **locally-free**: for any point x in X there is an open set $x \in U$ such that $\Gamma_E(U) \cong \mathcal{O}_X(U)^{\oplus(n)}$ where n is the rank of E. Consequently, note that for any point $x \in X$ one has an isomorphism of stalks $(\Gamma_E)_{(x)} \cong \mathcal{O}_{X,x}^n$.

Now assume $X = \operatorname{Spec} R$ is an affine scheme, and let M be an R-module. For any open set $U \subseteq X$ we have the map of rings $R = \mathcal{O}_X(X) \to \mathcal{O}_X(U)$, and so we can define

$$F_M(U) = \mathcal{O}_X(U) \otimes_R M.$$

This might not be a sheaf, but one can sheafify it to obtain $\widetilde{F_M}$. This will be a sheaf of $\mathcal{O}_X$-modules. We will write $\widetilde{M}$ instead of $\widetilde{F_M}$. For a prime ideal $P \in \operatorname{Spec} R$ one has a natural isomorphism $\widetilde{M}_{(P)} \cong M_P$. That is, the stalks of $\widetilde{M}$ are the usual localizations of M.

If X is a scheme, define a **quasi-coherent sheaf** on X to be a sheaf F of $\mathcal{O}_X$-modules such that for every $x \in X$ there is an affine open neighborhood $x \in U$ such that $F|_U$ is isomorphic to $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M. See Remark 19.5 below for a more intrinsic characterization. One can prove that when $X = \operatorname{Spec} R$ the category of quasi-coherent sheaves on X is equivalent to the category of R-modules, via the functor $M \mapsto \widetilde{M}$. A **coherent sheaf** is a quasi-coherent sheaf that is locally isomorphic to $\widetilde{M}$ for M a *finitely-generated* $\mathcal{O}_X(U)$-module (recall that by convention all of our rings are Noetherian).

Let qcMod_X and cMod_X denote the categories of quasi-coherent and coherent $\mathcal{O}_X$-modules, respectively. These are both abelian categories, for any scheme X.

Remark 19.5. Here are equivalent versions of the definitions that in some ways are more appealing in that they don't explicitly refer back to the categories of rings and modules on an affine chart. They also don't require any Noetherian conditions. A sheaf of $\mathcal{O}_X$-modules F is **quasi-coherent** if for each $x \in X$ there is an open neighborhood $x \in U$ such that $F|_U$ is a cokernel of free sheaves of $\mathcal{O}_X$-modules:

$$\bigoplus_{i \in \mathcal{I}} (\mathcal{O}_X)|_U \longrightarrow \bigoplus_{j \in \mathcal{J}} (\mathcal{O}_X)|_U \longrightarrow F|_U \longrightarrow 0.$$

A sheaf of $\mathcal{O}_X$-modules F is **finite-type** if for each $x \in X$ there is an open neighborhood $x \in U$, an $n \geq 0$, and a surjection $(\mathcal{O}_X)^n|_U \twoheadrightarrow F|_U$. Finally, a sheaf of $\mathcal{O}_X$-modules F is **coherent** if it is finite-type and for every open set U, every $n \geq 0$, and every surjection $(\mathcal{O}_X^n)|_U \twoheadrightarrow F$, the kernel is also finite-type.

We will not need these conditions, but include them to give the reader some assurance that the theory we are outlining can all be worked out in rigorous detail and in quite general settings.

The analog of Swan's theorem says that the assignment $E \mapsto \Gamma_E$ gives an equivalence of categories between algebraic vector bundles over X and the locally-free coherent sheaves on X. In this context it is often called the **Serre-Swan Theorem**. Note that when $X = \operatorname{Spec} R$ this is an equivalence between algebraic vector bundles on X and finitely-generated projective R-modules. Thus, one may regard the category of quasi-coherent sheaves as an expansion of the category of algebraic vector bundles into an abelian category, analogous to the way the category of R-modules is an expansion of the category of finitely-generated projectives. It is often useful to think of a quasi-coherent sheaf as being like a vector bundle but where there can be certain kinds of "jumps" in the fibers.

The category of quasi-coherent sheaves on a scheme X is the algebraic geometer's version of the category of modules over a ring. It is an abelian category with enough injectives, and so one can do homological algebra. When the scheme is not affine the locally free sheaves are not necessarily projective (see Exercise **??**), but they are flat and for all the schemes we will consider there are "enough" of them: any coherent sheaf is a quotient of a locally free sheaf, and so one can produce resolutions by locally free sheaves and thereby define left derived functors of tensor product. In short, most of the constructions of commutative algebra carry over from modules to quasi-coherent (or coherent) sheaves, though sometimes requiring a little extra care. See Exercises 20.13 through 20.15 at the end of the section for some experience with this.

At this point we must face up to the fact that we have not explicitly defined the category $\mathcal{S}ch$. Nevertheless, we have established the main ideas. The usual way to construct it is to take the basic properties we have learned and essentially force them into the definitions. One defines a *locally-ringed space* to be a topological space with a sheaf of rings for which the stalks are rings with unique maximal ideals. Given a ring R one can construct a locally-ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, and then a scheme is defined to be a locally-ringed space that is locally isomorphic to one of these affines. Consult any basic text on algebraic geometry for details, e.g. [H] or [Sha2].

Example 19.6. Like the category of topological spaces, the category of schemes admits certain pathological objects. Here is just one example. Take two copies of $\mathbb{A}^n$ and glue them together along the open set $\mathbb{A}^n - 0$, via the identity map. This gives a perfectly fine scheme: every point has an affine neighborhood, namely one of the two $\mathbb{A}^n$'s. But it is a weird object, usually referred to as "affine n-space with the origin doubled". Working over $\mathbb{C}$ and giving the closed points the analytic topology would yield a non-Hausdorff space, typically not something we want to consider. We will sometimes want to rule out such oddities, and the technical word here is that this is a "non-separated" scheme. See Section 19.10 below for this term and others.

19.7. **Projective space.** Continuing at a fairly intuitive and non-rigorous level, let us consider what $\mathbb{C}P^n$ will be like as a scheme. The closed points will of course correspond to points $[x_0 : \cdots : x_n]$ with $x_i \in \mathbb{C}$ and not all x_i equal to zero, subject to the usual relation that $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for any nonzero λ. There should be Zariski open sets U_i corresponding to the condition $x_i \neq 0$, and these will cover $\mathbb{C}P^n$.

Note that the fractions $\frac{x_j}{x_i}$ represent algebraic functions defined on U_i. We expect $\mathcal{O}(U_i)$ to be the polynomial ring $\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$, reflecting the fact that U_i is isomorphic to $\mathbb{A}^n$.

It is somewhat cumbersome, but one can define the scheme $\mathbb{P}^n$ to be the result of pasting the affine schemes $U_0, \ldots, U_n$ together along their evident intersections. For example, $\mathbb{P}^1$ consists of the two affines $\operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}]$ and $\operatorname{Spec} \mathbb{C}[\frac{x_1}{x_0}]$ glued together along the identity map of $\operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}, \frac{x_1}{x_0}]$; or said differently, this is two copies of $\operatorname{Spec} \mathbb{C}[t]$ glued together along the map $\operatorname{Spec} \mathbb{C}[t, t^{-1}] \to \operatorname{Spec} \mathbb{C}[t, t^{-1}]$ that sends $t \mapsto \frac{1}{t}$.

To give a quasi-coherent sheaf on $\mathbb{P}^n$ we need to give, for each i, an $\mathcal{O}(U_i)$-module, in such a way that certain compatibility conditions are satisfied. One evident way to do this is to start with a graded $\mathbb{C}[x_0, \ldots, x_n]$-module M, and to define $F_M(U_i)$ to be the degree zero elements in $M[x_i^{-1}]$. The compatibilities then become automatic.

Let R denote the graded ring $\mathbb{C}[x_0, \ldots, x_n]$. Consider the graded module $M = R(d)$, which is a copy of R but with the generator 1 shifted into degree $-d$ (so that $M_k = R_{k+d}$). Then the degree 0 piece of $M[x_i^{-1}]$ is the span of the degree d monomials in $x_0, \ldots, x_n, x_i^{-1}$. These monomials are the same as algebraic sections of the line bundle $(L^{\otimes d})^* \to \mathbb{C}P^n$, where $L \to \mathbb{C}P^n$ is the tautological bundle: the monomial m corresponds to the section that sends $[x_0 : \cdots : x_n]$ to the functional given by $\underline{x} \otimes \cdots \otimes \underline{x} \mapsto m(\underline{x})$, where $\underline{x} = (x_0, \ldots, x_n)$. Note that the functional that sends $\lambda\underline{x} \otimes \cdots \otimes \lambda\underline{x} \mapsto m(\lambda\underline{x})$ is the same functional, so that this is well-defined. We claim that the $\mathcal{O}_{\mathbb{C}P^n}$-module F_M is locally free and corresponds to the line bundle $(L^*)^{\otimes d}$.

Lkewise, now consider $M = R(-d)$, with the generator 1 up in degree d. Then the degree 0 piece of $M[x_i^{-1}]$ consists of degree $-d$ monomials in $x_0, \ldots, x_n, x_i^{-1}$, and these correspond to algebraic sections of $L^{\otimes d} \to \mathbb{C}P^n$: a monomial m corresponds to the section sending $[x_0 : \cdots : x_n]$ to the point $m(\underline{x}) \cdot \underline{x} \otimes \cdots \otimes \underline{x}$ (d tensor factors). Again, note that in this description $[\lambda x_0 : \cdots : \lambda x_n]$ is sent to the same point, because a λ^d from the tensors cancels the λ^{-d} from the monomial.

Algebraic geometers use the notation $\mathcal{O}(d)$ for the coherent sheaf corresponding to $R(d)$, which plays the role of $(L^*)^{\otimes d}$, and $\mathcal{O}(-d)$ for the sheaf corresponding to $R(-d)$, playing the role of $L^{\otimes d}$.

19.8. **The secret behind gradings.** Why do *graded* rings and modules appear when considering projective space? This is often a source of puzzlement when first learning this subject. Here is a sketch of an explanation.

Consider the scheme $\mathbb{A}^1 - 0 = \operatorname{Spec} \mathbb{C}[t^{\pm 1}]$. There is a multiplication map $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to \mathbb{A}^1 - 0$ making this into a group scheme: the dual map of rings is $\mathbb{C}[t^{\pm 1}] \mapsto \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ given by $t \mapsto t_1 t_2$. The unit is the point $\operatorname{Spec} \mathbb{C} \to \mathbb{A}^1 - 0$, corresponding to the map of rings $\mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ sending $t \to 1$. When thought of as a group one often writes this scheme as $\mathbb{G}_m$, for the "multiplicative group".

The scheme $\mathbb{C}P^n$ should be obtained from $\mathbb{A}^{n+1} - 0$ by quotienting out by the evident $\mathbb{G}_m$-action. So we can try to understand $\mathbb{C}P^n$ via the projection map $\pi\colon \mathbb{A}^{n+1} - 0 \to \mathbb{C}P^n$. For example, algebraic functions on $U \subseteq \mathbb{C}P^n$ should be algebraic functions on $\pi^{-1}(U)$ that are invariant under the $\mathbb{G}_m$-action.

Let $X = \operatorname{Spec} R$ be an affine scheme with a $\mathbb{G}_m$-action. We claim that the action induces a natural grading on R (and in fact is equivalent to a grading). The action is a map $\mu\colon \mathbb{G}_m \times X \to X$, which on the level of rings is $\mu^*\colon R \to R[t, t^{-1}]$. Write $\mu^*(r) = \sum r_n t^n$. The fact that the action is unital translates to the condition $\sum_n r_n = r$, and the action being associative translates into the condition that $(r_n)_k$ is r_n if $k = n$ and 0 otherwise. Setting $R_n = \{r_n | \, r \in R\}$ then gives a grading for R.

Let $f \in R$, and regard this as a function $f\colon X \to \mathbb{A}^1$. This map is equivariant (with $\mathbb{A}^1$ having the trivial action) if the following diagrams commute:

$$\begin{tikzcd}
\mathbb{G}_m \times X \arrow[r, "\mu"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & \mathbb{A}^1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R[t, t^{-1}] & R \arrow[l, "\mu^*"'] \\
R \arrow[u, "i"] & \mathbb{C}[t] \arrow[l, "f^*"] \arrow[u, "f^*"']
\end{tikzcd}$$

(the second diagram is the dual of the first). In the second diagram t maps to $\sum_n f_n t^n$ along the up-top route, and to f along the bottom-up route. So f is equivariant precisely when $f_n = 0$ for all $n \neq 0$, or equivalently $f \in R_0$. More generally, $f \in R_n$ corresponds to the condition that can be written as $f(\lambda x) = \lambda^n f(x)$ for all $\lambda \in \mathbb{C} - 0$ and $x \in X$.

Exercise 19.9. Verify the last sentence of the above remark.

Now let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring. Then $X = \operatorname{Spec} R$ has a $\mathbb{G}_m$-action and we can try to construct an associated projective space $\operatorname{Proj} R$. Note that R_0 is a retract of R and therefore $\operatorname{Spec} R_0$ is a retract of X. This subspace has trivial $\mathbb{G}_m$-action and plays the role of an "origin" inside X, which we will want to ignore when constructing the projective space. The subvarieties of $\operatorname{Proj} R$ will correspond to subvarieties of X that are invariant under the $\mathbb{G}_m$-action, but excluding the "irrelevant" subvarieties of $\operatorname{Spec} R_0$. This translates to prime ideals of R that are *graded* (equal to the direct sum of their homogeneous pieces) and that do not contain the ideal $R_+ = \bigoplus_{n \geq 1} R_n$. So as a set we define $\operatorname{Proj} R$ to be the set of such graded prime ideals, and we equip this with the usual Zariski topology where the closed sets are of the form $V(I)$ for $I \subseteq R$ a graded ideal. To describe $\operatorname{Proj} R$ as a scheme we need to give the sheaf of regular functions $\mathcal{O}_{\operatorname{Proj} R}$, but it is enough to describe this on a basis of open sets. For a homogeneous element $f \in R$ let $D(f)$ be the set of primes in $\operatorname{Proj} R$ that do not contain f, which is a Zariski open. The ring $\mathcal{O}_{\operatorname{Proj} R}(D(f))$ will then be defined to be the ring of degree 0 elements in the localization R_f (note that these correspond to functions on $\operatorname{Spec} R_f \subseteq X$ that are invariant under the $\mathbb{G}_m$-action).

We have gone quickly through the details of the construction of $\operatorname{Proj} R$ because we will only need that such a thing exists, and very little about the intricacies behind it.

As two basic examples we note that if k is a field and $R = k[x_0, \ldots, x_n]$ with the x_i in degree 1 then $\operatorname{Proj} R$ is projective space $\mathbb{P}^n_k$. If $R = k[x_1, \ldots, x_n, y_0, \ldots, y_r]$

with the x_i in degree 0 and the y_i in degree 1, then $\operatorname{Proj} R$ is $\mathbb{A}^n_k \times \mathbb{P}^r_k$. We will see further examples in Section 19.14 where we talk about blowups.

19.10. **Properties of schemes.** We collect here a few common properties of schemes that will be useful to us.

integral: every affine open is Spec of an integral domain (equivalently, X is nonempty, connected, and every point has an affine open neighborhood that is Spec of a domain).

locally Noetherian: every point has an affine open neighborhood that is Spec of a Noetherian ring (equivalently, every affine open is Spec of a Noetherian ring).

Noetherian: the scheme admits a finite cover by open affines that are each Spec of a Noetherian ring (equivalently, the scheme is locally Noetherian and also quasi-compact as defined below.)

regular: every point has an affine open neighborhood that is Spec of a regular Noetherian ring (equivalently, X is locally Noetherian and all its local rings are regular; equivalently, every affine open is Spec of a Noetherian regular ring).

Similarly to the category of topological spaces, the category of schemes admits plenty of pathological objects. The following terms are largely about eliminating pathologies of various types:

quasi-compact: the underlying topological space is compact (equivalently, X has a finite cover by affine opens).

quasi-compact morphism: $X \to Y$ is quasi-compact if the inverse image of every quasi-compact open is quasi-compact (equivalently, Y has an affine open covering $\{U_i\}$ such that each $f^{-1}(U_i)$ is quasi-compact).

separated morphism: $X \to Y$ is separated if the diagonal map $X \to X \times_Y X$ is a closed immersion. [This should be thought of as an analog of the Hausdorff condition in topology; it removes the pathology of having two points that are "infinitely close" to each other.]

quasi-separated morphism: $X \to Y$ is quasi-separated if the diagonal map $X \to X \times_Y X$ is quasi-compact.

finite-type morphism: $X \to Y$ is quasi-compact and for every $x \in X$ there is an affine neighborhood $\operatorname{Spec} S = U \subseteq X$ and an affine open $\operatorname{Spec} R = V \subseteq Y$ such that $f(U) \subseteq V$ and the associated map $R \to S$ exhibits S as a finitely-generated R-algebra.

The following term is not about ruling out pathologies, but rather is an attempt to capture the topological idea of a map with compact fibers:

proper morphism: a morphism that is separated, finite type, and universally closed.

Remark 19.11. The above definition of proper morphism feels esoteric, and it can be difficult to see the meaning here. It is a good idea to look up the topological notions of separated and proper maps in Appendix A for comparison. In topology "separated" is the relative form of the Hausdorff property, and "proper" is the relative form of compactness. Note, however, that in algebraic geometry the term "proper" is the analog of what topologists would call "separated and proper". This can be more than a little confusing.

Standard examples of proper morphisms are inclusions of closed subschemes, projection maps $\mathbb{P}^n_X \to X$, as well as maps built up from these via compositions or pullbacks.

Schemes over a field

When k is a field one can consider the overcategory $\mathcal{S}ch \downarrow \operatorname{Spec} k$. The objects are called schemes over the field k. The following terminology is used in this setting:

projective: a scheme that is isomorphic to a closed subscheme of some projective space $\mathbb{P}^n_k$.

quasi-projective: a scheme that is isomorphic to an open subscheme of a projective scheme.

variety: an integral scheme such that the map $X \to \operatorname{Spec} k$ is separated and of finite type.

complete variety: a variety such that $X \to \operatorname{Spec} k$ is proper.

smooth variety: this is a condition about nondegeneracy of certain Jacobian matrices formed from the equations defining the scheme, and can also be described in terms of sheaves of differentials. Over characteristic zero fields it is equivalent to the variety being regular, and so we will use the two terms interchangeably in this context.

Remark 19.12. At its most fundamental level, algebraic geometry is about understanding solutions sets of algebraic equations. This perhaps gives the impression that all schemes of interest are quasi-projective, and it should be warned that this is not the case. In dimension three and above it is very easy to glue quasi-projective schemes together in mild ways and end up with something that is not quasi-projective. See [Sha2, Chapter 6.2.3] for a nice discussion.

19.13. **Pulling back and pushing forward $\mathcal{O}_X$-modules.** If $R \to S$ is a map of rings then there are two associated functors on the categories of modules. The "extension-of-scalars" functor $(-) \otimes_R S\colon \operatorname{Mod}_R \to \operatorname{Mod}_S$ takes finitely-generated modules to finitely-generated modules and is right exact. The "restriction-of-scalars" functor $\operatorname{Mod}_S \to \operatorname{Mod}_R$, that takes an S-module and regards it as an R-module, is exact but only preserves finite-generation if S is module-finite over R.

For a map of schemes $f\colon Y \to X$ one has the pullback functor $f^*\colon \operatorname{qcMod}_X \to \operatorname{qcMod}_Y$, that locally is modeled by the tensor product. That is to say, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\operatorname{Spec} R = V \subseteq X$ is an affine open with an R-module M and an isomorphism $\widetilde{M} \cong \mathcal{F}|_V$, and $\operatorname{Spec} S = U \subseteq Y$ is an affine open such that $f(U) \subseteq V$, then there is an isomorphism $\widetilde{(S \otimes_R M)} \cong (f^*\mathcal{F})|_V$. The pullback functor is right exact, and when X and Y are locally Noetherian it sends coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_Y$-modules.

If $f\colon Y \to X$ is slightly nice (quasi-compact and quasi-separated, see Section 19.10)) there is also the corresponding direct-image functor $f_X\colon \mathrm{qcMod}_Y \to \mathrm{qcMod}_X$. In general this does not preserve coherence, but it does in the case where X is locally Noetherian and f is proper [EGA3, Theorem 3.2.1].

Schemes have open and closed subschemes, which restrict to the usual notions for affine schemes. If $i\colon Z \hookrightarrow X$ is a closed subscheme then in an affine chart $U = \operatorname{Spec} R$ for X, Z corresponds to $\operatorname{Spec} R/I$ for some ideal I. In this case the direct image functor i_* is the analog of taking an R/I-module and thinking of it as an R-module. If $\mathcal{F}$ is a quasi-coherent sheaf on Z then the stalks of $i_*\mathcal{F}$ satisfy

$$(i_*\mathcal{F})_{(x)} = \begin{cases} \mathcal{F}_{(x)} & \text{if } x \in Z, \\ 0 & \text{if } x \notin Z. \end{cases}$$

For this reason $i_*\mathcal{F}$ is also called the "extension-by-zero" of $\mathcal{F}$ (note that for inclusions of non-closed subschemes the direct image and the extension-by-zero are different, so be careful).

The structure sheaf $\mathcal{O}_Z$ is a sheaf on Z, and the direct image $i_*\mathcal{O}_Z$ is often also written as $\mathcal{O}_Z$, by abuse. (This is the analog of taking the ring R/I and denoting it by the same symbol when thinking of it as an R-module).

19.14. **Blowups.** In the next section we will need to use blowup varieties, so in preparation we discuss them here. Even if one is mostly interested in affine schemes, blowups are an important tool and the full power of general schemes is needed to discuss them.

Let X be a smooth complex variety and let $A \hookrightarrow X$ be a closed subvariety, which for convenience we assume to be smooth as well. The blow-up $\mathrm{Bl}_A(X)$ of X at A is an algebraic variety that homotopically corresponds to removing A and then sewing in a copy of the projectived normal bundle in its place. That is, let V be a tubular neighborhood of A, with associated homeomorphism $V \cong N_{X/A}$. Then there is a homotopy equivalence

$$\mathrm{Bl}_A(X) \simeq (X - A) \amalg_{(V-A)} \mathbb{P}(N) \tag{19.15}$$

Here the map $V - A \to \mathbb{P}(N)$ is the map $N - 0 \to \mathbb{P}(N)$ that sends any nonzero element of a fiber N_a to the corresponding line it spans, regarded as an element of $\mathbb{P}(N_a)$. The homotopy equivalence in (19.15) can in fact be upgraded to a homeomorphism, but the usual point-set model for the blowup is a little different and so this can create confusion. We describe the usual model next.

Let us start with the case $X = \mathbb{C}^n$ and $A = \{\underline{0}\}$. The idea of the blowup $\mathrm{Bl}_{\underline{0}}\,\mathbb{C}^n$ is that away from $\underline{0}$ it is exactly $\mathbb{C}^n - \{\underline{0}\}$, but the point $\underline{0}$ is "blown up" into a whole family of points representing the normal directions around $\underline{0}$. We can construct this blowup by doing something clever with projective space. Begin by observing that given a point $\underline{x} \in \mathbb{C}^n$ and a line $[\underline{Y}] \in \mathbb{C}P^{n-1}$ then $\underline{x} \in [\underline{Y}]$ if and only if $x_iY_j = x_jY_i$ for all pairs $1 \le i, j \le n$. To see this, note that for $\underline{x} = \underline{0}$ both conditions are vacuous; but if instead some $x_i \neq 0$ then the equations are equivalent to $Y_i \neq 0$ and $x_j = Y_j \cdot \frac{x_i}{Y_i}$ for all j.

Define $B = \mathrm{Bl}_{\underline{0}}\, \mathbb{C}^n$ to be the subset of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ consisting of pairs $(\underline{x}, [\underline{X}])$ such that $\underline{x} \in [\underline{X}]$, or equivalently $x_i X_j = x_j X_i$ for all i, j. Observe that when $\underline{x} \neq 0$ then $[\underline{X}]$ is determined uniquely and is equal to $[\underline{x}]$, whereas when $\underline{x} = 0$ then $[\underline{X}]$ can be any line in $\mathbb{C}^n$. The point $\underline{x} = 0$ has been "blown up" into an entire $\mathbb{C}P^{n-1}$ in B. Note that B, being defined by algebraic equations, is a subvariety of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$.

The projection map $\pi_1 \colon \mathbb{C}^n \times \mathbb{C}P^{n-1} \to \mathbb{C}^n$ restricts to give $\pi \colon B \to \mathbb{C}^n$, called the standard projection associated to the blowup. Observe that the fiber over $\underline{0}$ is $E = \mathbb{C}P^{n-1}$; this is usually called the **exceptional divisor** (we discuss this terminology in a moment). Also note that we have the pushout diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d, "\pi"] \\ \{\underline{0}\} \arrow[r] & \mathbb{C}^n. \end{tikzcd}$$

That is, squashing E back down to a point recovers the original space $\mathbb{C}^n$.

Exercise 19.16. It is useful to think about the case $n = 2$. Consider the sequence $p_n = (\frac{1}{n}, \frac{3}{n})$, which of course converges to $(0, 0)$ in $\mathbb{C}^2$. This sequence lifts uniquely into B. What is the limit in B?

Exercise 19.17. Let $\tilde{B} = \{\underline{x} \in \mathbb{C}^n \mid 1 \leq |\underline{x}|\} \cup_{S^{2n-1}} \mathbb{C}P^{n-1}$, where each $\underline{x} \in S^{2n-1}$ is identified with the associated point $[\underline{x}] \in \mathbb{C}P^{n-1}$. This is the first description we gave of the blowup, see (19.15), where one takes out a tubular neighborhood and sews in a copy of the projectivized normal bundle. Define a map $f \colon \tilde{B} \to B$ as follows:

$$\begin{cases} f(\underline{x}) = \big((|\underline{x}| - 1)\underline{x}, [\underline{x}]\big) & \text{for } \underline{x} \in \mathbb{C}^n \text{ with } |\underline{x}| \geq 1, \\ f([\underline{x}]) = (\underline{0}, [\underline{x}]) & \text{for } [\underline{x}] \in \mathbb{C}P^{n-1}. \end{cases}$$

Verify that f is a homeomorphism.

Returning to general properties of $B = \mathrm{Bl}_{\underline{0}}\, \mathbb{C}^n$, consider the region of B defined by $X_i \neq 0$. This is an open set U_i; it consists of all points $\underline{x}$ where $x_i \neq 0$ together with the pairs $(\underline{0}, [\underline{X}])$ where $X_i \neq 0$. The equation $x_i = 0$ therefore cuts out $E \cap U_i$ inside of U_i. That is to say, E is defined by a single equation inside of U_i. Since this holds for each i, we see that E is locally cut out by a single equation—algebraic geometers call such an object a *divisor*.

Remark 19.18. There is a clever way of encoding the equations $x_i X_j = x_j X_i$. Introduce a formal variable t and set $X_i = x_i t$; then the validity of those equations is automatic. This technique proves useful when talking about more general blowups, as we will see below. For now, let us just introduce the algebra behind it. If R is a ring and I is an ideal, the **Rees ring** is defined to be the graded ring

$$R[It] = R \oplus It \oplus I^2 t^2 \oplus \cdots \subseteq R[t]$$

That is, $R[It]$ consists of polynomials in t where the coefficient of each t^r lies in I^r. One can check that when $R = k[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_n)$ then $R[It]$ is isomorphic to the quotient ring $k[x_1, \ldots, x_n, X_1, \ldots, X_n]/(x_i X_j - x_j X_i \mid 1 \leq i, j \leq n)$ via $X_i = x_i t$, where each x_i has degree 0 and each X_i has degree 1. Taking Proj of this graded ring yields the blowup $\mathrm{Bl}_{\underline{0}}\, \mathbb{C}^n$.

Let us take another look at $\mathrm{Bl}_{\underline{0}}\,\mathbb{C}^3 \subseteq \mathbb{C}^3 \times \mathbb{C}P^2$ (taking $n = 3$ just to keep the numbers small), this time keeping the Rees ring in mind. Points of the blowup are pairs $((x, y, z), [X : Y : Z])$. We can think of x, y, and z as being functions on the original space $\mathbb{C}^3$, with the corresponding X, Y, and Z functions on the normal space to $\underline{0}$. What is confusing is that in some sense x and X are the same function, but with an artificial distinction being thrown in regarding their domains: x is a function on $\mathbb{C}^3$ whereas X is a function on $N_{\underline{0}}\mathbb{C}^3$ (which happens to be another copy of $\mathbb{C}^3$). As soon as one allows for this distinction one can write down functions like xX, x^2yZ^3, and so forth. Note that x^2, xX, and X^2 are all different.

The way this plays out in the Rees ring is that rather than distinguishing the two types of functions via lowercase and uppercase letters, one introduces the artificial symbol t that says "regard this as a function on the normal space". So we have the functions x and xt (formerly called X), and then (for example) the functions x^2, $x(xt) = x^2t$, and $(xt)(xt) = x^2t^2$. The Rees ring is simply the universal ring constructed in this way, consisting of the original elements of a ring R together with elements of an ideal I that are "promoted" to having a separate existence, here regarded as functions on the normal spaces to $V(I)$.

Now let us return to more advanced blowups. Increasing our level of sophistication, let us keep $X = \mathbb{C}^n$ and now let A be the linear subspace defined by the equations $x_1 = x_2 = \cdots = x_c = 0$. Each point on A has a normal space of dimension c, and we will blow up that normal space at the origin—in fact, we will do this simultaneously at all the points of A. Here define $\mathrm{Bl}_A\,\mathbb{C}^n$ to be the subset of $\mathbb{C}^n \times \mathbb{C}P^{c-1}$ consisting of pairs $(\underline{x}, [\underline{X}])$ such that $x_iX_j = x_jX_i$ for all $1 \leq i, j \leq c$. Again observe that for any $\underline{x}$ not in A there is exactly one possible $[\underline{X}]$, namely $[\underline{X}] = [x_1 : \cdots : x_c]$, whereas for $\underline{x}$ in A any $[\underline{X}]$ is possible.

Again, we can keep track of the equations $x_iX_j = x_jX_i$ by introducing a formal variable t and setting $X_i = tx_i$ for all i. The blowup $\mathrm{Bl}_A\,\mathbb{C}^n$ is Proj of the Rees ring $R[It]$ where $R = \mathbb{C}[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_c)$.

Finally, for an arbitrary pair (X, A) consisting of a smooth complex variety and a smooth subvariety of codimension c, at each point in A we can find local coordinates in which A is defined by the vanishing of the first c of them. So we can build the blowup $\mathrm{Bl}_A X$ by doing the above construction locally around each point in A and patching the results together. In an affine chart $\operatorname{Spec} R$ on X the subvariety is of the form $A = V(I)$, and the blowup is defined to be $\operatorname{Proj} R[It]$.

The key facts about $\mathrm{Bl}_A X$ are that it comes with a projection map $\pi\colon \mathrm{Bl}_A X \to X$, the map $\pi^{-1}(X - A) \to X - A$ is a homeomorphism, the fibers $\pi^{-1}(a)$ for $a \in A$ are projective spaces $\mathbb{C}P^{c-1}$, and moreover $\pi^{-1}(A)$ is precisely the projectivized normal bundle of A in X. The closed subscheme $\pi^{-1}(A) \subseteq \mathrm{Bl}_A X$ is locally cut out by a single equation (i.e., it is a divisor) and is called the *exceptional divisor*. Finally, we have a pushout square

$$\begin{tikzcd}
\mathbb{P}(N_{X/A}) \arrow[r, equal] & \pi^{-1}(A) \arrow[r] \arrow[d] & \mathrm{Bl}_A X \arrow[d, "\pi"] \\
& A \arrow[r] & X.
\end{tikzcd}$$

We have described the construction of the blowup in the setting of complex manifolds because that is the best for building intuition, but in fact all of this can be done entirely in the realm of algebraic geometry and without the smoothness conditions. If X is any scheme and $A \hookrightarrow X$ is a closed subscheme there is a blowup $\mathrm{Bl}_A X$ with similar properties to what we described above. Without smoothness one does not have a normal *bundle*, but one always has a normal *cone*. For $X = \operatorname{Spec} R$ and $A = V(I)$, I an ideal of R, the blowup $\mathrm{Bl}_A X$ is defined to be $\operatorname{Proj} R[It]$, and for general schemes one patches together these local constructions. The idea behind the Rees ring to keep in mind is that we have the functions on the original variety (this is the R) and then additional functions on the normal directions: these are the functions ft for $f \in I$, and products of such. Recall that 'ft' is not a true product, and the t just plays the formal role of distinguishing the f that is a function on the normal directions from the f that is a function on X.

20. Exactness and fiberwise exactness

Suppose X is a scheme and $\mathcal{L}_\bullet$ is a chain complex of locally free $\mathcal{O}_X$-modules of finite rank. We know this corresponds to a chain complex of algebraic vector bundles on X, and we have discussed the fact that exactness in a certain degree, when regarded as a complex of sheaves (or modules in the case X is affine), is not equivalent to fiberwise exactness in that degree. We now explore the relations between these concepts in more detail.

Start by recalling some algebra. If R is a commutative ring, M a finitely-generated R-module, and $P \subseteq R$ a prime ideal, then the following are equivalent:
(1) $M_P = 0$,
(2) $M_f = 0$ for some $f \in R - P$,
(3) $\operatorname{Ann}(M) \not\subseteq P$, i.e. $P \notin V(\operatorname{Ann}(M))$.
Thinking in terms of sheaves on $\operatorname{Spec} R$, the second condition is equivalent to saying that the sheaf is identically zero on some Zariski neighborhood of P; this is because the open sets $D(f) = \operatorname{Spec} R - V(f)$ for $f \in R - P$ are cofinal in all Zariski neighborhoods of P in $\operatorname{Spec} R$.

The **support** of M is the Zariski closed set

$$\operatorname{Supp}(M) = V(\operatorname{Ann}(M)) = \{P \in \operatorname{Spec} R \,|\, M_P \neq 0\}.$$

One readily extends this to the sheaf setting: if $\mathcal{F}$ is a sheaf on X define

$$\operatorname{Supp}(\mathcal{F}) = \{P \in X \,|\, \mathcal{F}_{(P)} \neq 0\}.$$

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then locally it corresponds to a finitely-generated module and so $\operatorname{Supp}(\mathcal{F})$ is closed in each piece of an affine cover of X—hence it is closed in X.

Returning to the setting of finitely-generated modules, here is another useful set of equivalent statements:
(1) $M = 0$,
(2) $M_P = 0$ for all prime ideals P,
(3) $M_m = 0$ for all maximal ideals m,
(4) $M/mM = 0$ for all maximal ideals m.
Note that the equivalence of (3) and (4) follows from $M/mM = M_m/mM_m$ and Nakayama's lemma. Rephrasing things geometrically and generalizing to schemes, if $\mathcal{F}$ is a coherent sheaf on a scheme X then the following are equivalent:
(1) $\mathcal{F}$ is the zero sheaf,

(2) All stalks of $\mathcal{F}$ are zero,
(3) All stalks of $\mathcal{F}$ at closed points are zero,
(4) All fibers of $\mathcal{F}$ at closed points are zero.

Now let us look at complexes of $\mathcal{O}_X$-modules and their homology sheaves. If $\mathcal{F}_\bullet$ is a complex of $\mathcal{O}_X$-modules and p is a point of X, we will say that $\mathcal{F}_\bullet$ is **exact at p in degree s** if the complex of stalks $(\mathcal{F}_\bullet)_{(p)}$ is exact in degree s.

Proposition 20.1. *Assume X is a locally Noetherian scheme and $\mathcal{F}_\bullet$ is a complex of coherent $\mathcal{O}_X$-modules. Let $n \in \mathbb{Z}$. Then the following statements are equivalent:*
(1) $H_n(\mathcal{F})_{(p)} = 0$;
(2) $\mathcal{F}_\bullet$ is exact at p in degree n;
(3) There exists a Zariski open affine neighborhood U of p such that the complex $\Gamma(U, \mathcal{F}_\bullet)$ is exact in degree n.

Proof. Stalks are computed by directed colimits, and such colimits commute with taking homology. So $H_n(\mathcal{F}_{(p)}) = H_n(\mathcal{F})_{(p)}$ and this yields (1) $\Longleftrightarrow$ (2). The implication (3)$\Rightarrow$(2) follows from the fact that localization is exact: if $M' \to M \to M''$ is a sequence of modules that is exact in the middle then so is $M'_p \to M_p \to M''_p$. For (2)$\Rightarrow$(3) we can take an affine neighborhood $V = \operatorname{Spec} R$ of p, where R is Noetherian, and our complex of sheaves corresponds to a complex $M_\bullet$ of finitely-generated R-modules. The assumption implies that $(M_{n+1})_p \to (M_n)_p \to (M_{n-1})_p$ is exact in the middle. Then $H_n(M)_p = 0$ and since $H_n(M)$ is finitely-generated there exists an $f \in R - P$ such that $H_n(M)_f = 0$. Therefore M_f is exact in degree n. But M_f is exactly the complex $\Gamma(U, \mathcal{F})$ where $U = D(f) = \operatorname{Spec} R - V(f)$, and U is open in $\operatorname{Spec} R$ and therefore also open in X. $\square$

Corollary 20.2. *Assume X is a locally Noetherian scheme and $\mathcal{F}_\bullet$ is a complex of coherent $\mathcal{O}_X$-modules. Let $n \in \mathbb{Z}$. Then the following statements are equivalent:*
(1) $\mathcal{F}_\bullet$ is exact in degree n;
(2) $\mathcal{F}_\bullet$ is exact in degree n at all points of X;
(3) $\mathcal{F}_\bullet$ is exact in degree n at all closed points of X.

Proof. These are just the statement that $H_i(\mathcal{F}) = 0$ if and only if $H_i(\mathcal{F})_{(p)} = 0$ for all points p, if and only if $H_i(\mathcal{F})_{(p)} = 0$ for all closed points p. $\square$

Now that we have a good understanding of exactness and its stalkwise-characterization, let us turn to fiberwise exactness. For complexes of algebraic vector bundles the slogan to remember is that "fiberwise exactness implies stalkwise exactness, but not vice versa".

Proposition 20.3. *Let X be a locally Noetherian scheme and let $\mathcal{L}_\bullet$ be a chain complex of locally free coherent $\mathcal{O}_X$-modules. Let x be a closed point of X. Then if $\mathcal{L}_\bullet$ is fiberwise-exact over x in degree s then $\mathcal{L}_\bullet$ is exact at x in degree s.*

Since the issues are local around x one immediately reduces to a problem in commutative algebra. Here is the version in local algebra:

Proposition 20.4. *Let (R, m) be a local ring, let A, B, and C be free modules of finite rank, and let $A \xrightarrow{f} B \xrightarrow{g} C$ be maps such that $gf = 0$. If $A/mA \to B/mB \to C/mC$ is exact in the middle then the original sequence is exact in the middle.*

Proof. By doing row and column operations we can assume that the matrix for g has the form $\left[\begin{smallmatrix} I & 0 \\ 0 & * \end{smallmatrix}\right]$ where the submatrix in the lower right has entries in m. In other words, we have isomorphisms $B \cong R^j \oplus B'$, $C \cong R^j \oplus C'$, under these isomorphisms g takes the form $\mathrm{id}_{\mathbb{R}^j} \oplus g'$, and the matrix for g' has entries in m. Note that B' and C' are free, being direct sums of free modules (and using that all finitely-generated projectives over local rings are free). Since $gf = 0$ it must be that $\mathrm{im}(f) \subseteq B'$. Again doing row and column operations just as above, we find that there are isomorphisms $A \cong R^s \oplus A'$, $B' \cong R^s \oplus B''$, with A' and B'' free, such that under these isomorphisms f takes the form $\mathrm{id}_{\mathbb{R}^s} \oplus f'$ with the entries for any matrix for f' having entries in m. This is all to say that our original sequence is isomorphic to one of the form

$$\begin{tikzcd}
R^s \arrow[r, "\mathrm{id}"] & R^s & \\
\oplus & \oplus & \\
A' \arrow[r, "f'"] & B'' \arrow[r, "g'"] & C' \\
& \oplus & \oplus \\
& R^j \arrow[r, "\mathrm{id}"] & R^j
\end{tikzcd}$$

with the matrices for f' and g' having entries in m. After reducing modulo m the maps f' and g' become zero, and so the only way this sequence could be exact in the middle is if $B'' = 0$. But this implies the original sequence is exact in the middle as well. $\square$

Proof of Proposition 20.4. Working locally around x we reduce the proposition to the following result: When $Q_\bullet$ is a complex of finitely-generated projectives over a Noetherian ring R and m is a maximal ideal, then if Q/mQ is exact in degree s then Q_m is also exact in degree s. This follows immediately from Proposition 20.4 using that $Q/mQ = Q_m/mQ_m$. $\square$

Exercise 20.5. Generalize Proposition 20.4 as follows. Given a bounded below chain complex of finitely-generated free modules over a local ring, prove that $P_\bullet$ is isomorphic to a direct sum of a split-exact sequence (i.e. a sum of elementary complexes) and a complex of finitely-generated free modules where the differentials have matrices in m.

When looking at a specific degree in a chain complex, we have seen that fiberwise exactness implies stalkwise exactness. The converse is not true, but it *is* true when the complexes are bounded below and exactness is considered in all degrees at once:

Proposition 20.6. *Let X be a locally Noetherian scheme and let $\mathcal{L}_\bullet$ be a bounded-below chain complex of locally free coherent $\mathcal{O}_X$-modules. Let x be a closed point of X. Then $\mathcal{L}_\bullet$ is fiberwise exact at x if and only if $\mathcal{L}_\bullet$ is exact at x.*

Proof. This reduces to the claim that if R is a Noetherian ring, m is a maximal ideal, and $Q_\bullet$ is a bounded-below complex of finitely-generated projectives, then Q/mQ is exact if and only if Q_m is exact. Since $Q/mQ = Q_m/mQ_m$ one may as well replace R with R_m and reduce to the case where R is local. The statement that if Q/mQ is exact then Q is exact follows from Proposition 20.4. For the other direction, if Q is exact then it decomposes as a direct sum of elementary complexes (see Lemma 6.3 and the remarks thereafter) and hence $Q \otimes_R (R/m)$ is still exact. $\square$

While we are on this topic, let us record the following:

Proposition 20.7. *Let X be a locally Noetherian scheme and let $\mathcal{L}_\bullet$ be a bounded chain complex of locally free coherent $\mathcal{O}_X$-modules. Then the set of points $p \in X$ such that $\mathcal{L}_\bullet$ is exact at p is Zariski open.*

Proof. The complement of the set in question is the intersection over all $n \in \mathbb{Z}$ of $\operatorname{Supp} H_n(\mathcal{L}_\bullet)$. Each $\operatorname{Supp} H_n(\mathcal{L}_\bullet)$ is closed, and since $\mathcal{L}_\bullet$ is bounded there are only finitely many n where this set is not equal to X. So we have a finite intersection of closed subsets, which is closed. $\square$

It is useful to define

$$\operatorname{Supp} \mathcal{L}_\bullet = \bigcap_n \operatorname{Supp} H_n(\mathcal{L}_\bullet) = \{p \in X \,\big|\, \mathcal{L}_\bullet \text{ is not exact at } p\}.$$

Note that ??? implies that the closed points in $\operatorname{Supp} \mathcal{L}$ coincide with with the closed points of X over which $\mathcal{L}_\bullet$ is fiberwise exact.

There are many interesting phenomena relating the geometry of $\operatorname{Supp} \mathcal{L}_\bullet$ to the algebraic features of $\mathcal{L}_\bullet$. Here is one result along these lines:

Theorem 20.8 (The New Intersection Theorem)**.** *Let X be a locally Noetherian scheme and let $0 \to \mathcal{L}_n \to \cdots \to \mathcal{L}_0 \to 0$ be a bounded complex of locally free coherent sheaves. If* $\operatorname{Supp} \mathcal{L}_\bullet$ *is zero-dimensional (in particular, nonempty) then $n \geq \dim X$.*

This result has an interesting history. It immediately reduces to the evident analog in local algebra, and in that context it was originally proved by Peskine-Szpiro for characteristic p rings [PS1]. Hochster's work on reduction to characteristic p allowed the result to be generalized to all equicharacteristic local rings. In the 1980s Roberts proved the theorem in general using local Chern characters [R1, R4], and in the 2000s a new proof was given by Piepmeyer and Walker using K-theory [PW].

20.9. **Pullbacks and Tor.** Let $\mathcal{L}_\bullet$ be a complex of locally free coherent sheaves on a scheme X. The complex of fibers over a closed point $x \in X$ is an example of a pullback complex. More generally, if $f\colon Y \to X$ is a map of schemes then we can consider $f^*\mathcal{L}_\bullet$ and ask how the homology of this pullback complex relates to the homology of $\mathcal{L}_\bullet$. The pullback complex is constructed by tensoring, and so the homology is computed by a spectral sequence involving Tor:

$$\operatorname{Tor}_p(f_*\mathcal{O}_Y, H_q(\mathcal{L}_\bullet)) \Rightarrow H_{p+q}(f^*\mathcal{L}_\bullet).$$

We will not need this spectral sequence, as our applications will only involve the simple case where $\mathcal{L}_\bullet$ is a resolution (i.e. only has homology in degree zero) and then the spectral sequence is just the single isomorphism

$$H_n(f^*\mathcal{L}_\bullet) \cong \operatorname{Tor}_n(f_*\mathcal{O}_Y, H_0(\mathcal{L}_\bullet)).$$

But this is motivation for us to develop the basics of Tor. This is done in the exercises at the end of the section.

◦ Exercises ◦

Proving real theorems in algebraic geometry requires extensive training, but for some simple things one can get by just with commutative algebra and the basic technique of working locally. For example, a map $\mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules is surjective if and only if it is surjective on all stalks, if and only if every point $x \in X$ has an affine open U such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. The corresponding statement for modules over a commutative ring is familiar from commutative algebra, and this is the evident globalization.

The following exercises lead up to the definition and properties of Tor for $\mathcal{O}_X$-modules, and they are good training for the technique of "do commutative algebra locally". We will use these results in the next section.

Exercise 20.10. Let X be a locally Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $p \in X$ is such that $\mathcal{F}_{(p)} \cong (\mathcal{O}_{X,x})^n$ as $\mathcal{O}_{X,x}$-modules, where $n \geq 0$. Prove that there exists an open neighborhood U of p such that $\mathcal{F}|_U \cong (\mathcal{O}_X|_U)^{\oplus(n)}$. Conclude that $\mathcal{F}$ is locally free if and only if all of its stalks are free. [Hint: Reduce to the following problem in commutative algebra: if R is Noetherian, $P \subseteq R$ is prime, and M is finitely-generated such that M_P is free over R_P, then there is an $f \in R - P$ such that M_f is free over R_f.]

Exercise 20.11. Let X be a locally Noetherian scheme and let $f\colon \mathcal{M} \to \mathcal{N}$ be a map of coherent $\mathcal{O}_X$-modules. Prove that if f is fiberwise surjective then f is surjective. [Hint: Reduce to the commutative algebra statement that if R is a Noetherian ring and $M \to N$ is a map of finitely-generated modules such that $M/mM \to N/mN$ is surjective for every maximal ideal m, then $M \to N$ is surjective.]

Exercise 20.12. Over affine schemes, locally free $\mathcal{O}_X$-modules are projectives in the category of quasi-coherent $\mathcal{O}_X$-modules. This need not be true for more general schemes; in fact, it need not even be true that $\mathcal{O}_X$ is itself projective. When $X = \mathbb{P}^1$ with coordinates $[x:y]$ prove that the diagram

$$\begin{tikzcd} & \mathcal{O}(-1) \oplus \mathcal{O}(-1) \arrow[d, "{(x,y)}"] \\ \mathcal{O} \arrow[r, "\mathrm{id}"] & \mathcal{O} \end{tikzcd}$$

does not have a lifting, despite the vertical map being a surjection.

Exercise 20.13. In module theory, every module is a quotient of a free module and every finitely-generated module is a quotient of a free module of finite rank. It turns out that the analogs in $\mathcal{O}_X$-modules do not always hold.

A scheme X is said to have **enough locally frees** if every quasi-coherent $\mathcal{O}_X$-module is a quotient of a locally free $\mathcal{O}_X$-module. (A scheme without this property will be given in Exercise 20.14 below).

Assume X has enough locally frees.

(a) Prove that every quasi-coherent $\mathcal{O}_X$-module has a resolution by locally free $\mathcal{O}_X$-modules.
(b) Locally free $\mathcal{O}_X$-modules need not be projective, so we do not have the usual lifting properties that we need to do homological algebra. Prove the following local version: given a diagram of quasi-coherent $\mathcal{O}_X$-modules

$$\begin{tikzcd} & \mathcal{F} \arrow[d, two heads, "v"] \\ \mathcal{L} \arrow[r, "u"] & \mathcal{G} \end{tikzcd}$$

where v is surjective and $\mathcal{L}$ is locally free, there exists a surjection $\mathcal{L}' \to \mathcal{L}$ with $\mathcal{L}'$ locally free, and a lifting $\mathcal{L}' \to \mathcal{F}$ making the evident triangle commute. [Hint: Form the pullback $u^*\mathcal{F}$ and then use the assumption on X.]

(c) Suppose that $\mathcal{M}$ is quasi-coherent and both $\mathcal{L}_\bullet \to \mathcal{M}$ and $\mathcal{J}_\bullet \to \mathcal{M}$ are resolutions by locally free $\mathcal{O}_X$-modules. If these were projective resolutions we would get a comparison map from one resolution to another, but that is too much to expect here. Instead prove that there is a third resolution $\mathcal{E}_\bullet \to \mathcal{M}$ by locally free $\mathcal{O}_X$-modules together with surjective quasi-isomorphisms $\mathcal{E} \twoheadrightarrow \mathcal{L}$ and $\mathcal{E} \twoheadrightarrow \mathcal{J}$. [Hint: This can be done using only the lifting property from (b). It might help to first find an $\mathcal{E}$ where only the quasi-isomorphism $\mathcal{E} \to \mathcal{L}$ is surjective.]

(d) Define $\mathrm{Tor}_i(\mathcal{M}, \mathcal{N}) = H_i(\mathcal{L}_\bullet \otimes \mathcal{N})$ where $\mathcal{L}_\bullet \to \mathcal{M}$ is any resolution by locally free $\mathcal{O}_X$-modules. We need to prove that this is independent of resolution.
   (i) Let $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{L}_\bullet$ be a surjective map of locally free resolutions of $\mathcal{M}$ and let $\mathcal{K}_\bullet$ be the kernel. Explain why $0 \to \mathcal{K} \otimes \mathcal{N} \to \mathcal{E} \otimes \mathcal{N} \to \mathcal{L} \otimes \mathcal{N} \to 0$ is still a short exact sequence of complexes. Prove that each $\mathcal{K}_i$ is locally free and that the complex $\mathcal{K}$ is locally split-exact. Conclude that $\mathcal{K} \otimes \mathcal{N}$ is exact, and therefore $\mathcal{E} \otimes \mathcal{N} \to \mathcal{L} \otimes \mathcal{N}$ is a quasi-isomorphism. [Hint: All of these are an exercise in the art of working locally.]
   (ii) Use (c) to complete the proof that $\mathrm{Tor}_i(\mathcal{M}, \mathcal{N})$ is well-defined.

(e) Think about the functoriality of Tor in this context and how that can be justified. Prove that short exact sequences give rise to long exact Tor-sequences, and that $\mathrm{Tor}_*(\mathcal{M}, \mathcal{N})$ can be computed by resolving either $\mathcal{M}$ or $\mathcal{N}$.

Exercise 20.14. Not all schemes have enough locally frees. The exercise is to think through, and fill in the details of, the following example. Take two copies of $\mathbb{A}^n$ and glue them together along $\mathbb{A}^n - 0$, with the resulting scheme denoted X. This is called "$\mathbb{A}^n$ with the origin doubled". On $\mathbb{A}^n$ all locally free modules are actually free (this is the Quillen-Suslin theorem), and so one readily sees that the same property holds for X. Any map of free $\mathcal{O}_X$-modules will have to have the same behavior on the fibers over the two origins, by continuity. So any $\mathcal{O}_X$-module that is a cokernel of a map of locally free modules must have the same stalks and fibers over the two origins. But it is easy to construct quasi-coherent $\mathcal{O}_X$-modules where this fails: for example, take the structure sheaf of one of the origins.

Exercise 20.15. A scheme X is said to have the **resolution property** if every quasi-coherent $\mathcal{O}_X$-module of finite type (see Remark 19.5) is a quotient of a locally free $\mathcal{O}_X$-module of finite rank. From now on we will abbreviate "locally free of finite rank" to "finite locally free". The name "resolution property" is somewhat inapt, as it is only in certain Noetherian settings that one can iterate the property to construct resolutions. The name was first used in [T1], in the Noetherian context, but it has since been appropriated for the more general situation.

(a) Assume that X is a locally Noetherian scheme (see Section 19.10). Assume as well that X has the resolution property. Prove that every quasi-coherent $\mathcal{O}_X$-module of finite type has a resolution by finite locally free modules.

(b) Let R be a regular local ring of dimension n, let M be a finitely-generated R-module, and let $F_\bullet \to M$ be any resolution of M by finitely-generated free modules. Let K_{n-1} be the kernel of $F_{n-1} \to F_{n-2}$. Prove that $\mathrm{Tor}_i(K_{n-1}, R/m) \cong \mathrm{Tor}_{i+n}(M, R/m)$ for all $i \geq 1$ and use this to deduce that K_{n-1} is free.

(c) Let X be a regular scheme with the resolution property, and let $n = \dim X$. If $\mathcal{M}$ is a coherent sheaf and $\mathcal{L}_\bullet \to \mathcal{M}$ is a resolution by finite locally free $\mathcal{O}_X$-modules, let $\mathcal{K}_{n-1}$ be the kernel of $\mathcal{L}_{n-1} \to \mathcal{L}_{n-2}$. Prove that $\mathcal{K}_{n-1}$ is locally free, so that

$$0 \to \mathcal{K}_{n-1} \to \mathcal{L}_{n-1} \to \mathcal{L}_{n-2} \to \cdots \to \mathcal{L}_0 \to \mathcal{M} \to 0$$

is also a resolution of $\mathcal{M}$ by finite locally free $\mathcal{O}_X$-modules.

The example from Exercise 20.14 is clearly pathological, in the sense that one is not likely to encounter that scheme in the course of ordinary life. In fact, most reasonable schemes have enough locally frees—and in particular this will include all schemes that we naturally encounter in this book. To make a precise statement, the following types of schemes all have enough locally frees:

(1) All affine schemes.
(2) All open subschemes of closed subschemes of projective space $\mathbb{P}^n_X$ where X is affine.
(3) Any scheme admitting an ample line bundle, as defined in [EGA2, Definition 4.5.3] (see also [EGA2, 4.5.5])
(4) Any scheme admitting an ample family of line bundles, as defined in [SGA6, II.2.2.4] (see also [TT, Definition 2.1.1]). Such schemes are called *divisorial* [SGA6, II.2.2.5].
(5) Any separated, Noetherian, locally factorial scheme (the latter condition includes regular schemes, since a regular local ring is a UFD).

Note that type (1) is included in (2), type (2) is included in (3), and type (3) is included in (4). Type (5) is also included in (4) by [SGA6, II.2.2.7]. For discussion of the fact that schemes of type (4) have enough locally frees, and also satisfy the resolution property, see [TT, Lemma 2.1.3]. The fact that type (5) schemes have the resolution property is also in [H, Exercise III.6.8].

21. The denouement: connecting algebra, topology, and geometry

Although we are far from the end of this book, we have reached the point where we finally have the tools to bring Serre's definition of intersection multiplicities into context. In this section we will show how Serre's definition is naturally linked, via K-theory, to topological intersection multiplicities. Conceptually this is fairly straightforward, but there is one hiccup: when trying to make links between the algebro-geometric and topological worlds, one inevitably runs into the issue that the former is much more rigid. Certain techniques that are easy in the topological world require clever geometric substitutes in the algebro-geometric world. In this section we will run into exactly this issue, and will encounter a technique called *deformation to the normal bundle* that appears as a substitute for topological tubular neighborhoods.

The work in this section breaks up into two basic pieces. The first is the local index theorem, which involves a very specialized analysis of the comparison between algebraic and topological K-theory. The second is a more global version of this comparison, where one connects resolutions in algebraic geometry to fundamental classes in topological K-theory. Once these tools are in place it becomes a simple matter to relate Serre's intersection multiplicities to the ones in topological K-theory.

21.1. **The local index.** Suppose that $E_\bullet$ is a bounded chain complex of vector bundles on $\mathbb{C}^n$ that is exact on $\mathbb{C}^n - 0$. Then we get a class $[E_\bullet] \in K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^{-2n}(pt)$. But the Bott calculations say that this group is cyclic, generated by β^n. Thus $[E_\bullet] = d \cdot \beta^n$ for a unique integer d. We call this integer the **local index** of the complex $E_\bullet$, and we will denote it $\operatorname{ind}_0(E_\bullet)$. The natural question is: how do we compute this invariant from the data in $E_\bullet$?

I don't know a simple answer to this question, but the question becomes more manageable if we assume that the complex $E_\bullet$ is algebraic: that is, if we assume that each E_i is an algebraic vector bundle and the maps $E_i \to E_{i-1}$ are algebraic. The phrase "complex of algebraic vector bundles" will always mean such an object.

The following result is an example of an "index theorem", where a topological invariant detected by K-theory is described in terms of more rigid (in this case algebraic) data.

Theorem 21.2. *If $E_\bullet$ is a bounded complex of algebraic vector bundles on $\mathbb{C}^n$ that is exact on $\mathbb{C}^n - 0$, then the local index is given by*

$$\operatorname{ind}_0(E_\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(P_\bullet)$$

where $P_\bullet$ is a complex of finitely-generated, projective $\mathbb{C}[x_1, \dots, x_n]$-modules such that $P_\bullet(\mathbb{C}) \cong E_\bullet$. (Here $P \mapsto P(\mathbb{C})$ is the functor that associates to every projective $\mathbb{C}[x_1, \dots, x_n]$-module the corresponding vector bundle over $\mathbb{C}^n$; see Section 10.10, where this was denoted $P \mapsto P_{\mathbb{C}}$.)

The proof of Theorem 21.2 comes down to a comparison between algebraic and topological K-theory groups. Once the machinery for this comparison is in place, the theorem follows by a simple computation. To set up this machinery we need to recall some ideas from Part 1 of these notes.

Now let $Z \subseteq \operatorname{Spec} R = X$ be Zariski closed. Define $K^0_{\mathrm{alg}}(X, X - Z)$ by taking the free abelian group on bounded chain complexes of algebraic vector bundles on

X that are exact at every point in $X - Z$ and quotienting by the following two relations:

(1) $[P_\bullet] = 0$ if $P_\bullet$ is exact on all of X, and
(2) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ for every short exact sequence of chain complexes $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

Note that $K^0_{\text{alg}}(X, X - Z)$ is exactly the same as the group denoted $K(R)_Z$ in Section 5.21; sometimes we will revert to that notation when we want to concentrate on the underlying algebraic perspective.

Exercise 21.3. Prove that if $P_\bullet \to Q_\bullet$ is a quasi-isomorphism of chain complexes that are exact on $X - Z$ then $[P_\bullet] = [Q_\bullet]$ in $K_{alg}(X, X - Z)$. (We have proven similar claims on more than one occasion, the exercise is just to remember how the proof works.)

The assignment $P \mapsto P(\mathbb{C})$ from algebraic to topological vector bundles induces a map of abelian groups

$$\phi\colon K^0_{\text{alg}}(X, X - Z) \to \mathcal{K}^0(X(\mathbb{C}), X(\mathbb{C}) - Z(\mathbb{C})).$$

This first uses Proposition 20.6 to know that if $P_\bullet$ is a chain complex that is exact on $X - Z$ then $P_\bullet(\mathbb{C})$ is fiberwise exact at all points in $X(\mathbb{C}) - Z(\mathbb{C})$. Then we need to check that relations (1) and (2) in the definition of $K^0_{\text{alg}}(X, X - Z)$ are preserved, but the first is trivial and the second follows from Proposition 18.12. To avoid cumbersome notation it will be convenient to write the target group of ϕ as $K^0_{top}(X, X - Z)$; it looks much more pleasant to write

$$\phi\colon K^0_{\text{alg}}(X, X - Z) \to K^0_{top}(X, X - Z).$$

Sometimes we will drop the "top" and just write $K^0(X, X - Z)$, but we will never drop the "alg".

If M is a finitely-generated R-module, recall that the **support** of M is

$$\operatorname{Supp} M = \{Q \subseteq R \,|\, Q \text{ is prime and } M_Q \neq 0\}.$$

This coincides with $V(\operatorname{Ann} M)$, the set of all primes containing $\operatorname{Ann} M$. In particular, $\operatorname{Supp} M$ is Zariski-closed. Let $G(X)_Z$ (or $G(R)_Z$) denote the Grothendieck group of finitely-generated R-modules M such that $\operatorname{Supp} M \subseteq Z$.

We have the usual Euler characteristic map $\chi\colon K^0_{\text{alg}}(X, X - Z) \to G(X)_Z$ that sends $[P_\bullet]$ to $\sum_i (-1)^i [H_i(P)]$. The following result should come as no surprise:

Theorem 21.4. *If R is regular then $\chi\colon K^0_{\text{alg}}(X, X - Z) \to G(X)_Z$ is an isomorphism, for any closed $Z \subseteq \operatorname{Spec} R$.*

Proof. The inverse sends a class $[M]$ to the class $[P_\bullet]$ for any finite projective resolution $P_\bullet$ for M over R. The proof that this is well-defined, and that the maps are inverses, is exactly the same as for Theorem 2.13. □

Our discussion so far has focused on the case where X is affine. Even if one only cares about this case, certain arguments are going to end up forcing us to consider more general schemes. To this end, let us just observe that the definition of the group $K^0(X, X - Z)$ can be repeated almost verbatim for any scheme X and closed subset $Z \subseteq X$, using bounded complexes of locally free sheaves that are exact (on stalks) at points in $X - Z$. Likewise, $G(X)_Z$ is defined to be the Grothendieck group of coherent sheaves on X whose support is contained in Z. For Noetherian schemes the Euler characteristic gives a map $\chi\colon K^0(X, X-Z) \to G(X)_Z$, and this is an isomorphism if X is regular and nice enough that we are guaranteed the existence of locally free resolutions.

Theorem 21.5. *Assume X is a separated, regular, Noetherian scheme and $Z \subseteq X$ is a closed subset. Then $\chi\colon K^0(X, X-Z) \to G(X)_Z$ is an isomorphism.*

Exercise 21.6. Prove the above theorem (consult Exercises 20.13, 20.15, and the subsequent remarks therein).

Now let us restrict to the case where $R = \mathbb{C}[x_1, \ldots, x_n]$, so that X is affine n-space $\mathbb{A}^n_{\mathbb{C}}$; we will just write $X = \mathbb{C}^n$ for convenience. Let $Z = \{0\} = V(x_1, \ldots, x_n)$ be the closed set consisting only of the origin. Our aim will be to calculate the group $K^0_{\mathrm{alg}}(X, X-Z) = K^0_{\mathrm{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0)$ in this case.

Let $m = (x_1, \ldots, x_n)$. It is easy to see that the following conditions on a finitely-generated R-module M are equivalent:

(1) $\operatorname{Supp} M = \{m\}$;
(2) $\operatorname{Ann} M$ is contained in only one maximal ideal, namely m;
(3) $\operatorname{Rad}(\operatorname{Ann} M) = m$;
(4) M is annihilated by a power of m.

Assuming M satisfies these conditions, consider the finite filtration

$$M \supseteq mM \supseteq m^2M \supseteq \cdots \supseteq m^kM \supseteq m^{k+1}M = 0.$$

Then in $G(R)_Z$ we have $[M] = \sum_{i=0}^{k}[m^iM/m^{i+1}M]$. But each quotient is a finite-dimensional R/m-vector space, so $[M]$ is just a multiple of $[R/m]$. This shows that $G(R)_Z$ is cyclic, generated by $[R/m]$. Moreover, each quotient $m^iM/m^{i+1}M$ is finite-dimensional as a $\mathbb{C}$-module (where the module structure is coming from $\mathbb{C} \subseteq R$). It follows that M is also finite-dimensional as a $\mathbb{C}$-module. Since dimension is additive it gives a function

$$\dim\colon G(R)_Z \to \mathbb{Z},$$

which is clearly surjective and hence an isomorphism.

The isomorphism $K^0_{\mathrm{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \to G(\mathbb{C}^n)_{\{0\}}$ from Theorem 21.4 sends the Koszul complex $K(x_1, \ldots, x_n; R)$ to $[R/m]$, but we have just seen that the target is isomorphic to $\mathbb{Z}$ and $[R/m]$ is a generator. So $K^0_{\mathrm{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \cong \mathbb{Z}$ and is generated by the Koszul complex.

Now consider the following diagram:

$$\begin{tikzcd} K^0_{\mathrm{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \arrow[r, "\phi"] \arrow[d, "\chi", "\cong"'] & K^0_{top}(\mathbb{C}^n, \mathbb{C}^n - 0) \arrow[r, "\cong"] & \mathbb{Z}\langle \beta^n \rangle \\ G(R)_{\{m\}} \arrow[rr, "\dim", "\cong"'] & & \mathbb{Z}. \end{tikzcd} \tag{21.7}$$

We know by Bott's calculations (Theorem 18.22) that the target of ϕ is isomorphic to $\mathbb{Z}$ and is generated by the Koszul complex. Likewise, we have just seen that the domain of ϕ is isomorphic to $\mathbb{Z}$ and is generated by the algebraic Koszul complex. Since ϕ clearly carries the algebraic Koszul complex to the topological one, ϕ is an isomorphism. This is important enough to record:

Proposition 21.8. *The comparison map $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \to K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$ is an isomorphism, and both the domain and codomain are infinite cyclic.*

Our desired local index theorem is an immediate consequence of the above discussion:

Proof of Theorem 21.2. Fill in diagram (21.7) with the map $\mathbb{Z}\langle\beta^n\rangle \to \mathbb{Z}$ that sends β^n to 1. The diagram then commutes, because one only has to check this on the Koszul complex that generates $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0)$; and here it is obvious. The commutativity of this diagram is exactly the statement of Theorem 21.2. $\square$

21.9. **Resolutions and fundamental classes.** Now we'll use these ideas to do something a bit more sophisticated. Let $Z \hookrightarrow \mathbb{C}^n$ be a closed algebraic subvariety and let $I \subseteq \mathbb{C}[x_1, \dots, x_n] = R$ be the ideal of functions vanishing on Z. So $Z = V(I)$. Assume that Z is smooth of codimension c. Then we have a relative fundamental class $[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$.

Let $P_\bullet$ be a bounded, projective resolution of R/I over $\mathbb{C}[x_1, \dots, x_n]$. Note that if $Q \in \operatorname{Spec} R$ then

$$Q \in Z \iff Q \supseteq I \iff (R/I)_Q \neq 0.$$

So if $Q \notin Z$ then $(R/I)_Q = 0$ and therefore $(P_\bullet)_Q$ is exact. Therefore $P_\bullet$ gives a class $[P_\bullet] \in K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z)$. Using our natural transformation $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z) \to K^0(\mathbb{C}^n, \mathbb{C}^n - Z)$ we get a corresponding class $[P_\bullet]$ in relative topological K-theory. We can promote this to a class in relative K^{2c} by multiplying by β^{-c}. It is reasonable to expect this class to be related to $[Z]_{rel}$, and that is indeed the case:

Theorem 21.10. *In the above situation we have $[Z]_{rel} = \beta^{-c} \cdot [P_\bullet]$.*

Note that the β^{-c} factor could be dropped if we regarded $[Z]_{rel}$ as a class in $K^0(\mathbb{C}^n, \mathbb{C}^n - Z)$ instead of $K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$.

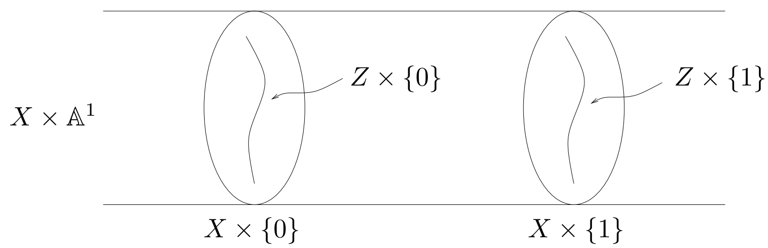
Theorem 21.10 gives one of the key connections between K-theory and homological algebra: projective resolutions give fundamental classes in K-theory. To prove this theorem, recall that $[Z]_{rel}$ is defined by choosing a tubular neighborhood U of Z in $\mathbb{C}^n$, together with an isomorphism between U and the normal bundle $N = N_{\mathbb{C}^n/Z}$. The class $[Z]_{rel}$ is the unique class that restricts to the Thom class $\mathcal{U}_N$. This is all encoded in the following diagram:

$$\begin{tikzcd}
{[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)} \arrow[d, "\text{excision}"', "\cong"] & K^0(\mathbb{C}^n, \mathbb{C}^n - Z) \arrow[l, "\cong"', "\beta^{-c}"] \arrow[d, "\cong"] & K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z) \arrow[l, "\phi"'] \\
K^{2c}(U, U - Z) \arrow[d, equal] & K^0(U, U - Z) \arrow[l, "\beta^{-c}"] \arrow[d, equal] & \\
\mathcal{U}_N \in K^{2c}(N, N - 0) & K^0(N, N - 0) \arrow[l, "\beta^{-c}"] & K^0_{\text{alg}}(N, N - 0). \arrow[l, "\phi"']
\end{tikzcd}$$

Our goal is to take $[P_\bullet] \in K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z)$, push it across the top row and then down, and show that the image is $\mathcal{U}_N$. But note that $\mathcal{U}_N$ is algebraic, as it is represented by the Koszul complex. So $\mathcal{U}_N$ lifts to a class in $K^0_{\text{alg}}(N, N - 0)$. In some sense the most natural idea for our proof would be to stay entirely in the right-most column, and to compare both $[P_\bullet]$ and $\mathcal{U}_N$ on the algebraic side of things.

Of course one immediately sees the trouble, which is that the neighborhood U is *not* algebraic—and so we have a missing group in the third column, obstructing our proof. With some work we will establish a clever way around this, using a technique from algebraic geometry called *deformation to the normal bundle*.

Let X be a smooth scheme over $\mathbb{C}$ and let $Z \hookrightarrow X$ be a smooth closed subscheme (feel free to only think about $X = \mathbb{C}^n$). We begin by considering $X \times \mathbb{A}^1$:



We will work with the blow-up of $X \times \mathbb{A}^1$ at the subvariety $Z \times \{0\}$:

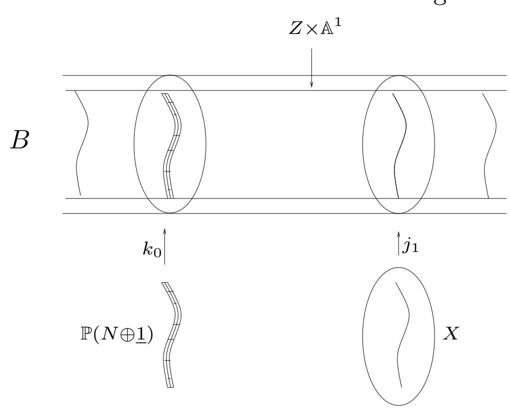
$$B = B\ell_{Z\times 0}(X \times \mathbb{A}^1).$$

Set $N = N_{X/Z}$ and $N' = N_{X\times\mathbb{A}^1/Z\times\{0\}}$. Note that $N' = N \oplus 1$. Topologically, we have a homeomorphism

$$B \cong \left[(X \times \mathbb{A}^1) - (Z \times \{0\})\right] \amalg_{(V-0)} \mathbb{P}(N')$$

where V is a tubular neighborhood of $Z \times \{0\}$ in $X \times \mathbb{A}^1$.

Let $\pi\colon B \to X \times \mathbb{A}^1$ be the projection associated to the blow-up. Let $j_1\colon X \hookrightarrow B$ be the map $x \mapsto (x, 1)$. Let $B_0 = \pi^{-1}(0)$ and let $j_0\colon B_0 \hookrightarrow B$ be the inclusion. Let $k_0\colon \mathbb{P}(N \oplus \underline{1}) \hookrightarrow B$ be the inclusion of the exceptional divisor, which actually lands inside B_0. These can be visualized via the following schematic picture:



We claim that $\pi\colon B \to X \times \mathbb{A}^1$ has a section f over $Z \times \mathbb{A}^1$. The definition of this section is completely clear (and unique) on $Z \times (\mathbb{A}^1 - 0)$, the only sublety is the definition on $Z \times \{0\}$; but here we use the canonical section of $\mathbb{P}(N \oplus \underline{1}) \to Z$. A little effort shows that this gives a well-defined map $Z \times \mathbb{A}^1 \to B$, and it is clearly a section (indicated in the above picture).

Example 21.11. Consider the case $X = \mathbb{C}^n$ and $Z = \{\underline{0}\}$. Then B is the blowup of $\mathbb{C}^n \times \mathbb{C}$ at the origin, and we can use the model as a subspace of $\mathbb{C}^{n+1} \times \mathbb{C}P^n$.

The map $\mathbb{Z} \times \mathbb{A}^1 \to B$ on closed points is given by

$$(\underline{0}, s) \mapsto \begin{cases} ((\underline{0}, s), [\underline{0} : s]) & \text{if } s \neq 0, \\ (\underline{0}, 0), [\underline{0} : 1]) & \text{if } s = 0. \end{cases}$$

Observe that this is continuous at $s = 0$.

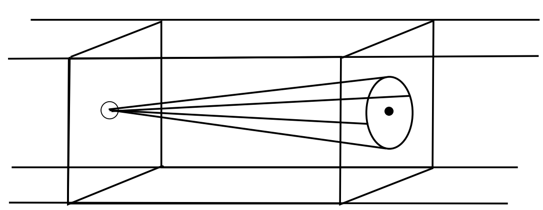
Algebraically, the blowup is $\operatorname{Proj} R[It]$ where $R = \mathbb{C}[x_1, \ldots, x_n, s]$ and $I = (x_1, \ldots, x_n, s)$. Let $J \subseteq R[It]$ be the ideal $J = (x_1, \ldots, x_n, x_1 t, \ldots, x_n t)$. Then $R[It]/J = \mathbb{C}[s][st]$ and Proj of this is the line $Z \times \mathbb{A}^1$, exhibiting this as a closed subscheme of B.

Observe that $B_0 = \mathrm{Bl}_Z(X) \amalg_{\mathbb{P}(N)} \mathbb{P}(N \oplus 1)$ is a decomposition as a union of closed subschemes. The $\mathbb{P}(N \oplus 1)$ is the exceptional divisor, and so of course is closed in B and therefore also in B_0 (since B_0 is itself closed in B). For $\mathrm{Bl}_Z(X)$ we work locally with $X = \operatorname{Spec} R$ and $Z = V(I)$, so that the blowup is Proj of the Rees ring $T = (R[s])[(I, s)t]$ (apologies for the horrible notation). Quotienting out the ideal (s, st) gives the Rees ring $R[It]$, demonstrating that $\mathrm{Bl}_Z(X)$ is closed in B (and therefore also in B_0).

Consider the following (non-commutative) diagram of pairs of spaces:

$$\begin{tikzcd}
(X, X - Z) \arrow[r, "j_1"] & (B, B - (Z \times \mathbb{A}^1)) \\
(U, U - Z) \arrow[u, "h"] \arrow[d, equal] & (B_0, B_0 - (Z \times \{0\})) \arrow[u, "j_0"'] \\
(N, N - 0) \arrow[r, "i"] & (\mathbb{P}(N \oplus \underline{1}), \mathbb{P}(N \oplus \underline{1}) - \mathbb{P}(\underline{1})) \arrow[u, "k_0"']
\end{tikzcd} \tag{21.12}$$

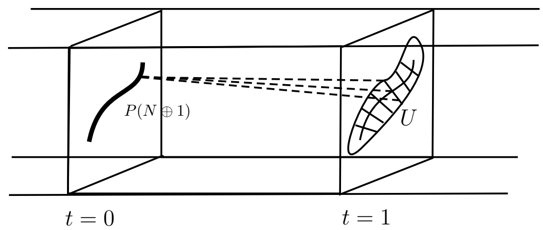
Here i is the canonical open inclusion that sends a point $v \in N$ to the line spanned by $(v, 1) \in N \oplus 1$. Note that all of the maps are algebraic except for the inclusion $h \colon N \hookrightarrow X$ (we implicitly identify N with U here). We claim that this diagram commutes up to homotopy—this homotopy is the so-called "deformation to the normal bundle". To see this, let us first examine the case $Z = \{\underline{0}\}$ where we can use the model for the blowup where $B \subseteq \mathbb{C}^{n+1} \times \mathbb{C}P^n$. Let U be the open unit disk around Z and consider the homotopy $U \times I \to B$ given by $(\underline{x}, t) \mapsto (t(\underline{x}, 1), [\underline{x} : 1])$. At $t = 1$ this is the embedding j_1, and at $t = 0$ it is the embedding k_0. The following picture depicts what is happening here:



$t = 0$ $\qquad\qquad$ $t = 1$

As a point $\underline{x}$ in the neighborhood U (at $t = 1$) moves towards the origin at $t = 0$, along the indicated line, it converges to the point $[\underline{x} : 1]$ in $\mathbb{C}P^n$.

For the case of a general Z, the homotopy is exactly the one above but taking place in the normal directions to Z. In reference to the crude picture below, each fiber of the normal disk bundle U (at $t = 1$) is collapsed—via the straight-line homotopy depicted above—to the corresponding point on Z at $t = 0$, but thereby converging to the associated points $[\underline{x} : 1]$ in the blowup.



Remark 21.13. The technique of "deformation to the normal bundle", which is essentially embodied in the diagram (21.12) and its homotopy commutativity, was used extensively in papers by Fulton and Macpherson in the 1970s, and has a prominent role in the book [Fu]. The technique gives a substitute in algebraic geometry for the role played by tubular neighborhoods in topology.

With the basics of deformation to the normal bundle under out belt, we are now ready to give the proof of our result:

Proof of Theorem 21.10. Write $\mathbb{C}^n = X$. The argument we will give doesn't use anything special about $\mathbb{C}^n$, and actually works for any smooth variety.

Apply $K^0(-)$ to the diagram (21.12) to obtain the commutative diagram

(21.14)
$$\begin{tikzcd} K^0(X, X-Z) \arrow[d, dotted, "h^*"'] & K^0(B, B-(Z\times\mathbb{A}^1)) \arrow[l, "j_1^*"'] \arrow[d, "k_0^*"] \\ K^0(N, N-0) & K^0(\mathbb{P}(N\oplus\underline{1}), \mathbb{P}(N\oplus\underline{1}) - \mathbb{P}(\underline{1})). \arrow[l, "i^*"'] \end{tikzcd}$$

The left vertical arrow is dotted only as a reminder that it is not algebraic. There is a similar diagram, without the dotted arrow, in which every $K^0(-)$ has been replaced with K^0_{alg}; and this new diagram maps to the one above.

Let $Q_\bullet$ be a resolution of $\mathcal{O}_{Z\times\mathbb{A}^1}$ by locally-free $\mathcal{O}_B$-modules (see Section 19 for terminology and notation here). Then we have the corresponding class $[Q_\bullet] \in K^0_{alg}(B, B-(Z\times\mathbb{A}^1))$. We will show that

(1) $j_1^*(Q_\bullet)$ is a resolution of $\mathcal{O}_Z$ on X, and
(2) $(k_0 \circ i)^*(Q_\bullet)$ is a resolution of the structure sheaf of the zero-section on N.

Statement (1) implies that there is a zig-zag of quasi-isomorphisms $j_1^*(Q_\bullet) \xleftarrow{\simeq} \mathcal{E}_\bullet \xrightarrow{\simeq} P_\bullet$, where $P_\bullet$ is our chosen resolution of $\mathcal{O}_Z$ on X and $\mathcal{E}_\bullet$ is some other bounded locally free resolution of $\mathcal{O}_Z$ (this follows from Exercises 20.13 and 20.15). Hence $j_1^*([Q_\bullet]) = [P_\bullet]$ in $K^0_{\mathrm{alg}}(X, X-Z)$. Statement (2) implies that there is a similar zig-zag of quasi-isomorphisms between $(k_0 \circ i)^*(Q_\bullet)$ and $J^*_{p^*N,\Delta}$, since both give resolutions of the structure sheaf of the zero section on N. Hence $i^*(k_0^*([Q_\bullet])) = \mathcal{U}_N$ in $K^0_{\mathrm{alg}}(N, N-0)$. Now push all of this into topological K^0 and use the commutativity of (21.14) to obtain that $h^*([P_\bullet]) = \mathcal{U}_N$. But h^* is an isomorphism, and $[Z]_{rel}$ was defined to be the unique class in $K^0(X, X-Z)$ that maps to $\mathcal{U}_N$ via h^*. So $[P_\bullet] = [Z]_{rel}$.

So the proof reduces to checking the algebraic facts (1) and (2). The geometric intuition here is that the subscheme $Z \times \mathbb{A}^1$ intersects the fibers B_0 and B_1 cleanly enough so that exactness of $Q_\bullet$ is maintained after pulling back to these fibers. Algebraically this boils down to certain Tor calculations that we now explain. If $\mathcal{L}_\bullet \to \mathcal{F}$ is a resolution by locally frees, then we have the left derived functor of j_1^* given by $(L_r j_1^*)(\mathcal{F}) = H_r(j_1^*\mathcal{L}_\bullet)$. Apply $(j_1)_*$ and use that this direct image functor is exact (since B_1 is a closed subscheme of B) to write

$$(j_1)_*(L_r j_1^*(\mathcal{F})) = (j_1)_* H_r(j_1^*\mathcal{L}_\bullet) = H_k((j_1)_* j_1^*\mathcal{L}_\bullet) = H_r(\mathcal{L}_\bullet \otimes_{\mathcal{O}_B} \mathcal{O}_{B_1}) = \operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_{B_1}). \tag{21.15}$$

The functor $(j_1)_*$ does not change the stalks at the points of B_1, and so $L_r j_1^*(\mathcal{F}) = 0$ if and only if $\operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_{B_1}) = 0$. In particular, (1) is equivalent to the statement that $\operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{O}_{Z\times\mathbb{A}^1}, \mathcal{O}_{B_1}) = 0$ for $r > 0$.

Observe that j_1 factors as $X \xrightarrow{\tilde{j}_1} B \setminus B_0 \xrightarrow{j} B$. Here $B \setminus B_0$ is open in B and so j^* is exact. The scheme $B \setminus B_0$ is just $X \times (\mathbb{A}^1 - 0)$ and $j^*(\mathcal{O}_{Z\times\mathbb{A}^1}) = \mathcal{O}_{Z\times(\mathbb{A}^1-0)}$. We therefore need to prove that $L_r\tilde{j}_1^*(\mathcal{O}_{Z\times(\mathbb{A}^1-0)}) = 0$ for $r > 0$, and by the analog of (21.15) this will be implied by $\operatorname{Tor}_r^{\mathcal{O}_{X\times(\mathbb{A}^1-0)}}(\mathcal{O}_{X\times 1}, \mathcal{O}_{Z\times(\mathbb{A}^1-0)}) = 0$. The vanishing of the Tor_r sheaf can be checked locally (e.g. on stalks) at each point of $X\times(\mathbb{A}^1-0)$, and so in particular we can check it locally on X. That is to say, it suffices to prove the vanishing where X is an open affine $X = \operatorname{Spec} R$, in which case $Z = V(I)$ and we are looking at $\operatorname{Tor}_r^{R[s,s^{-1}]}(R[s,s^{-1}]/(s-1), R/I[s,s^{-1}])$. But $R[s,s^{-1}]/(s-1)$ is resolved by the complex $R[s,s^{-1}] \xrightarrow{s-1} R[s,s^{-1}]$ and $s-1$ is a nonzerodivior on $R/I[s,s^{-1}]$, therefore the indicated Tor_r modules vanish for $r \geq 1$ and we are done.

For statement (2) we need to look at the composite

$$N \xrightarrow{i} \mathbb{P}(N \oplus 1) \xrightarrow{k_0} B_0 \xrightarrow{j} B$$

and compute the left derived functors of pulling back $\mathcal{O}_{Z\times\mathbb{A}^1}$ along this composite. Here i is an open inclusion and both k_0 and j are closed inclusions. We first claim that $k_0 i$ is actually an open inclusion: this is because $B_0 = \operatorname{Bl}_Z X \cup_{\mathbb{P}(N)} \mathbb{P}(N \oplus 1)$ and so $N = \mathbb{P}(N\oplus 1) - \mathbb{P}(N) = B_0 - \operatorname{Bl}_Z X$, together with the fact that $\operatorname{Bl}_Z X$ is closed in B_0. So $(k_0 i)^*$ is exact, and it is enough to prove that $L_r j^*(\mathcal{O}_{Z\times\mathbb{A}^1}) = 0$ for $r > 0$.

Now perform the same analysis from (21.15) but with j_1 replaced by j and deduce that $L_r j^*(\mathcal{O}_{Z\times\mathbb{A}^1}) = 0$ if and only if $\operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{O}_{Z\times\mathbb{A}^1}, \mathcal{O}_{B_0}) = 0$. Observe that B_0 sits in the pullback diagram

$$\begin{tikzcd} B_0 \arrow[r, "j_0"] \arrow[d] & B \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd} \tag{21.16}$$

So in an open affine chart $\operatorname{Spec} R$ on B, the subscheme B_0 is Spec of $R \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/(s) = R/(s)$. We need to compute $\operatorname{Tor}_r^R(R/I[s], R/(s))$. It will be important that B is flat over $\mathbb{A}^1$, as then the composite $\operatorname{Spec} R \to B \to \mathbb{A}^1$ is also flat (since the first map is an open inclusion, hence flat). For flatness over curves (like $\mathbb{A}^1$), [H, PropositioñIII.9.7] says this follows from just checking that every irreducible component has dense image in the curve; that is clearly the case for $B \to \mathbb{A}^1$.

To wrap up the proof, observe that $\mathbb{C}[s]/(s)$ is resolved by $\mathbb{C}[s] \xrightarrow{s} \mathbb{C}[s]$. Since $\mathbb{C}[s] \to R$ is flat, $R \xrightarrow{s} R$ is a resolution of $R/(s)$. But s is a nonzerodivisor on $R/I[s]$, so $\operatorname{Tor}_r^R(R/I[s], R/(s)) = 0$ for $r > 0$. This completes the calculation. $\square$

As a consequence of Theorem 21.10 we can now obtain Serre's formula for intersection multiplicities:

Corollary 21.17. *Let Z and W be smooth, closed subvarieties of $\mathbb{C}^n$ such that $Z \cap W = \{0\}$. Then $i(Z, W; 0) = \sum(-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)$, where $R = \mathbb{C}[x_1, \ldots, x_n]$ and I and J are the ideals of functions vanishing on Z and W, respectively.*

Proof. Start with the relative fundamental classes $[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$ and $[W]_{rel} \in K^{2d}(\mathbb{C}^n, \mathbb{C}^n - W)$, where c and d are the codimensions of Z and W inside of $\mathbb{C}^n$. Note that since 0 is an isolated point of intersection we must have $c + d \geq n$ (this is geometrically intuitive, but see Lemma 21.19 below for an algebraic proof). There are in some sense two cases, depending on whether $c + d = n$ or $c + d > n$. In the former case, multiplying our fundamental classes together we get

$$[Z]_{rel} \cdot [W]_{rel} \in K^{2n}(\mathbb{C}^n, \mathbb{C}^n - (Z \cap W)) = K^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0).$$

Note that $K^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0) \cong \mathbb{Z}$ and is generated by $[0]_{rel}$. The topological definition of $i(Z, W; 0)$ is that it is the unique integer for which

$$[Z]_{rel} \cdot [W]_{rel} = i(Z, W; 0) \cdot [0]_{rel}. \tag{21.18}$$

This definition works in any complex-oriented cohomology theory.

If $c + d > n$ then it is clear that Z and W may be moved near 0 (in the topological setting) so that they do not intersect at all, and therefore $[Z]_{rel} \cdot [W]_{rel} = 0$. Equation (21.18) is still a valid definition, it just yields that $i(Z, W; 0) = 0$ here.

The key to the proof is simply realizing that all of our constructions can be lifted back into K^0_{alg}. Let $P_\bullet \to R/I$ and $Q_\bullet \to R/J$ be bounded projective resolutions. Then $[Z]_{rel} = \beta^{-c} \cdot [P_\bullet]$ and $[W]_{rel} = \beta^{-d} \cdot [Q_\bullet]$ by Theorem 21.10. So $[Z]_{rel} \cdot [W]_{rel} = \beta^{-c-d} \cdot [P. \otimes_R Q] \in K^{2(c+d)}(\mathbb{C}^n, \mathbb{C}^n - 0)$. Recall from Theorem 18.22 that $K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong \mathbb{Z}$ and is generated by the Koszul complex J^*. Recall as well that $[0]_{rel} = \beta^{-n} \cdot [J^*]$, by definition. If we write $[P \otimes_R Q] = s[J^*]$ for $s \in \mathbb{Z}$, then we have the formula

$$s \cdot \beta^{-c-d} \cdot [J^*] = i(Z, W; 0) \cdot \beta^{-n} \cdot [J^*].$$

If $c + d = n$ then the formula implies $s = i(Z, W; 0)$. If $c + d \neq n$ then the only way the formula can be true is if both sides are zero, in which case $s = 0 = i(Z, W; 0)$. So $s = i(Z, W; 0)$ in either case.

To conclude the proof we just note that the Local Index Theorem (21.2) gives $s = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(P \otimes_R Q)$, and $H_i(P \otimes Q) \cong \operatorname{Tor}_i(R/I, R/J)$. $\square$

Lemma 21.19. *Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let P and Q be prime ideals such that* $\operatorname{Rad}(P + Q)$ *is a maximal ideal. Then* $\dim R/P + \dim R/Q \leq n$*, or equivalently* $\operatorname{codim} R/P + \operatorname{codim} R/Q \geq n$.

Proof. The proof is from [S, Proposition III.17], though we follow the arrangement given in [Hoc2]. Let $\mu \colon R \otimes_k R \to R$ be the multiplication map. This is surjective and the kernel is the ideal $\Delta = (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$, so we can identify R with the quotient $(R \otimes_k R)/\Delta$. Consider the string of isomorphisms

$$R/(P+Q) \cong R/P \otimes_R R/Q \cong (R/P \otimes_k R/Q) \otimes_{R \otimes_k R} R \cong (R/P \otimes_k R/Q)/\Delta.$$

Since Δ is generated by n elements, quotienting out by Δ drops the dimension by at most n. So we obtain

$$\dim R/(P+Q) \geq \dim(R/P \otimes_k R/Q) - n.$$

Since $\mathrm{Rad}(P+Q)$ is a maximal ideal we have that $\dim R/(P+Q) = 0$. So the proof will be completed if we show that $\dim(R/P \otimes_k R/Q) = \dim R/P + \dim R/Q$. For convenience let us prove the analogous statement for any domains D_1 and D_2 that are finitely-generated k-algebras.

By Noether normalization there exist polynomial rings $B_1 \subseteq D_1$ and $B_2 \subseteq D_2$ such that each D_i is module-finite over B_i, and dimension theory tells us that $\dim D_i = \dim B_i$. But then $B_1 \otimes_k B_2 \to D_1 \otimes_k D_2$ is an inclusion (since the tensors are taken over a field), the extension is module-finite, and $B_1 \otimes_k B_2$ is a polynomial ring over k. So $\dim(D_1 \otimes_k D_2) = \dim(B_1 \otimes_k B_2) = \dim B_1 + \dim B_2 = \dim D_1 + \dim D_2$. □

Exercise 21.20. Suppose that Z and W are smooth algebraic subvarieties of $\mathbb{C}^n$ such that $Z \cap W = \{p_1, \ldots, p_d\}$.

(a) Choose a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(p_i) = 0$ for $i > 1$ and $f(p_1) \neq 0$, and let $S = R_f$. Let $U = \operatorname{Spec} R_f \subseteq \mathbb{C}^n$ be the corresponding Zariski open set. Convince yourself that it is reasonable to define the intersection multiplicity $i(Z, W; p_1)$ by the formula

$$[Z]_{rel,U} \cdot [W]_{rel,U} = i(Z, W; p_1) \cdot \beta^{-n} \cdot [p_1]_{rel,U}$$

where $[Z]_{rel,U}$ is the image of $[Z]_{rel}$ under $K^*(\mathbb{C}^n, \mathbb{C}^n - Z) \to K^*(U, U - Z)$, and similarly for $[W]_{rel,U}$ and $[p_1]_{rel,U}$. In particular, convince yourself such and f exists and that $i(Z, W; p_1)$ is independent of the choice of f.

(b) Next, modify the proof of Corollary 21.17 to show that

$$i(Z, W; p_1) = \sum (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_f$$

where I and J are the ideals corresponding to Z and W.

(c) If M is a finitely-generated module over R_f such that $\operatorname{Supp} M = \{p\}$ (p a maximal ideal of R), prove that $M = M_p$. Deduce that

$$i(Z, W; p_1) = \sum (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_{p_1}.$$

Corollary 21.17 (and Exercise 21.20) in some sense brings to a close the main questions we raised at the beginning of this book. We have now seen why Serre's alternating sum of Tor definition of intersection multiplicity is a reasonable one, and how this ties in to the study of K-theory.

Our discussion in this section has been confined to the truly geometric case of affine algebras over $\mathbb{C}$, but recall that Serre's definition works over every regular local ring R. If M and N are finitely-generated R-modules such that $\ell(M \otimes N) < \infty$ then Serre defines $i(M, N) = \sum_{j=0}^{\dim R} (-1)^j \ell(\operatorname{Tor}_j(M, N))$. Serre proved that in this situation one always has $\dim M + \dim N \geq \dim R$ and asked if the the following are true:

(1) $i(M, N) \geq 0$;

(2) $i(M, N) = 0$ if $\dim M + \dim N < \dim R$;
(3) $i(M, N) > 0$ if $\dim M + \dim N = \dim R$.

Using prime filtrations of M and N one reduces easily to the case $M = R/P$ and $N = R/Q$ where P and Q are prime. In the case where R is a localization of a finite-type $\mathbb{C}$-algebra, the agreement with topological intersection multiplicities that we have established in this section more or less proves (1)–(3) (one has to do a little to reduced to the case where R/P and R/Q are smooth to tie in with our above arguments.) However, Serre came up with purely algebraic arguments that handle the cases where R is equicharacteristic or mixed characteristic and unramified. In the 1980s Roberts and Gillet-Soulé independently established (2) using ideas related to K-theory, and in the late 1990s Gabber proved (1) in general. Note that (3) remains open at the moment. We will return to a discussion of some of these ideas in ????.

22. More about relative K-theory

Our aim in this section is to revisit the topic of relative K-theory and give the proof of Theorem 18.16. Recall that this is a comparison between the geometric groups $\mathcal{K}(X, A)$, defined in terms of chain complexes of vector bundles that are exact on A, and the group $K^0(X, A)$ that comes to us from topology once we have the K-theory spectrum. The proof is somewhat clunky and frustrating in a way that requires some apology.

This kind of clunkiness often surfaces when one tries to compare the world of geometry to the world of homotopy theory. Somehow these subjects inhabit different universes, and bringing the homotopical world down to earth turns out to be an inherently messy undertaking. While we have this beautiful cohomology theory K^*, at the moment we only have one access to it: that is the connection between vector bundles and $K^*(X)$. Theorem 18.16 is essentially about building another access route, but our only technique is to go through the one we already have. Thus, the clunkiness.

The ideas behind the proof are interesting and have their own intrinsic appeal, but within the context of this book they hardly ever resurface outside the confines of this one argument. Once we have this new access route, we just use it and can largely forget about the trouble that went into constructing it. So this section should be regarded as giving some technical information that is important, but not necessary for anything later in the book.

Throughout this section (X, A) will be a pair of spaces. At various points we will need to assume that the spaces are paracompact Hausdorff, or that $A \hookrightarrow X$ is a cofibration (e.g. a relative cell complex), and sometimes we will assume that (X, A) is a finite CW-pair. Recall that the pair (X, A) is said to be cofibrant if A is cofibrant and $A \hookrightarrow X$ is a cofibration.

Recall the group $\mathcal{K}(X, A)$ introduced in Definition 18.6, made from bounded complexes of vector bundles on X that are exact on A. For specificity we focus on $\mathbb{C}$-vector bundles, but the same arguments work for the real and quaternionic cases. We take several steps aimed at analyzing these groups:

(1) Whereas $\mathcal{K}(X, A)$ is defined via generators and relations, we know that the additive inverse of a chain complex is just another chain complex (in fact, its suspension). So we can construct an isomorphic group just as certain equivalence classes of chain complexes, with the monoid structure induced by direct sum. We call this construction $\mathcal{M}_\infty(X, A)$. It has the advantage over $\mathcal{K}(X, A)$ of the elements being a bit easier to get our hands on.

(2) Rather than consider all bounded complexes, we consider complexes that are concentrated in degrees between 0 and n, for some fixed $n \geq 1$. This leads to sets (in fact, monoids) $\mathcal{M}_n(X, A)$. We prove that—for nice enough pairs (X, A)—all choices of n give rise to isomorphic monoids; so in some sense the use of chain complexes is overkill, as it suffices to just look at complexes of length 1. (The use of all chain complexes makes for a more natural theory, however—for example, the tensor product of two complexes of length 1 is a complex of length 2, and can only be turned back into a complex of length 1 by a cumbersome process).

(3) Finally, and most importantly, we replace our consideration of chain complexes by that of a related but different construct. Namely, we consider $\mathbb{Z}$-graded

collections of vector bundles $\{E_i\}$ together with an exact differential defined only over the set A (that is, a differential on $(E_\bullet)|_A$). Let us call such things "A-relative exact complexes". When (X, A) is sufficiently nice it turns out that every A-relative exact complex may be extended to give an ordinary chain complex, and the space of all possible extensions is contractible. So homotopically speaking there is no real difference between the theories obtained from the two notions. The A-relative exact complexes turn out to give a theory that is a bit easier to manipulate, essentially because one doesn't have to deal with extraneous data.

As one final remark before we get started, it is possible to simplify some of the arguments in this section by only working with finite CW-pairs (X, A). This is what is done in [ABS]. The book [At4] generalized the situation somewhat by working with pairs of compact Hausdorff spaces. Unfortunately, we have seen already in this book that in applications we often want to consider $(\mathbb{C}^n, \mathbb{C}^n - 0)$ or $(E, E - 0)$ (E a vector bundle) or $(X, X - Z)$ (X an algebraic variety and Z a subvariety). One can often model these pairs, up to homotopy, by finite CW-complexes, but that involves an extra step that we usually want to sweep under the rug. So we have endeavored in this section to use constructions that will apply to pairs (X, A) in as broad a generality as we could manage without making life too difficult.

In outline the arguments in this section largely follow Atiyah's in [At4, Section 2.6], which were in turn modelled on the ones in [ABS]. However, there are a few important differences that will be explained in ?????.

22.1. **Relative chain complexes.** Let (X, A) be a pair of spaces. Let $\mathcal{C}\mathrm{h}(X, A)$ denote the category whose objects are bounded chain complexes of vector bundles on X that are exact on A. A map in this category is simply a map of chain complexes of vector bundles. In contrast to this, let $\mathcal{C}\mathrm{h}(X, A)_A$ denote the category whose objects are collections $\{E_i\}$ of vector bundles on X, all but finitely-many of which are zero, together with maps $d\colon E_{i+1}|_A \to E_i|_A$ making the restriction $E_\bullet|_A$ into an exact chain complex of vector bundles on A. A map $E \to E'$ in $\mathcal{C}\mathrm{h}(X, A)_A$ is a collection of maps $E_i \to E'_i$ of vector bundles on X that commute with the maps d where defined (i.e., over the set A). An object in $\mathcal{C}\mathrm{h}(X, A)_A$ will be called an ***A*-relative exact complex**.

Note the difference between $\mathcal{C}\mathrm{h}(X, A)$ and $\mathcal{C}\mathrm{h}(X, A)_A$: in the former the differentials are defined on all of X, whereas in the latter they are only defined on A. In both cases the maps between chain complexes are defined on all of X. Observe that there is an evident functor $\mathcal{C}\mathrm{h}(X, A) \to \mathcal{C}\mathrm{h}(X, A)_A$, which we will denote $E \mapsto E(A)$; this just restricts all the differentials to A.

Say that two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$ are **homotopic** if there is an object $\mathcal{E}$ in $\mathcal{C}\mathrm{h}(X \times I, A \times I)$ together with isomorphisms $\mathcal{E}|_{X\times 0} \cong E_\bullet$ and $\mathcal{E}|_{X\times 1} \cong E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$. Write this relation as $E_\bullet \sim_h F_\bullet$. Likewise, define two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)_A$ to be homotopic if there is an object $\mathcal{E}$ in $\mathcal{C}\mathrm{h}(X \times I, A \times I)_{A\times I}$ together with isomorphisms $\mathcal{E}|_{X\times 0} \cong E_\bullet$ and $\mathcal{E}|_{X\times 1} \cong E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)_A$. In each of these two settings the notion of homotopy is readily seen to be an equivalence relation. Write $\sim_h$ for this relation.

Remark 22.2. Observe that the above notion of homotopy has isomorphisms built into it: we only required that $\mathcal{E}|_{X\times 0}$ was isomorphic to E, not equal to it. So with this definition isomorphic complexes are automatically homotopic, via a constant homotopy. The choice to bundle these two relations together rather than keeping them separate is just for convenience, as it makes some results easier to state and prove. It would have been equivalent for us to define "strict homotopy" (with equalities rather than isomorphisms) and then taken the equivalence relation generated by strict homotopies and isomorphisms.

We will also need a second equivalence relation on chain complexes. Say that two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}h(X,A)$ are **stably equivalent** if there exist elementary complexes (see Definition 18.9) $P_1,\dots,P_r,Q_1,\dots,Q_s$ in $\mathcal{C}h(X,A)$ such that

$$E \oplus P_1 \oplus \cdots \oplus P_r \cong E' \oplus Q_1 \oplus \cdots \oplus Q_s.$$

Write this as $E_\bullet \sim_{st} E'_\bullet$. We define a similar equivalence relation on the objects of $\mathcal{C}h(X,A)_A$.

Let $\mathcal{C}h_n(X,A)$ be the full subcategory of $\mathcal{C}h(X,A)$ consisting of chain complexes $E_\bullet$ such that $E_i = 0$ when $i \notin [0,n]$, and let $\mathcal{C}h_n(X,A)_A$ be the analogous subcategory of A-relative complexes. It will be convenient to allow the index n to be ∞ here; note that this still corresponds to bounded complexes, but where there is no fixed upper bound on the nonzero region.

Let $\mathcal{M}_n(X,A)$ denote the set of equivalence classes of objects in $\mathcal{C}h_n(X,A)$ under the equivalence relation generated by the homotopy relation and the stable-equivalence relation. Define $\mathcal{M}_n(X,A)_A$ analogously. The restriction functor $E \mapsto E(A)$ (restrict the maps to A, leave the objects alone) clearly induces a map of sets $\mathcal{M}_n(X,A) \to \mathcal{M}_n(X,A)_A$.

Note that direct sum of complexes makes $\mathcal{M}_n(X,A)$ into a monoid, and similarly for $\mathcal{M}_n(X,A)_A$. Also observe that we have the evident maps $\mathcal{M}_n(X,A) \to \mathcal{M}_{n+1}(X,A)$ obtained by regarding a complex concentrated in degrees $[0,n]$ as also concentrated in degrees $[0,n+1]$, and the direct limit of these maps is $\mathcal{M}_\infty(X,A)$. The same holds for the A-relative version.

Exercise 22.3. Suppose that X is paracompact Hausdorff. Prove that the map $\mathcal{M}_1(X,\emptyset) \to K^0_{Grt}(X)$ (the Grothendieck group of vector bundles on X) sending $E_\bullet$ to $[E_0]-[E_1]$ is an isomorphism of monoids. (One step in the proof will probably be to show that the domain is actually a group). Repeat for the map $\mathcal{M}_n(X,\emptyset) \to K^0_{Grt}(X)$ that sends $E_\bullet$ to $\sum(-1)^i[E_i]$.

Observe that there is a natural map $\mathcal{M}_\infty(X,A) \to \mathcal{K}(X,A)$ induced by sending a chain complex $E_\bullet$ to its class in $\mathcal{K}(X,A)$, since the definition of $\mathcal{K}(X,A)$ has the $\sim_h$ and $\sim_{st}$ relations built into it. This is a map of monoids, and in fact is an isomorphism under mild hypotheses:

Proposition 22.4. *The monoid $\mathcal{M}_\infty(X,A)$ is always a group, and when X is paracompact Hausdorff the natural map $\mathcal{M}_\infty(X,A) \to \mathcal{K}(X,A)$ is an isomorphism.*

Proof. Suppose $E_\bullet$ is a chain complex in $\mathcal{C}h_\infty(X,A)$ and let $C_\bullet(t)$ be the mapping cone of the multiplication by t map on $E_\bullet$. By the homotopy relation we have $C_\bullet(1) = C_\bullet(0)$ in $\mathcal{M}_\infty(X,A)$. But $C_\bullet(1)$ is exact on X and so is a direct sum of elementary complexes: if X is paracompact Hausdorff one can cite Proposition 18.10 for this, but $C_\bullet(1)$ is nice enough that one can also produce the splitting by hand without any assumptions on X at all (see Exercise 22.5 below). Hence $C_\bullet(1) \sim 0$ in $\mathcal{M}_\infty(X,A)$. Finally, observe that $C_\bullet(0) = E_\bullet \oplus \Sigma E_\bullet$. This shows that $E_\bullet$ has an additive inverse in $\mathcal{M}_\infty(X,A)$, namely $\Sigma E_\bullet$. Thus, $\mathcal{M}_\infty(X,A)$ is a group.

For the second claim, consider the assignment that sends a chain complex $E_\bullet \in \mathcal{C}h(X, A)$ whose lowest nonzero degree is n ($n \in \mathbb{Z}$) to $(-1)^n[\Sigma^n E_\bullet] \in \mathcal{M}_\infty(X, A)$. We will show that this induces a group homomorphism $\mathcal{K}(X, A) \to \mathcal{M}(X, A)$ by verifying that it respects the defining relations for $\mathcal{K}(X, A)$. Relation (3) (the homotopy relation) is automatic. Relation (2) says that exact complexes are zero, and here is where we need that X is paracompact Hausdorff: Proposition 18.10 says that in this setting all exact complexes are isomorphic to direct sums of elementary complexes.

Finally to check relation (1) we need to verify that if $E_\bullet$ and $J_\bullet$ are two complexes whose lowest degrees are $n \le k$ then

$$(-1)^n[\Sigma^n(E_\bullet \oplus J_\bullet)] = (-1)^n[\Sigma^n E_\bullet] + (-1)^k[\Sigma^k J_\bullet]$$

in $\mathcal{M}_\infty(X, A)$. This follows from what we proved in the last paragraph, since $[\Sigma^n J_\bullet] = (-1)^n[J_\bullet] = (-1)^{n-k}[\Sigma^k J_\bullet]$. Consequently, our assignment induces a group map $\mathcal{K}(X, A) \to \mathcal{M}_\infty(X, A)$. This is readily checked to be a two-sided inverse for the map from the statement of the proposition. $\square$

Exercise 22.5. Let $E_\bullet$ be any bounded complex of vector bundles on a space X, and let $C_\bullet$ be the mapping cone of the identity. Prove that $C_\bullet$ is a direct sum of elementary complexes.

Exercise 22.6. Below we will prove that $\mathcal{M}_n(X, A)$ is also a group when $n \ge 1$ and (X, A) is a cofibrant pair. Try to prove this just for $n = 1$ and see where you get stuck.

22.7. **Strategy of the proof.** Let $\mathcal{L}(X, A)$ and $\mathcal{L}(X, A)_A$ be defined similarly to the $\mathcal{M}$ sets except that we do not include the homotopy relation—so the equivalences are generated by isomorphisms and stable equivalences. There are canonical surjections from the $\mathcal{L}$-constructions to the corresponding $\mathcal{M}$-constructions.

Consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{M}_1(X,A) \arrow[r] \arrow[d, "(22.11)"'] & \mathcal{M}_n(X,A) \arrow[r] \arrow[d, "(22.11)"'] & \mathcal{M}_\infty(X,A) \arrow[r, "(22.4)"] \arrow[d, "(22.11)"', "③"] & \mathcal{K}(X,A) \\
K^0(X,A) & \mathcal{M}_1(X,A)_A \arrow[l, "\mathbb{D}"'] \arrow[r, "①", "(22.22)"'] & \mathcal{M}_n(X,A)_A \arrow[r, "②", "(22.22)"'] & \mathcal{M}_\infty(X,A)_A & \\
 & \mathcal{L}_1(X,A)_A \arrow[r] \arrow[u, "(22.46)"] & \mathcal{L}_n(X,A)_A \arrow[r] \arrow[u, "(22.46)"'] & \mathcal{L}_\infty(X,A)_A. \arrow[u, "(22.46)"'] &
\end{tikzcd}$$

All of the maps are the evident ones except for $\mathbb{D}$, which will need to be constructed below. By the end of this section we will have proven that all of the maps are isomorphisms when (X, A) is a cofibrant pair, except for $\mathbb{D}$ which requires that (X, A) is homotopy equivalent to a finite CW-pair. (Many of the isomorphisms hold in somewhat more generality). The references in the diagram are to the results that establish the indicated isomorphism.

Our natural transformation $\mathcal{K}(X, A) \to K^0(X, A)$ will be obtained by zig-zagging through this diagram. Note that we do not really need the third level with the $\mathcal{L}$-sets, but these are used in [ABS] and [At4] and it is convenient to include them as part of the story. As the $\mathcal{L}$ and $\mathcal{M}$ groups in the diagram end up being isomorphic (though this is not obvious) it is a matter of taste whether one uses $\mathcal{M}_n(X,A)_A$ or $\mathcal{L}_n(X,A)_A$.

Note that once we know that the maps labelled ① are isomorphisms for all n, we immediately get the same for ② by passing to the colimit. So the steps we will follow are:

- Show that map ③ is an isomorphism.
- Show that map ① is an isomorphism (for all n).
- Construct and analyze the natural transformation $\mathbb{D}\colon \mathcal{M}_1(X,A)_A \to K^0(X,A)$.

The isomorphisms $\mathcal{L}_n(X,A)_A \to \mathcal{M}_n(X,A)_A$ will fall out of this outline as a simple corollary.

Remark 22.8. A natural question is why we use $\mathcal{K}(X,A)$ or $\mathcal{M}_\infty(X,A)$ at all, when apparently the connection to $K^0(X,A)$ is easier with the A-relative groups $\mathcal{M}(X,A)_A$. So it is worth remarking up front that the tensor product of chain complexes gives $\mathcal{K}(X,A)\otimes\mathcal{K}(Y,B)\to\mathcal{K}(X\times Y, A\times Y\cup X\times B)$ (or the same with $\mathcal{M}_\infty$), whereas one does *not* get pairings like these with the A-relative constructions. If differentials are only defined on A and B then on the tensor product one only obtains differentials on $A\times B$. So $\mathcal{K}$ and $\mathcal{M}_\infty$ stand out as the spots in the diagram with naturally-defined pairings. Relating those pairings to the ones on K^0 will be another challenge, undertaken in Section 22.39.

Remark 22.9. Although not in the above diagram, one can introduce a final set of groups $\mathcal{L}_n(X,A)$ consisting of chain complexes defined on all of X, exact on A, up to isomorphism and stability. However, these are not isomorphic to the above groups, as shown by the following example. Let $X=\mathbb{R}$ and $A=\emptyset$. The chain complexes $\underline{1}\xrightarrow{0}\underline{1}$ and $\underline{1}\xrightarrow{\mathrm{id}}\underline{1}$ are homotopic and so represent the same element in $\mathcal{K}(X,A)$, but they do not become isomorphic after taking direct sums with elementary complexes. In particular, the former complex has homology in the fiber over 0 whereas the latter does not, and these properties will not change upon addition of exact complexes. In constrast, note that when we regard these as objects in $\mathcal{C}h(X,A)_A$ they are actually isomorphic, and therefore identified in $\mathcal{L}_\infty(X,A)_A$.

22.10. **Comparing complexes to A-relative complexes.** Our first important result is the following:

Proposition 22.11. *Assume that $A\hookrightarrow X$ is a cofibration. Then for any $1\le n\le\infty$ the map $\mathcal{M}_n(X,A)\to\mathcal{M}_n(X,A)_A$ is a bijection.*

We start with a lemma:

Lemma 22.12. *When $A\hookrightarrow X$ is a cofibration the functor $\mathcal{C}h(X,A)\to\mathcal{C}h(X,A)_A$ is surjective on objects. Additionally, if E_1 and E_2 are objects in $\mathcal{C}h(X,A)$ such that $E_1(A)\cong E_2(A)$ then E_1 and E_2 are homotopic.*

Proof. Given a $\mathbb{Z}$-indexed collection of vector spaces V_i, let Ch-struct$(V)\subseteq\prod_i \operatorname{Hom}(V_{i+1},V_i)$ denote the collection of sequences $(d_i)_{i\in\mathbb{Z}}$ satisfying $d_i\circ d_{i+1}=0$ for all i. Regard Ch-struct(V) as a topological space by giving it the subspace topology. Note that if $d\in$ Ch-struct(V) then $t\cdot d\in$ Ch-struct(V) for any $t\in\mathbb{C}$. Picking any path in $\mathbb{C}$ from 0 to 1 (e.g. the standard one along the real line) thereby gives a contracting homotopy showing that Ch-struct(V) is contractible.

Now suppose that $\{E_i\}_{i\in\mathbb{Z}}$ is a collection of vector bundles on X having the property that only finitely many are nonzero. For $x \in X$ write E_x for the collection $\{(E_i)_x\}_{i\in\mathbb{Z}}$. Let Ch-struct$(E)$ be the set of pairs $(x \in X, d \in \text{Ch-struct}(E_x))$, topologized so that the map $\pi_1\colon \text{Ch-struct}(E) \to X$ is a fiber bundle. The assumption that only finitely many E_i are nonzero implies that each x has a neighborhood over which all the bundles are trivial, and we use such neighborhoods in the usual way to define the topology on Ch-struct(E). As the fibers of Ch-struct$(E) \to X$ are contractible this map is a weak homotopy equivalence.

Note that making $\{E_i\}$ into a chain complex is precisely the same as giving a section of Ch-struct$(E) \to X$. Likewise, equipping $\{E_i\}$ with a chain complex differential *over* A is the same as giving a section defined over A.

Suppose given an object E in $\mathcal{C}h(X,A)_A$. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & \text{Ch-struct}(E) \arrow[d, two heads, "\simeq"] \\ X \arrow[r] & X \end{tikzcd} \tag{22.13}$$

where the top horizontal map encodes the differentials on E. Since $A \to X$ is a cofibration and Ch-struct$(E) \to X$ is an acyclic fibration, there is a lifting $X \to$ Ch-struct(E). This lifting precisely gives a chain complex structure on $\{E_i\}$, defined on all of X, that extends the one defined over A. This proves that the functor $E \mapsto E(A)$ is surjective on objects.

Next observe that if $f\colon Y \to X$ is any map then Ch-struct(f^*E) is canonically identified with the pullback of $Y \to X \leftarrow \text{Ch-struct}(E)$. This is an easy exercise. Suppose that (E,s) and (E,s') are two preimages in $\mathcal{C}h(X,A)$ for the same object E in $\mathcal{C}h(X,A)_A$. Then s and s' correspond to two liftings in the square (22.13). Given this data, form the new diagram

$$\begin{tikzcd} (X\times 0) \amalg_{(A\times I)} (X\times 1) \arrow[r] \arrow[d, tail] & \text{Ch-struct}(E) \arrow[d, two heads, "\simeq"] \\ X\times I \arrow[r, "\pi"] & X \end{tikzcd}$$

where $\pi\colon X \times I \to X$ is the projection, $A \times I \to \text{Ch-struct}(E)$ is the constant homotopy, and the top horizontal map equals s and s' on the two copies of X. Once again, the diagram has a lifting. The resulting map $X \times I \to \text{Ch-struct}(E)$ corresponds to a section of Ch-struct$(\pi^*E) \to X \times I$, and so specifies a complex of vector bundles on $X \times I$. The differentials are constant with respect to 'time' on $A \times I$, and so in particular the complex is exact on $A \times I$. So it lives in $\mathcal{C}h(X \times I, A \times I)$. By construction it restricts to the two liftings (E,s) and (E,s') at times 0 and 1.

Now suppose that E and F are objects in $\mathcal{C}h(X,A)$ and that $E(A) \cong F(A)$. So there are isomorphisms $u_i\colon E_i \to F_i$ of bundles over X which, when restricted to A, commute with the differentials d. Let d'_i be the composite

$$E_i \xrightarrow{u_i} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{u_{i-1}^{-1}} E_{i-1}.$$

Note that the d' maps give a chain complex structure on $\{E_i\}$; call this new chain complex E'. We have $E' \cong F$ as objects in $\mathcal{C}h(X,A)$. Observe that $d'|_A = d|_A$, and so $E(A) = E'(A)$. It follows by what we have already proven that E and E'

are homotopic in $\mathcal{C}h(X, A)$. Since E' and F are (trivially) homotopic, transitivity gives that E is homotopic to F. $\square$

Remark 22.14. The above proof was written in part to demonstrate the technique of translating a desired task into a lifting problem. This is a useful technique that we will need again later in this section. However, it is worth pointing out that in the case of relative cell complexes the lifting problems could be solved in a very concrete and simple way. If (Y, B) is a relative cell complex and $E_\bullet$ is an object in $\mathcal{C}h(Y, B)_B$, we extend the differentials from B to all of Y by an induction over the cells of $Y - B$. If e^n is such a cell, we assume inductively that the differentials have been defined over the boundary. Let $f\colon D^n \to Y$ be the characteristic map of the cell, and consider the pullback $f^*E_\bullet$. Choose trivializations for all of these bundles over the disk. Points in the interior of the disk have the form tx for $x \in \partial D^n$ and $t \in [0, 1)$. Define the differential over tx to be t times the differential over x. This clearly gives the required extension.

Proof of Proposition 22.11. We need to prove that $\mathcal{M}_n(X, A) \to \mathcal{M}_n(X, A)_A$ is bijective, and surjectivity is provided by Lemma 22.12. For injectivity, it will suffice to prove, given two objects E and F in $\mathcal{C}h_n(X, A)$, that

(1) if $E(A) \sim_h F(A)$ then $E = F$ in $\mathcal{M}_n(X, A)$, and

(2) if $E(A) \sim_{st} F(A)$ then $E = F$ in $\mathcal{M}_n(X, A)$.

For suppose that $E(A)$ and $F(A)$ are identified in $\mathcal{M}_n(X, A)_A$. Then there is a finite chain of objects $\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_r$ in $\mathcal{C}h_n(X, A)_A$ such that $\tilde{E}_1 = E(A)$, $\tilde{E}_r = F(A)$, and for each i either $\tilde{E}_i \sim_h \tilde{E}_{i+1}$ or $\tilde{E}_i \sim_{st} \tilde{E}_{i+1}$. By Lemma 22.12 there are chain complexes $E_i \in \mathcal{C}h_n(X, A)$ such that $E_i(A) = \tilde{E}_i$, for each i. We can choose $E_1 = E$ and $E_r = F$, and we do so. By iterated applications of (1) and (2) we then know that $E_1, E_2, \ldots, E_r$ are all identified in $\mathcal{M}_n(X, A)$, and in particular E and F are identified. This is what we needed to prove.

We turn to the proofs of (1) and (2). Suppose that $E(A) \sim_h F(A)$. Then there exists an $\tilde{\mathcal{E}} \in \mathcal{C}h(X \times I, A \times I)_{A \times I}$ together with isomorphisms $\tilde{\mathcal{E}}|_{X \times 0} \cong E(A)$ and $\tilde{\mathcal{E}}|_{X \times 1} \cong F(A)$. By Lemma 22.12 there is an $\mathcal{E} \in \mathcal{C}h(X \times I, A \times I)$ such that $\mathcal{E}(A) = \tilde{\mathcal{E}}$. Then $\mathcal{E}|_{X \times 0}(A) \cong E(A)$, and so by Lemma 22.12 $\mathcal{E}|_{X \times 0}$ is homotopic to E. The same reasoning gives that $\mathcal{E}|_{X \times 1}$ is homotopic to F. But $\mathcal{E}|_{X \times 0}$ and $\mathcal{E}|_{X \times 1}$ are homotopic, so by transitivity E is homotopic to F.

Finally, suppose that $E(A) \sim_{st} F(A)$. So there exist elementary complexes $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_s$ in $\mathcal{C}h_n(X, A)_A$ such that $E(A) \oplus \bigoplus_i P_i \cong F(A) \oplus \bigoplus_j Q_j$. Note that each elementary complex actually lives in $\mathcal{C}h_n(X, A)$, as its only nonzero differential is the identity and therefore is defined on all of X.

Let $P = \oplus_i P_i$ and $Q = \oplus_j Q_j$. Then $(E \oplus P)(A) \cong (F \oplus Q)(A)$, and so Lemma 22.12 tells us that $E \oplus P$ and $F \oplus Q$ are homotopic. Consequently, $E \oplus P$ is identified with $F \oplus Q$ in $\mathcal{M}_n(X, A)$. But the stability relation identifies E with $E \oplus P$ and F with $F \oplus Q$, hence E and F are themselves identified. $\square$

22.15. **An unusual construction from homological algebra.** Our next aim is to prove that the maps $\mathcal{M}_n(X, A)_A \to \mathcal{M}_{n+1}(X, A)_A$ are bijections when $n \geq 1$. This will be based on a strange technique for folding the top group of an exact complex two terms lower down, to construct a new complex that also happens to be exact. We now describe this unusual construction.

Let V be an exact complex of vector spaces and assume that $V_i = 0$ for $i > n$. (The complex could actually consist of projectives over some ring, but let us stick with the simpler setting). Since the complex is exact there exists a contracting homotopy: maps $e\colon V_i \to V_{i+1}$ such that $de + ed = \mathrm{id}$. Let ΓV be the following chain complex, concentrated in degrees smaller than n and agreeing with V in degrees smaller than $n-2$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & V_{n-1} & \longrightarrow & V_{n-2} \oplus V_n & \longrightarrow & V_{n-3} & \longrightarrow & V_{n-4} & \longrightarrow & \cdots \\
 & & x & \longmapsto & (dx, ex) & & & & & & \\
 & & & & (a,b) & \longmapsto & da. & & & &
\end{array}$$

It is an elementary exercise to prove that ΓV is exact, but this will also follow directly from the two decompositions we produce next.

Exercise 22.16. Check that the above construction is a chain complex and prove via naive means that it is exact. Additionally, prove that $V_{n-1} = d(V_n) \oplus ed(V_{n-1})$.

Remark 22.17. We know (e.g. Proposition 18.10) that an exact complex of vector spaces splits (non-canonically) as a direct sum of elementary complexes. The above construction is essentially taking the top piece of such a splitting and shifting it down one degree. The contracting homotopy e encodes the splitting.

For any vector space W and any $k \in \mathbb{Z}$, recall that $D_k(W)$ denotes the elementary chain complex consisting of W in degrees k and $k+1$, where the differential is the identity.

Exercise 22.18. Let V be a bounded chain complex of vector spaces and suppose e is a contracting homotopy for V.

(a) Prove that $V_i = e(\ker d_{i-1}) \oplus d(V_{i+1})$ and that $d_i\colon e(\ker d_{i-1}) \to d(V_i)$ is an isomorphism. So the contracting homotopy yields a decomposition of V into a direct sum of elementary complexes.

(b) Define a splitting for V to be a sequence of vector spaces L_i and an isomorphism $\bigoplus_i D_i(L_i) \to V$. Define two splittings (L, ϕ) and (L', ϕ') to be equivalent if there is a collection of isomorphisms $L_i \to L'_i$ making the expected triangle with ϕ and ϕ' commute. Prove that there is a bijection between the equivalence classes of splittings of V and the set of contracting homotopies of V.

Returning to our chain complex V with contracting homotopy e, note that for $x \in V_n$ one has $ed(x) = x$. Using this, we can write down a natural chain map $V \to D_{n-1}(V_n)$ that is the identity in degree n and the map $e\colon V_{n-1} \to V_n$ in degree $n-1$. Let $\widetilde{\Gamma} V$ be the desuspension of the mapping cone of this map; specifically, $\widetilde{\Gamma} V$ is the following chain complex, concentrated in dimensions at most n and agreeing with V in dimensions smaller than $n-2$:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & V_n & \longrightarrow & V_{n-1} \oplus V_n & \longrightarrow & V_{n-2} \oplus V_n & \longrightarrow & V_{n-3} & \longrightarrow & V_{n-4} & \longrightarrow & \cdots \\
 & & x & \longmapsto & (dx, -x) & & (x,y) & \longmapsto & dx & & & & \\
 & & & & (a,b) & \longmapsto & (da, ea+b) & & & & & &
\end{array}$$

There is a canonical inclusion $D_{n-2}(V_n) \hookrightarrow \widetilde{\Gamma} V$, and the quotient is V. Moreover, this inclusion has a canonical splitting χ defined by $\chi(a,b) = ea + b$ for $(a,b) \in$

$V_{n-1} \oplus V_n = (\widetilde{\Gamma} V)_{n-1}$ and $\chi(a,b) = b$ for $(a,b) \in V_{n-2} \oplus V_n = (\widetilde{\Gamma} V)_{n-2}$. This splitting gives an isomorphism $\widetilde{\Gamma} V \cong D_{n-2}(V_n) \oplus V$.

Notice as well that there is an evident map $D_{n-1}(V_n) \hookrightarrow \widetilde{\Gamma} V$: in degree n this equals the identity and in degree $n-1$ it equals the differential of $\widetilde{\Gamma} V$. The cokernel of this inclusion is precisely ΓV. Moreover, there is again a canonical splitting for the inclusion: in degree n it is equal to the identity, and in degree $n-1$ it is the negation of the projection map $V_{n-1} \oplus V_n \to V_n$. This splitting gives $\widetilde{\Gamma} V \cong D_{n-1}(V_n) \oplus \Gamma V$.

Summarizing, we have produced two split-exact sequences

$$0 \to D_{n-2}(V_n) \to \widetilde{\Gamma} V \to V \to 0 \quad \text{and} \quad 0 \to D_{n-1}(V_n) \to \widetilde{\Gamma} V \to \Gamma V \to 0$$

and these induce isomorphisms

$$\widetilde{\Gamma} V \cong D_{n-2}(V_n) \oplus V \quad \text{and} \quad \widetilde{\Gamma} V \cong D_{n-1}(V_n) \oplus \Gamma V. \tag{22.19}$$

An important point is that the maps in these short exact sequences, their splittings, and therefore the induced isomorphisms in (22.19) are all canonical in the pair (V, e).

By the way, notice that it follows immediately from the isomorphisms in (22.19) that the homology groups of ΓV and V coincide; therefore ΓV is exact.

The constructions from above depended on the choice of contracting homotopy e. As one last remark before getting back to topology, let us consider the space of *all* contracting homotopies on an arbitrary chain complex V. Denote this space as $\text{contr-h}(V) \subseteq \prod_i \text{Hom}(V_i, V_{i+1})$; an element of $\text{contr-h}(V)$ is a collection of maps $\{e_i : V_i \to V_{i+1}\}$ satisfying $de + ed = \text{id}$. Of course this space might be empty, but we claim that it is either empty or contractible. To see this, recall the internal Hom-complex $\underline{\text{Hom}}(V, V)$. In degree k this is $\prod_i \text{Hom}(V_i, V_{i+k})$, and given a collection $\{\alpha_i : V_i \to V_{i+k}\}$ the differential is the collection of maps $\{d \circ \alpha_i - (-1)^k \alpha_{i-1} \circ d\}$. A contracting homotopy for V is just an element $e \in \underline{\text{Hom}}(V, V)_1$ satisfying $de = \text{id}$, and the space of contracting homotopies is just $d^{-1}(\text{id})$. If this space is nonempty then it is homeomorphic to the space of 1-cycles in $\underline{\text{Hom}}(V, V)$, which is a vector space and hence contractible.

Remark 22.20. The Γ- and $\widetilde{\Gamma}$-constructions used in this section seem to have first appeared in [Do]. Notice that very little about the contracting homotopy e was ever used—in fact, all we really needed was the component of e in the top dimension, the map $e : V_{n-1} \to V_n$. And all that was important about this map was that it was a splitting for the differential $d_n : V_n \to V_{n-1}$. Rather than use the space of contracting homotopies as a parameter space, we could have used the (simpler) space of splittings for d_n. The reader can check that this is again an affine space, homeomorphic to the vector space of all maps $f : V_{n-1} \to V_n$ such that $fd = 0$; in particular, this parameter space is again contractible.

We have used the space of contracting homotopies because this approach generalizes a bit more easily to the situation of algebraic K-theory. See [FH] and [D3] for the importance of these contracting homotopies.

22.21. **Back to topology.** Let $E_\bullet$ be a bounded chain complex of vector bundles on a paracompact Hausdorff space Z, and assume that $E_\bullet$ is exact. One can prove by brute force that $E_\bullet$ has a contracting homotopy, by producing a splitting for $E_\bullet$ as in Proposition 18.10 (by successively splitting off the bottom vector bundle) and then taking the induced contracting homotopy as in Exercise 22.18. But as another argument, consider the map contr-h$(E) \to Z$ whose fiber over each $z \in Z$ is the space of contracting homotopies for $(E_\bullet)_z$. It is easy to see that contr-h$(E) \to Z$ is a fiber bundle, and our remarks in the last section show that the fibers are contractible. If Z is cofibrant then a lift is guaranteed in the diagram

$$\begin{tikzcd} & \text{contr-h}(E) \arrow[d, "\simeq"] \\ Z \arrow[r, "\text{id}"] & Z, \end{tikzcd}$$

and this lift precisely gives a contracting homotopy for $E_\bullet$. Moreover, if e and e' are two liftings then there is a homotopy $Z \times I \to$ contr-h(E) between them because the diagram

$$\begin{tikzcd} (Z \times 0) \amalg (Z \times 1) \arrow[r, "e \amalg e'"] \arrow[d] & \text{contr-h}(E) \arrow[d, "\simeq"] \\ Z \times I \arrow[r, "\pi"] & Z, \end{tikzcd}$$

admits a lifting. This is all we will need, but it is worth observing that one can say even more here: the space of all liftings, which is the space of contracting homotopies on $E_\bullet$, is contractible.

If e is a chosen contracting homotopy for $E_\bullet$ then we can form the associated chain complex ΓE by repeating the construction from Section 22.15 but in the bundle setting. This is a new chain complex of vector bundles that is still exact on Z. This construction of course depends on the choice of contracting homotopy e, and so we should probably write $\Gamma_e E$. But since any two choices for e are homotopic, it follows that ΓE is well-defined up to homotopy.

We will use the above construction to prove the following:

Proposition 22.22. *Assume that A is cofibrant and $A \hookrightarrow X$ is a cofibration. Then for any $n \geq 2$ the map $j\colon \mathcal{M}_{n-1}(X,A)_A \to \mathcal{M}_n(X,A)_A$ is a bijection.*

Proof. Let $E_\bullet$ be a chain complex in $\mathcal{C}h_n(X,A)_A$, where $n \geq 2$. By the preceding considerations, since A is cofibrant there exists a contracting homotopy for $(E_\bullet)|_A$. Using such a contraction e we can form $\Gamma_e E$, which is an object in $\mathcal{M}_{n-1}(X,A)_A$. Different choices for e are homotopic and therefore give rise to homotopic complexes, so we get a well-defined function

$$\Gamma\colon\ \mathcal{C}h_n(X,A)_A \to \mathcal{M}_{n-1}(X,A)_A.$$

It is an elementary exercise to check that $\Gamma(E \oplus J) = \Gamma(E) + \Gamma(J)$, and it follows from this that if $E \sim_{st} E'$ then $\Gamma E \sim_{st} \Gamma E'$. Likewise, if $E \sim_h E'$ then pick an $\mathcal{E} \in \mathcal{C}h_n(X \times I, A \times I)_{A \times I}$ with $\mathcal{E}|_{X \times 0} \cong E$ and $\mathcal{E}|_{X \times 1} \cong E'$. Choose a contracting homotopy e for $\mathcal{E}|_{A \times I}$ and note that $[(\Gamma_e \mathcal{E})_{X \times 0}] = [\Gamma E]$ and $[(\Gamma_e \mathcal{E})_{X \times 1}] = [\Gamma E']$. It follows at once that $[\Gamma E] = [\Gamma E']$. Putting everything together, we have shown that Γ gives a map

$$\Gamma\colon \mathcal{M}_n(X,A)_A \to \mathcal{M}_{n-1}(X,A)_A.$$

It is trivial that $\Gamma j = \text{id}$. We claim that $j\Gamma = \text{id}$ as well, thus establishing that j is a bijection. This claim almost follows directly from the canonical isomorphisms (22.19), except there is an important step that must be filled in. We would like to say that these isomorphisms globalize to give

$$\widetilde{\Gamma}E \cong D_{n-2}(E_n) \oplus E \quad \text{and} \quad \widetilde{\Gamma}E \cong D_{n-1}(E_n) \oplus \Gamma E. \tag{22.23}$$

This is certainly true if we restrict all the chain complexes to the subspace A. However, to give an isomorphism in $\mathcal{C}h(X,A)_A$ we actually need to give a collection of isomorphisms for bundles **over X** (they are only required to commute with the differentials over A, however). So we must verify that in each degree the isomorphisms from (22.19) globalize not just to A but to X. Those isomorphisms were obtained from split short exact sequences, so we must check that all the maps involved can be extended over X. But we have formulas for all of these maps, and most of them are inclusions of a summand or projections onto a summand—these obviously extend to all of X. The one exception is in one of the splittings, where we used the map $\chi\colon E_{n-1} \oplus E_n \to E_n$ given on fibers by $(a,b) \mapsto ea + b$ where e was part of the given contracting homotopy. Since e is only defined on A, this does not automatically make sense on all of X. However, the particular map e we are using in this formula is a section over A of the bundle $\underline{\mathrm{Hom}}(E_{n-1}, E_n) \to X$. The diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}(E_{n-1}, E_n) \arrow[d, "\simeq"] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd}$$

must have a lifting, and this gives an extension of e to a bundle map $\tilde{e}\colon E_{n-1} \to E_n$ defined on all of X. The formula $(a,b) \mapsto \tilde{e}a + b$ then gives the desired splitting that works on all of X. Note: it is important here that the splitting is only required to commute with the differentials on A, since this is all we have guaranteed.

To summarize, we have indeed justified the isomorphisms in (22.23). These imply that ΓE and E represent the same class in $\mathcal{M}_n(X,A)_A$. In other words, we have proven that $j \circ \Gamma = \mathrm{id}$, and so Γ is a two-sided inverse for j. $\square$

Remark 22.24 (Atiyah's proof)**.** Atiyah proves a version of Proposition 22.22 in [At4, Chapter 2.6]; the argument orginally comes from [ABS]. We will explain the basic ways his proof differs from ours, and why these differences are important.

Let $E_\bullet$ be a chain complex in $\mathcal{C}h_n(X,A)_A$. There is a canonical map $D_{n-1}(E_n) \to E_\bullet$ which in degree n equals the identity and in degree $n-1$ equals the differential $E_n \to E_{n-1}$. Let $E'_\bullet = E_\bullet \oplus D_{n-2}(E_n)$ and consider the composition

$$D_{n-1}(E_n) \to E_\bullet \hookrightarrow E'_\bullet.$$

The map in degree $n-1$ is $d \oplus 0\colon E_n \to E_{n-1} \oplus E_n$, which is defined only on A. Atiyah shows via a lifting argument that this can be extended to a monomorphism of bundles on X. Let Q be the quotient, and observe that by Proposition 9.2 the sequence $0 \to E_n \to E'_n \to Q \to 0$ is a split-exact sequence of bundles over X. A choice of splitting $\chi\colon E'_n \to E_n$ then shows that $E'_\bullet$ is the direct sum of $D_{n-1}(E_n)$ and a complex

$$0 \to Q \to E_{n-2} \oplus E_n \to E_{n-3} \to E_{n-4} \to \cdots \tag{22.25}$$

This last complex lies in $\mathcal{C}h_{n-1}(X,A)_A$, and it represents the same class as E in $\mathcal{M}_n(X,A)_A$. This shows that $j\colon \mathcal{M}_{n-1}(X,A)_A \to \mathcal{M}_n(X,A)_A$ is surjective.

This argument does not give an inverse for j, however. The complex in (22.25) depends on a choice (the extension of a certain map to all of X), and so it is not clear how to use this construction to make an inverse for j.

In our argument we gave a construction ΓE that **did not depend** on choosing any such extensions to X. Such extensions did appear, but only in the isomorphisms showing that our ΓE had the correct properties. By pushing these choices into the maps rather than the objects, we were able to write down an explicit inverse for j.

In Atiyah's case he found a clever way around his problem, by instead constructing a map $\mathcal{M}_n(X,A)_A \to \mathcal{M}_1(X,A)_A$ that is an inverse to the appropriate composition of j's. This is enough to deduce injectivity of all the j maps. Atiyah's construction proceeds by choosing Hermitian inner products on all of the bundles E_i, and then letting $\alpha\colon E_i \to E_{i+1}$ be the adjoint of the differential. The map $d+\alpha\colon \bigoplus_{i \text{ odd}} E_i \to \bigoplus_{i \text{ even}} E_i$ is seen to lie in $\mathcal{C}h_1(X,A)_A$, and it clearly has the desired properties. This element seems to again depend on choices, namely the choice of inner products; but the space of all such choices is contractible, and so one indeed gets a well-defined element of $\mathcal{M}_1(X,A)_A$. The exercises below will give you enough information to fill in the details of this approach.

The disappointing aspect of Atiyah's argument is that it does not work in the related context of algebraic K-theory. In that setting one cannot play a corresponding game with inner products. In contrast, our argument with contracting homotopies does generalize. See Appendix F.

Exercise 22.26. Assume V is an exact chain complex of finite-dimensional real vector spaces, and that $V_i = 0$ for $i < 0$ and for $i > n$. Choose an inner product on each V_i, and let $\alpha_i\colon V_i \to V_{i+1}$ be the adjoint of $d_{i+1}\colon V_{i+1} \to V_i$. That is, for each $x \in V_i$ and $y \in V_{i+1}$ one has $\langle \alpha x, y\rangle = \langle x, dy\rangle$.

(a) Prove that $\alpha^2 = 0$, and so (V,α) is a cochain complex. Observe that this is isomorphic to the dual complex V^*, and therefore is exact.

(b) For each i prove that $\ker\alpha_i$ is the orthogonal complement of $\operatorname{im} d_{i+1}$ inside of V_i. As a corollary, deduce that d restricts to an isomorphism $\ker\alpha_i \to \operatorname{im} d_i$ and α restricts to an isomorphism $\operatorname{im} d_i \to \ker\alpha_i$. Produce an example when $n = 1$ showing that these isomorphisms need not be inverses.

(c) Observe that $V_i = (\operatorname{im} d_{i+1}) \oplus (\ker\alpha_i)$ for each i, and confirm that the following picture shows V decomposing into a direct sum of length 1 complexes:

$$\begin{tikzcd}
\cdots & (\operatorname{im} d_{i+1}) \arrow[ddl, "\alpha"'] & (\operatorname{im} d_i) \arrow[ddl, "\alpha"'] & (\operatorname{im} d_{i-1}) \arrow[ddl, "\alpha"'] & \cdots \arrow[ddl, "\alpha"'] \\
 & \oplus & \oplus & \oplus & \\
\cdots \arrow[uur, "d"'] & (\ker\alpha_i) \arrow[uur, "d"'] & (\ker\alpha_{i-1}) \arrow[uur, "d"'] & (\ker\alpha_{i-2}) \arrow[uur, "d"'] & \cdots
\end{tikzcd}$$

(d) Let $V_{odd} = \oplus_{i \text{ odd}} V_i$ and $V_{ev} = \oplus_{i \text{ even}} V_i$. Observe that $d+\alpha\colon V_{odd} \to V_{ev}$ is an isomorphism, by the following diagram:

$$\begin{tikzcd}
{} \arrow[drrr, "\alpha"'] & \cdots & \oplus(\operatorname{im} d)_1 \arrow[drrr, "\alpha"] & \oplus\ (\ker\alpha)_1 \arrow[dl, "d"'] & \oplus\ (\operatorname{im} d)_3 \arrow[drrr, "\alpha"] & \oplus\ (\ker\alpha)_3 \oplus \arrow[dl, "d"'] & \cdots & \\
 & \cdots & \oplus(\operatorname{im} d)_0 & \oplus\ (\ker\alpha)_0 & \oplus\ (\operatorname{im} d)_2 & \oplus\ (\ker\alpha)_2 \oplus & \cdots & \cdots
\end{tikzcd}$$

Note: Here we have written $(\operatorname{im} d)_i$ for the component of $\operatorname{im} d$ contained in degree i; i.e., $(\operatorname{im} d)_i = \operatorname{im} d_{i+1}$, but $(\ker\alpha)_i = \ker\alpha_i$.

(e) Prove an analog of these results for exact complexes of $\mathbb{C}$-vector spaces, in which one chooses Hermitian inner products on all of the V_i's.

Exercise 22.27. Let IP_n denote the space of all inner products on $\mathbb{R}^n$. Note that this may be identified with the space of all positive-definite, symmetric $n \times n$ matrices, which we topologize as a subspace of $M_{n\times n}(\mathbb{R})$. We will prove that IP_n is contractible, for all n.

(a) Prove $\mathrm{IP}_1 \cong \mathbb{R}_{>0}$.
(b) Consider the process of extending an inner product from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$. Let H^n denote the upper half-space $\{x \in \mathbb{R}^n \,|\, x_n > 0\}$. Prove that $\mathrm{IP}_n \cong \mathrm{IP}_{n-1} \times H^n$, and deduce that IP_n is contractible for all $n \geq 1$.
(c) Use a similar line of argument to show that the space of Hermitian inner products on $\mathbb{C}^n$ is contractible. The analog of H^n is the space $(\mathbb{C}^n - \mathbb{C}^{n-1})/S^1$, where S^1 is the group of unit complex numbers acting via scalar multiplication on $\mathbb{C}^n$. As part of your argument you will have to show that this orbit space is contractible.

22.28. **Another interlude on where we are headed.** Recall that we are trying to produce a natural map of groups $\mathcal{K}(X, A) \to K^0(X, A)$ such that when $A = \emptyset$ the map sends $E_\bullet$ to $\sum_i (-1)^i[E_i]$. At this point we have established the chain of isomorphisms of monoids

$$\mathcal{M}_1(X, A)_A \xrightarrow{\cong} \mathcal{M}_\infty(X, A)_A \xleftarrow{\cong} \mathcal{M}_\infty(X, A) \xrightarrow{\cong} \mathcal{K}(X, A)$$

for cofibrant pairs (X, A) (recall that this hypothesis ensures, in particular, that X and A are paracompact and Hausdorff). So for this situation our problem is equivalent to producing a natural map from any of these gadgets into $\mathcal{K}(X, A)$ with the desired properties. We will call any such natural map an **Euler characteristic**.

In the next part of the argument we will construct an Euler characteristic on $\mathcal{M}_1(X, A)_A$ and prove that it is unique on cofibrant pairs where X and A are homotopically compact.

Let us make one simple but useful observation that falls out of what we have done already. *A priori* the sets $\mathcal{M}_n(X, A)_A$ are only monoids, but the above chain of isomorphisms shows they are actually groups.

Corollary 22.29. *Let (X, A) be a cofibrant pair. Then for all $1 \leq n \leq \infty$ the commutative monoid $\mathcal{M}_n(X, A)_A$ is a group.*

22.30. **The difference bundle construction.** Suppose given a pair (X, A). For this section we will need to assume that both X and A are paracompact Hausdorff, as we will need to use homotopy invariance of bundles. Our goal in this section is to construct a map

$$\chi_1 \colon \mathcal{M}_1(X, A)_A \to K^0(X, A),$$

natural in (X, A), and prove that it is an isomorphism when (X, A) is a finite CW-pair.

Write $j \colon A \hookrightarrow X$ for the inclusion. Write $\mathrm{Cyl}(j)$ for the mapping cylinder, A_T for the copy of A at the top of the mapping cylinder ('T' is for top), and $\mathrm{Cone}(j) = \mathrm{Cyl}(j)/A_T$ for the mapping cone. Let $D(X, A, X)$ denote the *double mapping cylinder*, namely

$$D(X, A, X) = \mathrm{Cyl}(j) \amalg_{A_T} \mathrm{Cyl}(j).$$

It will be convenient to denote the left copy of $\mathrm{Cyl}(j)$ inside $D(X,A,X)$ as Cyl_0 and the right copy as Cyl_1. Each of these mapping cylinders contains a copy of X, which will likewise be denoted X_0 and X_1. Let $j_i\colon X_i \hookrightarrow D(X,A,X)$ be the two inclusions, and let $\pi\colon D(X,A,X) \to X$ be the evident projection.

In the long exact sequence for the pair $(D(X,A,X),X_0)$ the map j_0^* is split by π^*, hence the sequence breaks up into short exact sequences. At the K^0 level this is

$$0 \longrightarrow K^0(D(X,A,X),X_0) \longrightarrow K^0(D(X,A,X)) \xrightarrow{j_0^*} K^0(X_0) \longrightarrow 0. \tag{22.31}$$

In addition, we have the sequence of isomorphisms

$$\begin{tikzcd} K^0(D(X,A,X),X_0) & K^0(D(X,A,X),\mathrm{Cyl}_0) \arrow[l, "j_0^*"', "\cong"] \arrow[r, "j_1^*", "\cong"'] & K^0(\mathrm{Cyl}_1,A_T) \\ & & K^0(X,A). \arrow[u, "\cong", "\pi^*"'] \end{tikzcd} \tag{22.32}$$

The first and last of these are isomorphisms by homotopy invariance, whereas the second is by excision and homotopy invariance (we can fatten up the two cylinders into open sets if we like, intersecting in $A\times(0,1)$). Note that all of our diagrams are natural in the pair (X,A).

Given an element in $\mathcal{C}\mathrm{h}_1(X,A)_A$ our strategy will be to produce an element in $K^0(D(X,A,X),X_0)$ via a geometric construction and then push it into $K^0(X,A)$ via the above string of isomorphisms.

Suppose $E_1 \xrightarrow{d} E_0$ is an object in $\mathcal{C}\mathrm{h}_1(X,A)_A$, and recall that the map d is only defined over A. Note that d is an isomorphism $(E_1)|_A \to (E_0)|_A$. Construct a bundle $\mathcal{B}(E_0,E_1,d)$ on $D(X,A,X)$ by taking $\pi_0^*(E_0)$ on Cyl_0 and $\pi_1^*(E_1)$ on Cyl_1 and gluing them along the isomorphism

$$\pi_1^*(E_1)|_{A_T} = E_1 \xrightarrow{d} E_0 = \pi_0^*(E_0)|_{A_T}.$$

Here we are using Corollary 8.27 for the gluing construction, which works because the inclusion $A_T \hookrightarrow D(X,A,X)$ has the required properties. The bundle $\mathcal{B}(E_0,E_1,d)$ is called the **difference bundle** corresponding to $E_1 \xrightarrow{d} E_0$.

Consider the element $\alpha(E_\bullet) = [\mathcal{B}(E_0,E_0,\mathrm{id})] - [\mathcal{B}(E_0,E_1,d)] \in K^0(D(X,A,X))$. Restricting to X_0 sends this class to $[E_0]-[E_0]=0$, therefore by (22.31) $\alpha(E_\bullet)$ lifts to a unique class in $\tilde\alpha(E_\bullet) \in K^0(D(X,A,X),X_0)$. Note that if we restrict $\alpha(E_\bullet)$ to X_1 (or equivalently, $\tilde\alpha(E_\bullet)$ to the pair $(X_1,\emptyset)$) then we get $[E_0]-[E_1]$.

It is easy to check that $D(X,A,X)$ is paracompact, because X and A are. It follows at once from Proposition 11.1 (bundles on $D(X,A,X)\times I$ restrict to isomorphic bundles on $D(X,A,X)\times\{0\}$ and $D(X,A,X)\times\{1\}$) that if $E_\bullet \sim_h E'_\bullet$ in $\mathcal{C}\mathrm{h}_1(X,A)_A$ then the corresponding α classes are equal.

We next check that if $E_\bullet \sim_{st} E'_\bullet$ then $\alpha(E_\bullet)=\alpha(E'_\bullet)$. To this end, suppose that J is a vector bundle on X. Then $\mathcal{B}(E_1\oplus J, E_0\oplus J, d\oplus\mathrm{id}) = \mathcal{B}(E_1,E_0,d)\oplus\pi^*J$, and likewise $\mathcal{B}(E_0\oplus J,E_0\oplus J,\mathrm{id}\oplus\mathrm{id}) = \mathcal{B}(E_0,E_0,\mathrm{id})\oplus\pi^*J$. Therefore

$$\alpha(E_\bullet\oplus D_0(J)) = [\mathcal{B}(E_0,E_0,\mathrm{id})] + [\pi^*J] - \big([\mathcal{B}(E_1,E_0,d)] + [\pi^*J]\big) = \alpha(E_\bullet).$$

We have therefore proven that α gives a well-defined map $\mathcal{M}_1(X,A)_A \to K^0(D(X,A,X))$, and it is clearly a group homomorphism. The composite $j_0^*\circ\alpha$ is zero, and so α factors through $K^0(D(X,A,X),X_0)$. Define $\chi_1\colon \mathcal{M}_1(X,A)_A \to K^0(X,A)$ to be the composition of this map with the sequence of isomorphisms from

(22.32). Note that some authors also call $\chi_1(E_\bullet)$ the "difference bundle" associated to $E_\bullet$.

Remark 22.33. Atiyah [At4] gives this construction for pairs (X, A) where both X and A are compact Hausdorff, and uses $X \amalg_A X$ instead of our $D(X, A, X)$. When $A \hookrightarrow X$ is a cofibration these two spaces are homotopy equivalent. In the case where we only have compactness, it perhaps takes a bit more thought to see that gluing two bundles over X together over A actually defines a bundle.

Our construction of χ_1 is clearly natural in pairs (X, A) of paracompact Hausdorff spaces.

Proposition 22.34. *The map $\chi_1\colon \mathcal{M}_1(X, A)_A \to K^0(X, A)$ is an isomorphism whenever X and A are cofibrant and also homotopically compact (e.g. (X, A) is a finite CW-pair).*

Proof. The proof proceeds in three steps.

Step 1: $A = \emptyset$. This step is trivial, since $\mathcal{M}_1(X, \emptyset)_\emptyset$ is readily identified with the Grothendieck group of vector bundles on X (see Exercise 22.3). Since X is both cofibrant and homotopically compact, it is homotopy equivalent to a finite CW-complex. So the natural map $K^0_{Grt}(X) \to K^0(X)$ is an isomorphism.

Step 2: $A = *$. Here we consider the diagram of groups

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_1(X,*)_* \arrow[r, "\alpha"] \arrow[d, "\chi_1"'] & \mathcal{M}_1(X,\emptyset)_\emptyset \arrow[r, "\beta"] \arrow[d, "\chi_1"] & \mathcal{M}_1(*,\emptyset)_\emptyset \arrow[r] \arrow[d, "\chi_1"] & 0 \\
0 \arrow[r] & K^0(X,*) \arrow[r] & K^0(X) \arrow[r] & K^0(*) \arrow[r] & 0.
\end{tikzcd}$$

The bottom row is exact, and the middle and right vertical maps are isomorphisms by Step 1. It will suffice to show that the top row is exact, since then the left vertical map is also an isomorphism.

For the remainder of this step let $x \in X$ denote the basepoint.

Surjectivity of β is trivial. For exactness at the middle spot, note that $\mathcal{M}_1(\{x\}, \emptyset)_\emptyset \cong \mathbb{Z}$ via $(V_1, V_0) \to \operatorname{rank} V_0 - \operatorname{rank} V_1$. So if $E_\bullet$ is in the kernel of β then $\operatorname{rank}_x E_1 = \operatorname{rank}_x E_0$, and therefore there certainly exists an isomorphism $(E_1)_x \cong (E_0)_x$. Such an isomorphism determines an element of $\mathcal{M}_1(X, x)_{\{x\}}$ that is a preimage for $E_\bullet$.

Finally, let $E_\bullet \in \mathcal{C}h(X, x)_{\{x\}}$ and assume that $E_\bullet$ is in the kernel of α. Using the isomorphism $\mathcal{M}_1(X, \emptyset)_\emptyset \cong K^0(X)$, this means that $[E_1] = [E_0]$ in $K^0(X)$. So there exists a vector bundle Q such that $E_1 \oplus Q \cong E_0 \oplus Q$. Adding $\mathrm{id}\colon Q \to Q$ to $E_\bullet$ and using this isomorphism, we may assume that $E_\bullet$ satisfies $E_1 = E_0$. Our isomorphism of fibers over x is then an element $\sigma \in \operatorname{Aut}((E_0)_x) \cong GL_n(\mathbb{C})$ for $n = \operatorname{rank}(E_0)$. As $GL_n(\mathbb{C})$ is connected, choose a path from the identity element to σ in $\operatorname{Aut}((E_0)_x)$. If $\pi\colon X \times I \to X$ is the projection, this path defines an isomorphism $(\pi^* E_0)|_{\{x\}\times I} \to (\pi^* E_0)|_{\{x\}\times I}$ and therefore an object in $\mathcal{C}h_1(X \times I, \{x\} \times I)$. Restricting this element to $X \times \{0\}$ and $X \times \{1\}$ shows that $E_\bullet$ equals $E_0 \xrightarrow{\mathrm{id}} E_0$ in $\mathcal{M}_1(X, x)_{\{x\}}$. But the latter element is zero, by the stability relation.

[NOTE: When proving the result for $\mathbb{R}$-vector bundles an extra argument is needed here, because $GL_n(\mathbb{R})$ is not path-connected. If σ is in the path component of the identity, the argument is the same as above. If not, add the elementary complex $D_0(\mathbb{R}^2)$ onto E and then use the isomorphism of complexes

$$\begin{tikzcd} E_0 \oplus \underline{2} \arrow[r, "\mathrm{id}\oplus t", "\cong"'] \arrow[d, "\sigma\oplus\mathrm{id}"] & E_0 \oplus \underline{2} \arrow[d, "\sigma\oplus t"] \\ E_0 \oplus \underline{2} \arrow[r, "\mathrm{id}", "\cong"'] & E_0 \oplus \underline{2} \end{tikzcd}$$

where $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the map that interchanges the standard basis vectors. The isomorphism $\sigma \oplus t$ is now in the path component of the identity and so we are back in the previous case; therefore, our complex represents zero in $\mathcal{M}_1(X,x)_{\{x\}}$.]

Step 3: General case. Here we will consider the two squares

$$\begin{tikzcd} \mathcal{M}_1(X,A)_A \arrow[r, "\pi^*"] \arrow[d, "\chi_1"'] & \mathcal{M}_1(\mathrm{Cyl}(j),A_T)_{A_T} \arrow[d, "\chi_1"] & \mathcal{M}_1(\mathrm{Cone}(j),*)_{\{*\}} \arrow[l, "p^*"'] \arrow[d, "\chi_1", "\cong"'] \\ K^0(X,A) \arrow[r, "\pi^*", "\cong"'] & K^0(\mathrm{Cyl}(j),A_T) & K^0(\mathrm{Cone}(j),*). \arrow[l, "p^*"', "\cong"] \end{tikzcd}$$

The right vertical map is an isomorphism by Step 2, since $\mathrm{Cone}(j)$ is cofibrant and homotopically compact because X and A were. Here p is the projection $\mathrm{Cyl}(j) \to \mathrm{Cone}(j)$, and the lower p^* is an isomorphism because K is a cohomology theory. Likewise, π is the projection $\mathrm{Cyl}(j) \to X$ and the lower π^* is an isomorphism because of homotopy invariance.

Since $p^* \circ \chi_1$ is an isomorphism, the upper p^* is injective. We next show it is surjective. Let $E_\bullet$ be an object in $\mathcal{C}h_1(\mathrm{Cyl}(j),A_T)_{A_T}$. The space $\mathrm{Cyl}(j)$ is cofibrant, since both A and X are. It is also homotopy equivalent to X, which is homotopically compact. It follows that there exists a bundle Q on $\mathrm{Cyl}(j)$ such that $E_1 \oplus Q$ is trivial. We can add the elementary complex $D_0(Q)$ to $E_\bullet$ without changing its class in $\mathcal{M}_1$, and in this way we can assume that E_1 is trivial, i.e. $E_1 = \underline{n}$ for some n. The differential $d\colon (E_1)|_{A_T} \to (E_0)|_{A_T}$ is therefore a trivialization $A_T \times \mathbb{C}^n \xrightarrow{\cong} (E_0)|_{A_T}$. Define J to be the pushout in $\mathcal{T}op$ of

$$\mathbb{C}^n \longleftarrow (E_0)|_{A_T} \hookrightarrow E_0$$

where the left map is the composite $(E_0)|_{A_T} \xrightarrow{d^{-1}} A_T \times \mathbb{C}^n \xrightarrow{\pi_2} \mathbb{C}^n$. In words, J is constructed by gluing all of the fibers of E_0 above A_T together into one, using the identification provided by the trivialization. Clearly we have the map $J \to \mathrm{Cone}(j)$, and this is a vector bundle (see Exercise 8.42).

Consider the element of $\mathcal{M}_1(\mathrm{Cone}(j),*)_{\{*\}}$ given by $\underline{n} \xrightarrow{d'} J$, where d' is the isomorphism $\mathbb{C}^n \to J_*$ that is built into the definition of J. We claim that $[E_\bullet] = p^*([\underline{n} \to J])$. The universal property of pullbacks gives us a dotted arrow in the diagram

$$\begin{tikzcd} E_0 \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & p^*J \arrow[r] \arrow[d] & J \arrow[d] \\ & \mathrm{Cyl}(j) \arrow[r, "p"] & \mathrm{Cone}(j), \end{tikzcd}$$

where all of the other maps are the evident ones. This arrow is a bundle map and a fiberwise isomorphism, hence an isomorphism of bundles. Now one readily checks that $\underline{n} \xrightarrow{d} E_0$ is isomorphic to p^* applied to $\underline{n} \xrightarrow{d'} J$, hence our complex is in the image of p^* and we have proven surjectivity. So the upper p^* is an isomorphism, and likewise for the middle χ_1.

As our last step we will prove that the upper π^* is an isomorphism, which implies the same for the left χ_1 map and completes the proof.

Since $\mathrm{Cyl}(j) \to X$ is part of a homotopy equivalence (and all the spaces are paracompact Hausdorff, being cofibrant), every bundle on $\mathrm{Cyl}(X)$ is isomorphic to one that is pulled back from X. So any element in $\mathcal{M}_1(\mathrm{Cyl}(j), A_T)_{A_T}$ is represented by a complex of the form $\pi^*E_1 \xrightarrow{d} \pi^*E_0$ for some bundles E_1 and E_0 on X. Since $(\pi^*E_i)|_{A_T}$ can be canonically identified with $(E_i)|_{A_T}$, the differential d can be regarded as a map $(E_1)|_A \to (E_0)_A$. Thus we have described an element of $\mathcal{M}_1(X, A)_A$, and it is clear that applying π^* yields our original class. So π^* is surjective.

To prove injectivity of π^* we will construct an explicit left inverse. Suppose given an object $d\colon E_1 \to E_0$ in $\mathcal{C}h_1(\mathrm{Cyl}(j), A_T)$. Restricting to the cylinder $A \times I \hookrightarrow \mathrm{Cyl}(j)$ gives two bundles on $A \times I$ and an isomorphism between them on $A \times \{0\}$ (we let time 0 denote the top of the cylinder, and time 1 the bottom). By Proposition 11.8 this isomorphism can be extended to an isomorphism D on $A \times I$, and then restricted to time 1 to give $D_1\colon (E_1)|_{A\times 1} \to (E_0)_{A\times 1}$. The bundle map $(E_1)|_X \xrightarrow{D_1} (E_0)|_X$ represents an element $\alpha(E_\bullet) \in \mathcal{M}_1(X, A)_A$. Proposition 11.8 also says that a different extension D' gives an isomorphism D'_1 that is homotopic to D_1, therefore $\alpha(E_\bullet)$ does not depend on the choice of D.

Adding an elementary complex to $E_\bullet$ clearly results in the addition of an elementary complex to $\alpha(E_\bullet)$. Finally, suppose given an element $\mathcal{E} \in \mathcal{C}h_1(X\times I, A\times I)_{A\times I}$ and set $E = \mathcal{E}|_{X\times 0}$ and $E' = \mathcal{E}_{X\times 1}$. As above we can extend the isomorphism $E|_{A_T\times I} \to E'|_{A_T\times I}$ to $E|_{\mathrm{Cyl}(A)\times I} \to E'|_{\mathrm{Cyl}(A)\times I}$ and then this defines an object in $\mathcal{C}h_1(X \times I, A \times I)_{A\times I}$ The restriction of this object to $X \times 0$ represents $\alpha(E_\bullet)$ and the restriction to $X \times 1$ represents $\alpha(E'_\bullet)$, and so $\alpha(E_\bullet) = \alpha(E'_\bullet)$ in $\mathcal{M}_1(X, A)_A$. This shows that α is a well-defined map $\mathcal{M}_1(\mathrm{Cyl}(j), A_T)_{A_T} \to \mathcal{M}_1(X, A)_A$. It is clear that $\alpha \circ \pi^* = \mathrm{id}$, so this completes the proof that π^* is a bijection. It follows that the left vertical χ_1 is a bijection, which is what we needed. □

22.35. **Wrapping things up.** We can now bring together all of the work in this section to prove the main results. The hard work has already been done, and now it is just a matter of fitting together some small details. Let us first start with a simple lemma:

Lemma 22.36. *Suppose that $\eta\colon K^0(X, A) \to K^0(X, A)$ is a natural transformation defined on all finite CW-pairs (X, A) that is equal to the identity when $A = \emptyset$. Then η is the identity on all pairs.*

Proof. Just use the natural isomorphism $\pi^*\colon K^0(X/A, *) \to K^0(X, A)$ together with the natural short exact sequence $0 \to K^0(X/A, *) \to K^0(X/A) \to K^0(*) \to 0$. □

When (X, A) is a cofibrant pair we have all the tools to construct a map $\mathcal{K}(X, A) \to K^0(X, A)$. Standard homotopical techniques allow us to reduce to this case, leading to the following result:

Proposition 22.37. *There is a natural transformation* $\chi\colon \mathcal{K}(X,A) \to K^0(X,A)$, *defined for all pairs* (X,A), *having the property that for* $A=\emptyset$ *one has* $\chi([E_\bullet]) = \sum_i (-1)^i [E_i]$. *All such natural transformations agree on pairs* (X,A) *where* X *and* A *are homotopically compact, and are an isomorphism on pairs that are homotopy equivalent to a finite CW-pair.*

Proof. We will use the functorial factorizations on the model category $\mathcal{T}op$, though this is not strictly necessary. Given a pair (X,A) factor $\emptyset \to A$ as $\emptyset \rightarrowtail \tilde{A} \xrightarrow{\sim}\!\!\!\!\!\rightarrow A$. Then factor the composite $\tilde{A} \to A \to X$ as $\tilde{A} \rightarrowtail \tilde{X} \xrightarrow{\sim}\!\!\!\!\!\rightarrow X$. In this way we functorially produce a map of pairs $(\tilde{X},\tilde{A}) \to (X,A)$ where the former is a cofibration between cofibrant spaces. Define $\chi\colon \mathcal{K}(X,A) \to K^0(X,A)$ to be the following composite:

$$\begin{tikzcd}
\mathcal{K}(X,A) \arrow[rrrr, dotted, "\chi"] \arrow[d] & & & & K^0(X,A) \arrow[d, "\cong"'] \\
\mathcal{K}(\tilde{X},\tilde{A}) & \mathcal{M}_\infty(\tilde{X},\tilde{A}) \arrow[l, "\cong"', "(22.4)"] \arrow[r, "\cong", "(22.11)"'] & \mathcal{M}_\infty(\tilde{X},\tilde{A})_{\tilde{A}} & \mathcal{M}_1(\tilde{X},\tilde{A})_{\tilde{A}} \arrow[l, "\cong"', "(22.22)"] \arrow[r, "\chi_1"] & K^0(\tilde{X},\tilde{A}).
\end{tikzcd}$$

This gives a natural transformation of functors. When $A=\emptyset$ then $\tilde{A}=\emptyset$ and it is easy to check that the composite across the bottom row sends a graded collection of vector bundles to their alternating sum. It follows that the top map must do the same thing.

If $A' \hookrightarrow X'$ is a cofibration between cofibrant objects with a map of pairs $(X',A') \to (X,A)$ that is a weak equivalence on both components, then there is a commutative triangle of weak equivalences

$$\begin{tikzcd}
(X',A') \arrow[r] \arrow[dr] & (\tilde{X},\tilde{A}) \arrow[d] \\
& (X,A).
\end{tikzcd}$$

One readily sees that the map χ defined above by passing through $(\tilde{X},\tilde{A})$ coincides with the similarly-defined map that passes through (X',A') (just chase around a big commutative diagram). In particular, the exact choice of $(\tilde{X},\tilde{A})$ does not affect the definition of χ. As a result, when (X,A) is a finite CW-pair we can take $\tilde{X}=X$ and $\tilde{A}=A$ and so χ is just the composite across the bottom row of the above diagram. But for finite CW-pairs χ_1 was proven to be an isomorphism in Proposition 22.34, and so χ is also an isomorphism here.

Now suppose that χ' is any other natural transformation that equals χ when $A=\emptyset$. Since χ is an isomorphism on finite CW-pairs we can define $\eta=\chi'\circ\chi^{-1}$ on such pairs, thereby obtaining natural maps $\eta\colon K^0(X,A)\to K^0(X,A)$. When $A=\emptyset$ these are identity maps by our assumption on χ', so by Lemma 22.36 we must have $\eta=\mathrm{id}$ on all pairs. Therefore $\chi=\chi'$ on all finite CW-pairs.

Now assume (X,A) is a pair where both X and A are homotopically compact. Then by Lemma E.2 there is a finite CW-pair (X',A') and a map of pairs $(X',A')\to$

(X, A) that is a weak equivalence on each term. The diagram

$$\begin{tikzcd} \mathcal{K}(X,A) \arrow[r, "\chi"] \arrow[d] & K^0(X,A) \arrow[d, "\cong"] \\ \mathcal{K}(X',A') \arrow[r, "\chi"] & K^0(X',A') \end{tikzcd}$$

must commute both for χ (shown) and for χ'. Since χ and χ' are equal for (X', A'), they must also be equal for (X, A).

The only thing left to check is that χ is an isomorphism on pairs (X, A) that are homotopy equivalent to a finite CW-pair (X', A'). Let $f\colon (X', A') \to (X, A)$ be part of such a homotopy equivalence and consider the diagram

$$\begin{tikzcd} \mathcal{K}(X,A) \arrow[r, "\chi"] \arrow[d, "f^*"'] & K^0(X,A) \arrow[d, "f^*"] \\ \mathcal{K}(X',A') \arrow[r, "\chi"] & K^0(X',A'). \end{tikzcd}$$

Both of the vertical maps are isomorphisms and we have already proven that the bottom χ is an isomorphism, therefore the top one is as well. $\square$

Remark 22.38. Note that we have not proven that χ is an isomorphism whenever X and A are homotopically compact. It is not clear whether this is true.

22.39. **Compatibility of products.** As we have observed before, if $E_\bullet \in \mathcal{C}h(X, A)$ and $F_\bullet \in \mathcal{C}h(Y, B)$ then $E_\bullet \hat{\otimes} F_\bullet \in \mathcal{C}h(X \times Y, (A \times Y) \cup (X \times B))$. This is readily seen to induce pairings

$$\mu\colon \mathcal{K}(X, A) \otimes \mathcal{K}(Y, B) \to \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)).$$

It is almost formal that these pairings are compatible (via χ) with the similar pairings on K^0: it is easy to check that they agree when $A = B = \emptyset$, and then the formal properties of cohomology theories allow one to bootstrap up to CW-pairs.

Proposition 22.40. *For any CW-pairs (X, A) and (Y, B) the diagram*

$$\begin{tikzcd} \mathcal{K}(X,A) \otimes \mathcal{K}(Y,B) \arrow[r, "\mu"] \arrow[d, "\chi\otimes\chi"] & \mathcal{K}(X\times Y, (A\times Y)\cup(X\times B)) \arrow[d, "\chi"] \\ K^0(X,A) \otimes K^0(Y,B) \arrow[r] & K^0(X\times Y, (A\times Y)\cup(X\times B)) \end{tikzcd}$$

is commutative, where the bottom horizontal map is the product on K-theory. The same diagram commutes if (X, A) and (Y, B) are more general pairs but where we assume A is open in X and B is open in Y (for a more general statement, see Remark 22.41).

Proof. We first check this when (X, A) and (Y, B) are CW-pairs. It is trivial to check that the diagram commutes when $A = B = \emptyset$. The general case follows formally from this one using naturality. First check that it works for $A = B = *$

using the diagram

$$\begin{tikzcd}
K^0(X,*) \otimes K^0(Y,*) \arrow[r] \arrow[d, tail] & K^0(X \times Y, (X \times *) \cup (* \times Y)) \arrow[d, tail] & K^0(X \wedge Y, *) \arrow[l, "\cong"'] \arrow[dl, tail] \\
K^0(X) \otimes K^0(Y) \arrow[r] & K^0(X \times Y) &
\end{tikzcd}$$

(and the similar one for $\mathcal{K}(-,-)$ that maps to it). Here the diagonal map is an injection (for any cohomology theory) because stably the product $X \times Y$ splits as $X \vee Y \vee (X \wedge Y)$, and consequently the central vertical map is an injection. The desired commutativity now follows by a diagram chase.

Finally, deduce the general case using the diagram

$$\begin{tikzcd}
K^0(X,A) \otimes K^0(Y,B) \arrow[r] & K^0(X \times Y, (A \times Y) \cup (X \times B)) & K^0(X/A \wedge Y/B, *) \arrow[l, "\cong"'] \arrow[dl, "\cong"] \\
K^0(X/A,*) \otimes K^0(Y/B,*) \arrow[r] \arrow[u, "\cong"'] & K^0(X/A \times Y/B, X/A \vee Y/B) \arrow[u, "\cong"'] &
\end{tikzcd}$$

and its $\mathcal{K}$-analog. The main point here is the square, but the portion of the diagram to the left of it is there to show that the central vertical map is an isomorphism. Again, the desired commutativity now follows from an easy diagram chase.

Now suppose that (X,A) and (Y,B) are more general pairs but where additionally A is open in X and B is open in Y. Let (X',A') and (Y',B') be cofibrant models for the two pairs, which exist as in the proof of Proposition 22.37. The diagram from the statement of the proposition for (X,A) and (Y,B) maps to the similar one for (X',A') and (Y',B'), and the maps $K^0(X,A) \to K^0(X',A')$ and $K^0(Y,B) \to K^0(Y',B')$ are isomorphisms. The desired result will follow from what has already been proven if we know that

$$K^0(X \times Y, A \times Y \cup X \times B) \to K^0(X' \times Y', A' \times Y' \cup X' \times B')$$

is an isomorphism. The map $X' \times Y' \to X \times Y$ is certainly a weak equivalence, but it is not clear that the same can be said for the relative term. This is the tricky point.

For convenience let $W = A \times Y \cup X \times B$ and let $W' = A' \times Y' \cup X' \times B'$. Consider the following two long exact sequences for the triple $(X \times Y, W, A \times Y)$ and its primed analog:

$$\begin{tikzcd}
\cdots \arrow[r] & K^0(X \times Y, W) \arrow[r] \arrow[d] & K^0(X \times Y, A \times Y) \arrow[r] \arrow[d, "\cong"] & K^0(W, A \times Y) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K^0(X' \times Y', W') \arrow[r] & K^0(X' \times Y', A' \times Y') \arrow[r] & K^0(W', A' \times Y') \arrow[r] & \cdots
\end{tikzcd}$$

The middle map is an isomorphism because $X' \times Y' \to X \times Y$ and $A' \times Y' \to A \times Y$ are weak equivalences. For the third map we consider the square

$$\begin{tikzcd}
K^0(W, A \times Y) \arrow[r] \arrow[d] & K^0(X \times B, A \times B) \arrow[d, "\cong"] \\
K^0(W', A' \times Y') \arrow[r] & K^0(X' \times B', A' \times B')
\end{tikzcd}$$

where the right vertical map is an isomorphism as above. Under the assumption that A is open in X and B is open in Y, the top horizontal map is an isomorphism by excision. The bottom map is an isomorphism for a similar reason, but here all of the pairs are CW-pairs and so the reasoning is just slightly different. It follows that the left vertical map is an isomorphism. Going back to our map of long exact sequences, we now know that two out of every three maps are isomorphisms and so the five lemma implies the last is also an isomorphism. This is what we needed. □

Remark 22.41. ????

Remark 22.42. Note that the tensor product of complexes does not give a pairing $\mathcal{M}_\infty(X,A)_A \otimes \mathcal{M}_\infty(Y,B)_B \to \mathcal{M}_\infty(X\times Y, A\times Y \cup X\times B)_{A\times Y\cup X\times B}$, as the resulting differential would only be defined on $A\times B$. So for products it is important to use $\mathcal{M}(X,A)$ rather than the A-relative version $\mathcal{M}(X,A)_A$.

22.43. **The proof of Theorem 18.16.** To finally close out our main goal for this section, let us just observe that Theorem 18.16 is a combination of Proposition 22.37 and Proposition 22.40. So the proof has now been completed.

22.44. **The role of homotopy invariance.** In [At4] Atiyah uses a version of our $\mathcal{M}_n(X,A)_A$ construction but where he leaves out the homotopy relation. To this end, let $\mathcal{L}_n(X,A)_A$ be this group: it consists of equivalence classes of objects in $\mathcal{C}h_n(X,A)_A$ up to isomorphism and stability. We have the natural surjection $\mathcal{L}_n(X,A)_A \twoheadrightarrow \mathcal{M}_n(X,A)_A$.

Proposition 22.45. *Assume that $A \hookrightarrow X$ is a cofibration and X is paracompact Hausdorff. Then the map $\mathcal{L}_1(X,A)_A \to \mathcal{M}_1(X,A)_A$ is a bijection.*

Proof. Suppose given an $\mathcal{E} \in \mathcal{C}h_1(X\times I, A\times I)_{A\times I}$ and let $E = \mathcal{E}|_{X\times 0}$ and $E' = \mathcal{E}|_{X\times 1}$. We need to show that $[E]=[E']$ in $\mathcal{L}_1(X,A)_A$. Since X is paracompact Hausdorff, the bundles on $X\times I$ are all isomorphic to ones pulled back along the projection $X\times I \to X$; so we can assume all the bundles in $\mathcal{E}$ are of this form. So $E_0 = E_0'$ and $E_1 = E_1'$.

Write d_A and d_A' for the differentials on E and E', and note that these are isomorphisms. The homotopy $d_A' \sim d_A$ induces a homotopy $\mathrm{id} \sim (d_A')^{-1}\circ d_A$. Consider the fiber bundle $\underline{\mathrm{Iso}}(E_1,E_1) \to X$ and the following diagram:

$$\begin{tikzcd}
(X\times 0)\cup(A\times I) \arrow[r, "\mathrm{id}\cup H"] \arrow[d, hook, "\simeq"'] & \underline{\mathrm{Iso}}(E_1,E_1) \arrow[d, two heads] \\
X\times I \arrow[r, "\pi"] & X
\end{tikzcd}$$

where id is the identity section and H is our homotopy on A. This diagram must have a lift, and restricting to time 1 gives an isomorphism $\phi\colon E_1 \to E_1$ that restricts to $(d_A')^{-1}d_A$ on A. Thus, we have the following isomorphism in $\mathcal{C}h_1(X,A)_A$:

$$\begin{tikzcd}
E_1 \arrow[r, "\phi", "\cong"'] \arrow[d, "d_A"'] & E_1 \arrow[d, "d_A'"] \\
E_0 \arrow[r, "\mathrm{id}", "\cong"'] & E_0.
\end{tikzcd}$$

This proves that $[E]=[E']$ in $\mathcal{L}_1(X,A)_A$ and we are done. □

Corollary 22.46. *Suppose that (X, A) is a cofibrant pair. Then for all $1 \leq n \leq \infty$ the map $\mathcal{L}_n(X, A)_A \to \mathcal{M}_n(X, A)_A$ is a bijection.*

Proof. Consider the commutative square

$$\begin{tikzcd}
\mathcal{L}_1(X,A)_A \arrow[r] \arrow[d, "\cong"'] & \mathcal{L}_n(X,A)_A \arrow[d] \\
\mathcal{M}_1(X,A)_A \arrow[r, "\cong"] & \mathcal{M}_n(X,A)_A.
\end{tikzcd}$$

The left vertical map is an isomorphism by Proposition 22.45 and the lower horizontal map is an isomorphism by Proposition 22.22. It follows that the upper horizontal map is an injection. But use of the Γ-construction shows easily that the upper horizontal map is surjective: the argument is a much-simplified form of the proof of Proposition 22.22, in that we don't actually need to construct the inverse map we just need to produce preimages for elements. So the upper horizontal map is a bijection, which implies the same for the right vertical map. $\square$

Consider now the following diagram:

$$\begin{tikzcd}
 & \mathcal{M}_1(X,A) \arrow[r] \arrow[d, "(22.11)"'] & \mathcal{M}_n(X,A) \arrow[r] \arrow[d, "(22.11)"'] & \mathcal{M}_\infty(X,A) \arrow[r, "(22.4)"] \arrow[d, "(22.11)"'] & \mathcal{K}(X,A) \\
K^0(X,A) & \mathcal{M}_1(X,A)_A \arrow[l, "D"'] \arrow[r, "(22.22)"'] & \mathcal{M}_n(X,A)_A \arrow[r, "(22.22)"'] & \mathcal{M}_\infty(X,A)_A & \\
 & \mathcal{L}_1(X,A)_A \arrow[r] \arrow[u, "(22.46)"] & \mathcal{L}_n(X,A)_A \arrow[r] \arrow[u, "(22.46)"'] & \mathcal{L}_\infty(X,A)_A. \arrow[u, "(22.46)"'] &
\end{tikzcd}$$

With the exception of D, all of the maps are isomorphisms whenever (X, A) is a cofibrant pair—the labels on the maps give the references, and for the unlabelled maps it follows by the commutative diagram. The map D is an isomorphism when X and A are cofibrant and homotopically compact.

Note that the product, defined in terms of tensor product of chain complexes, only makes sense for $\mathcal{M}_\infty(X, A)$ and $\mathcal{K}(X, A)$.

Part 4. K-theory and geometry II: Applications

In the last several sections we developed the basic connections between K-theory and geometry. We have seen that K-theory is a complex-oriented cohomology theory, and we understand "geometric" representatives for the Thom classes and fundamental classes that come with such a theory; in this case "geometric" means that we can write down explicit chain complexes of vector bundles representing the classes. In the following sections our aim is to further explore this general area: now that the basic picture is in place, where does it take us? The topics we cover are somewhat of a hodgepodge, but in some sense they they all revolve around the exploration of fundamental classes.

23. $K^*(\mathbb{C}P^n)$ and the K-theoretic analog of the degree

If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold then it has a fundamental class $[Z]$ in $H^*(\mathbb{C}P^n)$. Knowing this fundamental class comes down to knowing a single integer, called the degree of Z. The geometric interpretation of the degree is that it equals the number of points of intersection between Z and a generically chosen linear subspace of complementary dimension. In this section we will repeat this line of investigation but replacing H^* with K^*. So we must compute $K^*(\mathbb{C}P^n)$ and investigate what information is encoded in the fundamental class $[Z]_K$. We will find that knowing $[Z]_K$ amounts to knowing *several* integers (not just one); while we can give methods for computing these, their geometric interpretation is somewhat mysterious.

23.1. **Calculation of $K^*(\mathbb{C}P^n)$.** We begin with the following easy lemma:

Lemma 23.2. *Let E be any multiplicative cohomology theory. If $x_1, \dots, x_{n+1} \in \tilde{E}^*(\mathbb{C}P^n)$, then $x_1 \cdots x_{n+1} = 0$.*

Proof. The key is just that $\mathbb{C}P^n$ can be covered by $n+1$ contractible sets. To be explicit, let $U_i = \{[z_0 : \dots : z_n] \in \mathbb{C}P^n \,|\, z_i \neq 0\}$. Then U_i is open in $\mathbb{C}P^n$ and is homeomorphic to $\mathbb{C}^n$.

Choose our basepoint of $\mathbb{C}P^n$ to be $[1 : 1 : \cdots : 1]$ (or any other point in the intersection of all the U_i's). The contractibility of U_i implies that $E^*(\mathbb{C}P^n, U_i) \to E^*(\mathbb{C}P^n, *)$ is an isomorphism. So we may lift each x_i to a class $\tilde{x}_i \in E^*(\mathbb{C}P^n, U_i)$.

It now follows that $\tilde{x}_1\tilde{x}_2 \cdots \tilde{x}_{n+1}$ lifts $x_1 \cdots x_{n+1}$ in the map

$$E^*(\mathbb{C}P^n, U_1 \cup \cdots \cup U_{n+1}) \to E^*(\mathbb{C}P^n, *).$$

Since $U_1 \cup \cdots \cup U_{n+1} = \mathbb{C}P^n$, the domain is the zero group. So $x_1 \cdots x_{n+1} = 0$. $\square$

Recall that $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ where x is a generator in degree 2. It is not hard to see that we may take $x = [\mathbb{C}P^{n-1}]$. This calculation works for any complex-oriented cohomology theory:

Proposition 23.3. *Let E be a complex-oriented cohomology theory. There is an isomorphism of rings*

$$E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$$

sending x^i to $[\mathbb{C}P^{n-i}]$.

Proof. Consider the reduced Gysin sequence (as in Section 17.3) for the submanifold $\mathbb{C}P^{n-1} \overset{j}{\hookrightarrow} \mathbb{C}P^n$:

$$\begin{tikzcd} \cdots & \tilde{E}^k(\mathbb{C}P^n - \mathbb{C}P^{n-1}) \arrow[l] & \tilde{E}^k(\mathbb{C}P^n) \arrow[l] & & \\ & & E^{k-2}(\mathbb{C}P^{n-1}) \arrow[u, "j_!"] & \tilde{E}^{k-1}(\mathbb{C}P^n - \mathbb{C}P^{n-1}) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

Let $x = [\mathbb{C}P^{n-1}] = j_!(1) \in \tilde{E}^2(\mathbb{C}P^n)$. By Lemma 23.2 we know $x^{n+1} = 0$, so we get a map $E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$. We will show that this map is an isomorphism via induction on n. The case $n = 0$ is trivial.

Note that $x^2 = [\mathbb{C}P^{n-2}]$ by intersection theory (specifically, the E-theory analog of Theorem 17.9). The same reasoning gives $x^i = [\mathbb{C}P^{n-i}]$.

The spaces $\mathbb{C}P^n - \mathbb{C}P^{n-1}$ are homeomorphic to $\mathbb{C}^n$ and hence contractible. So the reduced Gysin sequence considered above breaks up into a collection of isomorphisms

$$j_! \colon E^{k-2}(\mathbb{C}P^{n-1}) \xrightarrow{\cong} \tilde{E}^k(\mathbb{C}P^n).$$

Taking all k's together, $j_!$ is a map of $E^*(pt)$-modules and therefore an isomorphism of such modules. By induction $E^*(\mathbb{C}P^{n-1})$ is a free $E^*(pt)$-module generated by the classes $1 = [\mathbb{C}P^{n-1}]$, $[\mathbb{C}P^{n-2}]$, $[\mathbb{C}P^{n-3}], \ldots, [\mathbb{C}P^0]$. Since the pushforward $j_!$ sends $[\mathbb{C}P^{n-i}]$ to $[\mathbb{C}P^{n-i}]$, we conclude that $\tilde{E}^*(\mathbb{C}P^n)$ is a free $E^*(pt)$-module on $[\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$. If we add 1 to this collection then we get a free basis for $E^*(\mathbb{C}P^n)$ over $E^*(pt)$. This proves that our map $E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$ is an isomorphism. $\square$

In the case of complex K-theory, we can rephrase the above result as saying that $K^{odd}(\mathbb{C}P^n) = 0$ and $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$, where $X = \beta \cdot [\mathbb{C}P^{n-1}]$. In particular, note that additively we have $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$ with free basis consisting of the powers of X. Ignoring powers of the Bott element as usual, we can write this free basis as $[\mathbb{C}P^n], [\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$.

23.4. **Fundamental classes in $K^*(\mathbb{C}P^n)$.** Now let $Z \hookrightarrow \mathbb{C}P^n$ be a complex, closed submanifold of codimension c. Recall that if we consider fundamental classes in singular cohomology then we have $[Z] = d \cdot [\mathbb{C}P^{n-c}]$ for a unique integer d that is called the **degree** of Z. Geometrically, d is the number of points on intersection of Z with a generically chosen copy of $\mathbb{C}P^c$.

We can now play this same game in the context of K-theory. We have a fundamental class $[Z]_K \in K^0(\mathbb{C}P^n)$, and we can therefore write

$$\begin{aligned} [Z]_K &= d_0 \cdot [\mathbb{C}P^n] + d_1 \cdot [\mathbb{C}P^{n-1}] + \cdots + d_n \cdot [\mathbb{C}P^0] \\ &= d_0 \cdot 1 + d_1 X + d_2 X^2 + \cdots + d_n X^n \end{aligned}$$

for unique integers d_i. These integers are topological invariants of the embedding $Z \hookrightarrow \mathbb{C}P^n$; our goal will be to explore them further. Multiply the above equation by X^n to obtain

$$d_0 X^n = [Z]_K \cdot X^n = [Z]_K \cdot [\mathbb{C}P^0] = [Z \cap \mathbb{C}P^0]$$

where in the last term we mean the intersection of Z with a generically chosen copy of $\mathbb{C}P^0$. But as long as Z is not all of $\mathbb{C}P^n$, this generic intersection will be empty—so if Z is codimension at least one then $d_0 = 0$. We then repeat this argument, but multiplying by X^{n-1} instead of X^n: we get

$$d_1 X^n = [Z]_K \cdot [\mathbb{C}P^1] = [Z \cap \mathbb{C}P^1].$$

Again, if the codimension of Z is at least 2 then the $\mathbb{C}P^1$ can be moved so that it doesn't intersect Z at all, hence $d_1 = 0$. Continuing this argument we find that

$$0 = d_0 = d_1 = \cdots = d_{c-1} \qquad \text{and} \qquad d_c = \deg(Z).$$

The last equality follows because Z intersects a generic $\mathbb{C}P^c$ in $\deg(Z)$ many points. The situation can be summarized as follows: the first non-vanishing d_i coincides with the classical degree of Z, but there is the possibility of the higher d_i's being nonzero. This is what we will investigate next.

Remark 23.5. Notice that what we have done so far works in any complex-oriented cohomology theory E^*. The fundamental class $[Z]$ can be written as a linear combination of the classes $[\mathbb{C}P^{n-i}]$ with coefficients from $E^*(pt)$. The first c of these coefficients vanish, until one gets to the coefficient of $[\mathbb{C}P^{n-c}]$—which must be equal to $\deg(Z)$. After this things become interesting, in the sense that one has invariants that are potentially not detected in singular cohomology.

To proceed further with our analysis of the higher d_i's in K-theory, we need to connect our fundamental classes with the vector bundle description of K-theory:

Proposition 23.6. *In $K^0(\mathbb{C}P^n)$ one has $[\mathbb{C}P^{n-1}] = 1 - L$ where $L \to \mathbb{C}P^n$ is the tautological line bundle. Consequently, $[\mathbb{C}P^{n-k}] = (1-L)^k$ for all k and $K^0(\mathbb{C}P^n) = \mathbb{Z}[L]/(1-L)^{n+1}$.*

Proof. We give two explanations. For the first, consider the map of vector bundles $f\colon L \to 1$ defined as follows: in the fiber over $x = [x_0 : x_1 : \cdots : x_n]$ we send $(x_0, \ldots, x_n)$ to x_0. Note that this is well-defined, since multiplying all the x_i's by a scalar λ yields the same homomorphism $L_x \to \mathbb{C}$.

The map f is exact on all fibers except those where $x_0 = 0$. The complex $0 \to L \to 1 \to 0$ is a resolution of the structure sheaf for $\mathbb{C}P^{n-1}$, and hence $1 - L$ represents the associated fundamental class in K-theory by Theorem 21.10.

Our second explanation takes place entirely in the topological world. The key fact is that the normal bundle to $\mathbb{C}P^{n-1}$ inside $\mathbb{C}P^n$ is L^*. Let U be a tubular neighborhood of $\mathbb{C}P^{n-1}$, and consider the chain of isomorphisms

$$K^0(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^{n-1}) \xrightarrow{\cong} K^0(U, U - \mathbb{C}P^{n-1}) \cong K^0(N, N-0).$$

The relative fundamental class $[\mathbb{C}P^{n-1}]_{rel}$ is the unique class that maps to the Thom class $\mathcal{U}_N$ under the above isomorphisms. But N is a line bundle, and recall from (18.23) that the Thom class is then the Koszul complex $[J^*] = [\pi^* N^* \to 1]$ where $\pi\colon N \to \mathbb{C}P^{n-1}$. A little thought shows that the complex $0 \to \pi^* N^* \to 1 \to 0$ is exactly the restriction of $0 \to L \to 1 \to 0$ on $\mathbb{C}P^n$. So this latter complex represents $[\mathbb{C}P^{n-1}]_{rel}$, and therefore $1 - L$ equals $[\mathbb{C}P^{n-1}]$.

The second set of statements in the proposition follow directly from Proposition 23.3 and Bott periodicity, since those give $K^0(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$ where $x = [\mathbb{C}P^{n-1}]$ and $x^k = [\mathbb{C}P^{n-1}]^k = [\mathbb{C}P^{n-k}]$. $\square$

Example 23.7. Note that it follows from the above lemma that $(1-L)^{n+1} = 0$ in $K^0(\mathbb{C}P^n)$. This relation comes up in many contexts, and it is useful to have a different perspective on it. Let $R = \mathbb{C}[x_0, \dots, x_n]$ and consider the Koszul complex for the regular sequence $x_0, \dots, x_n$. It has the form

$$0 \to R(-(n+1)) \to \cdots \to R(-2)^{\binom{n+1}{2}} \to R(-1)^{n+1} \to R \to 0,$$

and we know from Theorem 18.25 that this complex is exact except in degree zero where it has $H_0 = R/(x_0, \dots, x_n)$. This implies that if one takes the corresponding complex of vector bundles on $\mathbb{C}P^n$ and pulls back along $\mathbb{A}^{n+1} \to \mathbb{C}P^n$ then the resulting complex of bundles is exact except over the origin. So the original sequence of bundles on $\mathbb{C}P^n$ is exact, which tells us that

$$0 = 1 - (n+1)L + \tbinom{n+1}{2}L^2 - \cdots + (-1)^{n+1}L^{n+1}$$

in $K^0(\mathbb{C}P^n)$. Of course the expression on the right is precisely $(1-L)^{n+1}$.

As an alternative perspective, the above Koszul complex is the tensor product

$$(L \xrightarrow{x_0} \underline{1}) \otimes (L \xrightarrow{x_1} \underline{1}) \otimes \cdots \otimes (L \xrightarrow{x_n} \underline{1}).$$

A tensor product of complexes of vector spaces is exact as long as any factor is exact, and since any point $[x_0 : \cdots : x_n] \in \mathbb{C}P^n$ has some $x_i \neq 0$ we deduce that our complex of bundles is exact over every point.

We next compute the K-theoretic fundamental classes in a couple of simple examples:

Example 23.8. Let $Z = V(f) \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Let $R = \mathbb{C}[x_0, \dots, x_n]$. The homogeneous coordinate ring of Z is $R/(f)$, which has the short free resolution given by

$$0 \to R(-d) \xrightarrow{\cdot f} R \to R/(f) \to 0.$$

So $[Z] = 1 - L^d$ in $K^0(\mathbb{C}P^n)$. To write $[Z]$ as a linear combination of the $[\mathbb{C}P^{n-i}]$'s we need to write $1 - L^d$ in terms of powers of $X = 1 - L$. This is easy, of course:

$$\begin{aligned}[Z] = 1 - L^d = 1 - [1 - (1-L)]^d &= 1 - (1-X)^d \\ &= 1 - [1 - dX + \tbinom{d}{2}X^2 - \cdots + (-1)^d X^d] \\ &= dX - \tbinom{d}{2}X^2 + \tbinom{d}{3}X^3 - \cdots\end{aligned}$$

So we find that the higher d_i's for a hypersurface are all just (up to sign) binomial coefficients of d. This is somewhat disappointing, as we are not seeing new topological invariants—it is just the degree over and over again, encoded in different ways. This is not actually a surprise, though: it is known that all hypersurfaces of the same degree are actually diffeomorphic. See [La, 2.3.2], for example.

Things become more interesting in the next example:

Example 23.9. Consider $Z = V(f, g)$ where f, g is a regular sequence of homogeneous elements in $R = \mathbb{C}[x_0, \dots, x_n]$. Let $d = \deg(f)$ and $e = \deg(g)$. Because f, g is a regular sequence, $R/(f, g)$ is resolved by the Koszul complex:

$$0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/(f, g) \to 0.$$

We can now calculate

$$\begin{aligned}[Z]_K &= \Big[\, 0 \longrightarrow L^{d+e} \longrightarrow L^d \oplus L^e \longrightarrow 1 \longrightarrow 0 \,\Big] \\ &= 1 - (L^d + L^e) + L^{d+e} \\ &= 1 - (1-X)^d - (1-X)^e + (1-X)^{d+e} \\ &= \Big[\tbinom{d+e}{2} - \tbinom{d}{2} - \tbinom{e}{2}\Big] X^2 - \Big[\tbinom{d+e}{3} - \tbinom{d}{3} - \tbinom{e}{3}\Big] X^3 + \cdots \\ &= deX^2 + \tfrac{1}{2}de(2-d-e)X^3 + \ldots .\end{aligned}$$

The classical degree of Z is de, but our 'higher invariants' now see more than just this number. In fact, knowing the expansion of $[Z]_K$ as a linear combination of the X^i's implies that we know de and $de(2-d-e)$, which implies that we know $d+e$. But if we know de and $d+e$ then we know the polynomial $(\xi-d)(\xi-e) = \xi^2-(d+e)\xi+de$, which means we know its roots. So knowing the expansion of $[Z]_K$ is the same as knowing d and e. This example shows that the K-theoretic fundamental class sees more topological information than the singular cohomology fundamental class does.

Now that we have seen these simple examples we can return to our main question. Given $Z \hookrightarrow \mathbb{C}P^n$ of codimension c, how does one compute the d_i's in the equation

$$[Z]_K = (\deg Z)[\mathbb{C}P^{n-c}] + d_{c+1}[\mathbb{C}P^{n-c-1}] + d_{c+2}[\mathbb{C}P^{n-c-2}] + \cdots$$

And what do these d_i's mean in terms of geometry? We will soon see that one answer is given by the Hilbert polynomial.

23.10. **Review of the Hilbert polynomial.** Let $R = \mathbb{C}[x_0, \ldots, x_n]$, and regard this as a graded ring where each x_i has degree one. If M is a graded R-module write M_s for the graded piece in degree s. The **Poincaré series** of M is the formal Laurent series

$$P_M(\xi) = \sum_{s=-\infty}^{\infty} (\dim_{\mathbb{C}} M_s)\xi^s \in \mathbb{Z}[[\xi]]\,[\xi^{-1}]$$

(defined if M is finitely-generated over R, so that all the M_s are finite-dimensional). Note that this is evidently an additive invariant of finitely-generated, graded modules: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence then clearly $P_M(\xi) = P_{M'}(\xi) + P_{M''}(\xi)$. We may therefore regard the Poincaré series as a map of abelian groups

$$P\colon G_{grd}(R) \to \mathbb{Z}[[\xi]]\,[\xi^{-1}]$$

where $G_{grd}(R)$ is the Grothendieck group of finitely-generated *graded* modules over R.

We will calculate the Poincaré series of each $R(-k)$, but for this we need the following useful calculation:

Lemma 23.11. *The number of monomials of degree d in the variables $z_1, \ldots, z_n$ is equal to $\binom{d+n-1}{d}$.*

Proof. Monomials are in bijective correspondence with patterns of "dashes and slashes" that look like

$$- - -/- -//- - - -/- .$$

The above pattern corresponds to the monomial $z_1^3 z_2^2 z_4^4 z_5$, and from this the general form of the bijection should be clear. Monomials of degree d will correspond to patterns with d dashes, and if there are n variables then there will be $n-1$ slashes.

So we need to count patterns where there will be $d+n-1$ total symbols, of which d are dashes: the number of these is $\binom{d+n-1}{d}$. $\square$

It is an immediate consequence that $P_R(\xi) = \sum_{s\geq 0}\binom{s+n}{n}\xi^s$. For $R(-k)$ we simply shift the coefficients and obtain

$$P_{R(-k)}(\xi) = \sum_{s\geq k}\binom{s+n-k}{n}\xi^s = \xi^k P_R(\xi).$$

The power series $\{\xi^k P_R(\xi)\,|\,k\in\mathbb{Z}\}$ are obviously linearly independent over $\mathbb{Q}$, which shows that $G_{grd}(R)$ has infinite rank as an abelian group.

Proposition 23.12. *$G_{grd}(R)$ is isomorphic to $\mathbb{Z}^\infty$, with free basis the set of rank one, free modules $\{[R(-d)]\,|\,d\in\mathbb{Z}\}$.*

Proof. The key is the Hilbert Syzygy Theorem. Consider the diagram

$$\begin{tikzcd} \mathbb{Z}^\infty \arrow[r] \arrow[dr] & G_{grd}(R) \arrow[d, "P"] \\ & \mathbb{Z}[[\xi]]\,[\xi^{-1}] \end{tikzcd}$$

where the horizontal map sends the ith basis element to $[R(-i)]$ and the diagonal map is the composite. We have already seen that this composite is injective, because there is no $\mathbb{Z}$-linear relation amongst the images of the basis elements. So $\mathbb{Z}^\infty \to G_{grd}(R)$ is injective, and it only remains to prove surjectivity. But if M is any finitely-generated, graded R-module then the Syzygy Theorem guarantees a finite, graded, free resolution

$$0\to F_n\to F_{n-1}\to\cdots\to F_1\to F_0\to M\to 0.$$

So $[M]=\sum_i(-1)^i[F_i]$, and each F_i is a sum of elements $[R(-k)]$. This proves surjectivity. $\square$

The Hilbert polynomial is a variant of the Poincaré series that keeps track of less information. At first this might seem to be a bad thing, but we will see that what it does is give us a closer connection to geometry and topology. Here is the main result that gets things started:

Proposition 23.13. *Let M be a finitely-generated graded module over $R = \mathbb{C}[x_0,\ldots,x_n]$. Then there exists a unique polynomial $H_M(s)\in\mathbb{Q}[s]$ that agrees with the function $s\mapsto \dim M_s$ for $s\gg 0$. One has $\deg H_M(s)\leq n$. The polynomial $H_M(s)$ is called the **Hilbert polynomial** of M.*

Proof. Consider first the case $M=R$. A basis for M_s consists of all monomials in $x_0,\ldots,x_n$ of degree s, which Lemma 23.11 calculates to be $\binom{n+s}{s}=\binom{n+s}{n}$. This is a polynomial in s of degree n. Next consider $M=R(-k)$. The function $s\mapsto M_s$ is zero for $s<k$, and then for $s\geq k$ it coincides with $\binom{n+s-k}{n}$. This is again a polynomial in s of degree n.

Finally, consider the case of a general M. By the Hilbert Syzygy Theorem M has a finite, graded, free resolution

$$0\to F_n\to F_{n-1}\to\cdots\to F_1\to F_0\to M\to 0.$$

It follows that

$$\dim M_s = \dim(F_0)_s - \dim(F_1)_s + \cdots + (-1)^n\dim(F_n)_s.$$

But each F_i is a direct sum of finitely-many $R(-k)$'s, and so $s \mapsto \dim(F_i)_s$ has been shown to agree with a polynomial in s of degree at most n, for $s \gg 0$. The desired result follows at once. $\square$

The following calculation was given in the above proof, but we record it below because it comes up so often:

Corollary 23.14. *When $R = \mathbb{C}[x_0, \dots, x_n]$ and $k \in \mathbb{Z}$ the Hilbert polynomial for $R(-k)$ is $\binom{s+n-k}{n}$.*

Example 23.15. Consider a hypersurface $M = R/(f)$, where $f \in R$ is homogeneous of degree d. We then have the resolution

$$0 \longrightarrow R(-d) \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0$$

from which we find

$$H_{R/(f)} = H_R - H_{R(-d)} = \tbinom{s+n}{n} - \tbinom{s+n-d}{n} = \tfrac{(s+n)\cdots(s+1)}{n!} - \tfrac{(s+n-d)\cdots(s-d+1)}{n!}.$$

Note that the two binomial coefficients have leading terms $s^n/n!$, which therefore cancel. The coefficient of s^{n-1} is

$$\tfrac{1}{n!} \cdot \left[\tfrac{n(n+1)}{2} - \tfrac{(n-2d+1)n}{2} \right] = \tfrac{d}{(n-1)!}.$$

Note that the degree of $H_{R/(f)}$ is one less than the Krull dimension of $R/(f)$, and the leading coefficient is the degree of f (the geometric degree of the hypersurface) divided by $(n-1)!$. These are general facts that hold for any module: the degree of H_M is one less than the Krull dimension of M, and the leading coefficient times $(\deg H_M)!$ is always an integer—this integer is called the **multiplicity** of the module M. Proofs can be found in most commutative algebra texts.

The Hilbert polynomial is an additive invariant of finitely-generated modules: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence then clearly $H_M(s) = H_{M'}(s) + H_{M''}(s)$. We may therefore regard the Hilbert polynomial as a map of abelian groups

$$\mathrm{Hilb}\colon G_{grd}(R) \to \mathbb{Q}[s].$$

This map is clearly not injective, as it kills any module that is finite-dimensional as a $\mathbb{C}$-vector space. The subgroup of $G_{grd}(R)$ generated by such modules is $A = \langle [\mathbb{C}(-d)] \,\big|\, d \in \mathbb{Z} \rangle$, where $\mathbb{C}$ as an R-module is always interpreted as $R/(x_0, \dots, x_n)$. We may regard Hilb as a map

$$\mathrm{Hilb}\colon G_{grd}(R)/A \to \mathbb{Q}[s].$$

The domain of this map is calculated as follows:

Proposition 23.16. *The group $G_{grd}(R)/A$ is isomorphic to $\mathbb{Z}^{n+1}$, with free basis $R, R(-1), R(-2), \dots, R(-n)$. The map $\mathrm{Hilb}\colon G_{grd}(R)/A \to \mathbb{Q}[s]$ is an injection.*

Proof. Let B be the subgroup of $G_{grd}(R)/A$ generated by $[R], [R(-1)], \dots, [R(-n)]$. We will show that B is equal to the whole group.

The module $\mathbb{C} = R/(x_0, \dots, x_n)$ is resolved by the Koszul complex $K(x_0, \dots, x_n; R)$. This gives the relation in $G_{grd}(R)$ of

$$[\mathbb{C}] = [R] - (n+1)[R(-1)] + \tbinom{n+1}{2}[R(-2)] + \cdots + (-1)^{n+1}[R(-n-1)].$$

In $G_{grd}(R)/A$ we therefore have $[R(-n-1)] \in B$. For $d > 0$ we can tensor the Koszul complex with $R(-d)$ and then apply the same argument to find that

$[R(-n-1-d)] \in \langle [R(-d)], \dots, [R(-n-d)] \rangle \subseteq \langle [R], [R(-1)], \dots, [R(-n-d)] \rangle$. An inductive argument now shows that $[R(-k)] \in B$ for every $k \geq n+1$.

A similar induction works for $d < 0$ to show that $[R(-d)] \in B$ for all $d \in \mathbb{Z}$. In other words, $B = G_{grd}(R)/A$.

Now consider the map $\mathbb{Z}^{n+1} \to G_{grd}(R)/A$ that sends the ith basis element e_i to $[R(-i)]$ for $0 \leq i \leq n$. We have just shown that this map is surjective. Consider, then, the composite

$$\mathbb{Z}^{n+1} \longrightarrow G_{grd}(R)/A \xrightarrow{\text{Hilb}} \mathbb{Q}[s].$$

The images of our basis elements are the polynomials

$$\tbinom{s+n}{n},\ \tbinom{s+n-1}{n},\ \tbinom{s+n-2}{n},\ \dots,\ \tbinom{s}{n}.$$

Evaluating these polynomials at $s = 0$ gives the sequence $1, 0, 0, \dots, 0$. Evaluating at $s = 1$ gives $n+1, 1, 0, 0, \dots, 0$, and so forth. For $s = i$ the ith polynomial in the list evaluates to 1 and all the subsequent polynomials evaluate to 0. This proves that these polynomials are linearly independent over $\mathbb{Z}$, hence the above composite map is injective. So $\mathbb{Z}^{n+1} \to G_{grd}(R)/A$ is injective, and therefore is an isomorphism. Moreover, the map Hilb is injective. $\square$

The reader might suspect that the $\mathbb{Z}^{n+1}$ in the above result is really an incarnation of the group $K^0(\mathbb{C}P^n)$. This is the foundation for what we do in the next section.

Exercise 23.17. We will not need this, but it is a cute fact: Prove that the image of Hilb: $G(R)/A \to \mathbb{Q}[s]$ equals the set of polynomials $f(s) \in \mathbb{Q}[s]$ having the property that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. That is, the image consists of all rational polynomials that take integer values on integers. (See Section 24 if you get stuck.)

23.18. **K-theory and the Hilbert polynomial.** We now explain how the Hilbert polynomial encodes the same information as the K-theoretic fundamental class.

Given $Z \hookrightarrow \mathbb{C}P^n$ a smooth subvariety of codimension c, recall that we have the fundamental class $[Z] \in K^0(\mathbb{C}P^n)$ and that we may write

$$[Z] = d_c[\mathbb{C}P^{n-c}] + d_{c+1}[\mathbb{C}P^{n-c-1}] + \cdots + d_n[\mathbb{C}P^0].$$

We know that $d_c = \deg(Z)$, and our goal is to understand how to compute the higher d_i's.

Proposition 23.6 says that $[\mathbb{C}P^{n-k}] = (1-L)^k$ and that these classes for $k = 0, 1, \dots, n$ give a basis for $K^0(\mathbb{C}P^n)$. Evidently one can also use the basis $1, L, L^2, \dots, L^n$. We next introduce an algebraic analogue of $K^0(\mathbb{C}P^n)$. Let $R = \mathbb{C}[x_0, \dots, x_n]$. Take the Grothendieck group $K^0_{grd}(R)$ of finitely-generated, graded, projective R-modules (or equivalently, chain complexes of such modules) and quotient by the subgroup $\tilde{A}$ generated by all complexes $K(x_0, \dots, x_n; R) \otimes R(-d)$ for $d \in \mathbb{Z}$. We obtain a diagram

(23.19)
$$\begin{tikzcd} K^0_{\mathrm{grd}}(R)/\tilde{A} \arrow[r, "\phi", "\cong"'] \arrow[d, "\cong"] & K^0(\mathbb{C}P^n) \\ G_{grd}(R)/A \arrow[r, tail, "\text{Hilb}"] & \mathbb{Q}[s] \end{tikzcd}$$

where ϕ sends $[R(-d)]$ to L^d and where the vertical map is our usual 'Poincaré Duality' isomorphism, in this case sending the class of a projective module to the class of the same module in $G_{grd}(R)/A$. The map ϕ is an isomorphism by inspection: we have computed $G_{grd}(R)/A$ and $K^0(\mathbb{C}P^n)$, both are $\mathbb{Z}^{n+1}$, and ϕ clearly maps a basis to a basis.

Given $Z = V(I)$, we know by Theorem 21.10 that its fundamental class $[Z] \in K^0(\mathbb{C}P^n)$ is represented by a finite, graded, free resolution $F_\bullet \to R/I \to 0$. This resolution (or the alternating sum of its terms) lifts to $K^0_{grd}(R)/\tilde{A}$, and pushing this around the diagram into $\mathbb{Q}[s]$ just gives us $\mathrm{Hilb}_{R/I}(s)$. So the above diagram shows that knowing $\mathrm{Hilb}_{R/I}(s)$ is the same as knowing $[Z]$.

To say something more specific here, consider first the case $Z = \mathbb{C}P^{n-k}$. Then $R/I = R/(x_0, \dots, x_{k-1})$, which as a graded ring is just $\mathbb{C}[x_k, \dots, x_n]$. By Corollary 23.14 the Hilbert polynomial is then

$$\mathrm{Hilb}_{\mathbb{C}P^{n-k}}(s) = \tbinom{s+n-k}{n-k}.$$

So pushing our basis $[\mathbb{C}P^n], [\mathbb{C}P^{n-1}], \dots, [\mathbb{C}P^0]$ around diagram (23.19) into $\mathbb{Q}[s]$ yields the polynomials

$$\tbinom{s+n}{n},\ \tbinom{s+n-1}{n-1},\ \tbinom{s+n-2}{n-2},\ \dots,\ \tbinom{s}{0}.$$

If $Z = V(I) \hookrightarrow \mathbb{C}P^n$ is now arbitrary, then writing

$$\mathrm{Hilb}_{R/I}(s) = d_0\tbinom{s+n}{n} + d_1\tbinom{s+n-1}{n-1} + d_2\tbinom{s+n-2}{n-2} + \dots$$

implies that $[Z] = d_0[\mathbb{C}P^n] + d_1[\mathbb{C}P^{n-1}] + d_2[\mathbb{C}P^{n-2}] + \dots$. In other words, one obtains the expansion of $[Z]$ as a linear combination of the $[\mathbb{C}P^{n-i}]$'s by writing $\mathrm{Hilb}_Z(s)$ as a linear combination of the above-listed binomial functions. We record this fact for future reference:

Proposition 23.20. *Let $Z \hookrightarrow \mathbb{C}P^n$ be a smooth subvariety and let $I \subseteq R = \mathbb{C}[x_0, \dots, x_n]$ be the corresponding ideal of functions vanishing on Z. Then $\mathrm{Hilb}_{R/I}(s) = \sum_{i=0}^n d_i\binom{s+n-i}{n-i}$ if and only if $[Z]_K = \sum_{i=0}^n d_i[\mathbb{C}P^{n-i}]$.*

We have shown how to calculate the d_i's from the ideal I of equations defining Z: decompose the Hilbert polynomial into a sum of terms $\binom{s+n-i}{n-i}$, and take the resulting coefficients. This is still not exactly a 'geometric' description of the d_i's, although it is at least a description that takes place in the realm of algebraic geometry. We will get another perspective on this material via the Todd genus and the Grothendieck-Riemann-Roch Theorem, studied in Section 28. See especially Section 28.28.

24. Interlude on the calculus of finite differences

If we regard calculus as a set of tools built up from the study of $\frac{f(x+h)-f(x)}{h}$ where h is an infinitesimal, the calculus of finite differences is the analogous set of tools that come into play when h is a finite (non-infinitesimal) object. This is a classical subject whose roots run deep, though it is not exactly a standard part of modern lore. In this section we will discuss the theory as it applies to polynomials of one variable, and for us we will always have $h = 1$. It is somewhat of a surprise that this subject is so relevant to K-theory.

For a classical text on the calculus of finite differences see [MiT]. A quick introduction similar in sprit to what we do here can be found in [Sta, Section 1.9].

24.1. **The finite difference and sum operators.** If $f \in \mathbb{Q}[t]$ let Δf be the polynomial given by

$$(\Delta f)(t) = f(t+1) - f(t).$$

For example, $\Delta(t^2) = (t+1)^2 - t^2 = 2t+1$, and $\Delta(t^3) = (t+1)^3 - t^3 = 3t^2+3t+1$. Note that $\Delta\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$ is a linear map, and that it lowers degrees by one. We call Δ the **finite difference** operator, and we regard it as an analog of the familiar differentiation operator $D\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$.

The opposite of differentiation is integration, and there is an analogous operator that is the opposite of Δ. If $f \in \mathbb{Q}[t]$ define Sf to be the function $\mathbb{Z} \to \mathbb{Q}$ given by

$$(Sf)(t) = f(0) + f(1) + f(2) + \cdots + f(t-1).$$

For example,

$$(St)(t) = 0 + 1 + 2 + \cdots + (t-1) = \tbinom{t}{2} = \tfrac{t(t-1)}{2}$$

and

$$(St^2)(t) = 0^2 + 1^2 + \cdots + (t-1)^2 = \tfrac{(t-1)t(2t-1)}{6}.$$

(Note that in these formulas we are mixing the roles of t as a formal variable and t as a parameter, but this should not cause too much confusion.) One should think of the formula for $(Sf)(t)$ as giving a finite Riemann sum, based on intervals of width 1. It is not immediately clear that Sf is always a polynomial, nor that it raises degrees by one, but we will prove these things shortly. The following two facts are easy, though:

$$\text{(1)} \qquad \Delta S(f) = f \qquad\qquad D\textstyle\int(f) = f$$

$$\text{(2)} \qquad S\Delta(f) = f - f(0) \qquad\qquad \textstyle\int D(f) = f - f(0).$$

We have written each identity paired with the corresponding identity for classical differentiation/integration. Note that, like the usual integral, Sf will always have zero as its constant term since $(Sf)(0) = 0$ by definition.

Derivatives and integrals of polynomials are easy to compute because their values on the basic polynomials t^n are very simple. In fact, the point is really that the operators D and $\int$ act very simply on the sequence of polynomials

$$1, \quad t, \quad \tfrac{t^2}{2}, \quad \tfrac{t^3}{3!}, \quad \tfrac{t^4}{4!}, \quad \ldots$$

Of course D carries each polynomial in the sequence to the preceding one, and $\int$ carries each polynomial to the subsequent one. In contrast, the operators Δ and S are *not* very well-behaved on this sequence. It is better to use the sequence

$$1, \quad S(1), \quad S^2(1), \quad S^3(1), \quad \ldots$$

so let us compute these. It is easy to see that $S(1) = t$, and we previously saw that $S^2(1) = S(t) = \binom{t}{2}$. The following useful lemma implies that $S\binom{t}{k} = \binom{t}{k+1}$, where the binomial coefficient stands for the polynomial $\frac{1}{k!}t(t-1)(t-2)\cdots(t-k+1)$. And so starting with $1 = \binom{t}{0}$ we get $S^k(1) = \binom{t}{k}$,

Lemma 24.2. *For any $d, k \in \mathbb{Z}$ with $k \geq 0$ one has*

$$\Delta\binom{t+d}{k} = \binom{t+d}{k-1}, \qquad S\binom{t+d}{k} = \binom{t+d}{k+1} - \binom{d}{k+1}.$$

Proof. The statement about Δ follows from Pascal's Identity. Applying S to this equation gives

$$S\binom{t+d}{k-1} = S\Delta\binom{t+d}{k} = \binom{t+d}{k} - \binom{d}{k}$$

and changing k to $k+1$ gives the desired identity for S. □

Remark 24.3. The second statement in Lemma 24.2 is equivalent to the identity

$$\binom{d}{k+1} + \binom{d}{k} + \binom{d+1}{k} + \cdots + \binom{d+t-1}{k} = \binom{t+d}{k+1}.$$

For fun let us give a combinatorial proof of this. Imagine $t+d$ slots labelled 1 through $t+d$ where we are to place $k+1$ asterisks:

$$\underline{\quad}\ \underline{\ *\ }\ \underline{\quad}\ \underline{\ *\ }\ \underline{\ *\ }\ \underline{\quad}\ \underline{\ *\ }$$

We can sort these by where the rightmost asterisk is located. There are $\binom{d}{k+1}$ choices where the rightmost is in the first d slots. If the rightmost asterisk is in slight $d+1$, there are $\binom{d}{k}$ choices for placing the others. And more generally, there are $\binom{d+i-1}{k}$ configurations having the rightmost asterisk in spot $d+i$. Summing over $i \in [1, t]$ yields the desired formula.

Exercise 24.4. Pascal's triangle contains the binomial coefficients $\binom{n}{k}$ with $n \geq 0$ and $0 \leq k \leq n$. Extending the picture to include all $0 \leq k$ just adds an infinite trail of zeros to the right side of each row. Write down the first few rows of Pascal's triangle, then use Pascal's identity to work backwards to fill in the numbers $\binom{n}{k}$ for $n \leq 0$ and $k \geq 0$ (should this be called "Pascal's half-plane"?). In some ways this is a silly exercise, but it is worth doing this at least once in your life to watch the infinite sequences create themselves out of nothing.

The sequence of polynomials

$$1 = \binom{t}{0}, \quad \binom{t}{1}, \quad \binom{t}{2}, \quad \binom{t}{3}, \quad \ldots$$

is clearly a basis for $\mathbb{Q}[t]$, as the kth term has degree equal to k. In relation to the Δ and S operators, this basis plays the role classically taken by the polynomials $\frac{t^d}{d!}$ for D and $\int$. It is now clear that S applied to a polynomial of degree d yields a polynomial of degree $d+1$: a polynomial of degree d is a linear combination of $\binom{t}{d}, \binom{t}{d-1}, \ldots, \binom{t}{0}$ with the coefficient on the first term nonzero. Applying S changes each $\binom{t}{k}$ to $\binom{t}{k+1}$, and it is clear that this yields a polynomial of degree $d+1$.

For another example of the use of this binomial basis, note the following analog of the Taylor–Maclaurin explansion for writing polynomials in this basis:

Proposition 24.5. *Let $f \in \mathbb{Q}[t]$. Then*

$$f = \sum_{k=0}^{\infty} (\Delta^k f)(0) \cdot \binom{t}{k}.$$

Proof. We know $f = \sum_{k=0}^{N} a_k \binom{t}{k}$ for some N and some values $a_k \in \mathbb{Q}$. Plugging in $t = 0$ immediately gives $f(0) = a_0$. Now apply Δ to get $\Delta f = \sum_{k=0}^{N-1} a_{k+1}\binom{t}{k}$ and again plug in $t = 0$: this yields $(\Delta f)(0) = a_1$. Continue. □

Exercise 24.6. Let $\mathbb{Q}[t]_{int} \subseteq \mathbb{Q}[t]$ be the set of all polynomials $f(t)$ such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. Note that $\mathbb{Q}[t]_{int}$ is stable under Δ and S, and use this to prove that $\mathbb{Q}[t]_{int}$ is the $\mathbb{Z}$-linear span of the polynomials $\binom{t}{k}$, $k \geq 0$.

In contrast to the large number of similarities of the pair of operators (Δ, S) to $(D, \int)$, there is an important difference when it comes to the product rule. Of course we have $D(fg) = (Df)g + f(Dg)$, but one readily checks that this does **not** work for Δ. The correct rule is as follows:

Lemma 24.7. *For any $f, g \in \mathbb{Q}[t]$ one has $\Delta(fg) = (\Delta f)g + f(\Delta g) + (\Delta f)(\Delta g)$.*

Proof. A simple calculation, left to the reader. □

In the present section we will not have much use for this product rule, but it is a very important formula whose significance will become larger in subsequent sections. See Proposition 25.14 for a brief hint at this and Section 31 for a more thorough discussion.

24.8. **Translating between (Δ, S) and $(D, \int)$.** If $f \in \mathbb{Q}[t]$ and h is any integer (or even better, a formal variable) we have the Taylor fomula

$$f(t+h) = \sum_{k=0}^{\infty} f^{(k)}(t) \cdot \tfrac{h^k}{k!} = \sum_{k=0}^{\infty} (D^k f)(t) \cdot \tfrac{h^k}{k!}.$$

Note that the sum is really finite, since large enough derivatives of f are all zero. Taking $h = 1$ and rearranging somewhat we get

$$f(t+1) = \Big(\sum_{k=0}^{\infty} \tfrac{D^k}{k!}\Big) f = (e^D)f.$$

Let us be clear about what this means. The expression $\sum_{k=0}^{\infty} \frac{D^k}{k!}$ makes perfect sense as an *operator*, since evaluating this at any fixed polynomial gives a well-defined answer. It is sensible to denote this operator as e^D.

Taking one more step, we can write $(\Delta f)(t) = f(t+1) - f(t) = (e^D - 1)f$ where 1 denotes the identity operator. Or even more compactly,

$$\Delta = e^D - 1$$

is an identity of operators on $\mathbb{Q}[t]$. This identity gives us Δ as a linear combination of iterates of D.

We are next going to cook up a similar formula for the operator S, but this is a little harder. One does not wish for a formula using higher and higher powers of $\int$, as these operators become more and more complicated. Instead, we say to ourselves that S is very close to being an inverse for Δ (it is a right inverse, but not a left inverse). If it *were* an inverse we might want to write $S = \frac{1}{e^D - 1}$, but it is difficult to make sense of the latter expression as an operator—the trouble is that $e^D - 1$ has no constant term, otherwise we could expand as a power series. While this didn't work, we *can* make sense of the operator

$$\mathcal{B} = \tfrac{D}{e^D - 1}. \tag{24.9}$$

By this we mean write $\frac{x}{e^x-1} = \sum a_k x^k = 1 - \frac{x}{2} + \frac{x^2}{12} - \cdots$ as a formal power series, and set $\mathcal{B}$ equal to $\sum_{k=0}^{\infty} a_k D^k$. It follows purely formally that $\Delta \mathcal{B} f = Df$ for any polynomial f, and by applying S to both sides we find that

$$\mathcal{B}f - (\mathcal{B}f)(0) = SDf.$$

Replacing f by $\int f$ in the above formula, we get

$$Sf = SD(\textstyle\int f) = \mathcal{B}(\int f) - (\mathcal{B}\int f)(0).$$

Note that $\mathcal{B}(\int f)$ is equal to $\int f - \frac{1}{2}f + \frac{1}{12}Df - \cdots$. So this is as computable as the series $\frac{x}{e^x-1}$; the coefficients of this series are related to the Bernoulli numbers (see Appendix C). We have

$$\mathcal{B} = \sum_{k=0}^{\infty} \tfrac{B_k}{k!} D^k$$

where the B_k's are the Benoulli numbers.

24.10. **Sums of powers and Bernoulli numbers.** The (Δ, S) pair of operators is useful in a variety of mathematical situations. In a moment we will see an application to determining K-theoretic fundamental classes, but let us first look at a non-topological example. Almost every math student has seen the formulas

$$1 + 2 + \cdots + n = \tfrac{n(n+1)}{2} \quad \text{and} \quad 1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6}.$$

The proof of such formulas by mathematical induction is a common exercise in elementary proof courses. It turns out that there are also formulas for higher powers, of the form

$$1^k + 2^k + \cdots + n^k = p_k(n)$$

where $p_k(n)$ is a polynomial of degree $k+1$. How does one discover the appropriate polynomials? This seems to have first been done by Jacob Bernoulli, the coefficients in these polynomials being closely related to what are now called Bernoulli numbers.

Clearly we may rephrase the problem as that of computing $S(t^k)$, the exact connection being $S(t^k) = p_k(t-1)$. In the last section we developed a formula for S in terms of the usual derivative and integral operators, and we will now use that; but here it is easiest to use it in the form

$$SD(f) = \mathcal{B}f - (\mathcal{B}f)(0) = \left(\sum_{j=0}^{\infty} \tfrac{B_j}{j!} D^j\right) f - (\text{constant term of preceding expression}).$$

We obtain

$$\begin{aligned} S(t^k) = SD\Big(\tfrac{t^{k+1}}{k+1}\Big) &= \tfrac{1}{k+1} \sum_{j=0}^{\infty} \tfrac{B_j}{j!} D^j(t^{k+1}) - (\text{constant term}) \\ &= \tfrac{1}{k+1} \sum_{j=0}^{k} \tfrac{B_j}{j!} (k+1)(k)(k-1)\cdots(k+1-(j-1)) t^{k+1-j} \\ &= \tfrac{1}{k+1} \sum_{j=0}^{k} B_j \cdot \tbinom{k+1}{j} t^{k+1-j}. \end{aligned}$$

Note that the last expression is a polynomial in t, with all of its coefficients clearly visible.

Finally, recalling that $S(t^k) = p_k(t-1)$ we conclude that

$$1^k + 2^k + \cdots + n^k = p_k(n) = S(t^k)\big|_{t=n+1} = \tfrac{1}{k+1}\sum_{j=0}^{k} B_j \cdot \tbinom{k+1}{j}(n+1)^{k+1-j}.$$

Exercise 24.11. Check that the above formula gives the familiar identities in the cases $k = 1$ and $k = 2$, and then see what it gives when $k = 3$. Compare the above formula to (C.3).

24.12. **Back to K-theoretic fundamental classes.** Let $Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface defined by the homogeneous ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$. We saw in Section 23.18 that the coefficients in

$$[Z]_K = d_0[\mathbb{C}P^n]_K + d_1[\mathbb{C}P^{n-1}]_K + d_2[\mathbb{C}P^{n-2}]_K + \cdots$$

are the same as the coefficients in

$$\mathrm{Hilb}_Z(t) = d_0\tbinom{n+t}{n} + d_1\tbinom{n+t-1}{n-1} + d_2\tbinom{n+t-2}{n-2} + \cdots \tag{24.13}$$

The sequence of polynomials $\binom{n+t}{n}, \binom{n+t-1}{n-1}, \binom{n+t-2}{n-2}, \ldots$ is not quite the standard basis of binomial coefficients we used above, but it is close. Here is a small lemma about expanding polynomials in this basis (compare to Proposition 24.5):

Lemma 24.14. *If $f \in \mathbb{Q}[t]$ then $f = \sum_{k=0}^{\infty}(\Delta^k f)(-k-1) \cdot \binom{t+k}{k}$.*

Proof. The collection of polynomials $\binom{t+k}{k}$ for $k \geq 0$ is clearly a basis for $\mathbb{Q}[t]$ for degree reasons. Write $f = \sum c_k\binom{t+k}{k} = c_0\binom{t}{0} + c_1\binom{t+1}{1} + c_2\binom{t+2}{2} + \cdots$. Plugging in $t = -1$ immediately gives $f(-1) = c_0$. Apply Δ to both sides to get $\Delta f = \sum c_k\binom{t+k}{k-1} = c_1 + c_2\binom{t+2}{1} + \cdots$. Now plugging in $t = -2$ makes all the expressions vanish except the first, so $c_1 = (\Delta f)(-2)$. Continue in this way. □

The following corollary is immediate, by applying the above lemma to (24.13):

Corollary 24.15. *We have $[Z]_K = \sum_i d_i[\mathbb{C}P^{n-i}]_K$ where the coefficients are given by $d_i = (\Delta^{n-i}\,\mathrm{Hilb}_Z)(-n+i-1)$. Alternatively, the coefficient of $[\mathbb{C}P^j]_K$ in $[Z]_k$ is $(\Delta^j\,\mathrm{Hilb}_Z)(-j-1)$.*

Example 24.16. Let $Z \hookrightarrow \mathbb{C}P^n$ be a hypersurface of degree d, and write $Z = V(f)$. Then $R/(f)$ is resolved by $0 \to R(-d) \to R$ where the map is mutiplication by f. So

$$\mathrm{Hilb}_Z = \mathrm{Hilb}_R - \mathrm{Hilb}_{R(-d)} = \tbinom{n+t}{n} - \tbinom{n+t-d}{n}.$$

But then

$$\Delta^{n-i}\,\mathrm{Hilb}_Z = \tbinom{n+t}{n-(n-i)} - \tbinom{n+t-d}{n-(n-i)} = \tbinom{n+t}{i} - \tbinom{n+t-d}{i}$$

(using Lemma 24.2 for the first equality), and so

$$d_i = \tbinom{i-1}{i} - \tbinom{i-1-d}{i} = \tbinom{i-1}{i} + (-1)^{i+1}\tbinom{d}{i}$$

where in the last step we have used the identity $\binom{-s}{r} = (-1)^r\binom{r+s-1}{r}$. The expression $\binom{i-1}{i}$ is nonzero only when $i = 0$, so that we get

$$d_i = \begin{cases} 0 & \text{if } i = 0 \\ (-1)^{i+1}\binom{d}{i} & \text{if } i > 0. \end{cases}$$

This of course agrees with what we found in Example 23.8.

Just as in the last example, in practice Hilbert polynomials are often computed by first having a graded free resolution of R/I. Let us look at what happens in general here, so let the (finite) graded free resolution be

$$\cdots \oplus_j R(-e_{2j}) \to \oplus_i R(-e_{1i}) \to \oplus_u R(-e_{0u}) \to R/I \to 0.$$

Then the Hilbert series is given by

$$\mathrm{Hilb}_Z(t) = \sum_{k=0}^{\infty} (-1)^k \sum_j \tbinom{t+n-e_{kj}}{n}$$

(really a finite sum, of course). Then

$$d_i = (\Delta^{n-i}\,\mathrm{Hilb}_Z)(-n+i-1) = \sum_{k=0}^{\infty}(-1)^k \sum_j \tbinom{i-1-e_{kj}}{i} = (-1)^i \cdot \sum_{k=0}^{\infty}(-1)^k \sum_j \tbinom{e_{kj}}{i}$$

where in the last equality we are using $\binom{i-s}{i} = (-1)^i\binom{s-1}{i}$.

Example 24.17. Let $Z \hookrightarrow \mathbb{C}P^n$ be a complete intersection where the degrees of the defining equations are d and e. Then R/I is resolved by the Koszul complex, which looks like $0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/I \to 0$. We conclude that $d_i = (-1)^i\big[\binom{0}{i} - \binom{d}{i} - \binom{e}{i} + \binom{d+e}{i}\big]$, from which we see that $d_0 = d_1 = 0$ and for $i > 1$

$$d_i = (-1)^{i+1}\big[\tbinom{d}{i} + \tbinom{e}{i} - \tbinom{d+e}{i}\big].$$

Again, this recovers our calculation from Example 23.9.

Recall that if the codimension of $Z \hookrightarrow \mathbb{C}P^n$ is equal to c then we have $d_i = 0$ for $i < c$ and $d_c = \deg(Z)$. So any graded free resolution of R/I must satisfy the identities

$$\sum_k (-1)^k \sum_j \tbinom{e_{kj}}{i} = \begin{cases} 0 & \text{for } i < c \\ (-1)^c \deg(Z) & \text{for } i = c. \end{cases} \tag{24.18}$$

In this way we obtain topological conditions on what graded free resolutions can look like. Conditions such as these seem to have been first observed in [PS1].

Example 24.19. Show that the conditions (24.18) on the free resolution are equivalent to

$$\sum_k (-1)^k \sum_j e_{kj}^i = \begin{cases} 0 & \text{for } i < c \\ (-1)^c \deg(Z)\cdot c! & \text{for } i = c. \end{cases}$$

Hint: This is a purely numerical result. Given two lists $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ of real numbers, compare the conditions $\sum_j \binom{a_j}{i} = \sum_j \binom{b_j}{i}$, $0 \le i < c$, to the conditions $\sum_j a_j^i = \sum_j b_j^i$ in the same range. Apply this to our situation by dividing the e_{kj} into two lists according to the parity of k.

24.20. **A digression on the Riemann zeta function.** If $Z \hookrightarrow \mathbb{C}P^n$ then the values of the polynomial $\mathrm{Hilb}_Z(t)$ only have an *a priori* significance for $t \gg 0$; recall that these values represent the dimensions of the graded pieces of the homogeneous coordinate ring of Z. Yet, we saw in Corollary 24.15 that the values of $\mathrm{Hilb}_Z(t)$ at certain *negative* integers (encoded as the value of a particular finite difference $\Delta^?\,\mathrm{Hilb}_Z$) are equal to some naturally-occurring topological invariants of Z. The fact that these negative values have any significance at all is a bit surprising. This situation is somewhat reminiscent of one involving the Riemann zeta function, that coincidentally (or not) is also related to the story of the (Δ, S) operators. We are going to take a moment and talk about this, because of the feeling that it might be related to topology in a way that is not yet fully understood.

Recall that Riemann's $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Standard results from analysis show that this sum converges for $\mathrm{Re}(s) > 1$, and defines an analytic function in that range. It is a non-obvious fact that $\zeta(s)$ can be analytically continued to a meromorphic function on the complex plane with its only pole at $z = 1$. The values on negative numbers turn out to be computable and are related to the Bernoulli numbers. We will give an entirely non-rigorous treatment of this computation. Despite its failure to actually make sense, it is nevertheless intriguing.

If $f \in \mathbb{Q}[t]$ then we have seen that $e^D f$ makes sense and is equal to the polynomial $f(t+1)$. It readily follows that $e^{nD} f = f(t+n)$ for any integer $n \geq 0$. Now write

$$f(t) + f(t+1) + f(t+2) + \cdots = [I + e^D + e^{2D} + e^{3D} + \cdots]f = \left[\tfrac{1}{I - e^D}\right] f.$$

Of course none of the three expressions between the equals signs make any sense, but let us pretend for a moment that this is not a problem. Replacing f by Df we can then write

$$Df(t) + Df(t+1) + \cdots = \left[\tfrac{D}{I - e^D}\right] f = -\mathcal{B} f$$

where $\mathcal{B}$ is the Bernoulli operator of (24.9). Evaluating at $t = 0$ we would obtain

$$Df(0) + Df(1) + Df(2) + \cdots = -(\mathcal{B}f)(0).$$

Let us next try to apply this fanciful formula to compute $\zeta(-n) = 1^n + 2^n + 3^n + 4^n + \cdots$. We want $Df(t) = t^n$, and so should take $f(t) = \frac{t^{n+1}}{n+1}$. The above formula then suggests that

$$\zeta(-n) = -\mathcal{B}\big(\tfrac{t^{n+1}}{n+1}\big)(0) = -\left[\sum_{k=0}^{\infty} \tfrac{B_k}{k!} D^k \big(\tfrac{t^{n+1}}{n+1}\big)\right](0) = -\tfrac{B_{n+1}}{(n+1)!} \cdot \tfrac{(n+1)!}{n+1} = -\tfrac{B_{n+1}}{n+1}.$$

Amazingly, this is the correct answer—the same value can be deduced by rigorous arguments from complex analysis. The challenge is to find some explanation for why this wacky argument actually leads to something correct.

25. The Euler class

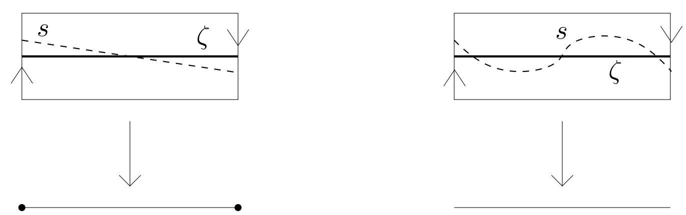
There is a general principle in algebraic topology that all rational cohomology theories detect the same information. The information is not necessarily detected in the same *way*, however, and this makes it hard to formulate the principle precisely. But here is a nice example of it. If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold then we have seen that knowing $[Z] \in K^0(\mathbb{C}P^n)$ is the same as knowing the integers d_i for which $[Z] = \sum d_i [\mathbb{C}P^{n-i}]$. Since $K^0(\mathbb{C}P^n)$ is free abelian, there is no loss of information in regarding this equation as taking place in $K^0(\mathbb{C}P^n) \otimes \mathbb{Q}$. By the above-mentioned principle, the numbers d_i should be able to be detected in rational singular cohomology. The Grothendieck-Riemann-Roch Theorem tells us how to do this, and that will be our next main goal.

To understand Riemann-Roch we need to first understand characteristic classes. We will give a very brief treatment, spread over the next two sections. For a more in-depth exploration some standard references are [MS] and [GH].

The present section deals with the Euler class, which is in some sense the most "primary" of characteristic classes. We discuss two versions: Euler classes in singular cohomology and Euler classes in K-theory.

25.1. **The Euler class for a vector bundle.** We will start with a geometric treatment that is lacking in rigor but shows the basic ideas, and then we will give a more rigorous treatment. Don't worry about verifying all the details in the following, just get the basic idea.

Start with a real vector bundle $E \to B$ of rank k, where B is a smooth manifold of dimension n. Let ζ denote the zero-section. We will try to construct something like an intersection product $\zeta \cdot \zeta$. To do this, we let s be a section of E that is a "small-perturbation" of ζ, chosen so that s and ζ intersect as little as possible. A good example to keep in mind is the Möbius bundle, shown below with two deformations of the zero section:



The zero locus $s^{-1}(0) \subseteq \mathrm{im}(\zeta)$ may, under good conditions, be given the structure of a cycle—part of this involves assigning multiplicities to the components in a certain way. If we want multiplicities in $\mathbb{Z}$ then we will need to assume the bundle is oriented, whereas if we are willing to use multiplicities only in $\mathbb{Z}/2$ then we can use any bundle. The dimension of this cycle is $\dim \zeta + \dim \zeta - \dim E$, which is $n + n - (n + k) = n - k$. This cycle clearly depends on the choice of s, but a different choice of s gives a homologous cycle. So the associated class in homology is independent of our choices, and is an invariant of E. We call it the **homology Euler class** of E: for orientable bundles we have

$$e_H(E) = \zeta \cdot \zeta = s^{-1}(0) \in H_{n-k}(B),$$

whereas for arbitrary bundles we have $e_H(E) \in H_{n-k}(B; \mathbb{Z}/2)$. The sections s used here are usually referred to as "generic sections" of E.

The following are easy properties of the Euler class construction:

(1) If E has a nonvanishing section, then $e_H(E) = 0$.
(2) $e_H(E \oplus F) = e_H(E) \cdot e_H(F)$ (where $\cdot$ is the intersection product).
(3) If L_1 and L_2 are line bundles on B then $e_H(L_1 \otimes L_2) = e_H(L_1) + e_H(L_2)$.

For (1) we simply note that if σ is a nonvanishing section then the deformation $t \mapsto t\sigma$ (for $t \in [0, 1]$) allows us to regard σ as a deformation of the zero-section. Taking $s = \epsilon\sigma$ for small ϵ, the vanishing locus of s is the same as the vanishing locus of σ—which is the empytset. So $e_H(E) = 0$.

For (2), if s is a generic section of E and s' is a generic section of F then $s \oplus s'$ is a generic section of $E \oplus F$. The vanishing locus of $s \oplus s'$ is the intersection of the vanishing loci of s and s'.

For (3), if s_1 and s_2 are generic sections of L_1 and L_2 then $s_1 \otimes s_2$ is a generic section of $L_1 \otimes L_2$. But $s_1 \otimes s_2$ vanishes at points in B where *either* s_1 or s_2 vanish. So the vanishing locus of $s_1 \otimes s_2$ is the union of the vanishing loci of s_1 and s_2.

Example 25.2.

(a) For the Möbius bundle $M \to S^1$ we have $e_H(M) = [*] \in H_0(S^1; \mathbb{Z}/2)$, as is clearly depicted in the pictures above. Note the necessity of $\mathbb{Z}/2$-coefficients here.
(b) If B is an orientable, smooth manifold of dimension d, then $e_H(TB) \in H_0(B)$. So $e_H(TB)$ is a multiple of $[*]$, and this multiple is precisely the Euler characteristic $\chi(B)$: a section of TB is just a vector field on B, and so this is the classical statement that a generic vector field on B vanishes at precisely $\chi(B)$ points. This connection with the Euler characteristic is why e_H is called the Euler class.
(c) Let $L \to \mathbb{R}P^n$ be the tautological line bundle, and recall that $L \cong L^*$ (Corollary 8.34). We examine L^* instead, since it is easier to write down formulas for sections. Generically choose a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and consider the section s_α whose value over $x = [x_0 : \cdots : x_n]$ is the functional sending $(x_0, \ldots, x_n)$ to $\alpha_0 x_0 + \cdots + \alpha_n x_n$. The vanishing locus for this section is a linear subspace of $\mathbb{R}P^n$, and of course we know that all such things are homotopic. So $e_H(L^*) = [\mathbb{R}P^{n-1}]$.
(d) A similar analysis allows one to calculare $e_H(L^*)$ where $L \to \mathbb{C}P^n$ is the tautological line bundle, but here one must be careful about getting the signs correct. It is clear enough that $e_H(L^*) = \pm[\mathbb{C}P^{n-1}]$, but determining the sign takes some thought. ?????
(e) Normal bundle of $Z \hookrightarrow \mathbb{C}P^n$????

25.3. **Cohomology version.** As is often the case, working in cohomology instead of homology allows us to make things completely rigorous while avoiding some of the thorny geometric issues that arose in our discussion above.

Let $E \to B$ be an orientable real bundle of rank k; that is, a bundle with a Thom class $\mathcal{U}_E \in H^k(E, E-0)$. Note that B need no longer be a manifold. Let $\zeta\colon B \to E$ denote the zero section, as usual. We may interpret ζ as a map of pairs $(B, \emptyset) \to (E, E-0)$, so that pulling back along ζ gives a cohomology class $\zeta^*(\mathcal{U}_E) \in H^*(B)$. Define this to be the **(cohomology) Euler class** of E:

$$e^H(E) = \zeta^*(\mathcal{U}_E).$$

As $\mathcal{U}_E$ plays the role of a (relative) fundamental class for the 0-section ζ, the pullback $\zeta^*(\mathcal{U}_E)$ captures the spirit of intersecting the 0-section with itself.

The main properties of the Euler class are as follows:

Proposition 25.4. *Let $E \to B$ be an oriented real bundle. Then*

(a) If E has a nonzero section then $e^H(E) = 0$.
(b) $e^H(E \oplus F) = e^H(E) \cup e^H(F)$ for any oriented bundle $F \to B$.
(c) For any map $f\colon Y \to B$ one has $e^H(f^*E) = f^*(e^H(E))$ (naturality under pullbacks).

Analogs of the above properties are all true for non-orientable bundles as long as one uses $\mathbb{Z}/2$-coefficients everywhere.

Proof. Properties (b) and (c) follow from the corresponding properties of Thom classes. To see (a), let s be a nonzero section. Consider the homotopy $H\colon I\times B \to E$ given by $H(t,b) = t\cdot s(b)$. This can be regarded as a homotopy between maps of pairs $(B,\emptyset) \to (E, E-0)$. It follows that $e^H(E) = \zeta^*(\mathcal{U}_E) = s^*(\mathcal{U}_E)$. But s factors through $E-0$, and so $s^*(\mathcal{U}_E) = 0$. $\square$

The reader might notice that the above result does not include a formula for $e^H(L_1\otimes L_2)$, as we had seen for the homology version. Don't worry, it's coming! See Proposition 25.8.

Example 25.5.

(1) For the Möbius bundle $M \to S^1$ we have $e^H(M) = [*] \in H^1(S^1;\mathbb{Z}/2)$. To prove this note that the Thom space of M is $D(M)/S(M) \cong \mathbb{R}P^2$, and the zero section $\zeta\colon S^1 \to \mathrm{Th}(M)$ is just a typical embedding of $\mathbb{R}P^1$ into $\mathbb{R}P^2$. The Thom class $\mathcal{U}_M \in H^1(\mathrm{Th}\, M;\mathbb{Z}/2) = H^1(\mathbb{R}P^2;\mathbb{Z}/2)$ is the unique nonzero class, and we know restricting along $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$ sends this class to the unique nonzero class in $H^1(\mathbb{R}P^1;\mathbb{Z}/2)$.

(2) Let $L \to \mathbb{R}P^n$ be the tautological bundle. If $j\colon \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ is the inclusion, then $j^*L \cong M$. So naturality gives $j^*e^H(L) = e^H(M)$. But $j^*\colon H^1(\mathbb{R}P^n;\mathbb{Z}/2) \to H^1(\mathbb{R}P^1;\mathbb{Z}/2)$ is an isomorphism, and so it follows from (1) that $e^H(L)$ must be the unique nonzero class in $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$.

(3) Let M be a smooth, oriented manifold of dimension n. Then the tangent bundle $TM \to M$ is a rank n oriented bundle. In this case, $e^H(TM) \in H^n(M;\mathbb{Z}) = \mathbb{Z}\langle[*]\rangle$ and so the problem is to determine the integer d for which $e^H(TM) = d[*]$. It is a fact from geometric topology ???? that there is a vector field $s\colon M \to TM$ with a finite number of vanishing points, and that when counted with appropriate signs this number is $\chi(M)$. Let $A = s^{-1}(0) = \{p_1,\dots,p_r\} \subseteq M$. The deformation $t \mapsto ts$ shows that s is homotopic to the zero section ζ, so that we get the following diagram

$$H^n(M) \xleftarrow{j^*} H^n(M, M-\{p_1,\dots,p_r\}) \xleftarrow{s^*} H^n(TM, TM-0)$$

where the composite map is $\zeta^* = s^*$. As is typical in these arguments, we next use that $H^n(M, M-\{p_1,\dots,p_r\}) \cong \oplus_i H^n(M, M-p_i)$ and that the orientation of M gives canonical generators $[*] \in H^n(M)$ and $[p_i] \in H^n(M, M-p_i)$. The map j^* sends each $[p_i]$ to $[*]$, and so it is really just a fold map $\mathbb{Z}^r \to \mathbb{Z}$. It remains to see how the Thom class $\mathcal{U}_{TM}$ maps to the canonical generators in $H^n(M, M-p_i)$ under s^*, but this is a local problem—by working through the definitions one sees that $\mathcal{U}_{TM} \mapsto d_i[p_i]$ where d_i is the local index of the vector field at p_i. We finally obtain that $e^H(TM) = (d_1+\cdots+d_r)[*]$, where $d_1+\cdots+d_r$ is the sum of the local indices and therefore equal to $\chi(M)$.

The following property of the Euler class is also useful:

Proposition 25.6. *Let M be an oriented manifold and let $j\colon Z \hookrightarrow M$ be a regularly embedded, oriented submanifold. Let $N_{M/Z}$ be the normal bundle. Then $e(N_{M/Z}) = j^*([Z])$.*

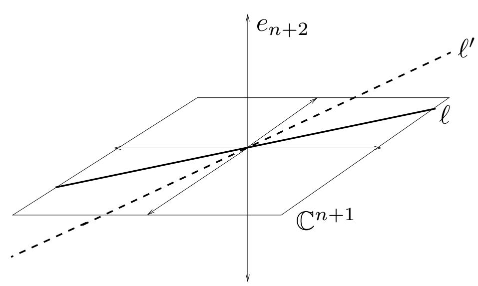
Proof. Intuitively the result should make sense, since both $e(N_{M/Z})$ and $j^*([Z])$ are modelled by the intersection product of Z with itself inside of M. To give a rigorous proof, let U be a tubular neighborhood of Z in M and $U \cong N$ be a regular homeomorphism. Note that the zero section $\zeta\colon Z \hookrightarrow N$ corresponds with the inclusion $j\colon Z \hookrightarrow U$ under this isomorphism.

Let c be the codimension of Z in M. Consider the commutative diagram

$$\begin{tikzcd}
H^0(Z) \arrow[r] & H^c(N,N-0) \arrow[r, "\cong"] \arrow[dr, "\zeta^*"'] & H^c(U,U-0) \arrow[d] & H^c(M,M-Z) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "j^*"] & H^c(M) \arrow[d, "j^*"] \\
 & & H^c(Z) \arrow[r, equal] & H^c(Z) \arrow[r, equal] & H^c(Z).
\end{tikzcd}$$

The image of 1 across the top row is $[Z]$, with $\mathcal{U}_N$ being an intermediate value in the composite. The image of $\mathcal{U}_N$ under ζ^* is $e(N)$, and so the diagram immediately yields $e(N) = j^*[Z]$. $\square$

Example 25.7. Consider the usual embedding $j\colon \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$. We claim that the normal bundle is $L^* \to \mathbb{C}P^n$, the dual of the tautological line bundle. The proof is that a linear functional ϕ on the line $\ell \subseteq \mathbb{C}^{n+1}$ determines a "nearby" line $\ell' = \{(x, \phi(x)) \,|\, x \in \ell\} \subseteq \mathbb{C}^{n+1}$, as shown below:



By Proposition 25.6 we find that $e(L^*) = j^*([\mathbb{C}P^n])$, and we know the latter is $[\mathbb{C}P^{n-1}]$ by intersection theory. We have shown that $e(L^* \to \mathbb{C}P^n) = [\mathbb{C}P^{n-1}]$.

We next wish to give a formula for the Euler class of a tensor product of line bundles. At the moment this might seem to be of limited interest, but it turns out to be very significant. We need to be careful about what context we are working in, however. All orientable real line bundles are trivial (one can write down an evident nonvanishing section), and so using the integral Euler class in this context doesn't lead to anything interesting. So we should work with mod 2 Euler classes and arbitrary real line bundles. Alternatively, if we use *complex* line bundles then they are automatically oriented and then we can indeed use the integral Euler class (of the underlying real plane bundle). So we really get two parallel results, one for the real and one for the complex case:

Proposition 25.8. *If L_1 and L_2 are real line bundles on B then the mod 2 Euler class satisfies*
$$e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2).$$
Likewise, if L_1 and L_2 are complex line bundles on B then the integral Euler class satisfies the same formula.

Proof. The proofs of the two parts are basically identical; we will do the complex case. Let $L \to \mathbb{C}P^\infty$ denote the tautological line bundle, and consider the bundle $\pi_1^*(L) \otimes \pi_2^*(L) \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, where $\pi_1, \pi_2 \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ are the two projections. Since L is the universal example of a line bundle, $\pi_1^*(L) \otimes \pi_2^*(L)$ is the universal example of a tensor product of line bundles. Write $E = \pi_1^*(L) \otimes \pi_2^*(L)$, for short. There is a classifying map $f \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ for E, giving a pullback diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & L \arrow[d] \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "f"] & \mathbb{C}P^\infty. \end{tikzcd}$$

By naturality $e^H(E) = f^*(e^H(L)) = f^*(x)$ where $x \in H^2(\mathbb{C}P^\infty)$ is the canonical generator.

If $*$ is a chosen basepoint in $\mathbb{C}P^\infty$ then observe that the diagram

$$\begin{tikzcd} \mathbb{C}P^\infty \times \{*\} \arrow[dr, "j_2"'] \arrow[drr, bend left, "\mathrm{id}"] & & \\ & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "f"] & \mathbb{C}P^\infty \\ \{*\} \times \mathbb{C}P^\infty \arrow[ur, "j_1"'] \arrow[urr, bend right, "\mathrm{id}"'] & & \end{tikzcd}$$

commutes up to homotopy. This is because fj_1 classifies $j_1^*(E)$, and this bundle is clearly isomorphic to L; similarly for fj_2.

Note that $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is the free abelian group generated by $x \otimes 1$ and $1 \otimes x$. So $f^*(x) = k(x \otimes 1) + l(1 \otimes x)$ for some integers k and l. The above homotopy commutative diagram forces $k = l = 1$. So we have proven that

$$e^H(E) = e^H(L) \otimes 1 + 1 \otimes e^H(L).$$

Now let L_1 and L_2 be two complex line bundles on a space B. There are maps $g_1, g_2 \colon B \to \mathbb{C}P^\infty$ such that $L_1 = g_1^*(L)$ and $L_2 = g_2^*(L)$. Then $L_1 \otimes L_2 = \gamma^*(E)$, where γ is the composite

$$B \xrightarrow{\Delta} B \times B \xrightarrow{g_1 \times g_2} \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

We obtain

$$\begin{aligned} e^H(L_1 \otimes L_2) = \gamma^*(e^H(E)) &= \gamma^*\big(e^H(L) \otimes 1 + 1 \otimes e^H(L)\big) \\ &= \Delta^*\big(e^H(L_1) \otimes 1 + 1 \otimes e^H(L_2)\big) = e^H(L_1) + e^H(L_2). \end{aligned}$$

□

Remark 25.9. One should note that the above proof is not geometric—this turns out to be important. Rather, the proof in some sense proceeds by showing that there is really not much choice for what $e^H(L_1 \otimes L_2)$ could be, given how small $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is; there is in fact only *one* possibility. We will shortly see that replacing H by other cohomology theories—ones with "more room", so to speak—allows for more to happen here. See Proposition 25.14 below.

Corollary 25.10. *Let $L \to \mathbb{C}P^n$ be the tautological line bundle. Then $e^H(L^*) = [\mathbb{C}P^{n-1}]$ and so*

$$e^H((L^*)^{\otimes k}) = k[\mathbb{C}P^{n-1}] \quad \textit{and} \quad e^H(L^{\otimes k}) = -k[\mathbb{C}P^{n-1}].$$

Proof. The first statement was proven in Example 25.7. All of the other statements follow directly from the first via Proposition 25.8. For $e^H(L)$ use that $L \otimes L^* \cong \underline{1}$ and so $0 = e^H(\underline{1}) = e^H(L) + e^H(L^*)$. □

A nice consequence of all of the above work is that for complex line bundles the Euler class gives a complete invariant:

Corollary 25.11. *Let L_1, L_2 be two complex line bundles over a space X. If $e(L_1) = e(L_2)$ then $L_1 \cong L_2$.*

Proof. Let $f_1, f_2 \colon X \to \mathbb{C}P^\infty$ be classifying maps for the two line bundles: so $f_1^*L \cong L_1$ and $f_2^*L \cong L_2$. Then $e(L_1) = f_1^*e(L)$ and $e(L_2) = f_2^*(e(L))$, by naturality of the Euler class. Our assumption is therefore equivalent to $f_1^*(e(L)) = f_2^*(e(L))$.

But $\mathbb{C}P^\infty$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, and so $[X, \mathbb{C}P^\infty]$ is naturally isomorphic to $H^2(X)$ via the map $f \mapsto f^*(z)$ where z is a chosen generator of $H^2(\mathbb{C}P^\infty)$. Corollary 25.10 gives that $e(L)$ is such a generator, so the fact that $f_1^*(e(L)) = f_2^*(e(L))$ implies that f_1 is homotopic to f_2. But this implies that L_1 is isomorphic to L_2. □

The following easy corollary will be needed often:

Corollary 25.12. *Let $j \colon Z \hookrightarrow \mathbb{C}P^n$ be a degree d hypersurface. Then the normal bundle is isomorphic to $j^*\big((L^*)^{\otimes d}\big)$.*

Proof. By Proposition 25.6 the Euler class of the normal bundle is $e(N) = j^*([Z])$. But we know $[Z] = d[\mathbb{C}P^{n-1}] = e((L^*)^{\otimes d})$, and so $j^*[Z]$ is the Euler class of $j^*((L^*)^{\otimes d})$. Now use Corollary 25.11. □

25.13. **Euler classes in K-theory.** Our construction of the Euler class $e^H(E)$ can be repeated verbatim in any complex-oriented cohomology theory. In particular, it can be repeated in K-theory. We explore this next.

Let $E \xrightarrow{\pi} X$ be a $\mathbb{C}$-bundle of rank k. We have a Thom class $\mathcal{U}_E \in K^0(E, E-0)$, and so we can mimic the above construction and define

$$e^K(E) = \zeta^*(\mathcal{U}_E) \in K^0(X).$$

This is the K**-theory Euler class** of E. [Note that it is slightly more appropriate to regard $\mathcal{U}_E$ as lying in $K^{2k}(E, E-0)$ and then $e^K(E) \in K^{2k}(X)$, but as usual we take advantage of Bott periodicity to work only in K^0.]

Let us unravel the above definition a bit. First, recall that $\mathcal{U}_E$ is the complex

$$[\ 0 \longrightarrow \textstyle\bigwedge^0(\pi^*E) \xrightarrow{\Delta\wedge -} \bigwedge^1(\pi^*E) \longrightarrow \dots \longrightarrow \bigwedge^k(\pi^*E) \longrightarrow 0\ ]^*$$

where Δ is the usual diagonal section as shown in the following diagram:

$$\begin{tikzcd}
E \arrow[r, equal] & \zeta^*(\pi^*E) \arrow[r] \arrow[d] & \pi^*E \arrow[r] \arrow[d] & E \arrow[d] \\
& X \arrow[r, "\zeta"'] & E \arrow[r, "\pi"'] \arrow[u, bend left, "\Delta"] & X.
\end{tikzcd}$$

Note that $\zeta^*(\pi^*(E)) = E$ because $\pi \circ \zeta = \mathrm{id}$. Next let us look at $\zeta^*(\mathcal{U}_E)$. For each j one has $\zeta^*(\bigwedge^j(\pi^*E)) = \bigwedge^j(E)$, and the restriction of Δ to the image of ζ is just the zero section! So the maps in the complex $\zeta^*(\mathcal{U}_E)$ are all zero. In other words, $\zeta^*(\mathcal{U}_E)$ is the Koszul complex for the zero section on E. Therefore

$$\begin{aligned}\zeta^*(\mathcal{U}_E) &= J^*_{E,0} \\ &= [\, 0 \longrightarrow \textstyle\bigwedge^0 E \xrightarrow{0} \bigwedge^1 E \xrightarrow{0} \dots \longrightarrow \bigwedge^k E \longrightarrow 0 \,]^* \\ &= \textstyle\sum (-1)^i [\bigwedge^i (E^*)].\end{aligned}$$

We see immediately that if E has a nonzero section then $e^K(E) = 0$. Indeed, if s is the nonzero section then we can deform ζ to s, and likewise deform $J^*_{E,\zeta}$ to $J^*_{E,s}$. But this latter complex is exact, and so represents zero in $K^0(X)$.

The analogs of properties (b) and (c) from Proposition 25.4 also hold, as these are simple consequences of corresponding properties of Thom classes. What about the analog of Proposition 25.8? If $L \to X$ is a complex line bundle then we have

$$e^K(L) = 1 - L^*.$$

So $e^K(L_1 \otimes L_2) = 1 - L_1^* L_2^*$, which is visibly *not* the same as $e^K(L_1) + e^K(L_2)$. Indeed, one can check the following more complicated formula:

Proposition 25.14. *Let L_1 and L_2 be complex line bundles on a space X. Then*

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2).$$

Proof. We simply observe that $1 - L_1^* L_2^* = (1 - L_1^*) + (1 - L_2^*) - (1 - L_1^*)(1 - L_2^*)$. $\square$

Remark 25.15. The difference between how e^H and e^K behave on tensor products of line bundles turns out to have much more significance than one might expect. In some sense it ends up accounting for all of the differences between H and K, at least in terms of how they encode geometry. See Section 31 for more discussion.

We end this section with some detailed computations of K-theoretic Euler classes:

Example 25.16. Let T be the complex tangent bundle to $\mathbb{C}P^n$. Our goal will be to compute $e^K(T)$ from first-principles. Let L denote the tautological line bundle over $\mathbb{C}P^n$. Note that L sits inside of the trivial bundle $\underline{n+1}$ in the evident way (if l is a line in $\mathbb{C}^{n+1}$, then points on l are defacto points in $\mathbb{C}^{n+1}$). Let $L^\perp$ be the orthogonal complement to L relative to the usual Hermitian metric on $\mathbb{C}^{n+1}$. So we have a short exact sequence of bundles

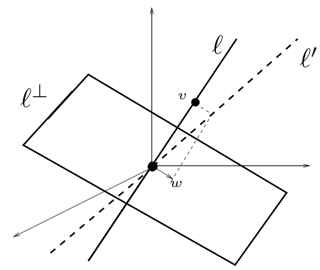
$$0 \to L \hookrightarrow \underline{n+1} \to L^\perp \to 0. \tag{25.17}$$

Since $\mathbb{C}P^n$ is paracompact this sequence is split, and hence $L \oplus L^\perp \cong \underline{n+1}$.

The basis of our computation is the following geometric fact:

$$T \cong \underline{\mathrm{Hom}}(L, L^\perp) \tag{25.18}$$

where $\underline{\mathrm{Hom}}(L, L^\perp)$ is the bundle over $\mathbb{C}P^n$ whose fiber over a point x is the vector space of linear maps $L_x \to L_x^\perp$. To understand this isomorphism, if ℓ is a point in $\mathbb{C}P^n$ then think of the tangent space T_ℓ as giving "local directions" for moving to all nearby points around ℓ. The following picture shows the line ℓ in $\mathbb{C}^{n+1}$ together with its orthogonal complement $\ell^\perp$ and a "nearby line" ℓ':



Note that ℓ' determines a linear map $\ell \to \ell^\perp$ as shown in the picture: a vector $v \in \ell$ is sent to the unique vector $w \in \ell^\perp$ such that $v + w \in \ell'$. This makes sense as long as ℓ' is not orthogonal to ℓ, which will be fine for all nearby lines. We clearly get a bijection between $\mathrm{Hom}(\ell, \ell^\perp)$ and a certain neighborhood of ℓ in $\mathbb{C}P^n$, and it is not hard to extrapolate from this to the isomorphism (25.18).

Now take the short exact sequence of (25.17) and apply $\underline{\mathrm{Hom}}(L, -)$ to get the short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(L, L) \longrightarrow \underline{\mathrm{Hom}}(L, \underline{n+1}) \longrightarrow \underline{\mathrm{Hom}}(L, L^\perp) \longrightarrow 0.$$

(To see that this sequence is exact, just check it on fibers—there it is obvious, because we are just dealing with vector spaces.) For any line bundle $\mathcal{L}$ one has the identity $\underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{L}) = \underline{1}$: for a one-dimensional vector space V the map $\mathbb{C} \to \mathrm{Hom}(V, V)$ mapping 1 to the identity is a canonical isomorphism, so we can do this fiberwise. Using this, together with the identification $T \cong \underline{\mathrm{Hom}}(L, L^\perp)$, the above short exact sequence can be written as

$$0 \to \underline{1} \to (n+1)L^* \to T \to 0. \tag{25.18}$$

Since there must be a splitting, $\underline{1} \oplus T \cong (n+1)L^*$. Dualizing, we obtain $\underline{1} \oplus T^* \cong (n+1)L$

Recall that $e^K(T) = \sum_i (-1)^i [\bigwedge^i (T^*)]$. We will compute $\bigwedge^i (1 \oplus T^*)$ and then extract formulas for $[\bigwedge^i (T^*)]$.

Of course $\bigwedge^0 \underline{1} = \bigwedge^1 \underline{1} = \underline{1}$ and $\bigwedge^j \underline{1} = 0$ for $j \geq 2$. This allows us to calculate

$$\begin{aligned} \textstyle\bigwedge^j (\underline{1} \oplus T^*) &= (\textstyle\bigwedge^0 \underline{1} \otimes \bigwedge^j T^*) \oplus (\bigwedge^1 \underline{1} \otimes \bigwedge^{j-1} T^*) \\ &= \textstyle\bigwedge^j T^* \oplus \bigwedge^{j-1} T^*. \end{aligned}$$

On the other hand, $\bigwedge^j (\underline{1} \oplus T^*) = \bigwedge^j ((n+1)L) = \binom{n+1}{j}(L^{\otimes j})$. So for every j we have

$$[\textstyle\bigwedge^j (T^*)] = \binom{n+1}{j} L^{\otimes j} - [\bigwedge^{j-1} (T^*)]$$

in K-theory.

The evident recursion now gives that

$$\begin{aligned} &[\textstyle\bigwedge^0 T^*] = 1 \\ &[\textstyle\bigwedge^1 T^*] = (n+1)[L] - [\underline{1}] \\ &[\textstyle\bigwedge^2 T^*] = \binom{n+1}{2}[L^{\otimes 2}] - (n+1)[L] + [\underline{1}] \\ &[\textstyle\bigwedge^3 T^*] = \binom{n+1}{3}[L^{\otimes 3}] - \binom{n+1}{2}[L^{\otimes 2}] + (n+1)[L] - [\underline{1}], \end{aligned}$$

and so on. The general formula, obtained by an easy induction, is

$$[\bigwedge^j T^*] = \sum_{k=0}^{j} (-1)^{k+j} \tbinom{n+1}{k} [L^{\otimes k}].$$

Taking the alternating sum of these expressions (from $j = 0$ to $j = n$) now yields

$$\begin{aligned} e^K(T) &= (n+1)[1] - n\tbinom{n+1}{1}[L] + (n-1)\tbinom{n+1}{2}[L]^2 - \cdots + (-1)^n 1 \cdot \tbinom{n+1}{n}[L]^n \\ &= (n+1)\Big([1] - n[L] + \tbinom{n}{2}[L]^2 - \tbinom{n}{3}[L]^3 + \cdots + (-1)^n[L]^n\Big) \\ &= (n+1)(1-[L])^n. \end{aligned}$$

Recall that $(1 - [L])^n = [*]$ (Proposition 23.6), and so we have determined that

$$e^K(T) = (n+1)[*].$$

Let us now confess that the result of this computation is not unexpected. Indeed, we saw previously that $e^H(TM) = \chi(M)[*]$ for any smooth manifold M, and in fact the same is true in any complex-oriented cohomology theory (by essentially the same proof). The "$n + 1$" in our formula for $e^K(T)$ is just $\chi(\mathbb{C}P^n)$. But note that we computed this without writing down anything remotely resembling a cell structure! In fact, the only geometry in the calculation was in the fact $T \cong \underline{\mathrm{Hom}}(L, L^{\perp})$; everything else was some simple linear algebra and then basic algebraic manipulation. It is useful to remember the overall theme of K-theory: do linear algebra fiberwise over a base space X, and see what this tells you about the topology of X. Our computation of $e^K(T_{\mathbb{C}P^n})$ gives an example of this.

The calculation in the above example is a little clunky. One way to streamline it is to introduce the formal power series

$$\lambda_t(E) = \sum_{i=0}^{\infty} t^i[\bigwedge^i(E)] = 1 + t[E] + t^2[\bigwedge^2(E)] + \cdots$$

which we regard as living in the ring $K(X)[[t]]$, where $E \to X$ was our vector bundle. Notice that this is actually a polynomial in t, since the exterior powers vanish beyond the rank of E; we consider it as a power series because in that context it has a multiplicative inverse, which we will shortly need.

If L is a line bundle then $\lambda_t(L) = 1 + t[L]$. Also, the formula $\bigwedge^k(E \oplus F) = \bigoplus_{i+j=k} \bigwedge^i(E) \otimes \bigwedge^j(F)$ yields the nice relation

$$\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F).$$

This is what ultimately simplifies our calculations. Finally, notice that the K-theoretic Euler characteristic can be written as $e^K(E) = \lambda_t(E^*)|_{t=-1}$.

Returning to the above calculation, the starting point for the algebra was the bundle isomorphism $\underline{1} \oplus T^* \cong (n+1)L$. Applying λ_t we obtain

$$\lambda_t(\underline{1})\lambda_t(T^*) = \lambda_t(\underline{1} \oplus T^*) = \lambda_t((n+1)L) = (\lambda_t(L))^{n+1}.$$

But $\lambda_t(\underline{1}) = 1 + t$ and $\lambda_t(L) = 1 + t[L]$, so we can write

$$\lambda_t(T^*) = \tfrac{(1+t[L])^{n+1}}{1+t}, \qquad \text{or} \qquad e^K(T) = \tfrac{(1+t[L])^{n+1}}{1+t}\Big|_{t=-1}$$

(and here is where we are using that our power series have multiplicative inverses). Our task is to expand the formula for $e^K(T)$ into powers of t, and then to set $t = -1$. The trick is to do this in a clever way.

We are going to ultimately want to write $e^K(T)$ in terms of powers of $(1-L)$, as they give our usual basis for $K^*(\mathbb{C}P^n)$. To this end, regard L as a formal variable and consider $f(L) = (1+tL)^{n+1}/(1+t)$, considered as a formal power series in two variables (but where we are choosing not to write t in the inputs of f). Let us expand this in powers of $(L-1)$ via the usual Taylor series:

$$f(L) = f(1) + f'(1)(L-1) + \tfrac{f''(1)}{2}(L-1)^2 + \cdots$$

It is simple to compute the kth derivative $f^{(k)}(L) = k!\binom{n+1}{k}\frac{(1+tL)^{n+1-k}t^k}{1+t}$. So the coefficient of $(L-1)^k$ is $\binom{n+1}{k}(1+t)^{n-k}t^k$, and we obtain

$$\lambda_t(T^*) = f(L) = \sum_{k=0}^{n+1} \tbinom{n+1}{k}(1+t)^{n-k}t^k(L-1)^k.$$

Notice that the substitution $t=-1$ will make the summands vanish for k smaller than n, and that the term $k=n+1$ vanishes because $(L-1)^{n+1}=0$ in $K(\mathbb{C}P^n)$. So only one term survives and we find that

$$e^K(T) = \lambda_t(T^*)|_{t=-1} = \tbinom{n+1}{n}t^n(L-1)^n|_{t=-1} = (n+1)(1-L)^n = (n+1)[*].$$

Example 25.19 (Euler characteristic of a hypersurface)**.** Using the λ_t operators introduced above, we will attempt a harder computation of an Euler class. Let $j\colon Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Our goal is to compute the Euler characteristic $\chi(Z)$. If T_Z and N_Z denote the tangent and normal bundles, respectively, then $T_Z \oplus N_Z \cong j^*T_{\mathbb{C}P^n}$. We know from Corollary 25.12 that $N_Z \cong j^*\mathcal{O}(d)$, so we have

$$\underline{1} \oplus T_Z \oplus j^*\mathcal{O}(d) \cong \underline{1} \oplus T_Z \oplus N_Z \cong \underline{1} \oplus j^*T_{\mathbb{C}P^n} \cong j^*(\underline{1} \oplus T_{\mathbb{C}P^n}) \cong (n+1)j^*\mathcal{O}(1),$$

the final isomorphism by (25.18). Taking duals and applying λ_t, we obtain

$$\lambda_t(\underline{1}) \cdot \lambda_t(T_Z^*) \cdot \lambda_t(j^*L^d) = \lambda_t(j^*L)^{n+1}.$$

Let $X = j^*L$. Then we can write

$$\lambda_t(T_Z^*) = \tfrac{(1+tX)^{n+1}}{(1+t)(1+tX^d)}.$$

We wish to compute the Euler class $e^K(T_Z) = \lambda_t(T_Z^*)|_{t=-1}$ and then write it in the form $(???)\cdot[*]$, in which case the mystery number in parentheses will be $\chi(Z)$. The trick is again to expand in powers of $(X-1)$, since the powers of $1-L$ are our standard generators for $K^0(\mathbb{C}P^n)$.

Consider the power series in X

$$\lambda_t(T^*) = f(X) = \tfrac{(1+tX)^{n+1}}{(1+t)(1+tX^d)} = f(1) + f'(1)(X-1) + \tfrac{f''(1)}{2}(X-1)^2 + \cdots$$

Note that $(1-L)^{n+1}$ is zero in $K^0(\mathbb{C}P^n)$ and therefore $(1-X)^{n+1}$ vanishes in $K^0(Z)$. Even better, $(1-L)^n = [*]$ in $K^0(\mathbb{C}P^n)$ and since $j^*[*]=0$ by intersection theory it follows that $(1-X)^n = 0$ in $K^0(Z)$. So we don't care about any terms in the above series beyond $(X-1)^{n-1}$. Let us also note at this point that $(1-L)^{n-1} = [\mathbb{C}P^1]$ in $K^0(\mathbb{C}P^n)$, and therefore $(1-X)^{n-1} = j^*\big((1-L)^{n-1}\big) = j^*([\mathbb{C}P^1]) = d[*]$; the last equality holds by intersection theory, since a generic $\mathbb{C}P^1$ will intersect Z in exactly d points.

Before tackling the general calculation let us do the first example, where $n = 2$. Here $f(X) = \frac{(1+tX)^3}{(1+t)(1+tX^d)}$ and we only need the first two terms of the series. Clearly $f(1) = 1 + t$ and an easy calculation gives

$$f'(X) = \frac{1}{1+t} \cdot \frac{(1+tX)^2}{(1+tX^d)^2} \Big[3t(1 + tX^d) - (1 + tX)tdX^{d-1} \Big]$$

so that

$$f'(1) = (3 - d)t.$$

Putting everything together,

$$f(X) = (1 + t) + (3 - d)t(X - 1)$$

and so

$$e^K(T_Z) = f(X)|_{t=-1} = 0 + (d - 3)(X - 1) = (3 - d)(1 - X) = (3 - d) \cdot d[*].$$

We conclude that $\chi(Z) = d(3 - d)$.

For larger n here is how things are going to work. First, we will calculate the derivatives $f^{(k)}(1)$ for $0 \le k \le n - 1$, and then substitute $t = -1$ into all of them. It will turn out (but is far from obvious) that the resulting expressions vanish for $k < n - 1$, so that

$$\begin{aligned} e^K(T_Z) &= f^{(n-1)}(1)|_{t=-1} \cdot (X - 1)^{n-1} \\ &= f^{(n-1)}(1)|_{t=-1} \cdot (-1)^{n-1} \cdot (1 - X)^{n-1} \\ &= f^{(n-1)}(1)|_{t=-1} \cdot (-1)^{n-1} \cdot d[*]. \end{aligned}$$

The conclusion will then be that

$$\chi(Z) = f^{(n-1)}(1)|_{t=-1} \cdot (-1)^{n-1} \cdot d. \tag{25.20}$$

Notice that everything comes down to computing the expressions $f^{(k)}(1)|_{t=-1}$, which is a purely algebraic problem. Unfortunately we cannot *first* plug in $t = -1$, since our formula for f has a $1 + t$ in the denominator; we have to first do the hard work of writing f as a polynomial in t before plugging in. At first glance this work looks very hairy! Already the formula for $f'(X)$ was quite complicated, and it only gets worse for the higher derivatives. The reader might wish to carry this out by brute force for $n = 3$, to get a feel for the difficulties.

We are going to sketch the completion of the calculations for the above example, but before diving into that we need to make a confession. It is possible to compute $\chi(Z)$ by doing a similar kind of calculation using singular cohomology instead of K-theory, and in that setting the algebra turns out to be much *easier*! There is a trade-off, which is that the computation cannot be done merely with Euler classes—one needs the complete theory of Chern classes, to be developed in the next section. See Example 26.8 for the computation of $\chi(Z)$ in that context. This situation is fairly typical of the relationship between K-theory and singular cohomology. For calculations that can be done in either theory, usually the singular cohomology version will involve simpler algebra, but more advanced geometric techniques; the K-theory version will involve more advanced algebra, but one needs less geometry. In some sense we saw this phenomenon already in the case of intersection multiplicities.

The completion of our calculation will proceed via the following steps:

(a) Suppose that $f(w) \in \mathbb{Q}[w]$ has degree k. Then there is an identity of formal power series
$$f(0) - tf(1) + t^2 f(2) - \cdots = \tfrac{u(t)}{(1+t)^{k+1}}$$
for a unique polynomial $u(t)$. Moreover, the degree of $u(t)$ is at most k and $u(-1)$ is $k!$ times the leading coefficient of f.

To justify the above claim, write $A_f = \sum_{k\geq 0}(-1)^k f(k)t^k$. Note the formula $(1+t)A_f = f(0) - tA_{\Delta f}$ where Δ is the finite difference operator from Section 24. Check the claim is true when $\deg f = 0$, and then do an induction on the degree. (Extra credit: Find a formula for $u(t)$ in terms of the numbers $\Delta^k f(0)$).

(b) By collecting terms notice that
$$\begin{aligned}\tfrac{1}{1+tX^d} &= 1 - tX^d + t^2X^{2d} - \cdots \\ &= 1 - t((X-1)+1)^d + t^2((X-1)+1)^{2d} - \cdots \\ &= \tfrac{1}{1+t} + (X-1)\Gamma_1 + (X-1)^2\Gamma_2 + \cdots\end{aligned}$$
where
$$\Gamma_k = -t\tbinom{d}{k} + t^2\tbinom{2d}{k} - t^3\tbinom{3d}{k} + \cdots$$
Applying (a) to $f(w) = \binom{dw}{k} = \frac{1}{k!}(dw)(dw-1)\cdots(dw-k+1)$ gives that $\Gamma_k = \frac{u_k(t)}{(1+t)^{k+1}}$ where $u_k(t)$ is a polynomial such that $u(-1) = d^k$.

(c) Set $f_r(X) = \frac{(1+tX)^r}{(1+t)(1+tX^d)}$. Our goal is to compute the derivatives $f_r^{(k)}(1)$ for $1 \leq k \leq r-2$, thus obtaining the coefficients in the Taylor expansion of f_r in powers of $(X-1)$. But remember we only need to know what happens to the answer after specializing at $t=-1$, which will simplify our task.

Notice the recursion relation
$$f_{r+1} = (1+tX)f_r = (1+t((X-1)+1))f_r = (1+t)f_r + (X-1)tf_r.$$
So if we know the expansion of f_r in terms of powers of $X-1$, it is easy to get the expansion of f_{r+1}. In the following table, row r shows the terms in the expansion for f_r (starting with f_0):

r	$(X-1)^0$	$(X-1)^1$	$(X-1)^2$	$(X-1)^3$	$\cdots$
0	$\frac{1}{(1+t)^2}$	$\frac{u_1}{(1+t)^3}$	$\frac{u_2}{(1+t)^4}$	$\frac{u_3}{(1+t)^5}$	$\cdots$
1	$\frac{1}{1+t}$	$\frac{u_1+t}{(1+t)^2}$	$\frac{u_2+tu_1}{(1+t)^3}$	$\frac{u_3+tu_2}{(1+t)^4}$	$\cdots$
2	1	$\frac{u_1+2t}{(1+t)}$	$\frac{u_2+2tu_1+t^2}{(1+t)^2}$	$\frac{u_3+2tu_2+t^2u_1}{(1+t)^3}$	$\cdots$
3	$1+t$	u_1+3t	$\frac{u_2+3tu_1+3t^2}{1+t}$	$\frac{u_3+3tu_2+3t^2u_1+t^3}{(1+t)^2}$	$\cdots$
4	$(1+t)^2$	$(1+t)(u_1+3t)+t(1+t)$	$u_2+4tu_1+6t^2$	$\cdots$	$\cdots$

The recursion is that to compute an entry of the table one multiplies the entry above it by $1+t$ and adds the result to t times the entry above-and-to-the-left.

(c) Ignoring the terrible-looking formulas, one important thing is evident from the table: in the column for $(X-1)^r$, after row $r+2$ we are getting polynomial multiples of $1+t$. So these entries will all vanish when we specialize to $t=-1$, and we see that in $f_r|_{t=-1}$ the first nonzero coefficient appears in the $(X-1)^{r-2}$

term. Our job is to calculate this coefficient. In order to do so, notice that it suffices to just look at all the numerators in the table; the denominators can basically be ignored. To this end, let $g_r(W)$ be the generating function for the numerators in row r of the table, specialized to $t=-1$. For example, $g_0(W) = 1 + dW + d^2W^2 + \cdots$ because $u_k(-1) = d^k$. The recursion relation for these numerators gives

$$g_{r+1} = g_r - Wg_r = (1-W)g_r.$$

So of course we will have

$$g_r = (1-W)^r g_0 = (1-W)^r \cdot \tfrac{1}{1-dW} = (1-W)^r \cdot (1 + dW + d^2W^2 + \cdots).$$

The number we are looking for is the coefficient of W^{r-2} in this power series, and it is a simple matter to compute it. The desired coeffient is

$$d^{r-2} - d^{r-3}\tbinom{r}{1} + d^{r-4}\tbinom{r}{2} - \cdots + (-1)^{r-2}\tbinom{r}{r-2}.$$

(d) Recalling (25.20) and using what we have just done, we have proven that

$$\begin{aligned}
\chi(Z_d \hookrightarrow \mathbb{C}P^n) &= f_{n+1}^{(n-1)}|_{t=-1} \cdot (-1)^{n-1} \cdot d \\
&= (-1)^{n+1} d\Big[d^{n-1} - d^{n-2}\tbinom{n+1}{1} + \cdots + (-1)^{n-1}\tbinom{n+1}{n-1}\Big] \\
&= (-1)^{n+1}\tfrac{1}{d} \cdot \Big[d^{n+1} - d^{n}\tbinom{n+1}{1} + \cdots + (-1)^{n-1}d^2\tbinom{n+1}{n-1}\Big] \\
&= (-1)^{n+1}\tfrac{1}{d} \cdot \Big[(d-1)^{n+1} - (-1)^n((n+1)d-1)\Big] \\
&= \tfrac{(1-d)^{n+1} + (n+1)d - 1}{d}.
\end{aligned}$$

None of the above formulas are particularly pleasant to look at, and they are difficult to remember. I like to encode the formula in a different way. Let ℓ denote the formal "lowering operator" that sends d^s to d^{s-1}, for each s. Then we may write

$$\chi(Z_d \hookrightarrow \mathbb{C}P^n) = d \cdot (\ell - I)^{n+1}(d^{n-1})$$

where I is the identity operator. For example,

$$\chi(Z_d \hookrightarrow \mathbb{C}P^3) = d \cdot (\ell^4 - 4\ell^3 + 6\ell^2 - 4\ell + I)(d^2) = d \cdot (6 - 4d^2 + d^3).$$

26. Chern classes

Fix a certain collection of vector bundles. A *characteristic class* for this collection assigns to each vector bundle $E \to X$ a cohomology class $b(E)$ belonging to some cohomology theory; the assignment is required to be natural. We have seen essentially two examples so far: for the collection of oriented, rank k vector bundles, we have the Euler classes e^H and e^K.

The Chern classes are characteristic classes for *complex* vector bundles, that generalize the Euler class in a certain way. Like the Euler class, they have close ties to geometry. Also like the Euler class, there are versions of Chern classes in both singular cohomology and K-theory—indeed, there are versions in any complex-oriented cohomology theory (defined in 18.3).

In this section we begin with a purely geometric look at the Chern classes, where we again forego all attempts at rigor. Afterwards we will pursue a more rigorous approach, which can even be done axiomatically.

26.1. **Geometric Chern classes in homology.** Let B be a complex manifold of dimension n, and let $E \to B$ be a complex vector bundle of rank k. If s is a generic section of E, then the locus where s vanishes gives a cycle in B that carries the Euler class $e_H(E) \in H_{2(n-k)}(B)$. This homology class will now be renamed as $C_k(E)$ and called the kth homology Chern class of E.

Now let s_1 and s_2 be *two* generic sections, chosen so that $s_1(x)$ and $s_2(x)$ are linearly independent on as large a subset of B as possible. We can now look at the degeneracy locus

$$D(s_1, s_2) = \{b \in B \,\big|\, s_1(b) \text{ and } s_2(b) \text{ are linearly dependent}\}.$$

Again, for generically chosen s_1 and s_2 it turns out this gives a cycle on B whose associated homology class is independent of any choices. The homology class lies in dimension $2(n-k+1)$, and we call it the $(k-1)$st homology Chern class $C_{k-1}(E)$.

At this point it is clear how to continue. For each j in the range $1 \leq j \leq k$, let $s_1, \dots, s_j$ be sections generically chosen to be as maximally linearly independent as possible. Consider the degeneracy locus

$$D(s_1, \dots, s_j) = \{b \in B \,\big|\, s_1(b), \dots, s_j(b) \text{ are linearly dependent}\},$$

which determines a homology class $C_{k-j+1} \in H_{2(n-k+j-1)}(B)$.

These homology classes can be thought of as obstructions to splitting off a trivial bundle. If E contains a trivial bundle of rank r then $0 = C_k(E) = C_{k-1}(E) = \cdots = C_{k-r+1}(E)$. This is clear, as by working inside the trivial subbundle we can choose our "generic sections" so that they are linearly independent everywhere.

The geometric details behind all of this can be found in [GH]. We are skipping them because they are somewhat difficult and are not needed for the cohomological version of the theory that we will describe next. The fact that cohomology gives a shortcut around the dicey geometric issues is part of the magic of homotopy theory, though it can also feel like part of the subject's curse.

26.2. **Chern classes in singular cohomology.** For the cohomological version of the theory we start by adopting an axiomatic approach. For any complex vector bundle $E \to X$ the Chern classes are cohomology classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ for $0 \leq i < \infty$ satisfying the following properties:

(1) $c_0(E) = 1$
(2) $c_i(E) = 0$ if $i > \text{rank } E$
(3) $c_i(f^*E) = f^*c_i(E)$ (naturality under pullback)
(4) The Whitney Formula: $c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E)c_{k-i}(F)$, for any k.
(5) $c_1\left(L^* \to \mathbb{C}P^1\right) = e^H(L^*) = [\mathbb{C}P^0] = [*]$, where $L \to \mathbb{C}P^1$ is the tautological line bundle.

Remark 26.3. The Whitney Formula can be written in a more convenient way using the *total Chern class*, namely

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \cdots \in H^*(X)$$

(notice that this is a finite sum by property (2)). Then the Whitney Formula becomes $c(E \oplus F) = c(E) \cdot c(F)$.

Note that if $E \to X$ is a trivial bundle then $c_i(E) = 0$ for $i > 0$. Indeed, E is the pullback of a bundle on a point: $E \cong \pi^*(\underline{n})$ where $\pi\colon X \to *$ and $n = \text{rank}(E)$. One has $c_i(\mathbb{C}^n \to *) = 0$ for $i > 0$ because a point has no cohomology in positive degrees. The fact that $c_i(E) = 0$ then follows from naturality.

Exercise 26.4. Show that given a theory of Chern classes satisfying the above axioms then for any $n \geq 1$ one has $c_1(L^*) = [\mathbb{C}P^{n-1}]$ for $L \to \mathbb{C}P^n$ the tautological bundle. Then prove that for $L \to \mathbb{C}P^\infty$ one has $c_1(L) = e(L)$, and moreoever the same is true for any line bundle on any space. (Axioms (3) and (5) should be enough for this).

Before showing the existence of the Chern classes, let us show that they are uniquely characterized by the above properties:

Proposition 26.5. *There is at most one collection of characteristic classes satisfying properties (1)–(5) above. Moreover, such a collection has the property that if $E \to X$ is a rank n bundle then the top Chern class $c_n(E)$ is equal to the Euler class $e(E)$.*

Proof. Let γ be the tautological k-plane bundle on $\mathrm{Gr}_k(\mathbb{C}^\infty)$. Consider the diagram

$$\begin{tikzcd}
\pi_1^*(L) \oplus \pi_2^*(L) \oplus \cdots \oplus \pi_k^*(L) \arrow[r] \arrow[d] & \gamma \arrow[d] \\
\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \arrow[r, "S"] & \mathrm{Gr}_k(\mathbb{C}^\infty \times \cdots \times \mathbb{C}^\infty) = \mathrm{Gr}_k(\mathbb{C}^\infty)
\end{tikzcd}$$

with the obvious maps, e.g. S is the map described in ???? that sends a collection of k lines in $\mathbb{C}^\infty$ to their direct sum in $(\mathbb{C}^\infty)^k$. Note that there are k copies of $\mathbb{C}P^\infty$ and $\mathbb{C}^\infty$ in the bottom row. Also, $\pi_i : \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the ith projection map. This diagram is a pullback, hence

$$S^*(\gamma) \cong (\pi_1^* L) \oplus \cdots \oplus (\pi_k^* L).$$

Applying cohomology to the map on the bottom row gives

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \xleftarrow{S^*} H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) .$$

By the Künneth Theorem,

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \cong H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x_1, \dots, x_k]$$

where $x_i = \pi_i^*(x)$ with $x \in H^2(\mathbb{C}P^\infty)$ being the canonical generator. There is an evident action on $\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$ by the symmetric group Σ_k. This action descends in cohomology to give the statement

$$\begin{aligned}
H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty)^{\Sigma_k} &\cong \Big[H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty)\Big]^{\Sigma_k} \\
&= \mathbb{Z}[x_1, \dots, x_k]^{\Sigma_k} \\
&= \mathbb{Z}[\sigma_1, \dots, \sigma_k]
\end{aligned}$$

where σ_i is the ith elementary symmetric function in the x_i's.

Recall that $[X, \mathrm{Gr}_k(\mathbb{C}^\infty)] \simeq \mathrm{Vect}_k(X)$. Under this bijection, S corresponds to the bundle $E = \oplus_i \pi_i^*(L)$. If $\alpha \in \Sigma_k$ then the map $S \circ \alpha$ corresponds to the direct sum of $\pi_i^*(L)$'s but where the sum is taken in a different order. Since this is isomorphic to the original bundle E, it must be that S and $S \circ \alpha$ are homotopic; in particular, they induce the same map on cohomology. Since this holds for all α, it follows that S^* lands inside the Σ_k invariants. That is, S^* can be regarded as a map

$$H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \xrightarrow{S^*} \Big[H^*(\mathbb{C}P^\infty)^{\otimes k}\Big]^{\Sigma_k} = \mathbb{Z}[\sigma_1, \dots, \sigma_k].$$

It is a theorem that the above map S^* is an isomorphism. We will not take the time to prove this, but the idea is simple enough. The Schubert cell decomposition of $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has all cells in even dimensions, and hence the coboundary maps are all zero; this computes $H^*(\mathrm{Gr}_k(\mathbb{C}^\infty))$ additively, and one readily checks that the groups have the same ranks as in $\mathbb{Z}[\sigma_1,\dots,\sigma_k]$. ????

Using the Whitney Formula (iteratively), we can see that

$$\begin{aligned} c_i(\pi_1^*L\oplus\cdots\oplus\pi_k^*L) &= \sum_\beta c_1(\pi^*_{\beta(1)}(L))\cdot c_1(\pi^*_{\beta(2)}(L))\dots c_1(\pi^*_{\beta(i)}(L)) \\ &= \sum_\beta \pi^*_{\beta(1)}(x)\cdot\pi^*_{\beta(2)}(x)\dots\pi^*_{\beta(i)}(x) \end{aligned}$$

where the sum ranges over strictly-increasing maps $\beta\colon\{1,\dots,i\}\to\{1,\dots,k\}$, and in the second sum $x=c_1(L)\in H^2(\mathbb{C}P^\infty)$. But note that if we write $x_j=\pi_j^*(x)$ then the second sum is simply the elementary symmetric function σ_i in the x_j's.

It follows from the above that $c_i(\gamma)$ is the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i under S^*.

Finally, suppose that $E\to X$ is any complex vector bundle, say of rank k. Then there is a map $f\colon X\to\mathrm{Gr}_k(\mathbb{C}^\infty)$ and an isomorphism $f^*\eta\cong E$. It follows that $c_i(E)=f^*(c_i(\eta))$.

To complete the proof, assume that c_* and c'_* are two sets of characteristic classes satisfying properties (1)–(5). Then $c_i(\eta)$ and $c'_i(\eta)$ must agree, for they each must be the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i. It then follows from naturality that $c_i(E)=c'_i(E)$ for all bundles E. □

By examining the above proof, one finds that we can *define* the Chern classes in the following way. First, when $\gamma\to\mathrm{Gr}_k(\mathbb{C}^\infty)$ is the tautological bundle then for $i\le k$ define $c_i(\gamma)$ to be the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i under S^*, and for $i>k$ define $c_i(\gamma)=0$. Second, for an arbitrary bundle $E\to X$ let $f\colon X\to\mathrm{Gr}_k(\mathbb{C}^\infty)$ be a classifying map and define $c_i(E)=f^*(c_i(\gamma))$.

Remark 26.6. For a bundle $E\to X$ one can also define K-theoretic Chern classes $c_i^K(E)\in K^0(X)$ (or really in $K^{2i}(X)$, but this is the same by periodicity). We will not pursue this at the moment, but see Section 33 below.

Example 26.7. Consider the tangent bundle $T=T\mathbb{C}P^n\to\mathbb{C}P^n$. We saw in Example 25.16 that $1\oplus T\cong(n+1)L^*$. Then by the Whitney Formula,

$$c(T)=c(1)\cdot c(T)=c(1\oplus T)=c(L^*)^{n+1}=(1+[\mathbb{C}P^{n-1}])^{n+1}.$$

Therefore,

$$c_i(T)=\tbinom{n+1}{i}[\mathbb{C}P^{n-i}]=\tbinom{n+1}{i}x^i$$

where $x\in H^2(\mathbb{C}P^n)$ is the canonical generator $[\mathbb{C}P^{n-1}]$. Note that the Euler class is $e(T)=c_n(T)=(n+1)x^n=(n+1)[*]$, and so this again calculates that $\chi(\mathbb{C}P^n)=n+1$.

Example 26.8. Consider a hypersurface $j\colon Z\hookrightarrow\mathbb{C}P^n$ of degree d. Recall from Corollary 25.12 that the normal bundle of this inclusion is $j^*\mathcal{O}(d)$, and we know $T_Z\oplus N_Z\cong j^*T_{\mathbb{C}P^n}$. Then applying total Chern classes we get

$$c(T_Z)\cdot c(N_Z)=j^*c(T_{\mathbb{C}P^n}).$$

But above we calculated that $c(T_{\mathbb{C}P^n}) = (1+x)^{n+1}$ where $x = [\mathbb{C}P^{n-1}]$, and $c(N_Z) = c(j^*\mathcal{O}(d)) = j^*(c(\mathcal{O}(d))) = 1 + d(j^*x)$. Let $z = j^*x$, so that we have

$$c(T_Z) = \tfrac{(1+z)^{n+1}}{1+dz} = (1 + (n+1)z + \tbinom{n+1}{2}z^2 + \cdots) \cdot (1 - dz + d^2z^2 - \cdots).$$

We can compute $\chi(Z)$ by finding the top Chern class (the Euler class), which in this case is $c_{n-1}(T_Z)$. A direct computation shows that

$$c_{n-1}(T_Z) = z^{n-1} \cdot \left(\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\right).$$

Finally, we need to remember that $x^{n-1} = [\mathbb{C}P^1]$ in $H^*(\mathbb{C}P^n)$ and therefore $z^{n-1} = j^*(x^{n-1}) = d[*]$, since a generic line intersects Z in d distinct points. So we have

$$c_{n-1}(T_Z) = [*] \cdot d \cdot (\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots),$$

thereby yielding

$$\chi(Z) = d \cdot \Big(\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\Big).$$

For example, a degree d hypersurface in $\mathbb{C}P^2$ has $\chi(Z) = d \cdot (3-d)$ and a degree d hypersurface in $\mathbb{C}P^3$ has $\chi(Z) = d \cdot (6 - 4d + d^2)$.

26.9. **Stiefel-Whitney classes.** One can repeat almost all of our above work in the setting of real vector bundles, but using $\mathbb{Z}/2$ coefficients everywhere. The analogs of the Chern classes in this setting are called **Stiefel-Whitney classes**. If $E \to X$ is a real vector bundle then the Stiefel-Whitney classes are cohomology classes $w_i(E) \in H^i(X;\mathbb{Z}/2)$, $0 \leq i < \infty$, satisfying the evident analogs of the axioms in Section 26.2. Geometrically, these are Poincaré Duals of certain cycles determined by degeneracy loci, just as in the complex case. In terms of our development, most things go through verbatim but the one exception is the computation $H^*(\mathrm{Gr}_k(\mathbb{R}^\infty);\mathbb{Z}/2) \cong \mathbb{Z}/2[\sigma_1,\ldots,\sigma_k]$. In the complex case this was fairly easy, because the standard cell structure on $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has cells only in even dimensions. This is of course not true for $\mathrm{Gr}_k(\mathbb{R}^\infty)$, and so one must work a bit harder here. We will not give details; see [MS].

Just as for the Chern classes, we will write $w(E)$ for the total Stiefel-Whitney class $1 + w_1(E) + w_2(E) + \cdots$.

Example 26.10. Here is an example where we can use Stiefel-Whitney classes to solve a problem that appeared earlier in these notes. Let $L \to \mathbb{R}P^n$ denote the tautological line bundle, and recall that once upon a time we needed to know whether $L \oplus L$ is stably trivial. This came up (for $n = 2$) in Section 14.10 during the course of trying to compute $KO(\mathbb{R}P^2)$.

If $(L \oplus L) \oplus \underline{N} \cong \underline{N+2}$ then applying total Stiefel-Whitney classes gives

$$1 = w(\underline{N+2}) = w(L \oplus L \oplus \underline{N}) = w(L) \cdot w(L) \cdot w(\underline{N}) = w(L)^2.$$

But L is a line bundle, so $w_i(L) = 0$ for $i > 1$ and $w_1(L)$ is the mod 2 Euler class, which we have previously computed to be the generator x on $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$. So $w(L) = 1 + x$ and therefore $w(L)^2 = 1 + x^2$. As long as $n \geq 2$ this is not equal to 1, and hence $L \oplus L$ cannot be stably trivial.

Example 26.11. Let T be the tangent bundle of $\mathbb{R}P^n$. Just as in Example 25.16 there is an isomorphism $\underline{1} \oplus T \cong (n+1)L^*$, where $L \to \mathbb{R}P^n$ is the tautological line bundle. But since we are now in the case of real bundles, $L \cong L^*$ by Corollary 8.34; so we will usually write $\underline{1} \oplus T \cong (n+1)L$. One of course finds that

$$w(T) = w(1 \oplus T) = w((n+1)L) = w(L)^{n+1} = (1+x)^{n+1},$$

similarly to the complex case.

26.12. **The projective bundle approach.** There is another approach to Chern classes that is due to Grothendieck ????. This approach adapts to any complex-orientable cohomology theory $\mathcal{E}$ with ease, so we now return to that level of generality.

Given a rank n bundle $p\colon M \to X$ consider the associated projective bundle $\pi\colon \mathbb{P}M \to X$. The pullback bundle π^*M has a canonical line-bundle inside of it; we will denote this L_M. To describe this precisely, recall that a point in $\mathbb{P}M$ is a pair (x, ℓ) where $\ell \subseteq M_x$ is a line. Then a point in π^*M is a triple (x, ℓ, v) where $\ell \subseteq M_x$ and $v \in M_x$. We define $L_M \subseteq \pi^*M$ to be the subspace of triples (x, ℓ, v) where additionally $v \in \ell$; that is, the subspace of triples (x, ℓ, v) where $v \in \ell \subseteq M_x$.

Since $L_M \subseteq \pi^*M$ we can form the quotient, which we will denote as Q_M. If X is paracompact and Hausdorff then by Proposition 9.2 we have $\pi^*M \cong L_M \oplus Q_M$ (though not canonically, of course).

Starting with $M \to X$ we have produced a map $\pi\colon \mathbb{P}M \to X$ with the property that π^*M splits off a line bundle. If we now pass to the space $\mathbb{P}Q_M \to \mathbb{P}M \to X$ then pulling back will result in Q_M splitting off a line bundle, so that M has split off a sum of two lines bundles. By iterating this process, after n steps it will be the case that when we pull back M it will split as a sum of line bundles:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2^*(L_M) \oplus L_{Q_M} \oplus Q_{Q_M} \arrow[r] \arrow[d] & L_M \oplus Q_M \arrow[r] \arrow[d] & M \arrow[d, "p"] \\
\cdots \arrow[r] & \mathbb{P}Q_M \arrow[r, "\pi_2"] & \mathbb{P}M \arrow[r, "\pi"] & X.
\end{tikzcd}$$

Now, this by itself is nothing to get excited about; after all, if we pull back M along a map $* \to X$ it also splits as a sum of line bundles. What is special about the present situation, though, is that each of the horizontal maps in the bottom row turns out to be an injection on $\mathcal{E}^*$ cohomology whenever $\mathcal{E}$ is complex orientable. So anything interesting about characteristic classes for M—which lie in $\mathcal{E}^*(X)$—will also be seen in the successive spaces as we pull back. This phenomenon is called the "splitting principle":

Proposition 26.13 (The Splitting Principle)**.** *Let $\mathcal{E}$ be a complex-oriented cohomology theory. Let X be paracompact and Hausdorff. Given any rank n complex bundle $M \to X$ there is a paracompact Hausdorff space Y_X and a map $\pi\colon Y_X \to X$ such that π^*M is an injection on $\mathcal{E}^*(-)$ and π^*M splits off a line bundle. By iterating, we can in fact assume that π^*M is isomorphic to a sum of n line bundles.*

In the above statement we have suppressed the precise identification of Y_X as the projective bundle $\mathbb{P}M$ because sometimes when invoking the splitting principle one doesn't actually care what the space Y_X is. But in fact, to prove Proposition 35.11 we will use special facts about $\mathbb{P}M$.

The bundle L_M has an Euler class $e(L_M) \in \mathcal{E}^2(\mathbb{P}M)$ (this is the $\mathcal{E}$-theory Euler class, but we suppress the $\mathcal{E}$ in the notation). In fact let us just write $e_M = e(L_M)$ for short.

Proposition 26.14. *Let $\mathcal{E}$ be a complex-oriented cohomology theory and let $M \to X$ be a rank n complex bundle. Then $\mathcal{E}^*(\mathbb{P}M)$ is free as an $\mathcal{E}^*(X)$-module on generators $1, e_M, e_M^2, \dots, e_M^{n-1}$.*

Given the above, there is a unique relation in $\mathcal{E}^*(\mathbb{P}M)$ of the form

$$e_M^n = b_n + b_{n-1}e_M + b_{n-2}e_M^2 + \cdots + b_1 e_M^{n-1}$$

where $b_i \in \mathcal{E}^{2i}(X)$. We define the ith $\mathcal{E}$-theoretic Chern class of M to be $c_i^{\mathcal{E}}(M) = (-1)^{i-1}b_i$. With this definition the following corollary is immediate:

Corollary 26.15. $\mathcal{E}^*(\mathbb{P}M) = \mathcal{E}^*(X)[e_M]/(e_M^n - c_1^{\mathcal{E}}(M)e^{n-1} + c_2^{\mathcal{E}}(M)e^{n-2} - \cdots)$.

Proof. Immediate from Proposition 26.14. □

Remark 26.16. Before proceeding let us discuss some motivation for the above definition, in particular why it is equivalent to our other definition in the case of singular cohomology. So let us return to the world where we already have the Chern classes in H^*. When we pull back M to $\mathbb{P}M$ we have the splitting $\pi^*M \cong L_M \oplus Q_M$, and taking total Chern classes gives

$$\pi^*c(M) = c(\pi^*M) = c(L_M \oplus Q_M) = c(L_M)c(Q_M) = (1 + e_M)c(Q_M).$$

Thus, we obtain

$$c(Q_M) = \frac{\pi^*c(M)}{1 + e_M} = \pi^*c(M) \cdot (1 - e_M + e_M^2 - \cdots).$$

The power series might be worrisome, and there are two ways to get past that: (1) assume that X is a finite-dimensonal CW complex, so that $\mathbb{P}M$ also is and therefore high enough powers of e_M will vanish; (2) move from $H^*(X) = \oplus_i H^i(X)$ into $\prod_i H^i(X)$ (the first injects into the second), and note that the power series makes sense in the second (all conclusions will be back in $\oplus_i H^i(X)$, though).

Since Q_M is rank $n-1$ we can now write

$$0 = c_n(Q_M) = c_n(M) - c_{n-1}(M)e_M + c_{n-2}(M)e_M^2 - \cdots + (-1)^n c_0(M)e_M^n$$

and rearranging gives

$$0 = e_M^n - c_1(M)e_M^{n-1} + c_2(M)e_M^{n-2} - \cdots$$

which is the desired formula (note that we are suppressing the π^* maps here, as $c_i(M)$ should really be $\pi^*c_i(M)$, but this practice is consistent with regarding $H^*(\mathbb{P}M)$ as an $H^*(X)$-module).

27. Comparing K-theory and singular cohomology

We have seen that singular cohomology and K-theory both encode geometry in similar ways: they have Thom classes, Euler classes, fundamental classes for submanifolds, etc. They can both be used to compute intersection multiplicities. One might hope for a natural transformation from one to the other, that allows one to directly compare what is happening in each theory. Our goal in this section is to construct such a natural transformation, with some caveats which we will discover along the way.

Let us imagine that we have a natural transformation $\phi\colon K^*(-) \to H^*(-)$, and that this is a ring homomorphism. Note first that ϕ cannot preserve the gradings, for $\beta \in K^{-2}(pt)$ is a unit whereas there is no unit in $H^{-2}(pt)$. We can fix this by formally adjoining a unit of the appropriate degree to H^*: let $H^*[t, t^{-1}]$ be the cohomology theory $X \mapsto H^*(X)[t, t^{-1}]$, where t is given degree -2. Then we can ask for a natural ring homomorphism $\phi\colon K^*(-) \to H^*(-)[t, t^{-1}]$. Restricting to $* = 0$ would give a natural ring homomorphism

$$\phi\colon K^0(-) \to H^{ev}(-) = \oplus_i H^{2i}(-).$$

We will investigate what this map can look like.

If $L \to X$ is a complex line bundle then we have the element $e^K(L) \in K^0(X)$, in some sense representing the intersection of the zero-section with itself. One's first guess would be that ϕ should send $e^K(L)$ to $e^H(L)$, as the latter represents the same 'geometry' inside of H^*. However, this hypothesis is not compatible with ϕ being a ring homomorphism. Recall from Proposition 25.14 that

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2),$$

whereas

$$e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2).$$

These formulas are incompatible.

So it cannot be that ϕ sends $e^K(L)$ to $e^H(L)$. However, it is *guaranteed* that $e^K(L)$ must be sent to *some* algebraic expression involving $e^H(L)$. Indeed, this is obviously so for the tautological bundle $L \to \mathbb{C}P^\infty$, since $e^H(L)$ is a generator of $H^2(\mathbb{C}P^\infty)$ and everything else in $H^*(\mathbb{C}P^\infty)$ is a polynomial in this generator; the case for general line bundles then follows from naturality.

So we know that ϕ will send $e^K(L) \mapsto f(e^H(L))$ for some $f(x) \in \mathbb{Z}[[x]]$. Note that when X is compact then sufficiently large powers of $e^H(L)$ will be zero, so in practice $f(e^H(L))$ is really just a *polynomial* in $e^H(L)$. Using a power series allows us to treat all spaces X at once, without assuming some uniform bound on their dimensions.

Let $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$ be the expansion for f. Note that if $L \to X$ is a trivial bundle then both $e^K(L)$ and $e^H(L)$ are zero, and from this it follows that $\alpha_0 = 0$. Next note that if ϕ is a ring homomorphism then we must have

$$\begin{aligned} f(e^H(L_1) + e^H(L_2)) &= f(e^H(L_1 \otimes L_2)) \\ &= \phi(e^K(L_1 \otimes L_2)) \\ &= \phi(e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2)) \\ &= f(e^H(L_1)) + f(e^H(L_2)) - f(e^H(L_1))f(e^H(L_2)). \end{aligned}$$

This suggests that we're looking for $f(x) \in \mathbb{Z}[[x]]$ such that

$$f(a+b) = f(a) + f(b) - f(a)f(b). \tag{27.1}$$

We can take two approaches to determine the coefficients of such an f.

Approach 1. Substitute $f(x) = \sum_i \alpha_i x^i$ into (27.1) to get

$$\text{LHS} = \alpha_1(a+b) + \alpha_2(a+b)^2 + \dots$$

and

$$\begin{aligned} RHS = [\alpha_1 a + \alpha_2 a^2 + \dots] &+ [\alpha_1 b + \alpha_2 b^2 + \dots] \\ &- [\alpha_1 a + \alpha_2 a^2 + \dots][\alpha_1 b + \alpha_2 b^2 + \dots]. \end{aligned}$$

By expanding and equating coefficients, we can try to determine the coefficients α_i. It turns out there is no equation determining α_1, but looking at the coefficient of

ab^{n-1} yields $n\alpha_n = -\alpha_1\alpha_{n-1}$, or $\alpha_n = -\frac{\alpha_1\alpha_{n-1}}{n}$. So by induction $\alpha_n = (-1)^{n-1}\frac{\alpha_1^n}{n!}$. Note, in particular, this last equation: it shows that f cannot have integral coefficients, as we were orginally guessing! So we can only make things work if the target of ϕ is $H^{ev}(-;\mathbb{Q})$.

We have been led to the conclusion $f(x) = 1-e^{-\alpha_1 x}$, and the reader may readily check that this does indeed yield a power series $f(x)$ satisfying (27.1).

Approach 2. In case you don't like the "equating coefficients" approach, one can also use some basic tools from differential equations to determine f. Recall that we want $f(a+b) = f(a)+f(b)-f(a)f(b)$. Define functions g and h by $g(a,b) = f(a+b)$ and $h(a,b) = f(a)+f(b)-f(a)f(b)$. The partial derivatives are readily computed to be

$$\frac{\partial g}{\partial a}(a,b) = f'(a+b) \quad \text{and} \quad \frac{\partial h}{\partial a}(a,b) = f'(a) - f'(a)f(b).$$

If $g(a,b)$ and $h(a,b)$ are the same function then the above partial derivatives are the same, so that $f'(a+b) = f'(a) - f'(a)f(b)$. Evaluating at $a = 0$ gives the ODE

$$f'(b) = f'(0)[1 - f(b)].$$

Setting $y = f(x)$, this becomes the separable ODE

$$\frac{dy}{dx} = f'(0)(1-y), \qquad \text{or} \qquad \frac{dy}{1-y} = f'(0)\,dx.$$

Integrating both sides yields

$$-\ln(1-y) = f'(0)x + C, \qquad \text{or} \qquad y = 1 - De^{-f'(0)x}$$

where C and D are constants. Since $f'(0) = \alpha_1$, we will write this solution as $f(x) = y = 1 - De^{-\alpha_1 x}$. We did lose some information in the differentiation process, so let's make sure this works by plugging this formula back into (27.1). We get

$$1 - De^{-\alpha_1(a+b)} = [1 - De^{-\alpha_1 a}] + [1 - De^{-\alpha_1 b}] - [1 - De^{-\alpha_1 a}][1 - De^{-\alpha_1 b}],$$

which reduces to

$$De^{-\alpha_1(a+b)} = D^2 e^{-\alpha_1(a+b)}.$$

This implies $D = D^2$, so $D = 0$ or $D = 1$. The case $D = 0$ is uninteresting to us (it corresponds to $f(x) = 1$, and we have already noted that the constant term must be zero for our application). So $D = 1$ and $f(x) = 1 - e^{-\alpha_1 x}$.

We now comment on the fact that α_1 seems to be able to take on any value whatsoever. Note that the presence of the grading on $H^*(X)$ immediately gives rise to a collection of endomorphisms on this theory. Indeed, for any $n \in \mathbb{Z}$ write $\psi_n \colon H^*(X) \to H^*(X)$ for the function that multiplies each $H^i(X)$ by n^i. This is clearly a ring homomorphism, and if we are using rational coefficients then it is even an isomorphism (provided $n \neq 0$). Note that with rational coefficients we actually have maps ψ_q for any $q \in \mathbb{Q}$.

So if we have a natural transformation $\phi \colon K^*(-) \to H^*(-;\mathbb{Q})[t,t^{-1}]$ we can compose it with the natural automorphisms ψ_q to makes lots of other natural transformations. We see that such a ϕ is far from unique. If we had a ϕ whose associated power series f was $f(x) = 1 - e^{-\alpha_1 x}$, then composing with ϕ_q gives one with associated power series $1 - e^{-q\alpha_1 x}$. This is why α_1 could not be explicitly determined.

We can turn these observations around and use them to our advantage. Since we can always compose with a ψ_q, we might as well do so in a way that simplifies things as much as possible. In particular, if we have a ϕ with associated power series $f(x) = 1 - e^{-\alpha_1 x}$ then we can compose with $\psi_{\alpha_1^{-1}}$ to get one with power series $1 - e^{-x}$. We might as well do this, to simplify matters.

Let us summarize what has happened so far. We knew that ϕ, if it exists, must send $e^K(L)$ to some power series in $e^H(L)$, for any line bundle $L \to X$. The different equations for $e(L_1 \otimes L_2)$ in K-theory versus singular cohomology then forced what this power series must be: $\phi(e^K(L)) = 1 - e^{-\alpha_1 x}|_{x=e^H(L)}$, for some $\alpha_1 \in \mathbb{Q}$. We then saw that we might as well assume $\alpha_1 = 1$, since by composing with a certain "trivial" automorphism one can arrange for this.

So now we are looking at an imagined natural transformation ϕ that sends $e^K(L)$ to $1 - e^{-x}|_{x=e^H(L)}$ for any line bundle $L \to X$. Recall that $e^K(L) = 1 - L^*$, and so $\phi(L^*) = e^{-x}|_{x=e^H(L)}$. But $e^H(L^*) = -e^H(L)$ (use that $L \otimes L^* \cong \underline{1}$, and so $e^H(L) + e^H(L^*) = e^H(\underline{1}) = 0$). So we have $\phi(L^*) = e^x|_{x=e^H(L^*)} = e^{c_1(L^*)}$. Since this must hold for any line bundle L, we might as well just write it as

$$\phi(L) = e^{c_1(L)}. \tag{27.2}$$

We next claim that ϕ is completely determined by formula (27.2). Recall the direct sum map $j\colon (\mathbb{C}P^\infty)^{\times k} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ from Section 13.14, classifying the bundle $\bigoplus_i \pi_i^*(L)$. Consider the diagram

$$\begin{tikzcd}
K^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \arrow[d, "\phi"] & K^0(\mathrm{Gr}_k(\mathbb{C}^\infty)) \arrow[l, "j_K^*"'] \arrow[d, "\phi"] \\
H^{ev}(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty; \mathbb{Q}) & H^{ev}(\mathrm{Gr}_k(\mathbb{C}^\infty); \mathbb{Q}). \arrow[l, "j_H^*"']
\end{tikzcd}$$

We know that $j^*\gamma \cong \pi_1^*(L) \oplus \cdots \oplus \pi_k^*(L)$, and therefore we see that

$$j_H^*(\phi(\gamma)) = \phi(j_K^*\gamma) = \sum_{i=1}^k \phi(\pi_i^*(L)) = \sum_{i=1}^k \pi_i^*(\phi(L)) = \sum_{i=1}^k e^{\pi_i^*(c_1(L))}.$$

Clearly this expression is invariant under the action of Σ_k, and we have said previously that j_H^* maps its domain isomorphically onto the subring of Σ_k-invariants. Thus, $\phi(\gamma)$ is determined by this formula.

Let $x_i = \pi_i^*(c_1(L))$. The power sum $x_1^r + \cdots + x_k^r$ can be written uniquely as a polynomial $S_r(\sigma_1, \ldots, \sigma_k)$ in the elementary symmetric functions of the x_i's. Here S_r is called the **rth Newton polynomial**; see Appendix D for a review of these. The first few Newton polynomials are

$$S_1 = \sigma_1, \quad S_2 = \sigma_1^2 - 2\sigma_2, \quad S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

If $E \to X$ is a vector bundle then define $s_r(E) = S_r(c_1(E), \ldots, c_k(E)) \in H^{2r}(X)$, where $k = \operatorname{rank} E$. This is a characteristic class for bundles, but it doesn't seem to have a common name. We have seen that

$$\phi(\gamma) = \sum_{k=0}^\infty \frac{1}{k!} s_k(\gamma).$$

But if $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is a classifying map for E then the commutative diagram

$$\begin{tikzcd}
K^0(X) \arrow[d, "\phi"'] & K^0(\mathrm{Gr}_k(\mathbb{C}^\infty)) \arrow[l, "f^*"'] \arrow[d, "\phi"] \\
H^{ex}(X;\mathbb{Q}) & H^{ev}(\mathrm{Gr}_k(\mathbb{C}^\infty);\mathbb{Q}) \arrow[l, "f^*"']
\end{tikzcd}$$

gives

$$\phi(E) = \phi(f^*(\gamma)) = f^*(\phi(\gamma)) = f^*\Bigl(\sum_{k=0}^{\infty} \tfrac{s_k(\gamma)}{k!}\Bigr) = \sum_{k=0}^{\infty} \tfrac{s_k(f^*\gamma)}{k!} = \sum_{k=0}^{\infty} \tfrac{s_k(E)}{k!}.$$

We have, at this point, reasoned as follows. IF there is a natural transformation of rings $\phi\colon K^0(-) \to H^{ev}(-;\mathbb{Q})[t,t^{-1}]$ THEN there is one that is given by the above formula. One can turn this around, by starting with the above formula and proving that it is a natural transformation of rings. This is not hard, and we will leave it to the reader. This natural transformation is called the **Chern character**, and is usually denoted $\mathrm{ch}\colon K^0(-) \to H^{ev}(-,\mathbb{Q})$. The defining formula is

$$\mathrm{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} s_k(E). \tag{27.3}$$

Exercise 27.4. Prove directly from the properties of Chern classes that the definition from (27.3) gives a ring homomorphism $K^0(X) \to H^{ev}(X;\mathbb{Q})$.

Since ch is a natural transformation, it of course maps $\widetilde{K}^0(X)$ into $\tilde{H}^{ev}(X;\mathbb{Q})$. Replacing X with ΣX and shifting indices, we get

$$\mathrm{ch}\colon K^{-1}(X) \to H^{odd}(X;\mathbb{Q}).$$

By periodicity we might as well regard the Chern character as giving maps

$$\mathrm{ch}\colon K^n(X) \to \oplus_p H^{n+2p}(X;\mathbb{Q}).$$

Perhaps more reasonably, we can regard ch as a map of graded rings $K^*(X) \to H^*(X)[t,t^{-1}]$ where t has degree 2.

Theorem 27.5. *The induced maps $K^n(X)\otimes\mathbb{Q} \to \oplus_p H^{n+2p}(X;\mathbb{Q})$ are isomorphisms, for all CW-complexes X.*

Proof. We have a natural transformation between two cohomology theories, so it is sufficient just to check that we have an isomorphism when X is a point. The isomorphism for general spaces then follows formally using long exact sequences, homotopy invariance, direct limits, etc.

When $s \geq 0$ the map $K^{-s}(pt) \to \oplus_p H^{-s+2p}(pt;\mathbb{Q})$ is the same, via the suspension isomorphism, as $\widetilde{K}^0(S^s) \to \widetilde{H}^{ev}(S^s;\mathbb{Q})$. When s is odd both the domain and codomain are zero, and when s is even the domain is a copy of $\mathbb{Z}$ generated by $\beta^{\frac{s}{2}}$ (by Theorem 18.22). Lemma 27.6 below (which is just a calculation) confirms that ch is a rational isomorphism in this case.

It remains to prove that $\mathrm{ch}\colon K^s(pt) \to \oplus_p H^{s+2p}(pt;\mathbb{Q})$ is a rational isomorphism when $s > 0$. If s is odd then both domain and codomain are zero. When s is even the generator of $K^s(pt)$ is $\beta^{-\frac{s}{2}}$, and the Chern character must map this to the multiplicative inverse of $\mathrm{ch}(\beta^{\frac{s}{2}})$. We have already analyzed this element, and it follows immediately that ch is a rational isomorphism in this case as well. □

Lemma 27.6. *For any $n \geq 1$, the image of* $\mathrm{ch}\colon \widetilde{K}^0(S^{2n}) \to \widetilde{H}^{ev}(S^{2n};\mathbb{Q})$ *is precisely* $\widetilde{H}^{ev}(S^{2n};\mathbb{Z})$.

Proof. Recall from Theorem 18.22 that $\widetilde{K}^0(S^{2n})$ is generated by $\beta^{\times n}$ where $\beta = 1-L \in \widetilde{K}^0(S^2)$. We can compute that $\mathrm{ch}(\beta) = 1-\mathrm{ch}(L) = 1-(1+c_1(L)) = -c_1(L)$, but this is a generator of $H^2(S^2;\mathbb{Z})$. Multiplicativity of the Chern character gives

$$\mathrm{ch}(\beta^{\times n}) = \big(\mathrm{ch}(\beta)\big)^{\times n} = c_1(L)^{\times n},$$

but the nth external product of a generator for $H^2(S^2;\mathbb{Z})$ gives a generator of $H^{2n}(S^{2n};\mathbb{Z})$. This completes the proof. $\square$

Of course a natural transformation of cohomology theories immediately extends to a natural transformation defined on pairs, using the standard techniques. If (X,A) is a pair let Cj denote the mapping cone of the inclusion $j\colon A \hookrightarrow X$, and let $\{U_j, V_j\}$ be the open cover of Cj where U_j is the mapping cylinder of j and V_j is the top half of the cone (so that $U_j \simeq X$, $V_j \simeq *$, and $U_j \cap V_j \simeq A$). Note that all of these are functorial in (X,A). We have the zig-zag of natural isomorphisms

$$K^*(X,A) \xrightarrow{\cong} K^*(U_j, U_j \cap V_j) \xleftarrow{\cong} K^*(C_j, V_j) \xrightarrow{\cong} K^*(C_j, pt) = \widetilde{K}^*(C_j)$$

and likewise for $H^*(-;\mathbb{Q})[t,t^{-1}]$. Starting with our existing map $\mathrm{ch}\colon \widetilde{K}^*(C_j) \to H^*(C_j;\mathbb{Q})[t,t^{-1}]$ we define $\mathrm{ch}\colon K^*(X,A) \to H^*(X,A;\mathbb{Q})[t,t^{-1}]$ to be the unique map that is compatible with the above isomorphisms.

This raises the following interesting question. Suppose $E_\bullet$ is a complex of vector bundles on X that is exact on A. Then $\mathrm{ch}([E_\bullet])$ is a well-defined element of $H^*(X,A;\mathbb{Q})$. How does one describe what this element is? We will return to this question in the future. ?????

The existence of the Chern character, as a multiplicative natural transformation between cohomology theories, immediately has an interesting and unexpected consequence:

Proposition 27.7. *If X is any $(2n-1)$-connected cofibrant space and $E \to X$ is a complex vector bundle then $c_n(E) \in H^{2n}(X;\mathbb{Z})$ is a multiple of $(n-1)!$.*

Proof. We first verify the result when $X = S^{2n}$. Note that if $E \to S^{2n}$ is a complex vector bundle then $\mathrm{ch}(E) = \frac{1}{n!} \cdot s_n(E)$, by definition of the Chern character. The Newton identities from Lemma D.1 show that $s_n(E) = (-1)^{n+1} n \cdot c_n(E)$, since in this case $c_1(E), \ldots, c_{n-1}(E)$ must all vanish. By Lemma 27.6 we know that ch takes its image in $H^{2n}(S^{2n};\mathbb{Z})$, and so $\frac{nc_n(E)}{n!}$ is integral; that is, $c_n(E)$ is a multiple of $(n-1)!$.

Now let X be any $(2n-1)$-connected cofibrant space. Replacing X by a homotopy equivalent space, we can assume X has a cell structure with no cells of degree smaller than $2n$; that is, the $2n$-skeleton is a wedge of $2n$-spheres. Consider the cofiber sequence $\vee S^{2n} \hookrightarrow X \to C$ where C is the cofiber, and the induced maps in cohomology:

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & \oplus\widetilde{K}^0(S^{2n}) & \longleftarrow & \widetilde{K}^0(X) & \longleftarrow & \widetilde{K}^0(C) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \\
\cdots & \longleftarrow & \oplus\widetilde{H}^{ev}(S^{2n};\mathbb{Q}) & \xleftarrow{j^*} & \widetilde{H}^{ev}(X;\mathbb{Q}) & \longleftarrow & \widetilde{H}^{ev}(C;\mathbb{Q}) & \longleftarrow & \cdots
\end{array}$$

Given a complex vector bundle $E \to X$ we know by the commutativity of the diagram that $j^*(\mathrm{ch}_n(E))$ lies in $\oplus \widetilde{H}^{ev}(S^{2n};\mathbb{Z})$. But since $H^{2n}(C;\mathbb{Q}) = 0$ the map j^* is injective in degree $2n$, and it is easy to see that $(j^*)^{-1}(H^{2n}(\vee S^{2n};\mathbb{Z})) = H^{2n}(X;\mathbb{Z})$ (if a cellular $2n$-cochain takes integral values on all of the $2n$-cells, it is integral). So $\mathrm{ch}_n(E)$ is an integral class. The same computation as in the previous paragraph shows that $\mathrm{ch}_n(E) = \pm\frac{c_n(E)}{(n-1)!}$, and this completes the proof. $\square$

The space $\mathbb{C}P^1$ is a complex manifold whose underlying topological manifold is S^2. Can any other spheres be given the structure of a complex manifold? Clearly this is only interesting for the even spheres. A simple corollary of the previous result rules out almost all possibilities:

Corollary 27.8. *If $n \geq 4$ then there is no complex structure on S^{2n}. Even more, there is no complex vector bundle whose underlying real bundle is the tangent bundle $T_{S^{2n}}$ (said differently, the tangent bundle $T_{S^{2n}}$ does not admit a complex structure).*

Proof. The second statement clearly implies the first. Let $T = T_{S^{2n}}$ and suppose that T has a complex structure. By Proposition 27.7 we know that $c_n(T)$ is a multiple of $(n-1)!$ in $H^{2n}(S^{2n};\mathbb{Z})$. But $c_n(T)$ is the Euler class of the underlying real bundle, and therefore it is twice a generator since $\chi(S^{2n}) = 2$. This implies that $\frac{2}{(n-1)!}$ is an integer, which clearly cannot happen if $n \geq 4$. $\square$

Remark 27.9. It is also known that S^4 cannot have a complex structure on its tangent bundle (and is therefore not a complex manifold); we will give a proof in Example 28.16 below, using the Todd genus. So the last remaining sphere we have not yet discussed is S^6.

It turns out that the tangent bundle to S^6 *does* admit a complex structure—one says that S^6 is an *almost complex manifold*. The construction is outlined in Exercise 27.12. Whether or not S^6 admits the structure of complex manifold is a famous open problem.

We close this section with an example showing how the Chern character can help us carry out the calculation of K-groups. This example will play an important role when we study the Atiyah-Hirzebruch spectral sequence.

Example 27.10. Recall that $\mathbb{C}P^2$ is the mapping cone on the Hopf map $\eta\colon S^3 \to S^2$. Since the suspension of η is 2-torsion ($\pi_4(S^3) \cong \mathbb{Z}/2$), a choice of null-homotopy for $\eta \circ 2$ gives a map $f\colon \Sigma^3\mathbb{R}P^2 \to S^3$ which coincides with η when restricted to the bottom cell. Let X be the cofiber of f; this is a cell complex with a 3-cell, a 5-cell, and a 6-cell. The 5-skeleton of X is $\Sigma\mathbb{C}P^2$. Our goal will be to compute the groups $\widetilde{K}^*(X)$. [Note: This choice of X, which seemingly has come out of nowhere, is motivated by the fact that this is in some sense the smallest space for which $\widetilde{K}^*(X)$ and $\widetilde{H}^*(X)$ have different torsion subgroups—see Remark 34.20 for a deeper perspective.]

There are two cofiber sequences that we can exploit: $S^3 \hookrightarrow X \to \Sigma^4\mathbb{R}P^2$ and $\Sigma\mathbb{C}P^2 \hookrightarrow X \to S^6$. We leave it to the reader to compute that $\widetilde{K}^0(\mathbb{R}P^2) = \mathbb{Z}/2$ and $K^1(\mathbb{R}P^2) = 0$, using that $\mathbb{R}P^2$ is the cofiber of $S^1 \xrightarrow{2} S^1$. Using this, the first cofiber sequence gives

$$0 = \widetilde{K}^0(S^3) \longleftarrow \widetilde{K}^0(X) \longleftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z} \longleftarrow K^1(X) \longleftarrow 0$$

(recall that $K^1(X) = K^{-1}(X)$ by periodicity). Note that we immediately deduce $K^1(X) \cong \mathbb{Z}$, and $\widetilde{K}^0(X)$ is either 0 or $\mathbb{Z}/2$. However, it is not clear how to analyze the map $\mathbb{Z} \to \mathbb{Z}/2$. The second cofiber sequence gives

$$0 \longleftarrow \widetilde{K}^0(X) \longleftarrow \mathbb{Z} \xleftarrow{\delta_K} \mathbb{Z}^2 \longleftarrow K^1(X) \longleftarrow 0$$

where the labelled map is the connecting homomorphism $\delta_K \colon \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2) \to \widetilde{K}^0(S^6)$. Again, we are left with the task of determining this map; agreement with the previous (partial) calculation demands that the cokernel either be 0 or $\mathbb{Z}/2$, and we need to determine which one. The good news is that because the domain and target are both torsion-free, there is a chance that the Chern character will give us the information we need. We will examine the commutative square

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "\cong"] & \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2) \arrow[r, "\delta_K"] \arrow[d, tail, "\mathrm{ch}"'] & \widetilde{K}^0(S^6) \arrow[r, "\cong"] \arrow[d, tail, "\mathrm{ch}"] & \mathbb{Z} \\
& \widetilde{H}^{odd}(\Sigma\mathbb{C}P^2;\mathbb{Q}) \arrow[r, "\delta_H"] & \widetilde{H}^{ev}(S^6;\mathbb{Q}).
\end{tikzcd}$$

The two Chern character maps are injective because they are rational isomorphisms and the domains are torsion-free.

Recall that $K^0(\mathbb{C}P^2) = \mathbb{Z}[Y]/(Y^3)$ where $Y = 1 - L$. Let u be the standard generator for $H^2(\mathbb{C}P^2)$, so that $c_1(L) = -u$. We have

$$ch(Y) = 1 - \mathrm{ch}(L) = 1 - (1 - u + \tfrac{u^2}{2}) = u - \tfrac{u^2}{2}, \qquad \mathrm{ch}(Y^2) = \mathrm{ch}(Y)^2 = u^2.$$

Write Y_1 and Y_2 for the suspensions of Y and Y^2, lying in $\widetilde{K}^1(\Sigma\mathbb{C}P^2)$; likewise, write u_1 and u_2 for the suspensions of u and u^2 in $H^*(\Sigma\mathbb{C}P^2;\mathbb{Z})$. Compatibility of the Chern character with suspension shows that

$$\mathrm{ch}(Y_1) = u_1 - \tfrac{u_2}{2}, \qquad \mathrm{ch}(Y_2) = u_2.$$

We must next compute the images of these classes under δ_H. But this is easy from the long exact sequence for $\Sigma\mathbb{C}P^2 \hookrightarrow X \to S^6$: one finds that $\delta(u_1) = 0$ and $\delta(u_2)$ is twice a generator in $H^6(S^6)$. So the subgroup $\langle \delta_H(\mathrm{ch}(Y_1)), \delta_H(\mathrm{ch}(Y_2))\rangle \subseteq H^*(S^6;\mathbb{Q})$ equals the subgroup $H^6(S^6;\mathbb{Z}) \subseteq H^6(S^6;\mathbb{Q})$. Finally, recall from Proposition 27.7 that the image of $\mathrm{ch}\colon \widetilde{K}^0(S^6) \to H^{ev}(S^6;\mathbb{Q})$ is also equal to $H^6(S^6;\mathbb{Z})$. It follows that δ_K is surjective, and so the cokernel of δ_K is zero. This completes our calculation: $\widetilde{K}^0(X) = 0$.

To appreciate the significance of this example, note that $\widetilde{H}^{ev}(X) \cong \mathbb{Z}/2$ (concentrated in degree 6) and $\widetilde{H}^{odd}(X) \cong \mathbb{Z}$ (concentrated in degree 3). The corresponding K-groups are $\widetilde{K}^0(X) \cong 0$ and $\widetilde{K}^1(X) \cong \mathbb{Z}$. It is a general fact that all torsion-free summands in $H^*(X)$ will also appaear in $K^*(X)$, as this follows from Theorem 27.5. But the present example demonstrates that the torsion subgroups of $H^*(X)$ and $K^*(X)$ can be different.

Exercise 27.11. If one is willing to work stably, there is a map $f\colon S^4 \to \mathbb{C}P^2$ whose cofiber is $\Sigma^{-1}X$. The induced map $f^*\colon \widetilde{K}^0(\mathbb{C}P^2) \to \widetilde{K}^0(S^4)$ is the map $\mathbb{Z}^2 \to \mathbb{Z}$ from the above example. Use the Chern character to give an alternative argument that this map is nonzero ("alternative" in the sense of not just quoting what we have already done, though the steps will be very similar).

◦ Exercises ◦

Exercise 27.12. In this exercise you will prove that TS^6 may be given a complex structure—that is, that S^6 is an almost complex manifold. The construction will make use of the $\mathbb{R}$-algebra $\mathbb{O}$ of octonions. This algebra is not associative but it is alternative: $a(ab) = (aa)b$ an $(ab)b = a(bb)$ for all a and b. The algebra comes with an involution $x \mapsto x^*$ that is $\mathbb{R}$-linear and satisfies $(xy)^* = y^*x^*$. For any $x \in \mathbb{O}$ the element xx^* is a nonnegative real multiple of the identity element 1, and is zero only when $x = 0$. Define $||x|| = \sqrt{xx^*}$

Define the real part of an octonion by $\mathrm{Re}(x) = \frac{x+x^*}{2}$, and the imaginary part by $\mathrm{Im}(x) = x - \mathrm{Re}(x)$. Note that $\mathrm{Re}(x^*) = \mathrm{Re}(x)$ for all x. Two important properties are $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ and $\mathrm{Re}(x(yz)) = \mathrm{Re}((xy)z)$ for all $x, y, z \in \mathbb{O}$. [Note that this does not imply, for example, that $\mathrm{Re}(a(bc)) = \mathrm{Re}(a(cb))$.]

(a) Define a bilinear form on $\mathbb{O}$ by $\langle x, y\rangle = \mathrm{Re}(xy^*)$. Check that this is symmetric and positive definite, and the associated norm is $|| - ||$. If p is an imaginary unit vector check that $\langle px, py\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{O}$.
(b) Verify that if $x \in \mathbb{O}$ is an imaginary unit vector then $x^2 = -1$.
(c) Regard S^6 as the unit sphere inside the imaginary part of $\mathbb{O}$. Define $J: TS^6 \to TS^6$ by $J_p(v) = pv$. Check that this is well-defined: if $p \in S^6$ and $v \in T_pS^6$ then $pv \in T_pS^6$ as well. (Hint: Given that $\langle 1, v\rangle = 0 = \langle p, v\rangle$ prove that $\langle 1, pv\rangle = 0 = \langle p, pv\rangle$.)
(d) Verify that $J^2 = -Id$, so that J gives a complex structure on the tangent bundle to S^6.

28. The topological Grothendieck-Riemann-Roch Theorem

As we present it here, the Grothendieck-Riemann-Roch (GRR) Theorem really has two components: one that is purely topological, and one that is algebro-geometric. The topological part is a comparison between the complex-oriented structures on K-theory and singular cohomology, and gives precise formulas for how they line up under the Chern character. From the perspective that we have adopted in these notes, this topological GRR theorem is fairly easy. The algebro-geometric component, on the other hand, is of a somewhat different nature; in our presentation it is a comparison between algebraic and topological K-theory, showing that certain topologically-defined maps are compatible with purely algebraic ones that at first glance appear quite different. This second part of the GRR theorem lets us see that certain algebraic constructions actually give topological invariants, whereas the first part leads to precise (although often complicated) topological formulas for these invariants.

In the present section we discuss the topological GRR theorem and some of its consequences. The next section will deal with the algebro-geometric version.

28.1. **The Todd class.** We have seen that for a line bundle $L \to X$ one has
$$\mathrm{ch}(e^K(L)) = (1 - e^{-x})|_{x=e^H(L)}.$$

The power series $1-e^{-x}$ is a multiple of x, which means that the right-hand-side can be written as $e^H(L)$ multiplied by a 'correction factor':

$$\mathrm{ch}(e^K(L)) = e^H(L)\cdot\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=e^H(L)}.$$

It is useful to have a name for this correction factor; for historical reasons, the name is actually attached to its inverse. We define the **Todd class** of L to be

$$\begin{aligned}\mathrm{Td}(L) = \Big(\frac{x}{1-e^{-x}}\Big)\Big|_{x=c_1(L)} &= \big(1-\tfrac{x}{2}+\tfrac{x^2}{6}-\tfrac{x^3}{24}+\cdots\big)\Big|_{x=c_1(L)}\\ &= \Big(1+\tfrac{x}{2}+\tfrac{x^2}{12}-\tfrac{x^4}{720}+\cdots\Big)\Big|_{x=c_1(L).}\end{aligned}$$

The coefficients in this power series are related to Bernoulli numbers, and we refer the reader to Appendix C for a review of the basics about these. The Bernoulli numbers are defined by $\frac{x}{e^x-1}=\sum_i \frac{B_i}{i!}x^i$, and so we have

$$\mathrm{Td}(L)=\sum_i (-1)^i\tfrac{B_i}{i!}c_1(L)^i.$$

Next observe that if $E\to X$ is a sum of line bundles $L_1\oplus\cdots\oplus L_k$ then

$$\begin{aligned}\mathrm{ch}(e^K(E)) &= \mathrm{ch}(e^K(L_1)\cdots e^K(L_k))\\ &= \mathrm{ch}(e^K(L_1))\cdots \mathrm{ch}(e^K(L_k))\\ &= [e^H(L_1)\cdots e^H(L_k)]\cdot\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(L_1)}\cdots\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(L_k)}\\ &= e^H(E)\cdot\prod_i\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(L_i)}.\end{aligned}$$

It therefore makes sense to define $\mathrm{Td}(E)$ to be the inverse of the product in the final formula. More generally, if E is a bundle of rank k then the Todd class of E is

$$\mathrm{Td}(E)=\prod_{i=1}^k\Big(\frac{x_i}{1-e^{-x_i}}\Big)$$

where $c_i(E)=\sigma_i(x_1,\dots,x_k)$. In other words, take the expression on the right and write each homogeneous piece as a polynomial in the elementary symmetric functions. Then replace those symmetric functions with the Chern classes of E, and one gets the Todd class. For example, if $\operatorname{rank} E=2$ then we would expand

$$\big(1+\tfrac{x}{2}+\tfrac{x^2}{12}-\tfrac{x^4}{720}+\cdots\big)\cdot\big(1+\tfrac{y}{2}+\tfrac{y^2}{12}-\tfrac{y^4}{720}+\cdots\big)$$

to get

$$1+\tfrac12(x+y)+\tfrac{1}{12}(x^2+y^2)+\tfrac14 xy+\tfrac{x^2y+xy^2}{24}-\tfrac{x^4+y^4}{720}+\tfrac{x^2y^2}{144}+\cdots$$

and then write this as

$$1+\tfrac12\sigma_1+\tfrac{1}{12}(\sigma_1^2-2\sigma_2)+\tfrac14\sigma_2+\tfrac{1}{24}(\sigma_1\sigma_2)+\tfrac{1}{720}(-\sigma_1^4+4\sigma_1^2\sigma_2+3\sigma_2^4).$$

Then replace each σ_i with $c_i(E)$ to get the formula for $\mathrm{Td}(E)$.

The first few terms of the Todd class of an arbitrary bundle are

$$\mathrm{Td}(E)=1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\frac{c_1c_2}{24}+\frac{-c_1^4+4c_1^2c_2-c_4+c_1c_3+3c_2^2}{720}+\cdots \tag{28.2}$$

The homogeneous components of this series are called the **Todd polynomials**. The fourth Todd polynomial, seen as the last term in the formula above, gives a sign of the growing complexity—particularly in the size of the denominators.

We will have more to say about computing the Todd class in Section 28.17 below. But let us now turn to the study of how it measures the comparison between K-theory and singular cohomology. As we have remarked above, one should think of the Todd class as a 'correction factor'. The most basic formula where it enters is

$$\mathrm{ch}(e^K(E)) = e^H(E) \cdot \mathrm{Td}(E)^{-1}.$$

In our above analysis we showed this when E is a sum of line bundles, but the general case readily follows from this one using the splitting principle. A very similar formula, which actually implies the above one, is the following:

Proposition 28.3. *Let $E \to X$ be a complex vector bundle. Then*

$$\mathrm{ch}(\mathcal{U}_E^K) = \mathcal{U}_E^H \cdot \mathrm{Td}(E)^{-1}.$$

That is to say, applying the Chern character to a K-theoretic Thom class does not quite give the H-theoretic Thom class—one needs the Todd class correction factor. Note, by the way, that it does not matter whether we write $\mathcal{U}_E^H \cdot \mathrm{Td}(E)^{-1}$ or $\mathrm{Td}(E)^{-1} \cdot \mathcal{U}_E^H$ in the above result, since both the Thom class and the Todd class are concentrated in even degrees.

Proof. The proof has four steps:

Step 1: If the result is true for sums of line bundles, it is true for all bundles.
Step 2: If the result is true for line bundles, it is true for all sums of line bundles.
Step 3: If the result is true for the tautological line bundle over $\mathbb{C}P^\infty$, it is true for all line bundles.
Step 4: The result is true for the tautological line bundle $L \to \mathbb{C}P^\infty$.

Step 1 is a direct consequence of the splitting principle. Indeed, if $E \to X$ is a line bundle then choose a map $p\colon \widetilde{X} \to X$ such that p^*E is a sum of line bundles and such that p^* induces monomorphisms in both singular cohomology and K-theory. If $\widetilde{E} = p^*E$, the claim follows at once from the commutative square

$$\begin{tikzcd}
K^0(E, E-0) \arrow[r, "\mathrm{ch}"] \arrow[d, tail] & H^*(E, E-0) \arrow[d, tail] \\
K^0(\widetilde{E}, \widetilde{E}-0) \arrow[r, "\mathrm{ch}"] & H^*(\widetilde{E}, \widetilde{E}-0).
\end{tikzcd}$$

Step 2 follows from the fact that $\mathcal{U}_{L_1 \oplus L_2 \oplus \cdots \oplus L_r} = \mathcal{U}_{L_1} \otimes \mathcal{U}_{L_2} \otimes \cdots \otimes \mathcal{U}_{L_r}$ and the fact that ch is multiplicative. Step 3 follows at once from naturality and the fact that every line bundle is pulled back from the tautological line bundle.

So we are reduced to Step 4, which is a calculation. Consider the zero section $\zeta\colon \mathbb{C}P^\infty \hookrightarrow L$ and the composite of natural maps

$$H^*(L, L-0) \xrightarrow{j^*} H^*(L) \xrightarrow{\zeta^*} H^*(\mathbb{C}P^\infty).$$

Recall that this composite sends $\mathcal{U}_L$ to the Euler class $e^H(L)$. The map ζ^* is an isomorphism by homotopy invariance, and the map j^* is also an isomorphism: the latter follows from the long exact sequence for the pair $(L, L-0)$ together with the fact that as spaces $L - 0 \cong \mathbb{C}^\infty - 0$ and is therefore contractible.

Consider the two elements $\mathrm{ch}(\mathcal{U}_L^K)$ and $\mathcal{U}_L^H \cdot \mathrm{Td}(L)^{-1}$ in $H^*(L, L-0)$. Applying the composite $\zeta^* \circ j^*$ sends the first to $\mathrm{ch}(e^K(L))$, by naturality. Likewise, the second is sent to $e^H(L) \cdot \mathrm{Td}(L)^{-1}$. We have already computed that these two images are the same (indeed, this is how we started off this section); since $\zeta^* \circ j^*$ is an isomorphism this means $\mathrm{ch}(\mathcal{U}_L^K) = \mathcal{U}_L^H \cdot \mathrm{Td}(L)^{-1}$. $\square$

28.4. **The Grothendieck-Riemann-Roch Theorem for embeddings.** Let $j\colon X \hookrightarrow Y$ be an embedding of complex manifolds of codimension c. We have seen that one can construct a push-forward map $j_!\colon K^*(X) \to K^{*+2c}(Y)$ and likewise in any complex-oriented cohomology theory (for example, in H^*). We will take advantage of Bott periodicity to write $j_!$ as a map $K^0(X) \to K^0(Y)$.

Consider the square

$$\begin{tikzcd} K^0(X) \arrow[r, "j_!"] \arrow[d, "\mathrm{ch}"] & K^0(Y) \arrow[d, "\mathrm{ch}"] \\ H^{ev}(X;\mathbb{Q}) \arrow[r, "j_!"] & H^{ev}(Y;\mathbb{Q}). \end{tikzcd}$$

This square does not commute; while this might seem strange, the point is just that the $j_!$ maps are defined using Thom classes and ch doesn't preserve these. But since ch almost preserves Thom classes, up to a correction factor, it follows that the above square almost commutes—up to the same factor. The precise result is as follows:

Proposition 28.5. *Let $j\colon X \hookrightarrow Y$ be an embedding of complex manifolds. Then for any $\alpha \in K^0(X)$ one has*

$$\mathrm{ch}(j_!\alpha) = j_!(\mathrm{Td}(N_{Y/X})^{-1} \cdot \mathrm{ch}(\alpha)).$$

Proof. Simply consider the diagram

$$\begin{tikzcd} K^0(X) \arrow[r] \arrow[d, "\mathrm{ch}"'] & K^0(N, N-0) \arrow[d, "\mathrm{ch}"'] & K^0(Y, Y-X) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "\mathrm{ch}"] & K^0(Y) \arrow[d, "\mathrm{ch}"] \\ H^{ev}(X) \arrow[r] & H^{ev}(N, N-0) & H^{ev}(Y, Y-X) \arrow[l, "\cong"'] \arrow[r] & H^{ev}(Y) \end{tikzcd}$$

where all singular cohomology groups have rational coefficients. The left horizontal arrows are the Thom isomorphism maps, and so the leftmost square does not commute; but the other two squares do. The compositions across the two rows are the pushforward maps $j_!$ in K-theory and singular cohomology, respectively. The desired result is now an easy application of Proposition 28.3. $\square$

The next result records how the Chern character behaves on fundamental classes:

Corollary 28.6. *If $j\colon X \hookrightarrow Y$ is an embedding of complex manifolds then $\mathrm{ch}([X]_K) = j_!(\mathrm{Td}(N_{Y/X})^{-1}) = [X]_H +$ (higher degree terms).*

Proof. Recall that $[X]_K = j_!(1)$, and so the first equality is just Proposition 28.5 applied to $\alpha = 1$. The second equality then follows directly from the fact that the Todd class of a bundle has the form $1 +$ higher degree terms together with $j_!(1) = [X]_H$. $\square$

Example 28.7. Let $j\colon Z \hookrightarrow \mathbb{C}P^n$ be a hypersurface of degree d, and consider the GRR square

$$\begin{tikzcd} K^0(Z) \arrow[r, "j_!"] \arrow[d, "\mathrm{ch}"] & K^0(\mathbb{C}P^n) \arrow[d, "\mathrm{ch}"] \\ H^{ev}(Z;\mathbb{Q}) \arrow[r, "j_!"] & H^{ev}(\mathbb{C}P^n;\mathbb{Q}). \end{tikzcd}$$

Recall that $K^0(\mathbb{C}P^n) = \mathbb{Z}[y]/(y^{n+1})$ where $y = 1 - L = [\mathbb{C}P^{n-1}]_K$, and that $H^*(\mathbb{C}P^n;\mathbb{Q}) = \mathbb{Q}[x]/(x^{n+1})$ where $x = [\mathbb{C}P^{n-1}]_H$. One has

$$\mathrm{ch}(y) = \mathrm{ch}(1-L) = \mathrm{ch}(1) - \mathrm{ch}(L) = 1 - e^{c_1(L)} = 1 - e^{-x}.$$

We will determine a formula for $[Z]_K$ by using the GRR statement $\mathrm{ch}([Z]_K) = j_!(\mathrm{Td}(N)^{-1})$. The normal bundle is $N = j^*\big((L^*)^{\otimes d}\big)$. So

$$\mathrm{Td}(N)^{-1} = j^*\left(\frac{1-e^{-dx}}{dx}\right) = j^*\big(1 - \tfrac{dx}{2} + \tfrac{d^2x^2}{6} - \tfrac{d^3x^3}{24} + \cdots\big).$$

Recall that for any α one has $j_!(j^*(\alpha)) = j_!(j^*(\alpha)\cdot 1) = \alpha \cdot j_!(1)$, and of course we know that $j_!(1) = dx$. We therefore conclude that

$$\mathrm{ch}([Z]_K) = dx \cdot \big(1 - \tfrac{dx}{2} + \tfrac{d^2x^2}{6} - \tfrac{d^3x^3}{24} + \cdots\big) = 1 - e^{-dx}.$$

We can now work backwards to determine $[Z]_K$. If we write

$$[Z]_K = a_1y + a_2y^2 + \cdots + a_ny^n$$

then

$$1 - e^{-dx} = \mathrm{ch}([Z]_K) = a_1(1-e^{-x}) + a_2(1-e^{-x})^2 + \cdots + a_n(1-e^{-x})^n.$$

Let $\alpha = e^{-x}$; then to determine the a_i's we need to expand $1 - \alpha^d$ in terms of powers of $1 - \alpha$. To do this, simply write

$$1 - \alpha^d = 1 - (1-(1-\alpha))^d = \sum_{k=1}^{d} (-1)^{k-1}\tbinom{d}{k}(1-\alpha)^k.$$

We conclude $[Z]_K = dy - \binom{d}{2}y^2 + \binom{d}{3}y^3 - \cdots$.

Of course, we have seen this calculation before in a slightly different form—see Example 23.8. But notice that GRR allowed us to carry it through without knowing anything about $[Z]_K$, whereas before we relied on the connection between K-theoretic fundamental classes and resolutions (and our ability to write down an appropriate resolution in this case).

28.8. **The general GRR theorem and some applications.** One can produce a version of the Grothendieck-Riemann-Roch theorem that works for arbitrary maps $f\colon X \to Y$ between compact, complex manifolds, not just embeddings. To do this, we first must extend our definition of pushforward maps. Note that for large enough N there is an embedding $j\colon X \hookrightarrow \mathbb{C}^N$, and therefore the map f can be factored as

$$\begin{tikzcd} & Y \times \mathbb{C}^N \arrow[d, "\pi"] \\ X \arrow[ur, dotted, "f\times j"] \arrow[r, "f"] & Y \end{tikzcd}$$

where π is projection onto the first factor. Let $B \subseteq \mathbb{C}^N$ be a large disk that contains $j(X)$. Now consider the composition

$$K^0(X) \xrightarrow{j_!} K^0(Y \times \mathbb{C}^N, Y \times (\mathbb{C}^N - B)) \cong \widetilde{K}^0(Y_+ \wedge S^{2N}) \cong K^{-2N}(Y).$$

Define this composite to be $f_!$. It requires some checking to see that this is independent of the choice of factorization of f.

Example 28.9. It is interesting to take $Y = *$. Then the pushforward $f_!$ is a map $K^0(X) \to K^{-2d}(*) \cong \mathbb{Z}$, so $f_!(1)$ gives an integer-valued invariant of the complex manifold X. It is called the **Todd genus** of X, and we will denote it Td-genus(X).

Example 28.10. We can duplicate the above definition of $f_!$ in any complex-oriented cohomology theory, and therefore we get an associated genus for complex manifolds (taking values in the coefficient ring of the theory). For singular cohomology let us call this the ***H*-genus**.

Note that if $f\colon X \to Y$ then $f_!$ is a map

$$f_!\colon H^i(X) \to H^{i+2(\dim Y - \dim X)}(Y).$$

If Y is a point then $f_!$ sends $H^i(X)$ to $H^{i-2\dim X}(pt)$, and so this is the zero map unless $i = 2\dim X$. In that dimension the cohomology of X is $\mathbb{Z}$, generated by $[*]$. But recall that $[*] = j_!(1)$ for any inclusion $j\colon * \hookrightarrow X$, and so

$$f_!([*]) = f_!(j_!(1)) = (f \circ j)_!(1) = \mathrm{id}_!(1) = 1.$$

We have therefore shown that $f_!\colon H^*(X) \to H^*(pt)$ sends an element $\alpha \in H^*(X)$ to the coefficient of $[*]$ appearing in its $2\dim(X)$-dimensional homogeneous piece. Usually it will be convenient to just say "$f_!(\alpha)$ is the top-dimensional piece of α".

As far as the H-genus is concerned, recall that it equals $f_!(1)$ for $f\colon X \to *$. But this will be zero unless $X = *$, in which case it is 1. So

$$H\text{-genus}(X) = \begin{cases} 0 & \text{if } \dim X > 0, \\ \#X & \text{if } \dim X = 0. \end{cases}$$

This is somewhat of a silly invariant, but it is what the theory gives us.

For the proof of our general version of GRR we will need to know the Todd genus of $\mathbb{C}P^n$, so let us compute this next:

Example 28.11 (Todd genus of $\mathbb{C}P^n$)**.** Recall that $\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \cdot \mathrm{Td}(F)$, and that $1 \oplus T_{\mathbb{C}P^n} \cong (n+1)L^*$. So

$$\mathrm{Td}(T_{\mathbb{C}P^n}) = \mathrm{Td}(T_{\mathbb{C}P^n}) \cdot \mathrm{Td}(1) = \mathrm{Td}\big((n+1)L^*\big) = \big[\mathrm{Td}(L^*)\big]^{n+1}.$$

Recall that $c_1(L^*) = [\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n)$. Call this generator x, for short. Then

$$\mathrm{Td}(T_{\mathbb{C}P^n}) = \Big(\frac{x}{1 - e^{-x}}\Big)^{n+1} = \Big(1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots\Big)^{n+1}.$$

Let's look at some examples. When $n = 1$ we have $x^2 = 0$ and so $\mathrm{Td}(T_{\mathbb{C}P^1}) = 1 + x$. When $n = 2$ we have $x^3 = 0$ and

$$\mathrm{Td}(T_{\mathbb{C}P^2}) = \Big(1 + \tfrac{x}{2} + \tfrac{x^2}{12}\Big)^3 = 1 + \tfrac{3}{2}x + x^2.$$

Finally, for $n = 3$ we have $x^4 = 0$ and

$$\mathrm{Td}(T_{\mathbb{C}P^3}) = \Big(1 + \tfrac{x}{2} + \tfrac{x^2}{12}\Big)^4 = 1 + 2x + \tfrac{11}{6}x^2 + x^3.$$

One discernible pattern in these polynomials is that the leading coefficient is always 1. This is an amusing exercise that we leave to the reader. It shows that $\text{Td-genus}(\mathbb{C}P^n) = 1$ for all n.

Exercise 28.12. Complete the above example by proving that the coefficient of x^n in $\left(\frac{x}{1-e^{-x}}\right)^{n+1}$ is equal to 1, for all n. One method is to interpret the coefficient as a residue:

$$\operatorname{Res}_{x=0}\left(\tfrac{1}{(1-e^{-x})^{n+1}}\right) = \tfrac{1}{2\pi i}\int_C \tfrac{1}{(1-e^{-x})^{n+1}}\,dx$$

where C is a small counterclockwise circle around the origin. Use the substitution $z = 1 - e^{-x}$ to convert this to a different residue that is easily computed. You will need to convince yourself that the mapping $x \mapsto 1 - e^{-x}$ takes C to another small counterclockwise loop around the origin.

Exercise 28.13. Let $Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Show that

$$\text{Td-genus}(Z) = d - \tbinom{d}{2} + \tbinom{d}{3} - \cdots + (-1)^n\tbinom{d}{n}.$$

Conclude that if $d \leq n$ then $\text{Td-genus}(Z) = 1$.

We now state the general GRR theorem:

Theorem 28.14 (Grothendieck-Riemann-Roch, full version)**.** *Let X and Y be compact, complex manifolds and let $f\colon X \to Y$ be a map. Then in the square*

$$\begin{tikzcd}
K^0(X) \arrow[r, "f_!"] \arrow[d, "\mathrm{ch}"] & K^0(Y) \arrow[d, "\mathrm{ch}"] \\
H^{ev}(X;\mathbb{Q}) \arrow[r, "f_!"] & H^{ev}(Y;\mathbb{Q}).
\end{tikzcd}$$

one has

$$\mathrm{ch}(f_!\alpha)\cdot \mathrm{Td}(T_Y) = f_!(\mathrm{ch}(\alpha)\cdot \mathrm{Td}(T_X))$$

for all $\alpha \in K^0(X)$, where T_X and T_Y are the tangent bundles of X and Y.

Exercise 28.15. Check that when $X \hookrightarrow Y$ is an embedding then the above version of GRR is equivalent to the version given in Proposition 28.5.

Proof of Theorem 28.14. The proof is via the steps listed below. We will outline arguments in each case, but leave some of the details to the reader.

Step 1: The result is true when f is an embedding.
Step 2: The result is true when f is $\mathbb{C}P^n \to *$.
Step 3: The result is true when f is the projection $Y \times \mathbb{C}P^n \to Y$, for any compact complex manifold Y.
Step 4: The result is true in general.

Step 1 was handled in Proposition 28.5. Step 2 is just a computation, where one computes both sides of the GRR formula and sees that they are the same. Use that $K^0(\mathbb{C}P^n)$ is generated by the classes $[\mathbb{C}P^i]$. For the left side of GRR use that $[\mathbb{C}P^i]$ is mapped to 1 via $f_!$, as $\text{Td-genus}(\mathbb{C}P^i) = 1$. For the right side use that $\mathrm{ch}([\mathbb{C}P^i]) = (1-e^{-x})^{n-i}$ and compute that the coefficient of x^n in the series $(1-e^{-x})^{n-i}\cdot\left(\frac{x}{1-e^{-x}}\right)^{n+1}$ is equal to 1. For this final piece use a method similar to what we did in Exercise 28.12 above.

For Step 3 consider the product map $K^0(Y) \times K^0(\mathbb{C}P^N) \to K^0(Y \times \mathbb{C}P^N)$. We claim that this is an isomorphism, for any CW-complex Y. Indeed, consider the functors $(X, A) \mapsto K^*(X, A) \otimes K^0(\mathbb{C}P^N)$ and $(X, A) \mapsto K^*(X \times \mathbb{C}P^N, A \times \mathbb{C}P^N)$. Our product map gives a natural transformation from the first to the second, and both functors are generalized cohomology theories (in the second case this is automatic, but in the first case this uses that $K^0(\mathbb{C}P^N)$ is free and therefore flat). One readily checks that the comparison map is an isomorphism when $(X, A) = (pt, \emptyset)$, and so it follows that it is an isomorphism for all CW-pairs (X, A).

To complete Step 3 it now suffices to verify the GRR formula on classes of the form $\alpha = (p_1)^*(\beta) \cdot (p_2)^*(\gamma)$ where $\beta \in K^0(Y)$, $\gamma \in K^0(\mathbb{C}P^N)$, and p_1 and p_2 are the projections of $Y \times \mathbb{C}P^N$ onto Y and $\mathbb{C}P^N$, respectively. Use the diagram

$$\begin{tikzcd} Y \times \mathbb{C}P^N \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbb{C}P^N \arrow[d, "\pi_1"] \\ Y \arrow[r, "\pi_2"] & * \end{tikzcd}$$

and the formulas

$$(p_1)_!\big[p_1^*\beta \cdot p_2^*\gamma\big] = \beta \cdot (p_1)_!(p_2^*\gamma) = \beta \cdot \pi_2^*\big((\pi_1)_!\gamma\big)$$

(as well as the analog of this in singular cohomology), together with Step 2.

Finally, for Step 4 factor $f \colon X \to Y$ as $X \xrightarrow{j} Y \times \mathbb{C}P^N \xrightarrow{\pi} Y$ where j is an embedding and π is projection. Use the diagram

$$\begin{tikzcd} K^0(X) \arrow[r, "j_!"] \arrow[d, "\mathrm{ch}"] & K^0(Y \times \mathbb{C}P^N) \arrow[r, "\pi_!"] \arrow[d, "\mathrm{ch}"] & K^0(Y) \arrow[d, "\mathrm{ch}"] \\ H^*(X; \mathbb{Q}) \arrow[r, "j_!"] & H^*(Y \times \mathbb{C}P^N; \mathbb{Q}) \arrow[r, "\pi_!"] & H^*(Y; \mathbb{Q}), \end{tikzcd}$$

where the horizontal composites are $f_!$. We established GRR for the two squares, by Steps 1 and 3. Deduce the general GRR by putting these two squares together. $\square$

To see one example of GRR, consider the case $Y = *$. Here the GRR theorem says $\mathrm{ch}(f_!(1)) = f_!(\mathrm{Td}(T_X))$. We will compute both sides independently, and then see what information this theorem is giving. On the left-hand-side, $f_!(1) = \text{Td-genus}(X) \cdot 1 \in K^0(*)$ and so $\mathrm{ch}(f_!(1)) = \text{Td-genus}(X) \cdot 1 \in H^0(pt; \mathbb{Q})$.

To analyze the right-hand-side we recall from Example 28.10 above that $f_! \colon H^{ev}(X) \to H^{ev}(pt)$ sends a class α to its top-dimensional piece (the component in dimension $2 \dim X$). So GRR says that

$$\text{Td-genus}(X) \cdot [*] = \text{top-dimensional piece of } \mathrm{Td}(T_X).$$

One of the surprises here is that the right-hand-side is not *a priori* an integer multiple of $[*]$: recall that the definition of the Todd class contains complicated denominators. The resulting integrality conditions can lead to some nonexistence results in topology, as demonstrated in the following example.

Example 28.16. We claim that there is no complex manifold whose underlying topological manifold is S^4; said differently, the space S^4 cannot be given a complex structure. If S^4 were a complex manifold then it would have a Todd genus, which we know will be an integer. But GRR tells us that the Todd genus is also the top-dimensional component of $\mathrm{Td}(T)$, where T denotes the complex tangent bundle of our fictitious complex manifold. But $c_1(T) = 0$ because $H^2(S^4) = 0$, and $c_2(T) = 2[*]$ because $c_2(T)$ is the Euler class and $\chi(S^4) = 2$. Plugging into (28.2) we find that $\mathrm{Td}(T) = 1 + \frac{c_2}{12}$ and so the top-dimensional piece is $\frac{1}{6}$. As this is not an integer, we have arrived at a contradiction.

A similar argument shows that S^{4n} is not a complex manifold for any n, although to follow through with this we will need to get better at computing terms in the Todd class. We return to this problem in Proposition 28.23 below.

28.17. **Computing the Todd class.** Let $a_1, a_2, \dots$ be indeterminates and write $Q(x) = 1 + a_1x + a_2x^3 + \cdots$. Let

$$Q(\underline{x}) = Q(x_1, \dots, x_n) = Q(x_1)Q(x_2)\cdots Q(x_n)$$

where the x_i's are formal variables of degree 1. This gives us a power series that is invariant under permutations of the x_i's, and so it may be written as a power series in the elementary symmetric functions $\sigma_i = \sigma_i(x_1, \dots, x_n)$, $1 \le i \le n$. Our goal will be to give a formula, in terms of the a_i's, for the coefficient of any given monomial $m = \sigma_1^{m_1} \cdots \sigma_n^{m_n}$.

Notice that the degree of m is $m_1 + 2m_2 + \cdots + nm_n$; call this number N. The coefficient of m in $Q(\underline{x})$ will not involve any terms a_i for $i > N$, so we might as well just assume that $a_i = 0$ for $i > N$. In this case write

$$Q(x) = 1 + a_1x + \cdots + a_Nx^N = (1 + t_1x)(1 + t_2x)\cdots(1 + t_Nx)$$

as a formal factorization of $Q(x)$ (or if you like, we are working in the algebraic closure of $\mathbb{Q}(a_1, \dots, a_n)$). Then

$$Q(\underline{x}) = \prod_{i=1}^{N} Q(x_i) = \prod_{i=1}^{N}\prod_{j=1}^{N}(1 + t_jx_i).$$

The evident next step is to reverse the order of the products, so let

$$Q_j = \prod_{i=1}^{N}(1 + t_jx_i) = 1 + t_j\sigma_1 + t_j^2\sigma_2 + \cdots + t_j^N\sigma_N$$

and observe that $Q(\underline{x}) = \prod_{j=1}^{N} Q_j$. Consider the process of multiplying out all factors in

$$(1 + \sigma_1t_1 + \cdots + \sigma_Nt_1^N) \cdot (1 + \sigma_1t_2 + \cdots + \sigma_Nt_2^N) \cdots (1 + \sigma_1t_N + \cdots + \sigma_Nt_N^N).$$

The first few terms are

$$1 + \sigma_1[t_1] + \sigma_2[t_1^2] + \sigma_1^2[t_1t_2] + \sigma_3[t_1^3] + \sigma_1\sigma_2[t_1t_2^2] + \sigma_1^3[t_1t_2t_3] + \cdots$$

where the bracket notation means to sum the terms in the Σ_N-orbit of the monomial inside the brackets (see Appendix D). Each of these brackets is a polynomial in the t_i's that is invariant under the symmetric group, and therefore can be written (in a unique way) as a polynomial in the variables $\sigma_i(t) = a_i$. These are the desired coefficients of $Q(\underline{x})$. The general result, whose proof has basically just been given, is the following:

Proposition 28.18. *The coefficient of $\sigma_1^{m_1} \cdots \sigma_n^{m_n}$ in $Q(\underline{x})$ is*

$$[t_1t_2\cdots t_{m_1}t_{m_1+1}^2\cdots t_{m_1+m_2}^2 t_{m_1+m_2+1}^3\cdots t_{m_1+m_2+m_3}^3\cdots t_{m_1+\cdots+m_n}^n].$$

The bracketed expression in the above result looks horrible, but it is simpler than it looks. The subscripts can basically be ignored. The idea is to write down a product of powers of the t_i's where no index i appears more than once and where the number of t_i's raised to the kth power is m_k. For example, here are a few σ-monomials and their associated coefficients:

$$\sigma_4 : [t_1^4], \qquad \sigma_1\sigma_3 : [t_1t_2^3], \qquad \sigma_2\sigma_3^2 : [t_1^2t_2^3t_3^3], \qquad \sigma_1\sigma_4^2\sigma_6 : [t_1t_2^4t_3^4t_4^6].$$

For Proposition 28.18 to be useful one has to write the bracketed expression as a polynomial in the elementary symmetric functions $\sigma_i(t) = a_i$. This is, of course, an unpleasant process. One case where it is not *so* bad is for the power sum $[t_1^n]$, since here we have the Newton polynomials S_n described in Appendix D.

Corollary 28.19. *For any $k \geq 1$, the coefficient of σ_k in $Q(\underline{x})$ is $S_k(a_1,\dots,a_k)$, where S_k is the kth Newton polynomial. By Proposition D.3 this is also equal to the two expressions*

$$(-1)^k \cdot \Big[\textit{coeff. of } x^k \textit{ in } 1 - x\frac{d}{dx}\big(\log Q(x)\big)\ \Big] = (-1)^{k-1}\cdot\Big[\textit{coeff. of } x^{k-1} \textit{ in } \frac{Q'(x)}{Q(x)}\Big].$$

Now let us specialize to $Q(x) = \frac{x}{1-e^{-x}} = \sum_i (-1)^i \frac{B_i}{i!}x^i$. Then writing $Q(x_1,\dots,x_n)$ as a power series in the elementary symmetric functions exactly yields an expression for the Todd class of a rank n vector bundle in terms of its Chern classes. Let us apply Corollary 28.19 to this situation; to do so we must compute $Q'(x)/Q(x)$. This is easy enough:

$$Q'(x) = \tfrac{1}{1-e^{-x}} - \tfrac{xe^{-x}}{(1-e^{-x})^2} = \tfrac{1}{x}Q(x) - \tfrac{e^{-x}}{1-e^{-x}}Q(x) = \big(\tfrac{1}{x} + 1 - \tfrac{1}{1-e^{-x}}\big)\cdot Q(x)$$

and so

$$1 - x\tfrac{Q'(x)}{Q(x)} = 1 - x\big(\tfrac{1}{x} + 1 - \tfrac{1}{1-e^{-x}}\big) = -x + Q(x).$$

Specializing Corollary 28.19 to the present situation now gives:

Corollary 28.20. *Let $E \to X$ be a rank n vector bundle. Then for any $2 \leq k \leq n$, the coefficient of c_k in the formula for $\mathrm{Td}(E)$ is equal to $\frac{B_k}{k!}$, whereas the coefficient of c_1 is $-\frac{B_1}{1!} = \frac{1}{2}$.*

Remark 28.21. Note that the above calculation reveals an interesting property of the coefficients of $\frac{x}{1-e^{-x}}$: when you put them into the Newton polynomials S_k the output is unaltered, at least for $k \geq 2$. That is, if $\frac{x}{1-e^{-x}} = \sum_i a_ix^i$ then $S_k(a_1,\dots,a_k) = a_k$ for $k \geq 2$. For example, $S_4 = a_1^4 - 4a_1^2 + 4a_1a_3 + 2a_2^a - 4a_4$ and the first few coeffcients of $\frac{x}{1-e^{-x}}$ are $\frac{1}{2}$, $\frac{1}{12}$, 0, and $-\frac{1}{720}$. Some grade-school arithmetic checks that indeed

$$S_4(\tfrac{1}{2}, \tfrac{1}{12}, 0, -\tfrac{1}{720}) = -\tfrac{1}{720}.$$

But this is hardly obvious from just looking at the formula for S_4.

28.22. **An application of GRR.** The following proposition is (mostly) weaker than what we proved back in Corollary 27.8. Still, we offer it as a sample application of the Todd genus.

Proposition 28.23. *No sphere S^{4n} admits the structure of a complex manifold.*

Proof. Assume S^{4n} is a complex manifold. Then it has a Todd genus, which is necessarily an integer by definition. This will be the coefficient of $[*]$ in the top component of the Todd class $\mathrm{Td}(T)$, where T denotes the complex tangent bundle to S^{4n}. To compute this Todd class directly from its definition, first note that $c_i(T) = 0$ for $i < 2n$, because $H^{2i}(S^{4n}) = 0$ in this range. We must also have $c_n(T) = e(T) = 2[*]$, because the Euler characteristic of an even sphere equals 2. These facts let us easily write down $\mathrm{Td}(T_{S^{4n}})$. Let us consider some examples of this.

When $n = 1$ we would have $\mathrm{Td}(T_{S^4}) = 1 + \frac{c_2}{12}$. The Todd genus of S^4 would then be $\frac{2}{12}$, which is not an integer. When $n = 2$ we would have $\mathrm{Td}(T_{S^8}) = 1 - \frac{c_4}{720}$, and so the Todd genus of S^8 would be $-\frac{2}{720}$; again, not an integer. These examples give the general idea, and the denominators only get worse as n gets larger.

To be specific, one will have $\mathrm{Td}_{S^{4n}} = 1 + Mc_{2n}$ where M is a mystery number that must be computed from the definition of the Todd class. It is a consequence of Corollary 28.20 that $M = \frac{B_{2n}}{(2n)!}$. The Todd genus of S^{4n} will then be $2 \cdot B_{2n}/(2n)!$. By Theorem C.5 the number 3 divides the lowest-terms-denominator of B_{2n}, and so this expression cannot be an integer. This is our contradiction. $\square$

Remark 28.24. Notice why our proof of Proposition 28.23 does not extend to cover spheres S^{4n+2}: the Chern class c_{2n+1} does not appear by itself in the formula for the Todd class, because the odd Bernoulli numbers are zero. If S^{4n+2} has a complex structure one can conclude that its Todd genus is zero, but this by itself does not produce a contradiction.

28.25. **The arithmetic genus.** We now discuss the problem of computing the Todd genus for smooth algebraic subvarieties $Z \hookrightarrow \mathbb{C}P^n$. We will see that it can be described entirely in terms of algebro-geometric data. This material foreshadows much of what we do in Section 29.

Let $p\colon \mathbb{C}P^n \to *$ and $q\colon Z \to *$ be the squash maps, and consider the composition

$$K^0(Z) \xrightarrow{j_!} K^0(\mathbb{C}P^n) \xrightarrow{p_!} K^0(pt).$$

The composite is $q_!$ and therefore sends 1 to $\mathrm{Td}\text{-genus}(Z) \cdot [*]$. On the other hand, if we write

$$j_!(1) = [Z] = a_{n-1}[\mathbb{C}P^{n-1}] + a_{n-2}[\mathbb{C}P^{n-2}] + \cdots + a_0[\mathbb{C}P^0]$$

then since $p_!([\mathbb{C}P^{n-i}]) = \mathrm{Td}\text{-genus}(\mathbb{C}P^{n-i}) \cdot [*] = [*]$ we have

$$p_!(j_!(1)) = (a_{n-1} + a_{n-2} + \cdots + a_0)[*].$$

So $\mathrm{Td}\text{-genus}(Z) = \sum_i a_i$.

Recall that knowing $[Z]$ is the same as knowing the Hilbert polynomial of Z. We wish to ask the question: how can the Todd genus be extracted from the Hilbert polynomial? To answer this, start by recalling the diagram

$$\begin{tikzcd}
K^0_{\mathrm{alg}}(\mathbb{C}P^n) \arrow[r, "\phi", "\cong"'] \arrow[d, "\cong"'] & K^0(\mathbb{C}P^n) \\
G_{grd}(\mathbb{C}[x_0, \ldots, x_n])/\langle[\mathbb{C}]\rangle \arrow[r, tail, "\mathrm{Hilb}"] & \mathbb{Q}[s]
\end{tikzcd}$$

from Section 23.18. The image of the function Hilb is the $\mathbb{Z}$-submodule of $\mathbb{Q}[s]$ generated by $\binom{s+n}{n}, \binom{s+n-1}{n-1}, \ldots, \binom{s}{0}$. If one takes $[\mathbb{C}P^{n-i}] \in K^0(\mathbb{C}P^n)$ and pushes it around the diagram, we have seen in Section 23.18 that the corresponding Hilbert polynomial is $\binom{s+n-i}{n-i}$.

The Todd genus can be thought of as the unique function $K^0(\mathbb{C}P^n) \to \mathbb{Z}$ sending all the classes $[\mathbb{C}P^{n-i}]$ to 1. We look for a similar function $\operatorname{im}(\text{Hilb}) \to \mathbb{Z}$ that sends $\binom{s+n-i}{n-i}$ to 1, for all i. A moment's thought shows that the map "evaluate at $s = 0$" has this property. We have therefore proven the following:

Proposition 28.26. *Let $Z \hookrightarrow \mathbb{C}P^n$ be an algebraic subvariety. Then the Todd genus of Z is* $\text{Hilb}_Z(0)$.

In algebraic geometry, the invariant $\text{Hilb}_Z(0)$ is sometimes called the **arithmetic genus**. So we have proven that the arithmetic genus and Todd genus coincide.

Remark 28.27. Many authors use the term *arithmetic genus* for the invariant $(-1)^{\dim Z}(\text{Hilb}_Z(0) - 1)$. This is the definition in both [H] and [GH], for example. Obviously the two definitions carry the same information, and the difference between them is only a matter of "normalization". The invariant $\text{Hilb}_Z(0)$ is sometimes called the *Hirzebruch genus*, or the *holomorphic Euler characteristic* (see Section ??? below for more information about this).

28.28. **Fundamental classes and the Todd genus.** Again let $Z \hookrightarrow \mathbb{C}P^n$ be a complex submanifold of codimension c and consider the fundamental class $[Z] \in K^0(\mathbb{C}P^n)$. Write

$$[Z] = a_{n-c}[\mathbb{C}P^{n-c}] + a_{n-c-1}[\mathbb{C}P^{n-c-1}] + \cdots + a_0[\mathbb{C}P^0].$$

We have seen that a_{n-c} is the degree of Z, which has a simple geometric interpretation: it is the number of intersection points of Z with a generic linear subspace of dimension c. But the question remains as to how to give a geometric interpretation for the other a_i's. We will now explain how the Todd genus gives an answer to this (although perhaps not an entirely satisfactory one).

As we saw in the last section, $\text{Td-genus}(Z) = \sum_{0 \leq i} a_i$. But note that multiplying the equation for $[Z]$ by $[\mathbb{C}P^{n-1}]$ gives $[Z] \cdot [\mathbb{C}P^{n-1}] = a_{n-c}[\mathbb{C}P^{n-c-1}] + \cdots + a_1[\mathbb{C}P^0]$ and therefore

$$\text{Td-genus}(Z \cap \mathbb{C}P^{n-1}) = \sum_{1 \leq i} a_i.$$

Here $Z \cap \mathbb{C}P^{n-1}$ indicates the intersection of Z with a generic hyperplane in $\mathbb{C}P^n$.

Likewise we have $[Z] \cdot [\mathbb{C}P^{n-j}] = a_{n-c}[\mathbb{C}P^{n-c-j}] + \cdots + a_j[\mathbb{C}P^0]$, and hence

$$\text{Td-genus}(Z \cap \mathbb{C}P^{n-j}) = \sum_{j \leq i} a_i.$$

So the partial sums $\sum_{j \leq i} a_i$ for $j = 0, 1, \ldots, n-c$ are the same as the Todd genera of $Z, Z \cap \mathbb{C}P^{n-1}, \ldots, Z \cap \mathbb{C}P^c$. (This gives another explanation for why a_{n-c} is the degree of Z). We immediately obtain the formulas

$$a_i = \text{Td-genus}(Z \cap \mathbb{C}P^{n-i}) - \text{Td-genus}(Z \cap \mathbb{C}P^{n-i-1}), \tag{28.29}$$

where again the intersections are interpreted to be generic. This is our desired geometric description of the a_i's. Note, however, that whether or not this is indeed "geometric" depends on whether one feels that this adjective applies to the Todd genus.

29. The algebro-geometric GRR theorem

Let X be a compact complex manifold and $E \to X$ a complex vector bundle. If π denotes the projection $X \to *$ then we get an element $\pi_!([E]) \in K^0(pt) = \mathbb{Z}$. This gives an integer-valued invariant of the bundle E, which we will call the **Todd number** of E. We will write it as

$$\text{Td-num}_X(E) = \pi_!([E]).$$

The topological GRR theorem identifies this number as $\Theta_X(\mathrm{Td}(T_X) \cdot \mathrm{ch}(E))$, and so we can calculate it in terms of the Chern classes of E and T_X. Note that the Todd number of the trivial bundle $\underline{1}$ is the Todd *genus* of X.

If X is an algebraic variety and $E \to X$ is an algebraic vector bundle then there is another way to compute the Todd number of E, in terms of algebro-geometric invariants. This identification of invariants is an example of the algebro-geometric GRR theorem. Although the theorem covers far more than just the Todd number, we will concentrate on this special case before stating the more general result.

29.1. Sheaf cohomology.

As we saw in ???? the algebraic vector bundle $E \to X$ gives rise to an associated coherent sheaf on the Zariski space X_{Zar}. We also call this sheaf E, by abuse. Modern algebraic geometry shows how to obtain sheaf cohomology groups $H^i(X; E)$. The general theory is technical (although not incredibly hard), and would take too long to recount here; the level of abstraction and technicality is roughly comparable to that of singular cohomology. But just as in the latter case, there are methods for *computing* the sheaf cohomology groups that do not require the high-tech definitions.

Suppose we have a Zariski open cover $\{U_\alpha\}$ of X with the property that each U_α is affine, and moreover assume that each iterated intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is affine (for each $k \geq 1$). Write $\Gamma(U_\alpha, E)$ for the algebraic sections of E defined over U_α. Then we may form the Čech complex

$$0 \to \bigoplus_\alpha \Gamma(U_\alpha, E) \to \bigoplus_{\alpha_1,\alpha_2} \Gamma(U_{\alpha_1} \cap U_{\alpha_2}, E) \to \cdots$$

and the sheaf cohomology group $H^i(X; E)$ is just isomorphic to the ith cohomology group of this complex.

Example 29.2 (Cohomology of $\mathcal{O}(k)$ on $\mathbb{C}P^n$)**.** Let $x_0, \ldots, x_{n+1}$ be homogeneous coordinates on $\mathbb{C}P^n$, and for each $0 \leq j \leq n$ let $U_j \subseteq \mathbb{C}P^n$ be the open subscheme defined by $x_j \neq 0$. Write $U_{j_1 \cdots j_r} = U_{j_1} \cap \cdots \cap U_{j_r}$, and note that all of these are affine. Indeed, U_j is the spectrum of $\mathbb{C}[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}]$, $U_{j,k}$ is the spectrum of the localization of this ring at x_k/x_j, and so forth.

Let $S = \mathbb{C}[x_0, \ldots, x_n]$, regarded as a graded ring where all x_i's have degree 1. Let $R_j = \Gamma(U_j, \mathcal{O}) = \mathbb{C}[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}]$. This is the degree zero homogeneous piece of the localization S_{x_j}. Further, observe that each $\mathcal{O}(k)$ is trivializable over U_j, and so $\Gamma(U_j, \mathcal{O}(k))$ will be a free R_j-module of rank 1. We can identify $\Gamma(U_j, \mathcal{O}(k))$ with the degree k homogeneous component of the localization S_{x_j}; that is, it is the $\mathbb{C}$-linear span of all monomials $x_0^{a_0} \cdots x_n^{a_n}$ where $a_j \in \mathbb{Z}$ and all other $a_i \geq 0$. This is readily checked to coincide with the cyclic R_j-module $x_j^k R_j$.

The analogs of the above facts work for any open set $U_{\underline{j}} = U_{j_1} \cap \cdots \cap U_{j_r}$. The sections $\Gamma(U_{\underline{j}}, \mathcal{O}(k))$ form the degree k homogeneous component of the ring

$S_{x_{j_1}\cdots x_{j_r}}$. We want to examine the Čech complex $\check{C}(U_\bullet, \mathcal{O}(k))$ for each value of k, but it is more convenient to take the direct sum over all values for k and consider them all at once.

For a collection of indices $\sigma \subseteq \{0, \dots, n\}$ let S_σ be the localization of S at the element $\prod_{i\in\sigma} x_i$. Then consider the augmented Čech complex

$$C^\bullet : \quad 0 \to S \to \oplus_i S_i \to \oplus_{i<j} S_{ij} \to \cdots \to S_{01\cdots n} \to 0$$

where the S is in degree -1 and where the differentials are all induced by the inclusions $S_\sigma \hookrightarrow S_{\sigma'}$ for $\sigma \subseteq \sigma'$. That is to say, if we have a tuple $\alpha = (\alpha_\sigma \in S_\sigma)_{\#\sigma=r}$ then $d\alpha$ is the tuple whose value at $\sigma' = \{i_0, \cdots, i_r\}$ (with the entries ordered from least to greatest) is

$$(d\alpha)_{\sigma'} = \sum_{k=0}^{r} (-1)^k \alpha_{i_0\cdots\widehat{i_k}\cdots i_r}.$$

The Čech complex for computing cohomology is obtained from $C^\bullet$ by omitting the S in degree -1, but we will quickly see why it is convenient to have that S around.

It is easy to compute the cohomology group $H^n(C^\bullet)$. The ring $S_{01\cdots\widehat{i}\cdots n}$ is generated as a vector space by monomials $x_0^{a_0}\cdots x_n^{a_n}$ where $a_i \geq 0$. So the image of $C^{n-1} \to C^n$ is the span of all monomials where *some* a_i is nonnegative. The monomials that are not in the image have the form $x_0^{-1}\cdots x_n^{-1} \cdot (x_0^{-b_0}\cdots x_n^{-b_n})$ where all $b_i \geq 0$. So $H^n(C^\bullet)$ only has terms in degree $k \leq -(n+1)$, and in such a degree the group is isomorphic to $S^{-k-(n+1)}(\mathbb{C}^{n+1})$. This computation will be subsumed by the more general one in the next paragraph, but it is useful to see this particular case by itself.

To compute the cohomology of $C^\bullet$ in all dimensions it is useful to regard S, and each of its localizations, as multigraded by the group $\mathbb{Z}^{n+1}$ (by the multidegrees of monomials). The maps in the complex preserve this multigrading, so we might as well look at one multidegree $\underline{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ at a time. Let $\tau = \{i_0, \dots, i_u\}$ be the complete list of indices for which $a_i < 0$. Note that S_σ is zero in multidegree $\underline{a}$ unless $\sigma \supseteq \tau$: that is, we will only have monomials of multidegree $\underline{a}$ if we have inverted the x_i's for $i \in \tau$. It is not hard to check that $C^\bullet_{\underline{a}}$ (the portion of $C^\bullet$ in multidegree $\underline{a}$) coincides with the augmented simplicial chain complex for $\Delta^{n-\#\tau}$ with coefficients in $\mathbb{C}$. The point is that the rings S_σ that are nonzero in degree $\underline{a}$ correspond to precisely those σ that contain τ, and these correspond in turn to subsets of $\{0, 1, \dots, n+1\} - \tau$. Subsets of $\{0, 1, \dots, n+1\} - \tau$ also index the simplices of $\Delta^{n-\#\tau}$, and we leave it to the reader to verify that the complexes do indeed coincide.

The augmented simplicial cochain complex for $\Delta^{n-\#\tau}$ has zero cohomology except in one extreme case—for when $\#\tau = n+1$ we have the augmented cochain complex of the emptyset, and this has a single $\mathbb{Z}$ in its cohomology. This corresponds to those multidegrees $\underline{a}$ in which all $a_i < 0$; for these the total degree satisfies $\sum a_i \leq -(n+1)$.

We have seen that $C^\bullet$ is exact except in cohomological degree n, and there we get a single copy of $\mathbb{C}$ in every multidegree $\underline{a}$ for which $\sum a_i \leq -(n+1)$. So for a fixed integral degree $k \leq -(n+1)$ the kth homogeneous component of $H^n(C)$ is the $\mathbb{C}$-linear span of monomials $\underline{x}^{\underline{a}}$ where all $a_i < 0$ and $\sum a_i = k$. This is what we saw earlier in the argument as well, and it gives us that $H^n(C)_k \cong S^{-k-(n+1)}(\mathbb{C}^{n+1})$.

Finally, let us turn to our original Čech complex by removing the S from degree -1 of $C^\bullet$. In doing so we introduce homology in degree 0, and the graded homology groups exactly coincide with the homogeneous components of S. That is, the kth graded piece of H^0 is isomorphic to $S^k(\mathbb{C}^{n+1})$.

We have now proven that

$$H^i(\mathbb{C}P^n; \mathcal{O}(k)) \cong \begin{cases} S^k(\mathbb{C}^{n+1}) & \text{if } i = 0, \\ S^{-k-(n+1)}(\mathbb{C}^{n+1}) & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

The following table shows these sheaf cohomology groups. Note that these vanish except for $0 \leq i \leq n$. In the table we write $\mathcal{O}_k$ instead of $\mathcal{O}(k)$, for typographical reasons, and we write $V = \mathbb{C}^{n+1}$. The dth symmetric power of V is denoted S^dV; note that this is isomorphic to the space of degree d homogeneous polynomials in $x_0, \dots, x_n$.

Table 29.2. Cohomology groups $H^i(\mathbb{C}P^n; \mathcal{O}(k))$

i	$\mathcal{O}_{-(n+3)}$	$\mathcal{O}_{-(n+2)}$	$\mathcal{O}_{-(n+1)}$	$\mathcal{O}_{-n}$	$\cdots$	$\mathcal{O}_{-1}$	$\mathcal{O}$	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$
0	0	0	0	0	$\cdots$	0	$\mathbb{C}$	S^1V	S^2V	S^3V
1	0	0	0	0	$\cdots$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	0	0	0	0	$\cdots$	0	0	0	0	0
n	S^2V	S^1V	$\mathbb{C}$	0	$\cdots$	0	0	0	0	0

29.3. **Sheaf cohomology and the Todd number.** When X is a projective variety the sheaf cohomology groups we introduced in the last section turn out to have the following properties:

- They are finite-dimensional over $\mathbb{C}$;
- $H^i(X; E)$ vanishes when $i > \dim X$;
- A short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ gives rise to a long exact sequence of sheaf cohomology groups.

The first two properties allow us to define the sheaf-theoretic Euler characteristic

$$\chi(X; E) = \sum_i (-1)^i \dim H^i(X; E).$$

and the third property yields that $\chi(X; E) = \chi(X; E') + \chi(X; E'')$. So $\chi(X; -)$ gives a homomorphism $K^0_{\mathrm{alg}}(X) \to \mathbb{Z}$. It will not come as a surprise that this agrees with the topologically-defined pushforward map $\pi_!$:

Proposition 29.4. *If X is a projective algebraic variety and $E \to X$ is an algebraic vector bundle then*

$$\text{Td-num}_X(E) = \chi(X; E).$$

This gives us our algebro-geometric interpretation of the Todd number. We postpone the proof for the moment, prefering to obtain this as a corollary of our general GRR theorem. But let us at least check the proposition in the important example of $\mathbb{C}P^n$. Recall that $K^0_{\mathrm{alg}}(\mathbb{C}P^n)$ is the free abelian group generated by $[\mathcal{O}], [\mathcal{O}(1)], [\mathcal{O}(2)], \dots, [\mathcal{O}(n)]$; so we can verify the result for all algebraic vector bundles E on $\mathbb{C}P^n$ by checking it for these particular $n+1$ cases. Luckily we have already computed the vector spaces $H^i(\mathbb{C}P^n; \mathcal{O}(k))$. From Table 29.1 we find that

$$\chi(\mathbb{C}P^n; \mathcal{O}(k)) = \begin{cases} \dim S^k\mathbb{C}^{n+1} & \text{if } k \geq 0, \\ 0 & \text{if } -n \leq k < 0, \\ (-1)^n \dim S^{-k-(n+1)}\mathbb{C}^{n+1} & \text{if } k < -n. \end{cases}$$

We leave the reader to check that all three cases in the above formula can be unified into the simple statement $\chi(\mathbb{C}P^n; \mathcal{O}(k)) = \binom{n+k}{n}$.

It remains to compute the Todd number of $\mathcal{O}(k)$. We use the by-now-familiar technique from Exercise 28.12:

$$\begin{aligned} \Theta_X\Big[\mathrm{Td}(T_X)\cdot \mathrm{ch}(\mathcal{O}(k))\Big] = \Theta_n\Big[\big(\tfrac{x}{1-e^{-x}}\big)^{n+1}\cdot e^{kx}\Big] &= \mathrm{Res}_{x=0}\big(\tfrac{1}{1-e^{-x}}\big)^{n+1}\cdot e^{kx}\,dx \\ &= \mathrm{Res}_{z=0}\big(\tfrac{1}{z^{n+1}}\cdot\tfrac{1}{(1-z)^{k+1}}\,dz\big) \\ &= \Theta_n\big((1-z)^{-(k+1)}\big) \\ &= (-1)^n\tbinom{-(k+1)}{n} \\ &= \tbinom{n+k}{n}. \end{aligned}$$

Note that the substitution $z = 1 - e^{-x}$ was used for the third equality.

30. Interlude: The classical Riemann-Roch theorem

The classical Riemann-Roch theorem relates to the problem of counting dimensions of spaces of meromorphic functions, on a fixed Riemann surface, with prescribed bounds on their zeros and poles. It is certainly not immediately clear how this is connected to the fancy Grothendieck-Riemann-Rochs theorems we encountered in the previous two sections. In this section we will explain the classical problems how their solutions fit into the modern perspective.

Let X be a Riemann surface, i.e. a compact, connected complex manifold of dimension one. A map $X \to \mathbb{C}$ is said to be holomorphic if it restricts to a holomorphic map on every open set in a coordinate chart for X. There aren't very many such maps! In fact we have:

Proposition 30.1. *On a Riemann surface all holomorphic maps are constant.*

Proof. Let $f\colon X \to \mathbb{C}$ be holomorphic. By continuity, the image of f is compact. By the Open Mapping Theorem from complex analysis, if f is not constant then its image is open in $\mathbb{C}$ (apply the theorem to each coordinate neighborhood of X). But the image of f cannot be both open and compact, so we have a contradiction. Hence f must be a constant map. $\square$

Instead of holomorphic functions let us instead consider meromorphic functions. There are uncountably many of these: for example, on $\mathbb{C}P^1$ we have the functions $\frac{1}{(z-a)^n}$ for every $a \in \mathbb{C}$ and $n \in \mathbb{Z}$. To cut things down to size, let us fix the locations of the possible poles and also the maximum degree of each pole. To this end, let us introduce some terminology. A **divisor** on X is a formal linear combination $D = \sum m_i[P_i]$ where the P_i are finitely-many points in X and each $m_i \in \mathbb{Z}$. We can also write $D = \sum_{P\in X} m_P[P]$ where it is understood that $m_P \neq 0$ only for finitely many P. Say that $D_1 \geq D_2$ if $m_P(D_1) \geq m_P(D_2)$ for all points P.

If $f\colon X \to \mathbb{C}$ is meromorphic and not identically zero then in local coordinates near a point P the function looks like $f(z) = a_k z^k + a_{k+1}z^{k+1} + \cdots$ with $a_k \neq 0$, for some $k \in \mathbb{Z}$. Set $\operatorname{ord}_P(f) = k$ and call this the "order of vanishing of f at P". Note that P is a zero of f if and only if $\operatorname{ord}_P(f) \geq 1$, and P is a pole of f if and only if $\operatorname{ord}_P(f) < 0$. When $n \geq 0$ it will be convenient to adopt the following language: $\operatorname{ord}_P(f) = n$ will be phrased as "f has n zeros at P", and $\operatorname{ord}_P(f) = -n$ will be phrased as "f has n poles at P".

Define

$$\operatorname{div}(f) = \sum_{P \in X} \operatorname{ord}_P(f)[P].$$

Note that $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$.

Definition 30.2. *For a divisor D, let*

$$L(D) = \{f\colon X \to \mathbb{C} \,|\, f \textit{ is meromorphic and } \operatorname{div}(f) \geq -D\}.$$

Set $\ell(D) = \dim L(D)$ (which a priori might be infinite).

For example, $L(5[P] - 2[Q])$ is the set of meromorphic functions having at most 5 poles at P, at least 2 zeros at Q, and no other poles. In general, functions in $L(D)$ have upper bounds on how many poles they have at each point and lower bounds on how many zeros they have at each point.

Exercise 30.3. Check that in this language f being holomorphic at a point P is equivalent to saying that "f has at least 0 zeros at P", and also equivalent to saying that "f has at most 0 poles at P". (The former version is a little awkward but technically okay.)

Proposition 30.4. *For any divisor D and point P one has $L(D) \subseteq L(D+[P])$, and moreover $\dim L(D+[P])/L(D) \leq 1$. Consequently, $\ell(D) \leq \ell(D+P) \leq \ell(D)+1$.*

Proof. If $f, g \in L(D+P)\backslash L(D)$ then f and g have the same order of vanishing at P, and so some linear combination $af + bg$ has a zero at P and is therefore in $L(D)$. $\square$

Corollary 30.5. *For every divisor D, $L(D)$ is finite-dimensionsal. Even more, $\ell(D) \leq 1 + \sum\limits_{m_P > 0} m_P$.*

Proof. If $D \leq 0$ then $L(D)$ consists entirely of holomorphic (hence constant) functions, and so $L(D)$ is either $\{0\}$ or $\mathbb{C}$. Adding summands $[P]$ to D can increase the dimension of $L(D)$ by at most one. $\square$

Our main goal will be to compute $\ell(D)$ for every divisor D. These numbers give interesting invariants of the Riemann surface. This is a tall order, though!

Define the **degree** of a divisor D by $\deg(D) = \sum_{P \in X} m_P(D)$.

Proposition 30.6. *For any meromorphic function f on X, $\deg(\operatorname{div} f) = 0$. (In other words, f has the same number of zeros and poles).*

Proof. This is really a theorem of topology. The map $f\colon X \to \mathbb{C}P^1$ has a topological degree $d \geq 0$, meaning that the induced map $f_*\colon H_2(X) \to H_2(\mathbb{C}P^1)$ sends the fundamental class $[X]$ to $d[\mathbb{C}P^1]$. Note that $d \geq 0$ because f is locally orientation-preserving. The degree can be computed locally by counting (with multiplicity) the preimages of any point. If m_P is the local degree of f at a point P then we have

$$\sum_{P\in f^{-1}(\infty)} m_P = d = \sum_{Q\in f^{-1}(0)} m_Q.$$

Now just observe that the local topological degree at a zero or pole agrees with the absolute value of its algebraic order, by a standard computation. So we have $\operatorname{div}(f) = \sum_{Q\in f^{-1}(0)} m_Q[Q] - \sum_{P\in f^{-1}(\infty)} m_P[P]$, and thus the degree of this advisor is zero. ☐

Corollary 30.7. *If* $\deg(D) < 0$ *then* $\ell(D) = 0$.

Proof. This is easiest to explain via an example. The space $L(P+Q-3R)$ consists of functions with at least 3 zeros at R, but possible poles only at P and Q and each of order at most 1. But if there are at least three zeros at R the function must have at least three poles as well, by Proposition 30.6. So there are no functions in $L(P+Q-3R)$ other than the zero function.

In general, if $\deg(D) < 0$ then $L(D)$ consists of functions with more zeros than poles, and by Proposition 30.6 no such (nonzero) functions exist. ☐

We also obtain the following improvement to the inequality in Corollary 30.5:

Corollary 30.8. *If* $\deg(D) \geq 0$ *then* $\ell(D) \leq \deg(D) + 1$.

Proof. Let P be a point in X and consider $E = D - (\deg(D) + 1)[P]$. Then $\deg(E) < 0$, so $\ell(E) = 0$ by Corollary 30.7. But then

$$\ell(D) = \ell(E + (\deg(D) + 1)[P]) \leq \ell(E) + \deg(D) + 1$$

where the inequality is by repeated application of Proposition 30.4. ☐

Let us compute $\ell(P)$ for any point P in our Riemann surface. Elements of $L(P)$ are meromorphic functions having at most a single pole, at P. So $\ell(P) \geq 1$, since $L(P)$ constains the constant functions. When $X = \mathbb{C}P^1$ then at a point $P = a$ we have $L(P) = \langle 1, \frac{1}{z-a}\rangle$ and thus $\ell(P) = 2$. The following result covers all other cases:

Proposition 30.9. *If the genus of* X *is positive then* $\ell(P) = 1$ *for all* $P \in X$. *Moreover,* $\ell(P - Q) = 0$ *for all* $P \neq Q$.

Proof. Suppose $f \in L(P)$ is non-constant. Then $f\colon X \to \mathbb{C}P^1$ has degree equal to 1, and is therefore injective (since the degree can be computed as the number of elements in the preimage of any point, counted with multiplicity). At the same time, the image of f must be open and compact (hence closed); therefore f is also surjective. So f is a homeomorphism and the genus of X is zero.

If X has positive genus we have now shown that $L(P)$ is the space of constant functions. But $L(P-Q) \subseteq L(P)$ is the subspace of functions that vanish at Q, and such a function is necessarily zero. ☐

Proposition 30.10. *Let* f *be a meromorphic function on* X. *Then for any divisor* D, *the assignment* $g \mapsto fg$ *is an isomorphism* $L(D) \xrightarrow{\cong} L(D - \operatorname{div}(f))$.

Proof. Immediate, as $h \mapsto \frac{h}{f}$ is the inverse. ☐

Definition 30.11. *Divisors* D_1 *and* D_2 *are* ***linearly equivalent****, written* $D_1 \sim D_2$, *if* $D_1 - D_2 = \operatorname{div}(f)$ *for some meromorphic function* f. *The* ***divisor class group*** *(often just called the class group) is* $\operatorname{Cl}(X) = \mathbb{Z}\langle X\rangle/\langle \operatorname{div}(f) \mid f \in \operatorname{Mer}(X)\rangle$.

Proposition 30.12. *If* $D_1 \sim D_2$ *then* $\ell(D_1) = \ell(D_2)$.

Proof. Immediate from Proposition 30.10. $\square$

The elements of $\mathrm{Cl}(X)$ can be thought of as parameterizing the different "compute $\ell(D)$" problems.

Example 30.13. For $X = \mathbb{C}P^1$ the function $\frac{1}{z-a}$ shows that $[a] \sim [\infty]$. It follows that $[a] \sim [b]$ for every $a, b \in \mathbb{C}P^1$, and so $\mathrm{Cl}(X) \cong \mathbb{Z}$ via the degree map. Hence, $\ell(D)$ depends only on $\deg(D)$. We can now compute:

$$\ell(D) = \begin{cases} 0 & \text{if } \deg(D) < 0, \\ 1 + \deg(D) & \text{if } \deg(D) \geq 0. \end{cases}$$

The case of $\deg(D) < 0$ is by Corollary 30.7. For the other case it suffices to compute $\ell(n[0])$, which counts meromorphic functions whose only pole is at 0 and has order at most n. One can write down a basis for such functions: $\frac{1}{z^k}$ for $0 \leq k \leq n$. So $\ell(n[0]) = n + 1$.

For X of positive genus it turns out that $\mathrm{Cl}(X)$ is always uncountable, so the $\mathbb{C}P^1$ case is deceptively simple.

30.14. **Meromorphic 1-forms.** 1-forms on X are constructs ω that look locally in a coordinate patch like $f(z)dz$. We will say that ω is holomorphic or meromorphic when the corresponding $f(z)$ functions are so. For a meromorphic 1-form ω define

$$\mathrm{div}(\omega) = \sum_P \mathrm{ord}_P(f)[P].$$

If ω and η are two meromorphic 1-forms then $\omega = h \cdot \eta$ for a unique meromorphic function h. Indeed, if in local coordinates $\omega = f(z)dz$ and $\eta = g(z)dz$ then we take $h = \frac{f(z)}{g(z)}$. Consequently, we obtain

$$\mathrm{div}(\omega) = \mathrm{div}(h) + \mathrm{div}(\eta)$$

and therefore $[\mathrm{div}(\omega)] = [\mathrm{div}(\eta)]$ in $\mathrm{Cl}(X)$. This is called the **canonical class** of X, and usually denoted K_X or just K.

Example 30.15. On $\mathbb{C}P^1$ we use the standard coordinates given by z near 0 and $w = \frac{1}{z}$ near ∞. We have $dw = -\frac{1}{z^2}dz$. Let η be the form that is $dz = -\frac{1}{w^2}dw$. Then $K = \mathrm{div}(\eta) = -2[\infty]$.

If f is any meromorphic function on X then one has the associated 1-form df. If f is not a constant map then df is nonzero, and so $\mathrm{div}(df)$ represents the canonical divisor class. For example, if $f(z) = z^2$ on $\mathbb{C}P^1$ then df is the 1-form that is $2zdz$ on the z-coordinate patch and $-\frac{2}{w^3}dw$ in the $w = \frac{1}{z}$ coordinate patch, so $\mathrm{div}(d(z^2)) = [0] - 3[\infty]$. Since $[0] = [\infty]$ in $\mathrm{Cl}(\mathbb{C}P^1)$ this divisor class is the same as $-2[\infty]$.

The degree of the canonical class is a numerical invariant of the Riemann surface X. It will come as no surprise that it is related to the genus:

Proposition 30.16. *If X has genus g then* $\deg(K) = 2g - 2 = -\chi(X)$.

Proof. This is another case where the result is topological. Let f be a non-constant meromorphic function, regarded as a map $f\colon X \to \mathbb{C}P^1$. Let d be the topological degree of f, and recall from the proof of Proposition 30.6 that we have

$$d = \sum_{P \text{ a pole}} (-\operatorname{ord}_P(f)).$$

The map f is a branched cover, so we can apply the Riemann-Hurwitz formula. Let BP be the set of points in X where f branches, and let $BL = f(BP) \subseteq \mathbb{C}P^1$ be the branch locus. For each $y \in BP$ let $\operatorname{ram}(y)$ denote the ramification index; intuitively this is the number of branches that are coming together at y. Then Riemann-Hurwitz says that

$$\chi(X) = 2d - \sum_{y \in BP} [\operatorname{ram}(y) - 1].$$

Let us recall where this formula comes from, since it is simple. If we remove BP and BL then $X - BP \to \mathbb{C}P^1 - BL$ is a d-fold cover, and so $\chi(X - BP) = d \cdot \chi(\mathbb{C}P^1 - BL)$. Since we are dealing with surfaces, removing a point decreases the Euler characteristic by 1 (it is like removing a triangle from a triangulation). So we get

$$\chi(X) - \#BP = \chi(X - BP) = d \cdot \chi(\mathbb{C}P^1 - BL) = d \cdot (2 - \#BL)$$

and this becomes

$$\chi(X) = 2d + \#BP - d \cdot \#BL = 2d + \#BP - \sum_{y \in BP} \operatorname{ram}(y) = 2d - \sum_{y \in BP} (\operatorname{ram}(y) - 1)$$

where in the second-to-last equality we have used that adding up the ramification indices in each fiber over a point in BL must yield the number d.

But if f is holomorphic at y then y is a branch point if and only if $f'(y) = 0$, and $\operatorname{ram}(y) = \operatorname{ord}_y(f') + 1$. Likewise, if y is a pole of f then y is a branch point if and only if $\operatorname{ord}_y(f) \leq -2$, and $\operatorname{ram}(y) = -\operatorname{ord}_y(f)$. So our Riemann-Hurwitz formula now becomes

$$\begin{aligned}
2 - 2g = \chi(X) &= 2d - \left[\sum_{\substack{P \text{ pole} \\ \operatorname{ord}_P(f) \leq -2}} (-\operatorname{ord}_P(f) - 1) + \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \right] \\
&= 2 \cdot \sum_{P \text{ pole}} \Big(-\operatorname{ord}_P(f) \Big) + \sum_{\substack{P \text{ pole} \\ \operatorname{ord}_P(f) \leq -2}} (\operatorname{ord}_P(f) + 1) - \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \\
&= \sum_{\substack{P \text{ simple} \\ \text{pole}}} 2 + \sum_{\substack{P \text{ pole} \\ \operatorname{ord}_P(f) \leq -2}} (-\operatorname{ord}_P(f) + 1) - \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \\
&= \sum_{P \text{ pole}} (-\operatorname{ord}_P(f) + 1) - \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \\
&= -\deg(\operatorname{div}(df)).
\end{aligned}$$

Since $\operatorname{div}(df)$ is a canonical divisor, this completes the proof. $\square$

We can now state the classical Riemann-Roch Theorem:

Theorem 30.17 (Riemann-Roch)**.** *For any divisor D on a Riemann surface X,*

$$\ell(D) - \ell(K - D) = 1 + \deg(D) - g = \deg(D) + \frac{\chi(X)}{2}.$$

The first equality, with $1 + \deg(D) - g$, is the classical statement. The second form of the equality connects better with modern versions of the Riemann-Roch theorem, with the two terms divided more clearly into contributions from D and contributions from X.

At first the Riemann-Roch theorem might not appear very useful for our problem, as it does not appear to calculate $\ell(D)$. It only gives us a relation between $\ell(D)$ and $\ell(K - D)$, and if we cannot calculate either one of them then we are out of luck. But sometimes we get lucky. For example, the following is immediate:

Corollary 30.18. *If $\deg(D) > \deg(K)$ then $\ell(D) = 1 + \deg(D) - g$.*

Proof. The hypothesis implies $\deg(K - D) < 0$, and so $\ell(K - D) = 0$ by Corollary 30.7. □

To give some additional perspective on the $\ell(K - D)$ term, consider the vector spaces

$$\Omega^1(D) = \{\text{meromorphic 1-forms } \omega \text{ such that } \operatorname{div}(\omega) \geq D\}.$$

For example, $\Omega^1(5[P] - 2[Q])$ consists of 1-forms that have at least an order 5 zero at P and at most an order 2 pole at Q (and no other poles).

If ω is a meromorphic 1-form, then the assignment $f \mapsto f\omega$ gives a map $L(E) \to \Omega^1(K - E)$. Indeed, if $\operatorname{div}(f) \geq -E$ then $\operatorname{div}(f\omega) = \operatorname{div}(f) + K \geq K - E$. This map is readily seen to be an isomorphism. Setting $E = K - D$ we can rephrase this result as follows:

Proposition 30.19. *For any divisor D, $L(K - D) \cong \Omega^1(D)$ via multiplication by any meromorphic 1-form ω.*

So the Riemann-Roch theorem for Riemann surfaces can also be stated as

$$\dim L(D) - \dim \Omega^1(D) = 1 + \deg(D) - g.$$

30.20. **Applications of Riemann-Roch.** We still need to prove the Riemann-Roch theorem, but let us come back to that after seeing a few applications.

Let X be a Riemann surface of genus 1 and choose $P \in X$. Note that $\deg(K) = -\chi(X) = 0$ here. Consequently we obtain $\ell(nP) = n$ for any $n > 0$, by Corollary 30.18. The following is therefore a sequence of proper inclusions:

$$L(P) \subsetneq L(2P) \subsetneq L(3P)$$

(the vector spaces having dimensions 1, 2, and 3, respectively). Choose $x \in L(2P)$ that is not a constant function, and choose $y \in L(3P) \backslash L(2P)$. Define $f\colon X \to \mathbb{C}P^2$ by

$$u \mapsto \begin{cases} [x(u) : y(u) : 1] & \text{if } u \neq P, \\ [0 : 1 : 0] & \text{if } u = P. \end{cases}$$

Note that $[x : y : 1] = [\frac{x}{y} : 1 : \frac{1}{y}]$ and the two fractions are holomorphic near P, in fact with zeros at P. So f is well-defined and holomorphic.

The vector space $L(6P)$ has dimension 6 but contains the functions y^2, x^3, x^2, xy, x, y, and 1. So there is a linear relation

$$ay^2 + bx^3 + cx^2 + dxy + ex + gy + h = 0$$

for some $a, b, c, d, e, g, h \in \mathbb{C}$. This shows that the image of f lies in the subspace of $\mathbb{C}P^2$ defined by the homogenization of the above equation, namely

$$az_1^2z_2 + bz_0^3 + cz_0^2z_2 + dz_0z_1z_2 + ez_0z_2^2 + gz_1z_2^2 + hz_2^3 = 0 \tag{30.21}$$

where $[z_0 : z_1 : z_2]$ are homogeneous coordinates on $\mathbb{C}P^2$.

We claim that f is also injective. It is clear from the definition that $f(P)$ cannot coincide with any $f(Q)$ for $Q \neq P$, since the former has vanishing third coordinate whereas the latter does not. So instead consider two distinct points Q and R in $X - \{P\}$. Consider the subspaces

$$\begin{tikzcd}
 & L(3P) & \\
L(3P-Q) \arrow[ur, hook] & & L(3P-R) \arrow[ul, hook'] \\
 & L(3P-Q-R). \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Corollary 30.18 calculates the dimensions as 3, 2, 2, and 1 (proceeding from top to bottom). So there exists a $g \in L(3P - Q)\backslash L(3P - Q - R)$. Then this function vanishes at Q but not R. Note that $1, x, y \in L(3P)$ and so $L(3P) = \langle 1, x, y\rangle$. Therefore we can write $g = m + nx + sy$ for $m, n, s \in \mathbb{C}$. The fact that $g(Q) = 0$ then implies that $f(Q)$ belongs to the subspace $nz_0 + sz_1 + mz_2 = 0$ of $\mathbb{C}P^2$, whereas $g(R) \neq 0$ implies that $f(R)$ does not belong to this subspace. So $f(Q) \neq f(R)$.

We have proven that f maps X injectively into the subvariety of $\mathbb{C}P^2$ defined by the cubic equation (30.21). Since the image of f is compact, and hence closed, it must be that f is a homeomorphism onto its image.

As our next application we use Riemann-Roch to prove that the points of X form an abelian group. Fix a point $\mathcal{O}$ in X.

For $P, Q \in X$ we will define a new point $P*Q$. We take $P*\mathcal{O} = P$ and $\mathcal{O}*Q = Q$, so assume $P \neq \mathcal{O}$ and $Q \neq \mathcal{O}$. Consider $L(P + Q - \mathcal{O})$, which has dimension 1 by Corollary 30.18. So there exists a meromorphic function g on X having a zero at $\mathcal{O}$ and possibly simple poles at P and/or Q (but no other poles). Recall from Proposition 30.9 that $L(P)$ has dimension 1 and therefore consists only of constant functions, and so $L(P - \mathcal{O}) = 0$ (the only constant function having a zero at $\mathcal{O}$ is the zero function). So $g \notin L(P - \mathcal{O})$, and therefore g has a simple pole at Q. Similarly, g has a simple pole at P. Since the number of zeros and poles for g must be equal (Proposition 30.6), this means g has one extra zero—call it S (and note that S might be $\mathcal{O}$). Define $P * Q = S$.

Since $\mathrm{div}(g) = [\mathcal{O}] + [S] - [P] - [Q]$ we have that $[P] + [Q] = [\mathcal{O}] + [S]$ in $\mathrm{Cl}(X)$, and in fact we could take this as the definition of S. For if $[P] + [Q] = [\mathcal{O}] + [T]$ in $\mathrm{Cl}(X)$ then there is a meromorphic function h having simple zeros at $\mathcal{O}$ and T and simple poles at P and Q. Then $h \in L(P + Q - \mathcal{O})$, but we have already remarked that this space has dimension 1. So h is a multiple of g, and therefore $S = T$.

To summarize: $P * Q$ is defined to be the unique point S such that $[P] + [Q] = [\mathcal{O}] + [S]$ in $\mathrm{Cl}(X)$. This definition also works when either P or Q is $\mathcal{O}$. Associativity of the $*$ operation is now immediate, as is the fact that $\mathcal{O}$ is the identity. We turn to the question of inverses.

Given a point P, consider $L(2\mathcal{O}-P)$. This has dimension 1 by Corollary 30.18, and $L(\mathcal{O}-P)=0$ by Proposition 30.9. So if $g\in L(2\mathcal{O}-P)$ is nonzero then g has a double pole at $\mathcal{O}$. This implies g has exactly two zeros, one at P and one at a point R (which possibly equals P). Then $[P]+[R]=[\mathcal{O}]+[\mathcal{O}]$ in $\mathrm{Cl}(X)$, and so $P*R=\mathcal{O}$.

Exercise 30.22. Write out the proof of associativity that was omitted above.

30.23. **Classical proof of the Riemann-Roch theorem.** Our aim here is just to give an outline of the main ideas, as some of the details require a good deal of work. The modern approach doesn't exactly short-cut this work, but instead sweeps it into the sheaf theory technology where some of it gets hidden.

We first concentrate on the case of a divisor $D=\sum m_P[P]$ where all $m_P\geq 0$. Let $\{P_1,\ldots,P_r\}$ be the set of points for which $m_P>0$, and write $m_i=m_{P_i}$. Pick a local coordinate near each P_i, and consider the vector space of "principal parts" of meromorphic functions having poles of at most order m_i at P_i. This is the vector space with basis $\frac{1}{z},\frac{1}{z^2},\cdots,\frac{1}{z^{m_i}}$, so taking the direct sum over all i we get a vector space PP of dimension $\deg(D)$. Consider the sequence of maps

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & L(D) \arrow[r] & PP \\ & & & \mathbb{C}^{\deg(D)} \arrow[u, "\cong"'] \end{tikzcd}$$

Here the leftmost map is the inclusion of the constant functions, and the next map sends a meromorphic function in $L(D)$ to the collection of its prinicpal parts around each P_i. Clearly we have exactness at $L(D)$. Riemann was interested in characterizing the image of $L(D)$ inside of PP. To do this, we need to recall a bit about residues.

If $\omega\in\Omega^1_{hol}(X)$ and $f\in L(D)$ then we can look at the residue

$$\mathrm{Res}_{z=P_i}\, f(z)\omega=\frac{1}{2\pi i}\int_{C_i} f\cdot\omega$$

where C_i is a small circle around P_i. Note that

$$\sum_i \mathrm{Res}_{z=P_i}\, f(z)\omega=0$$

by the Cauchy Integral Formula, since the C_i collectively bound a region on which $f(z)\omega$ is holomorphic. In this way be obtain from ω a linear equation satisfied by the image of $L(D)$ inside of PP. Specificially, we obtain a linear map $R_\omega\colon PP\to\mathbb{C}$ sending a collection of principal parts $(u_1,\ldots,u_r)$ to $\sum_i \mathrm{Res}_{P_i}(u_i\omega)$, and R_ω vanishes on the image of $L(D)$.

Riemann proved that $\Omega^1(X)\cong\mathbb{C}^g$, which means the above procedure yields g linear conditions on the image of $L(D)$ in PP. We can add this to our sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & L(D) \arrow[r] & P.P. \arrow[r, "\sum Res"] & \mathbb{C}^g \arrow[r] & Cok \arrow[r] & 0 \\ & & & \mathbb{C}^{\deg(D)} \arrow[u, "\cong"'] \end{tikzcd}$$

Riemann further proved exactness at the PP term: that is, a set of principal parts comes from a meromorphic function in $L(D)$ precisely when the g linear conditions coming from the elements of Ω^1 are satisfied. Note that the Cok term in the above sequence denotes the cokernel. From this sequence one gets that

$$0 = 1 - \ell(D) + \deg(D) - g + \dim(Cok)$$

or

$$\ell(D) = 1 + \deg(D) - g + \dim(Cok) \geq 1 + \deg(D) - g.$$

This is Riemann's part of the Riemann-Roch theorem. Roch, a student of Riemann, later identified $\dim(Cok)$ with $\ell(K-D)$.

30.24. **The modern approach to Riemann-Roch.** The vector space $L(D)$ is the space of holomorphic sections of a holomorphic line bundle $\mathcal{O}_X(D)$ (or equivalently, the space of sections of the corresponding rank one coherent sheaf). We construct this line bundle as follows. Choose a finite collection of coordinate patches $\{U_\alpha\}$ that cover X. On each U_α let g_α be the meromorphic function defined in the local coordinate z by

$$g_\alpha(z) = \prod_{P_i \in U_\alpha} (z - P_i)^{m_i}$$

where $D = \sum_i m_i[P_i]$. Note that this construction ensures that $\frac{g_\beta}{g_\alpha}$ is holomorphic and nonzero on $U_\alpha \cap U_\beta$. Let $\mathcal{O}_X(D)$ be the line bundle obtained by taking the trivial bundle on each U_α and gluing $(z, w) \in U_\alpha \times \mathbb{C}$ to $(z, \frac{g_\beta(z)}{g_\alpha(z)} \cdot w) \in U_\beta \times \mathbb{C}$ for $z \in U_\alpha \cap U_\beta$. This clearly gives a line bundle. If s is a holomorphic section of $\mathcal{O}_X(D)$ then let $s_\alpha \colon U_\alpha \to \mathbb{C}$ be the section written using the local trivialization on U_α. Then

$$\frac{g_\beta(z)}{g_\alpha(z)} s_\alpha(z) = s_\beta(z), \tag{30.25}$$

or equivalently

$$\frac{s_\alpha(z)}{g_\alpha(z)} = \frac{s_\beta(z)}{g_\beta(z)}.$$

for every $z \in U_\alpha \cap U_\beta$. This implies that the $\frac{s_\alpha}{g_\alpha}$ functions patch together to form a single meromorphic function $\frac{s}{g}$. Since the s_α were holomorphic, we have that $\mathrm{div}(\frac{s}{g}) \geq -D$. In this way we have obtained a map $\Gamma(X, \mathcal{O}_X(D)) \to L(D)$, and it is clearly injective. Surjectivity is just as easy: if $h \in L(D)$ then define $s_\alpha(z) = h|_{U_\alpha} \cdot g_\alpha$, which is necessarily holomorphic by the definition of g_α. Equation (30.25) is readily verified, so that the s_α patch together to give a holomorphic section of $\mathcal{O}_X(D)$.

Let us take a moment to consider the topological type of the line bundle $\mathcal{O}_X(D)$. If we imagine the points P_i in D moving around, this will not change the isomorphism type of the associated bundle. So we might as well let all the points come together, so that D approaches a divisor $(\deg D)[*]$ (it doesn't matter which point we choose here). This shows that the bundles $\mathcal{O}_X(D)$ and $\mathcal{O}(\deg D)$ are isomorphic as topological vector bundles. In particular, note that $c_1(\mathcal{O}_X(D)) = c_1(\mathcal{O}(\deg D)) = (\deg D)[*]$.

As we have seen in Proposition 29.4, the holomorphic Euler characteristic $\chi(X, \mathcal{O}_X(D))$ is a topological invariant:

$$\chi(X, \mathcal{O}_X(D)) = \pi_!([\mathcal{O}_X(D)]) \in K^0(pt) = \mathbb{Z}$$

where $\pi\colon X \to *$ and we are looking at the pushforward $\pi_!\colon K^0(X) \to K^0(*)$. The Grothendieck-Riemann-Roch theorem says that

$$\chi(X, \mathcal{O}_X(D)) = \Big(\mathrm{ch}(\mathcal{O}_X(D)) \cdot \mathrm{Td}(X)\Big)[*]$$

where the right-hand side denotes the coefficient of $[*]$ in the top-dimensional piece of $\mathrm{ch}(\mathcal{O}_X(D))\cdot \mathrm{Td}(X)$. Since X is a Riemann surface the top-dimensional cohomology is $H^2(X)$ and we have

$$\mathrm{ch}(\mathcal{O}_X(D)) = 1 + c_1(\mathcal{O}_X(D)), \qquad \mathrm{Td}(X) = \frac{c_1(T_X)}{1-e^{-c_1(T_X)}} = 1 + \frac{c_1(T_X)}{2}.$$

It follows that

$$\chi(X, \mathcal{O}_X(D)) = \left(c_1(\mathcal{O}_X(D)) + \frac{c_1(T_X)}{2}\right)[*] = \deg(D) + \frac{\chi(X)}{2}.$$

In the last equality we have used that $c_1(\mathcal{O}_X(D)) = \deg(D)[*]$ and $c_1(T_X) = \chi(X)[*]$.

This does not quite complete our story. The final piece is to identify $\dim H^1(X, \mathcal{O}_X(D))$ with Roch's $\ell(K - D)$ term, and for this we use Serre duality. The latter says that for any algebraic vector bundle E there is an isomorphism of vector spaces

$$H^i(X, E) \cong H^{1-i}(X, \mathcal{O}_X(K) \otimes E^*)^*$$

where $\mathcal{O}_X(K) = T_X^*$ is the canonical line bundle on X (the dual of the tangent bundle). This implies that

$$\begin{aligned}\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(K) \otimes \mathcal{O}_X(-D)) &= \dim H^0(X, \mathcal{O}_X(K - D)) \\ &= \ell(K - D).\end{aligned}$$

Putting everything together, we arrive at

$$\ell(D) - \ell(K - D) = \chi(X, \mathcal{O}_X(D)) = \deg(D) + \frac{\chi(X)}{2}$$

as desired.

Example 30.26. In the following tables we show the dimensions of $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$ as functions of $\deg(D)$ and the genus. Each table shows the results for a different genus. For convenience let $h^i(D)$ denote these dimensions, for $i \in \{0, 1\}$. Recall $h^0(D) = \ell(D)$ and $h^1(D) = \ell(K - D)$. The only information we need to make these tables is that $h^0 - h^1 = 1 + \deg(D) - g$, $h^0 = 0$ for $\deg(D) < 0$, and that $h^1 = 0$ for $\deg(D) > 2g - 2$ (since $h^1 = \ell(K - D)$).

Genus zero:

$\deg(D)$	-4	-3	-2	-1	0	1	2	3	4
$h^0(D)$	0	0	0	0	1	2	3	4	5
$h^1(D)$	3	2	1	0	0	0	0	0	0

Genus one:

$\deg(D)$	-4	-3	-2	-1	0	1	2	3	4
$h^0(D)$	0	0	0	0	A	1	2	3	4
$h^1(D)$	4	3	2	1	A	0	0	0	0

Here $0 \leq A \leq 1$ by Corollary 30.8, but the value of A will depend on the divisor. For $D = P - Q$ one has $A = 0$ by Proposition 30.9. If we fix P, Q, and R then for $D = P + Q - R - S$ one usually has $A = 0$, but for a unique S one gets $A = 1$.

Genus two:

$\deg(D)$	-4	-3	-2	-1	0	1	2	3	4
$h^0(D)$	0	0	0	0	A	B	C	2	3
$h^1(D)$	5	4	3	2	$A+1$	B	$C-1$	0	0

By Corollary 30.8 we know $A \leq 1$, $B \leq 2$, and $C \leq 3$. But in fact we can argue here that $C \leq 2$.

31. Formal group laws and complex-oriented cohomology theories

We have spent the past several sections exploring the differences in the way that geometric information is encoded in singular cohomology versus complex K-theory. Both of these theories have Thom classes for complex bundles and therefore have Gysin sequences, fundamental classes, Euler classes, and so forth. A key difference we observed was in the formula for the Euler class of a tensor product of line bundles: Proposition 25.8 versus Proposition 25.14. The construction of the Chern character was built around the differences between these formulas, and this difference also led directly to the Todd class and the topological Riemann-Roch theorem.

A natural question arises: Are there *other* complex-oriented cohomology theories lying around, which would then have these same kind of underlying connections to geometry? How does one understand the gamut of such theories? A full discussion of this story is outside the scope of this book, but we want to at least give an introduction to the key element: the purely algebraic notion of *formal group law*. In this section we outline some of the main theory behind what these are and how they connect to complex-oriented cohomology theories.

31.1. Formal group laws.

We start with a purely algebraic development. Let R be a fixed commutative ring.

Definition 31.2. *A (commutative, one-dimensional)* ***formal group law*** *over R is a power series $F(x, y) \in R[[x, y]]$ having the properties that*
(1) $F(x, y) = F(y, x)$
(2) $F(x, 0) = x = F(0, x)$
(3) $F(x, F(y, z)) = F(F(x, y), z)$.
Each part should be interpreted as an identity of formal power series.

It is sometimes useful to adopt the notation $x +_F y = F(x, y)$ and call this the "formal sum" of x and y. The above three properties are then written
(1) $x +_F y = y +_F x$
(2) $x +_F 0 = x = 0 +_F x$
(3) $x +_F (y +_F z) = (x +_F y) +_F z$.

One can talk about non-commutative formal group laws by omitting property (1), but we will not encounter them. Likewise, one can define an n-dimensional formal group law to be a power series in $R[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$ with properties similar to the above. But as we will not use these either, for us the phrase "formal group law" will always mean one that is commutative and one-dimensional.

The reader will note that will the above axioms encode commutativity, associativity, and the identity condition, there is no mention of inverses. It turns out that in this "formal" setting the existence of inverses is automatic; see Exercise 31.6 below.

Let us write $F(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \cdots$. The identity $F(x,y) = F(y,x)$ shows that $a_1 = a_2$, $a_3 = a_5$, and so on. The identity $F(x,0) = x$ shows that $0 = a_3$ and $1 = a_1$. So this type of reasonsing already shows that we must have

$$F(x,y) = x + y + a_4xy + a_5x^2y + a_5xy^2 + a_6x^3y + a_7x^2y^2 + a_6xy^3 + \cdots$$

Associativity will impose other conditions on the a_i, but we will return to that in a moment.

Example 31.3.

(a) $x +_F y = x + y$ is a formal group law, and certainly the simplest example. It is called the **additive formal group law**.

(b) Consider $x +_F y = x + y - xy = 1 - (1-x)(1-y)$. This is called the **multiplicative formal group law**. To verify associativity, observe that

$$1 - (x +_F y) = (1-x)(1-y)$$

and so

$$\begin{aligned} 1 - (x +_F (y +_F z)) = (1-x)(1-(y +_F z)) &= (1-x)(1-y)(1-z) \\ &= 1 - ((x +_F y) +_F z). \end{aligned}$$

Exercise 31.4.

(a) Verify that $x +_F y = x + y + axy$ is a formal group law, for any $a \in R$.

(b) For $a, b \in R$ verify that $x +_F y = x + y + axy + bx^2y + bxy^2$ is a formal group law only if $b = 0$.

(c) For $a, b, c, d \in R$ verify that $x +_F y = x + y + axy + bx^2y + bxy^2 + cx^3y + dx^2y^2 + cxy^3$ is a formal group law only if $b = c = d = 0$.

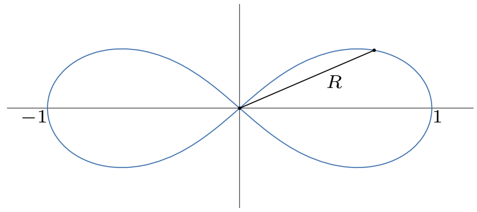
[Warning: Both (b) and (c) can be a bit tedious. Use of a computer algebra system is recommended.]

Exercise 31.5. Let $R = \mathbb{F}_3[\epsilon]/(\epsilon^2)$. Prove that $F(x,y) = x + y + \epsilon xy^3$ satisfies conditions (2) and (3) of Definition 31.2 but not condition (1). The fact that R has both torsion and nilpotents is important here: it is a theorem that if R is torsion-free and reduced then all power series satisfying (2) and (3) also satisfy (1).

Exercise 31.6. Let $F(x,y) = x + y + \sum a_{ij}x^iy^j$ be a formal group law over a commutative ring R. Prove that there exists a power series $I(x) \in R[[x]]$ such that $0 = F(x, I(x))$. [Hint: Set $I(x) = b_1x + b_2x^2 + \cdots$, then expand $F(x, I(x))$ and start solving the equations needed to make each coefficient vanish.]

In light of Exercise 31.4 one might get the impression that formal group laws with $F(x,y)$ a *polynomial* are quite rare, and that is indeed the case. So where do interesting examples come from? Historically, the first nontrivial examples probably came from the exploration of elliptic integrals. This is a long story and we will only give a very brief and streamlined introduction.

Consider the lemniscate curve given in polar coordinates by $r^2 = \cos(2\theta)$, drawn here:



This curve seems to have been first investigated in 1694 by Jacob Bernoulli, who realized that it had some properties similar to an ellipse. The arclength from the origin to the point with $r = R$, $0 \leq R \leq 1$, is given by the integral $\int_0^R \frac{1}{\sqrt{1-r^4}}\,dr$ (see Exercise 31.9). Work on these integrals led to the discovery of a surprising addition property, namely

$$\int_0^u \frac{1}{\sqrt{1-r^4}}\,dr + \int_0^v \frac{1}{\sqrt{1-r^4}}\,dr = \int_0^{f(u,v)} \frac{1}{\sqrt{1-r^4}}\,dr \tag{31.7}$$

where

$$f(u,v) = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}. \tag{31.8}$$

Such formulas were first established by Euler, though there were precursors in the work of Fagnano. In fact Euler proved such an addition property for any integral of the form $\int \frac{1}{\sqrt{P(r)}}\,dr$ where $P(r) = 1 + ar^2 - r^4$ for some constant a. These are examples of a certain class now known as "elliptic integrals", the name stemming from the fact that the integral for the arclength of an ellipse was one of the earliest examples.

One can expand $f(u,v)$ as a formal power series in u and v with rational coefficients, and equation (31.7) shows that this gives a formal group law over $\mathbb{Q}$. The symmetry and identity properties are obvious, and associativity follows from the fact that $f(u, f(v,w))$ and $f(f(u,v), w)$ both give the radius of the point on the curve where the arclength equals $\int_0^u \frac{1}{\sqrt{1-r^4}}\,dr + \int_0^v \frac{1}{\sqrt{1-r^4}}\,dr + \int_0^w \frac{1}{\sqrt{1-r^4}}\,dr$.

Exercise 31.9. Start with the arclength integral $\int \sqrt{dx^2+dy^2}$, translate into polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$, and finally use the lemniscate equation $r^2 = \cos^2(2\theta)$ to eliminate $d\theta$ and write the arclength entirely as a dr-integral. In this way verify that the arclength is given by $\int \frac{1}{\sqrt{1-r^4}}\,dr$.

Exercise 31.10. Use a computer algebra system to determine the power series $f(u,v)$ from (31.8) up through degree 12 (check: there end up being terms only in degrees 1, 5, and 9). Using only the terms up through degree 12, use the software to check that the identity $f(u, f(v,w)) = f(f(u,v), w)$ holds up through the same degree.

There are many more formal group laws than the few we have just seen, although it is not so easy just to guess what they look like. One source of these gadgets comes from looking at a 1-dimensional algebraic group in an analytic neighborhood of the identity. To understand this, let X be an algebraic curve over $\mathbb{C}$ (for simplicity) and let $O \in X$ be a point. Assume that there is a map of varieties $\mu\colon X \times X \to X$

making X into a group with O as identity. A choice of analytic coordinate near O gives an isomorphism of $\mathbb{C}[[z]]$ with the germs of analytic functions at O on X, and likewise an isomorphism of $\mathbb{C}[[z_1, z_2]]$ with the germs of analytic functions at (O, O) on $X \times X$. The map μ induces

$$\mu^*\colon \mathbb{C}[[z]] \to \mathbb{C}[[z_1, z_2]].$$

Let $F(z_1, z_2) = \mu^*(z)$. The group axioms for μ imply that $F(z_1, z_2)$ is a formal group law.

Example 31.11.

(1) Let $X = \mathbb{A}^1$, with group law $\mu\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $\mu(x, y) = x + y$. On germs of functions this becomes $\mathbb{C}[[z]] \to \mathbb{C}[[z_1, z_2]]$ given by $z \mapsto z_1 + z_2$, which is the additive formal group law.

(2) Let $X = \mathbb{A}^1 - 0$ with group law $\mu(x, y) = xy$. On rings of functions this is $\mathbb{C}[z, z^{-1}] \to \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ given by $z \to z_1 z_2$. The identity element of X is 1, and a local coordinate near 1 is $t = 1 - z$. One has

$$\mu^*(t) = 1 - \mu^*(z) = 1 - z_1 z_2 = 1 - (1 - t_1)(1 - t_2) = t_1 + t_2 - t_1 t_2.$$

This is the multiplicative formal group law.

(3) We saw in Section 30.20 that any Riemann surface of genus one (i.e., an elliptic curve) can be given a group multiplication. So any elliptic curve gives rise to a formal group law. (In particular, a certain elliptic curve gives rise to the formal group law from (31.8), though we will not explore this here.)

31.12. **The universal formal group law.** Given a ring homomorphism $\alpha\colon R \to S$ and a formal group law $F(x, y)$ over R, we get a formal group law $\alpha_*(F)$ by applying α to all the coefficients of F. If we write $\mathrm{FGL}(R)$ for the set of formal group laws over R, then we have just described a map $\alpha_*\colon \mathrm{FGL}(R) \to \mathrm{FGL}(S)$.

Proposition 31.13. *There is a universal formal group law. That is, there exists a ring L and a formal group law F on L having the property that for all rings R, the map*

$$\mathrm{Ring}(L, R) \longrightarrow \mathrm{FGL}(R), \qquad \alpha \mapsto \alpha_*(F)$$

is a bijection. That is to say, the functor $\mathrm{FGL}(-)$ *on the category of rings is corepresentable.*

Proof. This is easy. We already know that every formal group law looks like

$$F(x, y) = x + y + a_1 xy + a_2 x^2 y + a_2 xy^2 + a_3 x^3 y + a_4 x^2 y^2 + a_3 xy^3 + \cdots$$

Let $S = \mathbb{Z}[a_1, a_2, \ldots]$. Equating coefficients in the associativity identity $F(x, F(y, z)) = F(F(x, y), z)$ yields a collection of polynomial identities in the a_i. Let L be the quotient of S by the ideal generated by these equations. □

Remark 31.14. Note that the ring L can be given a natural grading. Give the formal variables x and y degree -2, and regard $F(x, y)$ as being homogeneous of degree -2. Then writing $F(x, y) = x + y + a_1 xy + a_2 x^2 y + a_2 xy^2 + a_3 x^3 y + a_4 x^2 y^2 + a_3 xy^3 + \cdots$ requires that a_1 have degree 2, a_2 have degree 4, a_3 and a_4 to have degree 6, and so forth. This defines a grading on S where $F(x, y)$ is homogeneous. The associativity identity is homogeneous, so it leads to homogeneous relations amongst the a_i. So L inherits the grading on S. (Note: We could have assigned x and y to have degree -1 instead of -2. The "unnatural" choice of -2 comes from the way all of this manifests in topology.]

The above result is rather formal, and gives us no indication of what the ring L looks like. A very nontrivial theorem of Lazard completes the picture:

Theorem 31.15 (Lazard)**.** *There is an isomorphism of graded rings* $L \cong \mathbb{Z}[u_1, u_2, \ldots]$ *where the degree of* u_i *is* $2i$*.*

Note that the u_i from Lazard's theorem are not equal to the a_i that appear in the proof of Proposition 31.13. We will not include a proof of Lazard's theorem, but ones can be found in ????.

31.16. **Isomorphism of formal group laws.**

Definition 31.17. *Suppose* F *and* G *are formal group laws over* R*. A map* $F \to G$ *is a power series* $f(x) \in R[[x]]$ *such that* $f(0) = 0$ *and* $f(x +_F y) = f(x) +_G f(y)$*.*

Note that if $f_1 \colon F \to G$ and $f_2 \colon G \to H$ then the composition $f_2 \circ f_1$ is the power series composition $f_2(f_1(x))$. This is defined because of the condition $f_1(x) = 0$. By an easy exercise in algebra, the power series f_1 has an inverse under composition if and only if $f_1'(0)$ is a unit. So a map of formal group laws $f \colon F \to G$ is an isomorphism if and only if $f'(0)$ in a unit in R.

Example 31.18. Let $F(x, y) = x+y-xy = 1-(1-x)(1-y)$ and $G(x, y) = x+y$. To produce a map from F to G we need a power series $f(x)$ such that

$$f(1 - (1 - x)(1 - y)) = f(x) + f(y).$$

This suggests the logarithm function, and a moment's thought shows that $f(x) = \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ works. However, this requires that we have $\mathbb{Q} \subseteq R$ for this power series to exist in $R[[x]]$. As the coefficient of x in $\log(1 - x)$ is 1, we actually have an isomorphism of formal group laws. So if $\mathbb{Q} \subseteq R$ then the additive and multiplicative formal group laws are isomorphic.

For future reference, we note that $f(x) = -\log(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ is also an isomorphism in this example.

Exercise 31.19. Let $F(x, y) = x+y-xy$ and $G(x, y) = x+y+axy$, where $a \in R$. If a is invertible, then $f(x) = -\frac{1}{a} \cdot x$ is an isomorphism between F and G. More generally, show that $F(x, y) = x+y+bxy$ and $G(x, y) = x+y+axy$ are isomorphic if $b = ua$ for some unit $u \in R$.

Over the rational numbers, there is really only one formal group law:

Proposition 31.20. *If* $\mathbb{Q} \subseteq R$ *then all formal group laws are isomorphic to the additive formal group law.*

Proof. Let F be a formal group law. We look for a power series $f(x) \in R[[x]]$ such that $f(F(x, y)) = f(x) + f(y)$ and $f'(0) = 1$. Apply $\frac{d}{dy}\big|_{y=0}$ to the first equation, to get:

$$\begin{aligned}
&f'(F(x, y))|_{y=0} \cdot F_y(x, 0) = f'(0) = 1 \\
&f'(F(x, 0)) \cdot F_y(x, 0) = 1 \\
&f'(x) = \frac{1}{F_y(x, 0)} \\
&f(x) = \int \frac{1}{F_y(x, 0)}\, dx
\end{aligned}$$

The power series $F_y(x,0)$ has constant term 1, and so is invertible in $R[[x]]$. As long as $R \supseteq \mathbb{Q}$ we can then integrate term-by-term to get $f(x)$, and reversing the above analysis shows that f is the desired isomorphism. $\square$

Example 31.21. Returning to the multiplicative formal group law $F(x,y) = x + y - xy$, we have $F_y(x,0) = 1 - x$, $\frac{1}{F_y(x,0)} = 1 + x + x^2 + \cdots$ and $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x)$.

31.22. **Formal group laws in topology.** Let E be a complex-oriented cohomology theory. Generalizing our discussion from Section 25 we will see that every complex line bundle $L \to X$ has an Euler class—also called the first Chern class—$c_1(L) \in E^2(X)$, and that there is a formal group law $F(x,y) \in E^*[[x,y]]$ such that $c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$.

Let us be clear about what we mean by the power series ring in the graded case. If R is a $\mathbb{Z}$-graded ring and X is a formal variable of degree d, then $R[[x]]$ will denote the linear span of homogeneous power series with coefficients in R. That is, $R[[x]] = \oplus_n P_n$ where $P_n = \{a_n + a_{n-d}X + a_{n-2d}X^2 + \cdots \mid a_i \in R_i\}$. One defines $R[[x,y]]$ similarly. This redefinition of $R[[x]]$ in the graded case takes some getting used to, but it is very convenient.

Exercise 31.23. Check that in the case of singular cohomology we have $H^*(pt)[[x]] = \mathbb{Z}[x]$ as a graded ring. So we can write $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[x]]$ or $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$, and they mean the same thing.

Likewise, check that if x is of degree 2 then $K^*(pt)[[x]] = \mathbb{Z}[[\beta x]]\,[\beta, \beta^{-1}]$ where here $\mathbb{Z}[[\beta x]]$ can be interpreted either as graded power series or ungraded power series (they are the same because both $\mathbb{Z}$ and βx are in degree 0).

Exercise 31.24. Chec that if R is a $\mathbb{Z}$-graded ring then our definition of $R[[x]]$ coincides with $\lim_n R[x]/(x^n)$ where the limit is taken in the category of *graded* rings.

Let $\xi \to \mathbb{C}P^\infty$ be the tautological line bundle and let $\zeta \colon \mathbb{C}P^\infty \hookrightarrow \operatorname{Th}\xi$ be the zero section. Consider the composite

$$\tilde{E}^2(\operatorname{Th}\xi) \xrightarrow{\zeta^*} \tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{E}^2(\mathbb{C}P^1) = \tilde{E}^2(S^2) = E^0(*).$$

Let $x = \zeta^*(U_\xi)$, which is an element in $\tilde{E}^2(\mathbb{C}P^\infty)$ that restricts to 1 under the above composition.

Given any complex line bundle $L \to X$, there is a classifying map $f \colon X \to \mathbb{C}P^\infty$ and a pullback diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \xi \arrow[d] \\ X \arrow[r] & \mathbb{C}P^\infty. \end{tikzcd}$$

Define the E-theory first Chern class (or Euler class) by $c_1^E(L) = f^*(x)$. We will drop the superscript and just write $c_1(L)$ when there is no possibility for confusion.

The class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ restricts to a class, which we will also denote by x, in $\tilde{E}^2(\mathbb{C}P^s)$ for each s. Since $x^{s+1} = 0$ in $E^*(\mathbb{C}P^s)$ by Lemma 23.2, we obtain an induced map $E^*[z]/(z^{n+1}) \to E^*(\mathbb{C}P^s)$ sending z to x. We have seen in Proposition 23.3 that this is an isomorphism. We wish to now go further and calculate

$E^*(\mathbb{C}P^\infty)$. Using that $\mathbb{C}P^\infty = \operatorname{colim}_s \mathbb{C}P^s$, the Milnor sequence gives us

$$0 \longrightarrow \lim\nolimits^1_s E^{k-1}(\mathbb{C}P^s) \longrightarrow E^k(\mathbb{C}P^\infty) \longrightarrow \lim\nolimits_s E^k(\mathbb{C}P^s) \longrightarrow 0.$$

Since $E^*(\mathbb{C}P^s) \to E^*(\mathbb{C}P^{s-1})$ is surjective for all s, the $\lim^1$ terms are always zero. So

$$E^*(\mathbb{C}P^\infty) = \lim_s E^*(\mathbb{C}P^s) = \lim_s E^*[x]/(x^{s+1}) = E^*[[x]]$$

(and recall that the right-most term means graded power series).

Similar arguments show (DO THEY?) that $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*[[x_1, x_2]]$ where $x_1 = \pi_1^*(x)$, $x_2 = \pi_2^*(x)$, and $\pi_1, \pi_2 \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ are the two projections.

Consider now the external tensor product $\xi \hat{\otimes} \xi \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. This is a line bundle, and so is classified by a map $g \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$:

$$\begin{tikzcd}
\xi\hat{\otimes}\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\
\mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "g"] & \mathbb{C}P^\infty
\end{tikzcd}$$

The map $g^* \colon E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is $E^*[[x]] \to E^*[[x_1, x_2]]$ and so sends x to a formal power series $F(x_1, x_2)$ of degree 2. By definition we have

$$F(x_1, x_2) = c_1(\xi \hat{\otimes} \xi).$$

So if $L_1 \to X$ and $L_2 \to X$ are any two line bundles, with classifying maps $f_1, f_2 \colon X \to \mathbb{C}P^\infty$, we have the diagram

$$\begin{tikzcd}
L_1 \otimes L_2 \arrow[r] \arrow[d] & L_1 \hat{\otimes} L_2 \arrow[r] \arrow[d] & \xi\hat{\otimes}\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\
X \arrow[r, "\Delta"] & X \times X \arrow[r, "f_1 \times f_2"] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "g"] & \mathbb{C}P^\infty.
\end{tikzcd}$$

Therefore

$$\begin{aligned}
c_1(L_1 \otimes L_2) = \Delta^*(f_1 \times f_2)^* g^*(x) &= \Delta^*(f_1 \times f_2)^*(F(x_1, x_2)) \\
&= F(f_1^*(x_1), f_2^*(x_2)) \\
&= F(c_1(L_1), c_1(L_2)).
\end{aligned}$$

It remains for us to understand why the power series $F(x_1, x_2)$ is a formal group law. The group axioms follow immediately from the following commutative diagrams in the homotopy category:

$$\begin{tikzcd}
\mathbb{C}P^\infty \arrow[r, "\mathrm{id} \times *"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "g"] & \mathbb{C}P^\infty
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "g"] \arrow[d, "t"'] & \mathbb{C}P^\infty \\
\mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[dl, "\mathrm{id} \times g"'] \arrow[dr, "g \times \mathrm{id}"] & \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[dr, "g"'] & & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[dl, "g"] \\ & \mathbb{C}P^\infty. & \end{tikzcd}$$

To see that these diagram commute (up to homotopy), use that $\mathbb{C}P^\infty$ is the classifying space for line bundles. The first diagram just reflects the isomorphism $L \otimes 1 \cong L$, the second diagram reflects that $L_1 \otimes L_2 \cong L_2 \otimes L_1$, and the third reflects that $L_1 \otimes (L_2 \otimes L_3) \cong (L_1 \otimes L_2) \otimes L_3$.

Summarizing, we have now proven the following:

Proposition 31.25. *Let E be a complex-oriented cohomology theory. Then there exists a formal group law F over E^* (of homogeneous degree 2 as a power series) having the property that $F(c_1(L_1), c_1(L_2)) = c_1(L_1 \otimes L_2)$ for all complex line bundles L_1 and L_2.*

We have seen in Proposition 25.8 that the formal group law for singular cohomology is the additive one, and in Proposition 25.14 that the formal group law for complex K-theory is the multiplicative one. To be more precise in the latter case, it is the formal group law over $\mathbb{Z}[\beta, \beta^{-1}]$ given by $x + y - \beta xy$.

We have not discussed it so far in this book, but there is a cohomology theory MU called complex cobordism. In a certain sense this is the universal complex-oriented cohomology theory. The coefficients are $MU^* = \mathbb{Z}[u_1, u_2, \ldots]$ where $|u_i| = -2i$, by a classical calculation due to Milnor. Quillen [Q2] proved that the formal group law for MU is the universal formal group law that we encountered in Proposition 31.13.

31.26. **Formal group laws and Leibniz rules.** There is another place that formal group laws appear in an unexpected way, but which relates closely to some of the K-theory calculations we have seen in earlier sections. The differentiation operator D from calculus obeys the Leibniz rule $D(fg) = (Df)g + f(Dg)$. One can write this as

$$D(fg) = \mu[Df \otimes g + f \otimes Dg] = \mu[D \otimes Id + Id \otimes D](f \otimes g).$$

Here $\mu(a \otimes b) = ab$, of course.

We saw in Lemma 24.7 that the finite-difference operator Δ obeys a different form of the Leibniz rule. Written in the above form, it is

$$\Delta(fg) = \mu[\Delta \otimes Id + Id \otimes \Delta + \Delta \otimes \Delta](f \otimes g).$$

One can feel the specter of formal group laws here, and a precise connection can be made as follows. For an operator J, define a "generalized Leibniz rule" to be a formula

$$J(fg) = \mu\Big[\sum_{i,j} a_{ij} J^i \otimes J^j\Big](f \otimes g)$$

where the a_{ij} are coefficients in some underlying ground ring. By setting $x = J \otimes Id$ and $y = Id \otimes J$, the expression inside the brackets can be written as $F(x, y)$ where $F = \sum_{i,j} a_{ij} x^i y^j$. The formal group law conditions on $F(x, y)$ are parallels of the following properties:

- $J(f \cdot 1) = J(f) = J(1 \cdot f)$
- $J(fg) = J(gf)$
- $J((fg)h) = J(f(gh))$

More precisely, the first FGL condition says that applying the Leibniz rule to $J(f \cdot 1)$ gives a formula that just reduces to $J(f)$. The second condition says that applying the Leibniz formula to $J(fg)$ and to $J(gf)$ lead to the same expression, and the third condition says something similar for applying the Leibniz rule twice to $J(fgh)$ (in the two different ways).

The connections between algebraic topology and these generalized differentiation operators has been explored in several papers by Nigel Ray, starting with [Ra1] and [Ra2]. (EXPLAIN THIS MORE?)

32. Algebraic cycles on complex varieties

Note: The material in this section requires the Atiyah-Hirzebruch spectral sequence from Section 34 below.

Let X be a smooth, projective algebraic variety over $\mathbb{C}$. As discussed in Section 17.3 every smooth subvariety Z of codimension q has a fundamental class $[Z] \in H^{2q}(X;\mathbb{Z})$. In fact it turns out that the smoothness of Z is not needed here: *every* subvariety $Z \hookrightarrow X$ of codimension q has such a fundamental class. This can be proven either by using resolution of singularities or by more naive methods—we will explain below.

Define $H^{2q}_{\text{alg}}(X;\mathbb{Z}) \subseteq H^{2q}(X;\mathbb{Z})$ to be the subgroup generated by the fundamental classes of all algebraic subvarieties. How large are these "algebraic" parts of the even cohomology groups? Are there examples of varieties X for which the algebraic part does not equal everything?

The answer to the latter question is provided by Hodge theory: yes, there do exist varieties X where not all of the even cohomology is algebraic. Hodge theory gives a decomposition of the cohomology groups $H^n(X;\mathbb{C}) \cong \oplus_{p+q=n} H^{p,q}(X)$, and the algebraic classes all lie in the $H^{i,i}$ pieces. The (even-dimensional) cohomology is entirely algebraic only when $H^{p,q} = 0$ for $p \neq q$ and $p+q$ even. But it is known that for an elliptic curve E one has $H^{1,0} = \mathbb{C} = H^{0,1}$, and the Künneth Theorem in this context says

$$H^{a,b}(X \times Y) = \bigoplus_{\substack{a=a_1+a_2 \\ b=b_1+b_2}} H^{a_1,b_1}(X) \otimes H^{a_2,b_2}(Y).$$

So for $E \times E$ one gets $H^{2,0}(E \times E) \cong \mathbb{C} \cong H^{0,2}(E \times E)$ and therefore not all of $H^2(E \times E)$ is algebraic.

The problem with Hodge theory is that it cannot see any torsion classes, as the coefficients of the cohomology groups need to be $\mathbb{C}$. Could it be true that torsion cohomology classes are always algebraic? A classical theorem of Lefschetz, reproved by Hodge, says that this holds for classes in H^2. But for higher cohomology groups the answer is again no, and the first proof was given by Atiyah and Hirzebruch [AH2]. There are three basic components to their proof:

(1) Every algebraic class must survive the Atiyah-Hirzebruch spectral sequence. The vanishing of the differentials therefore gives a sequence of obstructions for a given cohomology class to be algebraic.

(2) The differentials in the spectral sequence can be analyzed in terms of cohomology operations. If p is a prime then on a p^e-torsion class the first nonzero differential is d_{2p-1} and coincides with the operation $u \mapsto -\beta P^1(\bar{u})$ where $\bar{u}$ is reduction modulo p, P^1 is the first Steenrod pth power operation, and β is the mod p Bockstein. Note that this cohomology operation has degree $2p-1$.

(3) A clever construction of Serre's shows how to obtain smooth algebraic varieties whose cohomology contains that of BG through a range of dimensions, for G any finite group. Using this, one readily finds smooth varieties with even-dimensional, p-torsion cohomology classes for which βP^1 does not vanish; such a class cannot be algebraic.

The Atiyah-Hirzebruch proof is no longer the most efficient way to obtain conditions for torsion classes to be algebraic. Resolution of singularities shows that every algebraic class is actually a pushforward of the fundamental class of a manifold—i.e., every algebraic class lifts into complex cobordism. In the 1950s Thom had already obtained some necessary conditions for such a lifting to exist, in terms of Steenrod operations. Via this method K-theory is not needed at all, and moreover Thom's theory yields a stronger set of conditions: all odd degree mod p cohomology operation must vanish, for every prime p. Note that resolution of singularities was not proven by Hironaka until 1964 [Hir1], and so of course was not available at the time of [AH2].

Despite the modern shortcomings of Atiyah and Hirzebruch's method, we will spend this section describing it in detail. It sheds some light on the relationship between K-theory and singular cohomology, and also offers some interesting observations about algebraic varieties. We will also describe the approach via resolution of singularities, for comparison.

Remark 32.1. We should mention that [AH2] treats the case of *analytic* cycles in addition to algebraic cycles. The proofs are essentially the same, with one or two key differences. We will not cover the material on analytic cycles here.

Remark 32.2. The modern Hodge conjecture states that any class $\alpha \in H^{2n}(X;\mathbb{Q})$ whose image in $H^{2n}(X;\mathbb{C})$ lies in the Hodge group $H^{n,n}(X)$ is necessarily algebraic—that is, it lies in the subgroup $H^{2n}(X;\mathbb{Q})_{\mathrm{alg}}$. When Hodge originally raised this question he did not explicitly specify rational coefficients. Since any torsion class in $H^{2n}(X;\mathbb{Z})$ would map to zero in $H^{2n}(X;\mathbb{C})$, and therefore lie inside $H^{n,n}(X)$, an integral version of the Hodge conjecture would imply that all torsion classes are algebraic. One of the main points of [AH2] was to demonstrate that this integral form of the Hodge conjecture does not hold in general.

How complicated does a smooth complex variety need to be in order to violate the integral Hodge conjecture? The smallest such varieties produced by the methods of [AH2] were of complex dimension 7. Arguments in [BCC] showed the existence of such varieties in dimension 3 (in fact sufficiently general hypersurfaces in $\mathbb{C}P^4$ of large degree), but without producing explicit examples. Examples of dimension 5 were produced in [SV], and then explicit examples of dimension 3 were obtained in [T2] and later [BO]. This is just a sample of what is a very active area of research.

It turns out that the integral Hodge conjecture does hold for certain special classes of algebraic varieties, and this remains a topic of current interest. See [T3] and [V] as just two examples.

Up until Section 32.17 the material in this section closely follows [AH2].

32.3. **Fundamental classes for subvarieties.** If X is a smooth algebraic variety and $Y \subseteq X$ is a smooth subvariety of codimension q then we have seen that complex orientability yields a fundamental class $[Y] \in H^{2q}(X)$ and a relative fundamental class $[Y]_{rel} \in H^{2q}(X, X-Y)$. This comes about by choosing a tubular neighborhood U of Y that is homeomorphic to the normal bundle, and using the isomorphisms

$$H^{2q}(X, X-Y) \cong H^{2q}(U, U-Y) \cong H^{2q}(N, N-0).$$

Assuming Y is connected it follows that $H^{2q}(X, X-Y) \cong \mathbb{Z}$ (by Thom isomorphism), and $[Y]_{rel}$ is defined to be the image of the Thom class $\mathcal{U}_N \in H^{2q}(N, N-0)$. The fundamental class $[Y]$ is just the image of $[Y]_{rel}$ under $H^*(X, X-Y) \to H^*(X)$. Note that the argument shows that the choice of U and homeomorphism $U \cong N$ to be irrelevant: there are only two generators in $H^{2q}(X, X-Y)$, and ???

If Y is disconnected we define $[Y]$ to be $\sum_{i=1}^r [Y_i]$ where the Y_i are the connected components of Y. Note that in this case $H^{2q}(X, X-Y) \cong \mathbb{Z}^r$, with the generators corresponding to the images of the $[Y_i]_{rel}$ under $H^{2q}(X, X-Y_i) \to H^{2q}(X, X-Y)$.

We aim to show the existence of fundamental classes $[Y]$ even when the subvariety Y is not smooth. The approach we follow is to throw away the singular set and then observe that because this is in smaller dimension it didn't matter anyway.

Lemma 32.4. *Let X be a smooth algebraic variety and let $W \hookrightarrow Y \hookrightarrow X$ be closed subvarieties. Assume that Y has codimension $q \geq 1$ inside of X, and that W has codimension at least one inside of Y.*

(a) $H^*(X) \to H^*(X-Y)$ is an isomorphism for $* \leq 2q-2$.
(b) $H^*(X, X-Y) \to H^*(X-W, X-Y)$ is an isomorphism for $* \leq 2q$.
(c) If Y is irreducible then $H^{2q}(X, X-Y) \cong \mathbb{Z}$.

Proof. First note that part (a) is true when Y is smooth, using the long exact sequence for the pair $(X, X-Y)$, the isomorphism $H^*(X, X-Y) \cong H^*(N, N-0)$, and the Thom isomorphism theorem. For the general case, we can filter Y by subvarieties

$$\emptyset \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_s = Y$$

where each Y_i is the singular set of Y_{i+1} (so that $Y_{i+1} - Y_i$ is smooth). Since Y_0 is smooth, we can assume by induction that $H^*(X) \to H^*(X - Y_i)$ is an isomorphism for $* \leq 2(\operatorname{codim} Y_i) - 2$. Now look at the composition

$$H^*(X) \to H^*(X - Y_i) \to H^*(X - Y_{i+1}) = H^*\big((X - Y_i) - (Y_{i+1} - Y_i)\big).$$

Since $Y_{i+1} - Y_i$ is smooth in $X - Y_i$, the second map is an isomorphism for $* \leq 2(\operatorname{codim} Y_{i+1}) - 2$. It follows that the composite is an isomorphism for $* \leq 2(\operatorname{codim} Y_{i+1}) - 2$ as well, and now the desired result follows by induction.

Part (b) follows from part (a) via the long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^{*-1}(X-Y) \arrow[r] \arrow[d, "="] & H^*(X, X-Y) \arrow[r] \arrow[d] & H^*(X) \arrow[r] \arrow[d, "j^*"] & H^*(X-Y) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & H^{*-1}(X-Y) \arrow[r] & H^*(X-W, X-Y) \arrow[r] & H^*(X-W) \arrow[r] & H^*(X-Y) \arrow[r] & \cdots
\end{tikzcd}$$

By (a) the map labelled j^* is an isomorphism for $* \leq 2q$, and so the result follows by the five lemma.

Finally, for (c) we let Z be the singular set of Y. Then $H^{2q}(X, X-Y) \cong H^{2q}(X - Z, (X-Z) - (Y-Z))$ by (b). But $Y - Z$ is a smooth closed subvariety of $X - Z$ of codimension q. Since Y was irreducible, $Y-Z$ is also irreducible [H, Example 1.1.3]. By [Sha2, Theorem 7.1], $Y-Z$ is connected in the complex topology. The remarks from the beginning of this section then show that $H^{2q}(X-Z,(X-Z)-(Y-Z)$ is isomorphic to Z, with generator $[Y-Z]_{rel}$. □

Now let $Y \hookrightarrow X$ be an algebraic subvariety of codimension q, and let Z be the singular set. Since $Y-Z$ is a smooth subvariety of $X-Z$, we have a fundamental class $[Y-Z] \in H^{2q}(X-Z)$. But by Lemma 32.4(a) the map $H^{2q}(X) \to H^{2q}(X-Z)$ is an isomorphism. We define $[Y] \in H^{2q}(X)$ to be the preimage of $[Y-Z]$ under this map.

Define $H^*_{\text{alg}}(X) \subseteq H^*(X)$ to be the subgroup generated by the fundamental classes of all the algebraic subvarieties of X.

In the next section it will help to be able to focus on classes $[Y]$ where Y is irreducible. To this end, the following is useful:

Lemma 32.5. *If $Y \hookrightarrow X$ is a codimension q subvariety with irreducible components $Y_1, \ldots, Y_r$ then $[Y] = \sum_i [Y_i]$ in $H^{2q}(X)$. Therefore the subgroup $H^*_{\text{alg}}(X)$ is spanned by fundamental classes of irreducible subvarieties.*

Proof. Let $Z \subseteq Y$ be the singular set. This contains the points in each $Y_i \cap Y_j$, so $Y-Z$ is the disjoint union of the $Y_i - (Z \cap Y_i)$. So $[Y-Z] = \sum_i [Y_i - (Z \cap Y_i)]$ in $H^*(X-Z)$. Now consider the two maps

$$H^{2q}(X) \longrightarrow H^{2q}(X-Z) \longleftarrow H^{2q}(X-(Z\cap Y_i)) \longleftarrow H^{2q}(X).$$

On the left, $[Y]$ is the unique class that maps to $[Y-Z] = \sum_i [Y_i - (Z \cap Y_i)]$. On the right, $[Y_i]$ is the unique class that maps to $[Y_i - (Z \cap Y_i)]$. It follows at once that $[Y] = \sum_i [Y_i]$ in $H^{2q}(X)$. □

We need one more lemma before moving on:

Proposition 32.6. *Let M be a real manifold and let $N \hookrightarrow M$ be a codimension k real submanifold with a tubular neighborhood. Then $M - N \hookrightarrow M$ is $(k-1)$-connected.*

Sketch. ???? □

32.7. **Vanishing of differentials on algebraic classes.** Let X be a smooth algebraic variety, and let $Y \subseteq X$ be a subvariety. Let $\mathcal{F}_\bullet$ be a bounded resolution of $\mathcal{O}_Y$ by locally-free coherent $\mathcal{O}_X$-modules, and write $F_\bullet$ for the associated chain complex of $\mathbb{C}$-vector bundles on X. Then $[F_\bullet]$ defines a class in $K^0(X, X-Y)$ which we will denote $[Y]_{K,rel}$. Note that when Y is smooth this agrees with the relative fundamental class provided by the complex orientation of K-theory, by Theorem 21.10.

Proposition 32.8. *One has* $\text{ch}([Y]_{K,rel}) = [Y]_{H,rel} +$ *higher order terms.*

Proof. We first prove this when Y is smooth. If N denotes the normal bundle for Y in X, then the result will follow once we know $\text{ch}(\mathcal{U}_N^K) = \mathcal{U}_N^H +$ higher order terms, since both $[Y]_{K,rel}$ and $[Y]_{H,rel}$ are obtained from the Thom classes by applying natural maps. However, we have already seen in our discussion of Riemann-Roch that the complete formula is in fact

$$\text{ch}(\mathcal{U}_N^K) = \mathcal{U}_N^H \cdot \text{Td}(N)^{-1}$$

(see Proposition 28.3). Now just observe that $\text{Td}(N)^{-1} = 1 +$ higher order terms.

Now let Y be arbitrary. Let Z be the singular locus, and let q denote the codimension of Y. Consider the diagram

$$\begin{tikzcd} K^0(X, X-Y) \arrow[r, "j^*"] \arrow[d, "\mathrm{ch}"'] & K^0(X-Z, X-Y) \arrow[d, "\mathrm{ch}"] \\ H^*(X, X-Y) \arrow[r, "j^*"] & H^*(X-Z, X-Y). \end{tikzcd}$$

Both maps j^* are simply restriction to an open set, and so $j^*([Y]_{K,rel}) = [Y - Z]_{K,rel}$ and $j^*([Y]_{H,rel}) = [Y - Z]_{H,rel}$. By Lemma 32.4(b) the bottom map is an isomorphism for $* \leq 2q$. So the desired result for Y follows from the corresponding result for $Y - Z$, which has already been proven because $Y - Z$ is smooth in $X - Z$. $\square$

Theorem 32.9. *Let X be a smooth algebraic variety, and let $Y \subseteq X$ be a subvariety. Then $[Y]_H$ survives the Atiyah-Hirzebruch spectral sequence; that is, all differentials vanish on this class.*

Proof. This is now easy. The class $[Y]_H$ is the image of $[Y]_{H,rel}$ under the natural map $H^*(X, X-Y) \to H^*(X)$. By naturality of the Atiyah-Hirzebruch spectral sequence, it suffices to show that all differentials on $[Y]_{H,rel}$ are zero. This follows from Proposition 32.8 and ????. $\square$

Corollary 32.10. *Let X be a smooth algebraic variety, and let p be a fixed prime. If a p^e-torsion class $u \in H^{ev}(X)$ is algebraic then $\beta P^1(\overline{u}) = 0$.*

Proof. Proposition 34.21(b) identifies the first possible differential on a p^e-torsion class in the Atiyah-Hirzebruch spectral sequence as $u \mapsto \beta P^1(\bar{u})$. Using this, the result is immediate from Theorem 32.9. $\square$

32.11. **Construction of varieties with non-algebraic cohomology classes.** At this point our job is to construct a smooth, projective algebraic variety X that has a class $u \in H^{ev}(X)$ for which $\beta P^1 \bar{u} \neq 0$, for some prime p. Such a class cannot be algebraic by Corollary 32.10. It turns out that p can be any prime we like—that is, for any given p we can find an example of an X and a u. Moreover, u can be taken to lie in degree 4. The construction comes out of the following three results:

Theorem 32.12 (Serre)**.** *Let G be a finite group and let $n \geq 1$. Then there exists a linear action of G on a projective space $\mathbb{C}P^N$ together with an n-dimensional, closed, smooth subvariety $X \hookrightarrow \mathbb{C}P^N$ which is a complete intersection, invariant under G, and has G acting freely.*

Corollary 32.13. *If X is a variety having the properties in Theorem 32.12 then X/G is a smooth projective variety and there is an n-connected map $X/G \to \mathbb{C}P^\infty \times BG$. In particular, the homotopy $(n-1)$-type of X/G is the same as that of $K(\mathbb{Z}, 2) \times BG$.*

Proposition 32.14. *Let p be a prime. Then there exists a finite group G and a class $u \in H^4(BG; \mathbb{Z})$ that is killed by p and is such that $\beta P^1(\bar{u}) \neq 0$.*

We postpone the proofs for one moment so that we can observe the immediate consequence:

Corollary 32.15 (Atiyah-Hirzebruch)**.** *Fix a prime p. There exists a smooth, projective, complex algebraic variety X and a class $u \in H^{ev}(X)$ that is killed by p such that u is not algebraic.*

Proof. By Proposition 32.14 there exists a finite group G and a p-torsion class $u \in H^4(BG;\mathbb{Z})$ such that $\beta P^1(\bar{u}) \neq 0$. Note that P^1 has degree $2(p-1)$ and so $\beta P^1(\bar{u}) \in H^{2p+3}(BG;\mathbb{Z}/p)$.

By Corollary 32.13 there is a smooth, projective variety W of dimension $2p+3$ that admits a $(2p+3)$-connected map $W \to K(\mathbb{Z},2) \times BG$. But then $H^*(BG;\mathbb{Z})$ injects into $H^*(W)$ up through dimension $2p+3$, and likewise for $\mathbb{Z}/p$ coefficients. So there is a class $w \in H^4(W)$ such that $\beta P^1(\bar{w}) \neq 0$, and by Corollary 32.10 this class cannot be algebraic. $\square$

Theorem 32.12 requires some algebraic geometry, but the proofs of both Corollary 32.13 and Proposition 32.14 are purely topological. We tackle these in reverse order:

Proof of Proposition 32.14. We start by considering $p = 2$ (the odd case turns out to be extremely similar). As a first attempt we might try to take $G = \mathbb{Z}/2$. Then $B\mathbb{Z}/2 = \mathbb{R}P^\infty$, $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[x]$, and $H^*(\mathbb{R}P^\infty;\mathbb{Z}) = \mathbb{Z}/2[x^2]$. Note that we can regard the integral cohomology as being contained in the mod 2 cohomology. Unfortunately $\mathrm{Sq}^3 = \beta\,\mathrm{Sq}^2$ vanishes on all the integral classes, by an easy calculation. So this attempt doesn't work.

Next look at $B(\mathbb{Z}/2 \times \mathbb{Z}/2) = \mathbb{R}P^\infty \times \mathbb{R}P^\infty$. The mod 2 cohomology is $\mathbb{Z}/2[x,y]$, and because the integral cohomology is all 2-torsion it coincides with the subring of $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty;\mathbb{Z}/2)$ consisting of all elements whose Bockstein vanishes. Such elements of course include all polynomials in x^2 and y^2, but it also includes $\theta = x^2y + xy^2 = \beta(xy)$. In fact all the elements of the integral cohomology look like $x^{2i}y^{2j} \cdot \theta$. Another easy calculation shows that Sq^3 applied to such an element is $x^{2i}y^{2j}\,\mathrm{Sq}^3(\theta)$, and $\mathrm{Sq}^3(\theta) = x^4y^2 + x^2y^4 \neq 0$. This gives us lots of classes that are not killed by Sq^3, however they are all in odd dimensions. So this still doesn't solve our problem.

In the preceding paragraph, the reason things didn't work ultimately came down to the fact that $\beta(xy)$ had odd dimension. This gets fixed once we move to $B(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$. The mod 2 cohomology is $\mathbb{Z}/2[x,y,z]$, and again the integral cohomology is 2-torsion and so coincides with the elements in the mod 2 cohomology where the Bockstein vanishes. One such element is $\theta = \beta(xyz) = x^2yz + xy^2z + xyz^2$. It is easy to calculate that $\mathrm{Sq}^3\,\theta = x^4y^2z + x^4yz^2 + \cdots \neq 0$. So finally we have an even-dimensional integral cohomology class where Sq^3 vanishes.

The argument for odd primes works the same way. Recall that $H^*(B\mathbb{Z}/p;\mathbb{Z}/p) = \Lambda(u) \otimes \mathbb{F}_p[v]$, with $|u| = 1$, $|v| = 2$, $\beta(u) = v$ and $P^1(v) = v^p$. Take $G = \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$. The integral cohomology of BG is all p-torsion and so coincides with the subring of $H^*(BG;\mathbb{Z}/p)$ where the Bockstein vanishes. Look at $\theta = \beta(u_1u_2u_3) \in H^4(BG;\mathbb{Z})$. A simple calculation shows that $\beta P^1(\theta) \neq 0$. $\square$

Exercise 32.16. Verify all of the calculations outlined in the above proof.

Next we prove the corollary to Serre's theorem, again following [AH2]:

Proof of Corollary 32.13. Let $X \hookrightarrow \mathbb{C}P^N$ be the subvariety provided by Theorem 32.12(a). Since $X \hookrightarrow \mathbb{C}P^N$ is a complete intersection, the homotopical form of the Lefschetz Hyperplane Theorem (e.g. [Mi1, Theorem 7.4]) yields that $X \to \mathbb{C}P^N$ is an isomorphism on π_i for $i < \dim X = n$ and an epimorphism on $\pi_{\dim X}$. In other words, the map $X \to \mathbb{C}P^N$ is n-connected.

Let $L \to \mathbb{C}P^N$ be the tautological line bundle $\mathbb{C}^{N+1} - 0 \to \mathbb{C}P^N$. Note that G acts on $\mathbb{C}^{N+1}$, and so G acts on the bundle L. Hence G acts on the pullback bundle $j^*L \to X$. Since the action of G on X is free we get a line bundle $(j^*L)/G \to X/G$ which pulls back to j^*L along the projection $X \to X/G$. Let the classifying map for this line bundle be $X/G \to \mathbb{C}P^\infty$. The diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{C}P^N \\ \downarrow & & \downarrow \\ X/G & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

necessarily commutes up to homotopy, as the two compositions classify the same bundle j^*L. Since $X \to \mathbb{C}P^N$ and $\mathbb{C}P^N \to \mathbb{C}P^\infty$ are both n-connected, so is the composite $X \to \mathbb{C}P^\infty$.

Consider the composite map $X \to X/G \to \mathbb{C}P^\infty$. This is a G-equivariant map, where the target is given the trivial G-action. Since the map is n-connected, so is the map $X \times EG \to \mathbb{C}P^\infty \times EG$ (because EG is contractible). For any connected space Z the map $Z \times EG \to (Z \times EG)/G = Z_{hG}$ is a covering space and therefore an isomorphism on homotopy groups, so it follows that $(X \times EG)/G \to (\mathbb{C}P^\infty \times EG)/G$ is also a n-connected. That is, $X_{hG} \to \mathbb{C}P^\infty_{hG}$ is n-connected. But since the G-action on X is free one has $X_{hG} \simeq X/G$, and since the action on $\mathbb{C}P^\infty$ is trivial one has $(\mathbb{C}P^\infty)_{hG} \simeq \mathbb{C}P^\infty \times BG$. This completes the proof. $\square$

Finally we prove Serre's theorem, following [S1, Section 20]. Serre credits the method to Godeaux.

Proof of Theorem 32.12. We first give the proof for $G = \mathbb{Z}/2$. Even though the general case is basically the same, certain steps can be made more concrete by restricting to this case.

Recall that our goal is to construct a certain variety X of dimension n. For the moment, fix $M \geq 1$. Eventually we will narrow the choice of M in relation to n, but for now M is arbitrary.

Let $\mathbb{C}^{2M}$ have coordinates $x_1, \ldots, x_M, y_1, \ldots, y_M$, and let G act trivially on the x's and by negation on the y's. Consider the induced action on $P = \mathbb{P}(\mathbb{C}^{2M}) = \mathbb{C}P^{2M-1}$. The homogeneous coordinate ring is $R = \mathbb{C}[x_1, \ldots, x_M, y_1, \ldots, y_M]$, and the ring of invariants $S = R^G$ is the subring generated by the x_i's and the y_iy_j's (including $i = j$). An easy argument shows that every homogeneous element of S having even degree is a polynomial in the elements x_ix_j and y_iy_j. That is, if $S(2) \subseteq S$ is the $\mathbb{C}$-linear span of all even-dimensional homogeneous elements then $S(2)$ is generated (as a subring) by elements of degree 2.

Let $\alpha_1, \ldots, \alpha_s$ denote the degree 2 monomials in the x_i, in some order. Similarly, let $\beta_1, \ldots, \beta_s$ denote the degree 2 monomials in the y_j, and let $b = 2s - 1 = 2\binom{M+1}{2} - 1 = M^2 + M - 1$. Let $f \colon P \to \mathbb{C}P^b$ be the map

$$f([x_1 : \cdots : x_M : y_1 : \ldots : y_M]) = [\alpha_1 : \ldots : \alpha_s : \beta_1 : \ldots : \beta_s].$$

Clearly f induces a map $P/G \to \mathbb{C}P^b$. It is easy to see that this is a projective embedding (???). Its image Z is the closed subvariety whose homogeneous coordinate ring is $S(2)$, regarded as a quotient of the polynomial algebra $\mathbb{C}[\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_s]$. We have $\dim Z = \dim P = 2M-1$.

Let $A \subseteq P$ be the set of fixed points for the $\mathbb{Z}/2$-action. One readily checks that A is the disjoint union $\mathbb{C}P^{M-1} \amalg \mathbb{C}P^{M-1}$, where the first piece corresponds to the vanishing of the x_i's and the second piece to the vanishing of the y_j's. The fibers of f are sets with at most two elements, and so $f(A) \subseteq Z$ is a subvariety having the same dimension as A. Note that $P - A \to Z - f(A)$ is a covering space, and so $Z - f(A)$ is nonsingular.

Let $H \subseteq \mathbb{C}P^b$ be a generic subspace of dimension $b + n - (2M-1)$. The dimension has been chosen so that $H \cap Z$ will have dimension n, but note that we will need $n < 2M-1$ for H to exist. We can choose H so that it misses the singular set of Z and intersects Z transversely, and if $\dim f(A) + \dim H < b$ we can simultaneously require that H does not intersect $f(A)$. This dimension criterion is $M - 1 + b - (2M-1) + n < b$, or just $n < M$. So from now on we assume that M was chosen to be at least this large.

Our choice of H guarantees that $Z \cap H$ is a nonsingular variety of dimension n. Let $X = f^{-1}(Z \cap H)$. The criterion that $H \cap f(A) = \emptyset$ implies that $X \cap A = \emptyset$, and so G acts freely on X and the map $f|_X \colon X \to Z \cap H$ is a two-fold covering space. So X is also nonsingular of dimension n.

The subspace H is defined by the vanishing of linear elements $h_1,\dots,h_t$ in the ring S, where $t = (2M-1) - n$. Via the inclusion $S \subseteq R$ we can regard these as elements of R, where they are homogeneous of degree 2. The subvariety $X \subseteq P$ is the vanishing set of these polynomials. Given that X is of codimension t, we find that X is a set-theoretic complete intersection: it is the intersection of the t hypersurfaces defined by each of the h_i's. It remains to show that X is actually a scheme-theoretic complete intersection: i.e., that the ideal of functions vanishing on X is generated by a regular sequence.

Since X has codimension t is follows that $\operatorname{ht}(h_1,\dots,h_t) = t$. By ??? this implies that $h_1,\dots,h_t$ is a regular sequence. If we let $I = (h_1,\dots,h_t)$ then the ideal of functions vanishing on X is $\operatorname{Rad}(h_1,\dots,h_t)$. We will show that $\operatorname{Rad}(I) = I$, as this proves that X is a scheme-theoretic complete intersection.

By Macaulay's Unmixedness Theorem [E, Corollary 18.14], all associated primes of I are minimal primes of I. So I has a primary decomposition $I = Q_1 \cap \cdots \cap Q_k$ where each Q_i is primary and $\operatorname{Rad}(Q_i) = P_i$ is a minimal prime of I. The $V(P_i)$'s are the irreducible components of X, and so for each i we can choose a closed point $m \in V(P_i)$ that does not belong to any other component. So m is a maximal ideal containing P_i, and $IR_m = Q_iR_m$.

Let $m' = f(m)$ and consider the diagram of local rings

$$\begin{tikzcd}
\mathcal{O}_{X,m} \arrow[d] & \mathcal{O}_{Z,m'} \arrow[l] \arrow[d] \\
\widehat{\mathcal{O}}_{X,m} & \widehat{\mathcal{O}}_{Z,m'}. \arrow[l, "\cong"']
\end{tikzcd}$$

The bottom map is an isomorphism because $X \to Z$ is a two-fold covering space. The assumption that H meets Z transversely implies that $g_1,\dots,g_t$ is part of a regular system of parameters for $\mathcal{O}_{Z,m'}$; that is to say, their images in the Zariski cotangent space $(m')/(m')^2$ are independent. The same is therefore true in $\mathcal{O}_{X,m}$, because all maps in the above square induce isomorphisms on the Zariski cotangent space.

In particular, the fact that $h_1, \ldots, h_t$ is part of a regular system of parameters in $\mathcal{O}_{X,m}$ implies that IR_m is prime. So $Q_i R_m = IR_m = \mathrm{Rad}(IR_m) = P_i R_m$. This can only happen if $Q_i = P_i$ (if Q is primary with radical P and $P \subseteq m$, then $QR_m = PR_m$ if and only if $Q = P$). We have thus proven that $Q_i = P_i$ for every i, and this implies $I = \mathrm{Rad}(I)$.

This proof is now completed for the case $G = \mathbb{Z}/2$.

For a general finite group G let V be the regular representation and let $P = \mathbb{P}(V^M)$. Let R be the homogeneous coordinate ring of P, and let $S = R^G$. If $S(d) \subseteq S$ is the subring spanned by homogeneous elements in degrees a multiple of d, one can prove that for some value of d the ring $S(d)$ is generated as an algebra by its elements of degree d. Choose a $\mathbb{C}$-basis $f_0, \ldots, f_b$ for these generators and let $f\colon P \to \mathbb{C}P^b$ be the map $x \mapsto [f_0(x) : f_1(x) : \cdots : f_b(x)]$. This map induces a projective embedding $P/G \hookrightarrow \mathbb{C}P^b$; call the image Z. Let $A \subseteq P$ be the set of elements with nontrivial stabilizer under G. For $g \in G$ an easy argument shows that any eigenspace of g acting on V must have dimension equal to at most the number of right cosets of $\langle g \rangle$ in G. So the dimension of the eigenspace is at most $\#G/\#\langle g \rangle$, and therefore is bounded above by $\#G/2$. The eigenspaces of g acting on V^M thus have dimension at most $M \cdot \#G/2$, and from this one derives that $\dim A \leq (M \cdot \#G/2) - 1$. The rest of the argument proceeds almost identically to the $G = \mathbb{Z}/2$ case, the only change being that we take H to have dimension $b + n - (M \cdot \#G - 1)$ and that we only need to require $M > \frac{2n}{\#G}$ in order to choose H so that it avoids $f(A)$. $\square$

32.17. **Thom's theory.** Although this part of the story doesn't use K-theory, the obstructions to algebraicity obtained from resolution of singularities are so simple that they are worth discussing here.

Theorem 32.18. *Let X be a smooth algebraic variety, and let p be a fixed prime. If $u \in H^{ev}(X;\mathbb{Z})$ is algebraic then all odd-degree cohomology operations vanish on the mod p reduction $\bar{u} \in H^{ev}(X;\mathbb{Z}/p)$. In particular, all of the odd Steenrod squares vanish on the mod 2 reduction of u.*

The above theorem was probably folklore since the 1960s. It explicitly appears in the beautiful paper [T]. The key to this result is the cohomology theory called complex cobordism, denoted MU. This was first introduced and studied by Thom. The definitions are somewhat involved, but the corresponding homology groups $MU_*(X)$ are formed from classes of manifolds with a certain kind of complex structure mapping into X, with an equivalence relation coming from bordism of such things. See ??? for a general introduction. The spectrum MU can be constructed without going through the geometrical considerations, and the reader is taken through that in Exercise 32.22 below. Our discussion will sweep most of these details under the rug, as we only need a few carefully selected properties of MU.

First, MU is a complex-oriented cohomology theory. In fact it is the universal such one, in a certain sense. Consequently, there is a map of cohomology theories $MU^*(-) \to H^*(-)$ that is compatible with the complex orientations. So the map sends Thom classes to Thom classses (i.e. $\mathcal{U}_E^{MU} \mapsto \mathcal{U}_E^{H}$) and consequently if $Y \hookrightarrow X$

is a smooth subvariety it sends $[Y]_{MU}$ to $[Y]_H$. We will not explain this claim in detail, but it is not hard (see Exercise 32.22 below).

The next part of the argument is best explained using the language of spectra. The above map of cohomology theories comes from a map of spectra $MU \to H\mathbb{Z}$. If $Y \hookrightarrow X$ is a smooth subvariety of codimension q then $[Y]_{MU} \in MU^{2q}(X)$ is represented by a map $X \to \Sigma^{2q}MU$, and likewise $[Y]_H \in H^{2q}(X)$ is represented by a map $X \to \Sigma^{2q}H\mathbb{Z}$. In the homotopy category of spectra we have the commutative diagram

$$\begin{tikzcd} & \Sigma^{2q}MU \arrow[d] \\ X \arrow[ur, "{[Y]_{MU}}"] \arrow[r, "{[Y]_H}"'] & \Sigma^{2q}H\mathbb{Z}. \end{tikzcd}$$

Let θ be a cohomology operation of degree r on $H^*(-;\mathbb{Z}/p)$. This is a map of spectra $H\mathbb{Z}/p \to \Sigma^r H\mathbb{Z}/p$. The application of this operation to the mod p reduction of $[Y]_H$ enhances our diagram:

$$\begin{tikzcd} & \Sigma^{2q}MU \arrow[d] \arrow[rrd, dotted, "f"] & & \\ X \arrow[ur, "{[Y]_{MU}}"] \arrow[r, "{[Y]_H}"'] & \Sigma^{2q}H\mathbb{Z} \arrow[r] & \Sigma^{2q}H\mathbb{Z}/p \arrow[r, "\theta"'] & \Sigma^{2q+r}H\mathbb{Z}/p. \end{tikzcd}$$

The map labelled f is just the evident composite. Note that f is an element of $H^r(MU;\mathbb{Z}/p)$. The "miracle" is that we can easily compute this group. The spaces making up the spectrum MU are just Thom spaces of the universal bundles $\gamma_n \to BU(n)$, and their integral cohomology is known by the Thom isomorphism. We leave the details to the reader (see Exercise 32.22), but the trivial conclusion here is that $H^*(MU;\mathbb{Z})$ is free abelian and concentrated in even degrees. It follows that $H^*(MU;\mathbb{Z}/p)$ vanishes in all odd degrees. In particular, the map f is null when r is odd! Thus, we have proven Thom's theorem:

Proposition 32.19 (Thom)**.** *Let X be a smooth algebraic variety and let $Y \hookrightarrow X$ be a smooth subvariety. Fix a prime p. Then all odd degree cohomology operations vanish on the mod p reduction of the class $[Y]_H \in H^*(X)$.*

Remark 32.20. It is interesting to note how easy the language of spectra makes the above argument. As a challenge, try to unwind the argument and rephrase it without using spectra—it is not so pleasant.

One can deduce Theorem 32.18 from Proposition 32.19 using resolution of singularities and a little work. We are not going to give complete details, but we give a rough sketch. Complete details (and much more) can be found in [T].

Sketch of proof of Theorem 32.18. It suffices to prove Theorem 32.18 when u is the fundamental class of an irreducible subvariety $Y \hookrightarrow X$, say of codimension q. Such elements generate all algebraic cohomology classes. By Hironaka there is a resolution of singularities $\tilde{Y} \to Y$ obtained by successively blowing up Y at closed subschemes. Even more, one can successively blow up X at the same subschemes to produce a commutative square

$$\begin{tikzcd} \tilde{Y} \arrow[r, tail] \arrow[d] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, tail] & X \end{tikzcd}$$

where the horizontal maps are closed inclusions and the vertical maps are compositions of successive blow-ups (in particular, they are proper). We of course have the class $[\tilde{Y}]_{MU} \in MU^{2q}(\tilde{X})$, and with a little work one can construct a pushforward map $\pi_!\colon MU^{2q}(\tilde{X}) \to MU^{2q}(X)$. We claim that $\pi_!([\tilde{Y}]_{MU})$ is a lift of the class $[Y]_H$. By ???? this can be checked by applying $j^*\colon H^*(X) \to H^*(X-Z)$ where Z is the singular set of Y and seeing that $j^*(\pi_!([\tilde{Y}]_{MU})) = [Y-Z]$. This can in turn be deduced from an appropriate push-pull formula and the fact that $\tilde{Y} - \pi^{-1}(Y) \to Y - Z$ is a homeomorphism. In any case, if you accept this last point then we now have the diagram

$$\begin{tikzcd} & \Sigma^{2q}MU \arrow[d] \\ X \arrow[ur, "\pi_!([\tilde{Y}]_{MU})"] \arrow[r, "{[Y]_H}"'] & \Sigma^{2q}H\mathbb{Z} \end{tikzcd}$$

and at this point everything proceeds the same as before. This completes our sketch of a proof for Theorem 32.18. □

Note that even with this approach, as opposed to the K-theory approach of Atiyah-Hirzebruch, one still needs to gives examples of algebraic varieties with nontrivial odd-degree operations on even-dimensional cohomology classes. So the hard work that went on in Section 32.11 is still necessary.

◦ Exercises ◦

The following exercises introduce the reader to the cohomology theories MO and MU.

Exercise 32.21. Let $MO(n) = \mathrm{Th}(\gamma_n \to BO(n))$, and note that $MO(0) = S^0$.

(a) Confirm that there is a pullback diagram

$$\begin{tikzcd} \gamma_n \oplus \underline{1} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] & \\ \mathrm{Gr}_n(\mathbb{R}^\infty) \arrow[r] & \mathrm{Gr}_{n+1}(\mathbb{R} \oplus \mathbb{R}^\infty) \arrow[r, equal] & \mathrm{Gr}_{n+1}(\mathbb{R}^\infty) \end{tikzcd}$$

leading to a canonical (up to homotopy) map $\Sigma MO(n) \to MO(n+1)$. Define MO to be the spectrum made up of the sequence of spaces

$$MO(0),\ MO(1),\ MO(2),\ \cdots$$

and having the maps from (a) as structure maps. This is called the **real bordism spectrum**.

(b) Choose once and for all a basepoint in $BO(0)$ and take the corresponding basepoint in $BO(n)$ for all n. The fibers over these basepoints give canonical inclusions $S^n \hookrightarrow MO(n)$. Check that these assemble to give a map of spectra $S \to MO$.

(c) Each $MO(n)$ has a (canonical) Thom class $\mathcal{U}_n \in H^n(MO(n); \mathbb{Z}/2)$. These can be regarded as maps $MO(n) \to K(\mathbb{Z}/2, n)$. Verify that these assemble to give a map of spectra $MO \to H\mathbb{Z}/2$, and therefore a map of cohomology theories $MO^*(X) \to H^*(X; \mathbb{Z}/2)$.

Exercise 32.22. Let $MU(n) = \mathrm{Th}(\gamma_n \to BU(n))$.

(a) Confirm that there is a pullback diagram

$$\begin{tikzcd}
\gamma_n \oplus \underline{1} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] & \\
\mathrm{Gr}_n(\mathbb{C}^\infty) \arrow[r] & \mathrm{Gr}_{n+1}(\mathbb{C} \oplus \mathbb{C}^\infty) \arrow[r, equal] & \mathrm{Gr}_{n+1}(\mathbb{C}^\infty)
\end{tikzcd}$$

leading to a canonical (up to homotopy) map $\Sigma^2 MU(n) \to MU(n+1)$.

(b) Verify that there is a spectrum made up of the sequence of spaces

$$MU(0),\ \Omega MU(1),\ MU(1),\ \Omega MU(2),\ MU(2),\ \Omega MU(3),\ MU(3),\ \ldots$$

where the structure maps are induced by the maps in (a) and the standard counit maps $\Sigma\Omega X \to X$.

[Note that it feels more natural to just use the sequence $MU(0), MU(1), MU(2), \ldots$ but that this would give a spectrum-like object where the structure maps involve a two-fold suspension. One can develop such Σ^2-spectra and prove that they give an equivalent theory to regular spectra. While this is somehow the more pleasing approach, it is more common just to use the regular definition of spectra and throw in the "odd" spaces $\Omega MU(n)$ that are formally forced into the picture. But it is good to recognize that one can basically ignore those terms.]

(c) Verify that the Thom classes $\mathcal{U}_n \in H^{2n}(MU(n))$ give a map of spectra $MU \to H\mathbb{Z}$ (see Exercise 32.21(c) for the simpler case).

(d) There are canonical classes $\mathcal{U}_n \in MU^{2n}(MU(n))$, corresponding to the identity maps $MU(n) \to MU_{2n} = MU(n)$. Verify that these make MU into a complex-oriented spectrum and that the map of cohomology theories $MU \to H\mathbb{Z}$ sends the MU-Thom class for a bundle to its H-Thom class.

(e) If E is a spectrum then the homology group $H_n(E)$ is defined to be the direct limit

$$H_n(E_0) \to H_{n+1}(E_1) \to H_{n+2}(E_2) \to \cdots$$

For MU we can take the cofinal system of every other term and write

$$H_n(MU) = \mathrm{colim}\Big[H_n(MU(0)) \to H_{n+2}(MU(1)) \to H_{n+4}(MU(2)) \to \cdots\Big]$$

where the maps are induced by the maps in (a). Recall that the Thom isomorphism gives $\tilde{H}^i(MU(n)) \cong H^{i-n}(BU(n))$ and that $H^*(BU(n)) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where the c_i are the Chern classes. Verify that $H_*(MU)$ is concentrated only in even degrees and is free abelian, and conclude the same for $H^*(MU)$ by the universal coefficient theorem.

33. K-theoretic Chern classes: the γ-operations

Let $E \to X$ be a complex vector bundle of rank n. We will define certain natural classes $\tilde{\gamma}^k(E) \in \tilde{K}^0(X)$ with the following properties:

(1) $\tilde{\gamma}^k(E) = 0$ for $k > n$ and $\tilde{\gamma}^0(E) = 1$,
(2) $\tilde{\gamma}^k(E \oplus F) = \sum_{i+j=k} \tilde{\gamma}^i(E)\tilde{\gamma}^j(F)$,
(3) $\tilde{\gamma}^1(\underline{1}) = 0$,
(4) $\tilde{\gamma}^n(E^*) = (-1)^n e^K(E)$, where $e^K(E)$ is the K-theoretic Euler class defined in Section 25.13,
(5) $\tilde{\gamma}^1(E) = [E] - n$.

Properties (1)–(3), and almost (4), are familiar properties of the usual Chern classes in singular cohomology (in (4) we would not expect the dual or the sign to appear). But note that properties (1)–(3) are unaffected if we add in the duals and signs, so this suggests that we should define the K-theoretic Chern classes by

$$c_i^K(E) = (-1)^i \tilde{\gamma}^i(E^*).$$

These classes now satisfy all of the familiar properties. Below we will see some further justification of this definition.

There is a bit of a historical oddity here. One could change the historical definition of the $\tilde{\gamma}$-operations so that the dual is built in, and then the $\tilde{\gamma}$-operations would be precisely the K-theoretic Chern classes. On some levels this would be more satisfying, but we have chosen to follow the historical conventions.

The following is another familiar property of Chern classes, and is a consequence of the ones above:

Proposition 33.1. *Suppose that X is a paracompact Hausdorff space and the rank n bundle $E \to X$ admits r independent sections. Then $\tilde{\gamma}^{n+1-i}(E) = 0 = c^K_{n+1-i}(E)$ for $1 \le i \le r$.*

Proof. The r independent sections give an embedding $\underline{r} \hookrightarrow E$. If Q denotes the quotient, then the exact sequence $0 \to \underline{r} \hookrightarrow E \to Q \to 0$ is split by Proposition 9.2, and so $E \cong \underline{r} \oplus Q$. Then

$$\tilde{\gamma}^{n+1-i}(E) = \sum_{a+b=n+1-i} \tilde{\gamma}^a(\underline{r})\tilde{\gamma}^b(Q) = \tilde{\gamma}^{n+1-i}(Q).$$

But since Q has rank $n-r$, the latter expression vanishes if $n+1-i > n-r$, which is when $i < r+1$.

Since E splits off a copy of $\underline{r}$ if and only if E^* does, we can apply the above reasoning to E^* and deduce the vanishing of the c^K-classes. □

We still need to construct the $\tilde{\gamma}$ classes. We are going to write down a direct but rather unsatisfying definition; then we will spend the rest of the section explaining where the definition comes from.

Definition 33.2. *For $E \to X$ a rank n vector bundle define*

$$\begin{aligned}\tilde{\gamma}^k(E) &= (-1)^k \tbinom{n}{k}[\Lambda^0 E] + (-1)^{k-1}\tbinom{n-1}{k-1}[\Lambda^1 E] + (-1)^{k-2}\tbinom{n-2}{k-2}[\Lambda^2 E] + \cdots \\ &= \sum_{i=0}^{k} (-1)^{k-i}\tbinom{n-i}{k-i}[\Lambda^i E].\end{aligned}$$

Properties (1), (3), (4), and (5) are immediate from this definition. Property (2) can be deduced from the isomorphism $\Lambda^n(E \oplus F) \cong \bigoplus_{a+b=n} \Lambda^a E \otimes \Lambda^b F$ by a brute force calculation, but it requires some combinatorial cleverness. We will instead take a different route which helps us better understand where the above definition came from.

Proposition 33.3. *If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles then*

$$\tilde{\gamma}^k(E) = \bigoplus_{i_1 < \ldots < i_k} (L_{i_1} - 1) \cdots (L_{i_k} - 1).$$

Proof. Let $x_1, \ldots, x_n$ be formal variables, and consider the polynomial

$$P_k = \bigoplus_{i_1 < \cdots < i_k} (x_{i_1} - 1) \cdots (x_{i_k} - 1).$$

This is a symmetric function and therefore can be written as a polynomial in the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$. The constant term is clearly $\binom{n}{k}(-1)^k$ (put all $x_i = 0$). For $s \leq k$ a given monomial $x_{r_1} \cdots x_{r_s}$ appears in exactly $\binom{n-s}{k-s}$ summands (all k-subsets containing $r_1, \ldots, r_s$) with coefficient $(-1)^{k-s}$, so this gives a summand of P_k of the form $(-1)^{k-s}\binom{n-s}{k-s}\sigma_s$. Thus,

$$P_k = \sum_{i=0}^{k} (-1)^{k-i}\tbinom{n-i}{k-i}\sigma_i.$$

When we change x_i to L_i, the σ_i function is precisely $\Lambda^i E$. So this completes the proof. ☐

Corollary 33.4. *If E and F are sums of line bundles then $\tilde{\gamma}^k(E \oplus F) = \sum_{i+j=k} \tilde{\gamma}^i(E)\tilde{\gamma}^j(F)$.*

Proof. Immediate from Proposition 33.3. ☐

Remark 33.5. Properties (1)–(4) force the definition of the $\tilde{\gamma}$ classes on sums of line bundles. By applying the argument in the above proof one is led to Definition 33.2 in that special case, and it is natural to then extrapolate to the general case.

Note that we have still not proven property (2), though we are working up to it.

Let us wipe the slate at this point and assume we do not yet have a definition of the $\tilde{\gamma}$ classes. Recall the generating function

$$\lambda_t(E) = 1 + t[E] + t[\Lambda^2 E] + \cdots$$

and the fact that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$. We can introduction the analogous generating function

$$\tilde{\gamma}_t(E) = 1 + t\tilde{\gamma}^1(E) + t^2\tilde{\gamma}^2(E) + \cdots$$

and since we want property (2) to hold it is tempting to try to define $\tilde{\gamma}_t(E)$ in terms of $\lambda_t(E)$. For L a line bundle we know $\lambda_t(L) = 1 + t[L]$, whereas we want $\tilde{\gamma}_t(L) = 1 + t([L] - 1) = (1 - t) + t[L]$. This quickly leads us to write

$$\tilde{\gamma}_t(L) = (1 - t)\lambda_{\frac{t}{1-t}}(L)$$

and suggests that for a rank n bundle E we put

$$\tilde{\gamma}_t(E) = (1 - t)^n \lambda_{\frac{t}{1-t}}(E). \tag{33.6}$$

That is, we take this as the *definition* of the $\tilde{\gamma}^i(E)$ classes. With this defintion properties (1)—(3) are immediate; (4) is immediate for line bundles and then is deduced in general by applying property (2) and using that $e^K(E \oplus F) = e^K(E)e^K(F)$.

Unravelling equation (33.6) leads to

$$\tilde{\gamma}_t(E) = (1-t)^n + (1-t)^{n-1}t[E] + (1-t)^{n-2}t^2[\Lambda^2 E] + \cdots + t^n[\Lambda^n E].$$

The coefficient of t^k is therefore

$$\tilde{\gamma}^k(E) = (-1)^k\tbinom{n}{k} + (-1)^{k-1}\tbinom{n-1}{k-1}[E] + (-1)^{k-2}\tbinom{n-2}{k-2}[\Lambda^2 E] + \cdots$$

which recovers Definition 33.2.

Now we have finally confirmed that our definition of the $\tilde{\gamma}$ classes satisfies properties (1)–(5).

Remark 33.7. The more standard approach to this material is to first define classes $\gamma^i(E)$, and to get the $\tilde{\gamma}$ classes from those. We briefly recount this. For $x \in K^0(X)$ define

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x), \tag{33.8}$$

regarded as a power series in t with coefficients in $K(X)$. Note that $\gamma_t(x \oplus y) = \gamma_t(x)\gamma_t(y)$ because this property is inherited from the λ_t-construction. So γ_t is a group homomorphism $\gamma_t\colon K(X) \to 1 + tK(X)[[t]]$, where the target is the multiplicative group of power series with coefficients in $K(X)$ and having leading coefficient 1. Note that $\gamma_t(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t} = 1 + t + t^2 + \cdots$. One then defines

$$\tilde{\gamma}_t(x) = \gamma_t(x - \underline{\operatorname{rank} x}) = \frac{\gamma_t(x)}{\gamma_t(\underline{\operatorname{rank} x})} = (1-t)^{\operatorname{rank} x} \cdot \gamma_t(x) = (1-t)^{\operatorname{rank} x} \cdot \lambda_{\frac{t}{1-t}}(x).$$

An objection to this approach is that the formula (33.8) seems to come out of nowhere, whereas our development provided more motivation.

In the literature one almost never finds the $\tilde{\gamma}^i$ notation, so let us record the formula

$$\tilde{\gamma}^i(x) = \gamma^i(x - \operatorname{rank}(x)) \tag{33.9}$$

that translates from our notation into the classical one. In particular, if x has rank zero then $\gamma^i(x) = \tilde{\gamma}^i(x)$ for all i.

Exercise 33.10. If L is a line bundle calculate $\gamma_t(L)$ and $\tilde{\gamma}_t(L)$, and observe the differences. Then use the formula $\gamma_t(1-L) = \frac{\gamma_t(1)}{\gamma_t(L)}$ to calculate $\gamma^i(1-L) = (1-L)^i$ for all i. Repeat for $\tilde{\gamma}_t(1-L) = \frac{\tilde{\gamma}_1(1)}{\tilde{\gamma}_t(L)}$ to calculate $\tilde{\gamma}^i(1-L)$ for all i.

Exercise 33.11. Note that $\gamma_t(x)\gamma_t(-x) = \gamma_t(0) = 1$. Use this to calculate that $\gamma^1(-x) = -\gamma^1(x)$ and $\gamma^2(-x) = \gamma^1(x)^2 - \gamma^2(x)$. In general prove that $\gamma^i(-x)$ can be expressed as a homogeneous polynomial of degree i in the classes $1, \gamma^1(x), \gamma^2(x), \ldots, \gamma^i(x)$, where each $\gamma^j(x)$ is regarded as having degree j. In fact prove that $\gamma^i(y-x)$ can be expressed as a homogeneous polynomial of degree i in the $\gamma^a(y)$ and $\gamma^b(x)$ classes.

Observe that the analogous result holds for the $\tilde{\gamma}^i$ classes as well.

33.12. **Topology and the K-theoretic Chern classes.** The inclusion of the diagonal matrices into U_n takes the form $(S^1)^{\times n} \hookrightarrow U_n$, and induces $\alpha\colon (BS^1)^{\times n} \to BU_n$. This can be modeled geometrically by the map $(\mathbb{C}P^\infty)^{\times n} \to BU_n$ that classifies the direct sum $\pi_1^*L \oplus \cdots \oplus \pi_n^*L$ where $L \to \mathbb{C}P^\infty$ is the canonical line bundle. Since permuting the factors gives an isomorphic bundle, $\alpha\sigma \simeq \alpha$ for any $\sigma \in \Sigma_n$. For any cohomology theory E it follows that $a^*\colon E^*(BU_n) \to E^*((\mathbb{C}P^\infty)^{\times n})$ lands in the ring of invariants:

$$\alpha^*\colon E^*(BU_n) \to \left[E^*((\mathbb{C}P^\infty)^{\times n})\right]^{\Sigma_n}.$$

If E is complex-oriented then $E^*(\mathbb{C}P^\infty \times \cdots \mathbb{C}P^\infty) \cong E^*[[x_1, \dots, x_n]]$ (the ring of graded power series, as in Section 31.22), and so we have

$$\alpha^*\colon E^*(BU_n) \longrightarrow E^*[[\sigma_1, \dots, \sigma_n]]$$

Here x_i is the E-theory Euler class of $\pi_i^*(L)$.

Applying the above to complex K-theory, and pushing everything to K^0 using Bott periodicity, we have the map

$$K^0(BU_n) \longrightarrow \mathbb{Z}[[\sigma_1, \dots, \sigma_n]] \subseteq \mathbb{Z}[[x_1, \dots, x_n]] = K^0(\mathbb{C}P^n \times \cdots \times \mathbb{C}P^n)$$

where $x_i = e^K(\pi_i^*L) = 1 - \pi_i^*L^*$. If $\gamma \to BU_n$ is the tautological bundle, then $\alpha^*\gamma = \bigoplus_i \pi_i^*L$. Observe that

$$\begin{aligned}
\alpha^*\big(c_r^K(\gamma)\big) = c_r^K(\pi_1^*L \oplus \cdots \oplus \pi_n^*L) &= (-1)^r \tilde{\gamma}^r(\pi_1^*L^* \oplus \cdots \oplus \pi_n^*L^*) \\
&= (-1)^r \bigoplus_{i_1 < \cdots < i_r} (\pi_{i_1}^*L^* - 1) \cdots (\pi_{i_r}^*L^* - 1) \\
&= \bigoplus_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} \\
&= \sigma_r
\end{aligned}$$

where the third equality is by Proposition 33.3. This confirms that c_r^K is the "correct" analog of the classical rth Chern class in singular cohomology.

33.13. **Geometry and the K-theoretic Chern classes.** At this point we have understood the definition of c_r^K from combinatorial and topological perspectives. But one would hardly look at any of our definitions—e.g. Definition 33.2—and immediately see some interesting geometry underlying them. Our next aim is to see if we can identify some geometry secretly lurking here.

If C is a chain complex of vector spaces, define the dth symmetric product by the usual formula

$$\mathrm{Sym}_d(C) = (C^{\otimes(d)})/\Sigma_d$$

but note that the Σ_d-action on the tensor product is via the symmetric monoidal structure on chain complexes and so involves the Koszul sign rule. If V is a vector space and $V[i]$ denotes the chain complex with V in dimension i and zeros elsewhere, then

$$\mathrm{Sym}_d(V[i]) \cong \begin{cases} (\mathrm{Sym}_d V)[di] & \text{if } i \text{ is even,} \\ (\Lambda^d V)[di] & \text{if } i \text{ is odd.} \end{cases}$$

If C and D are chain complexes then we have the usual canonical isomorphism

$$\mathrm{Sym}_d(C \oplus D) \cong \bigoplus_{i+j=d} \mathrm{Sym}_i(C) \otimes \mathrm{Sym}_j(D). \tag{33.14}$$

If $f\colon V \to W$ is a map of vector spaces write Cf for the chain complex that has $(Cf)_0 = V$, $(CF)_{-1} = W$, all other chain groups zero, and the differential $(CF)_0 \to (CF)_{-1}$ is equal to f (this is essentially the desuspension of the mapping cone of f). Our next goal will be to compute the vector spaces $H_*(\mathrm{Sym}_d(Cf))$. To this end, observe that we can choose direct sum decompositions $V = V_1 \oplus V_2$ and $W = W_2 \oplus W_3$ in such a way that $f(V_1) = 0$, $f(V_2) = W_2$, and $f|_{V_2}\colon V_2 \to W_2$ is an isomorphism. For convenience let $g = f|_{V_2}$, so that $Cf \cong V_1[0] \oplus Cg \oplus W_3[-1]$. We then get that

$$\mathrm{Sym}_d(Cf) \cong \bigoplus_{i+j+k=d} \mathrm{Sym}_i(V_1)[0] \otimes \mathrm{Sym}_j(Cg) \otimes (\Lambda^k W_3)[-k].$$

The key observation is that $\mathrm{Sym}_j(Cg)$ is acyclic for all $j > 0$. Since the tensor product of an acyclic complex and anything is again acyclic, this means the only terms in the above sum that contribute to homology are the ones with $j = 0$. This gives

$$\mathrm{Sym}_d(Cf) \simeq \bigoplus_{i+k=d} \Big(\mathrm{Sym}_i(V_1) \otimes \Lambda^{d-i}(W_3)\Big)[-(d-i)]. \tag{33.15}$$

This decomposition simplifies dramatically if either V_1 or W_3 is zero:

Proposition 33.16. *Let $f\colon V \to W$ be a nonzero map of vector spaces.*

(a) If $\ker f = 0$ then the homology of $\mathrm{Sym}_d(Cf)$ is concentrated in dimension $-d$ and equals $\Lambda^d(\operatorname{coker} f)$. In particular, $\mathrm{Sym}_d(Cf)$ is exact for $d > \dim \operatorname{coker}(f)$.

(b) If $\operatorname{coker} f = 0$ then the homology of $\mathrm{Sym}_d(Cf)$ is concentrated in dimension 0 and equals $\mathrm{Sym}_d(\ker f)$.

Proof. Immediate from (33.15). □

For us the most important part of the above result is part (a). It shows that if a map is injective then some associated complexes will be exact. So that starts to feel like a context where K-theory might provide obstructions.

The above considerations all pass to vector bundles without any trouble. If $f\colon E_1 \to E_2$ is a map of bundles over a space X then one gets the complex of bundles $\mathrm{Sym}_d(Cf)$. The following result is immediate from the preceding one:

Corollary 33.17. *Let $f\colon E_1 \to E_2$ be an injective map of bundles. Then for any $d > \mathrm{rank}(E_2) - \mathrm{rank}(E_1)$ the complex $\mathrm{Sym}_d(Cf)$ is exact.*

This corollary allows us to write down a collection of K-theoretic obstructions to the existence of an embedding $E_1 \hookrightarrow E_2$. If such an embedding exists then for appropriate d we get that $\mathrm{Sym}_d(Cf)$ is exact, which implies that the alternating sum of the terms is zero in $K(X)$. This alternating sum is the same as for $\mathrm{Sym}_d(Cz)$ where $z\colon E_1 \to E_2$ is the zero map, in which case we can use the isomorphism

$$\mathrm{Sym}_d(Cz) \cong \bigoplus_{i=0}^{d} \Big(\mathrm{Sym}_i(E_1) \otimes \Lambda^{d-i}(E_2)\Big)[-(d-i)].$$

So define the class

$$\sigma_d(E_1, E_2) = \mathrm{Sym}_d(E_1 \xrightarrow{0} E_2) = \sum_{i=0}^{d} (-1)^{d-i}\, \mathrm{Sym}_i(E_1) \otimes \Lambda^{d-i}(E_2) \in K^0(X)$$

where the chain complex $E_1 \xrightarrow{0} E_2$ is in degrees 0 and -1. We have proven the following:

Proposition 33.18. *If there is an embedding of bundles $E_1 \hookrightarrow E_2$ then $\sigma_d(E_1, E_2) = 0$ in $K(X)$ for every $d > \operatorname{rank}(E_2) - \operatorname{rank}(E_1)$.*

If E is a vector bundle of rank n then the dth Chern class is supposed to be an obstruction to having $n - d + 1$ independent sections, so this suggests that we look at $\sigma_d(n - d + 1, E)$. Indeed, this unravels to

$$\begin{aligned}\sigma_d(n-d+1,E) &= \sum_{i=0}^{d}(-1)^{d-i}\operatorname{Sym}_i(n-d+1)\otimes\Lambda^{d-i}(E)\\ &= \sum_{i=0}^{d}(-1)^{d-i}\tbinom{n+i-d}{i}\Lambda^{d-i}(E)\end{aligned}$$

where we have used Lemmma 23.11 to calculate the dimension of the symmetric product. Comparison to Definition 33.2 gives that

$$\sigma_d(n-d+1,E) = (-1)^d\tilde{\gamma}^d(E) = c_d^K(E^*).$$

If we had started with $\tilde{\gamma}^d(E) = (-1)^d\sigma_d(n - d + 1, E)$ as the definition of the $\tilde{\gamma}$-classes, then of the properties we listed at the beginning of the section all but (2) are immediate. So is Proposition 33.1, even without (2). But what about property (2)? It seems natural to expect this to fall out of the behavior of symmetric products on sums (33.14), but it takes a little more work than that. We start with the following:

Lemma 33.19. *Write $\operatorname{Sym}_d(r)$ for the dimension of $\operatorname{Sym}_d(V)$ where V is a vector space of dimension r. Then*

$$\operatorname{Sym}_d(k+n-d-1) = \sum_{i+j=d}\operatorname{Sym}_i(k-i)\operatorname{Sym}_j(n-j)$$

for all values of d, k, and n where the formula makes sense.

Proof. Recall that $\operatorname{Sym}_d(r) = \binom{r+d-1}{d}$, by Lemma 23.11. Then the identity becomes

$$\tbinom{k+n-2}{d} = \sum_{i+j=d}\tbinom{k-1}{i}\tbinom{n-1}{j}$$

which is the standard Chu-Vandermonde identity for binomial coefficients, coming from the natural ways of breaking up a choice of d objects from a set of $k-1$ red objects and $n-1$ blue objects. □

Proposition 33.20. *Let E and F be bundles of ranks e and f. Then*

$$\sigma_d(e+f-d+1, E\oplus F) = \sum_{i+j=d}\sigma_i(e-i+1,E)\sigma_j(f-j+1,F).$$

Consequently, $\tilde{\gamma}^d(E\oplus F) = \sum_{i+j=d}\tilde{\gamma}^i(E)\tilde{\gamma}^j(F)$.

Proof. The definition of $\sigma_i(e - i + 1, E)$ may be rewritten as

$$\sigma_i(e-i+1,E) = \sum_{a=0}^{i}(-1)^a\operatorname{Sym}_{i-a}(e-i+1)\Lambda^a E.$$

We will use this version throughout the calculation. Observe that

$$\begin{aligned}
&\sum_{i+j=d} \sigma_i(e-i+1,E)\sigma_f(f-j+1,F)\\
&= \sum_{i+j=d}\sum_{a=0}^{i}\sum_{b=0}^{j}(-1)^{a+b}\operatorname{Sym}_{i-a}(e-i+1)\operatorname{Sym}_{j-b}(f-j+1)(\Lambda^a E)(\Lambda^b F)\\
&= \sum_{N=0}^{d}\sum_{a+b=N}\sum_{s+t=d-N}(-1)^N \operatorname{Sym}_s(e-a+1-s)\operatorname{Sym}_t(f-b+1-t)(\Lambda^a E)(\Lambda^b F)\\
&= \sum_{N=0}^{d}\sum_{a+b=N}(-1)^N \operatorname{Sym}_{d-N}(e-a+1+f-b+1-(d-N)-1)(\Lambda^a E)(\Lambda^b F)\\
&= \sum_{N=0}^{d}\sum_{a+b=N}(-1)^N \operatorname{Sym}_{d-N}(e+f-d+1)(\Lambda^a E)(\Lambda^b F)\\
&= \sum_{N=0}^{d}(-1)^N \operatorname{Sym}_{d-N}(e+f-d+1)\Lambda^N(E\oplus F)\\
&= \sigma_d(e+f-d+1,E\oplus F).
\end{aligned}$$

In the second equality we are changing the order of the summation and also substituting $s = i - a$ and $t = j - b$. In the third equality we used Lemma 33.19. $\square$

33.21. **The γ-filtration.** To any cohomology group $E^p(X)$ we may attach the Atiyah-Hirzebruch filtration, defined by

$$F_nE^p(X) = \{a \in E^p(X) \mid f^*(a) = 0 \text{ for every map } f\colon A \to X \text{ where } A \text{ is a CW-complex with } \dim(A) < n\}.$$

We have

$$E^p(X) = F_0E^p(X) \supseteq F_1E^p(X) \supseteq F_2E^p(X) \supseteq \cdots$$

with $F_{n+1}E^p(X) = 0$ if X is a CW-complex of dimension n. Note that when X is 0-connected one has $F_1E^p(X) = \tilde{E}^p(X)$.

Given a map $f\colon X \to Y$ the induced map $f^*\colon E^p(Y) \to E^p(X)$ clearly sends $F_nE^p(Y)$ into $F_nE^p(X)$. So the filtration is functorial. When f is a weak homotopy equivalence the induced map $F_nE^p(Y) \to F_nE^p(X)$ is readily checked to be an isomorphism.

When X is a CW-complex it follows from cellular approximation that

$$F_nE^p(X) = \ker(E^p(X) \to E^p(X_{n-1})).$$

But note that our original definition of the filtration shows that it is independent of the choice of CW-structure on X, and indeed does not require a CW-structure on X at all.

Exercise 33.22. If E is a multiplicative cohomomology theory verify that the Atiyah-Hirzebruch filtration is multiplicative, in the sense that the pairings

$$E^p(X) \otimes E^q(Y) \to E^{p+q}(X \times Y)$$

map the image of $F_nE^p(X) \otimes F_kE^q(Y)$ to $F_{n+k}E^{p+q}(X \times Y)$, for all n and k. Consequently, the filtration on $E^*(X)$ is also multiplicative.

Write $\mathcal{E}_p$ for the representing space for $E^p(-)$; that is, $E^p(X) = [X, \mathcal{E}_p]$. Then we can also define the Atiyah-Hirzebruch filtration using the Whitehead tower for the space $\mathcal{E}_p$. Write $\mathcal{E}_p\langle n, \infty\rangle$ for the $(n-1)$-connected cover of $\mathcal{E}_p$, which is the homotopy fiber of the natural map $\mathcal{E}_p \to P_{n-1}\mathcal{E}_p$ whose codomain is the $(n-1)$st Postnikov section of $\mathcal{E}_p$. We get a tower of homotopy fiber sequences

$$\begin{tikzcd}
 & K(\pi_2\mathcal{E}_p, 2) & K(\pi_1\mathcal{E}_p, 1) & K(\pi_0\mathcal{E}_p, 0) \\
\cdots \arrow[r] & \mathcal{E}_p\langle 2, \infty\rangle \arrow[u] \arrow[r] & \mathcal{E}_p\langle 1, \infty\rangle \arrow[u] \arrow[r] & \mathcal{E}_p \arrow[u]
\end{tikzcd}$$

and we can define

$$\mathcal{F}_n E^p(X) = \text{image of } [X, \mathcal{E}_p\langle n, \infty\rangle] \to [X, \mathcal{E}_p].$$

At first glance this looks totally different from the Atiyah-Hirzebruch construction, but in fact they are equal:

Proposition 33.23. *For every space X one has $\mathcal{F}_n E^p(X) = F_n E^p(X)$.*

Proof. It is immediate that $\mathcal{F}_n E^p(X) \subseteq F_n E^p(X)$, since if A is a CW-complex with $\dim A < n$ then $[A, \mathcal{E}_p\langle n, \infty\rangle] = 0$. To see the other direction, first replace X by a weakly equivalent CW-complex. Given an element $f \in F_n E^p(X)$, we can represent it by a map $X \to \mathcal{E}_p$ which is null on X_{n-1}. But then it extends to $\tilde{f} \colon X/X_{n-1} \to \mathcal{E}_p$. Using that $H^i(X/X_{n-1}; \mathcal{A}) = 0$ for $i < n$ and all coefficients $\mathcal{A}$, obstruction theory lets us inductively lift the map $\tilde{f}$ up the tower until it becomes a map into $\mathcal{E}_p\langle n, \infty\rangle$. From there we see that $f \in \mathcal{F}_n E^p(X)$. $\square$

Now let us apply these observations to K^0, whose representing space is $\mathbb{Z} \times BU$. Since $\pi_*(\mathbb{Z} \times BU)$ is nonzero only in even degrees, notice that the Whitehead tower approach gives that $F_{2n-1}K^0(X) = F_{2n}K^0(X)$ for all X and all n. Because of this, it is tempting to only consider the groups $F_{2n}K^0(X)$ and maybe even re-index them as F_n. We won't do that, but this observation is useful to keep in mind for what we are about to discuss.

Following Grothendieck, we will use the $\tilde{\gamma}$-operations to give an approximation to the Atiyah-Hirzebruch filtration on $K^0(X)$. This is the so-called γ-filtration. Grothendieck worked in the context of algebraic geometry, and there one has neither cell decompositions nor Whitehead towers. There are some geometric substitutes one can attempt to use in place of the former, but sometimes it is hard to prove things about those. The motivation for the γ-filtration is that it is defined purely algebraically, and it can be shown to have many of the properties of the Atiyah-Hirzebruch filtration. Rationally they turn out to be equal, as we will show below.

As a prelude to the γ-filtration we start with the following result:

Proposition 33.24. *For any vector bundle E on X one has $\tilde{\gamma}^k(E) \in F_{2k}K^0(X)$ for all k.*

Proof. First note that $\tilde{\gamma}^k$ is a stable operation, in the sense that $\tilde{\gamma}^k(E \oplus 1) = \tilde{\gamma}^k(E)$ for all E. So we may regard it as a map of sets $K^{st}(X) \to \widetilde{K}^0(X)$, or equivalently $\widetilde{K}^0(X) \to \widetilde{K}^0(X)$. This implies that the operation $\tilde{\gamma}^k$ is represented by a map $\tilde{\gamma}^k \colon BU \to BU$. If E is a bundle of rank less than k then $\tilde{\gamma}^k(E) = 0$, and from this it follows that the composite $BU(k-1) \hookrightarrow BU \xrightarrow{\tilde{\gamma}^k} BU$ is null. So $\tilde{\gamma}^k$ induces a map $g_k\colon BU/BU(k-1) \to BU$. But $\tilde{H}^*(BU/BU(k-1))$ vanishes for $* < 2k$, and so by obstruction theory g_k lifts to a map $BU/BU(k-1) \to (\mathbb{Z}\times BU)\langle 2k,\infty\rangle$. Therefore $\tilde{\gamma}^k$ lifts as well, and this is what we neeed to prove. $\square$

Define the γ-filtration on $K^0(X)$ to be the smallest multiplicative filtration having the property that each class $\tilde{\gamma}^n(E)$ is in F_n. More specifically, we set

$$F_n^\gamma K^0(X) = \langle \tilde{\gamma}^{i_1}(E_1)\tilde{\gamma}^{i_2}(E_2)\cdots\tilde{\gamma}^{i_r}(E_r)\,\big|\, r\geq 0,\ i_1+\cdots+i_r\geq n\rangle$$

where the generators range over all vector bundles $E_1,\dots,E_r$ on X. (Let us remind the reader once again that $\tilde{\gamma}^i(E)=\gamma^i(E-\mathrm{rank}(E))$ as in (33.9), and that the formulas in the literature are usually written in the latter notation). It follows at once that $F_\bullet^\gamma K^0(X)$ is a descending filtration that is functorial in X. Proposition 33.24 gives that $F_n^\gamma K^0(X)\subseteq F_{2n}K^0(X)$, for all n. Since $\tilde{\gamma}^1(E)=E-\mathrm{rank}(E)$ we also get that $F_1^\gamma K^0(X)=\widetilde{K}^0(X)$, and so $F_1^\gamma K^0(X)=\widetilde{K}^0(X)=F_1K^0(X)=F_2K^0(X)$.

Keep in mind when comparing the γ- and Atiyah-Hirzebruch filtrations that the former is indexed at double speed compared to the latter, i.e. we have $F_n^\gamma K^0(X)\subseteq F_{2n}K^0(X)$.

Computing the γ-filtration can feel daunting because one seems to need to know all of the vector bundles on the space X, which in fact almost never happens. The following result offers some simplifications:

Proposition 33.25. *It is always true that*

$$F_n^\gamma K^0(X) = \langle \tilde{\gamma}^{i_1}(x_1)\tilde{\gamma}^{i_2}(x_2)\cdots\tilde{\gamma}^{i_r}(x_r)\,\big|\, r\geq 0,\ i_1+\cdots+i_r\geq n, x_i\in\widetilde{K}^0(X)\rangle.$$

If $w_1,\dots,w_N$ are additive generators for $\widetilde{K}^0(X)$ then it is also true that

$$F_n^\gamma K^0(X) = \langle \tilde{\gamma}^{i_1}(w_{j_1})\tilde{\gamma}^{i_2}(w_{j_2})\cdots\tilde{\gamma}^{i_r}(w_{j_r})\,\big|\, r\geq 0,\ i_1+\cdots+i_r\geq n,\\ 1\leq j_1,\dots,j_r\leq N\rangle.$$

Proof. Note that the second statement is not immediate from the first, since the operations $\tilde{\gamma}^i$ are not additive.

For the first statement, write I_n for the group on the right. Since for a vector bundle E one has $\tilde{\gamma}^i(E)=\tilde{\gamma}^i(E-\mathrm{rank}(E))$ one certainly has $F_n^\gamma\subseteq I_n$. For the other direction, observe that a class $x\in\widetilde{K}^0(X)$ can be written as $x=[E]-[F]$ for some vector bundles E and F. By Exercise 33.11 the class $\gamma^i(x)$ can be written as a homogeneous polynomial of degree i in the classes $\gamma^a(E)$ and $\gamma^b(F)$ (where each γ^j class is assigned degree j), and from this it follows that $\gamma^i(x)\in F_i^\gamma$. It follows at once that $I_n\subseteq F_n^\gamma$, and so the two are equal.

For the second statement, let J_n be the group on the right. Note that $J_\bullet$ is the descending multiplicative filtration determined by the conditions that $\tilde{\gamma}^i(w_j)\in J_i$ for all i,j. We clearly have $J_n\subseteq I_n$. For the subset in the other direction, first note that by Exercise 33.11 the element $\tilde{\gamma}^i(-w_j)$ is a polynomial of degree i in the classes $\tilde{\gamma}^s(w_j)$. It follows at once that $\tilde{\gamma}^i(-w_j)\in J_i$, for all i and j.

Next we observe that if $x\in\widetilde{K}^0(X)$ and $\tilde{\gamma}^i(x)\in J_i$ for all i, then $\tilde{\gamma}^k(x\pm w_j)\in J_k$ for all k. This follows directly from the formulas

$$\tilde{\gamma}^k(x\pm w_j)=\sum_{t=0}^{k}\tilde{\gamma}^t(x)\tilde{\gamma}^{k-t}(\pm w_j)$$

together with what has already been proven.

Finally, a straighforward induction now proves that $\tilde{\gamma}^i(\sum s_j w_j) \in J_i$ for all i and all $s_1, \ldots, s_N \in \mathbb{Z}$. From this it follows directly that $I_n \subseteq J_n$, and so the two sets are equal. $\square$

Example 33.26. Let us compute the γ-filtration on $K^0(\mathbb{C}P^n)$. Recall from Proposition 23.6 that $K^0(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$ where $x = 1 - L$. As an abelian group this is $\mathbb{Z}^{n+1}$ generated by $1, L, L^2, \ldots, L^n$, and so $\widetilde{K}^0(\mathbb{C}P^n)$ is generated by the classes $1 - L^i$ for $i = 1, \ldots, n$.

By Exercise 33.10 we have $\tilde{\gamma}^s(1 - L^i) = (1 - L^i)^s$, which is a multiple of $(1 - L)^s = x^s$. It follows that $F_s^\gamma \subseteq (x^s)$, for all s (note that we are implicitly using Proposition 33.25 here). But we also have $x^s = (1 - L)^s = \tilde{\gamma}^s(1 - L) \in F_s^\gamma$, and so in fact $F_s^\gamma = (x^s)$ for all s.

Rationally, the γ- and Atiyah-Hirzebruch filtrations are the same up to this indexing issue:

Proposition 33.27. *For all $n \geq 0$, $F_n^\gamma K^0(X) \otimes \mathbb{Q} = F_{2n} K^0(X) \otimes \mathbb{Q}$.*

Proof. ??? $\square$

This completes our introduction to the γ-operations on $K^0(X)$. As the K-theoretic analogs of the Chern classes, they can be used in applications for some of the same purposes that the classical Chern classes are used. Some examples are discussed in Sections 39 and 40.

Part 5. Topological techniques and applications

In the next few sections we will mostly ignore the "geometric" perspective on K-theory that we have developed so far in these notes. Instead we will concentrate on the topological aspects of K-theory, in particular its use *as a cohomology theory* (forgetting about the complex-orientation). We will develop the basic topological tools for computing K-groups, use them to carry out some important computations, and then apply these computations to solve (or at least obtain information about) certain types of geometric, algebraic, and topological problems.

34. The Atiyah-Hirzebruch spectral sequence

Let $\mathcal{E}$ be a cohomology theory. Let X be a CW-complex with cellular filtration

$$\emptyset = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq X.$$

So each F_k/F_{k-1} is a wedge of k-spheres, and $X = \bigcup_k F_k$. Since one knows the cohomology groups $\mathcal{E}^*(F_k/F_{k-1})$, one can attempt to inductively determine the cohomology groups $\mathcal{E}^*(F_k)$ and thus to eventually determine $\mathcal{E}^*(X)$. A spectral sequence is a device for organizing all the information in such a calculation, and it has the surprising feature that one can determine $\mathcal{E}^*(X)$ without *explicitly* determining each of the steps $\mathcal{E}^*(F_k)$. It is somewhat magical that this can be done.

We will not try to teach the theory of spectral sequences from scratch here. For a thorough treatment the reader may refer to [Mc], for example. We will assume the reader has some familiarity with this theory, but at the same time we give a brief review.

34.1. **Generalities.** Each inclusion $F_{q-1} \hookrightarrow F_q$ yields a long exact sequence on cohomology, and these long exact sequences braid together to yield the following infinite diagram:

(34.2)

$$\begin{tikzcd}
 & {} \arrow[d, "i"] & & {} \arrow[d, "i"] & & \\
\cdots \arrow[r] & \boxed{\mathcal{E}^{p-1}(F_q)} \arrow[r, "j"] \arrow[d, "i"] & \mathcal{E}^p(F_{q+1},F_q) \arrow[r, "k"] & \mathcal{E}^p(F_{q+1}) \arrow[r, "j"] \arrow[d, "i"] & \mathcal{E}^{p+1}(F_{q+2},F_{q+1}) \arrow[r, "k"] & {} \\
\cdots \arrow[r] & \boxed{\mathcal{E}^{p-1}(F_{q-1})} \arrow[r, "j"] \arrow[d, "i"] & \boxed{\mathcal{E}^p(F_q,F_{q-1})} \arrow[r, "k"] & \boxed{\mathcal{E}^p(F_q)} \arrow[r, "j"] \arrow[d, "i"] & \mathcal{E}^{p+1}(F_{q+1},F_q) \arrow[r, "k"] & {} \\
\cdots \arrow[r] & \mathcal{E}^{p-1}(F_{q-2}) \arrow[r, "j"] \arrow[d, "i"] & \mathcal{E}^p(F_{q-1},F_{q-2}) \arrow[r, "k"] & \boxed{\mathcal{E}^p(F_{q-1})} \arrow[r, "j"] \arrow[d, "i"] & \boxed{\mathcal{E}^{p+1}(F_q,F_{q-1})} \arrow[r, "k"] & {} \\
 & {} & & {} & &
\end{tikzcd}$$

The terms in boxes constitute one long exact sequence: the one for the inclusion $F_{q-1} \hookrightarrow F_q$. Translating these terms vertically yields an infinite family of long exact sequences, each linked to the next via two of their three terms. A spectral sequence is a bookkeeping device for managing diagram chases in this setting. One obtains a spectral sequence of the form

$$E_1^{p,q} = \mathcal{E}^p(F_q, F_{q-1}) \Rightarrow \mathcal{E}^p(X).$$

Let us explain how this works, and in the course of doing so we will also explain what it means. By the way, notice that in our particular setup the columns of the diagram are eventually zero as one proceeds downward, because $F_i = \emptyset$ when i is negative. Notice as well that if $X = F_n$ for some value of n (so the filtration is finite) then the columns stabilize when moving upwards.

Remark 34.3. There seems to be no standard name for diagrams like the one above, consisting of families of long exact sequences braided together. But they are the source of essentially every spectral sequence. Massey defined the more general notion of an *exact couple*, but personally I have found that level of generality more harmful than useful. Briefly, define $D = \oplus_{p,q}\mathcal{E}^p(F_q)$ and $E = \oplus_{p,q}\mathcal{E}^p(F_q, F_{q-1})$. So the D groups form two-thirds of the columns in the above diagram, and the E-groups are the remaining third. The exact couple consists of the maps

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

and the requirement of exactness at each vertex. While this is an elegant abstraction of our situation, the suppression of the gradings can get in the way of understanding what is happening.

We will refer to our braided long exact sequences as an "exact couple", even though this is a slight abuse of terminology. "Bigraded exact couple" might be more appropriate.

We return to our development of the spectral sequence machinery:

(0). Here is the basic idea for how the spectral sequence operates. Consider an element $x \in \mathcal{E}^p(F_q, F_{q-1})$, and proceed as follows:

(i) Let $x_0 = kx$.

(ii) If $j(x_0) = 0$ then $x_0 = i(x_1)$ for some $x_1 \in \mathcal{E}^p(F_{q+1})$. We may then look at jx_1.

(iii) If $jx_1 = 0$ then $x_1 = i(x_2)$ for some $x_2 \in \mathcal{E}^p(F_{q+2})$. We may then look at jx_2.

(iv) Continuing in this way, we get a sequence of "obstructions" jx_u, $u = 0, 1, 2, \ldots$. Each one only exists if the previous one vanishes. Note that at each stage the vanishing of jx_u doesn't depend on the choice of x_u; however, it *may* depend on the choice of x_v made at some previous stage $v < u$. In this sense the obstructions are not unique: different choices of lifts may lead to different obstructions later down the line.

(v) Two questions to consider: (A) For each n, can we make a consistent choice of liftings $x_1, \ldots, x_n$? In other words, can we lift kx arbitrarily far up in the diagram? And (B) Can we make a consistent choice of liftings x_i for all $i \geq 1$? That is to say, can we lift kx *coherently* all the way up the diagram? Questions (A) and (B) are not equivalent, because it could be that extending from x_n to x_{n+1} is not possible even though it would have been possible with a different choice of the lower x_i. If one constantly has to be changing the lower choices in order to make each extension, it could be that there is no coherent choice of all the extensions as once.

The vanishing of our obstructions says something about these questions. If the filtration $F_\bullet$ was finite then the questions are the same, and imply that we have produced an element of $\mathcal{E}^p(X)$. In situation (B) something like this also works for infinite filtrations, although the resulting element of $\mathcal{E}^p(X)$ is only uniquely determined in good cases. The spectral sequence is a device for keeping track of these obstructions and liftings, and what they ultimately produce.

We will now go through all of the machinery needed to define and work with the spectral sequence associated to our exact couple.

(1). For $1 \leq r \leq \infty$ write

$$Z_r^{p,q} = \{x \in \mathcal{E}^p(F_q, F_{q-1}) \,\big|\, kx \text{ may be lifted at least } r \text{ times under } i\}.$$

This is called the group of ***r*-cycles** in the spectral sequence. The phrasing is ambiguous when $r = \infty$, but we mean $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$. We also define $B_r^{p,q} \subseteq Z_r^{p,q}$ to be the subgroup generated by all "obstructions" that arise from at most $r-1$ layers lower down in the diagram. To be precise, $B_r^{p,q}$ is spanned by the sets

$$ji^{-s}k(\mathcal{E}^{p-1}(F_{q-s-1}, F_{q-s-2}))$$

for $0 \leq s \leq r-1$. It is best to immediately forget this precise description and just remember the idea.

Notice that everything in $B_r^{p,q}$ maps to zero under k, and hence is contained in every $Z_t^{p,q}$. That is, we have

$$0 = B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_2^{p,q} \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} = \mathcal{E}^p(F_q, F_{q-1}).$$

Note that $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$ and $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$.

(2). It as an easy exercise to prove that $x \in B_r^{p,q}$ if and only if x can be written as $x = j(y_1 + y_2 + \cdots + y_r)$ for some y_u's such that $i^u(y_u) = 0$, for all u. As an immediate corollary, $B_\infty^{p,q}$ coincides with the image of j (or equivalently with the kernel of k).

Let $x_0 \in \mathcal{E}^p(F_q)$ and assume that we lifted x_0 a total of r times: that is, assume we have chosen elements $x_u \in \mathcal{E}^p(F_{q+u})$ for $1 \leq u \leq r$ such that $i(x_u) = x_{u-1}$. If $j(x_r) \in B_s$ for some s then (using the result of the previous paragraph) there exists a chain of elements $x'_u \in \mathcal{E}^p(F_{q+u})$ such that $i(x'_u) = x'_{u-1}$, $x'_u = x_u$ for $u \leq r-s$, and $j(x'_r) = 0$. That is to say, we can alter our chain of x_u's in the top $s-1$ spots and end up with a chain that can be extended upwards one more level. Indeed, just define $x'_u = x_u - i^{r-u}(y_{r-u+1} + \cdots + y_s)$ where $j(x_r) = j(y_1 + \cdots + y_s)$ and the y_i's are as in the preceding paragraph.

(3). Define $E_r^{p,q} = Z_{r-1}^{p,q}/B_{r-1}^{p,q}$. The process "apply k, lift $r-1$ times, then apply j" yields a well-defined map $d_r\colon E_r^{*,*} \to E_r^{*,*}$. This is our "obstruction" map, and it is now well-defined precisely because we are quotienting out by the subgroup $B_{r-1}^{*,*}$. Note that d_r shifts the bigrading, so that we have $d_r\colon E_r^{p,q} \to E_r^{p+1,q+r}$.

The map d_r satisfies $d_r^2 = 0$ because $kj = 0$. A little work shows that $E_{r+1}^{*,*}$ is precisely the homology of $E_r^{*,*}$ with respect to d_r. The sequence of chain complexes $E_1, E_2, E_3, \ldots$, each the homology of the previous one, is the **spectral sequence** associated to our exact couple.

(4). For fixed values of p and q we have "entering" and "exiting" differentials

$$E_r^{?,?} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{?,?}$$

for certain unimportant values of '?'. Using (2) it is easy to check that

(i) The exiting map d_r is zero if and only if $Z^{p,q}_{r-1} = Z^{p,q}_r$, and
(ii) The entering map d_r is zero if and only if $B^{p,q}_{r-1} = B^{p,q}_r$.

(5). It remains to interpret the E_∞-term. Notice that, strictly speaking, this is not one of the stages of our spectral sequence—it is not obtained as the homology of a previous complex. Still, in many examples one finds that E_∞ agrees with some finite stage E_r, at least through a range of dimensions.

Write $\mathcal{E}^p(F_q)_\infty$ for the set of all $x \in \mathcal{E}^p(F_q)$ such that $i(x) = 0$ and x lifts arbitrarily far up in the diagram (and note that this is *not* the same as saying that x lifts to the inverse limit). The map k induces a surjection $Z^{p,q}_\infty \twoheadrightarrow \mathcal{E}^p(F_q)_\infty$, and $B^{p,q}_\infty$ is clearly the kernel; so we have an induced isomorphism

$$\Gamma\colon E^{p,q}_\infty \xrightarrow{\cong} \mathcal{E}^p(F_q)_\infty.$$

Consider the groups $\mathcal{E}^p(\mathcal{F}) = \lim_q \mathcal{E}^p(F_q)$. As for any inverse limit these come with a filtration where we define

$$\mathcal{E}^p(\mathcal{F})_{ZPq} = \{\alpha \in \mathcal{E}^p(\mathcal{F}) \,|\, \text{the image of } \alpha \text{ in } \mathcal{E}^p(F_{q-1}) \text{ is zero}\}.$$

The "ZPq" subscript is supposed to remind us "zero past F_q". We call this the "ZP-filtration":

$$\mathcal{E}^p(\mathcal{F}) = \mathcal{E}^p(\mathcal{F})_{ZP0} \supseteq \mathcal{E}^p(\mathcal{F})_{ZP1} \supseteq \mathcal{E}^p(\mathcal{F})_{ZP2} \supseteq \cdots$$

There is an evident map $\mathcal{E}^p(\mathcal{F})_{ZPq} \to \mathcal{E}^p(F_q)_\infty$ which induces an injection

$$\mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)} := \mathcal{E}^p(\mathcal{F})_{ZPq}/\mathcal{E}^p(\mathcal{F})_{ZP(q+1)} \hookrightarrow \mathcal{E}^p(F_q)_\infty = E^{p,q}_\infty. \tag{34.4}$$

Notice our notation for the associated graded of the ZP-filtration.

(6). We are ultimately trying to get information about $\mathcal{E}^*(X)$. Observe that we have a natural map $\mathcal{E}^p(X) \to \mathcal{E}^p(\mathcal{F})$. This is always surjective; while not obvious, it is a consequence of the fact that if Z is any topological space then $[X, Z] \to \lim_q [X_q, Z]$ is surjective, which in turn is a routine application of the homotopy extension property for cellular inclusions. Let

$$\mathcal{E}^p(X)_{ZPq} = \ker\big(\mathcal{E}^p(X) \to \mathcal{E}^p(F_{q-1})\big)$$

and note that we have a map of filtrations $\mathcal{E}^p(X)_{ZP_\bullet} \to \mathcal{E}^p(\mathcal{F})_{ZP_\bullet}$. The map on associated graded groups

$$\mathcal{E}^p(X)_{ZP(q/q+1)} \to \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)}$$

is readily seen to be an isomorphism, for all q.

(7). If we are in "good" cases then the map in (34.4) will actually be an isomorphism. The question is whether an element of $\mathcal{E}^p(F_q)$ that can be lifted arbitrarily high in the diagram can also be lifted into the inverse limit—note that this is not automatic! It might be possible that higher and higher liftings exist but not "coherently"; that is, to get a higher lifting one needs to change arbitrarily many elements lower down in the chain.

Fix a p and consider the following condition:

(SSC_p) : There exists an N such that for all q the differentials entering and exiting the group $E^{p,q}_r$ are zero for all $r \geq N$.

If this "Spectral Sequence Convergence Condition" holds then by (4) we know $Z^{p,q}_{N-1} = Z^{p,q}_r$ and $B^{p,q}_{N-1} = B^{p,q}_r$ for all $r \geq N$. Therefore $Z^{p,q}_\infty = Z^{p,q}_{N-1}$,

$B_\infty^{p,q} = B_{N-1}^{p,q}$, and consequently $E_\infty^{p,q} = E_N^{p,q}$. These statements hold for all values of q. So the $E_\infty^{p,*}$ groups coincide with the stable values of the $E_r^{p,*}$ groups.

The convergence condition (SSC_p) gives us one more important consequence, namely that the map of (34.4) is an isomorphism:

$$(SSC_p) \Rightarrow \Big[\ \mathcal{E}^p(\mathcal{F})_{ZP(q/q+1)} \cong E_\infty^{p,q},\ \text{ for all } q\Big].$$

To prove this, let $x_0 \in \mathcal{E}^p(F_q)_\infty$. Then there exist elements $x_u \in \mathcal{E}^p(F_{q+u})$ for $1 \le u \le 2N$ such that $i(x_u) = x_{u-1}$ for each u (nothing special about $2N$ is being used here, it is just a convenient large number). The element $j(x_{2N})$ lies in B_{2N+1}, which we have seen equals B_{N-1} by (SSC_p). Therefore by (2) we may modify the x_u's in dimensions $x_{2N}, x_{2N-1}, \dots, x_{N+2}$ in such a way that the chain extends to an x_{2N+1}. It is important here that only the top $N-1$ elements are affected, for we can now continue by induction and build an element of $\mathcal{E}^p(\mathcal{F}) = \lim_s \mathcal{E}^p(F_s)$ that maps to x in $\mathcal{E}^p(F_q)$. This completes the proof.

To summarize, the statement that the spectral sequence "converges to $\mathcal{E}^p(X)$" is usually interpreted to mean that

- The groups $E_r^{p,*}$ stabilize, and equal $E_\infty^{p,*}$, at some value of r (usually one that is independent of the grading $*$); and,
- The maps in (34.4) are isomorphisms, for all values of q.

Under these conditions the stable values of the spectral sequence give the associated graded groups of the ZP-filtration on $\mathcal{E}^p(X)$. We have seen that (SSC_p) implies this kind of convergence.

(8). Sometimes it is convenient to have in mind a variation on the exact couple diagram (34.2). Fix an integer q and consider the following:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\boxed{\mathcal{E}^p(F_{q+2}, F_{q-1})} \arrow[r, "k"] \arrow[d] & \mathcal{E}^p(F_{q+2}) \arrow[r, "j"] \arrow[d] & \mathcal{E}^{p+1}(F_{q+3}, F_{q+2}) \\
\boxed{\mathcal{E}^p(F_{q+1}, F_{q-1})} \arrow[r, "k"] \arrow[d] & \mathcal{E}^p(F_{q+1}) \arrow[r, "j"] \arrow[d] & \mathcal{E}^{p+1}(F_{q+2}, F_{q+1}) \\
\mathcal{E}^p(F_q, F_{q-1}) \arrow[r, "k"] & \mathcal{E}^p(F_q) \arrow[r, "j"] & \mathcal{E}^{p+1}(F_{q+1}, F_q)
\end{tikzcd}$$

The groups in the boxes are "new", in the sense that they are not part of the exact couple (and they are being drawn in places which previously were occupied by other groups from the exact couple). Each trio of groups $\mathcal{E}^p(F_{q+r}, F_{q-1}) \to \mathcal{E}^p(F_{q+r-1}, F_{q-1}) \xrightarrow{jk} \mathcal{E}^{p+1}(F_{q+r}, F_{q+r-1})$ is part of the long exact sequence for a triple, and so is exact in the middle spot. From this one can see via a diagram chase that a class $u \in \mathcal{E}^p(F_q, F_{q-1}) = E_1^{p,q}$ lies in Z_r if and only if u lifts to a class in $\mathcal{E}^p(F_{q+r}, F_{q-1})$ (for this argument one needs to combine the above diagram with (34.2) and do the chase in both at once). So the differentials in the spectral sequence can be viewed as a sequence of obstructions for lifting u to a class in $\mathcal{E}^p(F_N, F_{q-1})$, for larger and larger N.

(9). (Summary of the general workings of spectral sequences). We have produced a sequence of bigraded chain complexes $E_1^{*,*}, E_2^{*,*}, \dots$ such that each equals the homology of the previous one. We also have a "limiting" collection of bigraded groups $E_\infty^{*,*}$, and we have seen that under certain convergence conditions these E_∞ groups really are the "stable values" in the sequence of E_r's, and moreover they give the associated graded groups for the ZP-filtration of $\mathcal{E}^*(X)$.

(10). Everything that we have said so far works for any increasing filtration $F_\bullet$ of X. Now we use the fact that a CW-filtration is very special. Notice that

$$E_1^{p,q} = \mathcal{E}^p(F_q, F_{q-1}) \cong \widetilde{\mathcal{E}}^p(F_q/F_{q-1}) \cong \widetilde{\mathcal{E}}^p\Big(\bigvee_\alpha S^q\Big) \cong \bigoplus_\alpha \widetilde{\mathcal{E}}^p(S^q) \cong \bigoplus_\alpha \mathcal{E}^{p-q}(pt)$$

where the wedges and direct sums are over the set of q-cells in X. We can identify this group with the cellular cochain group $C^q(X;\mathcal{E}^{p-q}(pt))$. The differential $d_1\colon E_1^{p,q} \to E_1^{p+1,q+1}$ is a map $C^q(X;\mathcal{E}^{p-q}(pt)) \to C^{q+1}(X;\mathcal{E}^{p-q}(pt))$ and it is readily checked to coincide with the differential in the cellular cochain complex. We conclude that

$$E_2^{p,q} \cong H^q(X;\mathcal{E}^{p-q}(pt)).$$

Notice that the E_2-term is a homotopy invariant of X, whereas the E_1-term was not.

(11). The bigraded groups forming each term of the spectral sequence can be reindexed in whatever way seems convenient, and topologists use various conventions in different settings. For the Atiyah-Hirzebruch spectral sequence the standard convention is to choose the grading $E_1^{p,q} = \mathcal{E}^{p+q}(F_p, F_{p-1})$ so that we get

$$E_2^{p,q} = H^p(X;\mathcal{E}^q(pt)).$$

Under this grading we have that the differential d_r is a map

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The groups in the spectral sequence are drawn on a grid where p is the horizontal axis and q the vertical one, with $E_2^{p,q}$ drawn in the (p,q)-spot. Finally, in this grading the Γ-map relating the E_∞-term to the associated graded of $\mathcal{E}^*(X)$ has the form

$$\Gamma\colon \mathcal{E}^{p+q}(X)_{ZP(p/p+1)} \hookrightarrow E_\infty^{p,q}.$$

In terms of the charts, the "total degree" lines are the diagonals where $p+q$ is constant. The (SSC_t) condition, translated into this new indexing, says that on the diagonal $p+q=t$ all entering and exiting differentials vanish past some finite stage of the spectral sequence. When this condition holds we are guaranteed convergence for the groups along this diagonal.

When one uses the Atiyah-Hirzebruch spectral sequence it is easy (after a while) to remember the indexing scheme for the groups and differentials, and the fact that the diagonals $p+q=n$ give the associated graded of $\mathcal{E}^n$. The other thing to keep track of is whether the $E_\infty^{p,q}$ groups give the pth or qth associated graded piece of the filtration (those are the only two obvious choices). To recall which one, just remember that the filtration is associated to cellular dimension, which is what also indexes cohomology H^*. The formula $E_2^{p,q} = H^p(X;\mathcal{E}^q)$ is enough to identify the p variable as the one that is relevant here, so that $E_\infty^{p,q}$ is the pth graded piece.

Remark 34.5 (**Warning about indexing**.)**.** For the rest of this book we will adopt the indexing conventions from (11) above, which are the standard ones for the Atiyah-Hirzebruch spectral sequence. This is **different** than the indexing we used in (0)–(10).

Remark 34.6 (Independence of cell structure)**.** As we have defined things, the spectral sequence depends on the chosen CW-structure on X. However, this dependence actually goes away from the E_2-term onward. Let X_1 and X_2 denote the same space but with two different CW-structures. The identity map $X_1 \to X_2$ is homotopic to a cellular map $f\colon X_1 \to X_2$, and f gives us a map of spectral sequences by naturality. Since f is a homotopy equivalence it induces an isomorphism on the E_2-terms, and therefore on all the finite pages of the spectral sequence as well.

The ZP-filtration on X was defined in terms of the CW-structure, but we can define it in a different way that doesn't make use of that. We leave it as an exercise to check that

$$\mathcal{E}^p(X)_{ZPq} = \{\alpha \in \mathcal{E}^p(X) \,\big|\, u^*(\alpha) = 0 \text{ for any map } u\colon A \to X \text{ where } A \text{ is a CW-complex of dimension less than } q\}.$$

These remarks show us that the spectral sequence from E_2-onwards may be regarded as a natural homotopy invariant of X. In particular, any map $g\colon X \to Y$ gives a map of spectral sequences in the opposite direction (by replacing g with a cellular map).

34.7. **The Postnikov tower approach.** Let $\mathcal{E}$ be a spectrum representing the cohomology theory $\mathcal{E}^*$ and let $P_n\mathcal{E}$ denote the nth Postnikov section for $\mathcal{E}$ (the spectrum obtained from $\mathcal{E}$ by attaching cells to kill off π_{n+1} and higher). There is a tower of fibrations

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & P_1\mathcal{E} \arrow[r, two heads] & P_0\mathcal{E} \arrow[r, two heads] & P_{-1}\mathcal{E} \arrow[r, two heads] & \cdots \\
& \Sigma H(\mathcal{E}_1) \arrow[u] & H(\mathcal{E}_0) \arrow[u] & \Sigma^{-1}H(\mathcal{E}_{-1}) \arrow[u] &
\end{tikzcd}$$

where $\mathcal{E}_i = \pi_i(\mathcal{E}) = \mathcal{E}^{-i}(pt)$ and HA denotes the Eilenberg-MacLane spectrum for the group A. If we apply function spectra $F(X, -)$ to all the spots in this diagram we get a new tower of fibrations

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & F(X, P_1\mathcal{E}) \arrow[r, two heads] & F(X, P_0\mathcal{E}) \arrow[r, two heads] & F(X, P_{-1}\mathcal{E}) \arrow[r, two heads] & \cdots \\
& F(X, \Sigma H(\mathcal{E}_1)) \arrow[u] & F(X, H(\mathcal{E}_0)) \arrow[u] & F(X, \Sigma^{-1}H(\mathcal{E}_{-1})) \arrow[u] &
\end{tikzcd}$$

Each fibration sequence $F(X, \Sigma^q H(\mathcal{E}_q)) \to F(X, P_q\mathcal{E}) \to F(X, P_{q-1}\mathcal{E})$ gives rise to a long exact sequence in homotopy groups, and these long exact sequences intertwine to form an exact couple. The associated spectral sequence has

$$E_1^{p,q} = \pi_{-p-q}F(X, \Sigma^q H(\mathcal{E}_q)) = H^{p+2q}(X; \mathcal{E}^{-q}),$$
$$d_r\colon E_r^{p,q} \longrightarrow E_r^{p-r+1,q+r},$$

and it is trying to converge to

$$\pi_{-p-q}F(X, \mathcal{E}) = \mathcal{E}^{p+q}(X).$$

It is not obvious, but with some trouble it can be seen that after re-indexing this is "the same" as the previously-constructed spectral sequence but with the E_1-term of this one corresponding to the E_2-term of the one constructed via CW-structures.

We won't really need this Postnikov version of the spectral sequence for anything, but it often provides a useful perspective. For example, note that this version of the spectral sequence is manifestly functorial in X and a homotopy invariant.

34.8. **Differentials.** The d_2-differential in the Atiyah-Hirzebruch spectral sequence is a map $H^p(X;\mathcal{E}^q) \to H^{p+2}(X;\mathcal{E}^{q-1})$. This is natural in X and it is also stable under the suspension isomorphism; so it is a stable cohomology operation. The d_3-differential is in some sense a secondary cohomology operation, and so on for all the differentials. This is often a useful perspective. For example, we can now prove the following general fact:

Proposition 34.9. *Suppose that the coefficient groups $\mathcal{E}^*(pt)$ are rational vector spaces. Then for any space X the differentials in the Atiyah-Hirzebruch spectral sequence all vanish. If X is homotopy equivalent to a finite-dimensional CW-complex then there are (non-canonical) isomorphisms*

$$\mathcal{E}^n(X) \cong \oplus_{p+q=n} H^p(X;\mathcal{E}^q(pt))$$

for every $n \in \mathbb{Z}$.

Proof. The point is that the only stable cohomology operation of nonzero degree on $H^*(-;\mathbb{Q})$ is the zero operation. This immediately yields that all d_2-differentials are zero. But then d_3 is a stable cohomology operation (not a secondary operation anymore) and therefore it also vanishes. Continue by induction.

Since $\mathcal{E}^n(X)$ is a rational vector space and has a finite filtration where the quotients are the $\mathbb{Q}$-vector spaces $H^p(X;\mathcal{E}^q)$, there are no extension problems and one obtains the isomorphism from the statement. □

Remark 34.10. In the Postnikov approach to the Atiyah-Hirzebruch spectral sequence one sees the d_2-differentials very explicitly. The Postnikov tower has "k-invariants" of the form

$$\Sigma^q H(\mathcal{E}_q) \to \Sigma^{q+2} H(\mathcal{E}_{q+1})$$

which desuspend to give $H(\mathcal{E}_q) \to \Sigma^2 H(\mathcal{E}_{q+1})$. These are quite visibly stable cohomology operations $H^*(-;\mathcal{E}^{-q}) \to H^{*+2}(-;\mathcal{E}^{-q-1})$. The connection between higher differentials and higher cohomology operations has a similar realization, but the details are too cumbersome to be worth discussing here.

It is worth observing that from the E_2-term onward there are never any differentials emanating from the $p = 0$ line of the spectral sequence. For convenience assume X is connected and choose a cell structure on X where $F_0 = \{*\}$. By the remarks in (8) above, such differentials would be the obstructions for a class in $\mathcal{E}^q(F_0, \emptyset)$ to lift to $\mathcal{E}^q(F_r, \emptyset)$; but such a lifting necessarily exists, because $F_0 \hookrightarrow F_r$ is split. Note that when X is connected $\mathcal{E}^q(X)_{ZP1} = \widetilde{\mathcal{E}}^q(X)$, and $\mathcal{E}^q(X)_{ZP(0/1)} = \mathcal{E}^q(X)/\widetilde{\mathcal{E}}^q(X) = \mathcal{E}^q(pt)$, giving further confirmation that the E_∞-term coincides with the E_2-term on the $p = 0$ line.

In the original exact couple (34.2) we could replace all the $\mathcal{E}^*$ groups with $\widetilde{\mathcal{E}}^*$ groups and still have an exact couple, with the resulting spectral sequence having the form

$$E_2^{p,q} = \widetilde{H}^p(X;\mathcal{E}^q) \Rightarrow \widetilde{\mathcal{E}}^{p+q}(X).$$

This just amounts to removing the entire $p = 0$ line from the Atiyah-Hirzebruch spectral sequence. Sometimes it is convenient to consider this reduced version of the spectral sequence.

34.11. **Multiplicativity.** Suppose that $\mathcal{E}$ is a multiplicative cohomology theory. Then for spaces X and Y we have the external product

$$\mathcal{E}^r(X) \otimes \mathcal{E}^s(Y) \to \mathcal{E}^{r+s}(X \times Y),$$

and this is readily checked to induce associated pairings on the ZP-filtration:

$$\mathcal{E}^r(X)_{ZPa} \otimes \mathcal{E}^s(Y)_{ZPb} \to \mathcal{E}^{r+s}(X \times Y)_{ZP(a+b)}$$

and

$$\mathcal{E}^r(X)_{ZP(a/a+1)} \otimes \mathcal{E}^s(Y)_{ZP(b/b+1)} \to \mathcal{E}^{r+s}(X \times Y)_{ZP(a+b/a+b+1)}. \tag{34.12}$$

The pairings $\mathcal{E}^q(pt) \otimes \mathcal{E}^{q'}(pt) \to \mathcal{E}^{q+q'}(pt)$ also can be fed into the cup product machinery to give

$$H^p(X; \mathcal{E}^q(pt)) \otimes H^{p'}(Y; \mathcal{E}^{q'}(pt)) \longrightarrow H^{p+p'}(X \times Y; \mathcal{E}^{q+q'}(pt)). \tag{34.13}$$

Since the Atiyah-Hirzebruch spectral sequence starts with the groups $E_2(-) = H^*(-; \mathcal{E}^*)$ and then converges to the groups $\mathcal{E}^*(-)$, it is natural to ask if the pairings of (34.13) and (34.12) are connected via this convergence process. The machinery for making this connection is somewhat cumbersome to write out, although in practice not so cumbersome to use.

To say that there is a **pairing of spectral sequences** $E_*(X) \otimes E_*(Y) \to E_*(X \times Y)$ is to say that

(i) For each r there is a product $E_r^{p,q}(X) \otimes E_r^{p',q'}(Y) \to E_r^{p+p',q+q'}(X \times Y)$;
(ii) The differential d_r satisfies the Leibniz rule
$$d_r(a \cdot b) = d_r(a) \cdot b + (-1)^p a \cdot d_r b$$
for all $a \in E_r^{p,q}(X)$ and $b \in E_r^{p',q'}(Y)$, and therefore induces a product on $H_*(E_r) = E_{r+1}$;
(iii) The product on the E_{r+1}-term equals the one induced by the product on the E_r-term, for all r;
(iv) There is a product on the E_∞-term which agrees with the products on the E_r-terms where defined (???);
(v) The maps $\mathcal{E}^p(-)_{ZP(q/q+1)} \to E_\infty^{q,p-q}(-)$ are compatible with the products, in the evident sense.

To give a decent treatment of pairings between Atiyah-Hirzebruch spectral sequences it is best to work at the level of spectra, and to work with a category of spectra where there is a well-behaved smash product. This introduces several layers of foundational technicalities that we do not wish to dwell on, so let us just say that these things can all be worked out. In this setting the right notion of "multiplicative cohomology theory" consists of a spectrum $\mathcal{E}$ together with a map $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$ that is associative and unital. Both the complex and real K-theory spectra can be given this structure. One has the following general result:

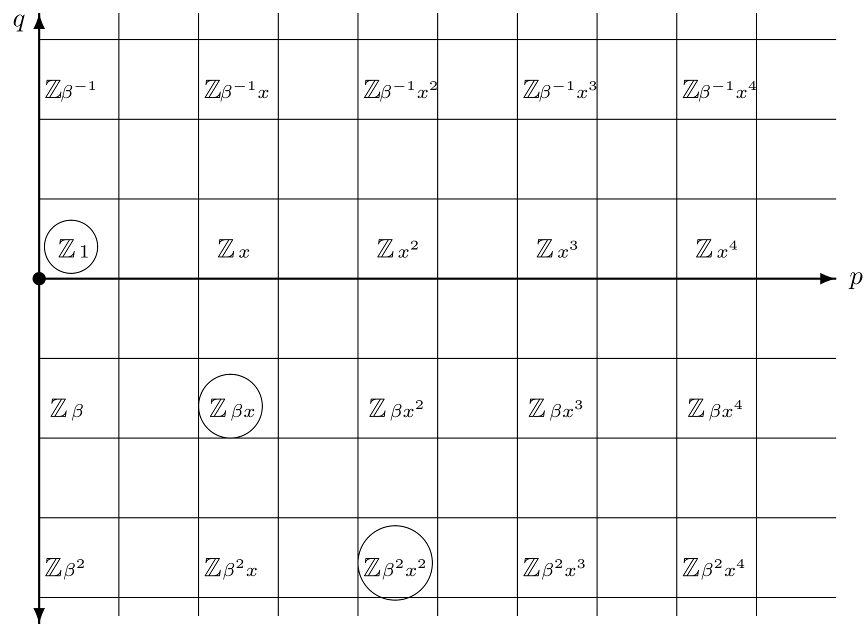
Theorem 34.14. *Let $\mathcal{E}$ be a spectrum with a product $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$. Then there is a pairing of Atiyah-Hirzebruch spectral sequences where the product on E_2-terms $H^p(X;\mathcal{E}^q) \otimes H^{p'}(Y;\mathcal{E}^{q'}) \to H^{p+p'}(X \times Y;\mathcal{E}^{q+q'})$ is equal to $(-1)^{p'q}$ times the cup product.*

We will not prove the above theorem here, as this would take us too far afield. For a proof, see [D2, Section 3]. What is more important is how to use the theorem; we will give some examples in the following section.

Remark 34.15. The signs in the above theorem cannot, in general, be neglected. See [D2, Section 2] for a complete discussion. However, notice that in the case of complex K-theory it is irrelevant because the groups $H^p(X;K^q)$ are only nonzero when q is even. This is a pleasant convenience. A similar convenience occurs for KO-theory: whereas the coefficients groups do have some nonzero terms in odd degrees, since these terms are all $\mathbb{Z}/2$'s one can once again neglect the signs.

34.16. **Some examples.** We now focus entirely on complex K-theory, examining two sample computations. Further examples, for both K and KO, are in Section 37.

Let us start by redoing the calculation $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$ where $X = L - 1$, now using the Atiyah-Hirzebruch spectral sequence. The following is the E_2-term:



Note that the E_2-term vanishes to the right of the line $p = 2n$, since $H^*(\mathbb{C}P^n)$ vanishes in this range. The circled groups (and others along the same diagonal) are the ones that contribute to $K^0(\mathbb{C}P^n)$. Note that there is no room for any differentials, because the nonzero groups only occur when both p and q are even. So the spectral sequence immediately collapses, and $E_2 = E_\infty$. It follows that the filtration quotients for the ZP-filtration on $K^0(\mathbb{C}P^n)$ are as follows:

$$K^0(\mathbb{C}P^n) \xleftarrow{\ \mathbb{Z}\ } K^0(\mathbb{C}P^n)_{ZP1} \xleftarrow{\ 0\ } K^0(\mathbb{C}P^n)_{ZP2} \xleftarrow{\ \mathbb{Z}\ } K^0(\mathbb{C}P^n)_{ZP3} \longleftarrow \cdots$$

with the $\mathbb{Z}$'s appearing exactly $n+1$ times. Since the quotients are free there are no extension problems and we conclude that $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$ as abelian groups.

The spectral sequence also gives information about the ring structure on $K^0(\mathbb{C}P^n)$. Note that $K^0(\mathbb{C}P^n)_{ZP1} = \widetilde{K}^0(\mathbb{C}P^n)$, and that $K^0(\mathbb{C}P^n)_{ZP2} = K^0(\mathbb{C}P^n)_{ZP1}$ by the previous paragraph. Consider the canonical map

$$K^0(\mathbb{C}P^n)_{ZP(2/3)} \xrightarrow{\cong} E_\infty^{2,-2} = \mathbb{Z}\langle \beta x\rangle.$$

Let α denote a preimage for βx under this isomorphism. The multiplicativity of the spectral sequence tells us that α^k maps to $\beta^k x^k$ under the corresponding map $K^0(\mathbb{C}P^n)_{ZP(2k/2k+1)} \to E_\infty^{2k,-2k}$. In particular, α^k is nonzero for $k < n+1$. We know $\alpha^{n+1} = 0$, either by Lemma 23.2 or by the fact that $\alpha^{n+1} \in K^0(\mathbb{C}P^n)_{ZP(2n+2)}$ and this filtration group is zero by the spectral sequence.

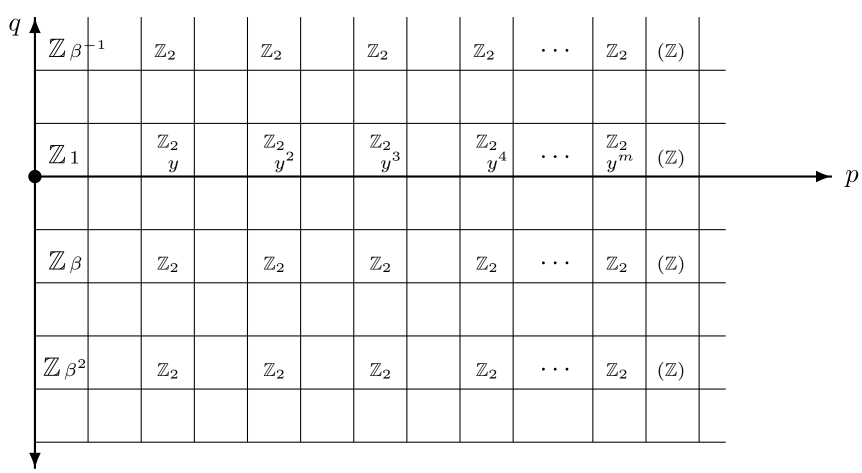
Now consider the map $\mathbb{Z}[\alpha]/(\alpha^{n+1}) \to K^0(\mathbb{C}P^n)$. This may be regarded as a map of filtered rings, where the domain is filtered by the powers of (α) and the target has the ZP-filtration. The spectral sequence tells us this is an isomorphism on the filtration quotients; but since there are only finitely many of these, it follows that the map is an isomorphism.

To complete our calculation is only remains to show that we may take $\alpha = \pm(1-[L])$. First verify this when $n=1$, where it just comes down to the fact that $1-[L]$ is a generator for $\widetilde{K}^0(S^2)$. For general n we can now use the naturality of the spectral sequence, applied to the inclusion $j\colon \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$. The spectral sequences tell us that the natural map

$$K^0(\mathbb{C}P^n)_{ZP(2/3)} \to K^0(\mathbb{C}P^1)_{ZP(2/3)}$$

is an isomorphism. The element $1-[L]$ represents an element in the domain, which must be a generator precisely because it maps to a generator in the target. This tells us that one of $\pm(1-[L])$ maps to βx and is therefore a candidate for α, and this is enough to conclude $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$ where $X = 1-[L]$.

For our next example let us consider $K^0(\mathbb{R}P^n)$. The E_2-term is very similar to the one before, with the difference that most $\mathbb{Z}$'s are changed to $\mathbb{Z}/2$'s:



The $\mathbb{Z}$'s in parentheses lie in degree $p = n$ when n is odd, and are not present when n is even. And $m = \lfloor \frac{n}{2} \rfloor$. As discussed in (34.8) above, there can be no differential emanating from the $p = 0$ column. So the only possible differentials allowed by the grading would occur when n is odd and would have a $\mathbb{Z}/2$ mapping into one of the $\mathbb{Z}$'s; but such a map must be zero. So all differentials vanish, and we again have $E_2 = E_\infty$. We conclude that the associated graded of the ZP-filtration on $\widetilde{K}^0(\mathbb{R}P^n)$ consists of $\lfloor \frac{n}{2} \rfloor$ copies of $\mathbb{Z}/2$, and so $\widetilde{K}^0(\mathbb{R}P^n)$ is an abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$. It remains to determine the group precisely. For this, use the map of spectral sequences induced by $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. The maps $\mathbb{Z} \to \mathbb{Z}/2$ are all surjections at E_2 and therefore also at E_∞. The surjection on E_∞-terms shows that $\alpha = 1 - [j^*L]$ generates the filtration quotient $K^0(\mathbb{R}P^n)_{ZP(2/3)}$, since $1 - [L]$ generates the corresponding quotient in the $\mathbb{C}P^n$ case. Note that $\alpha^2, \ldots, \alpha^{\lfloor \frac{n}{2} \rfloor}$ therefore generate the other filtration quotients, and so in particular are nonzero. But we can compute

$$\alpha^2 = (1 - [j^*L])^2 = 1 - 2[j^*L] + [j^*L]^2 = 1 - 2[j^*L] + 1 = 2(1 - [j^*L]) = 2\alpha,$$

where in the third equality we have used that the square of any real line bundle is trivial (Corollary 8.34). It follows that $\alpha^i = 2^{i-1}\alpha$. Since $\alpha^{\lfloor \frac{n}{2} \rfloor} \neq 0$ this gives $2^{(\lfloor \frac{n}{2} \rfloor - 1)}\alpha \neq 0$. The only abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$ that admits such an element is $\mathbb{Z}/(2^{\lfloor \frac{n}{2} \rfloor})$, and so $\widetilde{K}^0(\mathbb{R}P^n)$ is isomorphic to this cyclic group.

The spectral sequence also quickly shows that $K^1(\mathbb{R}P^n)$ is isomorphic to $\mathbb{Z}$ when n is odd, and 0 otherwise.

Remark 34.17. Note that we previously determined that $\widetilde{K}^0(\mathbb{R}P^2)$ was an abelian group of order 4, back in Section 14.10. Comparing the "brute force" approach used there to the spectral sequence machinery really shows the power of the latter: the argument is really the same, but the spectral sequence allows us to get at the conclusion much more quickly.

34.18. **More on differentials.** Since $K^{odd}(pt) = 0$ it follows for degree reasons that all differentials d_{2r} vanish in the Atiyah-Hirzebruch spectral sequence. So our first significant differential is d_3, which is a stable cohomology operation $H^*(-;\mathbb{Z}) \to H^{*+3}(-;\mathbb{Z})$. It is an easy matter to compute *all* such stable operations, as they are parameterized by the group $H^3(H\mathbb{Z})$ of stable homotopy classes $H\mathbb{Z} \to \Sigma^3 H\mathbb{Z}$; this can be computed as the cohomology group $H^{n+3}(K(\mathbb{Z},n))$ for $n \geq 3$. A routine calculation (say, with the Serre spectral sequence) shows that this group is $\mathbb{Z}/2$. The nonzero element is an operation α that is an integral lift of Sq^3, in the sense that if $u \in H^*(X;\mathbb{Z})$ then

$$\overline{\alpha(u)} = \mathrm{Sq}^3(\overline{u})$$

where $\overline{x}$ denotes the mod 2 reduction of a class x.

The above paragraph shows that our differential d_3 either equals zero (for all spaces X) or coincides with the operation α. The latter option is the correct one, and to see this it suffices to produce a single space X where d_3 is nonzero. For this we take the space X from Example 27.10, constructed as the cofiber of a map $\Sigma^3\mathbb{R}P^2 \to S^3$ that gives a null-homotopy for 2η. This space has $H^3(X) = \mathbb{Z}$ and $H^6(X) = \mathbb{Z}/2$, so there is a potential d_3 in the spectral sequence. If this d_3 were zero then we would have $\widetilde{K}^0(X) = \mathbb{Z}/2$, but we calculated in Example 27.10 (using the Chern character) that $\widetilde{K}^0(X) = 0$. So d_3 is nonzero here. We have therefore proven:

Proposition 34.19. *The differential d_3 in the Atiyah-Hirzebruch spectral sequence is the unique nonzero cohomology operation $H^*(-;\mathbb{Z}) \to H^{*+3}(-;\mathbb{Z})$. It satisfies $2d_3(x) = 0$, for all x.*

Remark 34.20. We have now explained the motivation for the space X from Example 27.10. It is literally the smallest space for which there is a nonzero α-operation in its cohomology.

The fact that $2d_3(x) = 0$ shows that d_3 must vanish on any class $x \in H^*(X)$ whose order is prime to 2. Alternatively, if we tensor the Atiyah-Hirzebruch spectral sequence with $\mathbb{Z}[\frac{1}{2}]$ then all d_3 differentials vanish. In that case d_5 is a cohomology operation $H^*(-;\mathbb{Z}[\frac{1}{2}]) \to H^{*+5}(-;\mathbb{Z}[\frac{1}{2}])$, and such things are classfied by $H^5(H\mathbb{Z};\mathbb{Z}[\frac{1}{2}])$. This group is readily calculated to be $\mathbb{Z}/3$. If one then also inverts 3 this will kill d_5, but it turns out to also kill d_7 because $H^7(H\mathbb{Z};\mathbb{Z}[\frac{1}{6}]) = 0$. The d_9 differential becomes a cohomology operation in $H^9(H\mathbb{Z};\mathbb{Z}[\frac{1}{6}]) \cong \mathbb{Z}/5$, and so one can kill it by inverting 5. This process continues, and shows that inverting all primes smaller than p kills all differentials below d_{2p-1}. Note that this gives another proof of Proposition 34.9 (but with more precise information), saying that after tensoring with $\mathbb{Q}$ all Atiyah-Hirzebruch differentials vanish.

The following result summarizes and expands the discussion in the last paragraph. The two parts are closely related and almost equivalent, but it is useful to have them both stated explicitly.

Proposition 34.21. *Fix a prime p.*

(a) Inverting $(p-1)!$ in the Atiyah-Hirzebruch spectral sequence results in $d_r = 0$ for $r < 2p-1$, together with $d_{2p-1}(u) = (-1)^{p+1}\beta P^1(\bar{u})$ for all classes u, where $\bar{u}$ is reduction modulo p, P^1 is Steenrod's first reduced power operation (for the prime p), and β is the Bockstein for the sequence $0 \to \mathbb{Z}[\frac{1}{(p-1)!}] \xrightarrow{p} \mathbb{Z}[\frac{1}{(p-1)!}] \to \mathbb{Z}/p \to 0$.

(b) Let $u \in H^*(X)$ be p^e-torsion, where p is a prime. Then in the Atiyah-Hirzebruch spectral sequence $d_i(u) = 0$ for $i < 2p-1$, and $d_{2p-1}(u) = (-1)^{p+1}\beta P^1(\bar{u})$ where β is the Bockstein for $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$ and P^1 is as in (a).

Proof. Fixing a prime p, the following is known about $H^*(K(\mathbb{Z},n))$ assuming $n > 2p-1$:

(i) $H^i(K(\mathbb{Z},n)) = 0$ for $0 < i < n$ and $H^n(K(\mathbb{Z},n)) \cong \mathbb{Z}$;
(ii) $H^i(K(\mathbb{Z},n))$ is torsion, with all orders prime to p, for $n < i < n+2p-1$;
(iii) $H^{n+2p-1}(K(\mathbb{Z},n))$ is isomorphic to a direct sum $\mathbb{Z}/p \oplus A$ where A is a torsion group whose order only has prime factors smaller than p. The $\mathbb{Z}/p$ summand is generated by $\beta P^1(\overline{u})$, where $u \in H^n(K(\mathbb{Z},n))$ is the fundamental class.

Alternatively, the above results say that

$$H^*\Big(K(\mathbb{Z},n);\mathbb{Z}[\tfrac{1}{(p-1)!}]\Big) \cong \begin{cases} 0 & \text{if } i < n, \\ \mathbb{Z}[\frac{1}{(p-1)!}] & \text{if } i = n, \\ 0 & \text{if } n < i < n+2p-1, \\ \mathbb{Z}/p & \text{if } i = n+2p-1. \end{cases}$$

These facts are easy calculations with the Serre spectral sequence, using the methods of [MT, Chapter 9].

Consider the Atiyah-Hirzebruch spectral sequence for $K(\mathbb{Z}, n)$, and in particular the differentials on the fundamental class u. By naturality of the spectral sequence this serves as a universal example for what happens on all spaces. Since inverting $(p-1)!$ kills all of the cohomology of $K(\mathbb{Z}, n)$ in dimensions strictly between n and $n + 2p - 1$, this shows that it also kills the differentials $d_r(u)$ for $r < 2p - 1$. By universality, inverting $(p-1)!$ kills these differentials for any space X.

If $u \in H^*(X)$ is a p^e-torsion class then $d_r(u)$ for $r < 2p-1$ is killed by a power of $(p-1)!$ by the preceding paragraph, but it is also killed by p^e; since these integers are relatively prime it follows that $d_r(u) = 0$ for $r < 2p - 1$.

It remains to identify d_{2p-1} in both (a) and (b). We know by the calculation of $H^*(K(\mathbb{Z}, n))$ for $n \gg 0$ that after localization at $(p-1)!$ one must have $d_{2p-1}(u) = \lambda \cdot \beta P^1(\bar{u})$, for some $\lambda \in \mathbb{Z}/p$. We need to determine λ, and for this we can examine a single well-chosen example space. The sample space we choose is a generalization of the one from Example 27.10. Consider the projection $\pi\colon \mathbb{C}P^p \to \mathbb{C}P^p/\mathbb{C}P^{p-1} \cong S^{2p}$. We claim that there is a stable map $f\colon S^{2p} \to \mathbb{C}P^p$ such that the composite $S^{2p} \to \mathbb{C}P^p \xrightarrow{\pi} S^{2p}$ has degree Mp for some integer M prime to p; for the proof, see Lemma 34.23 below. The "stable map" phrase is to indicate that f might only exist after suspensing some number of times, so really it is a map $f\colon \Sigma^r S^{2p} \to \Sigma^r \mathbb{C}P^p$. We can assume that r is even. Let X be the cofiber of f, and note that

$$\widetilde{H}^i(X) = \begin{cases} \mathbb{Z} & \text{if } r+2 \leq i < r+2p \text{ and } i \text{ is even,} \\ \mathbb{Z}/(Mp) & \text{if } i = r+2p+1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the cofiber sequence

$$\Sigma^r \mathbb{C}P^p \xrightarrow{j} X \xrightarrow{p} S^{r+2p+1}$$

Let $x \in H^2(\mathbb{C}P^p)$ be a chosen generator, and let $u \in H^{r+2}(X)$ be a class that j^* maps onto the suspension $\sigma^r(x)$. Let $v \in H^{r+2p+1}(X)$ be the image under p^* of the canonical generator from $H^{r+2p+1}(S^{r+2p+1})$. Note that j^* is an isomorphism on $H^*(-;\mathbb{Z}/p)$ for $* \leq 2p + r$, and so we can write $\bar{x}^k$ for the generator of $H^{2k+r}(X;\mathbb{Z}/p)$ that maps to the element of the same name in $H^{2k+r}(\mathbb{C}P^p;\mathbb{Z}/p)$. Observe that $\beta(\bar{x}^p) = Mv$ in $H^*(X;\mathbb{Z}/p)$.

In $H^*(\mathbb{C}P^p;\mathbb{Z}/p)$ we have $P^1(\bar{x}) = \bar{x}^p$, as this is how P^1 behaves on classes of degree 2. So in $H^*(\Sigma^r\mathbb{C}P^p;\mathbb{Z}/p)$ we have $P^1(\sigma^r\bar{x}) = \sigma^r(\bar{x}^p)$, by stability of the P^1 operation. It follows that $\beta P^1(\bar{u}) = Mv$.

In the Atiyah-Hirzebruch spectral sequence for X, after inverting $(p-1)!$ there is only one possible nonvanishing differential, namely $d_{2p-1}\colon \mathbb{Z}[\frac{1}{(p-1)!}] \to \mathbb{Z}/p$. We know that

$$d_{2p-1}(u) = \lambda \cdot \beta P^1(\bar{u}) = \lambda M v. \tag{34.22}$$

But we can also compute $d_{2p-1}(u)$ directly. The class u is represented by an element in $K^0(X_{r+2}, X_{r+1})$ (a cellular $(r+2)$-cochain). The differential is represented by choosing a lift of $u \in K^0(X_{r+2}, X_{r+1})$ to $\xi \in K^0(X_{2p+r}, X_{r+1})$ and then applying the connecting homomorphism from the long exact sequence for the triple

$(X_{2p+r+1}, X_{2p+r}, X_{r+1})$:

$$\begin{tikzcd} \xi \in K^0(X_{2p+r+1}, X_{r+1}) \arrow[r, "\delta"] \arrow[d] & K^1(X_{2p+r+1}, X_{2p+r}) \\ u \in K^0(X_{r+2}, X_{r+1}). \end{tikzcd}$$

Recall that $X_{r+1} = *$ and that u is represented by the class $\sigma^r(1-[L]) \in \widetilde{K}^0(\Sigma^r \mathbb{C}P^1)$. The element $1-[L] \in \widetilde{K}^0(\mathbb{C}P^1)$ lifts to the class with the same name in $\widetilde{K}^0(\mathbb{C}P^p)$, and so we may take $\xi = \sigma^r(1-[L]) \in K^0(\Sigma^r\mathbb{C}P^p, *)$. To compute $\delta(\xi)$ we can use the Chern character:

$$\begin{tikzcd} K^0(X_{2p+r}, *) \arrow[r, "\delta"] \arrow[d, "\mathrm{ch}"'] & K^1(X_{2p+r+1}, X_{2p+1}) \arrow[d, "\mathrm{ch}"] \\ H^{ev}(X_{2p+r}, *; \mathbb{Q}) \arrow[r, "\delta"] & H^{odd}(X_{2p+r+1}, X_{2p+1}; \mathbb{Q}). \end{tikzcd}$$

We know from Proposition 27.7 that the right vertical map is an injection and that its image is the integral subgroup $H^{odd}(X_{2p+r+1}, X_{2p+1}; \mathbb{Z})$. So we compute

$$\mathrm{ch}(\xi) = \mathrm{ch}\big(\sigma^r(1-[L])\big) = \sigma^r \,\mathrm{ch}(1-[L]) = \sigma^r(x - \tfrac{x^2}{2} + \tfrac{x^3}{6} - \cdots)$$

Applying δ to this expresstion kills everything except the class in degree $2p+r$, and we therefore get

$$\delta(\mathrm{ch}\,\xi) = (-1)^p \cdot \tfrac{1}{p!} \cdot Mp = (-1)^p \tfrac{M}{(p-1)!}$$

where the first two terms in the product come from $\mathrm{ch}(\xi)$ and the Mp comes from application of δ. Note that commutativity of the above square implies that $\frac{M}{(p-1)!}$ must be an integer, and that

$$\delta(\xi) = (-1)^p \tfrac{M}{(p-1)!} \cdot v$$

where v denotes the preferred generator of $K^1(X_{2p+r+1}, X_{2p+r}) \cong \mathbb{Z}$.

Putting everything together, we have just proven that in the Atiyah-Hirzebruch spectral sequence for X with $(p-1)!$ inverted one has

$$d_{2p-1}(u) = \Big[(-1)^p \tfrac{M}{(p-1)!}\Big]_p \cdot v$$

where $[-]_p$ denotes the residue modulo p. Wilson's Theorem says that $(p-1)! \equiv -1$ mod p, and so

$$d_{2p-1}(u) = \big[(-1)^{p+1} M\big]_p \cdot v.$$

Comparing to (34.22) gives $\lambda = (-1)^{p+1}$, and we are finished with (a).

Part (b) can be deduced from (a) using the commutative square

$$\begin{tikzcd} H^*(X) \arrow[r, "d_{2p-1}"] \arrow[d] & H^{*+2p-1}(X) \arrow[d] \\ H^*(X; \mathbb{Z}_f) \arrow[r, "d_{2p-1}"] & H^{*+2p-1}(X; \mathbb{Z}_f) \end{tikzcd}$$

where we are writing $\mathbb{Z}_f = \mathbb{Z}[\frac{1}{(p-1)!}]$. Let $B \subseteq H^{*+2p-1}(X)$ be the subgroup of elements killed by a power of p, and note that B injects into $H^{*+2p-1}(X; \mathbb{Z}_f)$. If

$p^e u = 0$ then both $d_{2p-1}u$ and $(-1)^{p+1}\beta P^1(\bar{u})$ belong to B. The commutative square, together with part (a), show that the two classes map to the same element of $H^{*+2p-1}(X;\mathbb{Z}_f)$; hence, they are the same. $\square$

Lemma 34.23. *Fix a prime p. Then for some $r > 0$ there exists a map $S^{2p+r} \to \Sigma^r\mathbb{C}P^p$ such that the composite*

$$S^{2p+r} \to \Sigma^r\mathbb{C}P^p \xrightarrow{\Sigma^r\pi} \Sigma^r S^{2p}$$

has degree equal to Mp for some M relatively prime to p. Here $\pi\colon \mathbb{C}P^p \to S^{2p}$ is the map that collapses $\mathbb{C}P^{p-1}$ to a point.

Proof. This is a computation with stable homotopy groups. Consider the homology theory $X \mapsto E_*(X) = \pi_*^s(X) \otimes \mathbb{Z}_{(p)}$. When $X = S^0$ the groups $E_*(X)$ are the p-components of the stable homotopy groups of spheres, and it is known that $E_i(X) = 0$ for $0 < i < 2p-3$ and $E_{2p-3}(X) \cong \mathbb{Z}/p$. An easy induction using the cofiber sequences $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n \to S^{2n}$ shows that $E_{2p-1}(\mathbb{C}P^n) \cong \mathbb{Z}/p$ for all $1 \leq n \leq p-1$. The long exact sequence for $\mathbb{C}P^{p-1} \hookrightarrow \mathbb{C}P^p \to S^{2p}$ then gives

$$\cdots \to \pi_{2p}^s(\mathbb{C}P^{p-1}) \to \pi_{2p}^s(\mathbb{C}P^p) \to \mathbb{Z} \to \pi_{2p-1}^s(\mathbb{C}P^{p-1}) \to \cdots$$

The element $p \in \mathbb{Z}$ necessarily maps to zero in $\pi_{2p-1}^s(\mathbb{C}P^{p-1}) \otimes \mathbb{Z}_{(p)}$, since the latter group is $\mathbb{Z}/p$. This means that there exists $M \in \mathbb{Z}$ prime to p such that Mp maps to zero in $\pi_{2p-1}^s(\mathbb{C}P^{p-1})$. But then Mp is the image of an element in $\pi_{2p}^s(\mathbb{C}P^p)$, and this element is what we were looking for. $\square$

34.24. **Differentials and the Chern character.** In the proof of Proposition 34.21 there was a key step where we used the Chern character to help compute a differential in the Atiyah-Hirzebruch spectral sequence. We will next explain a generalization of this technique.

The E_1-term of the Atiyah-Hirzebruch spectral sequence breaks up into chain complexes that look like

$$\cdots \to K^{-1}(F_{q-1}, F_{q-2}) \to K^0(F_q, F_{q-1}) \to K^1(F_{q+1}, F_q) \to \cdots$$

If q is odd this is the zero complex, and if q is even we have seen that it is isomorphic to the cellular chain complex for X with $\mathbb{Z}$ coefficients. The latter is via isomorphisms $K^0(F_q, F_{q-1}) \cong \widetilde{K}^0(\vee_\alpha S^q) \cong \bigoplus_\alpha \widetilde{K}^0(S^q) \cong \bigoplus_\alpha \mathbb{Z}$. Notice that we can also use the Chern character to obtain such an isomorphism, as we know $\mathrm{ch}\colon K^0(F_q, F_{q-1}) \to H^*(F_q, F_{q-1};\mathbb{Q})$ to be injective with image equal to $H^*(F_q, F_{q-1};\mathbb{Z})$ (Proposition 27.7). Since the Chern character is natural it actually gives us an isomorphism of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & K^{-1}(F_{q-1}, F_{q-2}) \arrow[r] \arrow[d, "\cong"'] & K^0(F_q, F_{q-1}) \arrow[r] \arrow[d, "\cong"'] & K^1(F_{q+1}, F_q) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & H^*(F_{q-1}, F_{q-2}) \arrow[r] & H^*(F_q, F_{q-1}) \arrow[r] & H^*(F_{q+1}, F_q) \arrow[r] & \cdots
\end{tikzcd}$$

Taken on its own, this is just a matter of convenience: we already had a natural isomorphism between these complexes, so identifying it as the Chern character just gives it a nice name. But using the fact that the Chern character is defined more globally (i.e., on all pairs (Y, B)) allows us to push this a bit further and obtain a description of the Atiyah-Hirzebruch differentials. The following result is a combination of [AH2, Lemmas 1.2 and 7.3]

Proposition 34.25. *Let X be a CW-complex and let $u \in H^p(X)$. Then in the Atiyah-Hirzebruch spectral sequence one has $d_i u = 0$ for all $2 \le i < r$ if and only if there exist $\tilde{u} \in H^p(F_{p+r-1}, F_{p-1})$ and $\xi \in K^*(F_{p+r-1}, F_{p-1})$ such that*

(i) $\tilde{u}$ is a lift for u under $H^p(F_{p+r-1}, F_{p-1}) \to H^p(F_{p+r-1}) \xleftarrow{\cong} H^p(X)$, and
(ii) $\mathrm{ch}(\xi) = \tilde{u} +$ higher order terms.

Moreover, if in the above situation α is a cellular cochain representating $\mathrm{ch}(\xi)_{p+r-1}$ then $\delta\alpha$ is integral and represents the differential $d_r u$.

Proof. To symplify some typography we assume throughout the proof that p is even, although the odd case is identical (or else one could just replace X with its suspension).

Assume that $u \in H^p(X)$ satisfies $d_i u = 0$ for $2 \le i < r$. Identifying u with an element in E_2, this condition says that u can be represented by a class $z \in E_1 = K^0(F_p, F_{p-1})$ with the property that $z \in Z_{r-1}$. The element associated to z by the isomorphism $K^0(F_p, F_{p-1}) \cong C^p_{cell}(X; \mathbb{Z})$ is a cellular p-cochain representative for u.

As remarked in (8) of Section 34.1, the condition $z \in Z_{r-1}$ is equivalent to saying that z lifts to a class $\tilde{z} \in K^0(F_{p+r-1}, F_{p-1})$. Now apply the Chern character to get the square

$$\begin{tikzcd} K^0(F_{p+r-1}, F_{p-1}) \arrow[r] \arrow[d, "\mathrm{ch}"'] & K^0(F_p, F_{p-1}) \arrow[d, "\mathrm{ch}"] \\ H^*(F_{p+r-1}, F_{p-1}; \mathbb{Q}) \arrow[r, "j^*"] & H^*(F_p, F_{p-1}; \mathbb{Q}), \end{tikzcd} \qquad \begin{tikzcd} \tilde{z} \arrow[r] \arrow[d] & z \arrow[d] \\ \mathrm{ch}(\tilde{z}) \arrow[r] & \mathrm{ch}(z). \end{tikzcd}$$

The element $\mathrm{ch}(z)$ is an (integral) cellular p-cochain that represents the class u. The groups $H^*(F_{p+r-1}, F_{p-1})$ are zero in degrees $* < p$, so $\mathrm{ch}(\tilde{z})$ is of the form $\mathrm{ch}(\tilde{z})_p +$ higher order terms. The fact that $\mathrm{ch}(z)$ is a cellular cochain representing u says that any lift of $\mathrm{ch}(z)$ into $H^p(F_{p+1}, F_{p-1})$ has the same image in $H^p(F_{p+1})$ as u:

$$H^p(X) \longrightarrow H^p(F_{p+1}) \longleftarrow H^p(F_{p+1}, F_{p-1})$$

It follows readily that $\mathrm{ch}(\tilde{z})_p$ has the same image in $H^p(F_{p+r-1})$ as u. This completes the $(\Rightarrow)$ direction of the first statement in the proposition. The $(\Leftarrow)$ direction follows in the same way, as all the steps are reversible.

For the final statement of the proposition we continue to assume (for convenience) that p is even. Consider the diagram

$$\begin{tikzcd} \xi \in K^0(F_{p+r-1}, F_{p-1}) \arrow[r, "\delta"] \arrow[d, "\mathrm{ch}"'] & K^1(F_{p+r}, F_{p+r-1}) \arrow[d, "\mathrm{ch}"] \\ H^*(F_{p+r-1}, F_{p-1}; \mathbb{Q}) \arrow[r, "\delta"] & H^{*+1}(F_{p+r}, F_{p+r-1}; \mathbb{Q}) \end{tikzcd}$$

where in both rows the map δ is the connecting homomorphism in the long exact sequence for the triple $(F_{p+r}, F_{p+r-1}, F_{p-1})$. Looking back on (8) of Section 34.1, the element $\delta(\xi)$ represents $d_r(u)$ in the E_r-term of the spectral sequence. But we know that the right vertical map is an injection whose image consists of the integral elements, and our isomorphism of the E_2-term with $H^*(X; \mathbb{Z})$ identifies $\delta(\xi)$ with $\mathrm{ch}(\delta(\xi))$. Commutativity of the square says this element is also $\delta((\mathrm{ch}\,\xi)_{q+r-1})$ (and also verifies that this class is integral). $\square$

35. Operations on K-theory

Experience has shown that when studying a cohomology theory it is useful to look not just at the cohomology groups themselves but also the natural *operations* on the cohomology groups. In the case of singular cohomology this is the theory of Steenrod operations. In the present section we will construct some useful operations $K^0(X) \to K^0(X)$. We start with the λ-operations, which are easy to define but have the drawback that they are not group homomorphisms. Then we modify these to obtain the Adams operations ψ^k, which are more nicely behaved.

35.1. **The lambda operations.** Fix a topological space X. We start with the exterior power constructions $E \mapsto \Lambda^k E$ on vector bundles over X. These are, of course, not additive: $\Lambda^k(E \oplus F) \not\cong \Lambda^k E \oplus \Lambda^k F$. So it is not immediately clear how these constructions induce maps on K-groups. The key lies in the formula

$$\Lambda^k(E \oplus F) \cong \bigoplus_{i+j=k} \Lambda^i E \otimes \Lambda^j F. \tag{35.2}$$

Construct a formal power series

$$\lambda_t(E) = \sum_{i=0}^{\infty} [\Lambda^i E] t^i = 1 + [E]t + [\Lambda^2 E]t^2 + \cdots \in K^0(X)[[t]].$$

Because the zero coefficient is 1, this power series is a unit in $K^0(X)[[t]]$. So λ_t is a function into the group of units inside $K^0(X)[[t]]$:

$$\begin{tikzcd}
\mathrm{Vect}(X) \arrow[r, "\lambda_t"] \arrow[d] & \Big(K^0(X)[[t]]\Big)^* \\
K^0(X) \arrow[ur, dotted] &
\end{tikzcd}$$

Formula (35.2) says that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$, and this implies the existence of the dotted-arrow group homomorphism in the above diagram. We will call this dotted arrow λ_t as well.

Finally, define $\lambda^k \colon K^0(X) \to K^0(X)$ by letting $\lambda^k(w)$ be the coefficient of t^k in $\lambda_t(w)$. Note that if E is a vector bundle over X then $\lambda^k([E]) = [\Lambda^k E]$. However, λ^k is not a group homomorphism; instead one has the formula

$$\lambda^k(u + v) = \sum_{i+j=k} \lambda^i(u)\lambda^j(v).$$

Example 35.3. To get a feeling for these operations let us compute $\lambda^k(-[E])$ for E a vector bundle over X. Note first that

$$\lambda_t(-[E]) = \tfrac{1}{\lambda_t([E])} = \tfrac{1}{1+[E]t+[\Lambda^2 E]t^2+\cdots}$$

For R a commutative ring and $a = 1 + a_1 t + a_2 t^2 + \cdots \in R[[t]]$, one has

$$\tfrac{1}{a} = 1 + P_1 t + P_2 t^2 + \cdots$$

where the P_i's are certain universal polynomials in the a_i's with coefficients in $\mathbb{Z}$. Equating coefficients in the identity $1 = (1 + a_1 t + a_2 t^2 + \cdots)(1 + P_1 t + P_2 t^2 + \cdots)$ gives

$$P_k + P_{k-1}a_1 + P_{k-2}a_2 + \cdots + P_1 a_{k-1} + a_k = 0,$$

which allows one to inductively determine each P_k. One finds that

$$P_1 = -a_1, \quad P_2 = a_1^2 - a_2, \quad P_3 = -a_1^3 + 2a_1a_2 + a_3$$

So we conclude that

$$\begin{aligned}
&\lambda^1(-[E]) = -[E],\\
&\lambda^2(-[E]) = [E]^2 - [\Lambda^2 E],\\
&\lambda^3(-[E]) = -[E]^3 + 2[E][\Lambda^2 E] + [\Lambda^3 E]
\end{aligned}$$

and so forth.

35.4. **Symmetric power operations.** One can repeat everything from the previous section using the symmetric product construction $E \mapsto \mathrm{Sym}^k E$ in place of the exterior product $\Lambda^k E$. One obtains a group homomorphism

$$\mathrm{sym}_t \colon K^0(X) \to \big(K^0(X)[[t]]\big)^*$$

and defines $\mathrm{sym}^k(w)$ to be the coefficient of t^k in $\mathrm{sym}_t(w)$. It turns out, however, that these operations do not give anything 'new'—they are related to the λ-operations by the formula

$$\mathrm{sym}^k(w) = (-1)^k \lambda^k(-w).$$

To explain this we need a brief detour on the deRham complex. The following material is taken from [FLS].

Let V be a vector space over a field F. Write $\mathrm{Sym}^*(V) = \oplus_k \mathrm{Sym}^k(V)$ and $\Lambda^*(V) = \oplus_k \Lambda^k V$. These each have a familiar algebra structure, and we have canonical isomorphisms

$$\mathrm{Sym}^*(V) \otimes \mathrm{Sym}^*(W) \xrightarrow{\cong} \mathrm{Sym}^*(V \oplus W), \qquad \Lambda^*(V) \otimes \Lambda^*(W) \xrightarrow{\cong} \Lambda^*(V \oplus W).$$

(In the former case, include $\mathrm{Sym}^*(V)$ and $\mathrm{Sym}^*(W)$ into $\mathrm{Sym}^*(V \oplus W)$ and then multiply there; likewise for the exterior algebra version). These isomorphisms allow us to equip both $\mathrm{Sym}^*(V)$ and $\Lambda^*(V)$ with coproducts Δ_{sym} and Δ_{ext} making them into Hopf algebras. The coproducts are

$$\mathrm{Sym}^*(V) \longrightarrow \mathrm{Sym}^*(V \oplus V) \xleftarrow{\cong} \mathrm{Sym}^*(V) \otimes \mathrm{Sym}^*(V)$$

and

$$\Lambda^*(V) \longrightarrow \Lambda^*(V \oplus V) \xleftarrow{\cong} \Lambda^*(V) \otimes \Lambda^*(V)$$

where in each case the first map is the one induced by the diagonal $\Delta \colon V \to V \oplus V$. If $v_1, \ldots, v_k \in V$ and $I \subseteq \{1, \ldots, k\}$ then write v_I for the monomial $v_{i_1} \cdots v_{i_r}$ where $I = \{i_1, \ldots, i_r\}$ and $i_1 < \cdots < i_r$. Then

$$\Delta_{sym}(v_1 \cdots v_k) = \sum_{A \dot{\cup} B = \{1, \ldots, k\}} v_A \otimes v_B$$

where the $\dot{\cup}$ symbol denotes disjoint union, and likewise

$$\Delta_{ext}(v_1 \cdots v_k) = \sum_{A \dot{\cup} B = \{1, \ldots, k\}} (-1)^{[A,B]} v_A \otimes v_B$$

where the sign $(-1)^{[A,B]}$ is the one that makes the equation $v_1 \cdots v_k = (-1)^{[A,B]} v_A v_B$ valid in $\Lambda^*(V)$.

Write $e\colon \operatorname{Sym}^k(V) \to \operatorname{Sym}^{k-1} V \otimes \operatorname{Sym}^1(V)$ and $e'\colon \Lambda^k(V) \to \Lambda^1 V \otimes \Lambda^{k-1}(V)$ for the projections of the coproduct onto the indicated factors. Finally, write d and κ for the composites

$$\begin{tikzcd}
\operatorname{Sym}^k(V)\otimes\Lambda^i(V) \arrow[r, "e\otimes\mathrm{id}"] & \operatorname{Sym}^{k-1}(V)\otimes\operatorname{Sym}^1(V)\otimes\Lambda^i(V) \arrow[d, equal] & \\
& \operatorname{Sym}^{k-1}(V)\otimes\Lambda^1(V)\otimes\Lambda^i(V) \arrow[r, "\mathrm{id}\otimes\mu"] & \operatorname{Sym}^{k-1}(V)\otimes\Lambda^{i+1}(V)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{Sym}^k(V)\otimes\Lambda^i(V) \arrow[r, "\mathrm{id}\otimes e'"] & \operatorname{Sym}^{k}(V)\otimes\Lambda^1(V)\otimes\Lambda^{i-1}(V) \arrow[d, equal] & \\
& \operatorname{Sym}^{k}(V)\otimes\operatorname{Sym}^1(V)\otimes\Lambda^{i-1}(V) \arrow[r, "\mu\otimes\mathrm{id}"] & \operatorname{Sym}^{k+1}(V)\otimes\Lambda^{i-1}(V).
\end{tikzcd}$$

The maps d and κ are called the **de Rham** and **Koszul** differentials, respectively; in a moment we will give concrete formulas for them and see that $d^2 = k^2 = 0$. The following diagram shows the maps d, and the maps κ go in the opposite direction:

(35.5) *The deRham and Koszul complexes:*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots \\
\operatorname{Sym}^3 V\otimes\Lambda^2 V \arrow[ur, "d"'] & \operatorname{Sym}^2 V\otimes\Lambda^2 V \arrow[ur, "d"'] & \operatorname{Sym}^1 V\otimes\Lambda^2 V \arrow[ur, "d"'] & \operatorname{Sym}^0 V\otimes\Lambda^2 V \\
\operatorname{Sym}^3 V\otimes\Lambda^1 V \arrow[ur, "d"'] & \operatorname{Sym}^2 V\otimes\Lambda^1 V \arrow[ur, "d"'] & \operatorname{Sym}^1 V\otimes\Lambda^1 V \arrow[ur, "d"'] & \operatorname{Sym}^0 V\otimes\Lambda^1 V \\
\operatorname{Sym}^3 V\otimes\Lambda^0 V \arrow[ur, "d"'] & \operatorname{Sym}^2 V\otimes\Lambda^0 V \arrow[ur, "d"'] & \operatorname{Sym}^1 V\otimes\Lambda^0 V \arrow[ur, "d"'] & \operatorname{Sym}^0 V\otimes\Lambda^0 V
\end{tikzcd}$$

Let $e_1, \dots, e_n$ be elements of V. It will be convenient to use the notation de_j for the element of $\Lambda^1(V)$ corresponding to e_j under the canonical isomorphism $\Lambda^1(V) \cong V$. Let $m = e_{i_1} \otimes \cdots \otimes e_{i_r} \in \operatorname{Sym}^r(V)$ and $\omega = de_{j_1} \wedge \cdots \wedge de_{j_s} \in \Lambda^s(V)$. It is an exercise to verify that

$$e(m) = \sum_u \big(e_{i_1} \otimes \cdots \otimes \widehat{e_{i_u}} \otimes \cdots \otimes e_{i_r}\big) \otimes e_{i_u}$$

and

$$e'(\omega) = \sum_u (-1)^{u-1} de_{j_u} \otimes \big(de_{j_1} \wedge \cdots \wedge \widehat{de_{j_u}} \wedge \cdots \wedge de_{j_s}\big).$$

So

$$d(m \otimes \omega) = \sum_u \big(e_{i_1} \otimes \cdots \otimes \widehat{e_{i_u}} \otimes \cdots \otimes e_{i_r}\big) \otimes (de_{i_u} \wedge \omega)$$

and

$$\kappa(m \otimes \omega) = \sum_u (-1)^{u-1}(m \otimes e_{j_u}) \otimes \big(de_{j_1} \wedge \cdots \wedge \widehat{de_{j_u}} \wedge \cdots \wedge de_{j_s}\big).$$

From these descriptions one readily sees that d is the usual deRham differential and κ is the usual Koszul differential. Consequently, $d^2 = 0$ and $\kappa^2 = 0$. In the two-dimensional array (35.5), if we take direct sums inside of each row then we get

$$\operatorname{Sym}^*(V) \otimes \Lambda^0(V) \xrightarrow{d} \operatorname{Sym}^*(V) \otimes \Lambda^1(V) \xrightarrow{d} \cdots$$

and this is an algebraic version of the deRham complex. We also get

$$\cdots \longrightarrow \operatorname{Sym}^*(V) \otimes \Lambda^2(V) \xrightarrow{\kappa} \operatorname{Sym}^*(V) \otimes \Lambda^1(V) \xrightarrow{\kappa} \operatorname{Sym}^*(V) \otimes \Lambda^0(V)$$

which is a Koszul complex.

Proposition 35.6.

(a) $d\kappa + \kappa d\colon \operatorname{Sym}^r(V) \otimes \Lambda^s(V) \to \operatorname{Sym}^r(V) \otimes \Lambda^s(V)$ *is multiplication by the total degree* $r + s$.

(b) In (35.5) every diagonal deRham chain complex is exact in dimensions where the total degree is prime to the characteristic of F.

(c) In (35.5) every diagonal Koszul chain complex is exact, regardless of the characteristic of the ground field, except for the diagonal in total degree 0.

Proof. Part (a) is a computation that is tedious but not particularly hard. If $v_1, \ldots, v_r, w_1, \ldots, w_s \in V$ then $\kappa d(v_1 \cdots v_r \otimes w_1 \cdots w_s)$ has two types of terms: "pure" terms where a v_i is moved to the right of the tensor and then moved back, and "mixed" terms where a v_i is moved to the right of the tensor and then a w_j is moved to the left. There are exactly r pure terms. Similarly $d\kappa$ has s pure terms where a w_j is moved to the left and then moved back, and also a bunch of mixed terms. One needs to check that the signs on the mixed terms in $d\kappa$ exactly cancel those in κd.

Part (b) follows from (a): the maps κ give a chain homotopy showing that multiplication by the total degree is homotopic to the zero map. If the total degree is invertible in the ground field, this implies that the homology must be zero in that dimension.

We do not actually need part (c) below, but we include it to complete the story. The maps d give a chain homotopy for the κ-complexes, much like in the proof of (b), but this gives exactness only for some spots in the complex. The proof of exactness at all spots is something we have already seen in a somewhat more general context, in Theorem 18.25(a). If we pick a basis $x_1, \ldots, x_n$ of V then $\operatorname{Sym}^*(V) = F[x_1, \ldots, x_n]$ and the κ-complex is the Koszul complex $K(x_1, \ldots, x_n)$. $\square$

Now we apply the above results to K-theory. Since the deRham and Koszul complexes were canonical constructions, we can apply them to vector bundles. The deRham complex gives us exact sequences

$$\begin{array}{l} 0 \longrightarrow \operatorname{Sym}^k E \otimes \Lambda^0 E \longrightarrow \operatorname{Sym}^{k-1} E \otimes \Lambda^1 E \longrightarrow \cdots \longrightarrow \operatorname{Sym}^1 E \otimes \Lambda^{k-1} E \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Sym}^0 E \otimes \Lambda^k E \longrightarrow 0. \end{array}$$

These show that in $K^0(X)$ one has

$$\sum_{a+b=k} (-1)^b [\mathrm{Sym}^a E] \cdot [\Lambda^b E] = 0.$$

Consequently, $\mathrm{sym}_t([E]) \cdot \lambda_{-t}([E]) = 1$. So

$$\mathrm{sym}_t([E]) = \tfrac{1}{\lambda_{-t}([E])} = \lambda_{-t}(-[E]) \qquad \text{and} \qquad \lambda_{-t}([E]) = \tfrac{1}{\mathrm{sym}_t([E])} = \mathrm{sym}_t(-[E]).$$

Any class $w \in K^0(X)$ has the form $w = [E] - [F]$ for some vector bundles E and F, and therefore

$$\begin{aligned} \mathrm{sym}_t(w) = \mathrm{sym}_t([E]) \cdot \mathrm{sym}_t(-[F]) = \lambda_{-t}(-[E]) \cdot \lambda_{-t}([F]) &= \lambda_{-t}([F] - [E]) \\ &= \lambda_{-t}(-w). \end{aligned}$$

So the sym^k and λ^k operations on K-theory are essentially the same: $\mathrm{sym}^k(w) = (-1)^k \lambda^k(-w)$.

This has been a long discussion with somewhat of a negative conclusion: the sym^k operations can be completely ignored in favor of the λ^k's (or vice versa). We have learned some useful things along the way, however.

35.7. **The Adams operations.** The usefulness of the λ^k operations is limited by the fact that they are not group homomorphisms. There is a clever method, however, for combining the λ-operations in a way that does produce a collection of group homomorphisms. This is originally due to Frank Adams [Ad2]. Before describing this construction we take a brief detour to develop the algebraic combinatorics that we will need.

Recall that we have a map $\lambda_t \colon K^0(X) \to \big(K^0(X)[[t]]\big)^*$ and that this is a group homomorphism:

$$\lambda_t(x+y) = \lambda_t(x) \cdot \lambda_t(y).$$

If we want additive maps $K^0(X) \to K^0(X)$ a natural idea is to apply logarithms to the above formula. To be precise, start with the formal power series

$$\log(1+z) = z - \tfrac{z^2}{2} + \tfrac{z^3}{3} - \cdots$$

Since $\lambda_t(x)$ has constant term equal to 1, we can use the above series to make sense of $\log(\lambda_t(x))$—but only provided that we add denominators into $K^0(X)$, say by tensoring with $\mathbb{Q}$. If we set $\mu_t(x) = \log(\lambda_t(x))$ then we would have

$$\mu_t(x+y) = \mu_t(x) + \mu_t(y).$$

The coefficients of powers of t in $\mu_t(x)$ then give additive operations, with the only difficulty being that they take values in $K^0(X) \otimes \mathbb{Q}$.

We can, however, eliminate the need for $\mathbb{Q}$-coefficients by applying the operator $\frac{d}{dt}$. Precisely, define

$$\nu_t(x) = \frac{d}{dt}\Big[\mu_t(x)\Big] = \tfrac{\lambda_t'(x)}{\lambda_t(x)} = (1 - z + z^2 - z^3 + \cdots)|_{z=\lambda_t(x)-1} \cdot \lambda_t'(x).$$

The two factors in the right-most expression both lie in $K^0(X)[[t]]$, and so this eliminates the problem with denominators: $\nu_t(x) \in K^0(X)[[t]]$ yet we still have $\nu_t(x+y) = \nu_t(x) + \nu_t(y)$. Taking coefficients of $\nu_t(x)$ thereby yields additive operations $\nu^k \colon K^0(X) \to K^0(X)$.

We could stop here, but there is one more modification that makes things a bit simpler later on. Suppose that L is a line bundle over X, and take $x = [L]$. Then $\lambda_t(x) = 1 + [L]t = 1 + xt$, hence

$$\nu_t(x) = \tfrac{x}{1+xt} = x(1 - xt + x^2t^2 - x^3t^3 + \cdots) = x - x^2t + x^3t^2 - \cdots .$$

The ν-operations simply give powers of x, together with certain signs: $\nu^k(x) = (-1)^k x^{k-1}$. It is easy to adopt a convention that makes these signs disappear, and we might as well do this; and while we are at it, let us shift the indexing on the ν's so that the kth operation sends x to x^k, since that will be easier to remember.

Putting everything together, we have arrived at the following definition:

$$\psi_t(x) = t\frac{d}{dt}\Big[\log(\lambda_{-t}(x))\Big] = t \cdot \tfrac{\lambda'_{-t}(x)}{\lambda_{-t}(x)},$$

and $\psi^k(x)$ is the coefficient of t^k in $\psi_t(x)$. The operations ψ^k are called **Adams operations**. We have proven that

(1) Each ψ^k is a group homomorphism $K^0(X) \to K^0(X)$, natural in X;
(2) If $x = [L]$ for L a line bundle then $\psi^k(x) = x^k$.

Conditions (1) and (2) actually completely characterize the Adams operations, although we will not need this.

If R is a commutative ring and $\alpha = \sum_{i\geq 1} \alpha_i t^i$ is a power series with no constant term, then there is an identity

$$\tfrac{\alpha'}{1+\alpha} = \sum_{k=0}^{\infty} (-1)^k S_k(\alpha_1, \dots, \alpha_k) t^{k-1}$$

where the S_k are certain universal polynomials called the *Newton polynomials*. See Appendix D for a discussion, and Proposition D.3 for the identity. Applying this with $\alpha = \lambda_{-t}(x) - 1$ yields part (a) of the following result. Part (b) is a standard recursive relation for the Newton polynomials, again derived in Appendix D.

Proposition 35.8.
(a) $\psi^k = S_k(\lambda^1, \dots, \lambda^k)$*, where* S_k *is the* k*th Newton polynomial;*
(b) $\psi^k = \lambda^1\psi^{k-1} - \lambda^2\psi^{k-1} + \cdots + (-1)^k\lambda^{k-1}\psi^1 + (-1)^{k+1}k\lambda^k$.

Proof. See Proposition D.3 and Lemma D.1. □

We record the first few Adams operations:

$$\psi^1 = \lambda^1, \quad \psi^2 = (\lambda^1)^2 - 2\lambda^2, \quad \psi^3 = (\lambda^1)^3 - 3\lambda^1\lambda^2 + 3\lambda^3.$$

For more of these, just see the list of Newton polynomials given in Table 4.2 of Appendix D.

35.9. **Properties of the Adams operations.**

Proposition 35.10. *Fix* $k, l \geq 1$ *and* $x, y \in K^0(X)$*. Then*
(a) $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$
(b) $\psi^k(xy) = \psi^k(x)\psi^k(y)$
(c) $\psi^k(\psi^l(x)) = \psi^{kl}(x)$
(d) If ℓ *is prime then* $\psi^\ell(x) \equiv x^\ell$ *mod* ℓ*.*

The proof will use the following terminology. A **line element** in $K^0(X)$ is any element $[L]$ where $L \to X$ is a line bundle. The span of the line elements consist of the classes of the form $[L_1] + \cdots + [L_a] - [L_1'] - \cdots - [L_b']$ where the L_i's and L_j''s are all line bundles.

Proof of Proposition 35.10. Note that if x and y are line elements then all of the above results are obvious because $\psi^r([L]) = [L^r]$. More generally, the results follow easily if x and y are in the span of the line elements. The general result now follows from the splitting principle in Proposition 35.11 below. Specifically, choose a $p\colon X_1 \to X$ such that p^* is injective on $K^0(X)$ and $p^*(x)$ is in the span of line elements. Then choose a $q\colon X_2 \to X_1$ such that q^* is injective and $q^*(p^*(y))$ is in the span of line elements. The identities in (a)–(d) all hold for $x' = (pq)^*(x)$ and $y' = (pq)^*(y)$, and so the injectivity of $(pq)^*$ shows they hold for x and y as well. □

Proposition 35.11 (The Splitting Principle)**.** *Let X be any space, and let $x \in K^0(X)$. Then there exists a space Y and a map $p\colon Y \to X$ such that $p^*\colon K^0(X) \to K^0(Y)$ is injective and $p^*(x)$ is in the span of line elements.*

Proof. Write $x = [E] - [F]$ for vector bundles E and F. Consider the map $\pi\colon \mathbb{P}(E) \to X$. Then $\pi^*E \cong E' \oplus L$ where L is a line bundle, and $\pi^*\colon K^0(X) \to K^0(\mathbb{P}(E))$ is injective (????). Iterating this procedure we obtain a map $f\colon Y \to X$ such that f^*E is a sum of line bundles and f^* is injective. Now use the same process to obtain a map $g\colon Y' \to Y$ such that g^* is injective and $g^*(f^*F)$ is a sum of line bundles. The composite $Y' \to X$ has the properties from the statement of the proposition. □

Corollary 35.12 (Characterization of Adams operations)**.** *Fix $k \geq 1$. Suppose that $F\colon K^0(-) \to K^0(-)$ is a natural ring homomorphism such that $F([L]) = [L^k]$ for any line bundle $L \to X$. Then $F = \psi^k$.*

Proof. Fix a space X, and let $\alpha \in K^0(X)$. Then $\alpha = [E] - [F]$ for some vector bundles E and F on X. By the Splitting Principle there exists a map $p\colon Y \to X$ such that p^*E and p^*F are direct sums of line bundles, and such that $p^*\colon K^0(X) \to K^0(Y)$ is injective. Our assumption on F implies at once that $F(p^*\alpha) = \psi^k(p^*\alpha)$, or equivalently $p^*(F\alpha) = p^*(\psi^k\alpha)$. Injectivity of p^* now gives $F(\alpha) = \psi^k(\alpha)$. □

The fact that ψ^k is natural and preserves (internal) products immediately yields that it also preserves external products. Recall that if $x \in K^0(X)$ and $y \in K^0(Y)$ then the external product can be written as $x \times y = \pi_1^*(x) \cdot \pi_2^*(y) \in K^0(X \times Y)$. Clearly $\psi^k(x \times y) = \psi^k(x) \times \psi^k(y)$. Using this, we easily obtain the following:

Proposition 35.13. *For $k \geq 1$, ψ^k acts on $\widetilde{K}^0(S^{2n})$ as multiplication by k^n.*

Proof. Let $\beta = 1 - [L]$ be the Bott element in $\widetilde{K}^0(S^2)$, and recall that the internal square β^2 is zero. From this it follows readily that

$$\psi^k(\beta) = 1 - L^k = 1 - (1-\beta)^k = 1 - (1 - k\beta) = k\beta.$$

Now recall that the external power $\beta^{(n)} = \beta \times \beta \times \cdots \times \beta$ generates $\widetilde{K}^0(S^{2n})$. But

$$\psi^k(\beta^{(n)}) = (\psi^k\beta)^{(n)} = (k\beta)^{(n)} = k^n\beta^{(n)}.$$

□

Recall the canonical filtration of $K^0(X)$ coming from the Atiyah-Hirzebruch spectral sequence, as discussed in Section 34 and especially in Remark 34.6. In particular, recall that $F^{2n-1}K^0(X) = F^{2n}K^0(X)$, for every n, as a consequence of $K^*(pt)$ being concentrated in even dimensions. The naturality of the Adams operations shows that they respect the filtration, and Proposition 35.13 shows that ψ^k acts as a scalar on the associated graded:

Proposition 35.14. *Let $k \geq 1$.*

(a) If $x \in F^{2n}K^0(X)$ then $\psi^k(x) = k^n x$ + terms of higher filtration. That is, $\psi^k(x) - k^n x \in F^{2n+2}K^0(X)$.

(b) If the induced filtration on $K^0(X)_{\mathbb{Q}}$ is finite (e.g., if X is a finite-dimensional CW-complex) then the operations ψ^k are diagonalizable on $K^0(X)_{\mathbb{Q}}$, with eigenvalues of the form k^r for $r \geq 0$. For $k \neq 1$ the decomposition

$$K^0(X)_{\mathbb{Q}} = \bigoplus_{r\geq 0} \mathrm{Eig}_{\psi^k}(k^r)$$

restricts to give

$$F^{2n}K^0(X)_{\mathbb{Q}} = \bigoplus_{r\geq n} \mathrm{Eig}_{\psi^k}(k^r),$$

and this decomposition is independent of k: for all $k, l \neq 1$ and $r \geq 0$ one has $\mathrm{Eig}_{\psi^k}(k^r) = \mathrm{Eig}_{\psi^l}(l^r)$.

Proof. For part (a) it suffices to replace X by a weakly equivalent CW-complex, so that $F^{2n}K^0(X) = \ker[K^0(X) \to K^0(X^{2n-1})]$. If $\alpha \in F^{2n}K^0(X)$ then let α_1 denote its image in $K^0(X^{2n})$. The cofiber sequence $X^{2n-1} \hookrightarrow X^{2n} \to X^{2n}/X^{2n-1}$ indcues a long exact sequence

$$\cdots \to \widetilde{K}^0(X^{2n}/X^{2n-1}) \to K^0(X^{2n}) \to K^0(X^{2n-1}) \to \cdots$$

The element $\alpha_1 \in K^0(X^{2n})$ maps to zero, and so it is the image of a class $\alpha_2 \in \widetilde{K}^0(X^{2n}/X^{2n-1})$. By Proposition 35.13 one knows $\psi^k(\alpha_2) = k^n\alpha_2$, and so $\psi^k\alpha_1 = k^n\alpha_1$. It follows that $\psi^k\alpha - k^n\alpha$ maps to zero in $K^0(X^{2n})$, and hence lies in $F^{2n+1}K^0(X^{2n}) = F^{2n+2}K^0(X^{2n})$.

For part (b), let $2n$ be the largest even integer such that $F^{2n}K^0(X)_{\mathbb{Q}} \neq 0$. It follows from (a) that ψ^k acts as multiplication by k^n on $F^{2n}K^0(X)_{\mathbb{Q}}$. But if $\psi^k(x) = k^n x$ for some $x \notin F^{2n}K^0(X)_{\mathbb{Q}}$ then choose $r < n$ largest so that $x \in F^{2r}K^0(X)_{\mathbb{Q}}$. Part (a) gives that $\psi^k(x) = k^r x + y$ where $y \in F^{2r+2}K^0(X)_{\mathbb{Q}}$, but then we get $(k^n - k^r)x = y$. Since $k^n \neq k^r$ we have $x \in F^{2r+2}K^0(X)_{\mathbb{Q}}$, which is a contradiction. So $F^{2n}K^0(X)_{\mathbb{Q}} = \mathrm{Eig}_{\psi^k}(k^n)$.

We now prove by reverse induction that $F^{2i}K^0(X)_{\mathbb{Q}}$ is the sum of k^j-eigenspaces of ψ^k, for $j \geq i$. Assume this holds for a particular value of i, and let $\alpha \in F^{2i-2}K^0(X)$. We know $\psi^k\alpha = k^{i-1}\alpha + \gamma$, where γ is some element of $F^{2i}K^0(X)_{\mathbb{Q}}$. The induction hypothesis says that $\gamma = \sigma_i + \sigma_{i+1} + \cdots + \sigma_n$ where each σ_r is an eigenvector for ψ^k with eigenvalue k^r. A routine calculation now shows that $\alpha - \sum_{r\geq i} \frac{1}{k^r - k^{i-1}}\sigma_r$ is a k^{i-1}-eigenvector for ψ^k, and hence α belongs to the sum of eigenspaces for eigenvalues k^r, $r \geq i-1$. This completes the induction step.

It remains to prove that the eigenspaces $\mathrm{Eig}_{\psi^k}(k^r)$ are independent of k. This is a small variant on the classical argument that commuting operators (in this case ψ^k and ψ^l) can be simultaneously diagonalized. First note that we have seen that

$\mathrm{Eig}_{\psi^k}(k^n) = F^{2n}K^0(X)_{\mathbb{Q}} = \mathrm{Eig}_{\psi^l}(l^n)$. We also have

$$\mathrm{Eig}_{\psi^k}(k^{n-1}) \oplus \mathrm{Eig}_{\psi^k}(k^n) = F^{2n-2}K^0(X)_{\mathbb{Q}} = \mathrm{Eig}_{\psi^l}(l^{n-1}) \oplus \mathrm{Eig}_{\psi^l}(l^n).$$

For this subspace choose a basis of ψ^k-eigenvectors $a_1, \ldots, a_s, b_1, \ldots, b_t$ (with the a's having eigenvalue k^{n-1} and the b's having eigenvalue k^n) that simultaneously diagonalizes ψ^l. By what was already proven, the b's span $\mathrm{Eig}_{\psi^l}(l^n)$; so the only possiblity for the a's is that their ψ^l-eigenvalue is l^{n-1}. It follows that the a's are a basis for $\mathrm{Eig}_{\psi^l}(l^{n-1})$, and therefore $\mathrm{Eig}_{\psi^k}(k^{n-1}) = \mathrm{Eig}_{\psi^l}(l^{n-1})$. Now repeat this argument inductively. $\square$

Remark 35.15. The Adams operations ψ^k are defined for $k \geq 1$. Someone once had the clever idea of taking the operation induced by $E \mapsto E^*$ and calling it ψ^{-1}. So $\psi^{-1}([E]-[F]) = [E^*]-[F^*]$. Then for $k \geq 1$ one checks that $\psi^{-1}\circ\psi^k = \psi^k\circ\psi^{-1}$, and so it is reasonable to define ψ^{-k} to be this common expression. With this definition parts (a)–(c) of Proposition 35.10 now hold for all $k, l \in \mathbb{Z} - \{0\}$.

35.16. **Adams operations for non-compact spaces.** ????

36. The Hopf invariant one problem

The Hopf invariant assigns an integer to every map $f\colon S^{2n-1} \to S^n$, giving a group homomorphism $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$. Elementary arguments show that 2 is always in the image, and the natural question is then whether 1 is also in the image. This is the Hopf invariant one problem—determine all values of n for which H is surjective (or said differently, all values of n for which there exists a map of Hopf invariant one).

It was known classically that H is surjective when $n \in \{1, 2, 4\}$, because the classical Hopf maps all have Hopf invariant equal to one. The question for other dimensions was first settled by Adams in [Ad1], who proved that no other Hopf invariant one maps exist. Adams's proof is not simple, even by modern standards, being based on secondary cohomology operations associated to the Steenrod squares. Several years after Adams gave his original proof, Adams and Atiyah [AA] used K-theory to give a much simpler solution to the Hopf invariant one problem. Their 'postcard proof' takes less than a page, in dramatic contrast to Adams's original method. This was seen—rightly so—as a huge demonstration of the power of K-theory.

Our goal in this section will be to present the Adams-Atiyah proof, although we will not quite do this in their style. Specifically, when Adams and Atiyah wrote their paper they clearly had an agenda: to write down the proof in as small a space as possible. If the goal is to accentuate how much the use of K-theory simplifies the solution, this makes perfect sense. But at the same time, writing the proof in this way results in a certain air of mystery: the proof involves a strange manipulation with the Adams operations ψ^2 and ψ^3 that comes out of nowhere—it seems like a magic trick.

In our presentation below we try to put this (ψ^2, ψ^3) trick into its proper context: it is part of a calculation of a certain Ext^1 group. The full calculation of this group is not hard, and quite interesting for other reasons—e.g., it connects deeply to the study of the J-homomorphism. Our presentation doesn't fit on a postcard, but by the time we are done we will have a good understanding of several neat and important things. Hopefully it won't seem like magic.

36.1. **Brief review of the problem.** Let $f\colon S^{2n-1} \to S^n$ and consider the mapping cone Cf. One readily computes that

$$H^i(Cf) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0, n, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Fix an orientation on the two spheres, and let a and b be corresponding generators for $H^n(Cf)$ and $H^{2n}(Cf)$. Then $b^2 = h \cdot a$ for a unique integer $h \in \mathbb{Z}$, and this integer is called the **Hopf invariant** of f: we write $h = H(f)$.

Note that if n is odd then $b^2 = -b^2$ and so $h = 0$. Therefore the Hopf invariant is only interesting when n is even.

Remark 36.2. We follow [Ha, Proposition 4B.1] to see that $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a group homomorphism. Given $f, g\colon S^{2n-1} \to S^n$ consider the diagram of mapping cones

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f+g"] \arrow[d] & S^n \arrow[r] \arrow[d, equal] & C_{f+g} \arrow[d] \\ S^{2n-1} \vee S^{2n-1} \arrow[r, "f\vee g"] & S^n \arrow[r] & X \end{tikzcd}$$

where the left vertical map is the equatorial collapse and $X = C_{f\vee g}$. Note that there are inclusions $Cf \hookrightarrow X$ and $Cg \hookrightarrow X$. The cohomology group $H^{2n}(X)$ has two generators a_1 and a_2, and naturality applied to those inclusions shows that $b^2 = H(f)a_1 + H(g)a_2$. But under the map $C_{f+g} \to X$ both a_1 and a_2 are sent to our usual generator a, and from this one gets that $H(f+g) = H(f) + H(g)$.

Remark 36.3. It is easy to see that 2 (and therefore any even integer) is always in the image of H. We again follow [Ha] here and let X be the pushout

$$\begin{tikzcd} S^n \vee S^n \arrow[r, tail] \arrow[d, "\nabla"'] & S^n \times S^n \arrow[d, dotted, "\alpha"] \\ S^n \arrow[r, dotted] & X. \end{tikzcd}$$

One readily checks that the cohomology of X consists of two copies of $\mathbb{Z}$, in degrees n and $2n$. So X is the mapping cone of a certain map $f\colon S^{2n-1} \to S^n$, the attaching map of the top cell. If $x \in H^n(S^n)$ is a fixed generator, then there is a generator $b \in H^n(X)$ that maps to $x \otimes 1 + 1 \otimes x$ under α^*. It follows that b^2 maps to

$$(x \otimes 1 + 1 \otimes x)^2 = 2(x \otimes x)$$

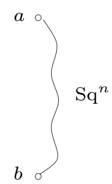
and therefore b^2 is twice a generator of $H^{2n}(X)$. One concludes that the Hopf invariant of f is ± 2, depending on one's sign choices.

The problem arises of determining the precise image of $H\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$, when n is even. By Remarks 36.2 and 36.3 the image is a subgroup that contains $2\mathbb{Z}$, so there are only two possibilities: either the image equals $2\mathbb{Z}$, or else it equals all of $\mathbb{Z}$. The latter happens if and only if there exists an element in $\pi_{2n-1}(S^n)$ having Hopf invariant equal to one. Thus, this is the "Hopf invariant one" problem.

The following several paragraphs involve the Steenrod squares. The results will not be needed later in this section, but they constitute an interesting part of the overall story.

As soon as one is versed in the Steenrod squares it is easy to obtain a necessary condition for the existence of a Hopf invariant one map $f\colon S^{2n-1} \to S^n$. In the mod

2 cohomology of Cf we have $\mathrm{Sq}^n(b) = b^2 = h \cdot a$. So if f has odd Hopf invariant then $\mathrm{Sq}^n(b) = a$, and the mod 2 cohomology of Cf looks like this:



This picture just says that the cohomology has generators a and b together with a Sq^n connecting b to a. As an immediate consequence we obtain Adem's theorem:

Proposition 36.4 (Adem)**.** *If $f\colon S^{2n-1} \to S^n$ has Hopf invariant one then n is a power of 2.*

Proof. The above picture represents a module over the Steenrod algebra only if Sq^n is indecomposable. But by our knowledge of the Steenrod algebra, the indecomposables all have degrees equal to a power of 2 (they are represented by the elements Sq^{2^i}). $\square$

The reader might have noticed that there actually seem to be *two* problems here, that are interelated. There is the Hopf invariant one problem, and there is the question of whether there exists a map $S^{k+n-1} \to S^k$ whose cofiber has a nonzero Sq^n in mod 2 cohomology. The first problem is inherently *unstable* in nature because it deals with the cup product, whereas the second problem is clearly stable. It is useful to note that the two problems are actually equivalent:

Proposition 36.5. *Fix $n \geq 1$. The following two statements are equivalent:*
(a) There exists a map $S^{2n-1} \to S^n$ of Hopf invariant one;
(b) There exists a $k \geq 0$ and a map $S^{k+n-1} \to S^k$ whose mapping cone has a nonzero Sq^n operation.

Proof. In the discussion preceding Adem's theorem we saw that (a) implies (b) by taking $k = n$. Conversely, if (b) holds for a certain map g then by suspending if necessary we can assume $k \geq n$. The Freudenthal Suspension Theorem guarantees that $\pi_{2n-1}(S^n) \to \pi_{k+n-1}(S^k)$ is surjective, so choose map a map $f\colon S^{2n-1} \to S^n$ that is a preimage of g. The spaces Cf and Cg are homotopy equivalent after appropriate suspensions, so the mod 2 cohomology of Cf has a nonzero Sq^n. It immediately follows that f has odd Hopf invariant, and consequently there exists a map of Hopf invariant one. $\square$

Adem's theorem is really an analysis of the stable problem, and it may be rephrased as follows. If $f\colon S^{n+k-1} \to S^k$ then there is a cofiber sequence $S^{n+k-1} \to S^k \to Cf$, and if $n \geq 2$ the long exact sequence on mod 2 cohomology breaks up into a family of short exact sequences

$$0 \leftarrow \tilde{H}^*(S^k; \mathbb{Z}/2) \leftarrow \tilde{H}^*(Cf; \mathbb{Z}/2) \leftarrow \tilde{H}^*(S^{n+k}; \mathbb{Z}/2) \leftarrow 0.$$

These are maps of modules over the Steenrod algebra $\mathcal{A}$, and both the left and right terms are isomorphic to the trivial $\mathcal{A}$-module $\mathbb{F}_2$ (graded to lie in the appropriate dimension). So the above short exact sequence represents an element of $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. Standard homological algebra identifies this Ext^1 with the module of indecomposables I/I^2, where I is the augmentation ideal of $\mathcal{A}$. Adem's Theorem works because we know this module of indecomposables precisely, and therefore can identify the Ext^1 groups precisely.

36.6. **An Ext calculation.** Most of this section will be spent in pursuit of a purely algebraic question, somewhat related to what we just saw. Let $(\mathbb{N}, \cdot)$ be the monoid of natural numbers under multiplication, and let $\mathcal{B} = \mathbb{Z}[\mathbb{N}]$ be the corresponding monoid ring. Write ψ^k for the element of $\mathcal{B}$ corresponding to $k \in \mathbb{N}$. Then $\mathcal{B}$ is simply the polynomial ring

$$\mathcal{B} = \mathbb{Z}[\psi^2, \psi^3, \psi^5, \ldots],$$

with one generator corresponding to each prime number. We think of $\mathcal{B}$ as the ring of formal Adams operations, and note that $K^0(X)$ is naturally a $\mathcal{B}$-module for any space X.

Let $\mathbb{Z}(r)$ denote the following module over $\mathcal{B}$: as an abelian group it is a copy of $\mathbb{Z}$, with chosen generator g, and the $\mathcal{B}$-module structure is $\psi^k.g = k^r g$. Our goal will be to compute the groups

$$\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$$

for all values of r and s.

Before exploring this algebraic problem let us quickly indicate the application to topology. Let $f\colon S^{n+k} \to S^n$ be a map of spheres, and write Cf for the mapping cone. The Puppe sequence looks like

$$S^{n+k} \to S^n \to Cf \to S^{n+k+1} \to S^{n+1} \to \cdots$$

and applying $\widetilde{K}^0(-)$ to this yields

$$\widetilde{K}^0(S^{n+k}) \leftarrow \widetilde{K}^0(S^n) \leftarrow \widetilde{K}^0(Cf) \leftarrow \widetilde{K}^0(S^{n+k+1}) \leftarrow \widetilde{K}^0(S^{n+1}) \leftarrow \cdots$$

Any $\widetilde{K}^0(-)$ group is naturally a $\mathcal{B}$-module, via the Adams operations; and all the maps in the above sequence are maps of $\mathcal{B}$-modules. Under the hypotheses that n is even and k is odd, the groups on the two ends vanish and we get a short exact sequence

$$0 \leftarrow \widetilde{K}^0(S^n) \leftarrow \widetilde{K}^0(Cf) \leftarrow \widetilde{K}^0(S^{n+k+1}) \leftarrow 0.$$

Proposition 35.13 says that as a $\mathcal{B}$-module $\widetilde{K}^0(S^{2r})$ is isomorphic to $\mathbb{Z}(r)$, and hence the above sequence yields an element

$$A(f) \in \mathrm{Ext}^1_{\mathcal{B}}\Big(\mathbb{Z}(\tfrac{n}{2}), \mathbb{Z}(\tfrac{n+k+1}{2})\Big).$$

That is to say, we have obtained a topological invariant of f taking values in this Ext group.

Now we begin our computation. Let X be a $\mathcal{B}$-module that sits in a short exact sequence

$$0 \to \mathbb{Z}(s) \to X \to \mathbb{Z}(r) \to 0. \tag{36.7}$$

Write a for a chosen generator of $\mathbb{Z}(s)$ (as well as its image in X) and $\tilde{b}$ for a chosen generator of $\mathbb{Z}(r)$. Write b for a preimage of $\tilde{b}$ in X. Then we have

$$\psi^k b = k^r b + P_k a$$

for a unique $P_k \in \mathbb{Z}$. The $\mathcal{B}$-module structure on X is completely determined by the ∞-tuple of integers $\mathbf{P} = (P_2, P_3, P_5, \ldots)$.

Does any choice of $\mathbf{P}$ correspond to a $\mathcal{B}$-module? To be a $\mathcal{B}$-module one must have $\psi^k\psi^l = \psi^l\psi^k$ on $\mathcal{B}$. But we can compute

$$\begin{aligned}\psi^k(\psi^l b) = \psi^k(l^r b + P_l a) &= l^r \cdot \psi^k(b) + P_l \cdot \psi^k a\\ &= l^r \cdot (k^r b + P_k a) + P_l k^s a\end{aligned}$$

and likewise

$$\begin{aligned}\psi^l(\psi^k b) = \psi^l(k^r b + P_k a) &= k^r \cdot \psi^l(b) + P_k \cdot \psi^l a\\ &= k^r \cdot (l^r b + P_l a) + P_k l^s a.\end{aligned}$$

Equating these expressions we find that $(l^r - l^s)P_k a = (k^r - k^s)P_l a$. Since a is infinite order in X it must be that

$$(l^r - l^s)P_k = (k^r - k^s)P_l,$$

and this holds for every two primes k and l. If $r = s$ this gives no condition and it is indeed true that any choice of $\mathbf{P}$ corresponds to a module X. But in the case $r \neq s$ we can write

$$\tfrac{P_l}{P_k} = \tfrac{l^r - l^s}{k^r - k^s}.$$

So once we fix a prime k, all other P_l's are determined by P_k. For convenience we take k to be the smallest prime, and obtain

$$P_l = P_2 \cdot \left(\tfrac{l^r - l^s}{2^r - 2^s}\right)$$

for every prime l. This shows that the module X depends on the single parameter P_2; however, it is still not true that all possible integral choices for P_2 correspond to $\mathcal{B}$-modules. Indeed, we will only get a $\mathcal{B}$-module if the above formula for P_l yields an *integer* for every choice of l. To this end define

$$Z_{r,s} = \left\{P \in \mathbb{Z} \,\middle|\, P \cdot \left(\tfrac{l^r - l^s}{2^r - 2^s}\right) \in \mathbb{Z}, \text{ for all primes } l\right\}.$$

For $P \in Z_{r,s}$ let X_P denote the corresponding $\mathcal{B}$-module for which $P_2 = P$.

Note that $Z_{r,s} \subseteq \mathbb{Z}$ is an ideal, and nonzero because it contains $2^r - 2^s$. In a moment we will compute this ideal in some examples. For now simply note that we have a map (in fact a surjection) $Z_{r,s} \to \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$ sending P to the extension (36.7) in which $X = X_P$. It is an exercise to check that this is indeed a map of abelian groups.

We next need to understand when X_P and X_Q are isomorphic as elements of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$. This is when there is a map of $\mathcal{B}$-modules $X_P \to X_Q$ yielding a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}(s) \arrow[r] \arrow[d, equal] & X_P \arrow[r] \arrow[d, "f"] & \mathbb{Z}(r) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}(s) \arrow[r] & X_Q \arrow[r] & \mathbb{Z}(r) \arrow[r] & 0
\end{tikzcd}$$

Such an f must satisfy $f(a) = a$, and $f(b) = b + Ja$ for some $J \in \mathbb{Z}$ (note that the symbols a and b are being used to simultaneously represent different elements of X_P and X_Q). The condition that f be a map of $\mathcal{B}$-modules is that $\psi^k(f(b)) = f(\psi^k b)$ for all primes k. For $k = 2$ the left-hand-side is

$$\psi^2(b + Ja) = 2^r b + Qa + J \cdot 2^s a$$

and the right-hand-side is

$$f(\psi^2 b) = f(2^r b + Pa) = 2^r(b + Ja) + Pa.$$

We obtain the condition

$$Q - P = (2^r - 2^s)J.$$

The reader may check as an exercise that the condition for the other ψ^k's follows as a consequence of this one.

The conclusion is that X_P and X_Q are isomorphic as elements of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$ precisely when $P - Q$ is a multiple of $2^r - 2^s$. The map $Z_{r,s} \to \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s))$ therefore descends to an isomorphism

$$Z_{r,s}/(2^r - 2^s) \xrightarrow{\cong} \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s)).$$

Finally, it remains to determine the group $Z_{r,s}/(2^r - 2^s)$. This is a cyclic group (since $Z_{r,s} \cong \mathbb{Z}$), and we need to find its order. To this end, note that the condition $P(\frac{l^r - l^s}{2^r - 2^s}) \in \mathbb{Z}$ is equivalent to

$$\tfrac{2^r - 2^s}{\gcd(2^r - 2^s, l^r - l^s)} \Big| P.$$

This is true for all l if and only if P is a multiple of

$$\mathrm{lcm}\Big\{ \tfrac{2^r - 2^s}{\gcd(2^r - 2^s, l^r - l^s)} \,\Big|\, l \text{ prime} \Big\} = \tfrac{2^r - 2^s}{\gcd(\{l^r - l^s \,|\, l \text{ prime}\})}.$$

So define

$$N_{r,s} = \gcd(2^r - 2^s, 3^r - 3^s, 5^r - 5^s, 7^r - 7^s, \ldots).$$

Then we have just determined that $Z_{r,s} = ((2^r - 2^s)/N_{r,s})$, and hence

$$\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s)) \cong (\tfrac{2^r - 2^s}{N_{r,s}})/(2^r - 2^s) \cong \mathbb{Z}/N_{r,s}.$$

We will explore the numbers $N_{r,s}$ in a moment, but we have already done enough to be able to solve the Hopf invariant one problem. So let us pause and tackle that first.

36.8. **Solution to Hopf invariant one.** We are ready to give the Adams-Atiyah [AA] solution to the Hopf invariant one problem:

Theorem 36.9. *If $f\colon S^{2n-1} \to S^n$ has Hopf invariant one then $n \in \{1, 2, 4, 8\}$.*

Proof. We assume $n > 1$ and prove that $n \in \{2, 4, 8\}$. We of course know that n is even, since otherwise the Hopf invariant is necessarily zero. Write $n = 2r$, and let X be the mapping cone of f. We have an exact sequence of $\mathcal{B}$-modules

$$0 \leftarrow \widetilde{K}^0(S^n) \leftarrow \widetilde{K}^0(X) \leftarrow \widetilde{K}^0(S^{2n}) \leftarrow 0$$

which has the form

$$0 \leftarrow \mathbb{Z}(r) \leftarrow \widetilde{K}^0(X) \leftarrow \mathbb{Z}(2r) \leftarrow 0.$$

Let $a \in \widetilde{K}^0(X)$ be the image of a chosen generator for $\widetilde{K}^0(S^{2n})$ and let $b \in \widetilde{K}^0(X)$ be an element that maps to a chosen generator of $\widetilde{K}^0(S^n)$. Then b^2 maps to 0 in $\widetilde{K}^0(S^n)$, so we have $b^2 = h \cdot a$ for a unique $h \in \mathbb{Z}$. A little thought shows that, up to sign, h is the Hopf invariant of the map f.

The key to the argument is the equivalence $\psi^2(b) \equiv b^2 \mod 2$ (Proposition 35.10(d)). Using our assumption that h is odd, this gives $\psi^2(b) \equiv a \mod$

2. However, recall our classification of extensions in $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(2r))$. Such extensions are determined by an integer $P_2 \in \mathbb{Z}$ satisfying

$$P_2 \cdot \left(\tfrac{l^{2r}-l^r}{2^{2r}-2^r}\right) \in \mathbb{Z} \tag{36.10}$$

for all primes l, where P_2 is defined by the equation $\psi^2 b = 2^r b + P_2 a$. Our assumption about the Hopf invariant of f now gives that P_2 is odd. But equation (36.10) says that

$$P_2 \cdot \tfrac{l^r}{2^r} \cdot \tfrac{l^r-1}{2^r-1} \in \mathbb{Z},$$

and if P_2 is odd and l is odd then this implies that $2^r | l^r - 1$.

Let us pause here and summarize. From the topology we have extracted a number-theoretic condition: if $n = 2r$ and $S^{2n-1} \to S^n$ has Hopf invariant one, then $2^r | l^r - 1$ for all odd primes l.

This number-theoretic condition is very restrictive, and it turns out just looking at $l = 3$ is enough to give us what we want. The lemma below shows that r lies in $\{1, 2, 4\}$, impying that our original n belongs to $\{2, 4, 8\}$ as desired. $\square$

Lemma 36.11. *If $2^r | 3^r - 1$ then $r \in \{0, 1, 2, 4\}$.*

Proof. Let $\nu(n)$ be the 2-adic valuation of an integer n: that is, $n = 2^{\nu(n)} \cdot (\text{odd})$. Here is a table showing the numbers $\nu(3^r - 1)$ for small values of r:

TABLE 36.12.

r	1	2	3	4	5	6	7	8	9	10	11	12
$\nu(3^r-1)$	1	3	1	4	1	3	1	5	1	3	1	4

The reader will reach the natural guess that $\nu(3^r - 1) = 1$ when r is odd, and this is easy to prove by working modulo 4. In $\mathbb{Z}/4$ we have $3 = -1$, and so $3^r = (-1)^r = -1$ when r is odd. Thus $3^r - 1 = 2$ in $\mathbb{Z}/4$, which confirms that $\nu(3^r - 1) < 2$.

When r is even the reader will note from the table that $\nu(3^r - 1)$ seems to grow quite slowly as a function of r. Again, this is easy enough to prove as soon as one has the idea to do so. If $r = 2u$ then

$$3^r - 1 = 3^{2u} - 1 = (3^u - 1)(3^u + 1).$$

Modulo 8 the powers of 3 are just 1 and 3, so the possible values for $3^u + 1$ are only 2 and 4. In particular, 8 does not divide $3^u + 1$: that is, $\nu(3^u + 1) < 3$ for all values of u. We therefore have $\nu(3^r - 1) \le \nu(3^u - 1) + 2$. If u is odd we stop here, otherwise we again divide by 2 and apply the same formula; a simple induction along these lines yields

$$\nu(3^r - 1) \le 1 + 2\nu(r).$$

The bound $1 + 2\nu(r)$ is generally substantially smaller than r. An easy exercise verifies that $r \le 1 + 2\nu(r)$ only when $r \in \{1, 2, 4\}$. So to summarize, we have shown that if $r \notin \{1, 2, 4\}$ then $1 + 2\nu(r) < r$; hence $\nu(3^r - 1) < r$, and so $2^r \nmid 3^r - 1$. $\square$

36.13. **Completion of the Ext calculation.** At this point we have finished with the solution to the Hopf invariant one problem. But there is another interesting problem that is still on the table, namely the exact computation of the groups

$$\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r), \mathbb{Z}(s)) \cong \mathbb{Z}/N_{r,s}.$$

We need to determine the numbers $N_{r,s} = \gcd(2^r - 2^s, 3^r - 3^s, 5^r - 5^s, \ldots)$. Note that $N_{r,s} = N_{s,r}$, so we can concentrate on the case $r \geq s$.

This calculation, of course, is intriguing from a purely algebraic perspective—when an answer comes down to finding one specific number, it would be difficult not to take the extra step and determine just what that number is. But the answer is also interesting for topological reasons. We have seen that if $f\colon S^{n+k-1} \to S^n$ where n and k are both even, then we get an extension $A(f) \in \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(\frac{n}{2}), \mathbb{Z}(\frac{n+k}{2}))$. A little work shows that this actually gives a group homomorphism

$$A\colon \pi_{n+k-1}(S^n) \to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathbb{Z}(\tfrac{n}{2}), \mathbb{Z}(\tfrac{n+k}{2})\big) \cong \mathbb{Z}/N_{\frac{n}{2},\frac{n+k}{2}}.$$

It is important to determine how large the target group is, and how close A is to being an isomorphism. This was investigated by Adams [A3].

We begin our investigation of the numbers $N_{r,s}$ by looking at $N_{r,r-1}$. Since $l^r - l^{r-1} = l^{r-1}(l-1)$ this is

$$N_{r,r-1} = \gcd(2^{r-1},\ 3^{r-1}\cdot 2,\ 5^{r-1}\cdot 4,\ 7^{r-1}\cdot 6,\ \ldots).$$

The 2^{r-1} in the first entry tells us that the gcd will be a power of 2, and the $3^{r-1}\cdot 2$ tells us that it will be at most 2^1. A moment's thought reveals that the gcd is precisely 2^1, as long as $r \geq 2$. When $r = 1$ the gcd is just 1:

$$N_{r,r-1} = \begin{cases} 1 & \text{if } r = 1, \\ 2 & \text{if } r \geq 2. \end{cases}$$

Next consider the numbers $N_{r,r-2}$, requiring us to look at $l^r - l^{r-2} = l^{r-2}(l^2-1)$. We have

$$\begin{aligned} N_{r,r-2} &= \gcd(2^{r-2}(2^2-1), 3^{r-2}(3^2-1), 5^{r-2}(5^2-1), 7^{r-2}(7^2-1), \ldots) \\ &= \gcd(2^{r-2}\cdot 3,\ 3^{r-2}\cdot 8,\ 5^{r-2}\cdot 24,\ 7^{r-2}\cdot 48,\ \ldots). \end{aligned}$$

From the first entry we see that the gcd will only have twos and threes in its factorization, with at most one 3. Later entries show that the gcd has at most 3 twos, and a brief inspection leads to the guess that the gcd is 24 as long as $r \geq 5$. To prove this we need to verify that $24 | l^2 - 1$ for primes $l > 3$. This is easy, though. Consider the numbers $l-1$, l, and $l+1$. At least one is a multiple of 3, and our hypotheses on l say that it isn't l. So 3 divides $(l-1)(l+1) = l^2 - 1$. Likwise, both $l-1$ and $l+1$ are even and at least one must be a multiple of 4: so $8 | l^2 - 1$ as well. The reader will now find it easy to check the following numbers:

$$N_{r,r-2} = \begin{cases} 1 & \text{if } r = 2, \\ 6 & \text{if } r = 3, \\ 12 & \text{if } r = 4, \\ 24 & \text{if } r \geq 5. \end{cases}$$

Remark 36.14. The two cases we have analyzed so far yield an evident conjecture: that $N_{r,r-t}$ is independent of r for $r \gg 0$. We will see below that this is indeed the case.

Let us work out two more cases before discussing the general pattern.

$$\begin{aligned} N_{r,r-3} &= \gcd(2^{r-3}(2^3-1),\ 3^{r-3}(3^3-1),\ 5^{r-3}(5^3-1),\ \ldots) \\ &= \gcd(2^{r-3}\cdot 7,\ 3^{r-3}\cdot 26,\ 5^{r-2}\cdot 124,\ \ldots). \end{aligned}$$

A very quick investigation shows that

$$N_{r,r-3} = \begin{cases} 1 & \text{if } r = 3, \\ 2 & \text{if } r \geq 4. \end{cases}$$

Moving to $N_{r,r-4}$ we have

$$\begin{aligned} N_{r,r-4} &= \gcd(2^{r-4}(2^4-1),\ 3^{r-4}(3^4-1),\ 5^{r-4}(5^4-1),\ \ldots) \\ &= \gcd(2^{r-4}\cdot 15,\ 3^{r-4}\cdot 80,\ 5^{r-4}\cdot 624,\ 7^{4-r}\cdot 2400,\ \ldots). \end{aligned}$$

The numbers are getting larger now, and it is harder to see the patterns. The relevant fact is that $l^4 - 1$ is a multiple of $2^4 \cdot 3 \cdot 5$ for all primes $l > 5$; and for $l = 2$ it is a multiple of $3 \cdot 5$, for $l = 3$ it is a multiple of $2^4 \cdot 5$, and for $l = 5$ it is a multiple of $2^4 \cdot 3$. We leave it as an exercise for the reader to prove this, using the factorization $l^4 - 1 = (l-1)(l+1)(l^2+1)$ and some easy number theory. The conclusion is that

$$N_{r,r-4} = \begin{cases} 1 & \text{if } r = 4, \\ 30 & \text{if } r = 5, \\ 60 & \text{if } r = 6, \\ 120 & \text{if } r = 7, \\ 240 & \text{if } r \geq 8. \end{cases}$$

By now it should be clear what the general pattern is, if not the specifics. To understand $N_{r,r-t}$ we consider the numbers

$$2^t - 1,\ 3^t - 1,\ 5^t - 1,\ 7^t - 1,\ \ldots$$

Excluding some finite set of primes at the beginning, there will be an "interesting" gcd to this set of numbers. When r is large the bad primes at the beginning become irrelevant to the computation, and so here $N_{r,r-t}$ is equal to the aforementioned "interesting" gcd. We encourage the reader to do some investigation on their own at this point. The "large r" values of $N_{r,r-t}$ are listed in the following table, together with their prime factorizations:

t	1	2	3	4	5	6	7	8	9	10	11	12
$N_{r,r-t}$	2	24	2	240	2	504	2	480	2	264	2	65520
p.f.	2	$2^3{\cdot}3$	2	$2^4{\cdot}5$	2	$2^3{\cdot}3^2{\cdot}7$	2	$2^5{\cdot}3{\cdot}5$	2	$2^3{\cdot}3{\cdot}11$	2	$2^4{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$

If you have been around the stable homotopy groups of spheres you will see some familiar numbers in this table, which might make you sit up and take notice. For example: $\pi_3^s \cong \mathbb{Z}/24$, $\pi_7^s \cong \mathbb{Z}/240$, $\pi_{11}^s \cong \mathbb{Z}/504$, and $\pi_{15}^s \cong \mathbb{Z}/960$ (note that the last one does not quite match). It is remarkable to have these numbers coming up in a purely algebraic computation! It turns out that what we are seeing here is the so-called "image of J". We will say more about this at a later time.

Fix $t \in \mathbb{Z}_+$. It turns out that there is a simple formula for the "stable" values of $N_{r,r-t}$, as a function of t. These stable values are also closely connected to the denominators of Bernoulli numbers. We close this section by explaining this.

Our examples have led to the hypothesis that the sequence of numbers

$$2^N(2^t-1),\ 3^N(3^t-1),\ 5^N(5^t-1),\ 7^N(7^t-1),\dots$$

has a greatest common divisor that is independent of N when $N \gg 0$. Our aim is to prove this, and to investigate this gcd. To this end, let $m_N(t)$ be this gcd:

$$m_N(t) = \gcd\{l^N(l^t-1) \,|\, l \text{ is prime}\}.$$

Notice the relation to our Ext-calculation is that when $r \geq s$ we have $N_{r,s} = m_s(r-s)$.

Also define

$$m'_N(t) = \gcd\{k^N(k^t-1) \,|\, k \in \mathbb{Z}_+\}.$$

Clearly $m'_N(t)$ divides $m_N(t)$, but in fact the two are equal:

Lemma 36.15. *For all t and N, $m'_N(t) = m_N(t)$.*

Proof. It will suffice to show that $m_N(t)$ divides $m'_N(t)$, or equivalently that every prime-power factor of the former is also a factor of the latter. So let p be a prime and suppose $p^e | m_N(t)$. Then $p^e | p^N(p^t-1)$, so $e \leq N$. For any l such that $(l,p)=1$ we have $p^e | l^t-1$, so $l^t = 1$ in $\mathbb{Z}/p^e$.

Now let $k \in \mathbb{Z}$ with $k \geq 2$. If $p \mid k$ then $p^e \mid k^N(k^t-1)$ since $e \leq N$. If $p \nmid k$ then write $k = l_1 l_2 \dots l_r$ where each l_r is a prime different from p. We know that $l_i^t = 1$ in $\mathbb{Z}/p^e$ for each i, and so $k^t = l_1^t l_2^t \dots l_r^t = 1$ in $\mathbb{Z}/p^e$ as well. That is, $p^e | k^t - 1$. We have therefore shown that $p^e | m'_N(t)$, which is what we wanted. $\square$

The next proposition proves that $m_N(t)$ stabilizes for $N \gg 0$, and it also determines an explicit formula for the stable value in terms of the prime factorization of t. Let $\nu_p(t)$ denote the exponent of the prime p in the prime factorization of t.

Proposition 36.16. *Let L be the supremum of all exponents in the prime factorization of t. Then $m_N(t)$ is independent of N for $N \geq L+2$. If we call this stable value $m(t)$ then*

(a) $m(t) = 2$ when t is odd;

(b) When t is even $m(t) = 2^{2+\nu_2(t)} \cdot \prod_{p \text{ odd},(p-1)|t} p^{1+\nu_p(t)}$.

(c) More generally,

$$m_N(t) = 2^{\min\{2+\nu_2(t),N\}} \cdot \prod_{p \text{ odd},(p-1)|t} p^{\min\{1+\nu_p(t),N\}}.$$

Remark 36.17. The notation $m(t)$ comes from Adams [A3]. Note that the proposition completes our Ext-calculations, via the formula $N_{r,s} = m_s(r-s)$. But see also ??? for more about the Ext groups.

Before proving the proposition let us look at a couple of examples. To compute $m(50)$ we write $50 = 2 \cdot 5^2$. Next we make a list of all odd primes p such that $p-1$ divides 50; these are 3 and 11. So

$$m(50) = 2^3 \cdot 3 \cdot 11 = 264.$$

For a harder example let us compute $m(12)$. We write $12 = 2^2 \cdot 3$, and our list of odd p such that $p-1$ divides 12 is 3, 5, 7, and 13. So

$$m(12) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 65520.$$

Further, we have $m(12) = m_4(12)$ and

$$\begin{aligned} m_3(12) &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \\ m_2(12) &= 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \\ m_1(12) &= 2^1 \cdot 3^1 \cdot 5 \cdot 7 \cdot 13 \\ m_0(12) &= 1. \end{aligned}$$

Here are several more values of m for the reader's curiosity (the numbers are, of course, better understood in terms of their prime factorizations):

t	2	4	6	8	10	12	14	16	18
$m(t)$	24	240	504	480	264	65520	24	16320	28728

To prove Proposition 36.16 we need a lemma from algebra. Most basic algebra courses prove that the group of units in $\mathbb{Z}/p$ is a cyclic group, necessarily isomorphic to $\mathbb{Z}/(p-1)$. One can also completely describe the group of units inside the ring $\mathbb{Z}/p^e$, for any e. Recall that this group simply consists of all residue classes of integers k such that $(p,k) = 1$. Here is the result:

Lemma 36.18. *Fix a prime p and consider the group of units $(\mathbb{Z}/p^e)^*$ inside the ring $\mathbb{Z}/p^e$.*

(a) If p is odd then $(\mathbb{Z}/p^e)^* \cong \mathbb{Z}/((p-1)p^{e-1}) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p^{e-1})$.

(b) If $e \geq 2$ then $(\mathbb{Z}/2^e)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/(2^{e-2})$. Here the $\mathbb{Z}/2$ is the subgroup $\{1, -1\}$ and the $\mathbb{Z}/(2^{e-2})$ is the subgroup of all numbers congruent to 1 mod 4.

(c) $(\mathbb{Z}/2)^* = \{1\}$.

Proof. We first recall the proof that $(\mathbb{Z}/p)^*$ is cyclic. If a finite abelian group is noncyclic, then it contains a subgroup isomorphic to $\mathbb{Z}/k \times \mathbb{Z}/k$, for some prime k (this follows readily from the structure theorem for finite abelian groups). But if this were true for $(\mathbb{Z}/p)^*$ then the field $\mathbb{Z}/p$ would have k^2 solutions to the polynomial $x^k - 1$, and this is a contradiction.

Assume that p is odd. Reduction modulo p gives a surjective map $(\mathbb{Z}/p^e)^* \to (\mathbb{Z}/p)^*$. Let K denote the kernel. Note that $(\mathbb{Z}/p^e)^*$ coincides with the set

$$\{1, 2, \ldots, p^e - 1\} - \{p, 2p, 3p, \ldots, (p^{e-1} - 1)p\}$$

and so has order $p^e - p^{e-1}$. Thus, $|K| = p^{e-1}$. It remains to show that K is cyclic, and for this it suffices to verify that K has exactly $p-1$ elements of order p. Let $a \in K - \{1\}$, and let the base p representation of a be

$$a = 1 + a_f p^f + a_{f+1} p^{f+1} + \cdots + a_{e-1} p^{e-1},$$

for $0 \leq a_i < p$ and $a_f \neq 0$ (note that $a_0 = 1$ by the definition of K). Write $b = a_f + a_{f+1} p + a_{f+2} p^2 + \cdots$, so that

$$a^p = (1 + p^f b)^p = 1 + p \cdot p^f b + \tbinom{p}{2} p^{2f} b^2 + \cdots$$

The terms after $p^{1+f} b$ all contain at least $f + 2$ factors of p; so modulo p^{f+2} one has $a^p \equiv 1 + p^{f+1} b \equiv 1 + a_f p^{f+1}$. So we can have $a^p = 1$ in $\mathbb{Z}/p^e$ only if $f = e - 1$.

Thus, we have shown that the elements of K that are pth roots of unity are precisely the elements $1 + ap^{e-1}$, for $0 \le a < p$. In particular, we have only $p - 1$ of these (excluding the identity element). This completes the proof.

The proof for $p = 2$ is similar. Of course $(\mathbb{Z}/4)^* \cong \mathbb{Z}/2$. For $e \ge 3$ consider the sequence $0 \to K \to (\mathbb{Z}/2^e)^* \to (\mathbb{Z}/4)^* \to 0$, where the right map is reduction modulo 4. This reduction map is split-surjective, with the splitting sending the generator of $(\mathbb{Z}/4)^*$ to -1. The proof that K is cyclic proceeds exactly as in the odd primary case. □

Proof of Proposition 36.16. Let p be an odd prime. Then one has

$$\begin{aligned} p^e \mid m_N(t) &\iff p^e \mid p^N(p^t - 1) \text{ and } p^e \mid l^N(l^t - 1) \text{ for all primes } l \neq p \\ &\iff e \le N \text{ and } p^e \mid l^t - 1 \text{ for all primes } l \neq p \\ &\iff e \le N \text{ and } p^e | k^t - 1 \text{ for all } k \in \mathbb{Z}_+ \text{ such that } p \nmid k \\ &\iff e \le N \text{ and all units in } \mathbb{Z}/p^e \text{ are } t\text{th roots of } 1 \\ &\iff e \le N \text{ and } (p-1)p^{e-1} | t. \end{aligned}$$

The third equivalence is by Lemma 36.15, and in the last equivalence we have used that $(\mathbb{Z}/p^e)^* \cong \mathbb{Z}/((p-1)p^{e-1})$. This last line shows why N is redundant when it is large enough: the condition $p^{e-1}|t$ already forces $e \le \nu_p(t) + 1$, and so $e \le N$ is redundant if $N \ge \nu_p(t) + 1$.

Assuming now that N is large enough so that we are in the stable case, the above equivalences show that $p|m(t)$ only when $p - 1|t$; and also that in this case $\nu_p(m(t)) = 1 + \nu_p(t)$.

The analysis of $p = 2$ is very similar. One finds

$$2^e \mid m_N(t) \iff e \le N \text{ and all units in } \mathbb{Z}/2^e \text{ are } t\text{th roots of } 1.$$

When $e = 1$ the latter condition is just $e \le N$. When $e > 1$ the latter condition is equivalent to $e \le N$ and $2^{e-2}|t$, using Lemma 36.18(b). This readily yields the desired result. □

Our final task is to make the connection between the numbers $m(t)$ and the Bernoulli numbers. For a review of the Bernoulli numbers and their basic properties, see Appendix A. The result we are after is the following:

Theorem 36.19. *When t is even, $m(t)$ is the denominator of the fraction $\frac{B_t}{2t}$ when expressed in lowest terms.*

The following table demonstrates this result in the first few cases:

t	2	4	6	8	10	12
$\lvert B_t \rvert$	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$
$\frac{\lvert B_t \rvert}{2t}$	$\frac{1}{24}$	$\frac{1}{240}$	$\frac{1}{504}$	$\frac{1}{480}$	$\frac{1}{264}$	$\frac{691}{65520}$
$m(t)$	24	240	504	480	264	65520

It should be remarked, perhaps, that this connection between $m(t)$ and Bernoulli numbers is worth more in effect than it is in practical value. The explicit formula for $m(t)$ from Proposition 36.16 is much more useful than its description as the denominator of $B_t/(2t)$. Moreover, the denominators of Bernoulli numbers are very simple: for B_n it is simply the product of all primes p such that $p-1$ divides n. So in the end the result of Theorem 36.19 is neither deep nor particularly useful. Still, it provides a nice-sounding connection between topology and number theory.

Proof of Proposition 36.19. By the theorem of von Staudt and Claussen (Theorem C.5) we know that the denominator of B_t (in lowest terms) is the product of all primes p such that $p-1$ divides t. Note that one such prime is $p=2$, so the denominator is even (in fact, congruent to 2 modulo 4) and the numerator is odd. Note also that these are the same primes appearing in the factorization of $m(t)$, by Proposition 36.16(b).

Consider now $\alpha = B_t/(2t)$. Since the numerator of B_t is odd, the number of twos in the denominator of α is $1+1+\nu_2(t)$. This is the same as the number of twos in the prime factorization of $m(t)$.

For every prime p such that $p-1|t$ we have one p appearing in the denominator of B_t and $\nu_p(t)$ of them appearing in t, so the total number of p's in the denominator of α is $1+\nu_p(t)$. At this point we have thereby shown that $m(t)|\operatorname{den}(\alpha)$.

It remains to check that if $p^e \mid t$ and $p-1 \nmid t$ then p^e divides the numerator of B_t and therefore disappears from the denominator of α. This is the nontrivial part of the proof.

Let p be an odd prime in the denominator of α, appearing with multiplicity e. Then p^e is also in the denominator of B_t/t. By Proposition C.6 we know that

$$\tfrac{k^t(k^t-1)B_t}{t} \in \mathbb{Z}$$

for all $k \in \mathbb{Z}$. Consequently we have $p^e|k^t(k^t-1)$ for all $k \in \mathbb{Z}$. But this exactly says that p^e divides the gcd $m(t)$. This verifies that the 'odd part' of $\operatorname{den}(\alpha)$ divides $m(t)$, and we have already checked the factors of 2 in a previous paragraph. So $\operatorname{den}(\alpha) \mid m(t)$, and therefore the two are equal. $\square$

Exercise 36.20. Let us return to our original motivation, which was to understand the groups $\operatorname{Ext}_{\mathcal{B}}(\mathbb{Z}(r),\mathbb{Z}(s))$. As part of our computation we discovered two surprises: the groups are symmetric in r and s, and there is a "stability" phenomenon related to s being large compared to $r-s$. In this exercise we will identify some algebraic structure behind these phenomena.

(a) If M is a $\mathcal{B}$-module then $M^* = \operatorname{Hom}_{\mathcal{A}b}(M,\mathbb{Z})$ becomes a $\mathcal{B}$-module by precomposition: if $f\colon M \to \mathbb{Z}$ then define $\psi^k.f = f \circ \psi^k$. Note that $\mathbb{Z}(r)^* \cong \mathbb{Z}(r)$, and that if $0 \to \mathbb{Z}(s) \to M \to \mathbb{Z}(r) \to 0$ is an exact sequence then so is $0 \leftarrow \mathbb{Z}(s)^* \leftarrow M^* \leftarrow \mathbb{Z}(r)^* \leftarrow 0$. Verify that this gives an isomorphism between $\operatorname{Ext}_{\mathcal{B}}(\mathbb{Z}(r),\mathbb{Z}(s))$ and $\operatorname{Ext}_{\mathcal{B}}(\mathbb{Z}(s),\mathbb{Z}(r))$.

(b) If M and N are $\mathcal{B}$-modules verify that the abelian group $M \otimes_{\mathbb{Z}} N$ becomes a $\mathcal{B}$-module via the formula $\psi^k(m \otimes n) = (\psi^k m) \otimes (\psi^k n)$. Check that $\mathbb{Z}(r)$ is a flat $\mathcal{B}$-module, so that we get maps

$$\operatorname{Ext}_{\mathcal{B}}(\mathbb{Z}(r),\mathbb{Z}(s)) \xrightarrow{\otimes\mathbb{Z}(1)} \operatorname{Ext}_{\mathcal{B}}(\mathbb{Z}(r+1),\mathbb{Z}(s+1)) \xrightarrow{\otimes\mathbb{Z}(1)} \cdots$$

Verify that the maps in this sequence eventually become isomorphisms.

37. Calculation of KO for stunted projective spaces

The goal in this section is to determine $KO^0(\mathbb{R}P^n)$ and $KO^0(\mathbb{R}P^n/\mathbb{R}P^a)$ for all values of n and a, together with the Adams operations on these groups. These computations are the key to solving the vector fields on spheres problem (introduced in

Section 15), which we do in the next section. As intermediate steps we also compute $K^*(\mathbb{R}P^n)$ and $K^*(\mathbb{R}P^n/\mathbb{R}P^a)$. The original written source for this material is Adams [Ad2] (although he acknowledges unpublished work of Atiyah-Todd and Bott-Shapiro for portions of the calculation). We follow Adams's approach very closely.

Some words of warning about this material are in order. The complete calculation of KO for stunted projective spaces is fairly involved. Several things end up going on at once, so that there is a bunch of stuff to keep track of. And calculations are just never very fun to read in the first place. We have attempted to structure our presentation to try to help with this, but of course it only goes so far. After some preliminary material we give a section which has just the statements of the results, with a minimal amount of discussion in between (and no proofs). The intent is to give the reader the general picture, and also a convenient reference section. All of the proofs are then given in a subsequent section.

Some readers might want to skip the proofs the first time through, and this is not a problem. Later applications in the text only need the results, not details from the proofs. However, I **highly recommend** that algebraic topologists go through the proofs carefully at an early stage in their career. I cannot stress this enough. Going through the proofs will teach you something important about this subject that I do not have words for, and it will open up doors for you down the road. Trust me that this is an important thing to do.

37.1. **Initial material.** The results in this section will make heavy use of the interplay between $\mathbb{R}P^n$ and $\mathbb{C}P^n$ in the homotopy category of spaces. We begin by reviewing the basics of what we will need.

Let η be the tautological complex line bundle on $\mathbb{C}P^n$, and let L be the tautological real line bundle on $\mathbb{R}P^n$. Let $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ be the inclusion.

Lemma 37.2. *The complexification of L is the pullback of η: that is, $cL \cong j^*\eta$.*

Proof. Complex line bundles on a space X are classified by homotopy classes in $[X, \mathbb{C}P^\infty]$. The real line bundle L is classified by the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$, so the complexificaton of L is classified by the composition $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$. The complex line bundle η is classified by $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$, so $j^*\eta$ is classified by the composition $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$. The result follows from the commutative diagram

$$\begin{tikzcd} \mathbb{R}P^n \arrow[r] \arrow[d] & \mathbb{C}P^n \arrow[d] \\ \mathbb{R}P^\infty \arrow[r] & \mathbb{C}P^\infty. \end{tikzcd}$$

□

Note that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so complex line bundles on X are classified by $[X, \mathbb{C}P^\infty] = H^2(X)$. If $n > 1$ then $H^2(\mathbb{R}P^n) = \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles: the trivial bundle and the nontrivial bundle. The bundle $cL = j^*\eta$ is nontrivial, since its classification map $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty \hookrightarrow \mathbb{C}P^\infty$ represents the generator of $H^2(\mathbb{R}P^n)$.

Remark 37.3. The powers η^k are all distinct line bundles on $\mathbb{C}P^n$ (e.g., the first Chern classes are $c_1(\eta^k) = kc_1(\eta) = kx$ where x is the generator of $H^2(\mathbb{C}P^n)$). The situation upon pulling back to $\mathbb{R}P^n$ is very different, however. We have

$$[j^*\eta]^2 = (cL)^2 = c(L^2) = c(1_{\mathbb{R}}) = 1_{\mathbb{C}}.$$

So the even powers of $j^*\eta$ are all trivial, and the odd powers are just $j^*\eta$. We will see that this accounts for the main difference between $K^0(\mathbb{C}P^n)$ and $K^0(\mathbb{R}P^n)$.

In addition to the inclusion $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ there is another interesting map from real to complex projective space. Every real line in $\mathbb{C}^{n+1}$ determines a complex line by taking the $\mathbb{C}$-linear span, and therefore we get a map $\mathbb{P}_{\mathbb{R}}(\mathbb{C}^{n+1}) \to \mathbb{P}_{\mathbb{C}}(\mathbb{C}^{n+1}) = \mathbb{C}P^n$. It is easy to see that this is a fiber bundle with fiber S^1 (the space of real lines in $\mathbb{C}$). Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ shows that the domain is homeomorphic to $\mathbb{R}P^{2n+1}$, giving us a fiber bundle

$$S^1 \longrightarrow \mathbb{R}P^{2n+1} \xrightarrow{q} \mathbb{C}P^n.$$

In terms of homogeneous coordinates, q sends the point $[x_0 : x_1 : \cdots : x_{2n} : x_{2n+1}]$ to $[x_0 + ix_1 : x_2 + ix_3 : \cdots : x_{2n} + ix_{2n+1}]$.

Lemma 37.4. *The diagram*

$$\begin{tikzcd} \mathbb{R}P^n \arrow[r, "i"] \arrow[dr, "j"'] & \mathbb{R}P^{2n+1} \arrow[d, "q"] \\ & \mathbb{C}P^n \end{tikzcd}$$

commutes up to homotopy, where i is the standard inclusion. Consequently, q^* sends $x \in H^2(\mathbb{C}P^n)$ to the nonzero element of $H^2(\mathbb{R}P^{2n+1})$. (The latter statement can also be proven via the Serre spectral sequence for q).

Proof. The diagram commutes on the nose if i is replaced by the inclusion sending $[x_0 : x_1 : \cdots : x_n]$ to $[x_0 : 0 : x_1 : 0 : \cdots : x_n : 0]$. But all linear inclusions from one projective space to another are homotopic. $\square$

Corollary 37.5. *There is an isomorphism of bundles $q^*\eta \cong cL$.*

Proof. As we have remarked before, there are only two isomorphism classes of complex bundles on $\mathbb{R}P^{2n+1}$. Since $j^*\eta = i^*(q^*\eta)$ is not trivial, the bundle $q^*\eta$ cannot be trivial. So the only possibility is $q^*\eta \cong cL$. $\square$

Note that for $a \le n$ one has the following commutative diagram

$$\begin{tikzcd} \mathbb{R}P^{2a+1} \arrow[r, "i"] \arrow[d, "q"'] & \mathbb{R}P^{2n+1} \arrow[d, "q"] \\ \mathbb{C}P^a \arrow[r, "i"] & \mathbb{C}P^n, \end{tikzcd}$$

and therefore q induces a map $\mathbb{R}P^{2n+1}/\mathbb{R}P^{2a+1} \to \mathbb{C}P^n/\mathbb{C}P^a$. We will also use q to denote this induced map on quotients, as well as various small modifications.

Note that these induced maps fit together to give a big commutative diagram

(37.6)
$$\begin{tikzcd}
S^{2a+2} \arrow[r, equal] & \mathbb{R}P^{2a+2}/\mathbb{R}P^{2a+1} \arrow[d, tail] & & \\
& \mathbb{R}P^{2a+3}/\mathbb{R}P^{2a+1} \arrow[r, "q"] \arrow[d, tail] & \mathbb{C}P^{a+1}/\mathbb{C}P^{a} \arrow[r, equal] \arrow[d, tail] & S^{2a+2} \\
& \vdots \arrow[d, tail] & \vdots \arrow[d, tail] & \\
& \mathbb{R}P^{2n+1}/\mathbb{R}P^{2a+1} \arrow[r, "q"] \arrow[d, tail] & \mathbb{C}P^{n}/\mathbb{C}P^{a} \arrow[dd, tail] & \\
& \mathbb{R}P^{2n+2}/\mathbb{R}P^{2a+1} \arrow[d, tail] & & \\
& \mathbb{R}P^{2n+3}/\mathbb{R}P^{2a+1} \arrow[r, "q"] \arrow[d, tail] & \mathbb{C}P^{n+1}/\mathbb{C}P^{a} \arrow[d, tail] & \\
& \vdots \arrow[d, tail] & \vdots \arrow[d, tail] & \\
& \mathbb{R}P^{\infty}/\mathbb{R}P^{2a+1} \arrow[r, "q"] & \mathbb{C}P^{\infty}/\mathbb{C}P^{a}. &
\end{tikzcd}$$

We will tend to use 'q' as a name for any composition of maps from this diagram that involves one horizontal step.

Notice that at the very top of the diagram we have a map from S^{2a+2} to itself. The next result identifies this map:

Lemma 37.7. *For $n \geq 1$ the composite*

$$S^{2n} \cong \mathbb{R}P^{2n}/\mathbb{R}P^{2n-1} \hookrightarrow \mathbb{R}P^{2n+1}/\mathbb{R}P^{2n-1} \xrightarrow{q} \mathbb{C}P^{n}/\mathbb{C}P^{n-1} \cong S^{2n}$$

is a homeomorphism.

Proof. The space $\mathbb{R}P^{2n}/\mathbb{R}P^{2n-1}$ consists of the basepoint and the affine space $\mathbb{R}^{2n}$ made up of points $[x_0 : x_1 : \cdots : x_{2n-2} : x_{2n-1} : 1]$. Likewise, the space $\mathbb{C}P^{n}/\mathbb{C}P^{n-1}$ consists of the basepoint and the affine space $\mathbb{C}^n$ made up of the points $[z_0 : z_1 : \cdots : z_{n-1} : 1]$. One readily uses the formula for q to see that it gives a continuous bijective correspondence, and is therefore a homeomorphism (the spaces involved being compact and Hausdorff). $\square$

37.8. **The main results.** Here we state the main theorems about the K-theory of real and complex projective spaces. The proofs will be deferred until the next section.

Theorem 37.9 (Complex K-theory of complex projective spaces and stunted projective spaces)**.** *Let η be the tautological line bundle on $\mathbb{C}P^n$, and write $\mu = [\eta]-1 \in \widetilde{K}^0(\mathbb{C}P^n)$.*

(a) $K^0(\mathbb{C}P^n) = \mathbb{Z}[\mu]/(\mu^{n+1})$ and $K^1(\mathbb{C}P^n) = 0$.

(b) The Adams operations on $K^0(\mathbb{C}P^n)$ are given by

$$\psi^k(\mu^s) = \left[(1+\mu)^k - 1\right]^s = k^s\mu^s + s\tbinom{k}{2}k^{s-1}\mu^{s+1} + \textit{(higher order terms)}.$$

(c) *The sequence* $0 \to \widetilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^a) \to \widetilde{K}^0(\mathbb{C}P^n) \to \widetilde{K}^0(\mathbb{C}P^a) \to 0$ *is exact, and identifies* $\widetilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^a)$ *as the free abelian group* $\mathbb{Z}\langle \mu^{a+1}, \mu^{a+2}, \dots, \mu^n\rangle \subseteq K^0(\mathbb{C}P^n)$. *The ring structure and action of the Adams operations are determined by the corresponding structures on* $K^0(\mathbb{C}P^n)$. *Also,* $K^1(\mathbb{C}P^n/\mathbb{C}P^a) = 0$.

Remark 37.10. Following Adams [Ad2] we write $\mu^{(i)}$ for the element of $K^0(\mathbb{C}P^n/\mathbb{C}P^a)$ that maps to μ^i in $K^0(\mathbb{C}P^n)$. The extra parentheses in the exponent remind us that this class is not a true ith power in the ring $K^0(\mathbb{C}P^n/\mathbb{C}P^a)$.

Theorem 37.11 (Complex K-theory of real projective spaces)**.** *Let* ν *be the element* $[j^*\eta]-1 \in \widetilde{K}^0(\mathbb{R}P^n)$, *where as usual* j *is the standard inclusion* $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$.

(a) $\widetilde{K}^0(\mathbb{R}P^n) \cong \mathbb{Z}/\big(2^{\lfloor \frac{n}{2}\rfloor}\big)$ *with generator* ν. *The ring structure has* $\nu^2 = -2\nu$ *and* $\nu^{f+1} = 0$, *where* $f = \lfloor \frac{n}{2} \rfloor$.

(b) $K^1(\mathbb{R}P^n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathbb{Z} & \text{if } n \text{ is odd.}\end{cases}$

(c) *The Adams operations on* $K^0(\mathbb{R}P^n)$ *are given by*

$$\psi^k(\nu^e) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \nu^e & \text{if } k \text{ is odd.}\end{cases}$$

Recall that we have calculated $\widetilde{K}^0(\mathbb{C}P^n/\mathbb{C}P^{a-1})$ to be the free abelian group $\mathbb{Z}\langle \mu^{(a)}, \mu^{(a+1)}, \dots, \mu^{(n)}\rangle$. For $k \leq 2n+1$ we may pull back these classes along the map $q\colon \mathbb{R}P^k/\mathbb{R}P^{2a-1} \to \mathbb{C}P^n/\mathbb{C}P^{a-1}$ to get elements of $\widetilde{K}^0(\mathbb{R}P^k/\mathbb{R}P^{2a-1})$. We again follow Adams [Ad2] and set

$$\bar{\nu}^{(t)} = q^*(\mu^{(t)}) \in \widetilde{K}^0(\mathbb{R}P^k/\mathbb{R}P^{2a-1})$$

for $a \leq t \leq n$. Note that these elements correspond nicely as k and n vary, due to the commutative diagram (37.6). We may as well take $n \mapsto \infty$ so that we have classes $\bar{\nu}^{(t)}$ for all $t \geq a$.

We claim that upon pulling back along the projection $\pi\colon \mathbb{R}P^k \to \mathbb{R}P^k/\mathbb{R}P^{2a-1}$ we have $\pi^*(\bar{\nu}^{(t)}) = \nu^t$; this explains our choice of notation. To prove this claim we can deal with the cases $k = 2u$ and $k = 2u+1$ simultaneously. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{R}P^{2u}/\mathbb{R}P^{2a-1} \arrow[r, tail] & \mathbb{R}P^{2u+1}/\mathbb{R}P^{2a-1} \arrow[r, "q"] & \mathbb{C}P^u/\mathbb{C}P^{a-1} \\
\mathbb{R}P^{2u} \arrow[r, tail] \arrow[u, "\pi"] & \mathbb{R}P^{2u+1} \arrow[r, "q"] \arrow[u, "\pi"] & \mathbb{C}P^u. \arrow[u, "\pi_{\mathbb{C}}"']
\end{tikzcd}$$

We have $\pi^*(q^*(\mu^{(t)})) = q^*(\pi_{\mathbb{C}}^*(\mu^{(t)})) = q^*(\mu^t) = (q^*\mu)^t = \nu^t$, where the last equality is by Corollary 37.5 and Lemma 37.2.

Observe that $\widetilde{K}^0(\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1}) = \widetilde{K}^0(S^{2a}) \cong \mathbb{Z}$ and the class $\bar{\nu}^{(a)}$ is a generator. This follows because $\mu^{(a)}$ generates $\widetilde{K}^0(\mathbb{C}P^a/\mathbb{C}P^{a-1}) \cong \mathbb{Z}$ and $\bar{\nu}^{(a)}$ is the pullback of $\mu^{(a)}$ along the map $\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1} \to \mathbb{C}P^a/\mathbb{C}P^{a-1}$, which by Lemma 37.7 is a homotopy equivalence.

For $n \geq 2a$ the inclusion $i\colon \mathbb{R}P^{2a}/\mathbb{R}P^{2a-1} \hookrightarrow \mathbb{R}P^n/\mathbb{R}P^{2a-1}$ induces a map

$$\mathbb{Z} = \widetilde{K}^0(S^{2a}) = \widetilde{K}^0(\mathbb{R}P^{2a}/\mathbb{R}P^{2a-1}) \xleftarrow{i^*} \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2a-1}).$$

This map is surjective, because the class $\bar{\nu}^{(a)}$ in the domain maps to the class $\bar{\nu}^{(a)}$ in the target, and the latter is a generator. In particular, not only do we know that the above map i^* is surjective but the class $\bar{\nu}^{(a)}$ in the domain gives a choice of splitting. This will be the main use of these $\bar{\nu}$ classes.

Theorem 37.12 (Complex K-theory of real stunted projective spaces)**.**
Suppose $n > a > 0$.

(a) *If $a = 2t$ then the sequence*

$$0 \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{K}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^a) \to 0$$

is exact. It identifies $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ with the subgroup of $\widetilde{K}^0(\mathbb{R}P^n)$ generated by $\nu^{t+1} = (-2)^t\nu$. As a group, $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z}/(2^g)$ where $g = \lfloor \frac{n-a}{2} \rfloor$. The ring structure and Adams operations are determined by the structures in $K^0(\mathbb{R}P^n)$. In particular, ψ^k acts as zero when k is even and as the identity when k is odd. We write $\nu^{(i)}$ for the element of $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ that maps to ν^i in $\widetilde{K}^0(\mathbb{R}P^n)$, so that $\nu^{(t+1)}$ is a generator.

(b) *Let $a = 2t - 1$. The cofiber sequence $\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1} \to \mathbb{R}P^n/\mathbb{R}P^{2t-1} \to \mathbb{R}P^n/\mathbb{R}P^{2t}$ induces a sequence*

$$0 \leftarrow \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) \leftarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \leftarrow \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t}) \leftarrow 0$$

that is short exact. Consequently, $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \cong \mathbb{Z} \oplus \mathbb{Z}/(2^f)$ where $f = \lfloor \frac{n}{2} \rfloor - t$; the former summand is generated by $\bar{\nu}^{(t)}$ and the latter summand is generated by $\nu^{(t+1)}$.

(c) *The action of ψ^k on $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1})$ is given by*

$$\psi^k(\nu^{(t+1)}) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \nu^{(t+1)} & \text{if } k \text{ is odd,} \end{cases}$$

and

$$\psi^k(\bar{\nu}^{(t)}) = k^t\bar{\nu}^{(t)} + \begin{cases} \frac{1}{2}k^t\nu^{(t+1)} & \text{if } k \text{ is even} \\ \frac{1}{2}(k^t - 1)\nu^{(t+1)} & \text{if } k \text{ is odd.} \end{cases}$$

Most of the content to the above theorem is represented in the following convoluted but useful diagram:

$$\begin{tikzcd}
 & \mathbb{Z}/\big(2^{\lfloor \frac{n}{2} \rfloor}\big) = \widetilde{K}(\mathbb{R}P^n) & & \widetilde{K}(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) = \mathbb{Z} & \\
\cdots & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{2t-2}) \arrow[l] \arrow[u, hook] & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \arrow[l] \arrow[ul] \arrow[ur, two heads] & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{2t}) \arrow[l, hook'] \arrow[ull] & \arrow[l] \\
 & \mathbb{Z}/(2^{B+1}) \arrow[u, equal] & \mathbb{Z} \oplus \mathbb{Z}/(2^B) \arrow[l, two heads] \arrow[u, equal] & \mathbb{Z}/(2^B) \arrow[l, tail] \arrow[u, equal] & {\scriptstyle B = \lfloor \frac{n}{2} \rfloor - t} \\
 & \nu^{(t)} & \bar{\nu}^{(t)}, \quad \nu^{(t+1)} \arrow[l] \arrow[dl, bend left] & \nu^{(t+1)} \arrow[l] & \\
 & \nu^{(t+1)} = -2\nu^{(t)} & & &
\end{tikzcd}$$

The indicated maps are injections/surjections, and our chosen generators of the groups are written in the bottom two lines. The generators $\nu^{(t+1)}$ and $\bar{\nu}^{(t)}$

map to the elements ν^{t+1} and ν^t in $\widetilde{K}(\mathbb{R}P^n)$, and $\bar{\nu}^{(t)}$ maps to a generator of $\widetilde{K}(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1}) = \widetilde{K}(S^{2t})$. The action of the Adams operations on $\nu^{(t+1)}$ is completely determined by what happens in $\widetilde{K}(\mathbb{R}P^n)$. Likewise, the action on $\bar{\nu}^{(t)}$ is completely determined by using the surjection onto $\widetilde{K}(S^{2t})$ together with the map into $\widetilde{K}(\mathbb{R}P^n)$. These are instructive exercises; but if necessary see the proofs in Section 37.22 for details.

Remark 37.13. The action of ψ^k on the element $\bar{\nu}^{(t)}$ is of crucial importance to the solution of the vector fields on spheres problem. See Section 38.

We now move from the realm of K-theory to KO-theory. Recall that $L \to \mathbb{R}P^n$ always denotes the tautological line bundle.

Theorem 37.14 (Real K-theory of real projective spaces)**.**
$\widetilde{KO}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^f)$ where $f = \#\{s \,|\, 0 < s \leq n,\ s \equiv 0, 1, 2,$ or 4 mod $8\}$. The group is generated by $\lambda = [L] - 1$, which satisfies $\lambda^2 = -2\lambda$ and $\lambda^{f+1} = 0$. The Adams operations are given by

$$\psi^k(\lambda) = \begin{cases} 0 & k \text{ even}, \\ \lambda & k \text{ odd}. \end{cases}$$

Remark 37.15. Because this number comes up so often, let us define

$$\varphi(n) = \#\{s \,|\, 0 < s \leq n,\ s \equiv 0, 1, 2, \text{ or } 4 \text{ mod } 8\}.$$

The following chart shows the groups $\widetilde{KO}^0(\mathbb{R}P^n)$ and $\widetilde{K}^0(\mathbb{R}P^n)$ as functions of n. To save space we write $\mathbb{Z}_n$ instead of $\mathbb{Z}/n$; but all the groups are cyclic, and so really one only needs to keep track of the order.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
KO	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{128}$	$\mathbb{Z}_{128}$	$\mathbb{Z}_{128}$
K	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{64}$	$\mathbb{Z}_{128}$

Observe that the $\widetilde{KO}^0(\mathbb{R}P^n)$ groups follow the by-now-familiar 8-fold pattern from Bott periodicity and Clifford algebras: starting in multiples of 8 the orders of the groups jump according to the pattern "jump-jump-nothing-jump-nothing-nothing-nothing-jump" (the first couple are not on the chart because the associated projective spaces are exceptions in some way). In particular, every eight steps a total of four jumps have occurred, resulting in the orders being multiplied by 16. This is the quasi-periodicity of the first line. The second line has the simpler quasi-periodocity of length 2, where every two steps the order of the group gets doubled. Note that the groups on the two lines coincide in dimensions congruent to 6, 7, and 8 modulo 8; in other dimensions there is a difference of a factor of 2.

For reference purposes we also Table 37.15 below showing the numbers $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$. Even though this is really the same information as in the previous table, it is very handy to have around. The table provides some useful information about the comparison between $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$, which we record in a proposition:

Proposition 37.16. *For every $n \geq 2$, the number $\varphi(n)$ equals either $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$. The former occurs precisely when n is congruent to 6, 7, or 8 modulo 8.*

Recall the complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^n)$. Both groups are cyclic, and by Lemma 37.2 the map sends the generator $\lambda = L - 1$ of the domain

TABLE 37.15. Comparison of $\varphi(n)$ and $\lfloor \frac{n}{2} \rfloor$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\varphi(n)$	0	1	2	2	3	3	3	3	4	5	6	6	7	7	7	7	8
$\lfloor \frac{n}{2} \rfloor$	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8

to the generator $\nu = j^*\eta - 1$ of the target. Hence, c is surjective. Our observations about the orders now proves part (a) of the following result. Part (b) follows at once from $c(\lambda) = \nu$ and the fact that $r_{\mathbb{R}} c = 2$.

Theorem 37.17. *Let $n \geq 2$.*

(a) The complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^n)$ is always surjective. It is an isomorphism if n is congruent to 6, 7, or 8 modulo 8, and it has kernel $\mathbb{Z}/2$ otherwise.

(b) The map $r_{\mathbb{R}}\colon \widetilde{K}^0(\mathbb{R}P^n) \to \widetilde{KO}^0(\mathbb{R}P^n)$ sends ν to 2λ.

Finally, we turn our attention to KO-theory of the spaces $\mathbb{R}P^n/\mathbb{R}P^a$. It is almost true that the Atiyah-Hirzebruch spectral sequence for $\mathbb{R}P^n/\mathbb{R}P^a$ is a truncation of the one for $\mathbb{R}P^n$. The mod 2 cohomology groups $H^*(\mathbb{R}P^n/\mathbb{R}P^a;\mathbb{Z}/2)$ are indeed a truncation of $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$, and the integral cohomology groups are a similar truncation when a is even. But when a is odd there is a $\mathbb{Z}$ in $H^{a+1}(\mathbb{R}P^n/\mathbb{R}P^a)$ that does not appear in $H^{a+1}(\mathbb{R}P^n)$. This new $\mathbb{Z}$ will contribute to $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ only if it shows up along the main diagonal in the Atiyah-Hirzebruch spectral sequence, which will happen precisely when $a+1$ is a multiple of 4. This explains the two cases in the following result:

Theorem 37.18 (Real K-theory of real, stunted projective spaces; part 1)**.**

(a) Suppose $a \not\equiv -1$ mod 4. Then the map $\pi^*\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{KO}^0(\mathbb{R}P^n)$ is an injection whose image is the subgroup generated by $\lambda^{\varphi(a)+1}$. So $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z}/(2^g)$ where $g = \varphi(n) - \varphi(a) = \#\{s \,|\, a < s \leq n, s \equiv 0, 1, 2, \text{ or } 4 \text{ mod } 8\}$. Let $\lambda^{(\varphi(a)+1)}$ be the preimage for $\lambda^{\varphi(a)+1}$ under π^*, which generates the group. Then

$$\psi^k(\lambda^{(u)}) = \begin{cases} 0 & k \text{ even}, \\ \lambda^{(u)} & k \text{ odd}. \end{cases}$$

(b) Assume that $a \equiv -1$ mod 4. The sequence

$$S^{a+1} = \mathbb{R}P^{a+1}/\mathbb{R}P^a \to \mathbb{R}P^n/\mathbb{R}P^a \to \mathbb{R}P^n/\mathbb{R}P^{a+1}$$

induces a split-exact sequence in KO-theory:

$$0 \leftarrow \widetilde{KO}^0(S^{a+1}) \leftarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \leftarrow \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{a+1}) \leftarrow 0.$$

Consequently,

$$\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \cong \mathbb{Z} \oplus \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{a+1}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^h$$

where $h = \varphi(n) - \varphi(a+1) = \#\{s \,|\, a+1 < s \leq n,\ s \equiv 0, 1, 2, \text{ or } 4 \text{ mod } 8\}$.

Notice that the above result does not give the action of the Adams operations in part (b). To do this we need to choose a specific generator for the $\mathbb{Z}$ summand, and this requires some explanation. It turns out (and this is not obvious) that the generator can always be chosen so that it maps to $\lambda^{\varphi(a+1)}$ in $\widetilde{KO}^0(\mathbb{R}P^n)$. This property is all that we will really need, but it is not so easy to prove; in fact there are always *two* such generators, and proving the desired existence seems to be best accomplished by having a method for systematically choosing a preferred generator out of the two possibilities. This is what we do next; in the chain

$$\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \leftarrow \widetilde{KO}(\mathbb{R}P^{4t+1}/\mathbb{R}P^{4t-1}) \leftarrow \cdots \leftarrow \widetilde{KO}(\mathbb{R}P^{n}/\mathbb{R}P^{4t-1}) \leftarrow \cdots$$

we wish to choose elements $\overline{\lambda}^{(\varphi(4t))}$ in each group with the property that they all map onto each other, they all map to a generator of the left-most group, and upon pulling back along the projection $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{4t-1}$ the element $\overline{\lambda}^{(\varphi(4t))}$ maps to $\lambda^{\varphi(4t)}$ (for any choice of n).

These elements will be produced by starting with the elements $\bar{\nu}^{(2t)}$ that we have already constructed, living in the bottom groups of the following diagram:

$$\begin{tikzcd}
\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \arrow[d, "c"'] & \cdots \arrow[l] & \widetilde{KO}(\mathbb{R}P^{n}/\mathbb{R}P^{4t-1}) \arrow[l] \arrow[d, "c"'] \arrow[dr] & \\
\widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \arrow[u, bend right, "r_{\mathbb{R}}"'] & \cdots \arrow[l] & \widetilde{K}(\mathbb{R}P^{n}/\mathbb{R}P^{4t-1}) \arrow[l] \arrow[u, bend right, "r_{\mathbb{R}}"'] \arrow[dr] & \widetilde{KO}(\mathbb{R}P^n) \arrow[d, "c"'] \\
& & & \widetilde{K}(\mathbb{R}P^n) \arrow[u, bend right, "r_{\mathbb{R}}"']
\end{tikzcd}$$

First note that it suffices to construct the $\overline{\lambda}^{(\varphi(4t))}$ classes for n sufficiently large, as we can then construct the classes for smaller n by naturality. In particular, we may assume that n is congruent to 6 (or 7 or 8) modulo 8. This forces the right-most vertical c map to be an isomorphism, by Theorem 37.17(a).

The rest of the argument breaks into two cases, depending on whether t is even or odd. When t is even, $\varphi(4t) = 2t - 1$ (see Table 37.15) and the vertical maps c in the above diagram are all isomorphisms; this will be proven carefully in Theorem 37.20(b) below, but for now we just accept it. Define

$$\overline{\lambda}^{(\varphi(4t))} = c^{-1}(\bar{\nu}^{(2t)}).$$

The desired properties of $\overline{\lambda}^{(\varphi(4t))}$ are immediate by the naturality of c and the known properties of $\bar{\nu}^{(2t)}$.

When t is odd one has $\varphi(4t) = 2t + 1$. The vertical maps c are no longer isomorphisms (except the rightmost one), but we can use the map $r_{\mathbb{R}}$ instead. The idea for this comes from the fact that when t is odd the map $c\colon \widetilde{KO}(S^{4t}) \to \widetilde{K}(S^{4t})$ is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, and $r_{\mathbb{R}}$ sends a generator to a generator. So $r_{\mathbb{R}}(\bar{\nu}^{(2t)})$ will gives us an element of $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ that maps to a generator in $\widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$.

However, note that $r_{\mathbb{R}}(\bar{\nu}^{2t})$ maps to $-\lambda^{(2t+1)}$ in $\widetilde{KO}(\mathbb{R}P^n)$. This follows at once from a simple calculation:

$$r_{\mathbb{R}}(\nu^{2t}) = r_{\mathbb{R}}((-2)^{2t-1}\cdot\nu) = (-2)^{2t-1}\cdot r_{\mathbb{R}}(\nu) = (-2)^{2t-1}\cdot 2\lambda = -(-2)^{2t}\cdot\lambda = -\lambda^{2t+1}.$$

The extra minus sign leads us to make the definition

$$\overline{\lambda}^{(\varphi(4t))} = -r_{\mathbb{R}}(\bar{\nu}^{(2t)})$$

in the case when t is odd.

We have now constructed the desired generators $\overline{\lambda}^{(\varphi(4t))}$. The group $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ is generated by the two elements $\lambda^{\varphi(4t+1)}$ (which is torsion) and $\overline{\lambda}^{(\varphi(4t))}$ (which is non-torsion). We next use these generators to describe the action of the Adams operations:

Theorem 37.19 (Real K-theory of real, stunted projective spaces; part 2)**.** *Let $t \geq 1$ and let $f = \varphi(4t)$. The Adams operations on $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$ are given by the formulas*

$$\psi^k\left(\lambda^{(f+1)}\right) = \begin{cases} 0 & k \text{ even}, \\ \lambda^{(f+1)} & k \text{ odd}; \end{cases}$$

$$\psi^k\left(\overline{\lambda}^{(f)}\right) = k^{2t}\overline{\lambda}^{(f)} + \begin{cases} \frac{1}{2}k^{2t}\lambda^{(f+1)} & k \text{ even}, \\ \frac{1}{2}(k^{2t}-1)\lambda^{(f+1)} & k \text{ odd}. \end{cases}$$

Just as we saw for $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^a)$, much of the information about KO-theory of stunted projective spaces is represented in the following useful diagram. The above formulas for the Adams operations are obtained easily by chasing information around the diagram.

$$\begin{tikzcd}
 & \mathbb{Z}/\left(2^{\varphi(n)}\right) = \widetilde{K}(\mathbb{R}P^n) & & \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) = \mathbb{Z} & \\
{} & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-2}) \arrow[l] \arrow[u, tail] \arrow[d, equal] & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \arrow[l] \arrow[ul] \arrow[ur] \arrow[d, equal] & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t}) \arrow[l] \arrow[ull, tail] \arrow[d, equal] & {} \arrow[l] \\
 & \mathbb{Z}/(2^{B+1}) & \mathbb{Z} \oplus \mathbb{Z}/(2^B) \arrow[l, two heads] & \mathbb{Z}/(2^B) \arrow[l, tail] & {\scriptstyle B=\varphi(n)-\varphi(4t)} \\
 & \lambda^{(f)} & \overline{\lambda}^{(f)},\quad \lambda^{(f+1)} \arrow[l] \arrow[dl] & \lambda^{(f+1)} \arrow[l] & {\scriptstyle f=\varphi(4t)} \\
 & \lambda^{(f+1)} = -2\lambda^{(f)} & & &
\end{tikzcd}$$

Due to lack of space the diagram does not show the groups $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-3})$, but these are similar to the $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-2})$ and $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t})$ cases in that these groups all inject into $\widetilde{KO}(\mathbb{R}P^n)$ and have the "expected" image.

Because we have needed this already in the process of defining the classes $\overline{\lambda}^{(\varphi(4t))}$, we also include some more information on the complexification map for stunted projective spaces. We want to investigate $c\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$, but results are awkward to state in this generality: one runs into a multitude of cases depending on the congruences classes of n and a modulo 8. We start with the observation that it is essentially enough to solve the problem for n large enough. If

$N \geq n$ then we have the diagram

$$\begin{tikzcd} \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \arrow[d, "c"'] & \widetilde{KO}^0(\mathbb{R}P^N/\mathbb{R}P^a) \arrow[l, two heads] \arrow[d, "c"] \\ \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) & \widetilde{K}^0(\mathbb{R}P^N/\mathbb{R}P^a). \arrow[l, two heads] \end{tikzcd}$$

So if we know the right vertical map then we can also figure out the left vertical map, using the horizontal surjections.

Notice that by choosing N so that it is congruent to 6 (or 7 or 8) modulo 8, we can get ourselves in the situation where $c\colon \widetilde{KO}^0(\mathbb{R}P^N) \to \widetilde{K}^0(\mathbb{R}P^N)$ is an isomorphism (see Theorem 37.17)—clearly this will simplify some matters in our analysis. This explains why we focus on this special case in the following result.

Here is a notational simplification that is very useful. At this point we have specified particular generators for the groups $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a)$, for all values of n and a. These are the elements $\lambda^{(i)}$ and $\overline{\lambda}^{(j)}$ for certain values of i and j that depend on a. To actually name i and j precisely requires separating various cases for a, and it is convenient to not always have to do this. We will write λ° and $\overline{\lambda}^\circ$ as abbreviations for our generators, but where we have not bothered to write the exact number in the exponent (it is uniquely specified, so we can be incautious about this). We also write ν° and $\bar{\nu}^\circ$ for our generators in $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$.

Theorem 37.20. *Consider the map $c\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a)$ and the map $r_{\mathbb{R}}$ going in the opposite direction. Assume that n is congruent to 6, 7, or 8 modulo 8.*

(a) Suppose a is even, so that both the groups are torsion.
 - *(i) If $a \equiv 6, 8 \bmod 8$ then c is an isomorphism, $c(\lambda^\circ) = \nu^\circ$, and $r_{\mathbb{R}}(\nu^\circ) = 2\lambda^\circ$.*
 - *(ii) If $a \equiv 2, 4 \bmod 8$ then c is an injection with cokernel $\mathbb{Z}/2$. One has $c(\lambda^\circ) = -2\nu^\circ$ and $r_{\mathbb{R}}(\nu^\circ) = -\lambda^\circ$.*

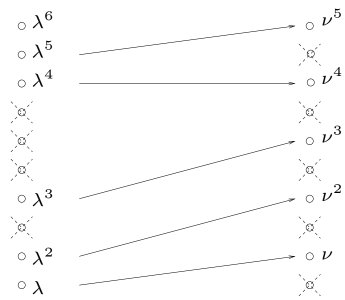
(b) Suppose that $a = 4t - 1$. Here both the domain and target of c have copies of $\mathbb{Z}$ inside them.
 - *(i) If $a \equiv 7 \bmod 8$ (i.e., t is even) then c is an isomorphism.*
 - *(ii) If $a \equiv 3 \bmod 8$ (i.e., t is odd) then c is a monomorphism, and the cokernel is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.*

(c) Suppose that $a = 4t + 1$. In this case c maps its domain isomorphically onto the torsion subgroup of the target of c. One has $c(\lambda^\circ) = \nu^\circ$ and $r_{\mathbb{R}}(\nu^\circ) = 2\lambda^\circ$.

37.21. **An extended example.** Let us demonstrate much of what we have learned by looking at a specific example. The Atiyah-Hirzebruch spectral sequence for computing $\widetilde{KO}(\mathbb{R}P^{10})$ gives one $\mathbb{Z}/2$ for every dimension from 1 through 10 that is congruent to 0, 1, 2, or 4 modulo 8. These are the dimensions 1, 2, 4, 8, 9, and 10—so we have six $\mathbb{Z}/2$'s, and $\widetilde{KO}(\mathbb{R}P^{10}) \cong \mathbb{Z}/(2^6)$. In comparison, $\widetilde{K}(\mathbb{R}P^{10})$ is just $\mathbb{Z}/(2^5)$ (as $5 = \frac{10}{2}$).

There is a visual way of representing this information that is useful, especially when it comes to the stunted projective spaces. Draw a cell diagram for $\mathbb{R}P^{10}$, leaving out the 0-cell. For $\widetilde{KO}(\mathbb{R}P^{10})$ discard all cells except the ones in dimensions congruent to 0, 1, 2, or 4 modulo 8; then label the remaining cells with ascending powers of λ. For $\widetilde{K}(\mathbb{R}P^{10})$ discard all the odd-dimensional cells and label the remaining ones with ascending powers of ν. Always remembering that $\lambda^2 = -2\lambda$

(and $\nu^2 = -2\nu$), the cells now represent the associated graded of $\widetilde{KO}(\mathbb{R}P^{10})$ (or $\widetilde{K}(\mathbb{R}P^{10})$) with respect to the 2-adic filtration. The picture below also shows the complexification map $c\colon \widetilde{KO}(\mathbb{R}P^{10}) \to \widetilde{K}(\mathbb{R}P^{10})$. Recall that this is a ring map and sends λ to ν:



$\widetilde{KO}(\mathbb{R}P^{10}) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10})$

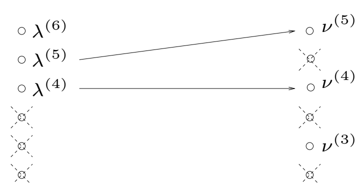
We see in this case that $c\colon \widetilde{KO}(\mathbb{R}P^{10}) \to \widetilde{K}(\mathbb{R}P^{10})$ is surjective with kernel $\mathbb{Z}/2$ (generated by $\lambda^6 = -32\lambda$). One has $r_{\mathbb{R}}(\nu) = r_{\mathbb{R}}(c(\lambda)) = 2\lambda$, and more generally

$$r_{\mathbb{R}}(\nu^k) = r_{\mathbb{R}}((-2)^{k-1}\nu) = (-2)^{k-1} r_{\mathbb{R}}(\nu) = (-2)^{k-1} \cdot 2\lambda = -(-2)^k\lambda = -\lambda^{k+1}.$$

Next let us consider the K-groups of $\mathbb{R}P^{10}/\mathbb{R}P^4$, referring to the diagram

$$\begin{tikzcd}
\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \arrow[r, tail] \arrow[d, "c"] & \widetilde{KO}(\mathbb{R}P^{10}) \arrow[d, "c"] \\
\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) \arrow[r, tail] & \widetilde{K}(\mathbb{R}P^{10}).
\end{tikzcd}$$

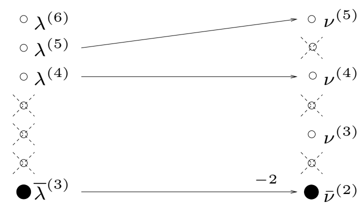
In relation to our cell-diagrams, the K-groups of $\mathbb{R}P^{10}/\mathbb{R}P^4$ are obtained by throwing away the bottom four cells. We obtain the picture



$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4)$

This picture tells us that $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \cong \mathbb{Z}/(2^3)$, generated by $\lambda^{(4)}$, and also $\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) \cong \mathbb{Z}/(2^3)$ with generator $\nu^{(3)}$. These each embed into the respective K-group of $\mathbb{R}P^{10}$. The complexification map therefore sends $\lambda^{(4)}$ to $\nu^{(4)}$ and $\lambda^{(5)}$ to $\nu^{(5)}$, and we find that this map has both kernel and cokernel isomorphic to $\mathbb{Z}/2$.

The situation is a little different if we consider the K-groups of $\mathbb{R}P^{10}/\mathbb{R}P^3$. Here the bottom cell of $\mathbb{R}P^{10}/\mathbb{R}P^3$ gives rise to a $\mathbb{Z}$ in singular cohomology, and a corresponding $\mathbb{Z}$ in the K-groups. The picture becomes as follows:



$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \xrightarrow{c} \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3)$

Here the black dots represent copies of $\mathbb{Z}$, so that $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ with the two summands generated by $\overline{\lambda}^{(3)}$ and $\lambda^{(4)}$, respectively. Likewise, $\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/8$ with the two summands generated by $\bar{\nu}^{(2)}$ and $\nu^{(3)}$. The various maps

$$\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \longrightarrow \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3) \longrightarrow \widetilde{KO}(\mathbb{R}P^{10}), \text{ and}$$
$$\widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^4) \longrightarrow \widetilde{K}(\mathbb{R}P^{10}/\mathbb{R}P^3) \longrightarrow \widetilde{K}(\mathbb{R}P^{10})$$

are the evident ones suggested by the diagrams. The only subtlety lies in determining the complexification map c. Of course $c(\lambda^{(i)}) = \nu^{(i)}$ for $i = 4, 5$, as this is forced by that happens on the subgroup $\widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^4) \subseteq \widetilde{KO}(\mathbb{R}P^{10}/\mathbb{R}P^3)$. To compute $c(\overline{\lambda}^{(3)})$ we must remember that $\overline{\lambda}^{(3)}$ is defined by the equation

$$\overline{\lambda}^{(3)} = -r_{\mathbb{R}}(\nu^{(2)})$$

(note that $3 = 4t - 1$ where $t = 1$, and so we are in the case where t is odd; in the case where t is even the definition of the $\overline{\lambda}$ classes is different). So we obtain

$$c\big(\overline{\lambda}^{(3)}\big) = -c(r_{\mathbb{R}}(\nu^{(2)})) = -(1+\psi^{-1})(\bar{\nu}^{(2)}) = -[\bar{\nu}^{(2)} + \bar{\nu}^{(2)}] = -2\bar{\nu}^{(2)}.$$

Here we have used Theorem 37.12(c) for evaluating $\psi^{-1}(\bar{\nu}^{(2)})$. Note that the pullback map induced by $\mathbb{R}P^{10} \to \mathbb{R}P^{10}/\mathbb{R}P^3$ sends $\overline{\lambda}^{(3)}$ to λ^3 and sends $-2\bar{\nu}^{(2)}$ to $-2\nu^2 = \nu^3$; hence the above formula is consistent with our previous computation of $c\colon \widetilde{KO}(\mathbb{R}P^{10}) \to \widetilde{K}(\mathbb{R}P^{10})$.

37.22. **The proofs.** We now give proofs for all of the results previously stated in this section.

Proof of Theorem 37.9. This is straightforward, and left to the reader. □

Proof of Theorem 37.11. There is no room for differentials in the Atiyah-Hirzebruch spectral sequence for $K^*(\mathbb{R}P^n)$, so it collapses at E_2. Part (b) follows immediately. It is also a direct consequence that $\widetilde{K}^0(\mathbb{R}P^n)$ is an abelian group of order $2^{\lfloor \frac{n}{2} \rfloor}$. It remains to solve the extension problems to determine precisely which group it is.

Observe that $L^2 = 1$, hence $(cL)^2 = c(L^2) = c(1) = 1$. So

$$\nu^2 = (cL-1)^2 = (cL)^2 - 2(cL) + 1 = 2(1 - c(L)) = -2\nu.$$

Note that an immediate consequence is $\nu^t = (-2)^{t-1}\nu$.

Let $F^i = \ker\big(\widetilde{K}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^{i-1})\big)$. So

$$\widetilde{K}^0(\mathbb{R}P^n) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots \supseteq F^{n+1} = 0.$$

The quotients F^i/F^{i+1} are the groups in the E_∞ term of the spectral sequence, and so are

$$F^i/F^{i+1} = \begin{cases} \mathbb{Z}/2 & 0 < i \leq 2\lfloor \frac{n}{2} \rfloor \text{ and } i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

So $F^0 = F^1 = F^2$ and $F^2/F^3 \cong \mathbb{Z}/2$. The element ν generates F^2/F^3: we know this by naturality of the spectral sequence, applied to the map $j\colon \mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$. The element μ generates F^2/F^3 for $\widetilde{K}^0(\mathbb{C}P^n)$, and $H^2(\mathbb{C}P^n) \to H^2(\mathbb{R}P^n)$ is the projection $\mathbb{Z} \to \mathbb{Z}/2$: the image of a generator is another generator. So $j^*\mu$ generates F^2/F^3 for $\widetilde{K}^0(\mathbb{R}P^n)$, and of course $\nu = j^*\mu$.

The multiplicativity of the spectral sequence then gives us that ν^2 generates F^4/F^5, and in general ν^j generates F^{2j}/F^{2j+1} for $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. In particular, ν^j is not equal to zero for j in this range. But $\nu^j = (-2)^{j-1}\nu$, so $2^{\lfloor \frac{n}{2} \rfloor - 1}\nu \neq 0$. This proves that the only possibility for $\widetilde{K}^0(\mathbb{R}P^n)$ is $\mathbb{Z}/2^{\lfloor \frac{n}{2} \rfloor}$, and that ν is a generator.

Note that we than have $0 = (-2)^{\lfloor \frac{n}{2} \rfloor}\nu = \nu^{\lfloor \frac{n}{2} \rfloor + 1}$.

For part (c) we just observe that $\psi^k(cL) = (cL)^k = c(L^k)$ and this equals 1 if k is even, and cL if k is odd. $\square$

Proof of Theorem 37.12. Parts (a) and (b) are trivial. In each case one writes down the evident long exact sequence and quickly sees that the given sequence is short exact. The only slight subtlety is seeing in the case $a = 2t - 1$ that $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \to \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1})$ is surjective, but this was explained when we constructed the element $\bar{\nu}^{(t)}$ (which maps to a generator in the target group).

For part (c), the action of ψ^k on $\nu^{(t+1)}$ is determined by the corresponding action in $\widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t})$; so there is nothing to prove here. The action on $\bar{\nu}^{(t)}$ is more interesting. We can, of course, write

$$\psi^k(\bar{\nu}^{(t)}) = A\bar{\nu}^{(t)} + B\nu^{(t+1)} \tag{37.23}$$

where A is a unique integer and B is unique modulo 2^f. Applying the map $i^*\colon \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t-1}) \to \widetilde{K}^0(\mathbb{R}P^{2t}/\mathbb{R}P^{2t-1})$ kills $\nu^{(t+1)}$ and sends $\bar{\nu}^{(t)}$ to a generator g, so this equation becomes $\psi^k(g) = Ag$. But we already know that ψ^k acts on such a generator by k^t, so $A = k^t$.

Next we apply the map $\pi^*\colon \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{2t}) \to \widetilde{K}^0(\mathbb{R}P^n)$ to equation (37.23). The map π^* sends $\bar{\nu}^{(t)}$ to ν^t and $\nu^{(t+1)}$ to ν^{t+1}, so using $A = k^t$ we obtain

$$\psi^k(\nu^t) = k^t\nu^t + B\nu^{t+1}$$

in $\widetilde{K}^0(\mathbb{R}P^n)$. Now use that ψ^k is a ring homomorphism, together with $\nu^{t+1} = -2\nu^t$. We get

$$(k^t - 2B)\nu^t = \big[\psi^k(\nu)\big]^t = \begin{cases} 0 & \text{if } k \text{ is even} \\ \nu^t & \text{if } k \text{ is odd.} \end{cases}$$

The group $\widetilde{K}^0(\mathbb{R}P^n)$ is $\mathbb{Z}/(2^g)$ with generator ν, and $\nu^t = (-2)^{t-1}\nu$. So the additive order of ν^t is 2^{g-t+1}, or equivalently 2^{f+1}. In the case that k is even it follows that $k^t - 2B$ is a multiple of 2^{f+1}, so that $\frac{k^t}{2} \equiv B \bmod 2^f$ (recall that B is only well-defined modulo 2^f in the first place).

In the remaining case where k is odd we get $k^t - 2B \equiv 1$ modulo 2^{f+1}. So $\frac{k^t-1}{2} \equiv B$ modulo 2^f, which is what we wanted. $\square$

Proof of Theorem 37.14. In the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(\mathbb{R}P^n)$, the diagonal of the E_2-term that is relevant to $\widetilde{KO}^0(\mathbb{R}P^n)$ consists of $\varphi(n)$ copies of $\mathbb{Z}/2$. The first concern is to determine if there are any differentials causing some of these copies to disappear by E_∞, and the second concern is the problem of extensions.

Observe that the complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{K}^0(\mathbb{R}P^n)$ is surjective, because ν generates the target and $\nu = c(L-1)$. So it follows from Theorem 37.11 that at least $\lfloor \frac{n}{2} \rfloor$ among our $\varphi(n)$ copies of $\mathbb{Z}/2$ must survive the spectral sequence.

The trick now is to not consider one n at a time, but rather to consider them all at once. When n is congruent to 6, 7, or 8 modulo 8 then we know $\varphi(n) = \lfloor \frac{n}{2} \rfloor$, and so here it must be that all the $\mathbb{Z}/2$'s along the main diagonal survive. That is, all differentials entering or exiting the main diagonal are zero. But then by naturality of the spectral sequence this is true for all n. We conclude that the order of $\widetilde{KO}^0(\mathbb{R}P^n)$ is $2^{\varphi(n)}$, no matter what n is.

When n is congruent to 6, 7, or 8 modulo 8 we now know that the orders of $\widetilde{KO}^0(\mathbb{R}P^n)$ and $\widetilde{K}^0(\mathbb{R}P^n)$ are the same. Since the complexification map is surjective, it is therefore an isomorphism. So $\widetilde{KO}^0(\mathbb{R}P^n)$ is cyclic. Since $c(\lambda) = \nu$ it follows that λ is a generator. In particular, λ is a generator for the quotient F^1/F^2.

But then by naturality of the spectral sequence (and with it, naturality of the filtration F^i) it follows that λ generates F^1/F^2 for *every* value of n. Since $L^2 = 1$ we of course have $\lambda^2 = -2\lambda$. At this point the argument follows the one in the proof of Theorem 37.11 to show that $\widetilde{KO}^0(\mathbb{R}P^n)$ is cyclic, for all values of n.

The computation of the Adams operations again follows from $\psi^k(L) = L^k$, which equals 1 if k is even and L if k is odd. $\square$

Proof of Theorem 37.17. This was given just prior to the statement of the theorem. $\square$

Proof of Theorem 37.18. For part (a) one examines the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}^*(\mathbb{R}P^n/\mathbb{R}P^a)$. Note that the quotient $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^a$ induces a map of spectral sequences in the other direction. The diagonal groups in the E_2-term for $\widetilde{KO}^*(\mathbb{R}P^n/\mathbb{R}P^a)$ are a truncation of the diagonal groups appearing in the E_2-term for $\widetilde{KO}^*(\mathbb{R}P^n)$. Since there are no entering or exiting differentials (along the diagonal) in the latter case, naturality of the spectral sequence guarantees there are no entering or exiting differentials for $\mathbb{R}P^n/\mathbb{R}P^a$. Passing to E_∞-terms now, we see that the associated graded groups for $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a)$ are a truncation of the associated graded groups for $\widetilde{KO}(\mathbb{R}P^n)$. Examining the map of filtered groups

$$\begin{tikzcd}
\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a) \arrow[d, "\pi^*"] & F_1 \arrow[l, tail] \arrow[d] & F_2 \arrow[l, tail] \arrow[d] & \cdots \arrow[l, tail] \\
\widetilde{KO}(\mathbb{R}P^n) & F_1' \arrow[l, tail] & F_2' \arrow[l, tail] & \cdots \arrow[l, tail]
\end{tikzcd}$$

we now find that $F_k \to F_k'$ is an isomorphism for $k \geq a+1$ and $F_k/F_{k+1} = 0$ for $k \leq a$. It follows that $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^a) \to \widetilde{KO}(\mathbb{R}P^n)$ is an injection, with image equal to F_{a+1}'. In our analysis of $KO(\mathbb{R}P^n)$ we have already seen that

$F'_{a+1} \subseteq \widetilde{KO}(\mathbb{R}P^n)$ is the subgroup generated by λ^{a+1}. Everything else in part (a) is then immediate.

For (b) we only need to prove that $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \to \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$ is surjective, since the latter group is isomorphic to $\widetilde{KO}^0(S^{4t}) \cong \mathbb{Z}$. Everything else in part (b) is routine. To do this, consider the following diagram:

$$\begin{tikzcd}
 & & \mathbb{Z}/2 \arrow[d, equal] & & \\
\cdots & \widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t}) \arrow[l] & \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2}) \arrow[l] & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2}) \arrow[l, two heads, "j_1^*"'] & \cdots \arrow[l] \\
\cdots & \widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t}) \arrow[l] \arrow[u] & \widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \arrow[l] \arrow[u, two heads, "i^*"] \arrow[d, equal] & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \arrow[l, "j_2^*"'] \arrow[u, two heads] & \cdots \arrow[l] \\
 & & \mathbb{Z} & &
\end{tikzcd}$$

The two indicated vertical maps are surjections because they sit inside long exact sequences where the third term is $\widetilde{KO}^0(\mathbb{R}P^{4t-1}/\mathbb{R}P^{4t-2}) = \widetilde{KO}^0(S^{4t-1}) = 0$. The indicated group is $\mathbb{Z}/2$ by part (a) of the theorem, which also yields the diagram

$$\begin{tikzcd}
\widetilde{KO}^0(\mathbb{R}P^{4t}) & \widetilde{KO}^0(\mathbb{R}P^n) \arrow[l, two heads] \\
\widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2}) \arrow[u, tail] & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2}). \arrow[l, "j_1^*"'] \arrow[u, tail]
\end{tikzcd}$$

The group $\widetilde{KO}^0(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-2})$ is the subgroup of $\widetilde{KO}^0(\mathbb{R}P^{4t})$ generated by $\nu^{1+\varphi(4t-2)}$, and $\widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-2})$ is the similarly-described subgroup of $\widetilde{KO}^0(\mathbb{R}P^n)$. It follows at once that j_1^* is surjective.

Returning to the earlier diagram, the image of j_2^* is an ideal (r) inside of $\mathbb{Z}$. The fact that $i^*j_2^*$ is surjective (readily observed from the diagram) proves that r must be odd. But the quotient $\mathbb{Z}/r$ will inject into $\widetilde{KO}^1(\mathbb{R}P^n/\mathbb{R}P^{4t})$, by the long exact sequence. The Atiyah-Hirzebruch spectral sequence shows that this latter group has no odd torsion, because it has a filtration where the quotients are only $\mathbb{Z}$'s and $\mathbb{Z}/2$'s. So the conclusion is that $r = 1$, hence j_2^* is surjective. $\square$

Proof of Theorem 37.19. The evaluation of $\psi^k\left(\lambda^{(f+1)}\right)$ is immediate using naturality and Theorem 37.14. For the evaluation of $\psi^k\left(\overline{\lambda}^{(f)}\right)$ one can repeat the proof of Theorem 37.12(c) almost verbatim. Alternatively, one can use the result of Theorem 37.12(c) together with the complexification map $c\colon \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \to \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^{4t-1})$, which is a monomorphism for n congruent to 6, 7, or 8 modulo 8; the result for other values of n can then be deduced by naturality. $\square$

Proof of Theorem 37.20. For part (a) we consider the two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{KO}^0(\mathbb{R}P^n/\mathbb{R}P^a) \arrow[r] \arrow[d, "c"] & \widetilde{KO}^0(\mathbb{R}P^n) \arrow[r] \arrow[d, "c", "\cong"'] & \widetilde{KO}^0(\mathbb{R}P^a) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \widetilde{K}^0(\mathbb{R}P^n/\mathbb{R}P^a) \arrow[r] & \widetilde{K}^0(\mathbb{R}P^n) \arrow[r] & \widetilde{K}^0(\mathbb{R}P^a) \arrow[r] & 0.
\end{tikzcd}$$

If a is congruent to 6 or 8 modulo 8 then the right vertical map is an isomorphism, which means the left vertical map is as well. The desired results are immediate.

If a is congruent to 2 or 4 modulo 8 then the right vertical map is a surjection with kernel $\mathbb{Z}/2$. It follows from the zig-zag lemma that the left vertical map is an injection with cokernel $\mathbb{Z}/2$. The generator for the domain is $\lambda^{(\varphi(a)+1)}$, and we are in the case where $\varphi(a) = \lfloor \frac{a}{2} \rfloor + 1$. So c maps this generator to $\nu^{(\lfloor \frac{a}{2} \rfloor + 2)} = -2\nu^{(\lfloor \frac{a}{2} \rfloor + 1)}$. The statement $r(\nu^\circ) = -\lambda^\circ$ then follows using that $rc = 2$.

For (b) we look at the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t}) \arrow[r] \arrow[d, "c"] & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \arrow[r] \arrow[d, "c"] & \widetilde{KO}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t}) \arrow[r] & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t-1}) \arrow[r] & \widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1}) \arrow[r] & 0. \end{tikzcd}$$

When t is even the right vertical map is an isomorphism by Bott's calculation, and the left vertical map is an isomorphism by (a). So the middle vertical map is also an isomorphism.

When t is odd the right vertical map is an injection with cokernel $\mathbb{Z}/2$, by Bott. The left vertical map is an injection with cokernel $\mathbb{Z}/2$ by part (a). So by the Snake Lemma the middle vertical map is also an injection, and its cokernel is either $(\mathbb{Z}/2)^2$ or $\mathbb{Z}/4$. The element $\bar{\nu}^{(2t)}$ maps to a generator for the right bottom group $\widetilde{K}(\mathbb{R}P^{4t}/\mathbb{R}P^{4t-1})$. If we verify that $2\bar{\nu}^{(2t)} = 0$ in the cokernel of c then we will have proven that this cokernel is $(\mathbb{Z}/2)^2$, not $\mathbb{Z}/4$. But note that

$$cr_{\mathbb{R}}(\bar{\nu}^{(2t)}) = (1 + \psi^{-1})(\bar{\nu}^{(2t)}) = 2\bar{\nu}^{(2t)}$$

where in the last equality we have used the formula for $\psi^{-1}(\bar{\nu}^{(2t)})$ from Theorem 37.12(c).

For (c) we consider the following:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+2}) \arrow[r, "p^*"] \arrow[d, "c"] & \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) \arrow[r] \arrow[d, "c"] & \widetilde{KO}(\mathbb{R}P^{4t+2}/\mathbb{R}P^{4t+1}) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+2}) \arrow[r] & \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) \arrow[r] & \widetilde{K}(\mathbb{R}P^{4t+2}/\mathbb{R}P^{4t+1}) \arrow[r] & 0. \end{tikzcd}$$

We have not yet discussed exactness of the top row. On the left end this follows because $\widetilde{KO}^{-1}(S^{4t+2}) = 0$. By our computations, the cokernel of p^* is a group of order $2^{\varphi(4t+2)-\varphi(4t+1)}$; but by inspection this number is equal to 1 when t is odd and 2 when t is even. As this is the same as the order of the group $\widetilde{KO}(S^{4t+2})$, this justifies exactness on the right.

If t is odd then the left vertical map is an isomorphism by (a), and the horizontal map p^* is an isomorphism; the desired claims follow at once. When t is even we must argue more carefully. Note that the image of $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+2})$ inside $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is precisely the torsion subgroup; let us call this image T. Since the group $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is torsion, its image under c is also torsion; so this image is a subgroup of T. Moreover, $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ is generated by $\lambda^{(\varphi(4t+1)+1)}$, and one readily computes that $\varphi(4t+1) = 2t+1$. Consider the square

$$\begin{array}{ccc} \widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \rightarrowtail & \widetilde{KO}(\mathbb{R}P^n) \\ {\scriptstyle c}\downarrow & & {\scriptstyle c}\downarrow{\scriptstyle \cong} \\ \widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1}) & \longrightarrow & \widetilde{K}(\mathbb{R}P^n). \end{array}$$

The left vertical map clearly must be injective. Compute that $\pi^*(c(\lambda^{(2t+2)})) = c(\lambda^{2t+2}) = \nu^{2t+2}$. There is only one element of the torsion subgroup of $\widetilde{K}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ that pulls back to ν^{2t+2}, namely $\nu^{(2t+2)}$. It follows that $c(\lambda^{(2t+2)}) = \nu^{(2t+2)}$. But $\nu^{(2t+2)}$ generates T, so c maps $\widetilde{KO}(\mathbb{R}P^n/\mathbb{R}P^{4t+1})$ isomorphically onto T. $\square$

38. Solution to the vector field problem

In this section we conclude our story of the vector field problem, following the original paper by Adams [Ad2]. Let us first recall the Hurwitz-Radon function $\rho(n)$: if $n = 2^{4b+a} \cdot (\text{odd})$ then $\rho(n) = 2^a + 8b - 1$. We have seen in Theorem 15.5 that one can construct $\rho(n)$ independent vector fields on S^{n-1}. The vector field problem will be settled once we prove the following:

Theorem 38.1 (Adams). *There do not exist $\rho(n)+1$ independent vector fields on S^{n-1}.*

Remark 38.2. There will inevitably come a time when the reader wishes to remember the formula for $\rho(n)$ but cannot immediately look it up. The key facts about the formula are:

(i) $\rho(n)$ only depends on the power of 2 in the prime factorization of n;
(ii) For $a \leq 3$ one has $\rho(2^a) = 2^a - 1$;
(iii) $\rho(16n) = \rho(n) + 8$.

These facts of course uniquely determine $\rho(n)$. Personally, I find the exact form of (iii) hard to remember when I haven't been working with this stuff for a while. What *is* able to stick in my head is that there are zero vector fields on S^0, one on S^1, three on S^3, seven on S^7—and then I have to remember that there are only eight on S^{15}. The jump from zero on S^0 to eight on S^{15} is the quasi-periodicity; so there are nine on S^{31}, eleven on S^{63}, fifteen on S^{127}, and so forth. From this it is easy to recover the formula $\rho(16n) = \rho(n) + 8$, and onward to the general formula for ρ.

The proof of Theorem 38.1 is quite involved—it requires a surprising amount of algebraic topology. KO-theory is usually regarded as the key tool in the proof, but one also needs Steenrod operations, James periodicity, and Atiyah duality in the stable homotopy category. This adds up to a sizable amount of material. We will take a modular approach to things; we start by giving an outline of the proof, and then we will fill in the details one by one.

38.3. **Outline of the proof.**

Step 1: We have the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv):

(i) There exist $k-1$ vector fields on S^{n-1}.

(ii) There exist $k-1$ vector fields on S^{un-1} for every $u \geq 1$.

(iii) The map $\pi_1 : V_k(\mathbb{R}^{un}) \to S^{un-1}$ has a section, for every $u \geq 1$.

(iv) The map $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1} \to S^{un-1}$ (projection onto the top cell) has a section in the homotopy category, for every u such that $un+2 > 2k$. That is to say, $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1}$ splits off the top cell in the stable homotopy category.

We have seen these implications back in Section 15, but let us briefly recall why they hold. For (i)⇒(ii) it is a direct construction: given $k-1$ orthogonal vector fields made from vectors with n coordinates, one can repeat those patterns in successive groups of coordinates to make $k-1$ orthogonal vector fields in un coordinates, for any u. The step (ii)⇒(iii) is a triviality, essentially just a restatement of the problem. Then for (iii)⇒(iv) it is because for $a+2 > 2b$ the space $V_b(\mathbb{R}^a)$ has a cell structure where the a-skeleton is $\mathbb{R}P^{a-1}/\mathbb{R}P^{a-b-1}$ (Proposition **??**).

Step 2: Steenrod operations allow one to prove that if $a+1 = 2^r \cdot (\text{odd})$ then $\mathbb{R}P^a/\mathbb{R}P^{a-b}$ does not split off the top cell for $b > 2^r$.

The proof will be described in detail below, but here is a short summary. The hypothesis says that $a = 2^r(2t+1) - 1 = 2^{r+1}t + 2^r - 1$, and this guarantees that there is a Sq^{2^r} operation in $H^*(\mathbb{R}P^\infty)$ connecting the class in degree $2^{r+1}t - 1$ to the class in degree a. If $b > 2^r$ then that Sq^{2^r} operation is still present in $\mathbb{R}P^a/\mathbb{R}P^{a-b}$, and this obstructs the splitting off of the top cell.

Step 3: Putting steps 1 and 2 together, we have that if $n = 2^m \cdot (\text{odd})$ then there do not exist 2^m vector fields on S^{n-1}.

For $m \leq 3$ this solves the vector field problem, because in this case $\rho(m) = 2^m - 1$.

Step 4: There are periodicities to the spaces $\mathbb{R}P^a/\mathbb{R}P^{a-b}$. If $L-1$ has finite order r_b in $\widetilde{KO}(\mathbb{R}P^{b-1})$, then

$$\mathbb{R}P^a/\mathbb{R}P^{a-b} \simeq \Sigma^{-sr_b}\Big(\mathbb{R}P^{a+sr_b}/\mathbb{R}P^{a+sr_b-b}\Big)$$

for every $s \geq 1$, where the homotopy equivalence is in the stable homotopy category. This is called *James periodicity*. The proof was given in Proposition 16.17.

Step 5: Now things get a bit more sophisticated. Atiyah proved that if M is any compact manifold and $E \to M$ is a real vector bundle, then $\mathrm{Th}(E \to M)$ is Spanier-Whitehead dual to $\mathrm{Th}(-E - T_M \to M)$, where T_M is the tangent bundle to M (for Thom spaces of negative bundles, see Section 16.13).

Recall from Example 16.10 that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \cong \mathrm{Th}((n-k)L \to \mathbb{R}P^{k-1})$. Also, for $M = \mathbb{R}P^{k-1}$ one has $T_M = kL - 1$ in $KO(M)$ (Example 26.11). So Atiyah Duality gives that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ is Spanier-Whitehead dual to

$$\mathrm{Th}\begin{pmatrix} -(n-k)L-T \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} = \mathrm{Th}\begin{pmatrix} -(n-k)L-kL+1 \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} = \mathrm{Th}\begin{pmatrix} -nL+1 \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} = \Sigma\,\mathrm{Th}\begin{pmatrix} -nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix}.$$

If r_k is the order of $L-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$ then the last spectrum may be interpreted as

$$\begin{aligned}\Sigma\,\mathrm{Th}\begin{pmatrix}-nL\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix} &\simeq \Sigma\Sigma^{-sr_k}\,\mathrm{Th}\begin{pmatrix}sr_kL-nL\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}\\ &= \Sigma^{1-sr_k}\,\mathrm{Th}\begin{pmatrix}(sr_k-n)L\\ \downarrow\\ \mathbb{R}P^{k-1}\end{pmatrix}\\ &= \Sigma^{1-sr_k}\left[\mathbb{R}P^{sr_k-n+k-1}/\mathbb{R}P^{sr_k-n-1}\right],\end{aligned}$$

where s is any integer sufficiently large so that $sr_k-n-1\geq 0$. We have therefore proven:

The Spanier-Whitehead dual of $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ is (up to suspension) $\mathbb{R}P^{sr_k-n+k-1}/\mathbb{R}P^{sr_k-n-1}$, where s is any integer such that $sr_k-n-1\geq 0$.

Step 6: A direct consequence of the previous statement is that

$\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$ splits off its top cell (stably)

if and only if

$\mathbb{R}P^{sr_k-n+k-1}/\mathbb{R}P^{sr_k-n-1}$ splits off its bottom cell (stably),

where $s\gg 0$ as above.

Step 7: Using step 6 we can add a condition onto the list of implications from step 1. Namely, we have (iv)⇒(v) where the latter is

(v) $\mathbb{R}P^{sr_k-un+k-1}/\mathbb{R}P^{sr_k-un-1}$ splits off its bottom cell (stably), for any $u\gg 0$ and any $s\gg 0$.

Step 8: Adams calculated $\widetilde{KO}(\mathbb{R}P^a)$ for all a, together with the Adams operations on these groups. He used this knowledge, together with Step 2 above, to prove the following:

For any $m\geq 0$, $\mathbb{R}P^{m+\rho(m)+1}/\mathbb{R}P^{m-1}$ does not split off its bottom cell in the stable homotopy category.

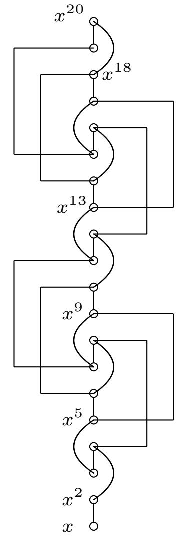
Step 9: Completion of the proof.

Proof of Theorem 38.1. Suppose there are $k-1$ vector fields on S^{n-1}. Then by Step 7 the space $\mathbb{R}P^{sr_k-un+k-1}/\mathbb{R}P^{sr_k-un-1}$ stably splits off its bottom cell for any u and s such that $un+2>2k$ and $sr_k-un-1\geq 0$. Choose u to be odd, and choose s to be a multiple of $2n$. Set $m=sr_k-un$, and note that m is an odd multiple of n; consequently, we have $\rho(m)=\rho(n)$.

We have that $\mathbb{R}P^{m+k-1}/\mathbb{R}P^{m-1}$ splits off its bottom cell. By Step 8 this implies that $k-1\leq\rho(m)=\rho(n)$. So there are at most $\rho(n)$ vector fields on S^{n-1}. □

The missing pieces from our outline are: **Step 2, Step 5, and Step 8**. We now fill in the details for these steps, one by one—but not quite in the above order. We save Atiyah duality for last, only because the other two pieces belong more to the same theme.

38.4. **Steenrod operations and stunted projective spaces (steps 2 and 3).** Let $x \in H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$ be the nonzero element. The Steenrod operations on $\mathbb{R}P^\infty$ are easily computed from the facts $\mathrm{Sq}^1(x) = x^2$, $\mathrm{Sq}^i(x) = 0$ for $i > 1$, and the Cartan formula. We leave this as an exercise for the reader. In the following picture we show the Sq^1, Sq^2, and Sq^4 operations on $H^*(\mathbb{R}P^{20};\mathbb{Z}/2)$:



The Sq^1 operations are depicted as vertical lines, the Sq^2 operations as curved lines, and the Sq^4s as "offseted vertical lines". For example, one can read off of the diagram that $\mathrm{Sq}^1(x^5) = x^6$, $\mathrm{Sq}^2(x^{10}) = x^{12}$, and $\mathrm{Sq}^4(x^{10}) = 0$ (in the latter case because the diagram does not have a Sq^4 emanating from the x^{10} class).

The pattern of Sq^{2^r} operations in $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$ is very simple. The first Sq^{2^r} operation occurs on x^{2^r} and thereafter they follow the pattern of "2^r on/2^r off". This is captured by the formula

$$\mathrm{Sq}^{2^r}(x^a) = \begin{cases} x^{a+2^r} & \text{if } a \geq 2^r \text{ and } a \equiv 2^r, 2^r+1, \ldots, 2^{r+1}-1 \bmod 2^{r+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Of course for $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$ the formulas must be truncated to account for the fact that classes above dimension n are not present.

To see how Steenrod operations give obstructions to stable splittings, consider $\mathbb{R}P^9/\mathbb{R}P^6$. Its cohomology has a Sq^2 connecting the class in degree 7 to the class in degree 9. Suppose the projection onto the top cell $p\colon \mathbb{R}P^9/\mathbb{R}P^6 \to S^9$ has a splitting χ in the stable homotopy category. Then the composite

$$H^*(S^9) \xrightarrow{p^*} H^*(\mathbb{R}P^9/\mathbb{R}P^6) \xrightarrow{\chi^*} H^*(S^9)$$

is an isomorphism. Write x_i for the generator in $H^i(\mathbb{R}P^9/\mathbb{R}P^6)$, so that in this notation we have $\mathrm{Sq}^2(x_7) = x_9$. Necessarily we must have $\chi^*(x_7) = 0$, therefore $0 = \mathrm{Sq}^2(\chi^* x_7) = \chi^*(\mathrm{Sq}^2 x_7) = \chi^*(x_9)$. But x_9 is in the image of p^*, so this is a contradiction.

Clearly this kind of argument will work for any $\mathbb{R}P^n/\mathbb{R}P^{n-b}$ where we have a nontrivial cohomology operation hitting the top class. Based on this, it is now easy to prove the following:

Proposition 38.5. *Write $n+1 = 2^s \cdot$ odd. If $\mathbb{R}P^n/\mathbb{R}P^{n-b}$ splits off its top cell stably then $b \leq 2^s$.*

Proof. If n is even then in $H^*(\mathbb{R}P^\infty)$ there is a Sq^1 hitting the class in degree n, and this operation will be present in $H^*(\mathbb{R}P^n/\mathbb{R}P^{n-b})$ as long as $b > 1$. So the top cell can not split off in this case. In other words, if $n+1 = 2^0 \cdot (\text{odd})$ then splitting of the top cell can only happen if $b \leq 2^0$.

Similarly, if $n = 4e+1$ then in $H^*(\mathbb{R}P^\infty)$ there is a Sq^2 hitting the class in degree n. This Sq^2 will be present in $H^*(\mathbb{R}P^n/\mathbb{R}P^{n-b})$ as long as $b > 2$, and again we find that under this criterion the top cell can not split off. So $n+1 = 2^1 \cdot (\text{odd})$ implies splitting of the top cell can only happen if $b \leq 2^1$.

The same style of argument continues. If $n = 2^r e + (2^{r-1} - 1)$ then there is a $\mathrm{Sq}^{2^{r-1}}$ hitting our class in degree n, and this obstructs the splitting of the top cell as long as $b > 2^{r-1}$. Rephrased, this says that if $n+1 = 2^{r-1}(2e+1)$ then the splitting does not exist when $b > 2^{r-1}$. Replacing $r-1$ with s, we have the desired result. $\square$

Corollary 38.6. *If $n = 2^s \cdot$ odd then there are at most $2^s - 1$ independent vector fields on S^{n-1}.*

Proof. If there are $k-1$ vector fields on S^{n-1} then $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-k-1}$ splits off its top cell for all $u \gg 0$. Choose a u that is odd, so that $un = 2^s \cdot$ odd. By Proposition 38.5 we conclude that $k \leq 2^s$. $\square$

The upper bounds provided by Corollary 38.6 agree with the Hurwitz-Radon lower bounds when $s \leq 3$. Note that these few cases cover quite a bit more than one first might think. For example, for spheres of dimension less than 50 it follows that the Hurwitz-Radon number of vector fields is the maximum possible for all but three cases, namely the spheres S^{15}, S^{31}, and S^{47} (multiples of 16 minus one). Corollary 38.6 yields that there exist at most 15 vector fields on S^{15}, 31 on S^{31}, and 15 on S^{47}, whereas the Hurwitz-Radon construction only gives 8 vector fields on S^{15} and S^{47}, and 9 vector fields on S^{31}. This demonstrates that our bounds for the spheres S^{16e-1} are still far away from our goal.

38.7. **Adams's Theorem and KO-theory (Step 8).** Next we move to the piece that finally cracked the proof, namely the following theorem of Adams [Ad2]:

Theorem 38.8. *Let $m \geq 1$. Then the space $\mathbb{R}P^{m+\rho(m)+1}/\mathbb{R}P^{m-1}$ does not split off its bottom cell in the stable homotopy category.*

Proof. Write $m = 2^u \cdot$odd, and consider $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ for $N \geq m$. Note that there is a Sq^{2^u} operation on $H^*(\mathbb{R}P^N/\mathbb{R}P^{m-1}; \mathbb{Z}/2)$ connecting the generator in degree m to the generator in degree $m + 2^u$. This proves that the bottom cell does not split off when $N \geq m + 2^u$. This settles the theorem in the case $u \leq 3$, as here $\rho(m) = 2^u - 1$ and so $m + 2^u = m + \rho(m) + 1$.

We will next do a similar argument—but using K-theory—in the case $u \geq 4$. Actually, the K-theory argument only uses $u \geq 3$ so for the sake of pedagogy let us just make this weaker assumption.

Since $m \equiv 0 \bmod 4$ we know by Theorem 37.18 that there is a short exact sequence
(38.9)

$$\begin{tikzcd} 0 & \widetilde{KO}(\mathbb{R}P^m/\mathbb{R}P^{m-1}) \arrow[l] \arrow[d, "\cong"] & \widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^{m-1}) \arrow[l] & \widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^m) \arrow[l] \arrow[d, "\cong"] & 0 \arrow[l] \\ & \mathbb{Z} & & \mathbb{Z}/(2^f) & \end{tikzcd}$$

where f is a certain integer we will recall in a moment. Each of the groups has Adams operations on it, and the maps are compatible with these operations. If we let $\mathcal{B} = \mathbb{Z}[\psi^2, \psi^3, \psi^5, \ldots]$, then (38.9) is an exact sequence of $\mathcal{B}$-modules (note that $\mathcal{B}$ is just the monoid ring $\mathbb{Z}[\mathbb{N}]$ from Section 36). If $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell then this extension is split; so we will attempt to algebraically analyze when such a splitting exists.

As a $\mathcal{B}$-module the group $\widetilde{KO}(\mathbb{R}P^m/\mathbb{R}P^{m-1})$ is $\mathbb{Z}(\frac{m}{2})$, meaning that each ψ^k acts as multiplication by $k^{\frac{m}{2}}$. It will be convenient to set $r = \frac{m}{2}$. Also, we know by Theorem 37.18(a) that on $\widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^m)$ the operation ψ^k acts as zero when k is even, and the identity when k is odd. We also have determined the action of the ψ^k's on the middle group, but let us ignore this for the moment and consider the situation in a bit more generality.

Let A be any abelian group, and let $A[2]$ be the $\mathcal{B}$-module whose underlying abelian group is A and where ψ^k acts as zero when k is even, and the identity when k is odd. Consider an extension of $\mathcal{B}$-modules

$$0 \leftarrow \mathbb{Z}(r) \leftarrow E \leftarrow A[2] \leftarrow 0.$$

Let g be an element of E that maps onto a generator for $\mathbb{Z}(r)$. Then we can write $\psi^k g = k^r g + \alpha_k$ for unique elements $\alpha_k \in A$, and the relations $\psi^k\psi^l = \psi^l\psi^k$ show that we must have

$$(38.10) \qquad \begin{cases} k^r\alpha_l = (l^r - 1)\alpha_k & \text{whenever } k \text{ is even and } l \text{ is odd,} \\ k^r\alpha_l = l^r\alpha_k & \text{whenever } k \text{ and } l \text{ are both even,} \\ (k^r - 1)\alpha_l = (l^r - 1)\alpha_k & \text{whenever } k \text{ and } l \text{ are both odd.} \end{cases}$$

So the extension E is determined by the elements $\alpha_k \in A$, for $k \geq 2$, satisfying the above equations.

To analyze when the sequence is split, let $\tilde{g}$ denote the image of g in $\mathbb{Z}(r)$. A splitting would send $\tilde{g}$ to an element $g + a$, for some $a \in A$. Since $\psi^k(\tilde{g}) = k^r\tilde{g}$ we find that

$$k^r(g + a) = \psi^k(g + a) = k^r g + \alpha_k + \begin{cases} 0 & \text{if } k \text{ is even,} \\ a & \text{if } k \text{ is odd.} \end{cases}$$

Rearranging to solve for α_k, we obtain

$$\alpha_k = \begin{cases} k^r a & \text{if } k \text{ is even,} \\ (k^r - 1)a & \text{if } k \text{ is odd.} \end{cases}$$

So these are the properties of an (α_k) sequence that are equivalent to the extension $0 \leftarrow \mathbb{Z}(r) \leftarrow E \leftarrow A[2] \leftarrow 0$ being split. Notice that such sequences are in some sense the "trivial" solutions of the relations (38.10).

We make one more general comment before returning to our specific situation. Let $a \in A$ and consider the sequence defined by

$$\alpha_k = \begin{cases} \frac{1}{2}k^r.a & k \text{ even} \\ \frac{1}{2}(k^r - 1).a & k \text{ odd.} \end{cases} \tag{38.11}$$

Note that the fractions multiplying a are in fact integers. So this sequence defines a valid extension E, and if a is not a multiple of 2 in A then the extension doesn't "look" split. Precisely, if A is torsion-free and $a \notin 2A$, then the extension is clearly nonsplit. We will see that the case where A is torsion is a bit more subtle.

Now let us return to the extension in (38.9). Here $A = \mathbb{Z}/(2^f)$, where $f = \varphi(N) - \varphi(m)$. Since m is a multiple of 8 (and this is the first place where we use this assumption), f also equals $\varphi(N - m)$.

In Theorem 37.19 we previously computed the action of the Adams operations on $\widetilde{KO}(\mathbb{R}P^N/\mathbb{R}P^{m-1})$. The corresponding α-sequence is precisely the one given by (38.11), where a is a generator for $\mathbb{Z}/(2^f)$. So the extension is split if and only if there exists a $B \in \mathbb{Z}$ such that

$$\tfrac{1}{2}k^r.a = k^r.(Ba) \quad (k \text{ even}), \qquad \tfrac{1}{2}(k^r - 1).a = (k^r - 1).(Ba) \quad (k \text{ odd})$$

for all k. Phrased differently, these say that

$$2^f \text{ divides } \quad \begin{cases} \frac{1}{2}k^r(1 - 2B) & \text{if } k \text{ is even,} \\ \frac{1}{2}(k^r - 1)(1 - 2B) & \text{if } k \text{ is odd.} \end{cases}$$

Since $1 - 2B$ is always odd, this is equivalent to

$$2^{f+1} \text{ divides } \quad \begin{cases} k^r & k \text{ even,} \\ k^r - 1 & k \text{ odd.} \end{cases}$$

Let $\nu(x)$ denote the 2-adic valuation of the integer x, and let us restate what we have now shown: If $8|m$ and $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell, then

$$\varphi(N - m) + 1 \le \min\bigl\{r,\ \nu(3^r - 1),\ \nu(5^r - 1),\ \nu(7^r - 1),\ \nu(9^r - 1),\ \ldots\bigr\}$$

where $r = m/2$. Here we have used $\nu(2^r) = r$ and have also left out $\nu(k^r)$ for even integers $k > 2$, as these numbers are at least r and hence irrelevant for the minimum.

Our next task is to consider the numbers $\nu(3^r - 1)$, and for these we refer to Lemma 38.12 below. Since in our case r is even one has $\nu(3^r - 1) = \nu(r) + 2$. Note that as $r \ge 4$ this term is no larger than r, and so the first term in the above minimum is irrelevant.

We could proceed to analyze the terms $\nu(k^r - 1)$ for odd $k > 3$, which is not hard, but in fact we have done enough to conclude the proof already. We have shown that if $8|m$ and $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ splits off its bottom cell, then

$$\varphi(N - m) + 1 \le \nu(r) + 2 = \nu(m) + 1,$$

or simply $\varphi(N - m) \le \nu(m)$. So our task is to find the largest x for which $\varphi(x) = \nu(m)$. To do this, write $\nu(m) = 4b + a$ where $0 \le a \le 3$. Let "φ-count" stand for counting the integers that are congruent to 0, 1, 2, or 4 modulo 8. Every cycle of 8 consecutive integers contributes 4 to the φ-count, and so for $\varphi(x) \ge 4b$ we would need $x \ge 8b$. The cases $a = 0, 1, 2, 3$ can now be analyzed by hand: for $a = 0$ we have $x = 8b$; for $a = 1$ we have $x = 8b + 1$; for $a = 2$ we have $x = 8b + 3$; and for

$a = 3$ we have $x = 8b + 7$. So in general the largest x such that $\varphi(x) = 4b + a$ is $x = 8b + 2^a - 1$.

Putting everything together, if $\varphi(N - m) \leq \nu(m) = 4b + a$ then $N - m \leq 8b + 2^a - 1 = \rho(m)$. So if $N \geq m + \rho(m) + 1$ then $\mathbb{R}P^N/\mathbb{R}P^{m-1}$ cannot split off its bottom cell. $\square$

Lemma 38.12. *If r is even then $\nu(3^r - 1) = \nu(r) + 2$. If r is odd then $\nu(3^r - 1) = 1$.*

Proof. If r is odd then modulo 4 we have $3^r = (-1)^r = -1$, so $3^r - 1 \equiv 2 \bmod 4$. This proves that $\nu(3^r - 1) = 1$.

If $r = 2^f \cdot u$ where u is odd, we prove by induction on f that $\nu(3^r - 1) = f + 2$. The base case is $f = 1$, and here we use $3^r - 1 = 3^{2u} - 1 = (3^u - 1)(3^u + 1)$. We know $\nu(3^u - 1) = 1$ by the preceding paragraph. Modulo 4 one has $3^u + 1 = (-1)^u + 1 = 0$, but modulo 8 one has $3^u + 1 = 4$. So $\nu(3^u + 1) = 2$, which confirms that $\nu(3^r - 1) = 3$.

For the inductive step, if $r = 2^{f+1}u$ where u is odd then write

$$3^r - 1 = (3^{2^f u} - 1)(3^{2^f u} + 1).$$

By induction we know $\nu(3^{2^f u} - 1) = f + 2$. Modulo 4 we have $3^{2^f u} + 1 = \big((-1)^{2^f}\big)^u + 1 = 1 + 1 = 2$. So $\nu(3^{2^f u} + 1) = 1$, hence $\nu(3^r - 1) = f + 3$. $\square$

Remark 38.13. It is intriguing that a number-theoretic analysis of $\nu(3^r - 1)$ was the ultimate step in both the Hopf invariant one problem and the vector fields on spheres problem. To my knowledge, there is no reason to suspect any connection between these two problems.

Exercise 38.14. If k is any odd number, prove that $\nu(k^r - 1) \geq \nu(r) + 2$ when r is even. This confirms that the terms for $k > 3$ were irrelevant for the minimum considered in the above proof.

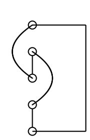
Example 38.15. To demonstrate the proof of Adams's Theorem, consider $m = 576 = 2^6 \cdot 9$. Starting strictly above 576, we mark off numbers until we have exceeded a φ-count of $\nu(m) = 6$. In the following sequence, the numbers contributing to the φ-count have boxes around them:

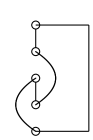
$$\mathbf{576} \mid \boxed{577}, \boxed{578}, 579, \boxed{580}, 581, 582, 583, \boxed{585}, \boxed{585}, \boxed{586}, 587, \boxed{588}$$

Adams's argument shows that $\mathbb{R}P^{588}/\mathbb{R}P^{575}$ does not split off its bottom cell. Note, of course, that $\rho(576) = 11$ and $588 = 576 + 11 + 1$. The point, however, is that one does not need to remember the awkward formula for $\rho(m)$; the procedure is simply to count past m until the φ-count exceeds $\nu(m)$.

38.16. **Atiyah duality (Step 5).** This is the final piece. The material in this section will complete our proof of Theorem 38.1.

Consider the space $\mathbb{R}P^9/\mathbb{R}P^4$. Its cohomology is shown below in the diagram on the left:

$H^*(\mathbb{R}P^9/\mathbb{R}P^4)$ 

 $H^*(????)$

We obtained the picture on the right by simply turning the left diagram upside down; is this also the cohomology of a space? It is easy to see that the answer is yes:

the right diagram is $H^*(\mathbb{R}P^{10}/\mathbb{R}P^5)$. This turns out to be a general phenomenon, first discovered by Atiyah. And the kind of 'duality' we are seeing actually takes place at a deeper level than just that of cohomology. It is essentially a geometric duality, taking place inside of the stable homotopy category.

The stable homotopy category is symmetric monoidal: the monoidal product is the smash $E, F \mapsto E \wedge F$, and the unit is the sphere spectrum S. It is also *closed* symmetric monoidal, meaning that there exist function objects $E, F \mapsto \mathcal{F}(E, F)$ and a natural adjunction

$$\mathrm{Hom}_{\mathrm{Ho}\,(Sp)}(E, \mathcal{F}(X, Y)) \cong \mathrm{Hom}_{\mathrm{Ho}\,(Sp)}(E \wedge X, Y).$$

Spanier-Whitehead duality has to do with the functor $X \mapsto DX = \mathcal{F}(X, S)$. This functor preserves cofiber sequences and it sends the n-sphere $S^n = \Sigma^\infty S^n$ to the $(-n)$-sphere S^{-n}. So if a certain spectrum X is built from cells in dimensions 0 through n, the spectrum DX is built from cells in dimensions $-n$ through 0.

For nice enough spectra X one has the property that $D(D(X)) \simeq X$; such spectra are called **dualizable**. All finite cell complexes are dualizable. One can prove that $H^*(DX)$ agrees with 'taking $H^*(X)$ and turning it upside down'.

The essentials of Spanier-Whitehead duality were known long before the details of the stable homotopy category had all been worked out (particularly the details behind the smash product). Here is the main result for finite complexes:

Proposition 38.17. *Let X be a finite cell complex that is embedded in S^n as a subcomplex (of some chosen cell structure on S^n). Then*

$$\Sigma^{n-1} D(\Sigma^\infty X) \simeq \Sigma^\infty (S^n - X).$$

Example 38.18. Let us check the above proposition in some very easy examples. We use the form $D(X) \simeq \Sigma^{-n+1}(S^n - X)$.

(a) $X = S^0$. Then $S^n - S^0 \simeq S^{n-1}$, and so we find $D(S^0) \simeq \Sigma^{-n+1} S^{n-1} \simeq S^0$. The 0-sphere is self-dual.

(b) Let $X = S^{n-1}$, embedded as the equator in S^n. The complement $S^n - X$ is S^0 (up to homotopy), so we have $D(S^{n-1}) \simeq \Sigma^{-n+1} S^0 = S^{-(n-1)}$. Again, this is as expected.

For some classical references on Spanier-Whitehead duality, see [A4, Chapter III.5] and [Swz, Chapter 14].

We can now state Atiyah's main theorem. The proof is taken directly from [At2]. See Section 16.13 for a discussion of Thom spaces of virtual bundles.

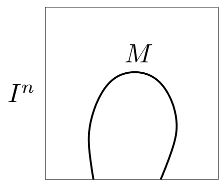
Theorem 38.19 (Atiyah Duality)**.** *Let M be a compact, smooth manifold.*

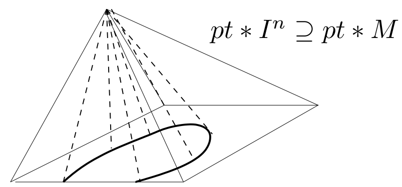
(a) If M has boundary then $D(M/\partial M) \simeq \mathrm{Th}(-T_M)$ where $T_M \to M$ is the tangent bundle.

(b) Now assume that M is closed, and let $E \to M$ be a real vector bundle. Then

$$D(\mathrm{Th}\, E) \simeq \mathrm{Th}(-E - T_M).$$

Proof. For (a), embed M into I^n (nicely) in such a way that ∂M maps into $I^{n-1} \times \{0\}$. (See [At2] for details). Consider the join $pt * I^n$, which is a pyramid; its boundary is homeomorphic to S^n. Refer to the following picture for an example:





Consider the subcomplex

$$X = M \cup (pt * \partial M) \subseteq \partial(pt * I^n) \cong S^n.$$

We have $M/\partial M \simeq M \cup (pt * \partial M)$, and so

$$D(M/\partial M) \simeq D(X) \simeq \Sigma^{-n+1}(S^n - X).$$

Projection away from pt gives a deformation retraction $S^n - X \xrightarrow{\sim} I^n - M$. Next observe $I^n - M \simeq I^n - U$, where U is a tubular neighborhood of M in I^n. Finally, notice that since I^n is contractible we have that $I^n/(I^n - U)$ is a model for $\Sigma(I^n - U)$ (up to homotopy). Putting everything together, we have

$$\begin{aligned} D(M/\partial M) \simeq \Sigma^{-n+1}(I^n - U) \simeq \Sigma^{-n}(I^n/(I^n - U)) &= \Sigma^{-n}\operatorname{Th}(N_{I^n/M}) \\ &\simeq \operatorname{Th}(N_{I^n/M} - \underline{n}). \end{aligned}$$

Now use that $T_M \oplus N_{I^n/M} \cong \underline{n}$.

For (b) use that $\operatorname{Th}(E) \cong Di(E)/S(E)$, where $Di(E)$ is the disk bundle and $S(E)$ is the sphere bundle of E. The disk bundle is a compact manifold with boundary $S(E)$, so by (a) one has

$$D(\operatorname{Th} E) = D\big(Di(E)/S(E)\big) \simeq \operatorname{Th}(-T_{Di(E)}).$$

If $\pi\colon Di(E) \to M$ is the bundle map then it is easy to see that $T_{Di(E)} \cong \pi^*(E \oplus T_M)$. Since π is a homotopy equivalence,

$$\operatorname{Th}(-T_{Di(E)}) = \operatorname{Th}(-\pi^*(E \oplus T_M)) \simeq \operatorname{Th}(-(E \oplus T_M)).$$

This finishes the proof. $\square$

We next apply what we just learned to stunted projective spaces. Recall from Example 16.10 that all stunted projective spaces are Thom spaces:

$$\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \cong \operatorname{Th}\begin{pmatrix} {\scriptstyle (n-k)L} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{k-1}} \end{pmatrix}.$$

Recall as well that the tangent bundle to $\mathbb{R}P^{k-1}$ satisfies $T \oplus 1 \cong kL$ (Example 26.11). Using these two facts, Atiyah Duality now gives that

$$D(\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}) \simeq \operatorname{Th}\begin{pmatrix} {\scriptstyle -(n-k)L-T} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{k-1}} \end{pmatrix} = \operatorname{Th}\begin{pmatrix} {\scriptstyle -nL+1} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{k-1}} \end{pmatrix} \simeq \Sigma\operatorname{Th}\begin{pmatrix} {\scriptstyle -nL} \\ \downarrow \\ {\scriptstyle \mathbb{R}P^{k-1}} \end{pmatrix}.$$

Let r_{k-1} be the additive order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$. Then $r_{k-1}L \cong \underline{r_{k-1}}$ (stably), and hence for any $s \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathrm{Th}\begin{pmatrix} -nL \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} \simeq \Sigma^{-sr_{k-1}} \mathrm{Th}\begin{pmatrix} -nL+sr_{k-1} \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} &\simeq \Sigma^{-sr_{k-1}} \mathrm{Th}\begin{pmatrix} -nL+sr_{k-1}L \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} \\
&\simeq \Sigma^{-sr_{k-1}} \mathrm{Th}\begin{pmatrix} (sr_{k-1}-n)L \\ \downarrow \\ \mathbb{R}P^{k-1} \end{pmatrix} \\
&\simeq \Sigma^{-sr_{k-1}} \Big[\mathbb{R}P^{sr_{k-1}-n+k-1}/\mathbb{R}P^{sr_{k-1}-n-1}\Big].
\end{aligned}$$

In the last line we imagine s chosen to be large enough so that $sr_{k-1}-n-1 \geq 0$. Putting everything together, we have proven the following:

Corollary 38.20. *Let r_{k-1} be the additive order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{k-1})$. Then there is a stable homotopy equivalence*

$$D(\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}) \simeq \Sigma^{1-sr_{k-1}} \Big[\mathbb{R}P^{sr_{k-1}-n+k-1}/\mathbb{R}P^{sr_{k-1}-n-1}\Big]$$

where s is any integer such that $sr_{k-1}-n-1 \geq 0$.

Example 38.21. Let us consider the Spanier-Whitehead dual of $\mathbb{R}P^9/\mathbb{R}P^4$, as in the beginning of this section. Relative to our above discussion, $n = 10$ and $k = 5$. By Theorem 37.14 we know $\widetilde{KO}(\mathbb{R}P^4) \cong \mathbb{Z}/8$, so the order of $[L]-1$ is 8. The above corollary gives

$$D(\mathbb{R}P^9/\mathbb{R}P^4) \simeq \Sigma^{1-8s}\Big[\mathbb{R}P^{8s-6}/\mathbb{R}P^{8s-11}\Big]$$

for any s where the right-hand-side makes sense. The smallest choice is $s = 2$, giving $D(\mathbb{R}P^9/\mathbb{R}P^4) \simeq \Sigma^{-15}(\mathbb{R}P^{10}/\mathbb{R}P^5)$.

39. The immersion problem for $\mathbb{R}P^n$

Let M be a compact, n-dimensional, real manifold. It is a classical theorem of Whitney from the 1940s that M can be immersed in $\mathbb{R}^{2n-1}$ and embedded into $\mathbb{R}^{2n}$ [Wh1, Wh2]. A much more difficult result, proved by R. Cohen [C] in 1985, says that M can be immersed in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the binary expansion of n. As a general result this is known to be the best possible, but for specific choices of M one could conceivably do better. Define the **immersion dimension** (resp. the **embedding dimension**) of M to be the smallest k such that M immerses (resp., embeds) into $\mathbb{R}^k$.

In general, determining the immersion and embedding dimensions of a given manifold seem to be difficult problems. Over the last 60+ years they have been extensively studied, particularly for the manifold $\mathbb{R}P^n$. The problem tends to involve two distinct components. As one aspect, clever geometric constructions are used to produce immersions (or embeddings) and therefore upper bounds on the immersion dimension. For lower bounds one must prove *non*-immersion results, and this is usually done by making use of some sort of homotopical invariants. Over the years the problem for $\mathbb{R}P^n$ has been used as a sort of testing ground for every new homotopical technique to come along.

Our intent here is not to give a complete survey of this problem, as this would take far too long. We will be content to give a small taste, entirely concentrating our focus on some easily-obtained lower bounds in the case of $\mathbb{R}P^n$. The methods involve Stiefel-Whitney classes (in singular cohomology) and some related characteristic classes in KO-theory.

39.1. **A short survey.** Before jumping into our analysis, let us give some sense of what is known about the problem. The following table shows the current knowledge (as of this writing) about the immersion and embedding dimensions for $\mathbb{R}P^n$ when $n \leq 24$:

TABLE 39.2. Immersion and embedding dimensions for $\mathbb{R}P^n$

$\mathbb{R}P^n$	2	3	4	5	6	7	8	9	10	11	12	13
imm. dim.	3	4	7	7	7	8	15	15	16	16	18	22
emb. dim.	4	5	8	9	$[9, 11]$	$[9, 12]$	16	17	17	$[17, 18]$	$[18, 21]$	$[22, 23]$

$\mathbb{R}P^n$	14	15	16	17	18	19	20	21	22	23	24
imm.	22	22	31	31	32	32	34	38	38	38	$[38, 39]$
emb.	$[22, 23]$	$[23, 24]$	32	33	33	$[33, 34]$	$[34, 37]$	39	39	39	$[39, 42]$

In the table, entries in brackets are given when the exact answer is not known. For example, the embedding dimension of $\mathbb{R}P^6$ is only known to lie in the interval $[9, 11]$. $\mathbb{R}P^6$ definitely embeds into $\mathbb{R}^{11}$ and does not embed into $\mathbb{R}^8$—but it is not known if $\mathbb{R}P^6$ embeds into $\mathbb{R}^9$ or $\mathbb{R}^{10}$. In comparison, we know much more about the immersion problem; the smallest unknown case is $\mathbb{R}P^{24}$.

The above data on the immersion and embedding dimensions was taken from a table compiled by Don Davis [Da2]. Davis's table contains substantially more data, covering slightly past $\mathbb{R}P^{100}$.

One of the earliest results is due to Milnor [MS]: if $n = 2^r$ then the immersion dimension of $\mathbb{R}P^n$ equals $2n-1$ (showing that the Whitney upper bound is sharp in this case). Peterson [P] proved that if $n = 2^r$ then the embedding dimension equals $2n$, again showing that the Whitney bound is sharp here. (We recount Milnor's proof in Corollary 39.11 below.) In general, if $n = 2^r + d$ for $0 \leq d < 2^r$ and d is relatively small, then one can expect the immersion and embedding dimensions to be $2^{r+1} + x$ where x is a known quantity or one that is tightly constrained. The following theorem encompasses most of what is known for $d \leq 10$:

Theorem 39.3. *Write $n = 2^i + d$ where $0 \leq d < 2^i$. Then the immersion dimension of $\mathbb{R}P^n$ equals $2^{i+1} + e$ and the embedding dimension equals $2^{i+1} + f$ where the following is known:*

d	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
i	≥ 1	≥ 2	≥ 3	≥ 3	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4	≥ 4
e	-1	-1	0	0	2	6	6	6	$[6, 7]$	14	14
f	0	1	1	$[1, 2]$	$[2, 5]$	7	7	7	$[7, 10]$	$[14, 15]$	$[14, 16]$

For example, for $\mathbb{R}P^{40}$ we write $40 = 2^5 + 8$ and consult the $d = 8$ column. It tells us that the immersion dimension is either $2^6 + 6$ or $2^6 + 7$ (70 or 71), and that the embedding dimension lies between $2^6 + 7$ and $2^6 + 10$ (71 through 74).

The above theorem is not credited because it represents the combined work over many years of several authors. Much credit should be given to Davis, who has brought all the results together and given complete references. The above theorem is just the first few lines of the table [Da2].

39.4. **Stiefel-Whitney techniques.** Suppose that M is a compact manifold of dimension n, and that M is immersed in $\mathbb{R}^{n+k}$. The immersion has a normal bundle ν, and there is an isomorphism of bundles $T_M \oplus \nu \cong \underline{n+k}$. Taking total Stiefel-Whitney classes of both sides gives

$$w(T_M) \cdot w(\nu) = w(T_M \oplus \nu) = w(\underline{n+k}) = 1.$$

Recall that the total Stiefel-Whitney class of a bundle E is $w(E) = 1 + w_1(E) + w_2(E) + \cdots$. Because the zero-coefficient is 1 we can formally invert this expression, and because $H^*(M)$ is zero in sufficiently large degrees this formal inverse actually makes sense as an element of $H^*(M)$. So we can feel free to write $w(E)^{-1}$, and we obtain

$$w(T_M)^{-1} = w(\nu).$$

We don't know anything about ν except its rank, which is equal to k. This guarantees that $w(\nu)$ does not have any terms of degree larger than k, and so we obtain the following simple result [MS, material preceding Theorem 4.8]:

Proposition 39.5. *Let M be a compact manifold of dimension n. If M immerses in $\mathbb{R}^{n+k}$ then $w(T_M)^{-1}$ vanishes in degrees larger than k.*

Let us apply this proposition to $\mathbb{R}P^n$. Here we have the identity $T_{\mathbb{R}P^n} \oplus 1 = (n+1)L$ (Example 26.11) and so

$$w(T_{\mathbb{R}P^n}) = w(T_{\mathbb{R}P^n} \oplus 1) = w((n+1)L) = w(L)^{n+1} = (1+x)^{n+1},$$

where x denotes the generator for $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. Taking inverses gives

$$w(T_{\mathbb{R}P^n})^{-1} = (1+x)^{-(n+1)} = \sum_{i=0}^{\infty} \tbinom{-(n+1)}{i} x^i.$$

We can rewrite the coefficient of x^i, since

$$\tbinom{-(n+1)}{i} = (-1)^i \tfrac{(n+1)\cdot(n+2)\cdots(n+i)}{i!} = (-1)^i \tbinom{n+i}{i}.$$

Putting everything together we obtain the following:

Corollary 39.6. *If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then $\binom{n+i}{i}$ is even for $k < i \leq n$.*

The above corollary yields very concrete non-immersion results, but to obtain these we need to be good at checking when binomial coefficients are even. Historically, topologists got pretty good at this because of the presence of binomial coefficients in the Adem relations. A key result is the following:

Lemma 39.7. *Let $n_j n_{j-1} \dots n_0$ be the base 2 representation for n; that is, each $n_j \in \{0,1\}$ and $n = \sum n_j 2^j$. Similarly, let $k_j k_{j-1} \dots k_0$ be the base 2 representation for k. Then*

$$\tbinom{n}{k} \equiv \prod_i \tbinom{n_i}{k_i} \quad \mod 2.$$

Proof. The result follows easily from four points: (i) $\binom{2n}{i}$ is even if i is odd; (ii) $\binom{2n}{2k} \equiv \binom{n}{k}$ (mod 2); (iii) $\binom{2n+1}{2k} \equiv \binom{2n}{2k}$ (mod 2); and (iv) $\binom{2n+1}{2k+1} \equiv \binom{n}{k}$ (mod 2). The four cases correspond to the four possibilities for n_0 and k_0, and set us up for a straightforward induction.

For (i) and (ii), imagine a column of the numbers 1 through n and a second "mirror" column containing the same entries. If i is odd, the i-element subsets of the two columns together may be partitioned into two classes: those which contain more elements from column A than column B, and those which contain less elements from column A. The operation of "switch entries between the two columns" gives a bijection between these two classes, thereby showing that $\binom{2n}{i}$ is even.

For (ii), note that the i-element subsets can be partitioned into groups determined by the number of elements from column A. This gives rise to the formula

$$\binom{2n}{2k} = \binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n}{k-1} + \cdots + \binom{n}{k-1}\binom{n}{1} + \binom{n}{k}\binom{n}{0}.$$

The terms on the right-hand-side are symmetric and so can be grouped in pairs, except for the middle term which is $\binom{n}{k}^2$. So working mod 2 we have

$$\binom{2n}{2k} \equiv \binom{n}{k}^2 \equiv \binom{n}{k}.$$

For (iii) just use Pascal's identity $\binom{2n+1}{2k} = \binom{2n}{2k} + \binom{2n}{2k-1}$ together with (i). For (iv) use $\binom{2n+1}{2k+1} = \binom{2n}{2k+1} + \binom{2n}{2k}$ together with (i) and (ii). $\square$

Example 39.8. To determine if $\binom{20}{9}$ is even then we note that 20 is 10100 in base 2, and 9 is 1001. Using the above lemma we compute

$$\binom{20}{9} = \binom{1\,0\,1\,0\,0}{0\,1\,0\,0\,1} \equiv \binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{1} = 1\cdot 0\cdot 1\cdot 1\cdot 0 = 0.$$

So $\binom{20}{9}$ is even.

Notice that $\binom{1}{1} = \binom{1}{0} = \binom{0}{0} = 1$, whereas $\binom{0}{1} = 0$. So in computations like the one above, the fnal answer is even if and only if $\binom{0}{1}$ appears at least once in the product—that is, if n has a certain bit turned "off" and the corresponding bit of k is "on". In particular, the following three statements are now obvious:

(i) If $n = 2^r$ then $\binom{n}{i}$ is even for all i in the range $0 < i < n$.
(ii) If $n = 2^r - 1$ then $\binom{n}{i}$ is odd for all $i \leq n$.
(iii) $\binom{2n}{n}$ is always even.

The point for the first two is that 2^r has all of its bits turned off except for the rth, whereas $2^r - 1$ has all of its bits turned on. For statement (iii) just consider the smallest bit of n that is turned on, and note that the corresponding bit is off in $2n$.

Corollary 39.6 is most often used in the form below. The proof is immediate from Corollary 39.6.

Corollary 39.9. *Fix $n \geq 2$, and let k be the largest integer such that $k \leq n$ and $\binom{n+k}{k}$ is odd. Then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{n+k-1}$.*

Note that k will be strictly less than n, as $\binom{2n}{n}$ is always even; this conforms with the Whitney immersion result. As an application of the above corollary, suppose we want to immerse $\mathbb{R}P^{10}$. We start with

$$\binom{20}{10} = \frac{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11}{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

We have 2^{10} dividing the numerator and 2^8 dividing the denominator. Start removing factors from the left, one by one from the numerator and denominator simultaneously, watching what happens to the number of 2's in each. The fraction does not become odd until we are looking at $\binom{15}{5}$. So the conclusion is that $\mathbb{R}P^{10}$ does not immerse into $\mathbb{R}P^{14}$.

The above process is cumbersome, and with a little investigation it is not hard to produce a shortcut.

Proposition 39.10. *Write $n = 2^i + d$ where $0 \le d < 2^i$. Then the largest k in the range $0 \le k \le n$ such that $\binom{n+k}{k}$ is odd is $k = 2^i - d - 1$.*

Proof. Note that if $j = 2^i - d - 1$ then $n+j = 2^{i+1} - 1$ and so $\binom{n+j}{j}$ is certainly odd. We must show that $\binom{n+j}{j}$ is even for j in the range $2^i - d \le j \le n$. This is the kind of analysis that is perhaps best left to the reader, but we will give a sketch. Suppose to the contrary that j is in this range and $\binom{n+j}{j}$ is odd. Let $e = j - (2^i - d - 1)$, so that $n + j = (2^i + d) + e + (2^i - d - 1) = e + (2^{i+1} - 1)$. Let the smallest bit of e that is turned on be the rth bit; this is also the smallest bit of $n + j$ that is turned off. Since $\binom{n+j}{n}$ is odd, this bit must be also off in n. If we write $e = e' + 2^r$, then $n + j = e' + (2^r + 2^{i+1} - 1)$. The term in parentheses has all bits off from the $(r + 1)$st through the ith, and so the bits of $n + j$ agree with the bits of e' (and of e) in this range. Since $\binom{n+j}{n}$ is odd, every bit of n that is turned on in this range must also be turned on in $n + j$—and therefore also in e. We have thus shown that

- The rth bit is off in n but on in e, and
- All bits greater than the rth that are on in n are also on in e.

These two facts show that $e > n$, which is not allowed since $e \le j \le n$. $\square$

Corollary 39.11. *If $n = 2^i + d$ where $0 \le d \le 2^i - 1$ then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{2^{i+1}-2}$. In particular, if $n = 2^i$ then the immersion dimension of $\mathbb{R}P^n$ equals $2n - 1$.*

Proof. The first line is immediate from Corollary 39.9 and Proposition 39.10. The second statement follows from the first together with the Whitney theorem saying that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-1}$. $\square$

Let us now change gears just a bit and consider embeddings. We can also use characteristic classes to give obstructions in this setting. The key result is the following:

Proposition 39.12. *Suppose M is a compact n-manifold that is embedded in $\mathbb{R}^{n+k}$. Then $w_k(\nu) = 0$ where ν is the normal bundle.*

Proof. Choose a metric on ν and let $S(\nu)$ be the sphere bundle. If $p\colon S(\nu) \to M$ denotes the projection map, then clearly $p^*\nu$ splits off a trivial bundle: $p^*\nu = \underline{1} \oplus E$ for some rank $k - 1$ bundle E on $S(\nu)$. The Whitney formula then gives $0 = w_k(p^*\nu) = p^*(w_k(\nu))$.

The proof will be completed by showing that $p^*\colon H^*(M; \mathbb{Z}/2) \to H^*(S(\nu); \mathbb{Z}/2)$ is injective. Let U be a tubular neighborhood of M in $\mathbb{R}^{n+k}$, arranged so that its closure $\overline{U}$ is homeomorphic to the disk bundle of ν. Write $\partial\overline{U}$ for the boundary, which is isomorphic to $S(\nu)$. We have the long exact sequence

$$\cdots \to H^i(\overline{U}, \partial\overline{U}) \to H^i(\overline{U}) \to H^i(\partial\overline{U}) \to \cdots$$

where all cohomology groups have $\mathbb{Z}/2$-coefficients. The projection $\overline{U} \to M$ is a homotopy equivalence, so our map p^* is isomorphic to $H^i(\overline{U}) \to H^i(\partial\overline{U})$. We can verify that this is injective by checking that the previous map in the long exact sequence is zero. We look only at $i > 0$, as the $i = 0$ case is trivial.

To this end, consider the diagram below:

$$\begin{tikzcd} H^i(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} - M) \arrow[r] \arrow[d, "\cong"'] & H^i(\mathbb{R}^{n+k}) \arrow[d] \\ H^i(\overline{U}, \partial\overline{U}) \arrow[r] & H^i(\overline{U}) \end{tikzcd}$$

The left vertical map is an isomorphism by excision, and the group in the upper right corner is zero. So the bottom horizontal map is zero, as we desired. $\square$

Compare the next result to Proposition 39.5:

Corollary 39.13. *Let M be a compact n-manifold. If M embeds into $\mathbb{R}^{n+k}$ then $w(T_M)^{-1}$ vanishes in degrees k and larger.*

Proof. We saw in the proof of Proposition 39.5 that $w(T_M)^{-1} = w(\nu)$. Since ν has rank k, this forced the Stiefel-Whitney classes to vanish in degrees larger than k. But now Proposition 39.12 also gives us the vanishing in degree k. $\square$

Corollary 39.14. *If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $\binom{n+i}{i}$ is even for $k \le i \le n$.*

Proof. Same as for Corollary 39.6. $\square$

Corollary 39.15. *Let $n = 2^i + d$ where $0 \le d \le 2^i - 1$. Then $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{2^{i+1}-1}$. In particular, if $n = 2^i$ then the embedding dimension of $\mathbb{R}P^n$ equals $2n$.*

Proof. Let k be the largest integer in the range $0 \le k \le n$ such that $\binom{n+k}{k}$ is odd. Then Corollary 39.14 shows that $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{n+k}$. But Proposition 39.10 identifies $k = 2^i - d - 1$, and so $n + k = 2^{i+1} - 1$. This proves the first statement. The second statement is then a consequence of the first together with the Whitney theorem that $\mathbb{R}P^n$ embeds into $\mathbb{R}^{2n}$. $\square$

To gauge the relative strength of Corollaries 39.11 and 39.15, see the table below and compare to Table 39.2. One gets a clear sense of how far algebraic topology has progressed since the early days of Stiefel-Whitney classes!

Table 39.16. Stiefel-Whitney lower bounds for the immersion and embedding dimensions of $\mathbb{R}P^n$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
imm$\ge$	3	4	7	7	7	8	15	15	15	15	15	15	15	16	31	31	31
emb$\ge$	4	4	8	8	8	8	16	16	16	16	16	16	16	16	32	32	32

39.17. ***K*-theoretic techniques.** One can readily imagine taking the basic approach from the last section and replacing the Stiefel-Whitney classes with characteristic classes taking values in some other cohomology theory. Atiyah [At3] pursued this idea using KO-theory, and certain constructions of Grothendieck provided the appropriate theory of characteristic classes. In this way he obtained some new non-immersion and non-embedding theorems. We explain this work next.

In Section 33 we described Chern classes in complex K-theory, which were defined using the $\tilde{\gamma}$-operations. One can basically repeat those constructions in KO-theory verbatim to obtain KO-characteristic classes for real vector bundles. Since the context is a little different, we give a very quick review here.

For an element $x \in \widetilde{KO}(X)$ define

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \tfrac{t}{1-t}[\lambda^1 x] + \left(\tfrac{t}{1-t}\right)^2[\lambda^2 x] + \cdots$$

Note that $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$. Define $\gamma^i(x)$ to be the coefficient of t^i in $\gamma_t(x)$. So we have

$$\gamma_k(x+y) = \sum_{i+j=k} \gamma^i(x)\gamma^j(y).$$

Note also that $\gamma_t(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t} = 1 + t + t^2 + \cdots$

For a vector bundle E over X define

$$\tilde{\gamma}_t(E) = \gamma_t(E - \underline{\operatorname{rank} E}) = \frac{\gamma_t(E)}{\gamma_t(\underline{\operatorname{rank} E})} = \frac{\gamma_t(E)}{\gamma_t(1)^{\operatorname{rank} E}} = \gamma_t(E)\cdot(1-t)^{\operatorname{rank} E}.$$

One should think of this as just being a renormalization of the γ_t construction; note that $\tilde{\gamma}_t(E) = 1$ if E is a trivial bundle. Observe that we still have the analog of the Whitney formula:

$$\begin{aligned}\tilde{\gamma}_t(E\oplus F) = \gamma_t(E\oplus F - \underline{\operatorname{rank}(E+F)}) &= \gamma_t\big((E - \underline{\operatorname{rank} E}) + (F - \underline{\operatorname{rank} F})\big)\\ &= \gamma_t(E - \underline{\operatorname{rank} E})\gamma_t(F - \underline{\operatorname{rank} F})\\ &= \tilde{\gamma}_t(E)\tilde{\gamma}_t(F).\end{aligned}$$

If L is a line bundle then

$$\tilde{\gamma}_t(L) = \gamma_t(L)\cdot(1-t) = \left[1 + \tfrac{t}{1-t}[L]\right]\cdot(1-t) = 1 - t + t[L] = 1 + t([L]-1).$$

So $\tilde{\gamma}_1(L) = [L] - 1$ and $\tilde{\gamma}_i(L) = 0$ for $i > 1$.

Finally, we observe that if E is a rank k bundle then $\tilde{\gamma}_i(E) = 0$ for all $i > k$. This is because

$$\tilde{\gamma}_t(E) = \lambda_{\frac{t}{1-t}}(E)(1-t)^k = \sum_{i=0}^{\infty}(\tfrac{t}{1-t})^i[\Lambda^i E]\cdot(1-t)^k = \sum_{i=0}^{k} t^i(1-t)^{k-i}[\Lambda^i E].$$

The final expression is a polynomial in t of degree at most k.

Compare the following result to Proposition 39.5 and Corollary 39.13.

Proposition 39.18. *Let M be a compact n-manifold. If M immerses into $\mathbb{R}^{n+k}$ then the power series $\tilde{\gamma}_t(T_M)^{-1}$ vanishes in degrees larger than k. If M embeds into $\mathbb{R}^{n+k}$ then $\tilde{\gamma}_t(T_M)^{-1}$ vanishes in degrees k and larger.*

Proof. The proofs are the same as before. If M immerses into $\mathbb{R}^{n+k}$ then $T_M \oplus \nu \cong \underline{n+k}$ where ν is the normal bundle. So $\tilde{\gamma}_t(T_M)\tilde{\gamma}_t(\nu) = \tilde{\gamma}_t(T_M\oplus\nu) = \tilde{\gamma}_t(\underline{n+k}) = 1$, and then $\tilde{\gamma}_t(T_M)^{-1} = \tilde{\gamma}_t(\nu)$. But since ν has rank k we have $\tilde{\gamma}_i(\nu) = 0$ for $i > k$.

For the second part of the proposition we need to prove that if M embeds into $\mathbb{R}^{n+k}$ then $\tilde{\gamma}_k(\nu) = 0$. The proof is exactly the same as for Proposition 39.12. $\square$

For the following proposition, recall from Remark 37.15 that $\varphi(n)$ denotes the number of integers s such that $0 < s \le n$ and s is congruent to 0, 1, 2, or 4 modulo 8.

Corollary 39.19 (Atiyah)**.**

(a) If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then $2^{\varphi(n)-j+1}$ divides $\binom{n+j}{j}$ for $k < j \leq \varphi(n)$.
(b) If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $2^{\varphi(n)-j+1}$ divides $\binom{n+j}{j}$ for $k \leq j \leq \varphi(n)$.

Proof. Recall that $1 \oplus T_{\mathbb{R}P^n} \cong (n+1)L$, as in Example 26.11. We get

$$\tilde{\gamma}_t(T_{\mathbb{R}P^n}) = \tilde{\gamma}_t(T_{\mathbb{R}P^n} \oplus 1) = \tilde{\gamma}_t((n+1)L) = \tilde{\gamma}_t(L)^{n+1} = \big(1 + t\lambda\big)^{n+1}$$

where $\lambda = [L] - 1 \in \widetilde{KO}^0(\mathbb{R}P^n)$. So

$$\tilde{\gamma}_t(T_{\mathbb{R}P^n})^{-1} = (1+t\lambda)^{-(n+1)} = \sum_{j=0}^{\infty} \binom{-(n+1)}{j}\lambda^j \cdot t^j = \sum_{j=0}^{\infty}(-1)^j\binom{n+j}{j}\lambda^j \cdot t^j.$$

If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$ then by Proposition 39.18 $\binom{n+j}{j}\lambda^j = 0$ in $\widetilde{KO}^0(\mathbb{R}P^n)$ for all $k < j$. If $\mathbb{R}P^n$ embeds into $\mathbb{R}^{n+k}$ then $\binom{n+j}{j}\lambda^j = 0$ for all $k \leq j$.

Now we recall from Theorem 37.14 that $\widetilde{KO}^0(\mathbb{R}P^n) \cong \mathbb{Z}/(2^{\varphi(n)})$ and that λ is a generator. Also recall that $\lambda^2 = -2\lambda$, or $\lambda^j = (-2)^{j-1}\lambda$. The desired conclusions follow immediately. □

Corollary 39.19 is best used in the following form. Let $\sigma(n)$ denote the largest value of j in the range $1 \leq j \leq \varphi(n)$ for which $\binom{n+j}{j}$ is not divisible by $2^{\varphi(n)+1-j}$; if no such j exists then set $\sigma(n) = 0$ by default. Then $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{n+\sigma(n)-1}$ and does not embed into $\mathbb{R}^{n+\sigma(n)}$.

For some values of n the result of Corollary 39.19 is stronger than what we obtained from Corollaries 39.6 and 39.15, and for some values of n it is weaker. We demonstrate some examples:

Example 39.20. For the question of immersions of $\mathbb{R}P^8$, we have $\varphi(8) = 4$ and $\sigma(8) = 4$. So Atiyah's result (39.19) gives that $\mathbb{R}P^8$ does not immerse into $\mathbb{R}^{11}$. The Stiefel-Whitney classes, however, told us that $\mathbb{R}P^8$ does not immerse into $\mathbb{R}^{14}$.

In contrast, for $\mathbb{R}P^{15}$ we have $\varphi(15) = 7$ and $\sigma(15) = 4$. So Atiyah's result tells us that $\mathbb{R}P^{15}$ does not immerse into $\mathbb{R}^{18}$. The method of Stiefel-Whitney classes (39.6) gives no information in this case.

The table below shows the lower bounds for the immersion dimension of $\mathbb{R}P^n$ obtained from Stiefel-Whitney techniques versus the KO-theoretic techniques. The reader will notice that the Stiefel-Whitney bounds are significantly better when $n = 2^i + d$ and d is small, whereas the KO-theoretic bounds are better for $n = 2^i + d$ when d is close to (but not exceeding) $2^i - 1$.

TABLE 39.21. Lower bounds for the immersion dimension of $\mathbb{R}P^n$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
S-W	3	4	7	7	7	8	15	15	15	15	15	15	15	16	31	31	31
KO	3	4	7	7	7	8	12	13	15	15	17	17	19	19	24	25	27

n	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
S-W	31	31	31	31	31	31	31	31	31	31	31	31	32	63	63
KO	27	31	31	31	31	34	35	38	39	40	41	42	43	48	49

As an example when n is much larger, the Stiefel-Whitney classes give no information on the immersion dimension of $\mathbb{R}P^{255}$. By contrast, the Atiyah result gives that the immersion dimension is at least 375. For $n = 2^i - 1$ the improvement of the Atiyah bound over the Hopf bound is on the order of $\frac{n}{2}$.

39.22. **Immersions and geometric dimension.** So far we have used various characteristic classes to give lower bounds on the immersion/embedding dimensions for $\mathbb{R}P^n$. To close this section we will show how to produce upper bounds for the *immersion* dimension, via a geometric result of Hirsch that translates this into a bundle-theoretic problem. The central tool again ends up being KO-theory. Using these methods we will completely determine the immersion dimension of $\mathbb{R}P^n$ for $n \leq 9$.

Let $E \to X$ be a vector bundle. Define the **virtual dimension** of E by the formula

$$\operatorname{v.dim} E = \min\{k \mid E \cong F \oplus \underline{(\operatorname{rank} E) - k} \text{ for some bundle } F \text{ of rank } k\}.$$

The virtual dimension is the smallest k such that E is isomorphic to a stabilized rank k bundle. If X is paracompact Hausdorff then by Proposition 9.2 we can also write

$$\operatorname{v.dim} E = \operatorname{rank} E - \max\{j \mid E \text{ has } j \text{ independent sections}\}.$$

Note that the virtual dimensions of E and $E \oplus \underline{1}$ might be different; the latter might be smaller than the former. With this in mind we can also introduce the **stable virtual dimension**:

$$\begin{aligned}\operatorname{sv.dim} E &= \min\{\operatorname{v.dim}(E \oplus \underline{r}) \mid r \geq 0\}\\ &= \min\{\operatorname{rank} F \mid F \text{ is a bundle that is stably equivalent to } E\}.\end{aligned}$$

Finally, we introduce the following related concept. For $\alpha \in \widetilde{KO}^0(X)$, define the **geometric dimension** of α to be

$$\operatorname{g.dim} \alpha = \min\{k \,\big|\, \alpha + \underline{k} = [F] \text{ for some vector bundle } F \text{ on } X\}.$$

The above three concepts are related as follows:

Proposition 39.23. *Let $E \to X$ be a vector bundle, where X is compact.*
(a) $\operatorname{g.dim}(E - \underline{\operatorname{rank} E}) = \operatorname{sv.dim}(E) \leq \operatorname{v.dim}(E)$.
(b) If X is a finite CW-complex and $\operatorname{rank} E > \dim X$ *then* $\operatorname{sv.dim}(E) = \operatorname{v.dim}(E)$.

Proof. For (a) only the first equality requires proof. This equality is almost a tautology: for any integer $d \geq 0$ we have

$$\begin{aligned}\operatorname{sv.dim}(E) \leq d &\iff \text{there exists an } F \text{ of rank } d \text{ such that } E \cong_{st} F\\ &\iff \text{there exists an } F \text{ of rank } d \text{ such that } [E] - \underline{\operatorname{rank} E} = [F] - \underline{d}\\ &\iff \text{there exists a bundle } F \text{ such that } [E] - \underline{\operatorname{rank} E} + \underline{d} = [F]\\ &\iff \operatorname{g.dim}(E - \underline{\operatorname{rank} E}) \leq d.\end{aligned}$$

The desired equality follows immediately.

For (b), let $r = \operatorname{sv.dim}(E)$, $k = \operatorname{rank}(E)$, and note that $r \leq k$. Then there exists a rank r bundle F such that E and F are stably isomorphic: $E \oplus \underline{N} \cong F \oplus \underline{(N + k - r)}$ for some $N > 0$. Since $\operatorname{rank}(E) > \dim X$ we can cancel the $\underline{N}$ factors to get $E \cong F \oplus \underline{(k - r)}$, by Proposition 13.19. Hence $\operatorname{v.dim}(E) \leq \operatorname{rank}(F) = r = \operatorname{sv.dim}(E)$. $\square$

The following result of Hirsch [Hi], and its corollary, translate the immersion problem into a purely homotopy-theoretic question. This is the key to why immersions are better understood than embeddings.

Theorem 39.24 (Hirsch). *Let M be a compact manifold of dimension n. For $k \geq 1$ the following statements are equivalent:*

(a) M can be immersed in $\mathbb{R}^{n+k}$
(b) There exists a bundle F of rank k such that $T_M \oplus F$ is trivial.
(c) There exists an O_n-equivariant map $\mathrm{Fr}(T_M) \to V_n(\mathbb{R}^{n+k})$, where $\mathrm{Fr}(T_M)$ is the bundle of n-frames in T_M.

Observe that (a) implies (b) by taking F to be the normal bundle of the immersion. Also, if $\phi\colon T_M \oplus F \to \underline{n+k}$ is an isomorphism then any n-frame in T_M yields an n-frame in $\mathbb{R}^{n+k}$ by applying ϕ; thus, one gets an equivariant map $\mathrm{Fr}(T_M) \to V_n(\mathbb{R}^{n+k})$. This shows (b) implies (c). So the content of the above theorem is really in (c)$\Rightarrow$(a); this is what was proven by Hirsch, via geometric arguments [Hi, Theorem 6.1 (taking $r = 0$ there)]. He actually showed much more, essentially proving that homotopy classes of immersions from M to $\mathbb{R}^{n+k}$ are in bijective correspondence with equivariant homotopy classes of maps $\mathrm{Fr}(T_M) \to V_n(\mathbb{R}^{n+k})$. We will not give Hirsch's proof here, but we will use the following corollary of his result. This corollary first appeared in [At3, Proposition 3.2] and in [Sa2, Theorem 2.1].

Corollary 39.25. *Let $k \geq 1$, and let M be a compact manifold of dimension n. Then M immerses in $\mathbb{R}^{n+k}$ if and only if $\mathrm{g.dim}(n - T_M) \leq k$.*

Proof. We have already seen the 'only if' direction when we obtained obstructions to immersions: if an immersion exists then $\underline{n+k} \cong T_M \oplus \nu$ where ν is the normal bundle, therefore $n - T_M = \nu - k$ and hence $\mathrm{g.dim}(n - T_M) = \mathrm{g.dim}(\nu - k) \leq k$.

For the other direction, assume $\mathrm{g.dim}(n - T_M) \leq k$. So there exists a rank k vector bundle F such that $n - T_M + k = F$ in $KO(M)$. This implies $n+k = T_M + F$ in $KO(M)$, which in turn yields that $\underline{n+k+N} \cong T_M \oplus F \oplus \underline{N}$ for some $N \geq 0$. Since the rank of $T_M \oplus F$ is larger than $\dim M$, it follows by Proposition 13.19 that we can cancel the $\underline{N}$ on both sides to get $\underline{n+k} \cong T_M \oplus F$. Then by Theorem 39.24 we know that M immerses into $\mathbb{R}^{n+k}$. □

Remark 39.26. Note in particular that if M is parallelizable then M immerses into $\mathbb{R}^{n+1}$ (taking $k = 1$ in Corollary 39.25, since $k = 0$ is not allowed).

We now specialize again to the case of $M = \mathbb{R}P^n$. Here we have

$$n - T_{\mathbb{R}P^n} = (n+1) - (1 \oplus T_{\mathbb{R}P^n}) = (n+1) - (n+1)L = (n+1)(1-L).$$

Proposition 39.27. *For $n \leq 8$ the immersion dimension of $\mathbb{R}P^n$ is as given in Table 39.2.*

Proof. Given the lower bounds given by Stiefel-Whitney classes (see Table 39.21), we only have to demonstrate the required immersions. The fact that $\mathbb{R}P^8$ immerses in $\mathbb{R}^{15}$ is a special case of Whitney's classical theorem. Both $\mathbb{R}P^3$ and $\mathbb{R}P^7$ have trivial tangent bundles (using the related fact for S^3 and S^7), and so by Remark 39.26 they immerse into $\mathbb{R}^4$ and $\mathbb{R}^8$, respectively.

For $\mathbb{R}P^6$ we must calculate the geometric dimension of $7(1-L) = -7(L-1)$. But $\varphi(6) = 3$, and so $\widetilde{KO}(\mathbb{R}P^6) \cong \mathbb{Z}/8$. Hence $8(L-1) = 0$, and so $-7(L-1) = L-1$.

The geometric dimension of $L-1$ is clearly at most 1, and so by Corollary 39.25 $\mathbb{R}P^6$ immerses into $\mathbb{R}^7$. Of course $\mathbb{R}P^4$ and $\mathbb{R}P^5$ therefore also immerse into $\mathbb{R}^7$.

A similar argument works to show that $\mathbb{R}P^2$ immerses into $\mathbb{R}^3$ (or one can just construct the immersion geometrically). $\square$

The reader should note why the above result stopped with $\mathbb{R}P^8$. For $\mathbb{R}P^9$ one finds that the immersion problem boils down to determining the geometric dimension of $-10(L-1) = 22(L-1)$ (here we used that $\varphi(9) = 5$ and so $\widetilde{KO}^0(\mathbb{R}P^9) = \mathbb{Z}/32$). The precise value of this geometric dimension is far from clear. We will close this section by analyzing it completely, following Sanderson [Sa2]. However, we take a short detour to illustrate some general principles.

Recall that for general n we have $n - T_{\mathbb{R}P^n} = (n+1)(1-L)$. We would like to interpret the geometric dimension of this class as being a stable virtual dimension, but for this we would need to be looking at a positive multiple of $L-1$ rather than $1-L$. There are two ways to get ourselves into this position. The first, which we have already seen, proceeds by recalling that $L-1$ has order $2^{\varphi(n)}$ in $\widetilde{KO}(\mathbb{R}P^n)$. So we can write

$$n - T_{\mathbb{R}P^n} = -(n+1)(L-1) = \big(2^{\varphi(n)} - (n+1)\big)(L-1)$$

and hence

$$\begin{aligned}\mathrm{g.dim}(n - T_{\mathbb{R}P^n}) &= \mathrm{g.dim}\Big(\big[2^{\varphi(n)} - (n+1)\big]L - \big[2^{\varphi(n)} - (n+1)\big]\Big)\\ &= \mathrm{sv.dim}\Big(\big[2^{\varphi(n)} - (n+1)\big]L\Big)\end{aligned}$$

(the last equality is by Proposition 39.23(b)). The second approach is from Sanderson [Sa1, Lemma 2.2]:

Proposition 39.28. *For the bundle $L \to \mathbb{R}P^n$, the statement* $\mathrm{g.dim}(a(L-1)) \le b$ *is equivalent to* $\mathrm{g.dim}((b-a)(L-1)) \le b$*, for any $a, b \in \mathbb{Z}$ with $b \ge 0$.*

Proof. It suffices to prove the implication in one direction, by symmetry. So suppose $\mathrm{g.dim}(a(L-1)) \le b$. This implies that $a(L-1) + b = E$ in $KO^0(\mathbb{R}P^n)$, for some rank b bundle E. Multiply by L to get $a(1-L) + bL = E \otimes L$, and then rearrange to find $(b-a)(L-1) + b = E \otimes L$. This yields that $\mathrm{g.dim}((b-a)(L-1)) \le b$. $\square$

Corollary 39.29. *Let $L \to \mathbb{R}P^n$ be the tautological bundle. For $k > 0$ the following statements are equivalent:*

(1) $\mathbb{R}P^n$ *immerses into* $\mathbb{R}^{n+k}$,
(2) $\mathrm{g.dim}(-(n+1)(L-1)) \le k$,
(3) $\mathrm{g.dim}((n+k+1)(L-1)) \le k$,
(4) $\mathrm{g.dim}\big([2^{\varphi(n)} - (n+1)]\,(L-1)\big) \le k$.

Proof. The equivalence (1) $\Longleftrightarrow$ (2) comes from Corollary 39.25, and (2) $\Longleftrightarrow$ (3) is by Proposition 39.28. Finally, (2) $\Longleftrightarrow$ (4) is true because $2^{\varphi(n)}(L-1) = 0$ in $\widetilde{KO}^0(\mathbb{R}P^n)$. $\square$

Part (3) of the above result, which is the part that comes from Proposition 39.28, will actually not be needed in the remainder of this section. But we record it here for later use (see Proposition 40.15).

We close this section by settling the immersion problem for $\mathbb{R}P^9$:

Proposition 39.30 (Sanderson)**.** *The immersion dimension of* $\mathbb{R}P^9$ *equals* 15.

This result is from [Sa2, Theorem 5.3]. Sanderson proves much more than this, for example that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-3}$ whenever n is odd. He also proved that $\mathbb{R}P^{11}$ immerses into $\mathbb{R}^{16}$, which ends up being the best result for both $\mathbb{R}P^{11}$ and $\mathbb{R}P^{10}$.

Proof. The lower bound of 15 is given by Stiefel-Whitney classes, as in Table 39.21. So we only need to prove that $\mathbb{R}P^9$ immerses into $\mathbb{R}^{15}$. By Corollary 39.29 this is equivalent to $\operatorname{g.dim}(22(L-1)) \le 6$, and is also equivalent to $\operatorname{g.dim}(16(L-1)) \le 6$. The proof below works for both statements, but for specificity we just prove the former. Note that what we must prove is equivalent to $\operatorname{sv.dim}(22L) \le 6$, by Proposition 39.23. We will outline the steps for this, and then give more details afterwards.

Step 1: There exists a rank 4 complex bundle E on $\mathbb{R}P^9$ such that $r_{\mathbb{R}}E$ is stably equivalent to $22L$ (recall that $r_{\mathbb{R}}E$ denotes the real bundle obtained from E by forgetting the complex structure).
Step 2: The bundle $E|_{\mathbb{R}P^8}$ has a nonzero section s.
Step 3: The bundle $r_{\mathbb{R}}E|_{\mathbb{R}P^8}$ has a field of (real) 2-frames.
Step 4: For $N \gg 0$ the field of real $(2+N)$-frames of $(r_{\mathbb{R}}E \oplus N)|_{\mathbb{R}P^8}$ may be extended over $\mathbb{R}P^9$.
Step 5: $\operatorname{sv.dim}(22L) = \operatorname{sv.dim}(r_{\mathbb{R}}E) \le 6$, hence $\mathbb{R}P^9$ can be immersed into $\mathbb{R}^{15}$.

We now justify each of these steps. For step 1 use that $r_{\mathbb{R}}\colon \widetilde{K}(\mathbb{R}P^n) \to \widetilde{KO}(\mathbb{R}P^n)$ has image equal to $\langle 2(L-1)\rangle$, by Theorems 37.14 and 37.17. Since $22(L-1)$ is therefore in the image, there is a complex bundle E on $\mathbb{R}P^9$ such that $r_{\mathbb{R}}E$ is stably equivalent to $22L$. The bundle E is represented by a map $\mathbb{R}P^9 \to BU$, and such a map necessarily factors up to homotopy through $BU(4)$: for this, use obstruction theory and the homotopy fiber sequences $S^{2n-1} \to BU(n-1) \to BU(n)$ (recall that $U(n)/U(n-1) \cong S^{2n-1}$). For a map $\mathbb{R}P^9 \to BU(n)$ to lift (up to homotopy) into $BU(n-1)$, one has obstructions in the groups $H^i(\mathbb{R}P^9; \pi_{i-1}S^{2n-1})$ for $0 \le i \le 9$. But as long as $5 \le n$ the homotopy groups $\pi_{i-1}S^{2n-1}$ vanish in this range, so all the obstruction groups are zero.

For step 2 we again proceed by obstruction theory. We have the sphere bundle $S(E|_{\mathbb{R}P^8}) \to \mathbb{R}P^8$ with fiber S^7, and all the obstruction groups vanish except for the last one: $H^8(\mathbb{R}P^8; \pi_7 S^7)$. The coefficents are untwisted because the complex structure gives a canonical orientation to each fiber. This final obstruction class is the same as the Euler class of $r_{\mathbb{R}}E$, or equivalently the top Chern class of E. Since $H^8(\mathbb{R}P^8;\mathbb{Z}) = \mathbb{Z}/2$ it will be sufficient to compute the mod 2 reduction of this class, which is the top Stiefel-Whitney class $w_8(r_{\mathbb{R}}E)$. Now we use that $r_{\mathbb{R}}E$ is stably isomorphic to $22L$, so the total Stiefel-Whitney class is $w(r_{\mathbb{R}}E) = w(22L) = w(L)^{22} = (1+x)^{22}$. Hence $w_8(r_{\mathbb{R}}E) = \binom{22}{8}x^8$. Since $\binom{22}{8}$ is even, the obstruction class vanishes and we indeed have a nonzero section.

Step 3 is trivial: the sections s and is give the field of real 2-frames.

Step 4 is obstruction theory yet again. If $F = r_{\mathbb{R}}E \oplus \underline{N}$ then we have a (partial) section of $V_{N+2}(F) \to \mathbb{R}P^9$ defined over $\mathbb{R}P^8$, and we must extend this to all of $\mathbb{R}P^9$. The obstruction lies in $H^9(\mathbb{R}P^9; \pi_8(V_{N+2}(\mathbb{R}^{N+8})))$. But the homotopy group in the coefficients has been computed [Wht2, BP] and is known to be zero for all $N \ge 2$ (see also Exercise 39.31 below for hints to this calculation).

Step 5 is now immediate: $r_{\mathbb{R}}E \oplus N$ has rank $8+N$ and has $2+N$ independent sections, hence $\mathrm{v.dim}(r_{\mathbb{R}}E \oplus N) \leq 6$. So $\mathrm{sv.dim}(r_{\mathbb{R}}E) \leq 6$ as well. Since $r_{\mathbb{R}}E$ is stably equivalent to $22L$, $\mathrm{sv.dim}(22L) = \mathrm{sv.dim}(r_{\mathbb{R}}E) \leq 6$. $\square$

Example 39.31. Compute the homotopy group needed in the above proof via the following steps:

(a) Use the fibration sequences $V_k(\mathbb{R}^n) \to V_{k+1}(\mathbb{R}^{n+1}) \to S^n$ to deduce that $\pi_8 V_{N+2}(\mathbb{R}^{N+8}) \cong \pi_8 V_4(\mathbb{R}^{10})$ for all $N \geq 2$.

(b) Using the cell structure from Proposition **??** reduce the problem to that of computing $\pi_8(\mathbb{R}P^9/\mathbb{R}P^5)$.

(c) Attempt to inductively compute $\pi_8(\mathbb{R}P^n/\mathbb{R}P^5)$ using the long exact homotopy sequences

$$\cdots \to \pi_8(\mathbb{R}P^n/\mathbb{R}P^5) \to \pi_8(\mathbb{R}P^{n+1}/\mathbb{R}P^5) \to \pi_8(\mathbb{R}P^{n+1}/\mathbb{R}P^5, \mathbb{R}P^n/\mathbb{R}P^5) \to \cdots$$

The Blakers-Massey theorem implies that $\pi_r(\mathbb{R}P^{n+1}/\mathbb{R}P^5, \mathbb{R}P^n/\mathbb{R}P^5) \cong \pi_r(D^{n+1}, S^n) \cong \pi_{r-1}S^n$ for $r \leq n+4$, so since $n \geq 5$ the long exact sequence becomes

$$\pi_8(S^n) \to \pi_8(\mathbb{R}P^n/\mathbb{R}P^5) \to \pi_8(\mathbb{R}P^{n+1}/\mathbb{R}P^5) \to \pi_7(S^n)$$

with the first map induced by $S^n \to \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^5$.

(d) Check that $\mathbb{R}P^7/\mathbb{R}P^5 \simeq S^7 \vee S^6$ (think about the attaching map of the 7-cell), so that $\pi_8(\mathbb{R}P^7/\mathbb{R}P^5) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with the generators corresponding to the maps η and η^2.

(e) By using the action of the Steenrod operations on $H^*(\mathbb{R}P^8/\mathbb{R}P^5; \mathbb{Z}/2)$, make some deductions about the attaching map for the 8-cell and use this to analyze the long exact sequence from (c). Conclude that $\pi_8(\mathbb{R}P^8/\mathbb{R}P^5) = \mathbb{Z}/2$ with the generator represented by the composite $S^8 \xrightarrow{\eta} S^7 \hookrightarrow S^7 \vee S^6 = \mathbb{R}P^7/\mathbb{R}P^5 \hookrightarrow \mathbb{R}P^8/\mathbb{R}P^5$.

(f) Finally, repeat the strategy from (e) to analyze the attaching map of the 9-cell in $\mathbb{R}P^9/RP^5$ and deduce that $\pi_8(\mathbb{R}P^9/RP^5) = 0$.

The arguments in this section naturally suggest the following problem, which is open:

Problem: Compute the geometric dimension of $k([L]-1) \in \widetilde{KO}(\mathbb{R}P^n)$, for any given k and n.

Over the years this problem has been extensively studied by Adams, Davis, Gitler, Lam, Mahowald, Randall, and many others. Just a few of the many references are [GM1, GM2, A5, As, DGM, LR1, LR2, BDM]. See also Section 40.14 for a bit more discussion.

39.32. **Summary.** In this section we obtained two sets of non-immersion/non-embedding results, one using Stiefel-Whitney classes in mod 2 singular cohomology and the other using the $\tilde{\gamma}$ classes in KO-theory. We also translated the immersion problem into a question about geometric dimension of reduced bundles, and for $\mathbb{R}P^n$ we completely solved this question for $n \leq 9$. As we have said before, this is far from the whole story—in fact it is just the very tip of a large and interesting iceberg. We refer the reader to the references cited in [Da2] for other pieces of the story, as well as to the survey paper [Da1].

40. The sums-of-squares problem and beyond

This section is in some ways an epilogue to the previous one. In the last section we started with a geometric problem, that of immersing $\mathbb{R}P^n$ into Euclidean space. We then used cohomology theories and characteristic classes to obtain necessary conditions for such an immersion to exist: we obtained two sets of conditions, one from mod 2 singular cohomology and one from KO-theory. In the present section we start with an *algebraic* problem, one that at first glance seems completely unrelated to immersions. It is the problem of finding sums-of-squares formulas in various dimensions, which we encountered already back in Section 15 (we will review the problem below). Once again we will use cohomology theories to obtain necessary conditions for the existence of such formulas. The surprise is that these conditions are basically the same as the ones that arose in the immersion problem! This is because both problems lead to the same homotopy-theoretic situation involving bundles over real projective space.

In theory the present section could be read completely independently of the last one. But because the underlying homotopy-theoretic problem is the same, we refer to the previous section for many details of its analysis.

40.1. **Review of the basic problem.** Recall from Section 15 that a **sums-of-squares formula of type $[r,s,n]$** (over $\mathbb{R}$) is a bilinear map $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ with the property that

$$|\phi(x,y)|^2 = |\phi(x)|^2 \cdot |\phi(y)|^2 \tag{40.2}$$

for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. If we write $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_s)$ then $\phi(x,y) = (z_1, \ldots, z_n)$ where each z_i is a bilinear expression in the x's and y's. Formula (40.2) becomes

$$(x_1^2 + \cdots + x_r^2) \cdot (y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2.$$

We will sometimes refer to this as an "$[r,s,n]$-formula" for short. Note that if an $[r,s,n]$-formula exists then one trivially has $[i,j,k]$-formulas for any $i \leq r$, $j \leq s$, and $k \geq n$.

For what values of r, s, and n does an $[r,s,n]$-formula exist? This is the **sums-of-squares** problem. Said differently, given a specific r and s what is the smallest value of n for which an $[r,s,n]$-formula exists? Call this number $r * s$. As with the immersion problem, there are two aspects here. One is the problem of constructing sums-of-squares formulas, thereby giving upper bounds for $r * s$; the other is the problem of finding necessary conditions for their existence, thereby giving lower bounds. The latter is the part that involves topology.

The sums-of-squares formulas that everyone knows are the ones coming from the multiplications on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. These haves types $[1,1,1]$, $[2,2,2]$, $[4,4,4]$, and $[8,8,8]$. Hurwitz proved than an $[n,n,n]$ formula only exists when $n \in \{1,2,4,8\}$, and the Hurwitz-Radon theorem generalizes this:

Theorem 40.3 (Hurwitz-Radon)**.** *A sums-of-squares formula of type $[r,n,n]$ exists if and only if $r \leq \rho(n) + 1$, where $\rho(n)$ is the Hurwitz-Radon number from Definition 15.4.*

This "Hurwitz-Radon theorem" is really the same as the one on vector fields on spheres that we saw as Theorem 15.5. The proof is the same as well:

Proof. Write $n = (\text{odd}) \cdot 2^{a+4b}$ with $0 \le a \le 3$. Recall from Theorem 15.13 that an $[r, n, n]$-formula exists if and only if there exists a Cl_{r-1}-module structure on $\mathbb{R}^n$. We saw in Section 15 that representations of Cl_{r-1} only exist on vector spaces whose dimension is a multiple of $2^{\varphi(r-1)}$. Thus, we have the chain of equivalences

$$\begin{aligned}
\text{an } [r,n,n]\text{-formula exists} &\iff \text{there exists a } \mathrm{Cl}_{r-1}\text{-module structure on } \mathbb{R}^n \\
&\iff 2^{\varphi(r-1)} \big| n \\
&\iff \varphi(r-1) \le a + 4b \\
&\iff r - 1 \le 2^a + 8b - 1 = \rho(n).
\end{aligned}$$

For the last equivalence note that $\varphi(2^a + 8b - 1) = \varphi(2^a - 1) + 4b = a + 4b$, where the first equality is the 8-fold periodicity of φ and the second is just a calculation for $0 \le a \le 3$. Moreover, $2^a + 8b - 1$ is the *largest* number whose φ-value is $a + 4b$; by periodicity this can again be checked just for $b = 0$ and $0 \le a \le 3$. So in general we have $\varphi(s) \le a + 4b = \varphi(2^a + 8b - 1)$ if and only if $s \le 2^a + 8b - 1$; this is what is used in the final equivalence. ☐

Remark 40.4. Note that if there exist formulas of type $[r, s_1, n_1]$ and $[r, s_2, n_2]$ then there is also a formula of type $[r, s_1 + s_2, n_1 + n_2]$ (by distribuitivity). This says that

$$r * (s_1 + s_2) \le r * s_1 + r * s_2.$$

Also notice that $r * s \le (r + a) * (s + b)$ whenever $a, b \ge 0$, because a formula of type $[r + a, s + b, n]$ automatically yields one of type $[r, s, n]$ by plugging in zeros for a of the x's and b of the y's.

The classical identities show that $2 * 2 = 2$, $4 * 4 = 4$, and $8 * 8 = 8$, and it is trival that $n * 1 = n$. Using these together with the observations of the previous paragraph, one can obtain upper bounds on $r * s$. For example, $3 * 10 \le 12$ because

$$3 * 10 \le 3 * 8 + 3 * 2 \le 8 * 8 + 4 * 4 = 8 + 4 = 12.$$

The following table shows what is known about $r * s$ for small values of r and s. For $r \le 8$ the values completely agree with the upper bounds obtained by the above methods.

TABLE 40.4. Values of $r * s$

$r \backslash s$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16	18
3		4	4	7	8	8	8	11	12	12	12	15	16	16	16	19
4			4	8	8	8	8	12	12	12	12	16	16	16	16	20
5				8	8	8	8	13	14	15	16	16	16	16	16	21
6					8	8	8	14	14	16	16	16	16	16	16	22
7						8	8	15	16	16	16	16	16	16	16	23
8							8	16	16	16	16	16	16	16	16	24
9								16	16	[16,17]	??					
10									16	[16,17]						
11										17						

To justify the numbers in the above table we have to produce lower bounds for $r * s$. For example, we have to explain why there do not exist formulas of type

[5, 10, 13]. Almost all the known lower bounds come from topological methods; we will describe some of these next.

40.5. **Lower bounds via topology.** Here is the key result that shows how a sums-of-squares formula gives rise to something homotopy-theoretic:

Proposition 40.6. *If an $[r, s, n]$-formula exists then there exists a rank $n-r$ bundle E on $\mathbb{R}P^{s-1}$ such that $rL \oplus E \cong \underline{n}$ (here rL is the direct sum of r copies of L).*

Proof. Let $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ be the map giving the sums-of-squares formula. If $u \in \mathbb{R}^s$ is a unit vector, check that $\phi(e_1, u), \ldots, \phi(e_r, u)$ is an orthonormal frame in $\mathbb{R}^n$. This is an easy consequence of the sums-of-squares identity; it is an exercise, but see the proof of Corollary 15.10 if you get stuck. In this way we obtain a map $f\colon S^{s-1} \to V_r(\mathbb{R}^n)$. Compose with the projection $V_r(\mathbb{R}^n) \to \mathrm{Gr}_r(\mathbb{R}^n)$ and then note that the map factors to give $F\colon \mathbb{R}P^{s-1} \to \mathrm{Gr}_r(\mathbb{R}^n)$. Precisely, given a line in $\mathbb{R}^s$ spanned by a vector u its image under F is the r-plane spanned by $\phi(e_1, u), \ldots, \phi(e_r, u)$.

Let η be the tautological r-plane bundle on $\mathrm{Gr}_r(\mathbb{R}^n)$. We claim that $F^*\eta = rL$. This follows from the commutative diagram

$$\begin{tikzcd} rL \arrow[r, "\tilde{F}"] \arrow[d] & \eta \arrow[d] \\ \mathbb{R}P^{s-1} \arrow[r, "F"] & \mathrm{Gr}_r(\mathbb{R}^n) \end{tikzcd}$$

where the top map is described as follows. Given r points on the same line $\langle u \rangle$ we write them as $\lambda_1 u, \ldots, \lambda_r u$ and then send them to the element $\lambda_1\phi(e_1, u) + \cdots + \lambda_r\phi(e_r, u)$ on the r-plane $F(\langle u \rangle)$. One readily checks that this does not depend on the choice of u; in fact, we could just say that points $z_1, \ldots, z_r$ on a common line ℓ are sent to the element $\phi(e_1, z_1) + \cdots + \phi(e_r, z_r)$ on $F(\ell)$. The fact that $\phi(e_1, u), \ldots, \phi(e_r, u)$ are orthonormal (hence independent) shows that $\tilde{F}$ is injective on fibers, hence an isomorphism on fibers. This shows that $F^*\eta \cong rL$.

We have now done all the hard work. To finish, just recall that η sits inside a short exact sequence $0 \to \eta \to \underline{n} \to Q \to 0$ where Q is the standard quotient bundle. This sequence is split because $\mathrm{Gr}_r(\mathbb{R}^n)$ is compact. Pulling back along F now gives $rL \oplus F^*Q \cong \underline{n}$, as desired. $\square$

The following result was originally proven independently by Hopf [Ho] and Stiefel [St]; Stiefel's method is the one we follow here.

Corollary 40.7 (Hopf-Stiefel)**.** *If an $[r, s, n]$-formula exists then the following two equivalent conditions hold:*
(1) $\binom{r+i-1}{i}$ is even for $n - r < i < s$;
(2) $\binom{n}{i}$ is even for $n - r < i < s$.

Proof. By Proposition 40.6 we know that $rL \oplus E \cong \underline{n}$ for some rank $n - r$ bundle on $\mathbb{R}P^{s-1}$. Applying total Stiefel-Whitney classes gives $w(rL)w(E) = w(rL \oplus E) = w(\underline{n}) = 1$, or $w(E) = w(rL)^{-1} = w(L)^{-r}$. So $w(L)^{-r}$ vanishes in degrees larger than $n - r$. But $w(L) = 1 + x$ where x is the generator for $H^1(\mathbb{R}P^{s-1}; \mathbb{Z}/2)$, and the coefficient of x^i in $(1 + x)^{-r}$ is $\binom{-r}{i} = \binom{r+i-1}{i}$ (recall that we are working modulo 2). So $\binom{r+i-1}{i}$ is even for $n - r < i < s$.

The equivalence of the conditions in (1) and (2) follows at once from the lemma below, taking $k = n - r + 1$ and $i = r + s - n - 2$ (note that conditions (1) and (2) are both vacuous unless $i \geq 0$). $\square$

Lemma 40.8. *For any non-negative integers n, k, and i, the following $\mathbb{Z}$-linear spans are the same inside of $\mathbb{Z}$ (that is, the lists on the two sides generate the same ideal):*

$$\mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \dots, \tbinom{n}{k+i} \Big\rangle = \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n+1}{k+1}, \dots, \tbinom{n+i}{k+i} \Big\rangle.$$

Consequently, an integer is a common divisor of the first set of binomial coefficients if and only if it is a common divisor of the second set.

Proof. Taking first differences and using Pascal's identity shows (via multiple iterations) that

$$\begin{aligned}
\mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n+1}{k+1}, \dots, \tbinom{n+i}{k+i} \Big\rangle &= \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \tbinom{n+1}{k+2}, \tbinom{n+2}{k+3}, \dots, \tbinom{n+i-1}{k+i} \Big\rangle \\
&= \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \tbinom{n}{k+2}, \tbinom{n+1}{k+3}, \dots, \tbinom{n+i-2}{k+i} \Big\rangle \\
&= \cdots \\
&= \mathbb{Z}\Big\langle \tbinom{n}{k}, \tbinom{n}{k+1}, \dots, \tbinom{n}{k+i} \Big\rangle.
\end{aligned}$$

$\square$

Example 40.9. Does a formula of type $[10, 10, 15]$ exist? If it did, statement (2) of Corollary 40.7 would imply that $\binom{15}{i}$ is even for $5 < i < 10$. But $\binom{15}{6}$ is odd.

The full power of the numerical conditions in Corollary 40.7 is subtle, and one really needs a computer to thoroughly investigate them. But the following consequence represents much of the information buried in those conditions:

Corollary 40.10. *If $r + s > 2^k$ then $r * s \geq 2^k$.*

Proof. We must show that if $r + s > 2^k$ then formulas of type $[r, s, 2^k - 1]$ do not exist. If they did, the Hopf-Stiefel conditions would imply that $\binom{2^k-1}{i}$ is even for i in the range $2^k - 1 - r < i < s$. But $\binom{2^k-1}{i}$ is odd no matter what i is, so the conditions are only consistent if the range is empty—or equivalently, if $2^k - 1 - r \geq s - 1$. The hypothesis $r + s > 2^k$ guarantees that this is not the case. $\square$

For example, sums-of-squares formulas of type $[16, 17, n]$ must all have $n \geq 32$.

For $r \leq 8$ the Hopf-Stiefel lower bounds for $r * s$ turn out to exactly match the upper bounds obtained via the constructive methods of Remark 40.4. So this justifies the numbers in Table 40.4 for the range $r \leq 8$.

40.11. **K-theoretic techniques.** We can also analyze the implications of Proposition 40.6 using KO-theory. This was first done by Yuzvinsky [Y]. Note that one could also use complex K-theory here, but KO-theory gives stronger results: the point is that $\widetilde{K}^0(\mathbb{R}P^m)$ and $\widetilde{KO}^0(\mathbb{R}P^m)$ are almost the same, but for certain values of m the latter group is slightly bigger (by a factor of 2).

Corollary 40.12 (Yuzvinsky)**.** *If an $[r, s, n]$-formula exists then the following two equivalent conditions hold:*

(1) $2^{\varphi(s-1)-i+1}$ divides $\binom{r+i-1}{i}$ for $n - r < i \leq \varphi(s-1)$;

(2) $2^{\varphi(s-1)-i+1}$ *divides* $\binom{n}{i}$ *for* $n-r<i\leq\varphi(s-1)$.

Proof. Proposition 40.6 gives that $rL\oplus E\cong \underline{n}$ for some bundle E on $\mathbb{R}P^{s-1}$. We again use characteristic classes, but this time the $\tilde{\gamma}$ classes in KO-theory. We find that $\tilde{\gamma}_t(E)=\tilde{\gamma}_t(L)^{-r}$, and so $\tilde{\gamma}_t(L)^{-r}$ must vanish in degrees larger than $n-r$. Recall that $\tilde{\gamma}_t(L)=1+t\lambda$ where $\lambda=[L]-1$, and so the coefficient of t^i in $\tilde{\gamma}_t(L)^{-r}$ is $\binom{-r}{i}\lambda^i=\pm\binom{r+i-1}{i}2^{i-1}\lambda$. Here we have used $\lambda^2=-2\lambda$.

Recalling that $\widetilde{KO}^0(\mathbb{R}P^{s-1})\cong\mathbb{Z}/(2^{\varphi(s-1)})$, we find that $2^{\varphi(s-1)}$ divides $2^{i-1}\binom{r+i-1}{i}$ for all $i>n-r$. This statement only has content for $i\leq\varphi(s-1)$, and thus we obtain the condition in (1).

The equivalence of (1) and (2) is an instance of the following general observation: the sequence of conditions

$$2^N\big|\tbinom{n}{k},\ 2^{N-1}\big|\tbinom{n}{k+1},\ 2^{N-2}\big|\tbinom{n}{k+2},\ldots,\ 2^{N-j}\big|\tbinom{n}{k+j}$$

is equivalent to the sequence of conditions

$$2^N\big|\tbinom{n}{k},\ 2^{N-1}\big|\tbinom{n+1}{k+1},\ 2^{N-2}\big|\tbinom{n+2}{k+2},\ldots,\ 2^{N-j}\big|\tbinom{n+j}{k+j}.$$

This follows at once by applying Lemma 40.8 multiple times, with $i=1$, $i=2$, ..., $i=j$. $\square$

The Hopf-Stiefel conditions are symmetric in r and s, but this is not true for the KO-theoretic conditions in the above proposition. For example, applying the conditions yields no information on $[3,6,n]$-formulas beyond $n\geq 5$ (which is trivial), whereas applying the conditions to $[6,3,n]$-formulas yields $n\geq 8$. One must therefore apply the conditions to both $[r,s,n]$ and $[s,r,n]$ to get the best information.

Example 40.13. Like we saw for the immersion problem, in some dimensions the KO-theoretic conditions are stronger than the Hopf-Stiefel conditions—and in some dimensions they are weaker. Neither result is strictly stronger than the other.

For example, the Hopf-Stiefel conditions show that $4*5\geq 8$ whereas the KO-conditions only show $4*5=5*4\geq 7$. The smallest dimension for which the KO-theoretic conditions are stronger is when $r=10$ and $s=15$. The Hopf-Stiefel conditions rule out the existence of $[10,15,15]$-formulas, but not $[10,15,16]$. The KO-conditions rule out $[15,10,16]$, however, and therefore also $[10,15,16]$ by symmetry.

To pick a larger example, the Hopf-Stiefel conditions show that $127*127\geq 128$ but the KO-conditions show that $127*127\geq 184$. The KO-conditions seem to give their greatest power when r and s are slightly less than a power of 2.

40.14. **Other problems.** The reader will probably have noticed that the above methods and results for the existence of sums-of-squares formulas are almost exactly the same as the ones in the last section on the existence of immersions for real projective spaces. It is natural to wonder if there is a closer connection between these two problems. There is! In fact there turns out to be a whole family of interconnected problems circling this area. We will explain this next.

A careful look at Proposition 40.6 shows that one can make the argument work with something much weaker than a sums-of-squares formula. Specifically, all we needed was a bilinear map $f\colon\mathbb{R}^r\otimes\mathbb{R}^s\to\mathbb{R}^n$ such that $f(x\otimes y)=0$ only when $x=0$ or $y=0$. Such a bilinear map is usually called **nonsingular**. Given such a map and a nonzero $u\in\mathbb{R}^r$, the elements $\phi(u,e_1),\ldots,\phi(u,e_r)$ are necessarily linearly independent—and this is really all that was needed in the proof of Proposition 40.6.

We can replace our sums-of-squares problem with the following: given r and s, for what values of n does there exist a nonsingular bilinear map of type $[r, s, n]$? The topological obstructions we found for sums-of-squares formulas are of course still valid in this new context.

The existence of nonsingular bilinear maps turns out to be related to the immersion problem for real projective spaces. More than this, both problems are connected to a number of similar questions that have been intently studied by algebraic topologists since the 1940s. Many of these problems were originally raised by Hopf [Ho]. This material takes us somewhat away from our main theme of K-theory, but it seems worthwhile to tell a bit of this story since we have encountered it.

To start with, let us introduce the following classes of statements:

$\mathrm{SS}[r, s, n]$:	there exists a sums-of-squares formula of type $[r, s, n]$
$\mathrm{NS}[r, s, n]$:	there exists a nonsingular bilinear map $\mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$
$\mathrm{RR}[r, s, n]$:	there exist $n \times s$ matrices $A_1, \dots, A_r$ with the property that every nonzero linear combination of them has rank s
$\mathrm{T}[r, s, n]$:	the tangent bundle $T_{\mathbb{R}P^n}$ has s independent sections when restricted to $\mathbb{R}P^r$
$\mathrm{IS}[r, s, n]$:	the bundle $nL \to \mathbb{R}P^r$ has s independent sections
$\mathrm{GD}[r, s, n]$:	over $\mathbb{R}P^r$ one has $\mathrm{g.dim}(n(L-1)) \le n - s$
$\mathrm{ES}[r, s, n]$:	the 'first-vector' map $p_1 \colon V_s(\mathbb{R}^{n+1}) \to S^n$ has a $\mathbb{Z}/2$-equivariant section over the subspace $S^r \subseteq S^n$. Here $\mathbb{Z}/2$ acts antipodally on $\mathbb{R}^{n+1}$, and both $V_s(\mathbb{R}^{n+1})$ and S^n get the induced action.
$\mathrm{AX}[r, s, n]$:	there exists an "axial map" $\mathbb{R}P^r \times \mathbb{R}P^s \to \mathbb{R}P^n$; this is a map with the property that the restrictions $\mathbb{R}P^r \times \{*\} \to \mathbb{R}P^n$ and $\{*\} \times \mathbb{R}P^s \to \mathbb{R}P^n$ are homotopic to linear embeddings, for some choice of basepoints in $\mathbb{R}P^r$ and $\mathbb{R}P^s$
$\mathrm{IM}[r, n]$:	$\mathbb{R}P^r$ immerses into $\mathbb{R}P^n$
$\mathrm{VF}[k, n]$:	there exist k independent vector fields on S^n.

The acronyms are mostly self-evident, except for a few: RR stands for "rigid rank", IS for "independent sections", and ES for "equivariant sections".

The above statements are closely interrelated, as the next result demonstrates. We should point out that very little from this result will be needed in our subsequent discussion. We are including it because most of the claims are easy to prove, and because the various statements get used almost interchangeably (often without much explanation) in the literature on the immersion problem.

Proposition 40.15.
(a) $\mathrm{GD}[r, s, n] \iff \mathrm{GD}[r, -n, -s]$
(b) $\mathrm{IM}[r, n] \iff \mathrm{GD}[r, -(n+1), -(r+1)] \iff \mathrm{GD}[r, r+1, n+1]$
(c) $T[n, k, n] \Rightarrow \mathrm{VF}[k, n]$
(d) One has the following implications:

$$\begin{tikzcd}
\mathrm{SS}[r,s,n] \arrow[r, Rightarrow, "1"] & \mathrm{NS}[r,s,n] \arrow[r, Leftrightarrow, "2"] & \mathrm{RR}[r,s,n] \arrow[d, Rightarrow, "3"] & \mathrm{ES}[r-1,s,n-1] \arrow[d, Rightarrow, "6"] \\
\mathrm{GD}[r-1,s,n] & \mathrm{IS}[r-1,s,n] \arrow[l, Rightarrow, "5"'] & \mathrm{T}[r-1,s-1,n-1] \arrow[l, Rightarrow, "4"'] \arrow[ur, Leftrightarrow] & \mathrm{AX}[r-1,s-1,n-1]
\end{tikzcd}$$

(e) If $r < n$ then implication 4 is reversible, and if $r \le n$ then 5 is reversible.

(f) If $r < n$ and $r \le 2(n-s)$ then implication 6 in (d) is also reversible.

Proof. Part (a) is Proposition 39.28, and part (b) is Corollary 39.29; we have seen these already. Part (c) follows from the fact that $T_{S^n} = p^*T_{\mathbb{R}P^n}$ where $p\colon S^n \to \mathbb{R}P^n$ is the projection.

For part (d), the first implication is obvious. The others we treat one by one.

$\mathbf{NS}[\boldsymbol{r,s,n}] \iff \mathbf{RR}[\boldsymbol{r,s,n}]$: The equivalence follows from adjointness, as bilinear maps $f\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ correspond bijectively to linear maps $F\colon \mathbb{R}^r \to \operatorname{Hom}(\mathbb{R}^s, \mathbb{R}^n)$. The linear map F is specified by the $n \times s$ matrices $F(e_1), \dots, F(e_r)$. It is easy to verify that f is nonsingular if and only if all nontrivial linear combinations of these matrices have rank s.

$\mathbf{RR}[\boldsymbol{r,s,n}] \Rightarrow \mathbf{T}[\boldsymbol{r-1,s-1,n-1}]$: First note that the tangent bundle of $\mathbb{R}P^n$ is the collection of pairs (x, v) such that $x, v \in \mathbb{R}^n$, $|x| = 1$, and $x \cdot v = 0$, modulo the identifications $(x, v) \sim (-x, -v)$. Secondly, note that since $\mathrm{NS}[r, s, n]$ is symmetric in r and s the same is true for $\mathrm{RR}[r, s, n]$. The condition $\mathrm{RR}[s, r, n]$ says that we have $n \times r$ matrices $A_1, \dots, A_s$ such that every nontrivial linear combination has rank r. For any $x \in \mathbb{R}^r - 0$ it follows that $A_1x, \dots, A_sx$ are independent: for if $\sum t_iA_ix = 0$ then $\sum t_iA_i$ has rank less than r, which only happens when all $t_i = 0$. Note that there is a matrix $P \in GL_n(\mathbb{R})$ such that the columns of PA_1 are the standard basis $e_1, \dots, e_r$, so by replacing each A_i with PA_i we can just assume these are the columns of A_1.

For every $x \in S^{r-1} \subseteq \mathbb{R}^r \subseteq \mathbb{R}^n$ consider the independent vectors $A_1x, A_2x, A_3x, \dots, A_sx$. Note that $A_1x = x$. Let $u_i(x) = A_ix - (A_ix \cdot x)x$. Then $u_2(x), \dots, u_r(x)$ are independent, and orthogonal to x. Since $u_i(-x) = -u_i(x)$ for each i, these give us $s-1$ independent sections of $T_{\mathbb{R}P^{n-1}}$ defined over $\mathbb{R}P^{r-1}$.

$\mathbf{T}[\boldsymbol{r-1,s-1,n-1}] \Rightarrow \mathbf{IS}[\boldsymbol{r-1,s,n}]$: This follows from the bundle isomorphism $T_{\mathbb{R}P^{n-1}} \oplus 1 \cong nL$ (cf. Example 25.16 and note that the same argument works in the real case, where we have $L \cong L^*$).

$\mathbf{IS}[\boldsymbol{r-1,s,n}] \Rightarrow \mathbf{GD}[\boldsymbol{r-1,s,n}]$: Trivial.

$\mathbf{T}[\boldsymbol{r-1,s-1,n-1}] \iff \mathbf{ES}[\boldsymbol{r-1,s,n-1}]$: First note that the frame bundle $V_{s-1}(T_{\mathbb{R}P^{n-1}})$ is homeomorphic to $V_s(\mathbb{R}^n)/\pm 1$ in an evident way. Under this homeomorphism the projection $V_{s-1}(T_{\mathbb{R}P^{n-1}}) \to \mathbb{R}P^{n-1}$ corresponds to the first-vector map $V_s(\mathbb{R}^n)/\pm 1 \to S^{n-1}/\pm 1$. So $T[r-1, s-1, n-1]$ is equivalent to the latter bundle having a section over $S^{r-1}/\pm 1$. But then consider the diagram

$$\begin{tikzcd}
V_s(\mathbb{R}^n) \arrow[r] \arrow[d] & V_s(\mathbb{R}^n)/\pm 1 \arrow[d] \\
S^{n-1} \arrow[r] & S^{n-1}/\pm 1
\end{tikzcd}$$

where the two horizontal maps are 2-fold covering spaces. This is a pullback square. It is easy to see that the right vertical map has a section defined over $S^{r-1}/\pm 1$ if and only if the left vertical map has a $\mathbb{Z}/2$-equivariant section defined over S^{r-1}.

$\mathbf{ES}[\boldsymbol{r-1,s,n-1}] \Rightarrow \mathbf{AX}[\boldsymbol{r-1,s-1,n-1}]$: Note that there is an evident map $V_s(\mathbb{R}^n) \to \mathcal{T}op(S^{s-1}, S^{n-1})$ that sends a frame $v_1, \dots, v_s$ to the map $(a_1, \dots, a_s) \mapsto a_1v_1 + \cdots + a_sv_s$. So a section $\chi\colon S^{r-1} \to V_s(\mathbb{R}^n)$ gives by composition a map $S^{r-1} \to \mathcal{T}op(S^{s-1}, S^{n-1})$, and then by adjointness a map $g\colon S^{r-1} \times S^{s-1} \to S^{n-1}$. The fact that χ was a section of the first-vector map shows that $g(x, e_1) = x$ for all $x \in S^{r-1}$. Also, it is clear that $g(e_1, -)$ is a linear inclusion $S^{s-1} \hookrightarrow S^{n-1}$

(there is nothing special about e_1 here). The $\mathbb{Z}/2$-equivariance of χ shows that $g(-x,y) = -g(x,y)$, and the similar identity $g(x,-y) = -g(x,y)$ is trivial. So g descends to give a map $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}$, and this map is axial.

For (e), the reversibility of both implications is governed by stability theory of vector bundles. For example, assume IS$[r-1, s, n]$. Then $nL \cong s \oplus E$ for some bundle E of rank $n-s$, where we are working over $\mathbb{R}P^{r-1}$. Recall that $j^*T_{\mathbb{R}P^{n-1}} \oplus 1 \cong nL$, where $j\colon \mathbb{R}P^{r-1} \hookrightarrow \mathbb{R}P^{n-1}$ is the inclusion. So $j^*T_{\mathbb{R}P^{n-1}} \oplus 1 \cong s \oplus E \cong 1 \oplus ((s-1) \oplus E)$. Since we are working over $\mathbb{R}P^{r-1}$ and $r-1 < n-1$, we can cancel the 1 on both sides to get $j^*T_{\mathbb{R}P^{n-1}} \cong (s-1) \oplus E$ (see Proposition 13.19). This says that T$[r-1, s-1, n-1]$ holds. The reversibility of implication 5 is very similar, and is left to the reader.

Part (f), on the reversibility of implication 6, is the only part of the proposition that is not elementary. Let $\mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ denote the space of $\mathbb{Z}/2$-equivariant maps, where the spheres have the antipodal action. This is a subspace of the usual function space $\mathcal{T}op(S^{s-1}, S^{n-1})$. Note that the space of equivariant maps has a $\mathbb{Z}/2$-action, given by composing (or equivalently, precomposing) a given map with the antipodal map. James [J1] shows that the evident map $\mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \to \mathcal{T}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1})$ is a principal $\mathbb{Z}/2$-bundle with respect to the above action.

We will also have need of the evaluation map $\mathrm{ev}\colon \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \to S^{n-1}$ sending $h \mapsto h(e_1)$. James [J1] shows that this is also a fibration.

Note that the Stiefel manifold $V_s(\mathbb{R}^n)$ is a subspace of $\mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ in the evident way, and that the following diagram commutes:

$$\begin{tikzcd} V_s(\mathbb{R}^n) \arrow[rr] \arrow[dr, "p_1"'] & & \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \arrow[dl, "\mathrm{ev}"] \\ & S^{n-1}. & \end{tikzcd}$$

James [J1, Theorem 6.5] proved that $V_s(\mathbb{R}^n) \hookrightarrow \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ is $[2(n-s)-1]$-connected, and this is the crucial point of the whole argument.

Suppose $f\colon \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}$ is an axial map. By the homotopy extension property we can assume that $f(*,-)$ and $f(-,*)$ are both equal to the canonical embeddings, where $*$ refers to some chosen basepoints in $\mathbb{R}P^{r-1}$ and $\mathbb{R}P^{s-1}$. The axial map f gives a map $F\colon \mathbb{R}P^{r-1} \to \mathcal{T}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1})$ by adjointness. Regard the target as pointed by the canonical embedding j, and note that F is a pointed map.

Covering space theory gives that there is a unique map $\tilde{F}\colon S^{r-1} \to \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1})$ such that $\tilde{F}(*)$ is the canonical embedding $S^{s-1} \hookrightarrow S^{n-1}$ and such that the diagram

$$\begin{tikzcd} S^{r-1} \arrow[r, "\tilde{F}"] \arrow[d] & \mathcal{T}op_{\mathbb{Z}/2}(S^{s-1}, S^{n-1}) \arrow[d] \\ \mathbb{R}P^{r-1} \arrow[r, "F"] & \mathcal{T}op(\mathbb{R}P^{s-1}, \mathbb{R}P^{n-1}) \end{tikzcd}$$

commutes. We claim that $\tilde{F}$ is $\mathbb{Z}/2$-equivariant, and that the composition $\mathrm{ev} \circ \tilde{F}$ is the standard inclusion $S^{r-1} \hookrightarrow S^{n-1}$. Both of these are easy exercises in covering space theory. The latter statement depends on the so-far unused portion of the axial map condition on f.

If $r-1 \leq 2(n-s)-1$ then by James's connectivity result the map $\tilde{F}$ can be factored up to homotopy through $V_s(\mathbb{R}^n)$. Moreover, this can be done in the category of $\mathbb{Z}/2$-equivariant pointed spaces over S^{n-1} (the only hard part here is the $\mathbb{Z}/2$-equivariance, and for this one uses that S^{r-1} has an equivariant cell decomposition made from free $\mathbb{Z}/2$-cells). In this way one produces the relevant equivariant section of the map $V_s(\mathbb{R}^n) \to S^{n-1}$. □

Remark 40.16. We have included the above proposition because it is very useful as a reference. However, it should be pointed out that there is something slightly deceptive about part (d). Some of the implications are more obvious than the long chains would suggest. For example, $\mathrm{NS}[r,s,n] \Rightarrow \mathrm{AX}[r-1,s-1,n-1]$ is a very easy argument of one or two lines. Likewise, $\mathrm{NS}[r,s,n] \Rightarrow \mathrm{GD}[r-1,s,n]$ is just the argument in Proposition 40.6. The picture in (d) is useful in showing *all* the relations at once, but it makes some of the statements seem more distant than they really are.

The reader will have noticed that Proposition 40.15 encodes several things that we have seen before. Some are transparently familiar, like parts (a) and (b). A less transparent example is

$$\mathrm{SS}[r,n,n] \iff \mathrm{SS}[n,r,n] \Rightarrow \mathrm{T}[n-1,r-1,n-1] \Rightarrow \mathrm{VF}[r-1,n-1].$$

This was the content of Corollary 15.10. In constrast, here is a similarly-obtained implication that we have *not* seen yet:

$$\mathrm{NS}[r,r,n] \Rightarrow \mathrm{GD}[r-1,r,n] \iff \mathrm{IM}[r-1,n-1].$$

From this we learn that immersion results for real projective space can be obtained by demonstrating the existence of nonsingular bilinear maps. This approach was successfully used by K.Y. Lam in [L1]. We briefly sketch his method simply to give the basic idea; for details the reader may consult [L1] and similar papers.

Recall that $\mathbb{O}$ denotes the octonions. Consider the map $f\colon \mathbb{O}^2 \times \mathbb{O}^2 \to \mathbb{O}^3$ given by

$$f\big((u_1,u_2),(x_1,x_2)\big) = (u_1x_1 - \overline{x}_2u_2, x_2u_1 + u_2\overline{x}_1, u_2x_2 - x_2u_2).$$

With a little work one can prove that this is nonsingular. Also, it is a general fact about the octonions that for any $a,b \in \mathbb{O}$ the commutator $[a,b] = ab - ba$ is imaginary. So the image of f actually lies in the 23-dimensional subspace of $\mathbb{O}^3$ where the real part of the third coordinate vanishes. So f gives a nonsingular bilinear map of type $[16,16,23]$. This shows that $\mathbb{R}P^{15}$ immerses into $\mathbb{R}^{22}$. By restricting f to appropriate subspaces Lam also obtained nonsingular bilinear maps of types $[11,11,17]$, $[13,13,19]$, and $[10,10,16]$, thereby proving that $\mathbb{R}P^{10}$ immerses into $\mathbb{R}^{16}$, $\mathbb{R}P^{12}$ immerses into $\mathbb{R}^{18}$, and $\mathbb{R}P^9$ immerses into $\mathbb{R}^{15}$.

40.17. **Summary.** In this section we examined the sums-of-squares problem, and saw how characteristic classes can be used to obtain lower bounds for the numbers $r*s$. Use of Stiefel-Whitney classes in singular cohomology yielded the Hopf-Stiefel lower bounds on $r*s$, whereas the use of the γ-classes in KO-theory gave the Yuzvinsky lower bounds. This story is very similar to the one for immersions of $\mathbb{R}P^n$ discussed in Section 39, and in fact the sums-of-squares problem is closely connected to this immersion problem. We closed the section by exploring the relations between these and a host of similar problems.

Part 6. Bott periodicity

Bott periodicity is a deep result, and we will spend several sections exploring it from different perspectives. What do I mean by "deep"? There are many instances in mathematics where, once you learn to look at a result in just the right way, you say to yourself "I could have guessed that." This doesn't mean it is easy to prove, but at least you have understood the result at some intuitive level. Then there are other results where, no matter how you look at it, you can't quite get to that "I could have guessed this" spot. For me, Bott periodicity is in this latter category.

A simpler example I like to mention of a "deep" result is the Pythagorean Theorem. Although I know several proofs of this fact—some of them quite simple!—down in my bones I suppose I have never felt like I truly know why the universe has to be that way. For me (sadly) the Pythgorean Theorem remains an unenlightening computation.

Of course, once one has the Pythagorean Theorem it becomes an absolutely crucial tool for exploring geometry, used at almost every turn in the road. It becomes part of one's thinking, and is a fundamental technique by which one understands more complicated results in geometry. There is a sense in which one gets to "understand" the Pythagorean Theorem just by learning to use it so well. Bott periodicity is also kind of like this. There are by now many different proofs, and once one has the result it opens entire worlds—one of these is the world of K-theory, that we have spent this whole book exploring. So one approach is just to accept Bott periodicity the same way as one accepts the Pythagorean Theorem, as a fundamental computation that will get used over and over again, and not worry to much about whether one could have "guessed" it. But I can't help feeling that one day we might have a deeper understanding.

While I do not know a way to "guess" the Bott periodicity theorems, I want to offer a very rough picture that the reader can keep in their head as a first approximation to the story. First, it is a basic fact in homotopy theory that $\Omega BG \simeq G$ for any topological group G; so we get $\Omega(\mathbb{Z}\times BU)\simeq U$ for free. Bott periodicity comes from a calculation of ΩU, or of $\Omega U(n)$ for large n. For (at the moment) unexplained reasons we will replace this with the homotopy equivalent space $\Omega_{I,-I}U(n)$ of paths from I to $-I$. If $H\subseteq \mathbb{C}^n$ is a complex subspace, then we can make an easy example of such a path by multiplying vectors in H by $e^{\pi i t}$ and vectors in $H^\perp$ by $e^{-\pi i t}$, as t goes from 0 to 1. So we are doing a (gradual) complex rotation on H and the opposite rotations on $H^\perp$. There are, of course, many types of paths on $U(n)$ other than these simple ones—but the fundamental calculation of Bott is that as n gets large these simple paths represent more and more of the homotopy type of $\Omega_{I,-I}U(n)$. So in a large range that grows with n we have that $\Omega_{I,-I}U(n)$ is weakly equivalent to the space of all H, which is the Grassmannian $\mathrm{Gr}_\bullet(\mathbb{C}^n)=\coprod_k \mathrm{Gr}_k(\mathbb{C}^n)$. This space has $n+1$ components, but in the stabilization process (which has both k and n going to infinity) it becomes a $\mathbb{Z}$'s worth of components and we get $\mathbb{Z}\times BU$. Thus we arrive at the Bott equivalence $\Omega U\simeq \mathbb{Z}\times BU$.

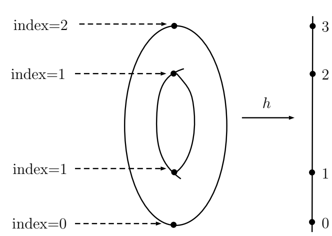
There are, of course, lots of questions to answer here. Why do these simple H-paths constitute a large range of the homotopy type of $\Omega_{I,-I}U(n)$? What exactly is going on in the stabilization process? (It is trickier than you might expect!) But the above paragraph does an adequate job of giving a very quick picture of what Bott periodicity is about.

What about periodicity for the orthogonal group O? Here one looks at the space $\Omega_{I,-I}O(2n)$ and observes that for every orthogonal complex structure J on $\mathbb{R}^{2n}$ (an orthogonal transformation such that $J^2 = -I$) the formula $t \mapsto e^{\pi t J}$ gives a very simple path from I to $-I$. Note that the condition $J^2 = -I$ gives that $e^{sJ} = \cos(s)I + \sin(s)J$. The Bott arguments show that the homotopy type of $\Omega_{I,-I}O(2n)$ is approximated in a large range by the space of these "simple" paths—i.e., by the space of all such J. This is the space of complex structures on $\mathbb{R}^{2n}$. As one iterates this procedure one finds that paths of complex structures (from a chosen J to $-J$) are approximated by spaces of quaternionic structures, and very quickly one finds the real Clifford algebras appearing. The eightfold periodicity of the Clifford algebras reappears in the analysis of these iterated loop spaces, so that $\Omega^8 O$ turns out to be O again. And that—in a nutshell—is real Bott periodicity. It is a complicated nutshell, but we will spend the next several sections filling in the details.

41. Periodicity via Bott-Morse theory

Bott's original approach to the periodicity theorems was through Morse theory and the analysis of geodesics on Lie groups and their homogeneous spaces. Later approaches have tried to eliminate as much of the Morse and Lie theory as possible, but I'm not sure how helpful this is in the end. There is something important hidden in the Bott-Morse arguments that is worth understanding, so this is where we are going to begin our journey.

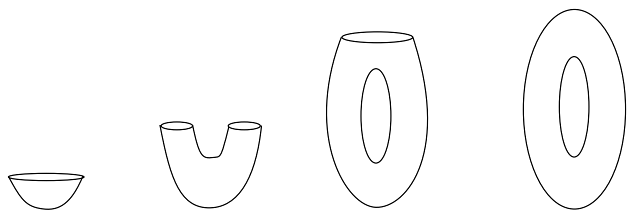
41.1. **Brief tour of Morse theory.** Let X be a smooth manifold, and let $h\colon X \to \mathbb{R}$ be a smooth function with nondegenerate critical points—this means that the Hessian matrix of second partial derivatives has full rank at these points. Such critical points are isolated (by the Taylor approximation), and to each one we can associate an **index**: geometrically, this is the number of directions on the manifold where the function h is *decreasing*. Algebraically, the index is the number of negative eigenvalues of the Hessian matrix. The canonical example is the height function on the torus, as depicted in the following picture:



In Morse theory one looks at the spaces $X_{\leq c} = h^{-1}((-\infty, c])$ and studies the changes as c increases. The homotopy type of $X_{\leq c}$ changes precisely when c is a critical value, and the change is equivalent to a cell attachment where the dimension of the cell is the index of the corresponding critical point. So in the example from the picture we get

$$\begin{array}{rl} c<0: & X_c \simeq \emptyset \\ 0 \le c < 1: & X_c \simeq pt \\ 1 \le c < 2: & X_c \simeq e^0 \cup e^1 = S^1 \\ 2 \le c < 3: & X_c \simeq e^0 \cup e^1 \cup e^1 = S^1 \vee S^1 \\ 3 \le c < \infty: & X_c \simeq e^0 \cup e^1 \cup e^1 \cup e^2 = X. \end{array}$$

Here are drawings depicting samples from the four nonempty stages:



Morse showed that when X is a Riemannian manifold one can apply similar techniques to study the loop space ΩX, taking h to be a certain "energy functional". The space ΩX is no longer a finite-dimensional manifold but with some care the techniques still work (this is similar to what happens in the classical calculus of variations). It will be useful to modify the loop space slightly, though. Choose $P, Q \in X$ and let $\Omega_{P,Q}X$ denote the space of paths which begin at P and end at Q. When $P = Q$ we will simplify the notation to $\Omega_P X$. If γ is any fixed path from Q to P, post-composition with γ and γ^{-1} gives the maps in a homotopy equivalence $\Omega_{P,Q}X \simeq \Omega_P X$.

We need to next restrict to the subspace $\Omega^{sm}_{P,Q}X \hookrightarrow \Omega_{P,Q}X$ consisting of piecewise C^∞ paths. The inclusion can be shown to be a homotopy equivalence [Mi1, Theorem 17.1]. We define the "energy functional" $E\colon \Omega^{sm}_{P,Q}(X) \to \mathbb{R}$ by

$$E(\gamma) = \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt$$

where the norm of $\frac{d\gamma}{dt}$ is of course defined via the Riemannian metric on X. The details of E are not so important here, but the crucial fact is that the critical points of E are the geodesics on X. The index of a geodesic is the dimension of a space of certain kinds of "small variations". We will demonstrate this concept with the example of $X = S^n$ for $n \geq 2$.

The main observation about the sphere is that there is a unique minimal geodesic between points P and Q except when they are antipodal. In the antipodal case, there is a whole family of minimal geodesics—and that family is precisely an S^{n-1}, parameterized by the midpoint of the geodesic (where it intersects the equator).

First consider the case where P and Q are generically positioned. Without loss of generality we assume P is the north pole, and Q is any point other than $-P$. There is exactly one minimal geodesic from P to Q, and there are no "variations" on this geodesic—so it has index 0. Next, there is the geodesic from P to Q going around the sphere in the opposite direction, passing through $-P$. The part of the geodesic going from P to $-P$ allows for "variation", though, as we discussed above—the dimension of such variations is $n-1$, so that is the index of this geodesic.

Continuing on, the next longest geodesic from P to Q is the one that takes the short path from P to Q but then contines all the way around the sphere—passing through $-P$ and P—before once again returning to Q. For this geodesic we get variations for the portion from P to $-P$, but then also for the portion from $-P$ back to P—so that is $(n-1)+(n-1)$ dimensions of variation. Proceeding in this way, one finds that the geodesics from P to Q have indices 0, $n-1$, $2(n-1)$, $3(n-1)$, and so forth. So Morse theory ends up telling us that

$$\Omega_{P,Q}S^n \simeq e^0 \cup e^{n-1} \cup e^{2(n-1)} \cup \cdots .$$

As we described above, Morse's original techniques assume that the critical points of h are isolated. In more general settings one might instead have that the inverse image of a critical value is a subspace of positive dimension. Bott extended Morse theory so that it could be applied in these settings; while it still gives a decomposition of the space X, the components are not just cells—they will be disk bundles of vector bundles over the critical subspaces instead. As an example, consider now the space $\Omega_{P,-P}S^n$. We don't have a single minimal geodesic from P to $-P$ like we did before—we have a whole family of minimal geodesics, forming the space S^{n-1}. This is our index 0 piece of the decomposition. The other geodesics all have index at least $2(n-1)$, so even without analyzing them in detail we find that

$$(*) \qquad \Omega_{P,-P}S^n \simeq S^{n-1} \cup (\text{cells of dimension at least } 2(n-1)).$$

Even this limited knowledge is substantial; for example, it is enough to prove the Freudenthal Suspension Theorem! Fix a minimal-length geodesic γ from P to $-P$, and consider the map $f\colon S^{n-1} \to \Omega_P S^n$ that sends a point x to the path "follow the geodesic from P to $-P$ that passes though x, then follow γ^{-1} back to P". This is essentially the map $S^{n-1} \to \Omega S^n$ that is adjoint to the identity $\Sigma S^{n-1} \to S^n$ (one should contract the image of γ to identify S^n with the reduced suspension of S^{n-1}). The Bott-Morse decomposition of (*) tells us that f induces isomorphisms on π_i for $i \le 2n-4$ and an epimorphism for $i = 2n-3$, which implies that the suspension map $\pi_i(S^{n-1}) \to \pi_{i+1}(S^n)$ has the same behavior. This is the Freudenthal Suspension Theorem.

Remark 41.2. Consider now the case of S^1. There are two minimal geodesics from a point P to $-P$, corresponding to the two ways of going around the circle. So the space of minimal geodesics is S^0, and we have the decomposition

$$\Omega_{P,-P}S^1 \simeq S^0 \cup (\text{other cells}).$$

Note that $\Omega_{P,-P}S^1 \simeq \mathbb{Z}$, so this decomposition gives an accurate approximation but only on two of the components. The issue here is that we have non-minimal geodesics that also have index zero, and so give rise to 0-cells. In fact, *all* of the geodesics have index 0, and the $\mathbb{Z}$ is exactly parameterizing the different geodesics.

This is not just an idle example. In our work below we will often enounter situations where ΩX is not connected and the minimal geodesics only tell us about some of the components. Of course in a loop space all of the components are homotopy equivalent, so as soon as we understand one we understand them all. But the existence of non-minimal geodesics of small index (in particular, index 0) will sometimes get in the way of us being able to specify how close an approximation the space of minimal geodesics is to ΩX.

41.3. **Applying Bott-Morse theory to the unitary group.** We will consider the group $U(n)$ and attempt to analyze geodesics from I to $-I$. Let us recall how the basic Lie theory works. For A in the tangent space at the identity, the path

$t \mapsto e^{tA}$ is a geodesic in $U(n)$ whose tangent vector at $t=0$ is A. The condition that e^{tA} is unitary is $I = e^{tA}(e^{tA})^\dagger = e^{tA}e^{tA^\dagger}$. Applying $\frac{d}{dt}\big|_{t=0}$ gives the equation $0 = A + A^\dagger$. Conversely, this equation implies that A and $A^\dagger$ commute and therefore $e^{tA}e^{tA^\dagger} = e^{t(A+A^\dagger)} = e^0 = I$. So the tangent space $T_I U(n)$ consists precisely of the skew-Hermitian matrices.

The maximal torus in $U(n)$ has rank n and consists of the diagonal matrices, so the tangent space of the torus is the subspace D_{skH} of skew-Hermitian diagonal matrices. We identify this with $\mathbb{R}^n$ via $(r_1, \ldots, r_n) \mapsto \mathrm{diag}(ir_1, \ldots, ir_n)$. Every element of $T_I U(n)$ is conjugate (via the adjoint action of $U(n)$, which is literally matrix conjugation) to an element of D_{skH}.

Finding geodesics from I to $-I$ is the same as finding all skew-Hermitian matrices A for which $t \mapsto e^{tA}$ passes through $-I$ when $t = 1$. By conjugating, we can assume that A is diagonal. Therefore $A = \mathrm{diag}(k_1\pi i, \ldots, k_n \pi i)$ for some odd integers k_j. One can check that the minimal geodesics are the ones for which all the k_j's are 1 or -1. Our space of minimal geodesics is the set of $U(n)$-conjugates of these. Since $U(n)$ contains the permutation matrices, we reduce to thinking about the conjugates of the matrices $D_r = \mathrm{diag}(\pi i, \ldots, \pi i, -\pi i, \ldots, -\pi i)$ where πi occurs r times and $-\pi i$ occurs $n-r$ times. The stabilizer of D_r is readily checked to be $U(r) \times U(n-r)$, and so our space of minimal geodesics is

$$\coprod_{r=0}^{n} U(n)/(U(r) \times U(n-r)) \cong \coprod_{r=0}^{n} \mathrm{Gr}_r(\mathbb{C}^n).$$

We can also think about this isomorphism in terms of eigenspaces. We are looking at skew-Hermitian matrices whose only eigenvalues are πi and $-\pi i$. Because they are skew-Hermitian, the eigenspaces must be orthogonal—hence knowing one eigenspace determines the other. So the conjugacy classes are determined by the πi-eigenspace.

In this way we have produced a map

$$\coprod_{r=0}^{n} \mathrm{Gr}_r(\mathbb{C}^n) \xrightarrow{\beta} \Omega_{I,-I} U(n)$$

that sends a subspace $H \subseteq \mathbb{C}^n$ to the geodesic $t \mapsto e^{tA}$ where A is the unique linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ which is multiplication by πi on H and multiplication by $-\pi i$ on $H^\perp$. This map β is called the **Bott map**. The appearance of the disjoint union in the domain is a little annoying at first, but in retrospect something like this is expected. Recall that $\pi_1 U(n) \cong \mathbb{Z}$, with the isomorphism induced by the determinant map $\det\colon U(n) \to S^1$. So $\pi_0(\Omega U(n)) \cong \mathbb{Z}$, meaning that $\Omega U(n)$ has countably many path components. Our map β is touching $n+1$ of these components. How do we understand which components? We use $\det\colon U(n) \to S^1$ to induce a map $\Omega_{I,-I} U(n) \to \Omega_{1,(-1)^n} S^1$, and this is a bijection on path components. The path components of the target can be indexed by integers k, with one possible correspondence having k associated to the path $t \mapsto e^{t\pi i(2k-n)}$. If $H \subseteq \mathbb{C}^n$ has dimension r then the map "multiplication by $e^{\pi i t}$ on H and multiplication by $e^{-\pi i t}$ on $H^\perp$" has determinant $e^{\pi i(2r-n)}$ and therefore $\beta(H)$ is in the path component of $\Omega_{I,-I} U(n)$ labelled by r.

The appearance of the disjoint union causes some issues in our Bott-Morse arguments, as in Remark 41.2. We will come back to this, but for the moment let us take an approach that gets us around that issue. Assume $n = 2k$ and consider only the portion $\mathrm{Gr}_k(\mathbb{C}^{2k}) \to \Omega_{I,-I}U(2k)$ of the Bott map. The image consists of conjugates of the geodesic $t \mapsto \mathrm{diag}(e^{\pi it}, \dots, e^{\pi it}, e^{-\pi it}, \dots, e^{-\pi it})$ where $e^{\pi it}$ and $e^{-\pi it}$ both appear k times. Such matrices (and their conjugates) all have determinant equal to 1, and so these are geodesics lying in $SU(2k)$. The loop space of $SU(2k)$ is path-connected, so we can save some trouble by just working here. We have found that the space of minimal geodesics from I to $-I$ is the space $\mathrm{Gr}_k(\mathbb{C}^{2k})$.

With a little effort one can analyze the non-minimal geodesics in $SU(2k)$ and calculate their index: for the geodesic corresponding to the matrix $A = \mathrm{diag}(\pi i r_1, \dots, \pi i r_{2k})$ (with all r_i odd and $\sum r_i = 0$) the index turns out to be

$$\text{index} = \sum_{r_i > r_j} (r_i - r_j - 2).$$

We refer the reader to [Bott] or [Mi1] for this calculation. When half the r's are 1 and the other half -1, this gives index 0—as expected for minimal geodesics. In all other cases one can readily prove that the index is at least $2k+2$ (a sequence giving this minimum is 3 followed by $k-2$ copies of 1, followed by $k+1$ copies of -1). So Bott-Morse theory gives us that

$$\Omega_{I,-I}SU(2k) \simeq \mathrm{Gr}_k(\mathbb{C}^{2k}) \cup (\text{cells of dimension } (2k+2) \text{ and higher}).$$

From here we get that $\pi_i \Omega SU(2k) \cong \pi_i \mathrm{Gr}_k(\mathbb{C}^{2k})$ for $i \leq 2k$. But then for each $i \geq 2$ we can choose a large enough k and argue that

$$\begin{aligned}
\pi_i U = \pi_i U(2k) = \pi_i SU(2k) = \pi_{i-1}\Omega SU(2k) &\cong \pi_{i-1} \mathrm{Gr}_k(\mathbb{C}^{2k}) \\
&= \pi_{i-1} \mathrm{Gr}_k(\mathbb{C}^{\infty}) \\
&= \pi_{i-1} BU(k) \\
&= \pi_{i-2}(\Omega BU(k)) \\
&= \pi_{i-2}(U(k)) \\
&= \pi_{i-2} U.
\end{aligned}$$

In the fifth isomorphism we have used that $\mathrm{Gr}_k(\mathbb{C}^{\infty})$ is obtained from $\mathrm{Gr}_k(\mathbb{C}^{2k})$ by adding $(2k+2)$-cells and higher, which follows from the standard cell decompositions for the Grassmannians. So we have now proven the most primitive version of Bott periodicity, namely the isomorphisms $\pi_t U \cong \pi_{t+2} U$ for $t \geq 0$.

We can perform the stabilization at the level of spaces rather than after taking homotopy groups, but this involves a little care. The issue is that we are dealing with paths from I to $-I$, and these do not map to similar paths under the inclusion $SU(2n) \hookrightarrow SU(2n+2)$. So we have to change back to paths from I to I.

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{C}^n$. Let $M^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$, where the basis for the first copy of $\mathbb{C}^n$ is denoted $e_1^+, \dots, e_n^+$ and the basis for the second copy is $e_1^-, \dots, e_n^-$. From now on when we write $U(2n)$ we mean the group of unitary automorphisms of M^{2n}. Let $\gamma\colon I \to SU(2n)$ be the path from I to $-I$ given by

$$\gamma(t) = \mathrm{diag}(e^{-\pi it}, \dots, e^{-\pi it}, e^{\pi it}, \dots, e^{\pi it})$$

(with n copies of $e^{-\pi it}$ and n copies of $e^{\pi it}$). For any other path α from I to $-I$, write $\alpha\gamma$ for the path $t \mapsto \alpha(t)\gamma(t)$, which is a loop at I. This gives a homeomorphism $\Gamma_n\colon \Omega_{I,-I}SU(2n) \to \Omega_I SU(2n)$.

Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Gr}_n(M^{2n}) \arrow[r, "\beta"] \arrow[d, "j_n"'] & \Omega_{I,-I}(SU(2n)) \arrow[r, "\Gamma_n"] \arrow[d, "g"'] & \Omega_I SU(2n) \arrow[d] \\
\mathrm{Gr}_{n+1}(M^{2(n+1)}) \arrow[r, "\beta"] & \Omega_{I,-I}(SU(2(n+1))) \arrow[r, "\Gamma_{n+1}"] & \Omega_I SU(2(n+1)).
\end{tikzcd}$$

Here j_n sends a subspace $W \subseteq M^{2n}$ to $W \oplus \langle e^+_{n+1}\rangle \subseteq M^{2(n+1)}$. For the map g, if $u\colon [0,1] \to SU(2n)$ is a path from I to $-I$ we define $(gu)(t) = u(t) \oplus \mathrm{diag}(e^{\pi it}, e^{-\pi it})$. The right vertical map is induced by the standard inclusion of $SU(2n)$ into $SU(2(n+1))$, namely $A \mapsto A \oplus \mathrm{id}$.

We need to check that the diagram commutes. Commutativity of the right-most square is easy. Commutativity of the left square follows after one remembers that the map β sends a subspace H to the path which at time t scales H by $e^{\pi it}$ and scales $H^\perp$ by $e^{-\pi it}$. Note that j_n can be thought of as "adding e^+_{n+1} to H and adding e^-_{n+1} to $H^\perp$".

Taking vertical colimits (as $n \to \infty$) in the above diagram, the first and third columns yield a map $\mathrm{colim}_n \mathrm{Gr}_n(M^{2n}) \to \Omega_I SU$. This also goes under the name "the Bott map", and will be denoted β_∞ (sometimes we will drop the subscript, though). Since the horizontal composites $\Gamma_n \circ \beta$ become arbitrarily highly connected as n goes to ∞, the map β_∞ is a weak equivalence. It only remains to understand the homotopy type of the colimit in the domain. For this, consider the doubly infinite diagram

$$\begin{tikzcd}
\mathrm{Gr}_1(\mathbb{C}^1_+ \oplus \mathbb{C}^1_-) \arrow[r] \arrow[d] & \mathrm{Gr}_1(\mathbb{C}^1_+ \oplus \mathbb{C}^2_-) \arrow[r] \arrow[d] & \mathrm{Gr}_1(\mathbb{C}^1_+ \oplus \mathbb{C}^3_-) \arrow[r] \arrow[d] & \cdots \arrow[r, Rightarrow] & \mathrm{Gr}_1(\mathbb{C}^\infty) \arrow[d] \\
\mathrm{Gr}_2(\mathbb{C}^2_+ \oplus \mathbb{C}^1_-) \arrow[r] \arrow[d] & \mathrm{Gr}_2(\mathbb{C}^2_+ \oplus \mathbb{C}^2_-) \arrow[r] \arrow[d] & \mathrm{Gr}_2(\mathbb{C}^2_+ \oplus \mathbb{C}^3_-) \arrow[r] \arrow[d] & \cdots \arrow[r, Rightarrow] & \mathrm{Gr}_2(\mathbb{C}^\infty) \arrow[d] \\
\vdots & \vdots & \vdots & & \vdots
\end{tikzcd}$$

The horizontal maps are induced by inclusions $\mathbb{C}^p_+ \oplus \mathbb{C}^q_- \hookrightarrow \mathbb{C}^p_+ \oplus \mathbb{C}^{q+1}_-$, whereas the vertical maps send a subspace $W \subseteq \mathbb{C}^p_+ \oplus \mathbb{C}^q_-$ to the subspace $W \oplus \langle e^+_{p+1}\rangle \subseteq \mathbb{C}^{p+1}_+ \oplus \mathbb{C}^q_-$. The space $\mathrm{colim}_n \mathrm{Gr}_n(M^{2n})$ is the colimit along the diagonal of the above diagram, and so is canonically isomorphic to the space obtained by first taking horizontal colimits and then vertical colimits. That is, $\mathrm{colim}_n \mathrm{Gr}_n(M^{2n}) \simeq \mathrm{Gr}_\infty(\mathbb{C}^\infty) \simeq BU$.

As the last topic of this section we describe an approach to understanding ΩU that handles all of the components at once, rather than just focusing on the $\Omega SU = (\Omega U)_0$ component.

Write $\mathbb{C}^{p,q}$ as an abbreviation for $\mathbb{C}^p \oplus \mathbb{C}^q$ where the first summand has basis $e^+_1, \dots, e^+_p$ and the second summand has basis $e^-_1, \dots, e^-_q$. It will be convenient to write $\mathbb{C}^p_+$ for the first summand and $\mathbb{C}^q_-$ for the second. For $p \le p'$ and $q \le q'$ we have the evident inclusions $\mathbb{C}^{p,q} \subseteq \mathbb{C}^{p',q'}$. Write $\mathrm{Gr}_\bullet \mathbb{C}^{p,q}$ for $\coprod_k \mathrm{Gr}_k(\mathbb{C}^{p,q})$. Let us also write $U(p,q)$ for the group of unitary automorphisms of $\mathbb{C}^{p,q}$, though note that there is an evident identification $U(p,q) \cong U(p+q)$.

Let $\gamma\colon [0,1] \to U(p,q)$ be the path $\gamma(t) = e^{-\pi it}|_{\mathbb{C}^p_+} \oplus e^{\pi it}|_{\mathbb{C}^q_-}$. That is, $\gamma(t)$ multiplies positive basis elements by $e^{-\pi it}$ and negative basis elements by $e^{\pi it}$. For a subspace $H \subseteq \mathbb{C}^{p,q}$, let $\Phi_H\colon [0,1] \to U(p,q)$ be the path where $\Phi_H(t)$ multiplies elements of H by $e^{\pi it}$ and elements of $H^\perp$ by $e^{-\pi it}$. Define $\beta_H(t) = \Phi_H(t)\cdot\gamma(t)$. Then β_H is a loop based at I, so that we have a map $\beta\colon \operatorname{Gr}_\bullet \mathbb{C}^{p,q} \to \Omega_I U(p+q)$. The following property will be important to us:

(**) For all t, the linear map $\beta_H(t)$ is the identity when restricted to $H \cap \mathbb{C}^p_+$ or to $H^\perp \cap \mathbb{C}^q_-$.

For $H \in \operatorname{Gr}_k(\mathbb{C}^{p,q})$ the unitary transformation $\Phi_H(t)$ has determinant equal to $e^{k\pi it}\cdot e^{-(p+q-k)\pi it} = e^{(2k-p-q)\pi it}$. The transformation $\gamma(t)$ has determinant $e^{-p\pi it}\cdot e^{q\pi it} = e^{(q-p)\pi it}$. So $\det\beta_H(t) = e^{2(k-p)\pi it}$. Therefore β maps $\operatorname{Gr}_k(\mathbb{C}^{p,q})$ into the component $[\Omega_I U(p,q)]_{k-p}$. For this reason, let us label the components of $\operatorname{Gr}_\bullet(\mathbb{C}^{p,q})$ so that Gr_k is the "$k-p$ component". So the components of $\operatorname{Gr}_\bullet(\mathbb{C}^{p,q})$ are labelled as $-p, -(p-1), \ldots, q-1, q$.

We now consider the following lattice diagram, consisting of two kinds of stabilizations:

$$
\begin{tikzcd}
\operatorname{Gr}_\bullet \mathbb{C}^{1,1} \arrow[r] \arrow[d] & \operatorname{Gr}_\bullet \mathbb{C}^{1,2} \arrow[r] \arrow[d] & \operatorname{Gr}_\bullet \mathbb{C}^{1,3} \arrow[r] \arrow[d] & \cdots \\
\operatorname{Gr}_\bullet \mathbb{C}^{2,1} \arrow[r] \arrow[d] & \operatorname{Gr}_\bullet \mathbb{C}^{2,2} \arrow[r] \arrow[d] & \operatorname{Gr}_\bullet \mathbb{C}^{2,3} \arrow[r] \arrow[d] & \cdots \\
\operatorname{Gr}_\bullet \mathbb{C}^{3,1} \arrow[r] \arrow[d] & \operatorname{Gr}_\bullet \mathbb{C}^{3,2} \arrow[r] \arrow[d] & \operatorname{Gr}_\bullet \mathbb{C}^{3,3} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
\tag{41.4}
$$

The horizontal maps consist of the evident inclusions $\operatorname{Gr}_k(\mathbb{C}^{p,q}) \hookrightarrow \operatorname{Gr}_k(\mathbb{C}^{p,q+1})$ ("add e^-_{q+1} to the complement of the subspace H"), whereas the vertical maps consist of the inclusions $\operatorname{Gr}_k(\mathbb{C}^{p,q}) \hookrightarrow \operatorname{Gr}_{k+1}(\mathbb{C}^{p+1,q})$ ("add e^+_{p+1} to the subspace H"). Note that both horizontal and vertical maps respect the labels on the components—that is, in all cases the component with label x is sent to the component with the same label x. The colimit across the top row is a disjoint union $\coprod_{k\geq 0}\operatorname{Gr}_k(\mathbb{C}^\infty)$, with components labelled -1 and higher. Taking colimits across each row gives the diagram

$$
\begin{tikzcd}
\cdots & \boxed{\text{-2}} & & \boxed{\text{-1}} & & \boxed{0} & & \boxed{1} & \cdots \\
 & & & \mathrm{Gr}_0(\mathbb{C}^\infty) \arrow[d] & \amalg & \mathrm{Gr}_1(\mathbb{C}^\infty) \arrow[d] & \amalg & \mathrm{Gr}_2(\mathbb{C}^\infty) \amalg \arrow[d] & \cdots \\
 & \mathrm{Gr}_0(\mathbb{C}^\infty) \arrow[d] & \amalg & \mathrm{Gr}_1(\mathbb{C}^\infty) \arrow[d] & \amalg & \mathrm{Gr}_2(\mathbb{C}^\infty) \arrow[d] & \amalg & \mathrm{Gr}_3(\mathbb{C}^\infty) \amalg \arrow[d] & \cdots \\
\iddots & \vdots & & \vdots & & \vdots & & \vdots &
\end{tikzcd}
$$

where in the first row the components are labelled with -1 and higher, in the second row with -2 and higher, and so on. So the colimit of the entire diagram is $\mathbb{Z} \times BU$.

Now consider the corresponding lattice of unitary groups:

$$
\begin{tikzcd}
U(1,1) \arrow[r] \arrow[d] & U(1,2) \arrow[r] \arrow[d] & U(1,3) \arrow[r] \arrow[d] & \cdots \\
U(2,1) \arrow[r] \arrow[d] & U(2,2) \arrow[r] \arrow[d] & U(2,3) \arrow[r] \arrow[d] & \cdots \\
U(3,1) \arrow[r] \arrow[d] & U(3,2) \arrow[r] \arrow[d] & U(3,3) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

Here both horizontal and vertical maps are induced by the inclusions $\mathbb{C}^{p,q} \hookrightarrow \mathbb{C}^{p',q'}$. The colimit of the diagram is $U(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)$ and therefore a model of the group U.

The Bott maps β give a map of diagrams from the array of Grassmannians to Ω_I applied to the array of unitary groups. One readily checks that property (**) above implies that all the squares commute. Taking colimits gives us the map $\mathbb{Z} \times BU \to \Omega_I U$. The Bott-Morse arguments imply that taking colimits in each row gives us a map that is a weak equivalence on the components that are hit, and then taking the colimit in the vertical direction allows us to hit all of the components. So our map $\mathbb{Z} \times BU \to \Omega_I U$ is a weak equivalence.

41.5. **Geodesics on the orthogonal group.** Analyzing geodesics on $O(n)$ is fairly similar to what we did for $U(n)$, but just different enough so that the space of minimal geodesics is not a Grassmannian anymore. This immediately dashes the chances for the kind of 2-fold periodicity we saw before. In a moment we will see precisely what the space of minimal geodesics is, but before diving into the analysis let us talk about the big picture.

The space of minimal geodesics turns out to be another Lie-theoretic space: not a Lie group, but a Riemannian submanifold of a Lie group having the property that geodesics that start tangent to the submanifold never leave it (like the equator of a sphere). So one can apply the Bott technique *again* to analyze its loop space. In this manner one gets yet another space of minimal geodesics (technically the minimal geodesics on the space of minimal geodesics), which again ends up Lie-theoretic in nature. So one continues in this process, and after seven times something magical happens in that one finally encounters the real Grassmannian—so that after looping one more time one gets back to the orthogonal group.

Part of what is challenging in navigating this argument is keeping track of the eight spaces and their various relations to each other. So let us spend some time on this topic before diving into details.

The key players are the orthogonal, unitary, and symplectic groups: $O(n)$, $U(n)$, and $Sp(n)$. These are the space of inner-product preserving linear automorphisms of $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$. As matrix groups they consist of $n \times n$ matrices A of (real, complex, quaternionic) numbers such that $A^\dagger A = I$ where $(-)^\dagger$ is the conjugate-transpose. The inclusions $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$ therefore induce standard inclusions of groups $O(n) \subseteq U(n) \subseteq Sp(n)$.

However, we can also think of $\mathbb{H}$ as $\mathbb{C}^2$ (or $\mathbb{C}$ as $\mathbb{R}^2$). When we do this, every symplectic map $\mathbb{H}^n \to \mathbb{H}^n$ is a unitary map $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$, and every unitary map $\mathbb{C}^n \to \mathbb{C}^n$ is an orthogonal map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. So we get inclusions $Sp(n) \hookrightarrow U(2n)$ and $U(n) \hookrightarrow O(2n)$.

Now let us discard the indices by passing to the direct limit groups O, U, and Sp. The following "clock" shows the six inclusions we have just mentioned and includes two extra terms we will explain presently:

$$\begin{tikzcd}
 & O \arrow[r] & U \arrow[dr] & \\
O \times O \arrow[ur] & & & Sp \arrow[d] \\
O \arrow[u] & & & Sp \times Sp \arrow[dl] \\
 & U \arrow[ul] & Sp \arrow[l] &
\end{tikzcd}$$

Here the inclusions $O \to O \times O$ and $Sp \to Sp \times Sp$ are the diagonals. The inclusion $O \times O \to O$ comes from writing $\mathbb{R}^\infty = \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ (think of separating the even and odd basis elements), and then sending a pair of orthogonal transformations (A, B) to the transformation that does A on the first copy of $\mathbb{R}^\infty$ and B on the second. The inclusion $Sp \times Sp \to Sp$ is defined similarly.

Okay, we are almost done. Now go back to the above clock and label each arrow with the corresponding group quotient, with the provisos that $(G \times G)/G = G$ and $G/(G \times G) = \mathbb{Z} \times BG$ (of course these are not really group quotients, cf. Remark 41.9):

$$\begin{tikzcd}
 & O \arrow[r, "U/O"] & U \arrow[dr, "Sp/U"] & \\
O \times O \arrow[ur, "\mathbb{Z}\times BO"] & & & Sp \arrow[d, "Sp"] \\
O \arrow[u, "O"] & & & Sp \times Sp \arrow[dl, "\mathbb{Z}\times BSp"] \\
 & U \arrow[ul, "O/U"] & Sp \arrow[l, "U/Sp"] &
\end{tikzcd}$$

What Bott's arguments will show is that applying loops to a label on an arrow gives the label on the preceding arrow:

$$\begin{array}{ll} \Omega(\mathbb{Z}\times BO)\simeq O & \Omega(\mathbb{Z}\times BSp)\simeq Sp \\ \Omega O\simeq O/U & \Omega Sp\simeq Sp/U \\ \Omega(O/U)\simeq U/Sp & \Omega(Sp/U)\simeq U/O \\ \Omega(U/Sp)\simeq \mathbb{Z}\times BSp & \Omega(U/O)\simeq \mathbb{Z}\times BO. \end{array}$$

Note that we can now get the homotopy groups for O (or really, any of the arrow labels) by simply knowing π_0 of all the spaces involved. We know that O has two path components, whereas U and Sp both have only one. So O/U still has two components, whereas U/O, U/Sp, and Sp/U are all connected. Starting with $\mathbb{Z}\times BO$ and moving counterclockwise, the π_0 values are therefore $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$. Voila!

Exercise 41.6. Use the above octagon to work out all the homotopy groups of U/O, Sp/U, and U/Sp.

Remark 41.7. Our circle of groups is called the "real Bott periodicity clock". It is very easy to remember, via the following method. Write down the groups O, U, Sp in order (which is the order of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$) and then on the other side of the clock write them in reverse order. Between the two O's write $O\times O$, and between the two Sp's write $Sp\times Sp$. It doesn't matter which direction you draw the arrows, as long as they all point in the same direction. The formula $\Omega(\mathbb{Z}\times BO)\simeq O$ will tell you which way the loop-relations go. If you were stuck on a desert island and needed to remember the spaces in the Ω-spectrum for KO, this is a way to do so.

Exercise 41.8. Work out the homotopy groups of U/O, O/U, and all of the other spaces in the Bott periodicity clock. Even better, work out a method for directly reading these off of the Bott periodicity clock that only requires remembering the number of components of O, U, and Sp.

Remark 41.9. It will take us a while to explain the meaning of the "quotients" $(G\times G)/G=G$ and $G/(G\times G)=\mathbb{Z}\times BG$. The first is perhaps not so strange, but the second begs some hint of an explanation. Consider $O(n)\times O(n)\hookrightarrow O(2n)$, where the former is the evident subgroup of block diagonal matrices. The quotient in spaces is $O(2n)/(O(n)\times O(n))\cong \mathrm{Gr}_n(\mathbb{R}^{2n})$. Stabilizing with n gives a familiar model of the space BO. This doesn't explain the extra $\mathbb{Z}$ factor, but it does (sort of) give a way of rationalizing the mnemonic $G/(G\times G)=\mathbb{Z}\times BG$.

Here are the precise statements that come out of the Bott-Morse argument with minimal geodesics:

$$\begin{aligned}
\Omega O(2n) &\simeq O(2n)/U(n)\cup(\text{cells of dimension}\geq 2n-2)\\
\Omega[O(4n)/U(2n)] &\simeq U(2n)/Sp(n)\cup(\text{cells of dimension}\geq 4n-2\ )\\
\Omega_0[U(4n)/Sp(2n)] &\simeq Sp(2n)/(Sp(n)\times Sp(n))\cup(\text{cells of dimension}\geq 4n+4\ )\\
\Omega Sp(n) &\simeq Sp(n)/U(n)\cup(\text{cells of dimension}\geq 2n+2)\\
\Omega[Sp(2n)/U(n)] &\simeq U(n)/O(n)\cup(\text{cells of dimension}\geq n+1)\\
\Omega_0[U(2n)/O(2n)] &\simeq O(2n)/(O(n)\times O(n))\cup(\text{cells of dimension}\geq n+1)
\end{aligned}$$

Here Ω_0 refers to the basepoint component of the loop space, whereas the actual loop space has $\pi_0 = \mathbb{Z}$ (coming from $\pi_1 U(n) = \mathbb{Z}$). Here is the extra $\mathbb{Z}$ we were missing in Remark 41.9.

Now that we have the lay of the land, let us take a look at the analysis of $\Omega_{I,-I}O(2n)$ (note that we need the dimension to be even in order for I and $-I$ to be in the same path component). The tangent space of $O(2n)$ at the identity consists of matrices A such that e^{At} lies in $O(2n)$ for all t. So we get $I = e^{At}(e^{At})^T = e^{(A+A^T)t}$ and this implies that $A + A^T = 0$. So the tangent space consists of the skew-symmetric matrices. We take as our maximal torus $T \subseteq O(2n)$ the set of rotation matrices of the form $R\theta_1 \oplus \cdots \oplus R\theta_n$ where $R\theta = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$. The corresponding tangent vectors are matrices of the form $D(r_1, \ldots, r_n) := r_1 J \oplus \cdots \oplus r_n J$ where $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $r_1, \ldots, r_n \in \mathbb{R}$. Lie theory tells us that every matrix $A \in T_I O(2n)$ is conjugate to one of the above $D(r_1, \ldots, r_n)$ matrices, but in fact we can do a little better. In $O(2)$ the matrix J is conjugate to $-J$, and so we can assume that all $r_i \geq 0$. (We can also use permutation matrices to ensure that $r_1 \geq r_2 \geq \cdots \geq r_n$, but will not need this).

Exercise 41.10. Check that J is conjugate to $-J$ in $O(2)$.

Our next task is to determine which matrices $A \in T_I O(2n)$ have the property that $t \mapsto e^{tA}$ is a path from I to $-I$. Since $e^A = Pe^{D(\underline{r})}P^{-1}$, this is the condition that $e^{D(\underline{r})} = -I$. But note that $e^{rJ} = \cos(r)I + \sin(r)J$, so the previous condition is equivalent to saying that each r_i is a (positive) odd multiple of π. It turns out (and is not surprising) that the *minimal* geoedesics are the ones where $r_i = \pi$ for all i. The space of minimal geodesics is then homeomorphic to the $O(2n)$-orbit space of $\pi D(1, \ldots, 1)$ under conjugation. If we think of $D(1, \ldots, 1)$ as giving a complex structure on $\mathbb{R}^{2n}$ (its square is $-I$, after all), the stabilizer of $D(1, \ldots, 1)$ is the subgroup of orthogonal transformations that commute with this complex structure—and this is precisely $U(n)$. So our space of minimal geodesics is $O(2n)/U(n)$.

The final task is to analyze the indices of the non-minimal geodesics. A calculation shows that for the geodesics corresponding to $\pi \cdot D(k_1, \ldots, k_n)$, with the k_i odd positive integers, the index is

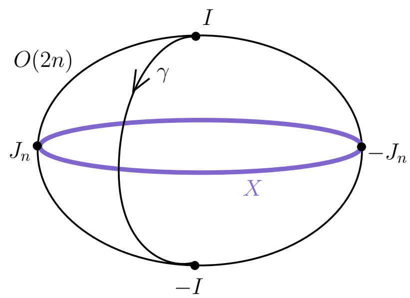
$$(n-1)\left[\sum_i k_i - n\right] + \sum_{k_i > k_j} (k_i - k_j - 2)$$

(see [Bott] or [Mi1] for this). The minimal case $(1, \ldots, 1)$ gives index 0 (as expected), and the next largest case will be $(3, 1, \ldots, 1)$ which gives index $2n-2$. So all non-minimal indices have index at least $2n-2$, and therefore the Bott-Morse theory gives that

$$\Omega O(2n) \simeq O(2n)/U(n) \cup (\text{cells of dimension} \geq 2n-2).$$

◦ Exercises ◦

Now we will use the same techniques to analyze $\Omega[O(2n)/U(n)]$. The first crucial observation is that $O(2n)/U(n)$—our space of minimal geodesics from the above argument—can be identified with a certain subspace X of $O(2n)$. Here is a picture to keep in mind:



The geodesics γ on $O(2n)$ starting at I are of the form $t \mapsto e^{tA}$, and therefore $\gamma(1) = \gamma(\frac{1}{2})^2$. So if the geodesic goes from I to $-I$, then its midpoint $\gamma(\frac{1}{2})$ squares to $-I$—that is, it gives a complex structure on $\mathbb{R}^{2n}$. This gives us our candidate for X:

$$X = CS_{2n} = \{J \in O(2n) \mid J^2 = -I\}.$$

This is the space of orthogonal complex structure on $\mathbb{R}^{2n}$, and it will play the role of the "equator" in the above picture. Let $\mathbb{J}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\mathbb{J}_n = \mathbb{J}_1^{\oplus(n)}$ denote the standard complex structures. The following diagram of homeomorphisms will be important for us:

$$\begin{tikzcd}
\{2n \times 2n \text{ skew-symm. } A \text{ with eigenvalues } \pm\pi i\} \arrow[r, "A \mapsto e^{\frac{1}{2}A} = \frac{1}{\pi}A"] \arrow[d, "A \mapsto (t \mapsto e^{tA})"'] & CS_{2n} \\
\{\text{minimal geodesics } I \to -I \text{ on } O(2n)\} \arrow[ur, "\gamma \mapsto \gamma(\frac{1}{2})"'] &
\end{tikzcd}$$

The vertical map is a homeomorphism because geodesics are determined by their tangent vector at I. For the horizontal map, note that if A is skew-symmetric with eigenvalues $\pm\pi i$ then there is a $P \in O(2n)$ such that $PAP^{-1} = \pi\mathbb{J}_n$. Then

$$e^{\frac{1}{2}A} = e^{\frac{\pi}{2}P^{-1}\mathbb{J}_n P} = P^{-1}e^{\frac{\pi}{2}\mathbb{J}_n}P = P^{-1}\mathbb{J}_n P = \tfrac{1}{\pi}A.$$

The map $A \mapsto \frac{1}{\pi}A$ is clearly a homeomorphism. The diagonal map is therefore also a homeomorphism, by commutativity of the diagram.

All of the maps in the diagram are compatible with conjugation by $O(2n)$. As we have remarked before, the space in the upper left corner is a single $O(2n)$-orbit: every such matrix A is conjugate to $\pi\mathbb{J}$. So in CS_{2n} every element is conjugate to $\mathbb{J}$. The stabilizer of $\mathbb{J}$ is—by definition—the unitary group $U(n) \subseteq O(2n)$. So we have $CS_{2n} \cong O(2n)/U(n)$.

We next want to compute the space of minimal geodesics in CS_{2n} from $\mathbb{J}$ to $-\mathbb{J}$. However, we first need to deal with the fact that CS_{2n} has two components (corresponding to the two components in $O(2n)$). The path components are represented by $\mathbb{J}$ and $-\mathbb{J}_1 \oplus \mathbb{J}_1^{\oplus(n-1)}$, and changing the sign on two of the $\mathbb{J}_1$ blocks doesn't change the path component. So $\mathbb{J}$ and $-\mathbb{J}$ are in the same path component precisely when n is even. From now on we make this extra assumption.

Exercise 41.11. Find a reflection in $O(2)$ that conjugates $\mathbb{J}_1$ to $-\mathbb{J}_1$, and use this to verify the assertions of the previous paragraph.

Observe that $T_{\mathbb{J}}O(2n) = \mathbb{J} \cdot T_I O(2n) = \{\mathbb{J}A \,|\, A \text{ skew-symmetric}\}$. Given a tangent vector $\mathbb{J}A \in T_{\mathbb{J}}O(2n)$, the corresponding geodesic is $t \mapsto \mathbb{J}e^{tA}$. For what values of A does this geodesic lie in CS_{2n}? We write

$$-I = \mathbb{J}e^{tA}\mathbb{J}e^{tA} = \mathbb{J}^2 + t(\mathbb{J}A\mathbb{J} + \mathbb{J}^2 A) + (\text{higher order terms in } t)$$

and we obtain the condition $\mathbb{J}A\mathbb{J} + \mathbb{J}^2 A = 0$, or $\mathbb{J}A = -A\mathbb{J}$. Conversely, if we have this anti-commuting condition then we can argue that

$$\mathbb{J}e^{tA} \cdot \mathbb{J}e^{tA} = -\mathbb{J}e^{tA}\mathbb{J}^{-1}e^{tA} = -e^{t\mathbb{J}A\mathbb{J}^{-1}}e^{tA} = -e^{-tA}e^{tA} = -I.$$

This computation shows that

$$T_{\mathbb{J}}CS_{2n} = \{\mathbb{J}A \,|\, A \text{ skew-symmetric and } \mathbb{J}A = -A\mathbb{J}\}.$$

From this we find that the geodesics in CS_{2n} from $\mathbb{J}$ to $-\mathbb{J}$ all come by applying left-multiplation-by $\mathbb{J}$ to *certain* geodesics in $O(2n)$ from I to $-I$—namely, ones of the form $t \mapsto e^{tA}$ where A anti-commutes with $\mathbb{J}$. The *minimal* such geodesics will again be where A has eigenvalues $\pm\pi i$, since that is the same condition for minimality in $O(2n)$.

We now get the following diagram of homeomorphisms:

$$\begin{tikzcd}
\left\{\begin{matrix} 2n \times 2n \text{ skew-sym } A \text{ with } \mathbb{J}A = -A\mathbb{J} \\ \text{and eigenvalues } \pm\pi i \end{matrix}\right\} \arrow[r, "A \mapsto \mathbb{J}e^{\frac{1}{2}A} = \frac{1}{\pi}\mathbb{J}A"] \arrow[d, "A \mapsto (t \mapsto \mathbb{J}e^{tA})"', "\cong"] & \left\{\begin{matrix} L \in O(2n),\ L^2 = -I, \\ \mathbb{J}L = -L\mathbb{J} \end{matrix}\right\} \\
\{\text{minimal geodesics } \mathbb{J} \to -\mathbb{J} \text{ in } CS_{2n}\} \arrow[ur, "\gamma \mapsto \gamma(\frac{1}{2})"'] &
\end{tikzcd}$$

We have already seen that the vertical map is a homeomorphism, and the horizontal map is clearly injective. So we only need to check that the horizontal map is surjective. But given an L in the target, we set $A = -\pi\mathbb{J}L$ and readily check that A has the required properties to be a preimage.

The space in the upper right corner of the above diagram can be described as the set of orthogonal quaternionic structures on $\mathbb{R}^{2n}$ (remember that n is even) that extend the fixed complex structure $\mathbb{J}$. Recalling that $U(n) \subseteq O(2n)$ is the subgroup of automorphisms that commute with $\mathbb{J}$, conjugation by $U(n)$ acts on all the spaces in the above diagram. By Lemma 41.12 below, the $U(n)$-action is transitive. For a fixed L, the stabilizer of L is precisely $Sp(\frac{n}{2})$. So the orbit of L is $U(n)/Sp(\frac{n}{2})$, and this is our space of minimal geodesics.

One can now proceed, as Bott did, to use the same techniques to analyze $\Omega[U(2n)/Sp(n)]$, and so on. At each stage something very similar happens, though the patterns are not perhaps totally manifest at first. This is where Clifford algebras enter the story, as they give a nice way of organizing what is happening in this process. We will pick this thread up in the next section.

The following result from linear algebra is a leftover piece from our above analysis:

Lemma 41.12. *Consider $\mathbb{C}^n$ with its standard Hermitian product $\langle -,-\rangle$. For $x,y \in \mathbb{C}^n$ let $x \cdot y = \mathrm{Re}(\langle x,y\rangle)$ be the associated real inner product. Suppose $L\colon \mathbb{C}^n \to \mathbb{C}^n$ is conjugate-linear (i.e. $L(ix) = -iL(x)$), orthogonal (preserves the dot product), and $L^2 = -I$. Then n is even and there exists a unitary map $P\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $P^{-1}LP$ equals the map $j\colon \mathbb{C}^n \to \mathbb{C}^n$ where* $j(e_k) = \begin{cases} e_{k+1} & \textit{if } k \textit{ is odd,} \\ -e_{k-1} & \textit{if } k \textit{ is even.} \end{cases}$

Proof. First note that the Hermitian inner product is recovered from the dot product by the formula $\langle x,y\rangle = (x\cdot y) + i(x \cdot iy)$. This follows at once from the formula $\langle x, iy\rangle = -i\langle x,y\rangle$. Consequently, we obtain

$$\begin{aligned}\langle Lx, Ly\rangle = (Lx \cdot Ly) + i(Lx \cdot iL(y)) &= (x\cdot y) + i(Lx \cdot -L(iy)) \\ &= (x \cdot y) - i(x \cdot iy) \\ &= \overline{\langle x,y\rangle}.\end{aligned}$$

Equivalently, $\langle x,y\rangle = \overline{\langle Lx, Ly\rangle}$. As a corollary we find that $\langle x, Lx\rangle = 0$ for all x, since

$$\langle x, Lx\rangle = \overline{\langle Lx, L^2x\rangle} = -\overline{\langle Lx, x\rangle} = -\langle x, Lx\rangle.$$

Let a_1 denote any unit vector in $\mathbb{C}^n$, and let $b_1 = L(a_1)$. Note that b_1 is also a unit vector, since L is orthogonal. Moreover, $\langle a_1, b_1\rangle = \langle a_1, La_1\rangle = 0$. Consequently, the vectors a_1, b_1 are $\mathbb{C}$-linearly independent.

Let $H = \mathrm{Span}\{a_1, b_1\}^\perp$, the orthogonal complement with respect to the Hermitian inner product. Then L restricts to a map $H \to H$, and we proceed by induction. In this way we establish that n is even and we produce a unitary basis $a_1, b_1, \ldots, a_r, b_r$ ($n = 2r$) where $L(a_k) = b_k$. Let $P\colon \mathbb{C}^n \to \mathbb{C}^n$ be the transformation given by $e_{2k-1} \mapsto a_k$, $e_{2k} \mapsto b_k$. Then $P^{-1}LP$ has the desired form, and since $\underline{a}, \underline{b}$ is a unitary basis the transformation P is unitary. ∎

◦ Exercises ◦

Exercise 41.13. Knowing the groups $\pi_*(U/O)$ and $\pi_*(O/U)$, analyze the long exact sequences for the fibrations $O \to U \to U/O$ and $U \to O \to O/U$ to determine what all of the maps are. Verify that

$$\pi_i(O) \to \pi_i(U) \text{ is } \begin{cases} \text{multiplication by } \pm 2 & \text{when } i \equiv 3 \bmod 8, \\ \text{multiplication by } \pm 1 & \text{when } i \equiv 7 \bmod 8, \\ \text{zero} & \text{otherwise.} \end{cases}$$

Likewise,

$$\pi_i(U) \to \pi_i(O) \text{ is } \begin{cases} \text{the projection } \mathbb{Z} \to \mathbb{Z}/2 & \text{when } i \equiv 1 \bmod 8, \\ \text{multiplication by } \pm 1 & \text{when } i \equiv 3 \bmod 8, \\ \text{multiplication by } \pm 2 & \text{when } i \equiv 7 \bmod 8, \\ \text{zero} & \text{otherwise.} \end{cases}$$

Note that these give us the maps $c\colon KO^*(pt) \to K^*(pt)$ and $r\colon K^*(pt) \to KO^*(pt)$ when $* \leq 0$.

Exercise 41.14. In our investigation of Bott periodicity we had to use the two homotopy-equivalent spaces $\Omega_{I,-I}SU(2n)$ and $\Omega_I SU(2n)$. Working more generally, let G be a topological group with identity e and let $x \in G$.

(a) For any path f from x to e, check that $\Omega_{e,x}G \to \Omega_e G$ given by post-concatenation with f is continuous.

(b) Let $\gamma\colon I \to G$ be a path from e to x, and let γ^{-1} denote the path $t \mapsto \gamma(t)^{-1}$ from e to x^{-1}. For $\lambda \in \Omega_{e,x}G$, let $\lambda \cdot \gamma^{-1}$ be the path $t \mapsto \lambda(t)\gamma^{-1}(t)$. Check that this defines a continuous map $\Omega_{e,x}G \to \Omega_e G$, and further that this map is homotopic to post-concatenation with the path $x\gamma^{-1}$. [Hint: In $I^2 \subseteq \mathbb{R}^2$ consider the diagonal path as well as the path with the same endpoints that moves counterclockwise around the boundary. Since I^2 is convex, these paths are homotopic relative to the endpoints via the straight-line homotopy. Use this fact to avoid having to write down any horrible formulas.]

42. Bott periodicity and Clifford algebras

Recall from Section 15.11 the real Clifford algebras $\mathrm{Cl}_r = \mathbb{R}\langle e_1, \ldots, e_n\rangle/(e_i^2 = -1, e_ie_j = -e_je_i)$. We previously denoted these Cl_r^+, but here we will drop the $+$ since we will not need to use the negative Clifford algebras. The Clifford algebras exhibit the pattern $\mathrm{Cl}_{r+8} \cong \mathrm{Cl}_r(16)$, where recall that $T(n)$ is shorthand for the matrix algebra $M_{n\times n}(T)$. The table below lists Cl_r for $0 \leq r \leq 8$. By inspection the module theory of these algebras is very easy to understand, in each case the module category being semi-simple. In the table we also list the irreducible modules for each Cl_r. Finally, in the last column we list the groups from the Bott periodicity clock, as discussed in the previous section. We have not yet explained the connection of these groups to the Clifford algebra story, but the table certainly demonstrates that *something* interesting is going on. Our task in this section will be to explain what it is.

Table 42.1. Clifford algebras and their modules

r	Cl_r	Irreducible modules	Bott group
0	$\mathbb{R}$	$\mathbb{R}$	O
1	$\mathbb{C}$	$\mathbb{C}$	U
2	$\mathbb{H}$	$\mathbb{H}$	Sp
3	$\mathbb{H}\times\mathbb{H}$	$\mathbb{H}_+$, $\mathbb{H}_-$	$Sp \times Sp$
4	$\mathbb{H}(2)$	$\mathbb{H}^2$	Sp
5	$\mathbb{C}(4)$	$\mathbb{C}^4$	U
6	$\mathbb{R}(8)$	$\mathbb{R}^8$	O
7	$\mathbb{R}(8)\times\mathbb{R}(8)$	$\mathbb{R}^8_+$, $\mathbb{R}^8_-$	$O \times O$
8	$\mathbb{R}(16)$	$\mathbb{R}^{16}$	O

Let M be a Cl_r-module. It will be useful to assume that all of our modules are equipped with a real inner product, denoted $\langle -, -\rangle$, with respect to which all of the e_i-multiplications are orthogonal. Any module can be equipped with such a structure: choose any inner product on the underlying vector space and then average it with respect to multiplication by the 2^r monomials in the e_i's.

Remark 42.2. For the bulk of our work from now on, all Clifford modules will be assumed to be "orthogonal" in the above sense—even when not said explicitly. The restriction to considering these orthogonal modules is not strictly necessary, but it has the effect of producing spaces that relate to the compact spaces $O(n)$ rather than the non-compact spaces $GL_n(\mathbb{R})$. See Remark 42.15 for another perspective.

Write $O(M)$ for the set of all orthogonal transformations of the underlying inner product space of M. For each $k \leq r+1$, define

$$E_k(M) = \{J \in O(M) \,\big|\, e_1, \ldots, e_{k-1}, J \text{ gives a } \mathrm{Cl}_k\text{-module structure on } M\}$$

Equip this set with the subspace topology from $O(M)$. This is the space of Cl_k-extension structures on the underlying Cl_{k-1}-structure of M, the "E" being for "Extension".

Note that $E_1(M)$ is the space of orthogonal complex structures on M, and $E_2(M)$ is the space of (orthogonal) quaternionic structures where the given e_1 action is multiplication by i. We saw both of these spaces coming up in our arguments from the last section. It is natural to set $E_0(M) = O(M)$. Note that we have the sequence of subsets

$$E_0(M) \supseteq E_1(M) \supseteq E_2(M) \supseteq \cdots$$

since if $e_1, \ldots, e_{k-1}, J$ satisfy the Clifford relations then so do $e_1, \ldots, e_{k-2}, J$.

The action of e_k on M gives a natural basepoint in $E_k(M)$, for each $k \leq r$. We also have the antipodal element $-e_k \in E_k(M)$. If the Cl_k-structure extends to a Cl_{k+1}-structure (e.g., if $k < r$) then e_k and $-e_k$ are in the same path component of $E_k(M)$: to see this, check that $t \mapsto \cos(\pi t)e_k + \sin(\pi t)e_{k+1}$ describes a path in $E_k(M)$. We will attempt to use Bott-Morse theory to approximate $\Omega_{e_k,-e_k}E_k(M)$ by the space of minimal geodesics.

The tangent space $T_I O(M)$ may be identified with the set of linear maps $A\colon M \to M$ that are skew-symmetric in the sense that $\langle Ax, y\rangle = -\langle x, Ay\rangle$ for all $x, y \in M$. Then a little legwork shows that

$$\begin{aligned} T_{e_k}E_k(M) = \{e_k \cdot A \,\big|\, & A \text{ is skew-symmetric, } Ae_k = -e_k A, \\ & \text{and } A \text{ commutes with } e_1, \ldots, e_{k-1}\}. \end{aligned}$$

One way to at least check the plausibility of this claim is to take a geodesic $t \mapsto e^{tA}$ in $O(M)$, left-multiply by e_k, and then see what conditions on A will result in the geodesic $t \mapsto e_k e^{tA}$ lying in $E_k(M)$. The above conditions fall out immediately.

Given such a tangent vector $e_k A$, the corresponding geodesic $t \mapsto e_k e^{tA}$ arrives at $-e_k$ when $t = 1$ precisely when the eigenvalues of A are odd multiples of πi. The *minimal* geodesics from e_k to $-e_k$ correspond to those A where the eigenvalues are $\pm\pi i$. We obtain the following diagram of homeomorphisms:

$$\begin{tikzcd} \left\{\begin{array}{l}\text{skew-sym } A \text{ with } e_kA = -Ae_k,\ A \\ \text{commutes with } e_1, \ldots, e_{k-1}, \text{ and has} \\ \text{eigenvalues } \pm\pi i\end{array}\right\} \arrow[r, "A \mapsto e_k e^{\frac{1}{2}A} = \frac{1}{\pi}e_k A"] \arrow[d, "A \mapsto (t \mapsto e_k e^{tA})"'] & E_{k+1}(M) \\ \{\text{minimal geodesics } e_k \to -e_k \text{ in } E_k(M)\} \arrow[ur, "\gamma \mapsto \gamma(\frac{1}{2})"'] & \end{tikzcd}$$

Exercise 42.3. Verify that for A as in the upper left corner one has $e^{\frac{1}{2}A} = \frac{1}{\pi}A$ (use diagonalization). Then check that the top horizontal map is a homeomorphism, so that the diagonal map is as well.

Let $\beta\colon E_{k+1}(M)\to\Omega_{e_k,-e_k}E_k(M)$ be the inverse to the diagonal map in the above diagram. In retrospect, β is easy to describe. It sends an element $\tilde{e}_{k+1}$ to the path $t\mapsto e_k e^{-\pi t e_k\tilde{e}_{k+1}}$, which is equal to the path

$$t\mapsto\cos(\pi t)e_k+\sin(\pi t)\tilde{e}_{k+1}$$

that we have seen before. The Bott-Morse arguments show that β is an n_k-equivalence for a certain integer n_k that goes to infinity as $\dim M\to\infty$. So we next stabilize.

Observe that if M' is another Cl_r-module then we have natural maps $E_k(M)\to E_k(M\oplus M')$ sending $\tilde{e}_k$ to $\tilde{e}_k\oplus e_k$. That is, we use the "new" e_k-structure on the M summand and the "old" e_k-structure (the one from the Cl_r-stucture) on the M' summand.

Let $\mathbb{M}=\{M_i\}$ be a sequence of representations of Cl_r, and let $E_k(\mathbb{M})$ be the colimit of the directed system

$$E_k(M_1)\to E_k(M_1\oplus M_2)\to E_k(M_1\oplus M_2\oplus M_3)\to\cdots\qquad\Rightarrow E_k(\mathbb{M}).$$

If we also write $\mathbb{M}$ for the infinite sum $\bigoplus_{i=1}^{\infty}M_i$, then $E_k(\mathbb{M})$ can be alternatively described as

$$E_k(\mathbb{M})=\{\tilde{e}_k\in O(\mathbb{M})\,\big|\,e_1,\ldots,e_{k-1},\tilde{e}_k\text{ is a }\mathrm{Cl}_k\text{-structure on }M\text{ and }\tilde{e}_k\text{ agrees with }e_k\text{ on a subspace of finite codimension}\}.$$

Here $O(\mathbb{M})$ refers to the group $\operatorname{colim}_k O(\bigoplus_{i=1}^{k}M_i)$.

Stabilizing the Bott maps here is a little tricky, because the map $j\colon E_k(M)\to E_k(M\oplus M')$ does not have the property that $j(-e_k)=-j(e_k)$. So it does not send paths $e_k\to -e_k$ to paths $j(e_k)\to -j(e_k)$. Some of the tricks we used in the previous section—in the case of U and O—to change from paths to loops don't readily solve the problem here, essentially because $E_k(M)$ doesn't have a multiplication on it. The following seems to be the best we can do.

Let $k\le r-1$ and consider the system

$$\begin{tikzcd}
E_{k+1}(M_1) \arrow[r]\arrow[d,"\beta"'] & E_{k+1}(M_1\oplus M_2) \arrow[r]\arrow[d,"\beta"'] & E_{k+1}(M_1\oplus M_2\oplus M_3) \arrow[r]\arrow[d,"\beta"'] & \cdots \\
\Omega_{e_k}(E_k(M_1)) \arrow[r] & \Omega_{e_k}(E_k(M_1\oplus M_2)) \arrow[r] & \Omega_{e_k}(E_k(M_1\oplus M_2\oplus M_3)) \arrow[r] & \cdots
\end{tikzcd}$$

The horizontal maps are all induced by the inclusions, but we need to explain the Bott maps β. In each setting, let γ denote the map $t\mapsto\cos(\pi t)e_k+\sin(\pi t)e_{k+1}$, and regard this as the canonical path from e_k to $-e_k$ in $E_k(M)$ (whatever M is). Define the Bott map β to send $\tilde{e}_{k+1}$ to the concantenation of paths

$$\beta(\tilde{e}_{k+1})=\gamma^{-1}*(t\mapsto\cos(\pi t)e_k+\sin(\pi t)\tilde{e}_{k+1})$$

($f*g$ means the path that travels g first and then follows with f). Note that this path moves from e_k to $-e_k$ via $\tilde{e}_{k+1}$, and then back to e_k via e_{k+1} (this is the γ^{-1} part).

The squares in the above diagram do not commute. Given an $\tilde{e}_{k+1}$, the two loops one gets by going around a square in the two different ways agree on the first part of the module, but disagree on the summand that is being added on. On this last summand one loop is constant, whereas the other loop follows the canonical path and then its inverse. So the squares commute up to homotopy, and in fact up to a very specific homotopy (the homotopy showing that a path followed by its inverse is null). It follows that we get a map from the telescope (or homotopy colimit) of the top sequence to the colimit of the bottom sequence. Since the telescope collapses down to the honest colimit, let us display all of this via the diagram

$$E_{k+1}(\mathbb{M}) \xleftarrow[\simeq]{\pi} \mathrm{Tel}_n\, E_{k+1}(\mathbb{M}_n) \xrightarrow{\beta_\infty} \mathrm{colim}_n\, \Omega_{e_k} E_k(\mathbb{M}_n) \xrightarrow{\simeq} \Omega_{e_k} E_k(\mathbb{M}).$$

Since the Bott maps $E_{k+1}(\mathbb{M}_n) \to \Omega_{e_k} E_k(\mathbb{M}_n)$ are equivalences in a range that goes to infinity with n, the map β_∞ is a weak equivalence. So we have shown that in the sequence of spaces

$$E_r(\mathbb{M}),\ E_{r-1}(\mathbb{M}),\ \ldots,\ E_0(\mathbb{M})$$

each space is equivalent to loops on the next one.

To eliminate the need to stop at E_r we do the following. Start with $\mathbb{M} = \mathbb{R}^\infty$, then group the summands by twos and give them the usual complex structure. Then group the $\mathbb{C}$-summands by twos and give the groupings the usual quaternionic structure. Take the first $\mathbb{H}$ and give it the Cl_3-structure of $\mathbb{H}_+$, the second $\mathbb{H}$ and give it the Cl_3-structure of $\mathbb{H}_-$, alternating back and forth forever. Continue inductively in the evident manner, so that in the end $\mathbb{R}^\infty$ is equipped with a Cl_∞-structure. Then we have the spaces $E_k(\mathbb{M})$ for all k, and our previous arguments give specific equivalences between loops on one and the next:

$$\ldots,\ E_2(\mathbb{M}),\ E_1(\mathbb{M}),\ E_0(\mathbb{M}) \qquad\qquad E_{k+1}(\mathbb{M}) \xrightarrow{\simeq} \Omega E_k(\mathbb{M}).$$

In this way we have obtained 'algebraic' models for all of the iterated loop spaces of $E_0(\mathbb{M}) \simeq O$.

Note that one can do these same constructions using the complex Clifford modules, thereby obtaining a sequence of algebraic models for the iterated loop spaces of U.

It remains to identify the homotopy types of the spaces $E_k(M)$ with something familiar (and similarly for the direct limit $E_k(\mathbb{M})$, of course). To this end, define the group of Clifford automorphisms

$$CA_k(M) = \{f \in O(M) \,\big|\, f \text{ is a } \mathrm{Cl}_k\text{-linear automorphism}\}.$$

Note that

$$O(M) = CA_0(M) \supseteq CA_1(M) \supseteq CA_2(M) \supseteq \cdots$$

as well as the evident identifications $CA_1(M) = U(M)$ and $CA_2(M) = Sp(M)$. The group $CA_{k-1}(M)$ acts on $E_k(M)$ by conjugation, and the stabilizer of the point e_k is $CA_k(M)$. So we get the injective map

$$CA_{k-1}(M)/CA_k(M) \longrightarrow E_k(M), \qquad [f] \mapsto f e_k f^{-1}. \tag{42.4}$$

We have seen these maps before in the cases $k = 1$ and $k = 2$, where they were actually homeomorphisms. But they are not always homeomorphisms, as the following example shows.

Example 42.5. Consider the Cl_3-module $M = \mathbb{H}$ where the action of e_1 and e_2 is that of i and j, respectively, and e_3 acts as e_1e_2. We will examine

$$CA_2(M)/CA_3(M) \hookrightarrow E_3(M).$$

An easy exercise shows that there are exactly two e_3-structures extending the Cl_2-action, namely $e_3 = \pm e_1e_2$. [The action of $e_1e_2e_3$ squares to 1 and is $\mathbb{H}$-linear, so it is either id or $-\mathrm{id}$]. So $E_3(M)$ consists of exactly two points.

The group $CA_2(M)$ is just $Sp(1)$, the group of unit quaternions, and $CA_3(M)$ is the same group since if a map respects the e_1- and e_2-actions on M then it also must respect e_1e_2. So $CA_2(M)/CA_3(M)$ is a single point. The image in $E_3(M)$ consists of the e_3-structure $e_3 = e_1e_2$.

For a similar example where the spaces are bigger, let $M = \mathbb{H}_+ \oplus \mathbb{H}_-$ where e_3 acts on the first factor as e_1e_2 and on the second factor as $-e_1e_2$. In this case it turns out that

$$E_3(M) = * \amalg S^4 \amalg *.$$

The first point is the structure $e_3 = e_1e_2$, and the last point is the structure $e_3 = -e_1e_2$. The S^4 is really $\mathbb{H}P^1$, and coincides with e_3-structures that are equal to e_1e_2 on a 1-dimensional $\mathbb{H}$-subspace V and equal to $-e_1e_2$ on $V^\perp$.

The group $CA_2(M)$ is $Sp(2)$, and the subgroup $CA_3(M)$ is $Sp(1) \times Sp(1)$: for the latter, an $\mathbb{H}$-linear map $f\colon M \to M$ commutes with the e_3-structure if and only if it preserves the subspaces $\mathbb{H}_+$ and $\mathbb{H}_-$. The map $CA_2(M)/CA_3(M) \to E_3(M)$ is a homeomorphism onto the second component, but of course is not surjective.

The above example reveals the key issue. Sometimes the Cl_{k-1}-module M admits different e_k-structures that lead to non-isomorphic Cl_k-modules, and when this happens the space $E_k(M)$ has multiple components. In fact, we know the group $CA_{k-1}(M)$ acts on $E_k(M)$ and therefore $E_k(M)$ will be a disjoint union of orbits, each of which is a homogeneous space for $CA_{k-1}(M)$. Two e_k-structures (call them e_k and e_k') are in the same orbit if and only if there is a Cl_k-isomorphism between M and M'. Thus, we obtain the following:

Proposition 42.6. *Let $\mathcal{I}(M)$ be the set of isomorphism classes of Cl_k-module structures on M extending the given Cl_{k-1}-structure. For each element of $\mathcal{I}(M)$ choose a specific e_k-structure representing it, and let $CA_k(M; e_k) \subseteq CA_{k-1}(M)$ be the subgroup of automorphisms that preserve this structure. Then there is a homeomorphism*

$$\coprod_{\mathcal{I}(M)} CA_{k-1}(M)/CA_k(M; e_k) \xrightarrow{\cong} E_k(M)$$

sending the coset $f \cdot CA_k(M; e_k)$ to fe_kf^{-1}, for all $f \in CA_{k-1}(M)$.

Exercise 42.7. Prove the above proposition.

We will eventually see that the spaces $E_k(M)$ have exactly one component except in the cases where $k \equiv 3 \bmod 4$ (Proposition 42.13), but in order to prove this we need to understand a bit more about Clifford modules.

42.8. **Understanding Clifford module structures.** Let M be a real vector space. Giving a Cl_2-structure on M is the same as giving M the structure of an $\mathbb{H}$-vector space, so we understand that fairly concretely. What does one need to specify in order to extend this to a Cl_3-structure? Rather than consider e_3, it is helpful to look instead at the element $y = e_1e_2e_3$. Note that giving the Cl_3-structure is the same as specifying left-multiplication by y. But $y^2 = 1$, so the eigenvalues of (multiplication by) y are ± 1; so M will decompose as the orthogonal sum of the two eigenspaces V_+ and V_-. Since y commutes with e_1 and e_2, each of V_+ and V_- is closed under the $\mathbb{H}$-action. One can readily reverse this reasoning and see that specifying V_+ is *equivalent* to specifying $y\cdot$ (note that $V_- = V_+^\perp$). So extending the Cl_2-structure to a Cl_3-structure is equivalent to giving an $\mathbb{H}$-subspace V of M.

If the Cl_3-structure extends to a Cl_4-structure then e_4—which must anticommute with $e_1e_2e_3$—must send V_+ to V_- and vice versa. So such an extension can only occur when $\dim V_+$ is exactly half of $\dim M$, and when this condition is satisfied giving the e_4-structure is equivalent to specifying an $\mathbb{H}$-linear isomorphism $V_+ \to V_-$.

One can repeat the above style of argument to inductively analyze what is required to extend a Cl_r-structure to a Cl_{r+1}-structure. There is some slight cleverness in finding an analog of y in each case, but essentially the first thing one tries always works. Table 42.8 below summarizes the results, and we leave it to the reader to work out the details as an exercise (but we give some hints). When some Clifford module structures do not extend, we indicate this—so if there is no notation, all extend. Also, we write $A^{\perp B}$ for the orthogonal complement of A inside of B.

TABLE 42.8. Giving Clifford module structures

r	Equivalent description of a Cl_r-structure on M
0	$\mathbb{R}$-module (extends only when dim is even)
1	$\mathbb{C}$-module (extends only when $\dim_{\mathbb{C}}$ is even)
2	$\mathbb{H}$-module
3	$\mathbb{H}$-module together with $\mathbb{H}$-submodule $V \subseteq M$ (extends only when $\dim V$ is half of $\dim M$) [V is the $+1$ eigenspace of $e_1e_2e_3$]
4	above together with an $\mathbb{H}$-linear isomorphism $V \to V^{\perp}$ (always extends) [the isomorphism is multiplication by e_4]
5	above together with a $\mathbb{C}$-submodule $W \subseteq V$ such that $e_2W = W^{\perp V}$ [W is the $+1$-eigenspace of $e_1e_4e_5$]
6	above together with a real subspace $U \subseteq W$ such that $e_1U = U^{\perp W}$ [U is the $+1$-eigenspace of $e_2e_4e_6$]
7	above together with a real subspace $T \subseteq U$ (extends only when $\dim T = \frac{1}{2}\dim U$) [$T$ is the $+1$-eigenspace of $e_1e_6e_7$]
8	above together with an $\mathbb{R}$-linear isomorphism $T \to T^{\perp U}$ (extends when $\dim T$ is even) [the isomorphism is multiplication by e_7e_8]
9	above together with a complex structure on T [given by e_8e_9] (extends when $\dim_{\mathbb{C}} T$ is even)
10	above together with a quaternionic structure on T extending the $\mathbb{C}$-structure [given by e_8e_{10}]

Remark 42.9. One can start to see the 8-fold quasi-periodicity in this chart. In row $8n$ we have a statement involving e_{8n}, and in row $8n+r$ for $0 < r \leq 7$ we take the statement in row r and change every word $e_{i_1}\cdots e_{i_s}$ to $(e_{8n}e_{8n+i_1})\cdots(e_{8n}e_{8n+i_s})$.

Exercise 42.10. Justify all of the lines in the above table. The following notes—also indexed by r to match the table—give short guides:

(3) $(e_1e_2e_3)^2 = 1$, so the eigenvalues of $e_1e_2e_3$ are ± 1 and M is the orthogonal sum of the eigenspaces. Since $e_1e_2e_3$ commutes with e_1 and e_2, these eigenspaces are closed under the $\mathbb{H}$-action.

(4) e_4 anticommutes with $e_1e_2e_3$, and so sends the $+1$-eigenspace V to the -1-eigenspace $V^{\perp}$. [In particular, an e_4-structure exists only when V and $V^{\perp}$ have the same dimension].

(5) $(e_1e_4e_5)^2 = 1$ and $e_1e_4e_5$ commutes with $e_1e_2e_3$.
(6) $e_2e_4e_6$ squares to 1 and commutes with $e_1e_4e_5$.
(7) $e_1e_6e_7$ squares to 1 and commutes with $e_2e_4e_6$.
(8) e_8 anticommutes with $e_1e_6e_7$.
Continue far enough that you believe the quasi-periodicity.

The diagram below gives another way of keeping track of what is happening here. Columns 0 through n show the deconstruction of a Cl_n-module M.

$$\begin{tikzcd}
M \arrow[r, dash] & M \arrow[r, dash] \arrow[loop above, "e_1"] & M \arrow[r, dash] \arrow[loop above, "e_1"] \arrow[loop below, "e_2"] \arrow[dr, dash] & V^{[e_1e_2e_3]} \arrow[rr, dash] \arrow[loop above, "e_1,e_2"] \arrow[d, bend left=60, "e_4"] \arrow[drr, dash] & & W^{[e_1e_4e_5]} \arrow[r, dash] \arrow[loop above, "e_1"] \arrow[d, "e_2"] \arrow[dr, dash] & U^{[e_2e_4e_6]} \arrow[r, dash] \arrow[d, "e_1"] \arrow[dr, dash] & T^{[e_1e_6e_7]} \arrow[rr, dash] \arrow[d, bend left=60, "e_7e_8"] & & T \arrow[r, dash] \arrow[loop above, "e_8e_9"] & T \arrow[loop above, "e_8e_9"] \arrow[loop below, "e_8e_{10}"] \\
& & & V^{\perp} \arrow[loop below, "e_1,e_2"] & & W^{\perp} \arrow[loop below, "e_1"] & U^{\perp} & T^{\perp} & & & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{tikzcd}$$

◦ Exercises ◦

Based on the above work, we can identify the spaces $E_k(M)$ for all k. To help with this, the following result will be useful:

Proposition 42.11. *One has the following homeomorphisms:*

$\{$*orthogonal complex structures on* $\mathbb{R}^{2n}\} \cong O(2n)/U(n)$

$\{$*rank n real subspaces H of $\mathbb{C}^n$ s.t. $iH = H^{\perp}$*$\} \cong U(n)/O(n)$

$\{$*orthogonal $\mathbb{H}$-structures on $\mathbb{C}^{2n}$ extending the complex structure*$\} \cong U(2n)/Sp(n)$

$\{$*rank n complex subspaces H of $\mathbb{H}^n$ s.t. $jH = H^{\perp}$*$\} \cong Sp(n)/U(n)$

Proof. The first and the third have similar proofs. For the third, $U(2n)$ acts on the space of quaternionic structures by conjugation. By Lemma 41.12 the action is transitive. The stabilizer of a fixed quaternionic structure is precisely $Sp(n)$, so this completes the identification.

For the second homeomorphism, observe that $U(n)$ acts on the left space. We will show the action is transitive and calculate the stabilizer of $\mathbb{R}^n$. Let V and W be two rank n real subspaces of $\mathbb{C}^n$ such that $iV = V^{\perp}$ amd $iW = W^{\perp}$. Note that $\mathbb{C}^n = V \oplus iV = W \oplus iW$. Choose orthonormal $\mathbb{R}$-bases for each of V and W, and note that these are necessarily unitary $\mathbb{C}$-bases for $\mathbb{C}^n$. Therefore the $\mathbb{C}$-linear transformation $L\colon \mathbb{C}^n \to \mathbb{C}^n$ sending the first basis to the second is unitary. This proves that $U(n)$ acts transitively on the given space. The stabilizer of the subspace $\mathbb{R}^n \subseteq \mathbb{C}^n$ is precisely $O(n)$. The proof for the fourth homeomorphism is identical. $\square$

Remark 42.12. For a Hermitian inner product space V, a real subspace H such that $iH = H^{\perp}$ is sometimes called a "real structure" on V. Likewise, if V is a quaternionic inner product space then a complex subspace $H \subseteq V$ such that

$jH = H^\perp$ is called a "complex structure" on V. Using this terminology, the results of Proposition 42.11 take on a nice symmetry.

Proposition 42.13. *Let M be a Cl_{k-1}-module, with $\dim_{\mathbb{R}} M = n$. Then one has the following identifications:*

k	$E_k(M)$	*conditions*	$CA_k(M)$
0	$O(n)$		$O(n)$
1	$O(n)/U(\frac{n}{2})$	n *is even*	$U(\frac{n}{2})$
2	$U(\frac{n}{2})/Sp(\frac{n}{4})$	n *is a multiple of* 4	$Sp(\frac{n}{4})$
3	$\mathrm{Gr}^{\mathbb{H}}_{\bullet}(\mathbb{H}^{\frac{n}{4}})$	n *is a multiple of* 4	$Sp(V) \times Sp(V^\perp)$
4	$Sp(\frac{n}{8})$	n *is a multiple of* 8	$Sp(V) = Sp(\frac{n}{8})$
5	$Sp(\frac{n}{8})/U(\frac{n}{8})$	n *is a multiple of* 8	$U(W) = U(\frac{n}{8})$
6	$U(\frac{n}{8})/O(\frac{n}{8})$	n *is a multiple of* 8	$O(U) = O(\frac{n}{8})$
7	$\mathrm{Gr}^{\mathbb{R}}_{\bullet}(\mathbb{R}^{\frac{n}{8}})$	n *is a multiple of* 8	$O(T) \times O(T^{\perp U})$
8	$O(\frac{n}{16})$	n *is a multiple of* 16	$O(T) = O(\frac{n}{16})$

Here $\mathrm{Gr}^F_{\bullet}(N)$ denotes the space of all F-subspaces of N, and so is a disjoint union with components indexed by the possible F-dimensions of such subspaces. The spaces V, W, U, and T are as in Table 42.8, and the space $E_k(M)$ is empty when the given conditions on n are not satisfied.

Corollary 42.14. *Let M be a Cl_k-module. The map $CA_{k-1}(M)/CA_k(M) \to E_k(M)$ of (42.4) is a homeomorphism when $k \not\equiv 3$ mod 4, and the inclusion of a connected component otherwise.*

Proof of Proposition 42.13. The identification of the $E_k(M)$ spaces follows immediately from the information in Table 42.8. The $k = 1, 2, 5, 6$ cases use Proposition 42.11, whereas $k = 3, 7$ are self-evident. For $k = 4$ one notes that the space of $\mathbb{H}$-linear isomorphisms $V \to V^\perp$ is (when nonempty) a torsor for the group $Sp(V)$ via precomposition. Similar for $k = 8$.

The identification of the groups in $CA_k(M)$ is also mostly immediate from Table 42.8, with the exceptions of $k = 4, 8$. For $k = 4$, the automorphisms of V and $V^\perp$ from $CA_3(M)$ must be compatible with the isomorphism $e_4\cdot\colon V \to V^\perp$, which is equivalent to saying that the automorphism of $V^\perp$ is determined by the one on V. So $CA_4(M) = Sp(V)$ and the map $CA_4(M) \to CA_3(M)$ is $f \mapsto (f, x \mapsto -e_4 f(e_4 x))$. The analysis for $k = 8$ is entirely similar. □

Taking the results of Proposition 42.13 and stabilizing, we can now complete the identification of the iterated loop spaces of O. In the following chart we start with the rightmost O and taking loops moves us to the left:

$\cdots$	$E_8\mathbb{M}$	$E_7\mathbb{M}$	$E_6\mathbb{M}$	$E_5\mathbb{M}$	$E_4\mathbb{M}$	$E_3\mathbb{M}$	$E_2\mathbb{M}$	$E_1\mathbb{M}$	$E_0\mathbb{M}$
$\cdots$	O	$\mathbb{Z}\times BO$	U/O	Sp/U	Sp	$\mathbb{Z}\times BSp$	U/Sp	O/U	O

The only slightly subtle point in the stabilization is what happens for $k = 3, 7$. But here one gets a bigraded lattice as in (41.4) and the analysis exactly follows what was done there. See Section 43.5 below, though, for a less ad hoc approach to stabilization.

Remark 42.15 ($*$-algebras and modules)**.** Throughout this section we always concentrated on orthogonal Clifford module structures. This was not absolutely necessary, but is convenient because it produces objects that connect to the compact groups O_n, U_n, and Sp_n rather than $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, and $GL_n(\mathbb{H})$. But the constant insistence on orthogonal actions can feel a little clunky.

Here is some different language that accomplishes the same thing. Let M be a finite-dimensional real inner product space, and let $\mathrm{End}_{\mathbb{R}}(M)$ be the algebra of $\mathbb{R}$-linear endomorphisms equipped with the involution $f \mapsto f^\dagger$. Recall that the adjoint $f^\dagger$ is the linear map uniquely characterized by $\langle f^\dagger x, y\rangle = \langle x, fy\rangle$, and in terms of matrices with respect to an orthonormal basis this just amounts to taking the transpose.

The pair $(\mathrm{End}_{\mathbb{R}}(M), \dagger)$ is a prototypical example of a $*$-algebra, which is an $\mathbb{R}$-algebra A equipped with a conjugation $*\colon A \to A$ that is an $\mathbb{R}$-linear anti-involution; the "anti" part is the identity $(xy)^* = y^*x^*$. As it turns out, the Clifford algebras come with natural structures of $*$-algebras where one sets $e_i^* = -e_i$ (note that this determines the $*$-operation on all other elements). We say that M is a $*$-module for Cl_k if the action map $\rho\colon \mathrm{Cl}_k \to \mathrm{End}_{\mathbb{R}}(M)$ is a map of $*$-algebras; in addition to being a module in the ordinary sense this amounts to the extra condition that $\rho(e_i^*) = \rho(e_i)^\dagger$ for all i. But because $e_i^* = -e_i = e_i^{-1}$, this is the condition that $\rho(e_i)^{-1} = \rho(e_i)^\dagger$—equivalently, $\rho(e_i)$ is orthogonal in the sense of preserving the inner product.

The upshot is that the "orthogonal Clifford modules" we have been considering can also be described as the $*$-modules for the Clifford $*$-algebras. There is nothing deep here—it is just a shift of language—but the theory of $*$-algebras and $*$-modules can feel a bit less ad hoc, and connects to the vast theory of C^*-algebras in analysis.

Exercise 42.16. For the $*$-operation on Cl_n defined in Remark 42.15, check that $(e_{i_1}\cdots e_{i_k})^* = (-1)^{\binom{k+1}{2}} e_{i_1}\cdots e_{i_k}$ when the i_j are all distinct.

43. The rich pageant of Clifford algebras

One thing that is missing from our story so far is a "natural" construction of the K-theory spectra K and KO. Clifford algebras have led us to models for the iterated loop spaces on U and O, but to write down a spectrum one needs iterated *de*loopings of $\mathbb{Z} \times BU$ and $\mathbb{Z} \times BO$. The periodicity theorems allow us to recast the iterated loops on O as iterated deloopings of $\mathbb{Z} \times BO$, but this approach feels a bit unsatisfying.

Now that the specter of Clifford algebras has been allowed in the door, though, it turns out there is much more one can do with them. Instead of the singly-indexed family of Clifford algebras we have studied so far, it is natural to look at a bigraded family. Karoubi showed how to use these to produce the required deloopings and the associated models of K-theory spectra. In this section we recount this theory.

The Clifford algebra construction is something that can be applied to any vector space equipped with a quadratic form, over any field. Given (V, q) we can define the Clifford algebra as a quotient of the tensor algebra $T(V)$:

$$\mathrm{Cl}(V, q) = T(V)/\langle v \otimes v = -q(v) \mid v \in V\rangle$$

Over the real numbers every quadratic form is isomorphic to one of the form

$$q(x_1,\ldots,x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)$$

for integers $p, q \geq 0$. The Clifford algebra associated to this form will be denoted $\mathrm{Cl}_{p,q}$. Note that $\mathrm{Cl}_{p,0}$ is the algebra that we have previously been calling Cl_p, whereas $\mathrm{Cl}_{0,q}$ is an algebra we saw in Section 15 and denoted Cl_q^- there.

One can readily check that $\mathrm{Cl}_{p,q}$ is isomorphic to the quotient of the tensor algebra $\mathbb{R}\langle e_1,\ldots,e_p,f_1,\ldots,f_q\rangle$ by the relations saying that $e_i^2=-1$, $f_j^2=1$, and any two distinct e- or f-variables anti-commute with each other. We will often find this model particularly convenient.

Computing the $\mathrm{Cl}_{p,q}$ algebras explicitly is just as easy as the computation of the $\mathrm{Cl}_k^{\pm}$ algebras that we did in Section 15. We can do it with only one additional fact:

Proposition 43.1. *For any $p, q \geq 0$ there is an isomorphism of $\mathbb{R}$-algebras $\mathrm{Cl}_{p+1,q+1} \to \mathrm{Cl}_{p,q}(2)$ given by $e_{p+1} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $f_{q+1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $e_i \mapsto \begin{bmatrix} 0 & e_i \\ e_i & 0 \end{bmatrix}$, $f_j \mapsto \begin{bmatrix} 0 & f_j \\ f_j & 0 \end{bmatrix}$ for $i \leq p$ and $j \leq q$.*

The proof is straightforward once one has the above formulas for the isomorphism, but before giving the proof let us explain where those formulas come from. Let N be a $\mathrm{Cl}_{p+1,q+1}$-module. The difference between $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{p+1,q+1}$ lies in the two generators e_{p+1} and f_{q+1}. Since $f_{q+1}^2 = 1$ the underlying vector space of N splits as $V_+ \oplus V_-$, where the summands are the $+1$ and -1 eigenspaces for f_{q+1}. The other e's and f's anticommute with f_{q+1} and so they map V_+ to V_- and vice versa. But we can use e_{p+1} to identify V_+ and V_-, and in that way we can think of the generators of $\mathrm{Cl}_{p,q}$ as operating on just one of these—V_+, say. More precisely, define $\tilde{e}_i = e_ie_{p+1}$ and $\tilde{f}_j = f_je_{p+1}$ and check that these elements define an action of $\mathrm{Cl}_{p,q}$ on V_+. Set $M = V_+$ with this action. Then $N = M \oplus e_{p+1}M$, and if we identify the underlying vector space with $M \oplus M$ in the evident way then f_{q+1} acts as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, e_{p+1} acts as the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, e_i acts as $\begin{bmatrix} 0 & \tilde{e}_i \\ \tilde{e}_i & 0 \end{bmatrix}$, and similarly for the f_j. Since it is easy to get confused about signs here, let us justify the nonzero entries in the matrix for e_i as an example: $e_im = e_{p+1}e_ie_{p+1}m = e_{p+1}(\tilde{e}_im)$ and $e_i(e_{p+1}m) = \tilde{e}_im$.

Proof of Proposition 43.1. It is routine to check that the given matrices satisfy the defining relations for the Clifford algebras, so that we do indeed get a map of $\mathbb{R}$-algebras. Taking products of the given matrices readily shows that we get all matrices of forms

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \quad \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}, \quad \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$$

where x is a product of e's and f's. Such matrices generate $\mathrm{Cl}_{p,q}(2)$ as a vector space, and so our map is surjective. Both the domain and codomain have dimension 2^{p+q+2} as vector spaces, and so the map is injective as well. $\square$

Morita theory tells us that the module categories of $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{p,q}(2) \cong \mathrm{Cl}_{p+1,q+1}$ are equivalent, though we basically proved this by hand in our above discussion. But let us record the result for future reference:

Proposition 43.2. *There is an equivalence of categories* $\mathrm{Cl}_{p,q}-\mathrm{Mod} \to \mathrm{Cl}_{p+1,q+1}-\mathrm{Mod}$ *that sends a module* M *to* $M \oplus M$ *with the action of the Clifford generators given by the formulas from Proposition 43.1.*

Recall that we already computed the algebras $\mathrm{Cl}_{p,0}$ and $\mathrm{Cl}_{0,q}$. Proposition 43.1 then lets us fill out the table below:

Clifford algebras $\mathrm{Cl}_{p,q}$

$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(32)\times\mathbb{H}(32)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\times\mathbb{R}(128)$	$\mathbb{R}(256)$
$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(16)\times\mathbb{H}(16)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\times\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
$\mathbb{H}(4)$	$\mathbb{H}(8)\times\mathbb{H}(8)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\times\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
$\mathbb{H}(4)\times\mathbb{H}(4)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\times\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\times\mathbb{H}(32)$
$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\times\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\times\mathbb{H}(16)$	$\mathbb{H}(32)$
$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\times\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\times\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
$\mathbb{R}(2)$	$\mathbb{R}(2)\times\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\times\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
$\mathbb{R}\times\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\times\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\times\mathbb{R}(16)$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\times\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\times\mathbb{R}(8)$	$\mathbb{R}(16)$
$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$

This table shows the algebras $\mathrm{Cl}_{p,q}$ where p is on the horizontal axis and q is on the vertical. Note that if we write $\mathrm{Cl}_{p,q} \cong A(n)$ where n is maximal possible, then the "root" A only depends on $p - q \bmod 8$. This is a combination of the $(1,1)$-periodicity from Proposition 43.1, the 8-fold periodicity of the $\mathrm{Cl}_{p,0}$ and $\mathrm{Cl}_{0,q}$ algebras, and a certain kind of 'duality' between the $\mathrm{Cl}_{p,0}$ and $\mathrm{Cl}_{0,q}$ families (they have the same roots but in reverse order).

Given an orthogonal $\mathrm{Cl}_{p,q}$-module M, let us now write $E^{(0,1)}_{p,q}(M)$ for the space of orthogonal $\mathrm{Cl}_{p,q+1}$-extension structures on M. That is, an element of this space is an $f_{q+1} \in O(M)$ such that $f_{q+1}^2 = \mathrm{id}_M$ and such that f_{q+1} anti-commutes with the action of the generators of $\mathrm{Cl}_{p,q}$ on M. We will likewise write $E^{(1,0)}_{p,q}(M)$ for the space of orthogonal $\mathrm{Cl}_{p+1,q}$-extension structures on M. The spaces $E^{(1,0)}_{p,0}(M)$ were studied in Section 42 and there called $E_{p+1}(M)$.

The following result is almost an immediate consequence of Proposition 43.2.

Proposition 43.3 ((1,1)-periodicity)**.** *For any* $\mathrm{Cl}_{p,q}$*-module* M *there are homeomorphisms* $E^{\epsilon}_{p,q}(M) \xrightarrow{\cong} E^{\epsilon}_{p+1,q+1}(M \oplus M)$*, where* ϵ *is either* $(1,0)$ *or* $(0,1)$ *and* $M \oplus M$ *is given the* $\mathrm{Cl}_{p+1,q+1}$*-structure described in Proposition 43.2.*

Proof. A linear map $M \oplus M \to M \oplus M$ is of the form $(x, y) \mapsto (Ax + By, Cx + Dy)$ for linear maps $A, B, C, D \colon M \to M$. The key observation is that such a map anti-commutes with the actions of e_{p+1} and f_{q+1} if and only if $A = D = 0$ and $B = C$. This is an easy computation. Based on this, we define the map

$$E^{(0,1)}_{p,q}(M) \xrightarrow{\cong} E^{(0,1)}_{p+1,q+1}(M \oplus M)$$

to send an f_{q+1}-structure f to $(x, y) \mapsto (fy, fx)$, which is readily checked to be an f_{q+2}-structure. The above key observation shows that any f_{q+2}-structure on $M \oplus M$ is of this form, and the inverse map is readily constructed.

The exact same proof works for $E^{(1,0)}$ in place of $E^{(0,1)}$. □

Remark 43.4. We have set up the definition of $E^{\epsilon}_{p,q}$ (where ϵ is $(1,0)$ or $(0,1)$) so that it involves *orthogonal* Clifford modules structures. Although this is convenient, it also leads to some issues. Dropping the orthogonality conditions yields spaces that are different but that turn out to have the same homotopy type, analogous to $GL_n(\mathbb{R})$ and $O(n)$.

One motivation for droppping the orthogonality condition is to generalize the setup as follows. Let $R \subseteq S$ be an extension of $\mathbb{R}$-algebras and let $\{s_i\}$ be a set of elements that generates S as an R-algebra. If M is an R-module one can define a space $E_{R\to S}(M)$ of S-module extensions of M. We regard this as a subset of $\prod_i \operatorname{Hom}_{\mathbb{R}}(M,M)$ by recording the action of each s_i, and we give $E_{R\to S}(M)$ the subspace topology induced by the product topology. (Note that in most applications the $\{s_i\}$ will be a finite collection and M will be finite-dimensional over $\mathbb{R}$, so this product is just a finite-dimensional Euclidean space).

One nice benefit of this generalization is that certain results become entirely algebraic. For example, Morita theory shows that the evident map

$$E_{R\to S}(M) \longrightarrow E_{R(n)\to S(n)}(M^{\oplus n})$$

is a homeomorphism. To see this, observe that we can construct a map in the other direction as follows. Given an $S(n)$-extension structure on $M^{\oplus n}$, give M the S-extension structure defined by

$$s.m = \pi_1\left(\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} m \\ 0 \end{bmatrix}\right)$$

(showing the case $n = 2$ for brevity). One of the compositions is then clearly the identity, and demonstrating that the other is the identity is a nice exercise in algebra.

For another result along these lines, consider the diagonal map $R \to R \times R$. A generator for the target is $f = (1,0)$, and in fact the target can be identified with $R[f]/(f^2-1)$. Giving an R-module M the structure of $R[f]/(f^2-1)$-algebra is the same as specifying an idempotent R-linear map $M \to M$. But by linear algebra any such map is diagonalizable with possible eigenvalues 1 and -1, so this is the same as giving two R-submodules M_+ and M_- of M such that $M = M_+ \oplus M_-$ (the eigenspace decomposition of multiplication by f). In this way we identify $E_{R\to R\times R}(M)$ with a subspace of $\operatorname{Gr}_\bullet(M)\times \operatorname{Gr}_\bullet(M)$, where $\operatorname{Gr}_\bullet(M) = \coprod_k \operatorname{Gr}_k(M)$. It is the space of R-module splittings of M.

As a final example, consider $R\times R$ regarded as the subring of diagonal matrices in the matrix algebra $M_{2\times 2}(R) = R(2)$. Let $f = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and write $\operatorname{diag}(a,b) = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$. Observe that $f^2 = 1$ and $f.\operatorname{diag}(a,b) = \operatorname{diag}(b,a).f$ for all $a,b \in R$. Moreover, f generates $R(2)$ as an $R\times R$-algebra (algebraically this is true as long as $\operatorname{char}(R) \neq 2$, but recall that for us R is an $\mathbb{R}$-algebra).

Every $R\times R$-module has the form $M \oplus N$ where M and N are R-modules and $(a,b).(m,n) = (am,bn)$; so $R\times R$ acts on M via the first coordinate and N via the second. We claim that extending this to an $R(2)$-structure is the same as specifying an R-linear isomorphism $\phi\colon M \to N$. Given such a map, define $f.(m,n) = (\phi^{-1}(n), \phi(m))$ and check that this defines an $R(2)$-structure. In the other direction, given an $R(2)$ structure one recovers ϕ by $\phi(m) = \pi_2(f.\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right])$ and it is routine to check that ϕ is an isomorphism.

We conclude that $E_{R\times R\to R(2)}(M\oplus N)$ is the space of R-linear isomorphisms from M to N.

While we will not develop this theory in complete detail, sometimes we will go back and forth between the "orthogonal" situation and the "algebraic" situation when convenient and without much comment.

43.5. **Stabilization.** Stabilization works as follows. We want to have maps $E^{\epsilon}_{p,q}(M)\to E^{\epsilon}_{p,q}(M\oplus N)$, where ϵ stands for either $(1,0)$ or $(0,1)$. In the latter case, for example, the map should take an f_{q+1}-structure on M to....? It is clear what to do on the M factor, but for N we need to be given an f_{q+1}-structure to start with. So let us do the following. Let $H_1,\dots,H_r$ be a complete list of irreducible $\mathrm{Cl}_{p,q+1}$-modules. We know from our identification of the Clifford algebras that r will always be 1 or 2, but that is not important. For each i we obtain maps

$$\theta_i\colon E^{(0,1)}_{p,q}(M)\to E^{(0,1)}_{p,q}(M\oplus H_i)$$

as we described above: $f_{q+1}\mapsto(f_{q+1},f^{H_i}_{q+1})$ where $f^{H_i}_{q+1}$ comes from the $\mathrm{Cl}_{p,q+1}$-structure on H_i. We can then form an infinite r-dimensional lattice where the nodes are indexed by the modules $H_1^{e_1}\oplus\cdots\oplus H_r^{e_r}$ and we have the maps $\theta_1,\dots,\theta_r$ leaving each node. Define the colimit of this diagram to be $E^{(0,1)}_{p,q}(\underline{H})$.

Remark 43.6. It was not important that the H_i themselves be irreducible, but rather that every irreducible representation be contained as a summand of some H_i. As long as we have this condition then our lattice will be cofinal in the one above, and so will have the same colimit. This observation will be important a bit later. Since under these hypotheses the colimit does not depend on the specific choice of the H_i's, we will also write $E^{(0,1)}_{p,q}(\infty)$ instead of $E^{(0,1)}_{p,q}(\underline{H})$.

If we also use $\underline{H}$ to denote the infinite $\mathrm{Cl}_{p,q+1}$-module $\bigoplus_i H_i^{\oplus\infty}$, then we can think of $E^{(0,1)}_{p,q}(\underline{H})$ as f_{q+1}-structures on $\underline{H}$ having the property that $f_{q+1}-f^{\underline{H}}_{q+1}$ has finite rank—or equivalently, the two structures f_{q+1} and $f^{\underline{H}}_{q+1}$ agree on a subspace of finite codimension. The topology is the subspace topology inherited from $O(\underline{H})$, which in turn is topologized as $\mathrm{colim}_s\, O(\bigoplus_{i=1}^r H_i^{\oplus s})$.

Observe that the same approach works to define the stabilized spaces $E^{(1,0)}_{p,q}(\infty)$ except that here the H's need to be irreducible representations of $\mathrm{Cl}_{p+1,q}$.

Example 43.7. Let us consider $E^{(0,1)}_{0,0}(\infty)$. Here $\mathrm{Cl}_{0,1}\cong\mathbb{R}\times\mathbb{R}$, and so we take $H_1=\mathbb{R}_+$ and $H_2=\mathbb{R}_-$ where the actions are $(a,b).r=ar$ and $(a,b).r=br$, respectively. Giving an f_1-structure on a vector space V is the same as giving a decomposition $V=V_+\oplus V_-$ where V_+ is the $+1$-eigenspace of f_1 and V_- is the -1-eigenspace. This explains the notation $\mathbb{R}_+$ and $\mathbb{R}_-$. Note that when V has an inner product and we are specifying an orthogonal f_1-structure then we only need to give V_+, as V_- will be the orthogonal complement.

To ease notation let us just write $E^{(0,1)}_{0,0}=G$. By the above remarks, $G(V)$ is the space of subspaces $V_+\subseteq V$—that is, $G(V)=\coprod_{k=0}^{\dim V}\mathrm{Gr}_k(V)$. To stabilize we form the bigraded array

$$\begin{tikzcd}
G(0) \arrow[r] \arrow[d] & G(\mathbb{R}_+) \arrow[r] \arrow[d] & G(\mathbb{R}_+^2) \arrow[r] \arrow[d] & \cdots \\
G(\mathbb{R}_-) \arrow[r] \arrow[d] & G(\mathbb{R}_+ \oplus \mathbb{R}_-) \arrow[r] \arrow[d] & G(\mathbb{R}_+^2 \oplus \mathbb{R}_-) \arrow[r] \arrow[d] & \cdots \\
G(\mathbb{R}_-^2) \arrow[r] \arrow[d] & G(\mathbb{R}_+ \oplus \mathbb{R}_-^2) \arrow[r] \arrow[d] & G(\mathbb{R}_+^2 \oplus \mathbb{R}_-^2) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

and $G(\underline{H})$ is the colimit. Since our model for G tracks the positive eigenspace of f_1, the horizontal maps

$$G(\mathbb{R}_+^a \oplus \mathbb{R}_-^b) \longrightarrow G(\mathbb{R}_+^{a+1} \oplus \mathbb{R}_-^b)$$

send $W \mapsto W \oplus e_{a+1}$ (here e_{a+1} denotes an element of the standard basis, not a Clifford algebra generator) whereas the vertical maps

$$G(\mathbb{R}_+^a \oplus \mathbb{R}_-^b) \longrightarrow G(\mathbb{R}_+^a \oplus \mathbb{R}_-^{b+1})$$

send W to W. Note that in this way we recover the same bigraded diagram that we found by ad hoc methods back in Section 41.3. Our conclusion is that $E_{(0,0)}^{(0,1)}(\infty) \cong \mathbb{Z} \times BO$.

43.8. **The Bott maps.**

If M is a $\mathrm{Cl}_{p,q+1}$-module then there is a map

$$\beta\colon E_{p,q+1}^{(0,1)}(M) \longrightarrow \Omega_{f_{q+1},-f_{q+1}} E_{p,q}^{(0,1)}(M)$$

that sends f_{q+2} to the path $t \mapsto \cos(\pi t) f_{q+1} + \sin(\pi t) f_{q+2}$. Similarly, if M is a $\mathrm{Cl}_{p+1,q}$-module then we have the map

$$\beta\colon E_{p+1,q}^{(1,0)}(M) \longrightarrow \Omega_{e_{p+1},-e_{p+1}} E_{p,q}^{(1,0)}(M)$$

that sends e_{p+2} to the map $t \mapsto \cos(\pi t) e_{p+1} + \sin(\pi t) e_{p+2}$. The Bott arguments show that these are equivalences in a range that goes to infinity as $\dim M$ increases.

Passing to colimits is mostly straightforward. We can choose a collection of finitely-generated $\mathrm{Cl}_{p,q+2}$-modules $H_1, \ldots, H_r$ such that every irreducible is a summand of some H_i. But even more, we can ensure that when we restrict to the $\mathrm{Cl}_{p,q+1}$-structure it is still true that every irreducible $\mathrm{Cl}_{p,q+1}$-module is a summand of some H_i. This is essentially automatic because of the way the Clifford modules work, but it is worth a moment's thought. We get a commutative lattice of Bott maps indexed by the modules $H_1^{a_1} \oplus \cdots \oplus H_r^{a_r}$, and passing to the colimits gives a weak equivalence

$$E_{p,q+1}^{(0,1)}(\infty) \xrightarrow{\simeq} \Omega E_{p,q}^{(0,1)}(\infty). \tag{43.9}$$

Note that we are using Remark 43.6 here, to know that the colimit in the target is what we want it to be. We are also using that Ω commutes with filtered colimits, due to the compactness of S^1.

A similar construction where $\underline{H}$ is a system of $\mathrm{Cl}_{p+1,q}$-modules gives a weak equivalence

$$E_{p+1,q}^{(1,0)}(\infty) \xrightarrow{\simeq} \Omega E_{p,q}^{(1,0)}(\infty).$$

43.10. **The periodicity theorems and the representing spectrum for K-theory.**

The Bott theorems give us equivalences $E^{(0,1)}_{p,q+1}(\infty) \xrightarrow{\simeq} \Omega E^{(0,1)}_{p,q}(\infty)$, and $(1,1)$-periodicity gives us homeomorphisms $E^{(0,1)}_{p,q+1}(\infty) \cong E^{(0,1)}_{p-1,q}(\infty)$. Putting these together gives weak equivalences

$$E^{(0,1)}_{p-1,q}(\infty) \xrightarrow{\simeq} \Omega E^{(0,1)}_{p,q}(\infty).$$

Fixing q therefore gives us an Ω-spectrum

$$K^{\mathbb{R}}_{(q)} = \Big[E^{(0,1)}_{0,q}(\infty), \quad E^{(0,1)}_{1,q}(\infty), \quad E^{(0,1)}_{2,q}(\infty), \ \ldots \Big]$$

These spectra are transparently 8-fold periodic by the 8-fold periodicity of the Clifford algebras, together with Morita theory. The spectrum $K^{\mathbb{R}}_{(0)}$ is more typically called KO. Observe that the (1,1)-periodicity in the table of Clifford algebras shows that the spaces in the spectrum for $K^{\mathbb{R}}_{(q)}$ are obtained from those in $K^{\mathbb{R}}_{(0)}$ by shifting to the right q spots. That is,

$$K^{\mathbb{R}}_{(q)} \simeq \Sigma^{-q} K^{\mathbb{R}}_{(0)} \simeq \Sigma^{-q} KO.$$

Bott's original arguments were about the equivalences $E^{(1,0)}_{p+1,0}(\infty) \xrightarrow{\simeq} \Omega E^{(1,0)}_{p,0}(\infty)$. In our current indexing we have $E^{(1,0)}_{-1,0}(\infty) \cong O \cong E^{(0,1)}_{-1,0}(\infty)$.

Now consider the first two rows of our table of Clifford algebras—that is, the Clifford algebras $\mathrm{Cl}_{p,0}$ and $\mathrm{Cl}_{p,1}$. The diagram below shows these together with the (horizontal) inclusions $\mathrm{Cl}_{p,0} \hookrightarrow \mathrm{Cl}_{p+1,0}$ and the (vertical) inclusions $\mathrm{Cl}_{p,0} \hookrightarrow \mathrm{Cl}_{p,1}$.

$$\begin{tikzcd}
\mathbb{R}\times\mathbb{R} \arrow[r] & \mathbb{R}(2) \arrow[r] & \mathbb{C}(2) \arrow[r] & \mathbb{H}(2) \arrow[r] & \mathbb{H}(2)\times\mathbb{H}(2) \arrow[r] & \mathbb{H}(4) \arrow[r] & \mathbb{C}(8) \arrow[r] & \mathbb{R}(16) \arrow[r] & \mathbb{R}(16)\times\mathbb{R}(16) \\
\mathbb{R} \arrow[u, "\eta"] \arrow[r, "\alpha"'] & \mathbb{C} \arrow[u, "\zeta"] \arrow[r, "\beta"'] & \mathbb{H} \arrow[u, "\epsilon"] \arrow[r, "\gamma"'] & \mathbb{H}\times\mathbb{H} \arrow[u, "\delta"] \arrow[r, "\delta"'] & \mathbb{H}(2) \arrow[u, "\gamma"] \arrow[r, "\epsilon"'] & \mathbb{C}(4) \arrow[u, "\beta"] \arrow[r, "\zeta"'] & \mathbb{R}(8) \arrow[u, "\alpha"] \arrow[r, "\eta"'] & \mathbb{R}(8)\times\mathbb{R}(8) \arrow[u, "\theta"] \arrow[r, "\theta"'] & \mathbb{R}(16) \arrow[u, "\eta"] \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tikzcd}$$

The maps with the same label—for example, the two maps labelled α—are not identical, but are related to each other in the evident way. Recall that the $E^{(1,0)}$-spaces involve the difference across a horizontal map, whereas the $E^{(0,1)}$-spaces involve the difference across a vertical map. Making use of Morita invariance, the above diagram yields the unexpected relation $E^{(1,0)}_{0,0}(\infty) \cong E^{(0,1)}_{6,0}(\infty)$, and more generally

$$(*) \qquad E^{(1,0)}_{6-p,0}(\infty) \cong E^{(0,1)}_{p,0}(\infty)$$

for $0 \le p \le 6$ together with $E^{(1,0)}_{7,0}(\infty) \cong E^{(0,1)}_{7,0}(\infty)$. This is another manifestation of Bott periodicity, although the exact connection might not be immediately clear. Recall that in the following sequence each space is loops on the one immediately after it:

$$\ldots \quad E^{(1,0)}_{2,0} \quad E^{(1,0)}_{1,0} \quad E^{(1,0)}_{0,0} \quad O \quad E^{(0,1)}_{0,0} \quad E^{(0,1)}_{1,0} \quad E^{(0,1)}_{2,0} \quad \ldots$$

(we are now leaving out the ∞ notation for brevity). The $E^{(1,0)}_{7,0}$ and $E^{(0,1)}_{7,0}$ terms are coincident with the O in the 8-fold pattern, and one can now see that (*) is also part of this pattern. Although (*) might at first feel like a 6-fold phenomenon, it is secretly 8-fold because of the two $(0,0)$ terms and the extra O.

Here is a mnemonic for our explicit descriptions of the various E-spaces. When R and S are successive Clifford algebras the space $E_{R\to S}$ often takes the form $G(R)/G(S)$ for appropriate groups $G(R)$ and $G(S)$ that depend only on the 'roots' of the Clifford algebras. For example, $E_{\mathbb{R}\to\mathbb{C}} \cong O/U$. In addition, $E_{R\to R\times R}$ is the Grassmannian $\mathbb{Z}\times BG(R)$, where $G(R)$ is that same group; e.g., $E_{\mathbb{R}\to\mathbb{R}\times\mathbb{R}} \cong \mathbb{Z}\times BO$. And $E_{R\times R\to R(2)}$ is $G(R)$. Using these mnemonics, one readily identifies the above sequence of spaces as

$$\ldots \quad \mathbb{Z}\times BSp \quad U/Sp \quad O/U \quad \boxed{\mathrm{O}} \quad \mathbb{Z}\times BO \quad U/O \quad Sp/U \quad Sp \quad \mathbb{Z}\times BSp\ldots$$

We can remember all of this with another version of the Bott periodicity clock:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r, "O/U"] & \mathbb{C} \arrow[dr, "U/Sp"] & \\
\mathbb{R}\times\mathbb{R} \arrow[ur, "O"] & & & \mathbb{H} \arrow[d, "Sp"] \\
\mathbb{R} \arrow[u, "\mathbb{Z}\times BO"] & & & \mathbb{H}\times\mathbb{H} \arrow[dl, "Sp"] \\
 & \mathbb{C} \arrow[ul, "U/O"] & \mathbb{H} \arrow[l, "Sp/U"] &
\end{tikzcd}$$

There are two ways to read this clock. Starting with $\mathbb{R}\to\mathbb{C}$ and going around clockwise, these are the maps $\mathrm{Cl}_{p,0}\to\mathrm{Cl}_{p+1,0}$ (but remember we only record the roots) and the labels on the arrows are the $E^{(1,0)}_{p,0}$ spaces. Reading this way, loops on each label gives the label on the next arrow. This is the left half of the above sequence. Alternatively, we can start with $\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ and regard this as $\mathrm{Cl}_{0,0}\to\mathrm{Cl}_{0,1}$ (the first vertical map in the above array), and then going counterclockwise gives the successive maps $\mathrm{Cl}_{p,0}\to\mathrm{Cl}_{p,1}$ (up to roots, of course). In this version the labels on the arrows are the $E^{(0,1)}_{p,0}$ spaces, and traveling counterclockwise gives *deloopings*. This is the right half of the above sequence.

Remark 43.11 (Symmetric spaces)**.** A **symmetric space** is a Riemannian manifold M having the property that for every $p\in M$ there is an isometry $I_p\colon M\to M$ that fixes p and where the derivative acts on T_pM as $-Id$. The map I_p is called an "inversion" about p. A simple example of a symmetric space is S^1, where for I_p we take the reflection in the diameter of the circle that passes through p. More generally, every Lie group G equipped with a bi-invariant metric is a symmetric space: inversion at the identity e is the map $g\mapsto g^{-1}$, and inversions at other points can be obtained by conjugation (e.g. $I_g(x) = g\cdot I_e(g^{-1}x)$). However, there exist symmetric spaces that are not Lie groups.

If G is a connected Lie group and $\sigma\in\mathrm{Aut}(G)$ is an involution, let H be an open subgroup of $G^\sigma = \{g\in G\,|\,\sigma(g)=g\}$. Then G/H is a symmetric space: σ induces $G/H\to G/H$ given by $gH\mapsto\sigma(g)H$, and one readily checks that $D_{eH}\sigma$ is $-Id$. Inversions at other points are obtained using the group action. As one example, consider $G=U(n)$ with σ complex conjugation, and $H=O(n)=G^\sigma$. Then the symmetric space is $U(n)/O(n)$. It is not immediately obvious, but it turns out that every symmetric space arises as such a G/H.

There is a complete classification of symmetric spaces, due to Elie Cartan. Similar to the classification of compact Lie groups, there are a handful of infinite families and then several isolated exceptional cases. The infinite families turn out to all be related to Clifford algebras—in fact, they are precisely the spaces that come up in Bott periodicity of the unitary and orthogonal groups.

We will not need to use symmetric spaces in the rest of this book, but it is good to know a little about how they fit into the story of Bott periodicity.

44. Pinning down Bott periodicity

At this point we have proven the isomorphisms $\pi_i U \cong \pi_{i+2} U$ for all $i \geq 0$, but we have not as yet produced explicit generators for these groups. Relatedly, the periodicity isomorphism itself is difficult to unravel—it is not as if we have an explicit formula for it. Somehow these things are buried in the Bott-Morse arguments, but it takes a bit to tease them out. That is our goal in this present section, for both the unitary and orthogonal groups.

We will start by investigating some relations between $U(n)$ and $U(2n)$. The identification $\mathbb{C}^{2n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n$ yields some easy connections. We will eventually approach this in very categorical ways, but let us begin by taking a path where everything is very concrete.

Observe that whenever $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$ then we have the element

$$E_{\alpha,\beta} = \begin{bmatrix} \alpha I_n & -\bar{\beta} I_n \\ \beta I_n & \bar{\alpha} I_n \end{bmatrix} \in SU(2n).$$

Identifying S^3 with the unit sphere in $\mathbb{C}^2$, then when $n = 1$ this is the standard isomorphism $S^3 \xrightarrow{\cong} SU(2)$. For $n > 1$ we can think of E as arising as a composite $S^3 \to SU(2) \to SU(2n)$ (the second map is basically tensoring with $Id_{\mathbb{C}^n}$, by the way).

If $A \in U(n)$ then we also get the element $F_A = \begin{bmatrix} I_n & 0 \\ 0 & A \end{bmatrix} \in U(2n)$. By conjugating, we obtain elements

$$G_{\alpha,\beta,A} = F_A E_{\alpha,\beta} F_A^{-1} = \begin{bmatrix} \alpha I_n & -\bar{\beta} A^\dagger \\ \beta A & \bar{\alpha} I \end{bmatrix} \in SU(2n).$$

Here we have written $A^\dagger$, but we could also have written A^{-1} since they are equal. This formula describes a map $G\colon S^3 \times U(n) \to SU(2n)$.

Observe that

(44.1)
$$G_{\alpha,\beta,A} = \mathrm{Re}(\alpha) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \mathrm{Im}(\alpha) \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + \mathrm{Re}(\beta) \begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix} + \mathrm{Im}(\beta) \begin{bmatrix} 0 & iA^\dagger \\ iA & 0 \end{bmatrix}$$

where each of the four matrices appearing in the linear combination is in $SU(2n)$. The coefficients of our linear combination are a point in S^3, and we have seen that any such "unital" linear combination gives a matrix in $SU(2n)$. This surprising property turns out to be very useful and rare, and we will see in Section 45.7 that such situations always come from Clifford algebra representations. In fact you might notice that the four matrices in our linear combination can be used to give a quaternionic structure on $\mathbb{C}^{2n}$ and that $\mathbb{H} \cong \mathrm{Cl}_2(\mathbb{R})$ (the four matrices gives the action of 1, i, j, and $-k$).

For what is about to happen we will not need the full S^3 parameter space, and it will be enough to restrict down to a 2-disk. But there is not a canonical choice for such a disk, and this leads to a bit of a mess where different viewpoints call for different choices. Most of the work in this section amounts to navigating this.

Exercise 44.2. Verify that an element $X \in U(2n)$ anticommutes with $\left[\begin{smallmatrix} iI & 0 \\ 0 & -iI \end{smallmatrix}\right]$ if and only if it has the form $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ where $B, C \in U(n)$. If in addition $X^2 = -I$ verify that $B = -C^\dagger$ (and note that this forces $X \in SU(2n)$). So if one went looking for quaternionic structures extending $\left[\begin{smallmatrix} iI & 0 \\ 0 & -iI \end{smallmatrix}\right]$ then one would quickly discover the above formulas.

We will only need the portion of G defined on S^2_{upp}, the upper hemisphere of S^2, regarded as sitting in S^3 as the points $(x, y, z, 0)$ with $x, y, z \in \mathbb{R}$ and $z \geq 0$. In relation to the complex coordinates used earlier, $\alpha = x + iy$ and $z = \mathrm{Re}(\beta)$. Let G' denote the map $S^2_{\text{upp}} \times U(n) \to SU(2n)$ given by

$$(x, y, z, A) \mapsto xI + y \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + z \begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix}. \tag{44.3}$$

When $z = 0$ the output does not depend on A, which says that G' respects the quotient relation $(u, A) \sim (u, B)$ for all $u \in \partial S^2_{\text{upp}}$ and $A, B \in U(n)$. So we obtain an induced map

$$\mathcal{B} \colon (S^2_{\text{upp}} \times U(n))/{\sim} \longrightarrow SU(2n).$$

The domain is a model for $\Sigma^2 U(n)$ (see Exercise 44.11(g)), so we will also write this as $\mathcal{B} \colon \Sigma^2 U(n) \to SU(2n)$. We call $\mathcal{B}$ the "Bott suspension map".

Exercise 44.4. For any space W the double suspension $\Sigma^2 W$ can be modelled by the construction $(S^2_{\text{upp}} \times W)/{\sim}$ where $(a, w) \sim (a, w')$ for all $a \in \partial S^2_{\text{upp}}$ and $w, w' \in W$. Verify this explicitly for $W = S^r$ by checking that a homeomorphism $S^{2+r} \longrightarrow (S^2_{\text{upp}} \times S^r)/{\sim}$ is given by

$$(u, x) \mapsto \Big((u_1, u_2, |x|), \tfrac{x}{|x|} \Big)$$

for $(u, x) \in \mathbb{R}^2 \times \mathbb{R}^{r+1}$ such that $|u|^2 + |x|^2 = 1$. Here the image point has the evident interpretation when $|x| = 0$. [Suggestion: It is easier to work with the inverse.]

Exercise 44.5. Verify that the square

$$\begin{tikzcd}
(S^2_{\text{upp}} \times U(n))/{\sim} \arrow[r, "\mathcal{B}"] \arrow[d, "\mathrm{id} \times i"] & SU(2n) \arrow[d, "i"] \\
(S^2_{\text{upp}} \times U(n+1))/{\sim} \arrow[r, "\mathcal{B}"] & SU(2n+2)
\end{tikzcd}$$

commutes up to homotopy.

Remark 44.6. Notice that we made a choice in (44.3) to use the matrix $\left[\begin{smallmatrix} 0 & -A^\dagger \\ A & 0 \end{smallmatrix}\right]$ from (44.1) rather than $\left[\begin{smallmatrix} 0 & iA^\dagger \\ iA & 0 \end{smallmatrix}\right]$. It turns out that this choice doesn't matter. We can replace $\left[\begin{smallmatrix} 0 & -A^\dagger \\ A & 0 \end{smallmatrix}\right]$ with the matrix $\left[\begin{smallmatrix} 0 & -\bar{s}A^\dagger \\ sA & 0 \end{smallmatrix}\right]$ for any $s \in S^1$ and obtain a map G'_s. Taking a path in S^1 from 1 to s gives a homotopy between our G' and any other G'_s. The formulas corresponding to G_1, G_{-1}, G_i, and G_{-i} are all commonly found in the literature, with different sources making different choices.

Remark 44.7. Another choice commonly found in the literature is to switch the locations of A and $A^\dagger$ in formula (44.3). This time the switch does alter the homotopy type of the maps G' and $\mathcal{B}$, but in a predictable way: it essentially precomposes them with the automorphism of $U(n)$ given by $A \mapsto A^\dagger$. We will not have to deal with this issue, but want to alert the reader to differences that might be encountered in other sources. The profusion of different choices here makes the literature a bit of a nightmare.

Given $f\colon S^r \to U(n)$ let $\mathcal{B}_*\Sigma^2(f)$ denote the composite

$$S^{2+r} = (S^2_{\text{upp}} \times S^r)/\!\sim \xrightarrow{\text{id}\times f} (S^2_{\text{upp}} \times U(n))/\!\sim \xrightarrow{\mathcal{B}} SU(2n)$$

where the first isomorphism is the one from Exercise 44.4. The assignment $f \mapsto \mathcal{B}_*\Sigma^2(f)$ gives a map of groups $\pi_r U(n) \to \pi_{r+2} SU(2n)$, as it is the composite

$$\pi_r U(n) \xrightarrow{\Sigma^2} \pi_{r+2}(\Sigma^2 U(n)) = \pi_{r+2}((S^2_{\text{upp}} \times U(n))/\!\sim) \xrightarrow{\mathcal{B}_*} \pi_{r+2} SU(2n).$$

This also explains the symbology $\mathcal{B}_*\Sigma^2$.

In coordinates $(u,x) \in \mathbb{R}^2 \times \mathbb{R}^{r+1}$ with $|u|^2 + |x|^2 = 1$ we have

$$(\mathcal{B}_*\Sigma^2 f)(u,x) = u_1 I + u_2 \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + |x| \begin{bmatrix} 0 & -f(\frac{x}{|x|})^\dagger \\ f(\frac{x}{|x|}) & 0 \end{bmatrix}.$$

Note that when $|x| = 0$ the last term is interpreted to be the zero matrix. Sometimes it is useful to absorb u_1 and u_2 into a single complex coordinate α. Here we identify S^{2+r} with $S(\mathbb{C} \times \mathbb{R}^{r+1})$ and then write

$$(\mathcal{B}_*\Sigma^2 f)(\alpha, x) = \begin{bmatrix} \alpha I & -|x| f(\frac{x}{|x|})^\dagger \\ |x| f(\frac{x}{|x|}) & \bar{\alpha} I \end{bmatrix} = G_{\alpha, |x|, f\left(\frac{x}{|x|}\right)}.$$

We regard $(1,0)$ as the basepoint of $S(\mathbb{C}\times\mathbb{R}^{r+1})$ and note that $(\mathcal{B}_*\Sigma^2 f)(1,0) = Id$.

The following result is a very concrete version of Bott periodicity:

Theorem 44.8. *For $r < 2n$ the map $\mathcal{B}_*\Sigma^2\colon \pi_r U(n) \longrightarrow \pi_{r+2} SU(2n)$ is an isomorphism. Moreover, the stabilization square*

$$\begin{tikzcd} \pi_r U(n) \arrow[r, "i_*"] \arrow[d, "\mathcal{B}_*\Sigma^2"'] & \pi_r U(n+1) \arrow[d, "\mathcal{B}_*\Sigma^2"'] \\ \pi_{r+2} SU(2n) \arrow[r, "i_*"] & \pi_{r+2} SU(2n+2) \end{tikzcd}$$

is commutative for all r and n.

The proof is not deep, and consists mostly of just carefully comparing our formula for $\mathcal{B}_*\Sigma^2$ to the maps that came up in the Bott-Morse arguments. But this is easier said than done and does take some care. The remainder of the section will be focused on this proof, but very little of what goes into this proof will be used later. So readers who are more interested in seeing applications of Theorem 44.8 should feel free to jump ahead to Section 45.

Exercise 44.9. Use the isomorphisms in Theorem 44.8 together with the long exact sequences for $SU(r-1) \hookrightarrow SU(r) \to S^{2r-1}$ to prove that for all $k \leq 2n-1$

one has

$$\pi_k(U(n)) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

That is to say, the Bott periodic groups in $\pi_* U(n)$ persist up through $* = 2n-1$. Also prove that $\pi_{2n}U(n)$ (the first non-periodic group) is always cyclic.

44.10. **Joins.** If X and Y are spaces then the join $X*Y$ is defined to be the quotient space $(X \times I \times Y)/\sim$ where the equivalence relation is genereated by $(x,0,y) \sim (x,0,y')$ and $(x,1,y) \sim (x',1,y)$ for all $x,x' \in X$ and $y,y' \in Y$. Intuitively one thinks of the join as the space of lines from points in X to points in Y. It will be convenient to adopt the notation

$$(x,t,y) = (1-t)[x] + t[y],$$

so that $X * Y = \{a[x] + b[y] \,\big|\, x \in X, y \in Y, a, b \in I, a+b=1\}$.

Note that $S^0 * Y$ is the (unreduced) suspension of Y. We will often give the points of S^0 names like $\pm P$, in which case points of $S^0 * Y$ will be written $a[\pm P] + (1-a)[y]$ where $y \in Y$ and $a \in I$.

When X and Y are pointed CW-complexes then the homotopy type of $X * Y$ coincides with $\Sigma(X*Y)$. We will not need this general fact, but see ??? for a proof.

The following exercise establishes several basic facts we will need about the behavior of the join:

Exercise 44.11. Justify all of the following claims.

(a) For the unit circle $S^1 \subseteq \mathbb{C}$, a homeomorphism $S^1 \cong S^0 * S^0 = \{\pm 1\} * \{\pm i\}$ is given by

$$\alpha \mapsto \operatorname{Re}(\alpha)^2[\operatorname{sgn}(\operatorname{Re}\alpha)] + \operatorname{Im}(\alpha)^2[\operatorname{sgn}(\operatorname{Im}\alpha)i]$$

where $\operatorname{sgn}(r)$ is $+1$ when $r > 0$ and -1 when $r < 0$ (the value when $r = 0$ doesn't actually matter for the above formula).

(b) The inverse of the map in (a) is given by $a[\pm 1] + b[\pm i] \mapsto (\pm\sqrt{a}) \pm (\sqrt{b} \cdot i)$, where the first $\pm$ follows the sign on 1 and the second follows the sign on i. Verify that

$$a[\pm 1] + b[\pm i] \mapsto \tfrac{1}{\sqrt{a^2+b^2}}[\pm a \pm bi]$$

also gives a homeomorphism $S^0 * S^0 \to S^1$, and that this map is homotopic to the first one.

In fact, suppose $\gamma\colon I \to \mathbb{R}^2$ is any homeomorphism onto the first quadrant of the unit circle $(x, y \geq 0)$ such that $\gamma(0) = (0,1)$ and $\gamma(1) = (1,0)$. Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Then

$$a[\pm 1] + b[\pm i] \mapsto \pm\gamma_1(a) \pm \gamma_2(a) \cdot i$$

is a homeomorphism and is homotopic to the above maps.

There is a moral here: *because joins are made up of straight lines, maps out of them can often be represented by many different formulas that are the same up to homotopy.*

(c) For the standard spheres $S^r \subseteq \mathbb{R}^{r+1}$ and $S^s \subseteq \mathbb{R}^{s+1}$, there is a homeomorphism $S^{r+s+1} \cong S^r * S^s$ given by

$$(x,y) \mapsto |x|^2\left[\tfrac{x}{|x|}\right] + |y|^2\left[\tfrac{y}{|y|}\right]$$

where $x \in \mathbb{R}^{r+1}$, $y \in \mathbb{R}^{s+1}$, and $|x|^2 + |y|^2 = 1$.

(d) When X, Y, and Z are compact Hausdorff the triple join $X*(Y*Z)$ is homeomorphic to the quotient space obtained from the set of formal sums

$$a[x]+b[y]+c[z]$$

where $a,b,c\geq 0$ and $a+b+c=1$ (topologized as $X\times Y\times Z\times\Delta^2$) subject to the quotient relation generated by $0[x]+b[y]+c[z]\sim 0[x']+b[y]+c[z]$ and the two analogous relations corresponding to $b=0$ and $c=0$. [Note: This result is non-obvious because of the need to commute products with the quotienting processes. Possibly the hypotheses that the spaces are compact and Hausdorff are unnecessarily strong, but they allow one to use adjunction arguments and other simplifying techniques. If $Q=(X\times Y\times Z\times\Delta^2)/\sim$ start by producing a map $X\times(Y\times Z\times I)\times I\to Q$, then get $(Y\times Z\times I)/\sim\longrightarrow Q^{X\times I}$, and continue to arrive at a map $X*(Y*Z)\to Q$.]

(e) A homeomorphism $S(\mathbb{C}\times\mathbb{R}^{r+1})\cong S^0*S^0*S^r$ is given by

$$(\alpha,x)\mapsto \mathrm{Re}(\alpha)^2\big[\mathrm{sgn}(\mathrm{Re}(\alpha))\big]+\mathrm{Im}(\alpha)^2\big[\mathrm{sgn}(\mathrm{Im}(\alpha))i\big]+|x|^2\Big[\tfrac{x}{|x|}\Big].$$

(f) $S^1*X\cong(D^2\times X)/\sim$ where the equivalence relation has $(a,x)\sim(a,x')$ for all $a\in\partial D^2$ and $x,x'\in X$. One such homeomorphism is given by $t[\alpha]+(1-t)[x]\mapsto(t\alpha,x)$, and another (homotopic) one is given by $t[\alpha]+(1-t)[x]\mapsto(\sqrt{t}\cdot\alpha,x)$. [This does not need X to be compact Hausdorff, though the proof is easier in that case. Likewise for the next part.]

(g) There is a homeomorphism

$$S^0*S^0*X\longrightarrow(S^2_{\mathrm{upp}}\times X)/\sim$$

given by $a[\pm M]+b[\pm N]+c[x]\mapsto((\pm_M\sqrt{a},\pm_N\sqrt{b},\sqrt{c}),x)$. Here we are using S^2_{upp} to line up with discussion from the beginning of the section, but note that we could just as well use D^2 instead (but getting slightly different formulas).

(h) Suppose given a map $f\colon Z\to\Omega_{P,Q}W$, and recall that the target denotes the space of paths that begin at P and end at Q. Check that one gets an "adjoint" $\mathrm{adj}(f)\colon S^0*Z\to W$ via the formula

$$(1-t)[\pm N]+t[z]\mapsto\begin{cases}[f(z)](\frac{t}{2}) & \text{for } +\\ [f(z)](1-\frac{t}{2}) & \text{for } -,\end{cases}$$

having the property that $[+N]\to P$ and $[-N]\to Q$. Moreover, if Z has a basepoint z and $\gamma=f(z)$ then the following diagram commutes

$$\begin{tikzcd}
\pi_r(Z,z) \arrow[r, "f_*"] \arrow[d, "\Sigma"'] & \pi_r(\Omega_{P,Q}W,\gamma) \arrow[d, "\mathrm{Adj}", "\cong"'] \\
\pi_{r+1}(S^0*Z,[+N]) \arrow[r, "\mathrm{adj}(f)_*"'] & \pi_{r+1}(W,P)
\end{tikzcd}$$

where Adj sends a homotopy element $\alpha\colon S^r\to\Omega_{P,Q}W$ to $\mathrm{adj}(\alpha)\colon S^0*S^r\to W$ and S^0*S^r has the north pole $[+N]$ as basepoint. As part of this, check that Σ and Adj are well-defined, group maps, and that Adj is an isomorphism.

Using Exercise 44.11(g) we now see that the Bott suspension map $\mathcal{B}$ can be reinterpreted as the map $\mathcal{B}\colon S^0 * S^0 * U(n) \to U(2n)$ given by the formula

$$a[\pm M] + b[\pm N] + c[A] \mapsto \pm_M \sqrt{a} I + \pm_N \sqrt{b} \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + \sqrt{c} \begin{bmatrix} 0 & -A^\dagger \\ A & 0 \end{bmatrix}.$$

44.12. **Proof of Theorem 44.8.**

Recall the Bott map $\beta\colon \operatorname{Gr}_n(\mathbb{C}^n_+ \oplus \mathbb{C}^n_-) \to \Omega_{I,-I} SU(2n)$ sending a subspace H to the path β_H given by

$$\beta_H(t) = [\text{the transformation that is } \cdot e^{\pi i t} \text{ on } H \text{ and } \cdot e^{-\pi i t} \text{ on } H^\perp].$$

For stabilization we need to pass to $\Omega_I SU(2n)$ instead, so let σ be the path from I to $-I$ where

$$\sigma(t) = [\text{the transformation that is } \cdot e^{-\pi i t} \text{ on } \mathbb{C}^n_+ \text{ and } \cdot e^{\pi i t} \text{ on } \mathbb{C}^n_-].$$

We use the homeomorphism $\Omega_{I,-I} SU(2n) \to \Omega_I SU(2n)$ that sends a path λ to the loop $t \mapsto \lambda(t) \cdot \sigma(t)$ (see Exercise 41.14), and will write $\tilde\beta$ for the composite

$$\operatorname{Gr}_n(\mathbb{C}^n_+ \oplus \mathbb{C}^n_-) \xrightarrow{\beta} \Omega_{I,-I} SU(2n) \longrightarrow \Omega_I SU(2n).$$

The proof centers on analyzing the following large diagram. All of the "action" in this diagram is in the two pentagons at the bottom; the composition running along the top of those pentagons is the Bott isomorphism we saw in Section 41. The six rectangles forming the top of the diagram are only there to show the stabilization and can be ignored.

$$\begin{tikzcd}
\pi_r U \arrow[r, "\cong", "c"'] & \operatorname{Vect}^{st}(S^{r+1}) & \pi_{r+1} \operatorname{Gr}_\infty(\mathbb{C}^{2\infty}) \arrow[l, dotted] \arrow[r, "\tilde\beta_*", "\cong"'] & \pi_{r+1}\Omega_I SU \arrow[r, equal] & \pi_{r+2} SU \\
C_k \pi_r U_k \arrow[u, "\cong"] \arrow[r, "\cong", "c"'] & C_k \operatorname{Vect}_k(S^{r+1}) \arrow[u, "\cong"'] & C_k \pi_{r+1} \operatorname{Gr}_k(\mathbb{C}^k_+ \oplus \mathbb{C}^\infty_-) \arrow[u, "\cong"'] \arrow[l, "\cong"'] & & \\
\pi_r U_n \arrow[u, "{[r\leq 2n-1]}"'] \arrow[r, "\cong", "c"'] \arrow[dr, "\Sigma"'] & \operatorname{Vect}_n(S^{r+1}) \arrow[u] & \pi_{r+1} \operatorname{Gr}_n(\mathbb{C}^n_+ \oplus \mathbb{C}^\infty_-) \arrow[u, "{[r\leq 2n-1]}"'] \arrow[l, "\cong"'] & & \\
& \pi_{r+1} \Sigma U_n \arrow[r, "\Theta_*"] \arrow[dr, "\Sigma"'] & \pi_{r+1} \operatorname{Gr}_n(\mathbb{C}^n_+ \oplus \mathbb{C}^n_-) \arrow[u, "{[r\leq 2n-1]}"'] \arrow[r, "\tilde\beta_*", "{[r\leq 2n-1]}"'] & \pi_{r+1}\Omega_I SU_{2n} \arrow[uuu] \arrow[r, equal] & \pi_{r+2} SU_{2n} \arrow[uuu, "{[r<4n-2]}"] \\
& & \pi_{r+2}(S^0 * S^0 * U_n) \arrow[urr, "\mathcal{B}_*"'] & &
\end{tikzcd}$$

One part of this diagram—the map Θ_*—has not been explained yet, but we will get to it shortly. We instead begin with several remarks to help the reader parse the above monstrosity:

(1) The composition across the top row is the stable form of the Bott periodicity isomorphism. The space $\operatorname{Gr}_\infty(\mathbb{C}^{2\infty})$ is the construction $\operatorname{colim}_k \operatorname{Gr}_k(\mathbb{C}^k_+ \oplus \mathbb{C}^k_-)$ that we encountered in Section 41. Recall that this space is a model for BU.

(2) The composition across the bottom of the diagram ($\mathcal{B}_* \circ \Sigma \circ \Sigma$) is the subject of Theorem 44.8 and our main goal.

(3) C_k is shorthand for colim_k, and we have written U_k as shorthand for $U(k)$.

(4) The dotted arrow in the top row is dotted because it is not naturally occuring; rather, it is defined to be the composite of the other isomorphisms in that square. All other maps in the diagram are the evident ones.
(5) Recall that the maps $\mathrm{Gr}_n(\mathbb{C}^n_+ \oplus \mathbb{C}^n_-) \to \mathrm{Gr}_{n+1}(\mathbb{C}^{n+1}_+ \oplus \mathbb{C}^{n+1}_-)$ send an n-plane H to $H \oplus \langle e_{n+1,+}\rangle$. That is, both $e_{n+1,+}$ and $e_{n+1,-}$ are added to the ambient space, but only the former is added to the subspace. Going to the colimit can be broken up as first adding in all of the negative basis vectors to the ambient space, and then adding the positive basis vectors one by one into both the ambient space and the subspace. This is what is going on at the top of the central column. We are doing this because $\mathrm{Gr}_k(\mathbb{C}^k_+ \oplus \mathbb{C}^\infty_-)$ is a model for $BU(k)$ and hence classifies rank k vector bundles.
(6) The maps labelled c are given by the clutching construction. The inequalities in brackets are ranges where the given maps are isomorphisms. There are three sources of these: the fiber sequences $U_k \hookrightarrow U_{k+1} \hookrightarrow S^{2k+1}$, the fact that $\mathrm{Gr}_n(\mathbb{C}^\infty)$ is obtained from $\mathrm{Gr}_n(\mathbb{C}^{2n})$ by attaching cells of dimension $2n+2$ and higher, and the Bott-Morse arguments giving the connectivity of the lower $\tilde{\beta}$ map.
(7) The four squares in the upper left obviously commute.
(8) The maps $\tilde{\beta}$ send a subspace H to the path of transformations

$$t \mapsto \ (e^{\pi i t} \text{ on } H,\, e^{-\pi i t} \text{ on } H^\perp) \cdot (e^{-\pi i t} \text{ on } \mathbb{C}^n_+,\, e^{\pi i t} \text{ on } \mathbb{C}^n_-)$$

where the $\cdot$ denotes multiplication of matrices or composition of transformations. If we denote this path $\tilde{\beta}_H$, notice that for $x \in H \cap \mathbb{C}^n_+$ or $x \in H^\perp \cap \mathbb{C}^n_-$ we have $(\tilde{\beta}_H)_t(x) = x$ for all t. In particular, this explains why $\tilde{\beta}$ extends to the colimit. This is the commutativity of the leftmost of the two upper right rectangles. The commutativity of the rightmost one is self-evident.

At this point we have analyzed the top portion of the diagram, made up of the six rectangles. It is difficult to directly relate the composition across the bottom of these rectangles to the $\mathcal{B}_* \circ \Sigma^2$ composite because some of the maps go in the "wrong" direction. We will correct for this by introducing the map $\Theta\colon \Sigma U(n) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$, which helps connect the two pieces. This map is in some sense a geometric object underlying the isomorphism $\pi_r U \cong \pi_{r+1} BU$.

Let $\mathcal{G}\colon U(n) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$ be the map that sends a matrix A to its graph $\mathcal{G}A = \{(x, Ax) \mid x \in \mathbb{C}^n\}$. There is an extension of $\mathcal{G}$ to a map $\Sigma U(n) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$ that deforms $\mathcal{G}A$ to each of the two "coordinate axes" in $\mathbb{C}^n \oplus \mathbb{C}^n$. Precisely, we have $\Theta\colon S^0 * U(n) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$ given by

$$\begin{aligned}(1-t)[N] + t[A] &\mapsto \{(x, tAx) \mid x \in \mathbb{C}^n\}\\ (1-t)[-N] + t[A] &\mapsto \{(tx, Ax) \mid x \in \mathbb{C}^n\}.\end{aligned}$$

Note that $[N] \mapsto \mathbb{C}^n \oplus 0$ and $[-N] \mapsto 0 \oplus \mathbb{C}^n$.

Pulling back the canonical bundle η on $\mathrm{Gr}_n(\mathbb{C}^{2n})$ along Θ gives a rank n bundle on $\Sigma U(n)$. As we have seen in Section 12.1, vector bundles on such a suspension are characterized by a clutching function $U(n) \to U(n)$, and we claim that in this case the clutching function is the identity. To see this, observe that the formulas

$$\begin{aligned}(1-t)[N] + t[A] &\mapsto [(e_1, tAe_1), \dots, (e_n, tAe_n)]\\ (1-t)[-N] + t[A] &\mapsto [(tA^{-1}e_1, e_1), \dots, (tA^{-1}e_n, e_n)]\end{aligned}$$

give trivializing sections over the upper and lower cones, respectively. When $t = 1$ the matrix that writes the first basis in terms of the second basis is precisely A (just look in the second coordinates).

Continuing now with our analysis of the big diagram:

(9) The trapezoidal pentagon in the lower left commutes because, given $f\colon S^r \to U(n)$, the vector bundle on S^{r+1} constructed with this clutching function is precisely the pullback in the diagram

$$\begin{tikzcd} E \arrow[rr, dotted] \arrow[d, dotted] & & \eta_n \arrow[d] \\ \Sigma S^r \arrow[r, "\Sigma f"] & \Sigma U(n) \arrow[r, "\Theta"] & \mathrm{Gr}_n(\mathbb{C}^{2n}). \end{tikzcd}$$

This is due to our construction of Θ, and the fact that $\Theta^*\eta_n$ is the bundle on $\Sigma U(n)$ whose clutching function is the identity.

(10) We observe now that the theorem will follow immediately once we have proven the commutativity of the bottom pentagon (which looks more like a triangle in the diagram), using that the indicated maps are isomorphisms in the given ranges.

To prove commutativity of the pentagon we need to unpack the Bott map β. Let $X \subseteq SU(2n)$ denote the subspace of matrices A such that $A^2 = -I$. All such matrices are diagonalizable with eigenvalues $\pm i$ and orthogonal eigenspaces, so there is a homeomorphism $\mathrm{Gr}_n(\mathbb{C}^{2n}) \longrightarrow X$ sending a subspace $H \subseteq \mathbb{C}^{2n}$ to the matrix for the transformation T_H which is multiplication by i on H and multiplication by $-i$ on $H^\perp$. We will move back and forth between $\mathrm{Gr}_n(\mathbb{C}^{2n})$ and X at will via this homeomorphism. For example, the Bott map $\beta\colon \mathrm{Gr}_n(\mathbb{C}^{2n}) \to \Omega_{I,-I}SU(2n)$ is also the map $\beta\colon X \to \Omega_{I,-I}SU(2n)$ given by

$$A \mapsto [t \mapsto e^{\pi t A} = \cos(\pi t)I + \sin(\pi t)A].$$

We next want to re-interpret our map $\Theta\colon S^0 * U(n) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$ (see above) as a map $\Theta'\colon S^0 * U(n) \to X$. This involves finding matrices in X whose i-eigenspaces are the given subspaces of $\mathbb{C}^{2n}$. A little legwork/inspiration suggests that to have (x, tAx) be an eigenvector one should look at unitary matrices of the form $Q = \begin{bmatrix} \alpha I & \beta A^\dagger \\ -\bar{\beta} A & \bar{\alpha} I \end{bmatrix}$, and we leave the reader to check that

- Q is unitary precisely when $|\alpha|^2 + |\beta|^2 = 1$.
- When Q is unitary one automatically has $\det(Q) = 1$, so $Q \in SU(2n)$.
- When Q is unitary, $Q^2 = -I$ if and only if $\alpha = ri$ for some $r \in \mathbb{R}$.
- When Q is unitary and $Q^2 = -I$, Q has $\{(x, tAx) \mid x \in \mathbb{C}^n\}$ as its i-eigenspace precisely when $r = \frac{1-t^2}{1+t^2}$ and $\beta = si$ where $s = \frac{2t}{1+t^2}$.
- When Q is unitary and $Q^2 = -I$, Q has $\{(tx, Ax) \mid x \in \mathbb{C}^n\}$ as its i-eigenspace precisely when $r = \frac{t^2-1}{1+t^2}$ and $\beta = si$ where $s = \frac{2t}{1+t^2}$ (just change t to t^{-1} in the previous bullet point).

These are routine exercises in linear algebra. The upshot is that the formula for the map $\Theta'\colon S^0 * U(n) \to X$ is

$$(1-t)[\pm N] + t[A] \mapsto \begin{bmatrix} \pm riI & siA^\dagger \\ siA & \mp riI \end{bmatrix} = \pm r \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix} + s \begin{bmatrix} 0 & iA^\dagger \\ iA & 0 \end{bmatrix}$$

where
$$r = \tfrac{1-t^2}{1+t^2}, \qquad s = \tfrac{2t}{1+t^2}.$$
Note that $r^2 + s^2 = 1$ here, for all values of t.

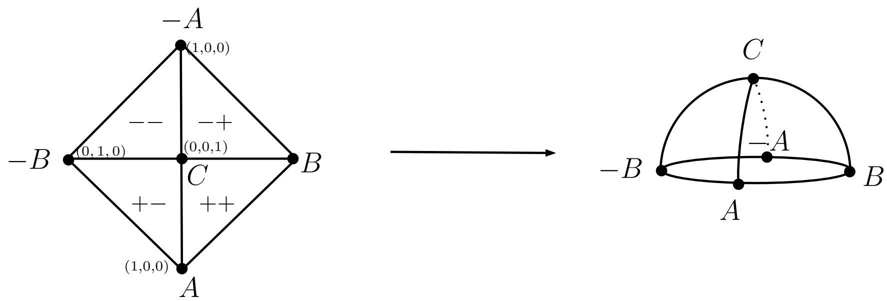
The above formulas for r and s are a little awkward, so we make the following observation. All that is important here is that the assignment $t \mapsto (r, s)$ maps I to the first quadrant of the unit circle, with $0 \mapsto (1, 0)$ and $1 \mapsto (0, 1)$. All such maps are homotopic, and using any such map to give the coefficients for r and s in the above formula gives a map $S^0 * U(n) \to X$ that is homotopic to the specific Θ' above. For example, one could use $(r, s) = (\sqrt{1-t}, \sqrt{t})$ or $(r, s) = (\cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t))$, and these lead to more pleasant-looking formulas.

Finally, we consider the composite
$$S^0 * U(n) \xrightarrow{\Theta'} X \xrightarrow{\beta} \Omega_{I,-I} SU(2n)$$
that is given by the formula
$$(1-t)[\pm N] + t[A] \mapsto \Big[u \mapsto \cos(\pi u)I + \sin(\pi u)\Theta'\big({\scriptstyle (1-t)[\pm N]+t[A]}\big)\Big].$$
We again take note that the coefficients $r_2(u) = \cos(\pi u)$ and $s_2(u) = \sin(\pi u)$ can be varied somewhat. All that is important is that $u \mapsto (r_2(u), s_2(u))$ maps I to the top of the unit circle and sends 0 to $(1, 0)$ and 1 to $(-1, 0)$. All such paths are homotopic, so we can change the coefficients in the above formula to any such $r_2(u)$ and $s_2(u)$ without changing the overall homotopy class of the map. In a moment it will also be convenient that $u = \frac{1}{2}$ is mapped to $(0, 1)$, so we will also insist on that.

Taking the "adjoint" of the above composite, as in Exercise 44.11(h), gives the map $\operatorname{adj}(\beta\Theta') \colon S^0 * S^0 * U(n) \to SU(2n)$ whose formula is
$$(*) \qquad a[\pm M] + b[\pm N] + c[A] \mapsto r_2\left(\tfrac{1 \mp_M a}{2}\right) I \pm_N s_2\left(\tfrac{1 \mp_M a}{2}\right) r\left(\tfrac{c}{1-a}\right) \left[\begin{smallmatrix} iI & 0 \\ 0 & -iI \end{smallmatrix}\right] + s_2\left(\tfrac{1 \mp_M a}{2}\right) s\left(\tfrac{c}{1-a}\right) \left[\begin{smallmatrix} 0 & iA^\dagger \\ iA & 0 \end{smallmatrix}\right].$$
Recall that the $\frac{1 \mp_M a}{2}$ term is mapping the "suspension coordinate" given by a to a "path coordinate" that goes from 0 to 1, as shown in the following picture:

$$\begin{array}{ccc} [+M]\ (a=1) & \longrightarrow & 0 \\ \downarrow & & \\ (a=0) & \longrightarrow & \tfrac{1}{2} \\ \uparrow & & \\ [-M]\ (a=1) & \longrightarrow & 1. \end{array}$$

The coefficients in $(*)$ are off-putting, but what is happening here is that we have four maps from $\Delta^2 = \{(a, b, c) \in I^3 \mid a + b + c = 1\}$ to S^2_{upp} (corresponding to the four choices of signs) that patch together to give the map shown in the following picture:



The diamond is really $\Delta^2 \subseteq \mathbb{R}^3$ together with its three images under the different reflections $(a, b, c) \mapsto (\pm a, \pm b, c)$. The points labelled $\pm A$, $\pm B$, C in the diamond map to the corresponding points in the sphere, with indicated lines mapping onto the indicated arcs of great circles. Each triangle in the diamond is identified with our standard Δ^2 by having the $\pm A$ vertex map to $(1, 0, 0)$, the $\pm B$ vertex map to $(0, 1, 0)$, and the C vertex map to $(0, 0, 1)$ (notice that, conveniently, these are also related to the coordinates of their images in the sphere).

While this map from the diamond to the upper half-sphere is not too hard to understand from a geometric perspective, the algebraic formula in $(*)$ is unpleasant. As we have seen before, it can largely be ignored: all that is important here is that the map sends the boundary of the diamond homeomorphically onto the equator and the four corners to the four corresponding equatorial poles—all such maps are homotopic. So we can instead use, for example, $(\pm a, \pm b, c) \mapsto (\pm\sqrt{a}, \pm\sqrt{b}, \sqrt{c})$ if we prefer. This shows that $\operatorname{adj}(\beta\Theta')$ is homotopic to the map

$$(**) \qquad a[\pm M] + b[\pm N] + c[A] \mapsto \pm_M \sqrt{a} I \pm_N \sqrt{b} \left[\begin{smallmatrix} iI & 0 \\ 0 & -iI \end{smallmatrix}\right] + \sqrt{c} \left[\begin{smallmatrix} 0 & iA^\dagger \\ iA & 0 \end{smallmatrix}\right].$$

It then follows that we have a homotopy-commutative diagram

$$(44.13) \qquad \begin{tikzcd} S^0 * S^0 * U(n) \arrow[rr, "\operatorname{adj}(\beta\Theta')"] \arrow[dr, "\cong"'] & & SU(2n) \\ & (S^2_{\mathrm{upp}} \times U(n))/{\sim} \arrow[ur, "\mathcal{B}"'] & \end{tikzcd}$$

where the isomorphism is the one from Exercise 44.11(g), namely $a[\pm M] + b[\pm N] + cA \mapsto ((\pm_M \sqrt{a}, \pm_N \sqrt{b}, \sqrt{c}), A)$. This requires one more comment, because our formula from $(**)$ does not exactly match the formula for $\mathcal{B}$ from (44.3) due to the presence of the i's in the final matrix. However, by Remark 44.6 $\mathcal{B}$ is homotopic to the version with the i's.

Exercise 44.14. For those who prefer algebra to geometry, check that the functions $r(t) = \sqrt{1 - t}$, $s(t) = \sqrt{t}$, and

$$r_2(u) = \begin{cases} \sqrt{1 - 2u} & \text{if } 0 \le u \le \frac{1}{2}, \\ -\sqrt{2u - 1} & \text{if } \frac{1}{2} \le u \le 1 \end{cases}, \qquad s_2(u) = \sqrt{1 - |2u - 1|}$$

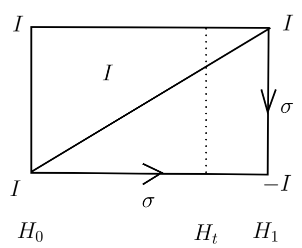
satisfy our various requirements and turn formula $(*)$ exactly into formula $(**)$.

We can now complete the last step of the proof:

(11) To see commutativity of the bottom pentagon in our large diagram we expand the picture as follows:

$$\begin{tikzcd}
\pi_{r+1}\Sigma U_n \arrow[r, "\Theta_*"] \arrow[dr, "\Theta'_*"'] \arrow[ddrrr, bend right, "\Sigma"] & \pi_{r+1}\,\mathrm{Gr}_n(\mathbb{C}^{2n}) \arrow[r, "\tilde{\beta}_*"] & \pi_{r+1}\Omega_I SU_{2n} \arrow[r, equal] & \pi_{r+2}SU_{2n} \\
 & \pi_{r+1}X \arrow[u, "\cong"] \arrow[r, "\beta_*"] & \pi_{r+1}\Omega_{I,-I}SU_{2n} \arrow[u, "\cong"', "K_*"] \arrow[ur, "\mathrm{Adj}", "\cong"'] & \\
 & & & \pi_{r+2}(S^0 * S^0 * U_n) \arrow[uu, shift right, "\mathrm{adj}(\beta\Theta')_*"] \arrow[uu, shift left, "\mathcal{B}_*"']
\end{tikzcd}$$

Here $K\colon \Omega_{I,-I}SU(2n) \to \Omega_I SU(2n)$ is the map that right-multiplies a loop by the canonical path $\sigma\colon t \mapsto \mathrm{diag}(e^{-\pi it}, \dots, e^{-\pi it}, e^{\pi it}, \dots, e^{\pi it})$, and Adj is the map that sends $S^{r+1} \to \Omega_{I,-I}SU_{2n}$ to its adjoint $S^{r+2} \cong S^0 * S^{r+1} \to SU_{2n}$ (see Exercise 44.11(h)) where the basepoint of the domain is the north pole. The upper left triangle and the upper rectangle commute by definition. The upper right triangle is checked to commute as follows. Let H_t be the homotopy from the constant path at I to σ rel the initial point shown in the following diagram:



(on the top triangle this is constant and on the lower one it is—looked at the right way—the homotopy showing that σ contatenated with its reverse is homotopic to the constant path). If $\alpha\colon S^{r+1} \to \Omega_{I,-I}SU_{2n}$ then $\mathrm{Adj}(\alpha)\colon S^0 * S^{r+1} \to SU_{2n}$ is the map that for each $x \in S^{r+1}$ sends the path "suspension of x" (from north to south pole) to $\alpha(x)$. Let $\mathrm{Adj}(\alpha)_t\colon S^0 * S^{r+1} \to SU_{2n}$ be the similar map that sends the path "suspension of x" to $\alpha(x) \cdot H_t$, where the multiplication is pointwise, using the group structure on SU_{2n}. This gives a homotopy, pointed at the north pole, from $\mathrm{Adj}(\alpha)_0 = \mathrm{Adj}(\alpha)$ to $\mathrm{Adj}(\alpha)_1 = K_*(\alpha)$.

The map $\mathrm{adj}(\beta\Theta')$ is the "adjoint" of the composite $\beta\Theta'$, and the bottom pentagon containing this map commutes by Exercise 44.11(h). Finally, the maps $\mathrm{adj}(\beta\Theta')$ and $\mathcal{B}$ are homotopic by (44.13), and hence induce the same map on homotopy groups.

◦ Exercises ◦

Summary: The main goal of this section was the very concrete version of Bott periodicity for U given by Theorem 44.8. The proof was a longish chase through the Bott periodicity map that comes out of Morse theory, but it was mostly tedious rather than having any groundbreaking ideas. Attempting to do something similar for O, however, presents some real challenges. To do this in a navigable manner will require us to step back and develop some more structure to help organize all that is happening.

For versions of Theorem 44.8 in the literature see ?????.

45. Pinning down Bott periodicity, Part 2

In the last section we uncovered a very down-to-earth version of Bott periodicity. We produced a family of maps $\mathcal{B}\colon (S^2_{\text{upp}} \times U(n))/\sim \longrightarrow SU(2n)$, with the domain a model for $\Sigma^2 U(n)$ and given by very explicit matrix formulas. Then we proved in Theorem 44.8 that the composition

$$\pi_r U(n) \xrightarrow{\Sigma^2} \pi_{r+2}\Sigma^2 U(n) \xrightarrow{\mathcal{B}_*} \pi_{r+2} SU(2n)$$

is an isomorphism for $r < 2n$ and is compatible with the Bott map β. The composition is also denoted $\mathcal{B}$. This theorem opens the door to several questions:

(1) We can now produce explicit generators for the groups $\pi_* U$ by starting with known generators in low dimensions and applying $\mathcal{B}$. What do these generators look like?
(2) The generators of $\pi_* U$ translate into generators for $\pi_*(BU) = K^{-*}(pt)$. Can we use these explicit descriptions to understand the ring structure?
(3) The construction of the map $\mathcal{B}$ was somewhat ad hoc. How do we better understand it?
(4) How do we generalize all of this to O instead of U?

Many of these questions are connected to each other. Our goal in this section is to answer them as best as we can.

45.1. **Generators for $\pi_* U$.** Let us use $\mathcal{B}$ to produce some interesting elements in the homotopy of the unitary groups. Start with the standard generator $f_1 \in \pi_1 U(1)$, namely the map $(x, y) \in S^1 \mapsto x + iy$. Then $\mathcal{B} f_1$ is the map $S^3 \to SU(2)$ that sends (z, x, y) to $\begin{bmatrix} z & -x+iy \\ x+iy & \bar{z} \end{bmatrix}$. Note that this could also be described using only complex coordinates, as

$$f_3\colon (z, w) \mapsto \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix},$$

which is the standard isomorphism from S^3 to $SU(2)$. Applying $\mathcal{B}$ one more time gives the map $f_5 \in \pi_5 SU(4)$ with formula

$$f_5\colon (u, z, w) \mapsto \begin{bmatrix} u & 0 & -\bar{z} & -\bar{w} \\ 0 & u & w & -z \\ z & -\bar{w} & \bar{u} & 0 \\ w & \bar{z} & 0 & \bar{u} \end{bmatrix}. \tag{45.2}$$

Clearly it is a triviality to keep writing these down in higher and higher dimensions to produce explicit formulas for $f_{2n+1} \in \pi_{2n+1} SU(2^n)$.

This procedure produces generators for $\pi_{2s+1} U$ that come from $\pi_{2s+1} U(2^s)$. But the stability isomorphisms shows that in fact a generator can be lifted all the way to $\pi_{2s+1} U(s+1)$. This is a substantial difference! For example, the above technique lets us write down an explicit lifting of the generator for $\pi_{15} U$ to an element of $\pi_{15} U(128)$, when in fact we know that a lifting actually exists into $\pi_{15} U(8)$.

One thing that is notable about the generators produced by Theorem 44.8 is that they are *linear*. For example, f_3 is given by the formula

$$(z_0, z_1, w_0, w_1) \mapsto z_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + w_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + w_1 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{45.3}$$

Liftings of these generators into smaller $U(n)$'s tend not to be linear anymore (in fact, we will see in Corollary 45.9 that they are *never* linear). For example, here is a lifting of f_5 into $\pi_5 SU(3)$:

$$(u, z, w) \mapsto \begin{bmatrix} u - \bar{w}z & \bar{w}^2 & -\bar{z} - \bar{w}\bar{u} \\ z^2 & u + z\bar{w} & w - z\bar{u} \\ w + \bar{u}z & \bar{z} - \bar{u}\bar{w} & \bar{u}^2 \end{bmatrix}. \tag{45.4}$$

How did we get this map? There is an interesting technique from [Lu], later refined and simplified in [PR], that we will now describe. Let $X \subseteq SU(d)$ be the subspace of matrices A having $a_{dd} = 0$. These are unitary transformations having the property that $A(e_d)$ lies in $\langle e_d \rangle^\perp$. Set $b = A(e_d)$. Under these circumstances we can follow A with the rotation in the plane $\langle e_d, b\rangle$ that rotates b back to e_d and e_d to $-b$. In this way we deform the original transformation into one that fixes e_d, and therefore lives in $SU(d-1)$. Here is a more precise statement:

Proposition 45.5. *There is a homotopy $H\colon X \times I \to SU(d)$ where H_0 is the inclusion and H_1 factors through $SU(d-1)$, given by*

$$H_t\colon \begin{bmatrix} E & b \\ c & 0 \end{bmatrix} \mapsto \begin{bmatrix} E - bc\sin(\frac{\pi}{2}t) & b\cos(\frac{\pi}{2}t) \\ c\cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \end{bmatrix}.$$

Proof. On one level the proof is just a computation: one simply checks that the given formula always gives a matrix in $SU(d)$, and then the stated properties are immediate. Checking that the formula gives a unitary matrix is routine algebra, though verifying that the determinant is 1 is a little trickier.

The result is clearer when we interpret the given formula as post-composing the original unitary transformation $A = \begin{bmatrix} E & b \\ c & 0 \end{bmatrix}$ with a rotation. The fact that A is unitary implies that $E(e_1), \dots, E(e_{d-1})$ are orthogonal to $A(e_d) = b$. Write R_t for the rotation that fixes $\langle e_d, b\rangle^\perp$ and sends $e_d \mapsto \cos(\frac{\pi}{2}t)e_d - \sin(\frac{\pi}{2}t)b$ and $b \mapsto \sin(\frac{\pi}{2}t)e_d + \cos(\frac{\pi}{2}t)b$. Then R_t clearly lies in $SU(d)$. The composite R_tA has the following behavior:

$$\begin{aligned} \text{For } 1 \le i \le d-1\text{:} \quad & e_i \mapsto Ee_i + c_ie_d \mapsto Ee_i + c_i(\cos(\tfrac{\pi}{2}t)e_d - \sin(\tfrac{\pi}{2}t)b) \\ & e_d \mapsto b \mapsto \cos(\tfrac{\pi}{2}t)b + \sin(\tfrac{\pi}{2}t)e_d. \end{aligned}$$

Note that this exactly reproduces the matrix formula in the statement of the proposition. $\square$

If a map $f\colon S^r \to SU(d)$ factors through X, Proposition 45.5 gives an explicit way to homotop it down into $SU(d-1)$. The condition about factoring through X seems rather specialized, but anytime there is a fixed matrix entry which vanishes for the entire image of f then there is a way to move that entry to the (d,d)-spot via rotations, so that in fact we land back inside of X. For example, start with the formula for f_5 in (45.2) and note the 0 in the $(3,4)$-entry. Left-multiply with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ 0 & 0 & \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{bmatrix}$$

to get a homotopy between the f_5-map and the similar map where the last two rows have been switched and the (new) third row negated. Now we have the zero in the $(4,4)$-entry, and applying the formula from Proposition 45.5 spits out (45.4).

Exercise 45.6. Start with the formula for f_5 in (45.2) and apply $\mathcal{B}$ to get a linear element $f_7 \in \pi_7 SU(8)$. You will notice that the formula has many zeros in it. Convince yourself that after using the above method to reduce to an element in $\pi_7 SU(7)$ with quadratic entries one can apply the method *again* to reduce to an element in $\pi_7 SU(6)$ with quartic entries. Challenge: For this example, how long can one continue this process?

45.7. **Linear homotopy classes.** The formula of (45.3) is very reminiscent of similar formulas we saw when talking about vector fields on spheres and the Hurwitz-Radon problem, as well as in the (related) theory of Clifford modules. This is, of course, no surprise since Clifford modules are somehow fundamental to Bott periodicity. But let us tease this out a bit more from an elementary perspective.

Say that a map $S^r \to U(n)$ is **linear** if it has the form

$$(x_0, \dots, x_r) \mapsto x_0 A_0 + x_1 A_1 + \cdots + x_r A_r$$

for some $A_0, \dots, A_r \in U(n)$. Note that given a linear map like this we can always left-multiply by A_0^{-1} and obtain a related map where $A_0 = Id$. From now on we will always assume that linear maps have been normalized in this way.

Proposition 45.8. *The elements $A_0, \dots, A_r \in U(n)$, with $A_0 = Id$, give a linear map if and only if $A_i^2 = -I$ and $A_i A_j = -A_j A_i$ for all $1 \le i < j \le r$. That is, the matrices $A_1, \dots, A_r$ give a unitary representation of the complex Clifford algebra $\mathrm{Cl}_r(\mathbb{C})$ on $\mathbb{C}^n$. [And the same result holds with $U(n)$ replaced by $O(n)$ and the complex Clifford algebras replaced by the real ones.]*

Proof. The "if" direction is a simple verification that we leave to the reader. For the "only if" direction, assume we have a linear map $S^r \to U(n)$. Then, in particular, $x_0 I + x_1 A_1 \in U(n)$ whenever $x_0^2 + x_1^2 = 1$. So

$$I = (x_0 I + x_1 A_1)(x_0 I + x_1 A_1^\dagger) = (x_0^2 + x_1^2)I + x_0 x_1 (A_1 + A_1^\dagger),$$

and since this holds whenever $x_0^2 + x_1^2 = 1$ we conclude that $A_1 + A_1^\dagger = 0$. Then $A_1^{-1} = A_1^\dagger = -A_1$, so $A_1^2 = -I$. Similarly, $A_i^2 = -I$ for $i \ge 1$.

By the same kind of analysis one uses that $x_i A_i + x_j A_j \in U(n)$ whenever $x_i^2 + x_j^2 = 1$ to conclude that $A_i A_j^\dagger + A_j A_i^\dagger = 0$. Then use that $A_i^\dagger = A_i^{-1} = -A_i$ (and similarly for A_j) to conclude that A_i and A_j anti-commute. □

We will call a collection of matrices $A_1, \dots, A_r$ satisfying the above conditions a set of **Hurwitz-Radon matrices**, or equivalently a set of **Clifford matrices**.

Corollary 45.9. *There exists a linear map $S^r \to U(n)$ if and only if n is a multiple of $2^{\lfloor \frac{r}{2} \rfloor}$. Likewise, there is a linear map $S^r \to O(n)$ if and only if n is a multiple of $2^{\sigma(r)}$ where $\sigma(r) = \#\{s \,|\, 0 < s \le r$ and $s \equiv 0, 1, 2,$ or $4 \bmod 8\}$.*

Proof. This follows immediately from the representation theory of the Clifford algebras. All of the irreducible modules for Cl_r have the same dimension: $2^{\lfloor \frac{r}{2} \rfloor}$ in the complex case, and $2^{\sigma(r)}$ in the real case. Since the representation theory is semisimple, the dimension of any representation will be a sum of the dimensions of the irreducibles. □

By just checking that the dimensions line up, we see from Corollary 45.9 that the linear maps $S^r \to U(n)$ produced by the $\mathcal{B}$ map—as in the discussion after Theorem 44.8—are the best possible. For example, the linear map $S^5 \to SU(4)$ cannot be lifted to a linear map into $SU(3)$, even though it does lift to a non-linear map into $SU(3)$ as we have seen. (We have not yet considered the orthogonal case but will do so shortly!)

We now understand that a unitary $\mathrm{Cl}_r(\mathbb{C})$-action on $\mathbb{C}^n$ gives a linear map $f\colon S^r \to U(n)$. If the action extends to a $\mathrm{Cl}_{r+1}(\mathbb{C})$-action then f extends to $S^{r+1} \to U(n)$, which means that f was null-homotopic (because the inclusion of S^r into S^{r+1} is null). We want to systemize this way of passing from Clifford modules to linear homotopy classes in order to get a better handle on it.

Let G_r be the Grothendieck group of finite-dimensional unitary $\mathrm{Cl}_r(\mathbb{C})$-modules. We can also make this construction for $\mathrm{Cl}_r(\mathbb{R})$, and when necessary we will write $G_r(\mathbb{C})$ and $G_r(\mathbb{R})$ to distinguish between the complex and real cases. The G_r groups are all free abelian with generators corresponding to isomorphism classes of irreducibles, since the representation theory of the Clifford algebras is semisimple.

The inclusion $\mathrm{Cl}_r(\mathbb{C}) \hookrightarrow \mathrm{Cl}_{r+1}(\mathbb{C})$ gives rise to a forgetful map from $\mathrm{Cl}_{r+1}(\mathbb{C})$-modules to $\mathrm{Cl}_r(\mathbb{C})$-modules, which in turn gives a map of abelian groups $G_{r+1} \to G_r$. Let $\mathcal{A}_r$ denote the cokernel. Our next goal will be to construct group homomorphisms

$$\mathcal{A}_r(\mathbb{C}) \to \pi_r U, \qquad \mathcal{A}_r(\mathbb{R}) \to \pi_r O.$$

Let M be a finite-dimensional unitary $\mathrm{Cl}_r(\mathbb{C})$-module (recall this means that M comes equipped with a Hermitian inner product and that the Clifford generators act as unitary transformations). By picking a unitary basis $\epsilon_1, \ldots, \epsilon_n$ for M we can represent each Clifford generator as acting by an element of $U(n)$, and thereby get a linear map $f_M\colon S^r \to U(n)$ as above. This depends on our choice of basis, though. Another choice of basis would change this to $P \cdot f_M \cdot P^{-1}$, where $P \in U(n)$ is the change-of-basis matrix. Fortunately, $U(n)$ is connected: a choice of path from Id to P will give a basepoint-preserving homotopy from f_M to Pf_MP^{-1}, and so we find that the element $f_M \in \pi_r U(n)$ is independent of the choice of basis.

The analogous argument in the real case requires an extra step, because of the fact that the groups $O(n)$ are not connected. The matter again reduces to comparing a map $f_M\colon S^r \to O(n)$ to a conjugate Pf_MP^{-1} where $P \in O(n)$. There is a path in $O(n)$ from P to either Id or R, where R is the reflection $\mathrm{diag}(-1, 1, 1, \ldots, 1)$, and as before such a path gives a homotopy. So we reduce to comparing f_M to Rf_MR^{-1}. We can re-interpret conjugation by R as conjugation by $-R$, and if n is odd then $-R$ has determinant 1 and so is path-connected to Id. So if n is odd then we are okay. But if n is even then we can compose with the inclusion $O(n) \hookrightarrow O(n+1)$ to reduce to the odd case. So the conclusion here is that, while $f_M \in \pi_r O(n)$ might depend on the choice of basis, the image in $\pi_r O$ does not.

Proposition 45.10. *$f_{M\oplus N} = f_M + f_N$ in $\pi_r U$ (or $\pi_r O$ for the real case). Consequently, we obtain maps of abelian groups $G_r(\mathbb{C}) \to \pi_r(U)$ and $G_r(\mathbb{R}) \to \pi_r(O)$ as well as induced maps $\mathcal{A}_r(\mathbb{C}) \to \pi_r(U)$ and $\mathcal{A}_r(\mathbb{R}) \to \pi_r(O)$.*

Proof. This boils down to checking that if $f, g\colon S^r \to U(n)$ then the product $fg\colon x \mapsto f(x)g(x)$ is homotopic to

$$f \oplus g\colon x \mapsto \begin{bmatrix} f(x) & 0 \\ 0 & g(x) \end{bmatrix}$$

when regarded as maps $S^r \to U(2n)$ (in the first case, via the inclusion $U(n) \hookrightarrow U(2n)$). That statement in turn follows from the claim that the two maps $U(n) \to U(2n)$ given by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \qquad A \mapsto \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \tag{45.11}$$

are homotopic. To justify this claim, consider the path in $U(2n)$ given by

$$t \mapsto \begin{bmatrix} \cos(t)I & -\sin(t)I \\ \sin(t)I & \cos(t)I \end{bmatrix}$$

from $t = 0$ to $t = \frac{\pi}{2}$. Left-multiplication by this path shows that the identity on $U(2n)$ is homotopic to the map $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}$. Right-multiplication by the path gives a similar homotopy between the identity and a map that involves switching column blocks instead of row blocks. Applying these two homotopies in succession gives the desired homotopy between the two maps of (45.11).

Check O case and basepoints. □

At this point we have constructed maps of graded groups $\mathcal{A}_*(\mathbb{C}) \to \pi_*(U)$ and $\mathcal{A}_*(\mathbb{R}) \mapsto \pi_* O$. We want to next understand how periodicity of the two sides interact. In each case, periodicity stems from a certain kind of multiplication that we now explain.

Let us first observe a clever "doubling" construction that one can apply to Clifford modules. If M is a Cl_r-module then consider the real vector space $M \oplus M$. This comes equipped with the map $f = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ which has $f^2 = -I$. Check that f anticommutes with $\left[\begin{smallmatrix} 0 & A \\ uA & 0 \end{smallmatrix}\right]$ (u a scalar constant) if and only if $u = 1$, and that f anticommutes with $\left[\begin{smallmatrix} B & 0 \\ 0 & uB \end{smallmatrix}\right]$ if and only if $u = -1$. This gives us a couple of ways of regarding $M \oplus M$ as a Cl_{r+1}-module, via the formulas $e_i \mapsto \left[\begin{smallmatrix} 0 & e_i \\ e_i & 0 \end{smallmatrix}\right]$, $e_{r+1} \mapsto f$ or via the formulas $e_i \mapsto \left[\begin{smallmatrix} e_i & 0 \\ 0 & -e_i \end{smallmatrix}\right]$, $e_{r+1} \mapsto f$.

Suppose given a Cl_r-module M and a Cl_s-module N. Can we merge these two actions into something like a Cl_{r+s}-module? Write the generators of Cl_r as $e_1, \dots, e_r$ and write the generators of Cl_s as $g_{r+2}, \dots, g_{r+s+1}$ (the strange indexing will make more sense in a moment) . We need to construct a new module where the e's and g's act simultaneously and anticommute. If we had said "commute" rather than "anticommute", there would be a standard solution: take $M \otimes N$ and consider the actions given by $e_i \otimes \mathrm{id}_N$ and $\mathrm{id}_M \otimes g_j$. To get anticommutativity instead we can take the double $(M \otimes N) \oplus (M \otimes N)$ and intertwine with the f-map in the following way:

$$\begin{aligned} e_i &\mapsto \begin{bmatrix} e_i \otimes \mathrm{id}_N & 0 \\ 0 & -e_i \otimes \mathrm{id}_N \end{bmatrix} \quad \text{for } 1 \le i \le r \\ e_{r+1} &\mapsto \begin{bmatrix} 0 & \mathrm{id}_M \otimes \mathrm{id}_N \\ -\mathrm{id}_M \otimes \mathrm{id}_N & 0 \end{bmatrix} \\ e_j &\mapsto \begin{bmatrix} 0 & \mathrm{id}_M \otimes g_j \\ \mathrm{id}_M \otimes g_j & 0 \end{bmatrix} \quad \text{for } r+2 \le j \le r+s+1. \end{aligned}$$

Note that we actually get a Cl_{r+s+1}-action in this way, not just a Cl_{r+s}-action. Write $M \otimes_c N$ for $(M \otimes N) \oplus (M \otimes N)$ with this Cl_{r+s+1}-action. We will call it the "Clifford tensor product".

Remark 45.12. If the Clifford algebra Cl_r is thought of as containing generators $e_0, e_1, \ldots, e_r$ with $e_0 = 1$, then the generators for the two Clifford algebras Cl_r and Cl_s merge to give

$$e_0, \ldots, e_r, e_{r+1}, \ldots, e_{r+s+2}$$

with e_{r+i} corresponding to e_i in Cl_s. In particular, e_{r+1} corresponds to the e_0 in Cl_s. Keeping this in mind helps parse the formulas in the definition of $M \otimes_c N$.

Remark 45.13. There were choices made of where to put the signs in the matrices defining the Clifford action on the tensor product. One is of course free to multiply any of the matrices by -1 and this produces a different module. The sign choices for the first and third sets of matrices seem reasonably "natural", but the sign on the e_{r+1} matrix seems arbitrary (in fact, thinking about the construction of $\mathbb{C}$ from $\mathbb{R}$ even suggests the opposite sign from ours). There is a good reason for this particular choice of sign, though: it is the one that makes Proposition 45.16(iv) work out. We do not claim this to be clear indication that the universe really wants this particular sign choice, but it was the deciding factor for us.

The Clifford tensor product induces a multiplication on the groups G_*, with the unfortunate grading $G_s \otimes G_t \to G_{s+t+1}$ (if this really bothers you, keep heart because we will eventually repair this oddity). It likewise induces pairings $\mathcal{A}_s \otimes \mathcal{A}_t \to \mathcal{A}_{s+t+1}$. We want to use this extra information to help understand the structure of the $\mathcal{A}_*$ groups. To this end, we need notation for talking about the irreducible Cl_n-modules before we go to the next stage of computing tensor products. When n is even there is a unique (up to isomorphism) irreducible Cl_n-module, so there are no issues there: let us just denote it F_n. But when n is odd there are two non-isomorphic irreducibles, which we will call $F_{n,+}$ and $F_{n,-}$. In order to explain what we mean by these, let us recall how Clifford modules work via the following diagram:

$$
\begin{tikzcd}
V \arrow[r, dash] \arrow[dr, dash] & V_-^{e_1} & & & & & \\
& V_+^{e_1} \arrow[u, bend right, "e_2"'] \arrow[rr, dash] \arrow[drr, dash] & & V_{+-}^{e_1,e_2e_3} & & & \\
& & & V_{++}^{e_1,e_2e_3} \arrow[u, bend right, "e_2e_4"'] \arrow[rr, dash] \arrow[drr, dash] & & V_{++-}^{e_1,e_2e_3,e_4e_5} & \\
& & & & & V_{+++}^{e_1,e_2e_3,e_4e_5} \arrow[u, bend right, "e_4e_6"'] & \\
0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

Here are the instructions for parsing this:

- A Cl_0-module is just a complex vector space V.
- To make V into a Cl_1-module one must give the action of e_1. But since $e_1^2 = -1$, the action will be diagonalizable and will split V into the $+i$ and $-i$ eigenspaces, denoted $V_+^{e_1}$ and $V_-^{e_1}$ (sometimes we will drop the superscripts for convenience).
- If the Cl_1-action extends to a Cl_2-action then since e_2 anti-commutes with e_1 it will have to interchange the two eigenspaces V_+ and V_-. In fact, mutiplication by e_2 will have to give an isomorphism $V_+ \to V_-$ and so the Cl_1-action extends if and only if V_+ and V_- have the same dimension.
- If the Cl_2-action extends to a Cl_3-action, then since e_2e_3 commutes with e_1 it will perserve each of the V_+ and V_- eigenspaces. But $(e_2e_3)^2 = -1$ so it will decompose V_+ (for example) into $+i$ and $-i$ eigenspaces, denoted here V_{++} and V_{+-}. Note that there will be a similar decomposition of V_-, but we don't need to include that because we obtain it from the V_+-decomposition via multiplication by e_2.
- To extend to a Cl_4-action, the element e_2e_4 commutes with e_1 and so will have to preserve V_+; but it anticommutes with e_2e_3 and so will interchange V_{++} and V_{+-} and give an isomorphism between them.
- At the next stage, e_4e_5 commutes with e_1 and e_2e_3 and so will preserve the V_{++} eigenspace, and since $(e_4e_5)^2 = -1$ it splits it into $+i$ and $-i$ eigenspaces.
- At this point the pattern is clear: at each stage one writes down a certain element e_je_k which commutes with the previously chosen elements, or else anticommutes with the one immediately preceding it. In the first case the element splits a previous eigenspace into two, whereas in the second case it gives an isomorphism between the two previous eigenspaces.
- Note that the diagram has been set up so that columns 0 through n show the information that constitutes a Cl_n-module, with the understanding that the curved isomorphisms all lie in the column containing their label. And when we say "show", of course the diagram only shows the splittings for

the $V_{+\cdots+}$ eigenspaces, since the splittings of the other eigenspaces can be obtained from these.

One can reverse the above decomposition process to *construct* Clifford modules very explicitly. Briefly, one starts with some chosen information at level n and then lets this freely generate the rest of the module by moving to the left in the diagram. For example, a Cl_3-module is determined by the choice of V_{++} and V_{+-}. Let $F_{3,+}$ be the module where $V_{++} = \mathbb{C}$ and $V_{+-} = 0$ (and $F_{3,-}$ the opposite). Note that these two spaces determine V_+, as it is their direct sum, which then determines V_- (as it is e_2V_+), which then determines V. So for example, we see that V is 2-dimensional in this case. For a Cl_4-module we would specify V_{++}, then set $V_{+-} = e_2e_4V_{++}$, and then continue as above—this produces a 4-dimensional Cl_4-module, and it is clear that when we restrict to Cl_3 we get $F_{3,+} \oplus F_{3,-}$.

Exercise 45.14. The module $F_{3,+}$ can be described by letting x be a generator of V_{++}, so that $e_1x = e_2e_3x = ix$. Check that x, e_2x gives a basis for V and that $e_3x = -ie_2x$. Verify that $e_1e_2e_3$ acts on $F_{3,+}$ as multiplication by -1. Then do this for $F_{3,-}$ and show that $e_1e_2e_3$ acts as the identity instead.

Next do something similar for $F_{5,+}$: if x is a generator for V_{+++} show that x, e_2x, e_4x, and e_2e_4x form a basis for V, and compute the e_j multiplications on these basis elements for all j. Check that $e_1e_2e_3e_4e_5$ acts on $F_{5,+}$ as multiplication by $-i$. Likewise, check that for $F_{3,-}$ the action is by $+i$.

It is often the case that one has a Cl_{2n+1}-module M and by a dimension count one knows it is isomorphic to either $F_{2n+1,+}$ or $F_{2n+1,-}$; but how do we decide which one? The above exercise gives a clue. Note that $(e_1e_2\cdots e_{2n+1})^2 = (-1)^{n+1}$ and so multiplication by $e_1e_2\cdots e_{2n+1}$ will have eigenvalues $\pm i^{n+1}$ (that is, ± 1 when n is odd and $\pm i$ when n is even).

Proposition 45.15. *The element $e_1e_2\cdots e_{2n+1}$ acts as multiplication by i^{n+1} on $F_{2n+1,+}$ and as multiplication by $-i^{n+1}$ on $F_{2n+1,-}$.*

Proof. Set $V = F_{2n+1,+}$ and let $\alpha = e_1e_2\cdots e_{2n+1}$. If x is a generator for $V_{++\cdots+}$ then a $\mathbb{C}$-basis for V is given by the set of elements $e_{2j_1}e_{2j_2}\cdots e_{2j_k}x$ for $j_1 < j_2 < \cdots < j_k$ (exercise). Each Clifford generator e_r commutes with α, therefore if we prove $\alpha x = \lambda x$ for some $\lambda \in \mathbb{C}$ then α acts by multiplication by λ on all of V.

Now we just observe that $\alpha = (1\cdot e_1)(e_2e_3)(e_4e_5)\cdots(e_{2n}e_{2n+1})$ and each pair in parentheses acts on x as multiplication by i. So $\alpha x = i^{n+1}x$, as desired.

In the case of $V = F_{2n+1,-}$ we have $x \in V_{++\cdots+-}$ and so each pair in the above decomposition of α acts on x as multiplication by i except for the final pair, which acts as multiplication by $-i$. □

With these considerations in mind we can proceed to analyze some tensor products:

Proposition 45.16. *One has isomorphisms*

(i) $F_0 \otimes_c F_{2n} \cong F_{2n+1,+} \oplus F_{2n+1,-} \cong F_{2n} \otimes_c F_0$
(ii) $F_0 \otimes_c F_{2n+1,+} \cong F_{2n+2} \cong F_{2n+1,+} \otimes_c F_0$
(iii) $F_0 \otimes_c F_{2n+1,-} \cong F_{2n+2} \cong F_{2n+1,-} \otimes_c F_0$
(iv) $F_{1,+} \otimes_c F_{2n+1,\pm} \cong F_{2n+3,\pm}$
(v) $F_{1,-} \otimes_c F_{2n+1,\pm} \cong F_{2n+3,\mp}$.

Before proving all of these isomorphisms let us give the immediate consequence:

Corollary 45.17. *The group $\mathcal{A}_{2n+1}$ is generated by the element $(F_{1,+})^n$.*

Proof. We know that $\mathcal{A}_{2n+1}$ is generated by $F_{2n+1,+}$ (one could also use $F_{2n+1,-}$ here, since they are negatives of each other). Proposition 45.16(iv) shows that this module is isomorphic to the n-fold Clifford tensor product of $F_{1,+}$ with itself. $\square$

Proof of Proposition 45.16. Note that F_{2n} has dimension 2^n as a $\mathbb{C}$-vector space, as does $F_{2n+1,\pm}$. Statements (ii) and (iii) are easy, since even Clifford algebras have exactly one irreducible and so the isomorphism class of a module is determined by its dimension over $\mathbb{C}$. So, for example, $F_0 \otimes_c F_{2n+1,+}$ and F_{2n+2} both have dimension 2^{n+1} and hence are isomorphic.

For $F_0 \otimes_c F_{2n}$, the dimension over $\mathbb{C}$ is 2^{n+1} and so the module can only be isomorphic to one of $F_{2n+1,+} \oplus F_{2n+1,+}$, $F_{2n+1,+} \oplus F_{2n+1,-}$, and $F_{2n+1,-} \oplus F_{2n+1,-}$. But note that the Cl_0-action on F_0 extends to a Cl_1-action, and so the Cl_{2n+1}-action on $F_0 \otimes_c F_{2n}$ extends to a Cl_{2n+2}-action. That is, the tensor product is in the image of $G_{2n+2} \to G_{2n+1}$. This is enough to identify it is as the second of the above options.

Finally, we tackle part (iv) (the proof for (v) is similar). A dimension calculation shows that the Cl_{2n+3}-module $F_{1,+} \otimes_c F_{2n+1,+}$ must be isomorphic to either $F_{2n+3,+}$ or $F_{2n+3,-}$, so we use Proposition 45.15 to decide which one. The action of the Clifford generators on the tensor product is given by the matrices

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & e_1 \\ e_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & e_2 \\ e_2 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & e_{2n+1} \\ e_{2n+1} & 0 \end{bmatrix}$$

and the product of these matrices is $\alpha = \left[\begin{smallmatrix} ie_1\cdots e_{2n+1} & 0 \\ 0 & ie_1\cdots e_{2n+1} \end{smallmatrix}\right]$. But $e_1 \cdots e_{2n+1}$ acts on $F_{2n+1,+}$ as multiplication by i^{n+1}, so the action of α on $F_{1,+} \otimes_c F_{2n+1,+}$ is multiplication by i^{n+2}. This shows we must have $F_{1,+} \otimes_c F_{2n+1,+} \cong F_{2n+3,+}$, and the analysis of $F_{1,+} \otimes_c F_{2n+1,-}$ proceeds similarly. $\square$

We are in the midst of analyzing $\mathcal{A}_* \to \pi_* U$, and we have now introduced and understood a ring structure on the domain. The same types of formulas lead to a ring structure on the target. Indeed, for $f\colon S^s \to U(n)$ and $g\colon S^t \to U(q)$ define $f * g\colon S^{s+t+1} \to U(2nq)$ to be given by

$$(f*g)(u,x) = \begin{bmatrix} |u|f(\hat{u}) \otimes I & I \otimes |x|g(\hat{x}) \\ -I \otimes |x|g(\hat{x})^\dagger & |u|f(\hat{u})^\dagger \otimes I \end{bmatrix}$$

for $(u,x) \in \mathbb{R}^{s+1} \oplus \mathbb{R}^{t+1}$ such that $|u|^2 + |x|^2 = 1$. Here we have written $\hat{u} = \frac{u}{|u|}$ etc, and are interpreting the matrix as a transformation of $(\mathbb{C}^n \otimes \mathbb{C}^q) \oplus (\mathbb{C}^n \otimes \mathbb{C}^q)$. One readily computes that the transformation is unitary.

It is clear that $f * g$ only depends on the homotopy classes $f \in \pi_s U(n)$ and $g \in \pi_t U(q)$, since for example a homotopy from f to f' could be essentially substituted into the above formula to give a homotopy from $f * g$ to $f' * g$. It is also clear that the definition of $f * g$ is compatible with stabilization, so that it induces a pairing $\pi_s U \times \pi_t U \to \pi_{s+t+1} U$. To check bilinearity use the model of addition in $\pi_* U$ which is induced from block sum of matrices.

The definitions have been set up so that it follows immediately that $\mathcal{A}_* \to \pi_* U$ is a ring map (and similarly for the orthogonal case). [CHECK THIS]

A short summary of our situation is:

- From algebraic computation we know all of the groups $\mathcal{A}_*$, the ring structure, and the fact that multiplication by the generator in $\mathcal{A}_1$ gives periodicity isomorphisms.
- If we had reason to know that $\mathcal{A}_* \to \pi_* U$ is an isomorphism of groups then we would deduce Bott periodicity, but unfortunately we have no a priori explanation of that fact.
- We do know the groups $\pi_* U$ by the Bott-Morse arguments from previous sections. The content of Theorem 44.8 shows that in the ring $\pi_* U$ multiplication by the generator in degree 1 gives a periodicity isomorphism. It then *follows* from algebra that $\mathcal{A}_* \to \pi_* U$ is an isomorphism in all degrees. We record this important fact for later use:

Theorem 45.18. *The map $\mathcal{A}_* \to \pi_* U$ is an isomorphism of graded rings.*

Proof. Follows from Theorem 44.8 by the above reasoning. $\square$

Let us next take a look at the orthogonal case, starting by determining the ring structure on $G_*(\mathbb{R})$. Much of the hard work can be avoided by using the complexification map $G_*(\mathbb{R}) \to G_*(\mathbb{C})$ (which is a ring homomorphism) together with what we already know about the codomain, but we still need to get our hands dirty in a few cases.

Proposition 45.19. *Generators of $G_*(\mathbb{R})$ and $G_*(\mathbb{C})$, together with the behavior of the complexification map on these generators, are as shown in the following table. The bracketed formulas on the left and right give relations in the two rings.*

$$
\begin{array}{cccc}
 & G_*(\mathbb{R}) & G_*(\mathbb{C}) & \\
 & x(1) \xrightarrow{\qquad} & a(1) & \\
 & x^2(2) \xrightarrow{\ \Delta\ } & b_+, b_-(1) & [a^2 = b_+ + b_-] \\
 & x^3(4) \xrightarrow{\ 2\ } & ab_+ & [ab_- = ab_+] \\
{[x^4 = y_+ + y_-]} & y_+, y_-(4) \xrightarrow{\ 2,2\ } & b_+^2, b_+b_- & [b_-^2 = b_+^2] \\
{[xy_+ = xy_-]} & xy_+ \xrightarrow{\ 2\ } & ab_+^2 & \\
{[x^2y_+ = 2g_5]} & g_5(8) \xrightarrow{\ \Delta\ } & b_+^3, b_+^2 b_- & \\
{[xg_5 = 2g_6]} & g_6(8) \xrightarrow{\qquad} & ab_+^3 & \\
\begin{array}{c}[xg_6 = z_+ + z_-] \\ {[y_+^2 = y_-^2 = 4z_+, y_+y_- = 4z_-]}\end{array} & z_+, z_-(8) \xrightarrow{\qquad} & b_+^4, b_+^3 b_- & \\
{[xz_- = xz_+]} & xz_+ \xrightarrow{\qquad} & ab_+^4. &
\end{array}
$$

Here numbers in parentheses indicate the dimension of the module over the underlying ground field (in places where this is not given it is easily deduced). Δ indicates that a generator maps to the sum of the two shown generators, 2 indicates that a generator is sent to twice the indicated generator of the target, and in all other cases the shown generators map to the shown generators. Finally, $(z_+)^2 = (z_-)^2$ and multiplication by z_+ is an isomorphism $G_i(\mathbb{R}) \to G_{i+8}(\mathbb{R})$ for all i.

Proof. Note that the relations in $G_*(\mathbb{C})$ were already established by Proposition 45.16. Let x be the irreducible Cl_0-module. Modules over Cl_1 and Cl_2 are determined by their dimension, so one readily sees that x^2 and x^3 are the irreducibles there. If c denotes complexification then we start with $c(x) = a$ and then get $c(x^2) = a^2 = b_+ + b_-$ and $c(x^3) = a^3 = a(b_+ + b_-) = 2ab_+$.

Recall that $y_\pm$ has the property that $e_1e_2e_3$ acts as ± 1. Since x^2 is $\mathbb{C}$ with e_1 acting as $-i$, x^4 is $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ with Cl_3-structure given by the matrices

$$\begin{bmatrix} i\otimes 1 & 0 \\ 0 & -i\otimes 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1\otimes i \\ 1\otimes i & 0 \end{bmatrix}.$$

So the product $e_1e_2e_3$ acts as $\left[\begin{smallmatrix} i\otimes i & 0 \\ 0 & i\otimes i \end{smallmatrix}\right]$, which has a 4-dimensional eigenspace for eigenvalue $+1$ and a 4-dimensional eigenspace for eigenvalue -1. So $x^4 = y_+ + y_-$.

The complexification $c(y_+)$ has $e_1e_2e_3$ acting as 1, so $c(y_+) = 2b_+^2$. Likewise, $c(y_-) = 2b_+b_-$.

Both xy_+ and xy_- must be the generator of $G_4(\mathbb{R})$ for dimension reasons, and $c(xy_+) = a(b_+^2 + b_+b_-) = 2ab_+^2$.

Call the irreducible Cl_5-module g_5. It has dimension 8 over $\mathbb{R}$, so dimension arguments show $x^2y_+ = 2g_5$. Since $c(x^2y_+) = 2a^2b_+^2 = 2b_+^3 + 2b_+^2b_-$, we conclude that $c(g_5) = b_+^3 + b_+^2b_-$. The same kind of reasoning shows that $xg_5 = g_6$ and $c(g_6) = ab_+^3$.

The module z_+ has dimension 8 and $\alpha = e_1e_2\cdots e_7$ acting as $+1$, so $c(z_+)$ has those same properties. Hence $c(z_+) = b_+^4$. The same reasoning shows $c(z_-) = b_+^3b_-$.

We know the dimensions of the generators for $G_s(\mathbb{R})$ for all s, and a dimension count shows that multiplication by z_+ must map generators to generators in all degrees except possibly those congruent to 3 modulo 4. In those execeptional degrees one must give a bit more argument. If M is a Cl_r-module then the Clifford generators act on $M \otimes_c z_+$ via the matrices

$$\begin{bmatrix} e_1I & 0 \\ 0 & -e_1I \end{bmatrix}, \ldots, \begin{bmatrix} e_rI & 0 \\ 0 & -e_rI \end{bmatrix}, \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \begin{bmatrix} 0 & Ie_1 \\ Ie_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & Ie_7 \\ Ie_7 & 0 \end{bmatrix},$$

where we have written e_1I instead of $e_1 \otimes I$ for typographical reasons. The product of these matrices is

$$\begin{bmatrix} \omega_r \otimes I & 0 \\ 0 & \omega_r \otimes I \end{bmatrix}.$$

So if ω_r acts with eigenvalue λ on M, then ω_{r+8} acts with eigenvalue λ on $M \otimes_c z_+$. From this it immediately follows that multiplication by z_+ is an isomorphism $G_{4s+3}(\mathbb{R}) \to G_{4s+11}(\mathbb{R})$ for all $s \geq 0$. The similar analysis of the behavior of ω_{r+8} on $M \otimes_c z_-$ reveals that $(z_-)^2 = (z_+)^2$. □

Corollary 45.20. *The ring $\mathcal{A}_*(\mathbb{R})$ is generated by $x \in \mathcal{A}_1(\mathbb{R})$, $y \in \mathcal{A}_3(\mathbb{R})$, and $z \in \mathcal{A}_7(\mathbb{R})$ subject to the relations*

$$2x = 0, \quad x^3 = 0, \quad xy = 0, \quad y^2 = 4z$$

together with commutativity.

Proof. Take $y = y_+$ and $z = z_+$. The rest is left to the reader. □

Now consider the square

$$\begin{array}{ccc} \mathcal{A}_*(\mathbb{R}) & \longrightarrow & \mathcal{A}_*(\mathbb{C}) \\ \downarrow & & \downarrow \\ \pi_*O & \longrightarrow & \pi_*U. \end{array}$$

We know all of the groups involved. We also know the top maps by Proposition 45.19, we know the bottom maps by Exercise 41.13, and we know the right vertical map is an isomorphism by Theorem 45.18. Thinking it through, this immediately yields that $\mathcal{A}_*(\mathbb{R}) \to \pi_* O$ is an isomorphism except possibly in the case where $* \equiv 1 \pmod 8$. It only takes a little more legwork to cover this case as well:

Theorem 45.21. *The map $\mathcal{A}_*(\mathbb{R}) \to \pi_* O$ is an isomorphism of graded rings.*

Proof. We already know this is a ring map. As remarked above, by comparison square to the complex case shows immediately that the map is an isomorphism in degrees not congruent to 1 mod 8. It remains to check this final case. For this, consider instead the square

$$\begin{tikzcd} \mathcal{A}_*(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{A}_*(\mathbb{R}) \arrow[d] \\ \pi_* U \arrow[r, "r"] & \pi_* O. \end{tikzcd}$$

Here the top map is induced by restriction of scalars along the evident maps of algebras $\mathrm{Cl}_i(\mathbb{R}) \to \mathrm{Cl}_i(\mathbb{C})$. It is a quick check that this square commutes. When $*$ is equivalent to 1 mod 8 the square takes the form

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}/2 \arrow[d] \\ \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}/2. \end{tikzcd}$$

The left vertical map is an isomorphism by Theorem 45.18 and the bottom horizontal map is projection by Exercise 41.13. It follows at once that the right vertical map must be nonzero, hence an isomorphism. □

Remark 45.22. The above result demonstrates a useful principle: often some hard work regarding real K-theory can be avoided by instead pulling the information from the complex case via the two maps relating them. It is almost as if the quadruple (O, U, c, r) should be regarded as one object. This technique comes up often enough that it is worth keeping near the top of one's toolbox.

Theorem 45.21 now lets us (finally) give the real analog of Theorem 44.8:

Corollary 45.23. *Let $\beta_{\mathbb{R}}$ be a generator for $\pi_7 O \cong \mathbb{Z}$, for example the generator arising from either of the fundamental representations of $\mathrm{Cl}_7(\mathbb{R})$ on $\mathbb{R}^7$. Then the maps*

$$\pi_i O \xrightarrow{\beta_{\mathbb{R}} \cdot (-)} \pi_{i+8} O$$

are isomorphisms for every $i \geq 0$.

Proof. Immediate from Theorem 45.21. □

Let us also explicitly describe the generators for $\pi_* O$ in low degrees. These are obtained directly from $\mathcal{A}_*(\mathbb{R}) \to \pi_* O$.

$i = 0$: $S^0 \to O(1) = \{+1, -1\}$, basepoint $\mapsto 1$ and nonbasepoint $\mapsto -1$

$i = 1$: $S^1 \to O(2)$, $\underline{a} = (a_0, a_1) \mapsto$ multiplication by $a_0 + ia_1$ on $\mathbb{C}$

$i = 3$: $S^3 \to O(4)$, $\underline{a} \mapsto$ left multiplication by $a_0 + a_1 i + a_2 j + a_3 k$ on $\mathbb{H}$.

$i = 7$: $S^7 \to O(8)$, $\underline{a} \mapsto$ left multiplication by $\underline{a}$ on $\mathbb{O}$.

The last line takes a little explanation. The octonions can be constructed by starting with $\mathbb{R}^8$ and defining products of basis elements by explicit formulas. Let the basis elements be denoted $e_0, \ldots, e_7$ with $e_0 = 1$. The products satisfy $e_i^2 = -1$ for $i \geq 1$ and $e_i e_j = -e_j e_i$ for $1 \leq i < j$. It turns out that one gets a map of algebras $\mathrm{Cl}_7 \to \mathrm{End}_{\mathbb{R}}(\mathbb{O})$ by sending e_i to left multiplication by e_i. This would be immediate if $\mathbb{O}$ were associative, but lack of associativity forces one to work a little harder. The key observation is that $e_i(e_j x) = -e_j(e_i x)$ for all $x \in \mathbb{O}$ and for all distinct $i, j > 0$. This property can be checked by brute force from the defining relations of $\mathbb{O}$, or deduced with less trouble from the Cayley-Dickson construction of $\mathbb{O}$. (Note that we are not claiming $e_i(e_j x) = (e_i e_j)x$, which would give associativity.) Since $\dim_{\mathbb{R}} \mathbb{O} = 8$, the only possibility is that $\mathbb{O}$ is one of the two irreducible representations of Cl_7. It doesn't matter which one it is, as they both give generators for $\mathcal{A}_7(\mathbb{R})$.

Theorem 45.21 almost brings to a close our long journey over the last several sections. We have at this point computed the groups $\pi_* U$ and $\pi_* O$ together with their ring structure, and moreover have seen how Clifford algebras give rise to a nice algebraic model for these rings via $\mathcal{A}_*(\mathbb{C})$ and $\mathcal{A}_*(\mathbb{R})$. The only thing that remains is for us to connect all of this back to K-theory.

A slight complaint is that the ring structures on $\mathcal{A}_*$ and $\pi_* U$ (resp. $\mathcal{A}_*(\mathbb{R})$ and $\pi_* O$) feel a bit strange and ad hoc. As we connect back to K-theory we will be able to fix this and understand the true origins of those structures. This will be our task in the next section.

◦ Exercises ◦

Exercise 45.24. Let M be a Cl_s-module. Recall that $D(M)$ is defined to be $M \oplus M$ with a certain Cl_{s+1}-action. Check that an isomorphism $D(M) \to \mathrm{Cl}_{s+1} \otimes_{\mathrm{Cl}_s} M$ is given by sending m in the first summand of $D(M)$ to $1 \otimes m + e_{s+1} \otimes m$, and m in the second summand to $1 \otimes m - e_{s+1} \otimes m$.

Exercise 45.25. Let M be a Cl_s-module and N be a Cl_t-module. Regard Cl_s as a subalgebra of Cl_{s+t+1} in the usual way, and let $j\colon \mathrm{Cl}_t \hookrightarrow \mathrm{Cl}_{s+t+1}$ send $e_i \mapsto e_{s+1} e_{s+1+i}$ for $1 \leq i \leq t$. Check that j is an inclusion of algebras and that it gives rise to an embedding $\mathrm{Cl}_s \otimes \mathrm{Cl}_t \hookrightarrow \mathrm{Cl}_{s+t+1}$.

Verify that an isomorphism $M \otimes_c N \longrightarrow \mathrm{Cl}_{s+t+1} \otimes_{(\mathrm{Cl}_s \otimes \mathrm{Cl}_t)} (M \otimes N)$ is given by sending w in the first summand of $M \otimes_c N$ to $1 \otimes w$ and w in the second summand to $e_{s+1} \otimes w$.

46. Graded Clifford modules and K-theory

The homotopy groups of U and BU are the same but with a shift. We have seen that the groups $\pi_* U$ are intimately related to Clifford modules, but it turns out that the groups $\pi_* BU$ are better described using *graded* Clifford modules. This might seem odd: given that the two sets of groups are essentially the same, why not just use the ungraded Clifford modules to describe both? But the algebra of graded modules turns out to be a better fit for the algebra of $\pi_* BU$, most notably the ring structure.

46.1. **Gradings.** A vector space V is said to be **$\mathbb{Z}/2$-graded** if it is equipped with a decomposition $V = V_0 \oplus V_1$ into an "even" part and an "odd" part, labelled by 0 and 1. The degree is also called the parity in this context. If V and W are two $\mathbb{Z}/2$-graded vector spaces then $\operatorname{Hom}(V,W)$ becomes $\mathbb{Z}/2$-graded by the formulas

$$\begin{aligned}\underline{\operatorname{Hom}}_0(V,W) &= \operatorname{Hom}(V_0,W_0) \oplus \operatorname{Hom}(V_1,W_1),\\ \underline{\operatorname{Hom}}_1(V,W) &= \operatorname{Hom}(V_0,W_1) \oplus \operatorname{Hom}(V_1,W_0).\end{aligned}$$

So the even maps are parity-preserving, and the odd maps are parity-reversing. There is a corresponding tensor product, called the **graded tensor product**:

$$(V\hat{\otimes}W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \quad (V\hat{\otimes}W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

Exercise 46.2. Check that these definitions yield a natural adjunction isomorphism $\underline{\operatorname{Hom}}(V\hat{\otimes}W, Z) \cong \underline{\operatorname{Hom}}(V, \underline{\operatorname{Hom}}(W,Z))$.

For any ungraded vector space V, write $V[j]$ ($j = 0,1$) for V regarded as a $\mathbb{Z}/2$-vector space concentrated entirely in degree j. Note the natural isomorphism $V[j]\hat{\otimes}W[k] \cong (V \otimes W)[j+k]$, where $j+k$ is interpreted modulo 2. From now on we will stop saying "modulo 2" when it is clear from context. Also note that $\mathbb{R}[0]$ is the unit for the graded tensor product.

A $\mathbb{Z}/2$-graded algebra A is a $\mathbb{Z}/2$-graded vector space equipped with a unit $\eta\colon \mathbb{R}[0] \to A$ and a multiplication map $\mu\colon A\hat{\otimes}A \to A$ that is associative and unital. In down to earth terms, A comes with a grading $A = A_0 \oplus A_1$ where the unit lives in degree 0 and the product of two homogenous elements adds the degree. One likewise defines $\mathbb{Z}/2$-graded modules over A in the expected way.

If A and B are $\mathbb{Z}/2$-graded algebras then $A\hat{\otimes}B$ inherits a $\mathbb{Z}/2$-graded algebra structure, but this happens in more than one way: there are some choices regarding signs. The choice that turns out to be most useful for us is the one following the Koszul sign rule, namely

$$(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1a_2 \otimes b_1b_2. \tag{46.3}$$

The maps $A \to A\hat{\otimes}B$ and $B \to A\hat{\otimes}B$ given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ are maps of algebras, the even elements of A and B map to elements that commute with each other, whereas the odd elements of A and B map to elements that anti-commute with each other. If M is an A-module and N is a B-module then $M\hat{\otimes}N$ becomes an $A\hat{\otimes}B$-module via the analogous formula.

Clifford algebras are naturally $\mathbb{Z}/2$-graded. If (V,q) is a vector space with quadratic form then the tensor algebra $T(V)$ is $\mathbb{N}$-graded and the relations $v^2 = -q(v)$ is even; so the quotient is $\mathbb{Z}/2$-graded. So $\operatorname{Cl}(V,q)_0$ consists of products of an even number of vectors in V, and $\operatorname{Cl}(V,q)_1$ consists of products of an odd number. We will write $\operatorname{Cl}(V,q)^{grd}$ when we need to accentuate that we are thinking of the Clifford algebra as $\mathbb{Z}/2$-graded.

The biggest advantage of introducing these $\mathbb{Z}/2$-gradings is that the graded tensor product gives us the most natural way of relating small Clifford algebras to bigger ones:

Proposition 46.4. *There is a unique map of graded algebras*

$$\mathrm{Cl}(V, q_V)\hat{\otimes}\,\mathrm{Cl}(W, q_W) \longrightarrow \mathrm{Cl}(V \oplus W, q_V \oplus q_W)$$

that sends $v \otimes 1$ to $(v, 0)$ for $v \in V$ and $1 \otimes w$ to $(0, w)$ for $w \in W$, and moreover this map is an isomorphism.

Proof. Left as an exercise. □

Remark 46.5. When applied to the standard Clifford algebras for the sum-of-squares form this becomes the isomorphism $\mathrm{Cl}_{s+t} \longrightarrow \mathrm{Cl}_s \hat{\otimes} \mathrm{Cl}_t$ given by

$$e_i \mapsto \begin{cases} e_i \otimes 1 & \text{if } 1 \le i \le s, \\ 1 \otimes e_{i-s} & \text{if } s+1 \le i \le s+t. \end{cases}$$

Using this, we see that if M is a graded Cl_s-module and N is a graded Cl_t-module then $M\hat{\otimes}N$ becomes a $\mathrm{Cl}_s \hat{\otimes} \mathrm{Cl}_t$-module and therefore a graded Cl_{s+t}-module via the above isomorphism.

There are interesting ways of moving back and forth between graded and ungraded Clifford algebras and their modules. We begin by describing a functor $D\colon \mathrm{Cl}_s\text{-Mod} \to \mathrm{Cl}^{grd}_{s+1}\text{-Mod}$. For M a Cl_s-module let DM be the graded vector space with $(DM)_0 = (DM)_1 = M$. If $m \in M$ write $m_{[0]}$ for $m \in (DM)_0$ and $m_{[1]}$ for $m \in (DM)_1$. Define the Cl_{s+1}-structure on DM by

$$\begin{aligned} e_i(m_{[j]}) &= (e_i m)_{[j+1]}, \quad \text{for } 1 \le j \le s, \\ e_{s+1}(m_{[j]}) &= (-1)^{j+1} m_{[j+1]} \end{aligned}$$

for all $m \in M$ and $j \in \{0, 1\}$. Note that this is the "doubling construction" that we introduced back in ???, and is isomorphic to $\mathrm{Cl}_{s+1} \otimes_{\mathrm{Cl}_s} M$ by Exercise 45.24. Now we are just observing that the doubling construction comes with a natural grading.

Exercise 46.6. Check that the following diagram

$$\begin{tikzcd} \mathrm{Cl}_{s+1}\text{-Mod} \arrow[r, "U"] \arrow[d, "D"'] & \mathrm{Cl}_s\text{-Mod} \arrow[d, "D"] \\ \mathrm{Cl}^{grd}_{s+2}\text{-Mod} \arrow[r, "U"] & \mathrm{Cl}^{grd}_{s+1}\text{-Mod} \end{tikzcd}$$

does NOT commute, where the U maps are forgetful functors induced by the inclusions $\mathrm{Cl}_s \hookrightarrow \mathrm{Cl}_{s+1}$ and so forth.

Proposition 46.7. *The functor $D\colon \mathrm{Cl}_s\text{-Mod} \to \mathrm{Cl}^{grd}_{s+1}\text{-Mod}$ is an equivalence of categories, for all s.*

Proof. To give an inverse functor $\Phi\colon \mathrm{Cl}_{s+1}\text{-Mod} \to \mathrm{Cl}_s\text{-Mod}$ we would like to send $M = M_0 \oplus M_1$ to just M_0, but the e_i's don't act on M_0—they instead interchange M_0 and M_1. So we have to massage this basic idea a bit.

Define $u\colon \mathrm{Cl}_s \to (\mathrm{Cl}^{grd}_{s+1})_0$ by $e_i \mapsto e_{s+1}e_i$. This is an isomorphism of algebras. Define $\Phi(M)$ to be M_0 regarded as a Cl_s-module via u. It is routine to now check that D and Φ are inverses of each other, up to natural isomorphism. □

Exercise 46.8. Let M be an ungraded Cl_s-module and N be an ungraded Cl_t-module. Then DM is a graded Cl_{s+1}-module and DN is a graded Cl_{t+1}-module, therefore $DM\hat{\otimes}DN$ is a graded Cl_{s+t+2}-module. So $\Phi(DM\hat{\otimes}DN)$ is an ungraded

Cl_{s+t+1}-module. Check that $\Phi(DM\hat{\otimes}DN)$ coincides with what we previously called $M\otimes_c N$ (DOES IT?)

When we introduced the Clifford tensor product we remarked that its behavior with respect to the Clifford indexing was unfortunate. Notice that shifting focus to the graded tensor product fixes this issue.

When we defined the Clifford tensor product back in ??? the construction seemed somewhat ad hoc, but we can now see it as transporting with the (very natural) graded tensor product across the equivalences provided by Proposition 46.7. Specifically, we have the following:

Proposition 46.9. *Let M be a Cl_{s-1}-module and N be a Cl_{t-1}-module. There is a natural isomorphism of graded Cl_{s+t}-modules of the form*

$$D(M\otimes_c N)\xrightarrow{\cong} DM\hat{\otimes}DN.$$

Consequently, the map $D\colon \bigoplus_s G_{s-1}\to\bigoplus_s G_s^{grd}$ is a ring homomorphism.

Proof. We give both an abstract proof and a more concrete version. The abstract proof is to use the algebraic interpretations of D and $\otimes_c$ given in Exercises 45.24 and 45.25, together with the evident algebraic isomorphisms

$$\begin{aligned}\mathrm{Cl}_{s+t}\otimes_{\mathrm{Cl}_{s+t-1}}\Big(\mathrm{Cl}_{s+t-1}\otimes_{(\mathrm{Cl}_{s-1}\otimes\mathrm{Cl}_{t-1})}(M\otimes N)\Big)&\cong \mathrm{Cl}_{s+t}\otimes_{(\mathrm{Cl}_{s-1}\otimes\mathrm{Cl}_{t-1})}(M\otimes N)\\ &\cong(\mathrm{Cl}_s\otimes_{\mathrm{Cl}_{s-1}}M)\hat{\otimes}(\mathrm{Cl}_t\otimes_{\mathrm{Cl}_{t-1}}N).\end{aligned}$$

The leftmost term is $D(M\otimes_c N)$ and the rightmost term is $(DM)\hat{\otimes}(DN)$. We leave the details to the reader.

Here is the more concrete version. The underlying vector space of $M\otimes_c N$ is defined to be $(M\otimes N)\oplus(M\otimes N)$, and the underlying vector space of $D(M\otimes_c N)$ is then defined to be $(M\otimes N)^{\oplus 4}$. Label the copies as $1--4$ so that the copies in $M\otimes_c N$ are 1 and 2, and the corresponding copies in the second summand of $D(M\otimes_c N)$ are 3 and 4.

Note that $DM\hat{\otimes}DN$ also has four copies of $M\otimes N$, in this case naturally indexed as 00, 01, 10, and 11 (corresponding to the gradings in DM and DN).

Define a map of vector spaces $F\colon D(M\otimes_c N)\longrightarrow DM\hat{\otimes}DN$ as follows:

$$\begin{aligned}w_1&\mapsto w_{00}-w_{01}\\ w_2&\mapsto -w_{10}-w_{11}\\ w_3&\mapsto w_{00}+w_{01}\\ w_4&\mapsto w_{10}-w_{11}.\end{aligned}$$

Here w represents element of $M\otimes N$ and the subscripts specify the summand it lies in. The map F is clearly a vector space isomorphism. It is routine, though slightly tedious, to check that it is compatiable with the two Cl_{s+t}^{grd}-actions. Again, we leave this to the reader. ∎

Recall that in the case of ungraded modules we defined $\mathcal{A}_s$ to be the cokernel of the restriction of scalars map $U\colon G_{s+1}\to G_s$. We make the analogous definition of $\mathcal{A}_s^{grd}$. We might guess that the isomorphisms $D\colon G_{*-1}\to G_*^{grd}$ induce a similar isomorphism between $\mathcal{A}_*$ groups, but recall from Exercise 46.6 that the necessary square does not commute. We can still get such an isomorphism, but we use the Φ functors instead:

Proposition 46.10. *The square of functors*

$$\begin{tikzcd} \mathrm{Cl}^{grd}_{s+1}\text{-Mod} \arrow[r, "U"] \arrow[d, "\Phi"'] & \mathrm{Cl}^{grd}_{s}\text{-Mod} \arrow[d, "\Phi"] \\ \mathrm{Cl}_s\text{-Mod} \arrow[r, "U"] & \mathrm{Cl}_{s-1}\text{-Mod} \end{tikzcd}$$

commutes up to natural isomorphism. Consequently, the induced square on Grothendieck groups commutes on the nose and therefore Φ induces a map $\Phi\colon \mathcal{A}^{grd}_s \to \mathcal{A}_{s-1}$, which is an isomorphism.

Proof. The latter two statements follow immediately from the first and the fact that Φ is an equivalence of categories. To check the first statement, recall that if M is a Cl^{grd}_{s+1}-module then $\Phi(UM)$ is M_0 with each e_i $(1 \leq i \leq s-1)$ acting as $e_s e_i$. Likewise, $U\Phi(M)$ is M_0 with each e_i acting as $e_{s+1}e_i$. The map $f\colon M_0 \to M_0$ given by $f(x) = (e_s - e_{s+1})x$ (using the original Cl^{grd}_{s+1}-structure) is readily checked to be an isomorphism $\Phi(UM) \to U(\Phi M)$. $\square$

Using graded modules gives us another useful tool that has not been mentioned yet. Given a graded Cl_s-module M, let ΠM denote M but with the parity shifted: $(\Pi M)_i = M_{i+1}$.

Proposition 46.11. *The induced map $\Pi\colon G^{grd}_s(\mathbb{C}) \to G^{grd}_s(\mathbb{C})$ is the identity when s is even, and interchanges the two positive generators when s is odd. Likewise, the induced map $\Pi\colon G^{grd}_s(\mathbb{R}) \to G^{grd}_s(\mathbb{R})$ is the identity when $s \not\equiv 0$ mod 4, and interchanges the two positive generators otherwise.*

Proof. Clearly Π must take irreducibles to irreducibles. In the non-exceptional cases there is a unique isomorphism class of irreducible module, and thus the action of Π is trivial. So the only work is in checking the exceptional cases where there are two isomorphism classes of irreducibles.

We give the argument in the real case, with the complex case being similar. Let $s = 4k$ and let M be an irreducible (ungraded) Cl_{s-1}-module where $\omega = e_1 \cdots e_{s-1}$ acts as $+1$. We will compute the corresponding actions of ω on $\Phi(DM)$ and on $\Phi(\Pi(DM))$.

For $\Phi(DM)$ we have ω acting as $(e_1e_2)(e_1e_3)\cdots(e_1e_s)$ on $(DM)_0$. But $e_1\colon (DM)_0 \to (DM)_1$ is -1, so this is $(-1)^{s-1}$ times the original action of ω on M. But s is even, so the action is by -1.

For $\Phi(\Pi(DM))$ we have the same formula, but now $e_1\colon (\Pi DM)_0 \to (\Pi DM)_1$ is $+1$, so the action of ω coincides with the original $+1$ action on M.

We have therefore proven that M and ΠM become different irreducibles after application of Φ, so they are the two positive generators of $G^{grd}_s(\mathbb{R})$. $\square$

46.12. **The coefficient rings of K and KO.** At this point in the book we have developed a number of ways to access the groups $K^*(pt)$ and $KO^*(pt)$, together with their ring structure, but we have not as yet carefully verified the consistency of these different approaches. We finally turn to this issue now. The work will focus on the following large diagram (note the use of red and blue colors, which we will discuss shortly):

$$\begin{tikzcd}
 & \pi_{s-1}O \arrow[d, "\text{clutch}"', "\cong"] \arrow[dr, "1"] & \mathcal{A}_{s-1} \arrow[l, "\cong_R"'] \arrow[r, "D", "\cong_R"'] \arrow[d, "2"] & \mathcal{A}_s^{grd} \arrow[dl, "3"'] \arrow[d, "4"] \\
 & KO^0_{st}(S^s) \arrow[d, "\cong"'] & L^1(D^s, \partial D^s) \arrow[l, "\text{clutch}"', "\cong"] \arrow[r, "\cong"'] & KO^0_{cplx}(D^s, \partial D^s) \\
\pi_s BO \arrow[r, "\cong"] & \widetilde{KO}^0(S^s) & L^1(S^s, *) \arrow[l, "\chi"'] \arrow[r, "\cong"] \arrow[u, "\pi^*"'] & KO^0_{cplx}(S^s, *) \arrow[u, "\pi^*", "\cong_R"'] \arrow[ll, bend left, "\chi", "\cong_R"']
\end{tikzcd}$$

The following several paragraphs explain how to parse this diagram. Note that we have drawn the diagram for KO^*, but the diagram for K^* is completely analogous.

All of the maps in this diagram turn out to be isomorphisms, but the arrows are drawn in the direction where the maps are most natural to define. For example, the two maps labelled "clutch" are clutching constructions. The vertical one is induced by the construction that takes a map $f\colon S^{s-1} \to O(n)$ to the rank n bundle on S^s having this as its clutching function. The inverse of this map feels a bit less natural, as it can only be constructed by choosing trivializations of a bundle on the upper and lower hemispheres (though to be fair, even the original map involves a choice of representing element f for the homotopy class).

The horizontal clutching map sends a map $g\colon E_1 \to E_0$ of bundles on D^s, exact on ∂D^s, to the bundle on S^s obtained by taking E^1 on the upper hemisphere, E_0 on the lower hemisphere, and then using g as the clutching function.

The two maps labelled χ sends a complex of vector bundles E_* (possibly just of length one) to $\sum_i (-1)^i [E_i]$. The maps labelled π^* are the pullbacks induced by the projection $\pi\colon (D^s, \partial D^s) \to (S^s, *)$.

Taking the direct sum over s gives a graded ring in each spot of the diagram. The red spots are the ones where we can describe this ring structure with a precise algebraic formula; e.g., on $K^0_{cplx}(D^s, \partial D^s)$ it is given by the external tensor product of chain complexes (note that this uses the canonical isomorphisms $D^s \times D^t \cong D^{s+t}$). The blue terms are ones where we have a ring structure for formal topological reasons, but it is much less clear how to actually compute it. For example, The external tensor product $K^0(S^s) \otimes K^0(S^t) \to K^0(S^s \times S^t)$ can be shown to send $\widetilde{K}^0(S^s) \otimes \widetilde{K}^0(S^t)$ into the image of $\widetilde{K}^0(S^s \wedge S^t)$ in the target, and in this way defines an element of $\widetilde{K}^0(S^s \wedge S^t)$. This is the ring structure on $\bigoplus_s \widetilde{K}^0(S^s)$. But while it is true that the external tensor product is a "formula" that we understand, re-interpreting it as an element of $\widetilde{K}^0(S^s \wedge S^t)$ is not formulaic—there is no evident way of digesting the formal difference of bundles on $S^s \times S^t$ into a formal difference of bundles on $S^s \wedge S^t$.

The terms that are neither red nor blue are ones where there is no reasonable ring structure to consider on the direct sum over s. That is to say, one can artifically put a ring structure by transplanting the ones we know across the isomorphisms, but the result is not useful.

One can start to see some of the difficulties in navigating elements of this diagram. Although there is a very direct map from $\pi_{*-1}O$ to $\widetilde{KO}^0(S^*)$, it is challenging to relate the algebraic multiplication on the former to the topological multiplication on the latter. To compare these two it ends up being easiest to go the long way around the diagram, as we will soon see.

Maps labelled $\cong_R$ are ring isomorphisms. Note that the rightmost π_* is a ring isomorphism by naturality and excision, whereas the bottom χ is a ring isomorphism by Theorem 18.16.

The maps labelled 1–4 are defined as follows:

(1) Given $f\colon S^{s-1} \to O(n)$, for $x \in D^s$ define $\hat{f}(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\hat{f}(x) = |x| f(\frac{x}{|x|})$ with the understanding that this is the zero map when $x = 0$. Send f to the map of bundles $\underline{n} \to \underline{n}$ that is multiplication by $\hat{f}(x)$ over the point $x \in D^s$.
(2) Identify $\mathbb{R}^s$ with the subspace of Cl_{s-1} spanned by $1, e_1, \dots, e_n$. Given a Cl_{s-1}-module M, send this to the map of trivial bundles $\underline{M} \to \underline{M}$ on D^s that over $x \in D^s$ is multiplication by x.
(3) Regard $\mathbb{R}^s$ as a subspace of Cl_s in the usual way, i.e. as the span of the elements $e_1, \dots, e_s$. Given a graded Cl_s-module M, send this to the map of trivial bundles $\underline{M_1} \to \underline{M_0}$ on D^s which over $x \in D^s$ is multiplication by x.
(4) Same as (3).

All of the triangles in the top part of the diagram are readily checked to commute. All of the maps labelled as isomorphisms are known to be so by previous results in the text. It then follows from commutativity of the diagram that the maps 1–4 are also isomorphisms.

The bottom right rectangle is clearly commutative, as is the triangle at the very bottom of the diagram. Commutativity of the bottom left rectangle takes a moment's thought. Take an element $x = [\alpha\colon E_1 \to E_0] \in L^1(S^s, *)$. By adding a complement to E_1 to both the domain and codomain, we can assume that E_1 is free. The fact that α is an isomorphism over the basepoint implies that E_0 and E_1 have the same rank, say n. We have $\chi(x) = [E_0] - [E_1] = [E_0] - n$. Going the other way around the square, we consider π^*E_0 and π^*E_1 on D^s and clutch them together using the isomorphism α. Using that E_1 is trivial, the resulting vector bundle is clearly isomorphic to E_0. So going the other way around the square gives $[E_0] - n$ again, hence the square commutes.

That the top horizontal maps are ring homomorphisms has been verified in Theorem 45.18 and Proposition 46.9. It remains to check this property for the vertical map:

Proposition 46.13. *The map $\mathcal{A}_s^{grd} \to K^0_{cplx}(D^s, \partial D^s)$ is a ring isomorphism.*

Proof. Note that the "isomorphism" part follows from the commutativity of the big diagram, so the only question is whether the map is a ring homomorphism. The main issue here is that the tensor product on graded modules is $\mathbb{Z}/2$-graded whereas the one on chain complexes is $\mathbb{Z}$-graded, so one must work a little to fit these together. The key step is the process described in ??? for folding a chain complex (in this case of length two) down into a length-one chain complex—this is the argument we used to prove that $K^0_{cplx}(X, A)$ is isomorphic to $L^1(X, A)$.

Let M and N be graded Cl_s-modules, and recall that $\underline{M_i}$ denotes the trivial bundle on D^s with fiber M_i. Let $\theta(M)$ denote the chain complex that has $\underline{M_i}$ in degree i and where the differential is left multiplication by x on the fiber over x.

Let $\alpha_M : \underline{M_0} \to \underline{M_1}$ be the map that is left multiplication by $\underline{x}$ on the fiber over x. This gives a contracting homotopy for the map of chain complexes $\theta(M) \to \theta(M)$

that is multiplication by $-|x|^2$:

$$\begin{tikzcd} \underline{M_1} \arrow[r, "-|x|^2"] \arrow[d, "x"'] & \underline{M_1} \arrow[d, "x"] \\ \underline{M_0} \arrow[r, "-|x|^2"'] \arrow[ur, "\alpha_M"] & \underline{M_0}. \end{tikzcd}$$

Consider the following map of complexes, where the domain is $M \otimes_{cplx} N$:

$$\begin{tikzcd} M_1 \otimes N_1 \arrow[r, "\mathrm{id}"] \arrow[d] & M_1 \otimes N_1 \arrow[d, "-2|x|^2"] \\ M_0 \otimes N_1 \oplus M_1 \otimes N_0 \arrow[r, "-\alpha_M \otimes 1 + 1 \otimes \alpha_N"] \arrow[d] & M_1 \otimes N_1 \\ M_0 \otimes N_0 & \end{tikzcd}$$

Let Γ denote the desuspension of the mapping cone. Explicitly, Γ is the following chain complex concentrated in degrees 2, 1, and 0 (for typographical reasons we have omitted the tensor symbols between M_i and N_j):

$$M_1N_1 \xrightarrow{\begin{bmatrix} d\otimes 1 \\ -1\otimes d \\ -1\otimes 1 \end{bmatrix}} M_0N_1 \oplus M_1N_0 \oplus M_1N_1 \xrightarrow{\begin{bmatrix} 1\otimes d & d\otimes 1 & 0 \\ \alpha_M\otimes 1 & -1\otimes\alpha_N & 2|x|^2 \end{bmatrix}} M_0N_0 \oplus M_1N_1.$$

If E is a vector bundle let $D(E)$ be the chain complex that has E in degrees 0 and 1 and where the differential is the identity. There are short exact sequences

$$0 \longrightarrow \theta(M\hat{\otimes}N) \longrightarrow \Gamma \longrightarrow \Sigma D(\underline{M_1} \otimes \underline{N_1}) \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma X \longrightarrow \Gamma \longrightarrow \theta M \otimes_{cplx} \theta N \longrightarrow 0$$

where X has $M_1 \otimes N_1$ in degrees 0 and 1 and the differential is multiplication by $-2|x|^2$. These exact sequences are more or less self-evident as soon as one goes looking for them (remember that Σ changes the sign on the differentials in addition to shifting them up).

The short exact sequences give us the K-theory relations

$$[\theta(M\hat{\otimes}N)] = [\Gamma] = [\Sigma X] + [\theta M \otimes_{cplx} \theta N].$$

As the final step we observe that the differential in X can be homotoped to one that is exact, via the homotopy $-2|x|^2 - t$, $0 \le t \le 1$. So X (and also ΣX) represents 0 in K-theory. $\square$

46.14. **MISC STUFF WAITING RECYCLING.** If A is a $\mathbb{Z}/2$-graded algebra then A^{op} is the same underlying graded vector space but with the product $a \odot b = (-1)^{|a|\cdot|b|}ba$. If M is a right A-module then $\mathrm{End}_A(M)$ is $\underline{\mathrm{Hom}}_A(M,M)$ with the multiplication given by composition. To clarify this a bit, note that $\mathrm{Hom}_A(M,N)$ may be canonically identified with $\mathrm{Hom}_A(\Pi M, \Pi N)$ in the evident way (no sign changes come into play here). We have $\mathrm{End}_A(M)_0 = \mathrm{Hom}_A(M,M) = \mathrm{Hom}_A(\Pi M, \Pi M)$ and $\mathrm{End}_A(M)_1 = \mathrm{Hom}_A(\Pi M, M) = \mathrm{Hom}_A(M, \Pi M)$. Given $f, g \in \mathrm{Hom}_A(\Pi M, M)$ we interpret fg as either the composition $\Pi(f) \circ g \in \mathrm{Hom}_A(\Pi M, \Pi M)$ or $f \circ \Pi(g) \in \mathrm{Hom}_A(M,M)$, both of which specify the same element in $\mathrm{End}_A(M)_0$.

Exercise 46.15. For each $a \in A$ let $\lambda_a : A \to A$ be left multiplication by a. Check that $a \mapsto \lambda_a$ gives an isomorphism of algebras $A \cong \operatorname{End}_A(A)$.

For $p, q \geq 0$ define $A^{p|q} = A^{\oplus p} \oplus (\Pi A)^{\oplus q}$ and $A(p|q) = \operatorname{End}_A(A^{p|q})$. Note that these are the $\mathbb{Z}/2$-graded analogs of matrix algebras. Observe that there are canonical isomorphisms

- $A^{a|b} \hat{\otimes} A^{p|q} \cong A^{ap+bq|aq+bp}$.
- $A(p|q) \cong \mathbb{R}(p|q) \hat{\otimes}_{\mathbb{R}} A$.
- $A(p|q)(a|b) \cong A(pa+qb|pb+qa)$.

Exercise 46.16. Let A be an R-algebra, let M be an R-module, and let N be an A-module. All of this is taking place in the $\mathbb{Z}/2$-graded setting. There is a canonical map

$$\theta \colon \operatorname{End}_R(M) \hat{\otimes} \operatorname{End}_A(N) \longrightarrow \operatorname{End}_{R \otimes_R A}(M \hat{\otimes}_R N), \qquad f \otimes g \mapsto f \otimes g.$$

(a) Check that θ is a homomorphism of graded rings.
(b) Check that the map θ is an isomorphism when M is finitely-generated and free, meaning that M is a finite direct sum of copies of R and ΠR.
(c) If A is an $\mathbb{R}$-algebra note that $A^{p|q} = \mathbb{R}^{p|q} \hat{\otimes}_{\mathbb{R}} A$. Use (b) to conclude that $\mathbb{R}(p|q) \hat{\otimes}_{\mathbb{R}} A \cong A(p|q)$.
(d) Use (b) for $\operatorname{End}_{\mathbb{R}}(\mathbb{R}^{p|q}) \hat{\otimes}_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(\mathbb{R}^{a|b}) \longrightarrow \operatorname{End}_{\mathbb{R}}(\mathbb{R}^{p|q} \hat{\otimes} \mathbb{R}^{a|b})$ to conclude that $\mathbb{R}(p|q) \hat{\otimes} \mathbb{R}(a|b) \cong \mathbb{R}(pa+qb|pb+qa)$, and deduce the analogous result with $\mathbb{R}$ replaced by A.

We will often find ourselves using the operation $A \mapsto A(1|1)$. Note that iterating this operation k times gives $A \mapsto A(2^{k-1}|2^{k-1})$. We will use the abbreviation $A(\!(n)\!)$ for $A(n|n)$.

A $\mathbb{Z}/2$-graded algebra A is a **graded division algebra** if ρ_a is a bijection for every homogeneous element $a \in A$. The **center** of a $\mathbb{Z}/2$-graded algebra A is the subalgebra of A_0 consisting of elements that commute with everything in A. Let F be a field. The algebra A is a **central simple F-algebra** if A is a graded division algebra whose center is F. Let $\operatorname{Br}_{gr}(F)$ be the set of isomorphism classes of finite-dimensional division algebras over F whose center is precisely F. This set can be equipped with a monoid structure defined as follows. Given finite-dimensional division F-algebras A and B with center F, the algebra $A \hat{\otimes} B$ is isomorphic to $D(p|q)$ for a unique central simple F-algebra D: one defines $[A] \cdot [B] = [D]$ in $\operatorname{Br}_{gr}(F)$. This makes $\operatorname{Br}_{gr}(F)$ into an abelian group, called the **graded Brauer group** of F.

Remark 46.17. A **graded two-sided ideal** of a $\mathbb{Z}/2$-graded algebra A is a graded subspace $I \subseteq A$ that is a two-sided ideal in the usual sense. A $\mathbb{Z}/2$-graded algebra is **simple** if the only graded two-sided ideals are 0 and A. A **graded central simple** F-algebra is a graded F-algebra A that is simple and whose center is F. One can prove that every central simple F-algebra has the form $D(p|q)$ for some graded division algebra over F and some $p, q \geq 0$. One can also define $\operatorname{Br}_{gr}(F)$ to be the isomorphism classes of graded central simple F-algebras modulo the equivalence relation generated by $A \sim A(p|q)$ for every $p, q \geq 0$. The tensor product of two graded central simple F-algebras is another graded central simple F-algebra, and this preserves the equivalence relation; so it induces a product on $\operatorname{Br}_{gr}(F)$.

With a little work one can classify all of the finite-dimensional graded division algebras over $\mathbb{R}$. In the following discussion we will stop saying "finite-dimensional" although it is always in the background. If A is such an algebra note that A_0 will be an (ungraded) division algebra over $\mathbb{R}$, and so will be either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. These give all of the graded division algebras where $A_1 = 0$. If $f \in A_1$ is nonzero then right multiplication by f will give an isomorphism $A_0 \to A_1$. So our algebra will take the form

$$A = A_0 \oplus A_0.f$$

and will be determined by two other pieces of information: for every $x \in A_0$ one must have $fx = \phi(x)f$ for some unique $\phi(x) \in A_0$, and $f^2 = \lambda$ for some $\lambda \in A_0$. The function $\phi\colon A_0 \to A_0$ must be additive and unital, and for $\mathbb{R}$ to be in the center of A it must be $\mathbb{R}$-linear. Associativity is equivalent to $\phi\colon A_0 \to A_0$ being an algebra isomorphism and $\phi(\lambda) = \lambda$, the latter coming from $\lambda f = f^2 \cdot f = f \cdot f^2 = f\lambda = \phi(\lambda)f$. So graded division algebras over $\mathbb{R}$ with $A_1 \neq 0$ are determined (not necessarily uniquely) by the data (ϕ, λ). Note that we can always replace f with rf for some $r \in \mathbb{R} - \{0\}$, which will replace λ by $r^2\lambda$ while describing the same graded algebra.

For $A_0 = \mathbb{R}$ we must have A_0 acting centrally on f, and up to squares we have only two possibilities: $f^2 = 1$ and $f^2 = -1$. So there are two possible algebras here:

$$\mathbb{R}_+ = \mathbb{R}[f]/(f^2 - 1) \qquad \text{and} \qquad \mathbb{R}_- = \mathbb{R}[f]/(f^2 + 1).$$

For $A_0 = \mathbb{C}$ we can have either ϕ equal to id or complex conjugation. In the first case we can rescale f by any complex number to reduce to the case $f^2 = 1$ (this uses commutativity of $\mathbb{C}$), so our unique isomorphism class of division algebra is represented by

$$\mathbb{C}_+ = \mathbb{C}[f]/(f^2 - 1).$$

The alternative is to have ϕ equal to complex conjugation, so we will have $zf = f\bar{z}$ for all $z \in \mathbb{C}$. Since λ must be fixed by complex conjugation, up to scaling by squares of real numbers we again have the two possibilities $f^2 = \pm 1$. So our two possibilities are

$$\overline{\mathbb{C}}_+ = \mathbb{C} \oplus \mathbb{C}.f, \quad [f^2 = 1, zf = f\bar{z}] \qquad \text{and} \qquad \overline{\mathbb{C}}_- = \mathbb{C} \oplus \mathbb{C}.f, \quad [f^2 = -1, zf = f\bar{z}]$$

where in each case z ranges over all complex numbers.

For $A_1 = \mathbb{H}$ we can again have $\phi = \text{id}$. ???? As in the above analysis we reduce to $f^2 = 1$ and $f^2 = -1$, obtaining the two $\mathbb{Z}/2$-graded algebras

$$\mathbb{H}_+ = \mathbb{H}[f]/(f^2 + 1) \qquad \text{and} \qquad \mathbb{H}_- = \mathbb{H}[f]/(f^2 - 1).$$

There are also abundant examples when $\phi \neq \text{id}$, since $\mathbb{H}$ has lots of automorphisms. However, these all turn out to be isomorphic to one of the above two (WHY?).

So the complete list of $\mathbb{Z}/2$-graded division algebras over consists of the following ten algebras:

$$\mathbb{R},\ \mathbb{C},\ \mathbb{H},\ \mathbb{R}_+,\ \mathbb{R}_-,\ \mathbb{C}_+,\ \overline{\mathbb{C}}_+,\ \overline{\mathbb{C}}_-,\ \mathbb{H}_+,\ \mathbb{H}_-.$$

If we restrict to the ones whose center is exactly $\mathbb{R}$ we get down to eight, namely

$$\mathbb{R},\ \mathbb{H},\ \mathbb{R}_+,\ \mathbb{R}_-,\ \overline{\mathbb{C}}_+,\ \overline{\mathbb{C}}_-,\ \mathbb{H}_+,\ \mathbb{H}_-.$$

The graded Brauer group over $\mathbb{R}$ will then be one of $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, and $(\mathbb{Z}/2)^3$, and we can determine which one by analyzing the elements of order 2. Recall that the inverse of A in the Brauer group is A^{op}. One readily checks that $(\mathbb{R}_+)^{op} = \mathbb{R}_-$ and analogously for $\overline{\mathbb{C}}_\pm$ and $\mathbb{H}_\pm$, but $\mathbb{H}^{op} \cong \mathbb{H}$ (via conjugation). So the only element of order 2 is $\mathbb{H}$, and the graded Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8$.

If we set $x = \mathbb{R}_+$ then $x^2 = \overline{\mathbb{C}}_+$ and $x^3 = \mathbb{H}_-$. We deduce that x has order 8 (since an order of 2 or 4 is not possible) and that the group structure is as depicted in this circle:

$$\begin{tikzcd}
& \mathbb{R} \arrow[r] & \mathbb{R}_+ \arrow[dr] & \\
\mathbb{R}_- \arrow[ur] & & & \overline{\mathbb{C}}_+ \arrow[d] \\
\overline{\mathbb{C}}_- \arrow[u] & & & \mathbb{H}_- \arrow[dl] \\
& \mathbb{H}_+ \arrow[ul] & \mathbb{H} \arrow[l] &
\end{tikzcd}$$

(the arrows depict multiplication by $\mathbb{R}_+$). In this version of the Bott periodicity clock we can interpret the reflective symmetry about the line joining $\mathbb{R}$ and $\mathbb{H}$ as representing the inverses in the group.

If we look instead at graded division algebras over $\mathbb{C}$ there are just two of them: $\mathbb{C}$ and $\mathbb{C}_+$. The graded Brauer group is $\mathbb{Z}/2$ in this case.

Alternatively, one can look at Morita equivalence classes of graded division algebras over $\mathbb{R}$ with the tensor product as a monoid with ten elements (???). The graded Brauer group of $\mathbb{R}$ is a submonoid (which happens to be a group). ?????????????????????????

Graded Clifford algebras $\mathrm{Cl}^g_{p,q}$

$\mathbb{R}(8\|8)$	$\mathbb{R}_+(8\|8)$	$\overline{\mathbb{C}}_+(8\|8)$	$\mathbb{H}_-(8\|8)$	$\mathbb{H}(16\|16)$	$\mathbb{H}_+(16\|16)$	$\overline{\mathbb{C}}_-(32\|32)$	$\mathbb{R}_-(64\|64)$	$\mathbb{R}(128\|128)$
$\mathbb{R}_+(4\|4)$	$\overline{\mathbb{C}}_+(4\|4)$	$\mathbb{H}_-(4\|4)$	$\mathbb{H}(8\|8)$	$\mathbb{H}_+(8\|8)$	$\overline{\mathbb{C}}_-(16\|16)$	$\mathbb{R}_-(32\|32)$	$\mathbb{R}(64\|64)$	$\mathbb{R}_+(64\|64)$
$\overline{\mathbb{C}}_+(2\|2)$	$\mathbb{H}_-(2\|2)$	$\mathbb{H}(4\|4)$	$\mathbb{H}_+(4\|4)$	$\overline{\mathbb{C}}_-(8\|8)$	$\mathbb{R}_-(16\|16)$	$\mathbb{R}(32\|32)$	$\mathbb{R}_+(32\|32)$	$\overline{\mathbb{C}}_+(32\|32)$
$\mathbb{H}_-(1\|1)$	$\mathbb{H}(2\|2)$	$\mathbb{H}_+(2\|2)$	$\overline{\mathbb{C}}_-(4\|4)$	$\mathbb{R}_-(8\|8)$	$\mathbb{R}(16\|16)$	$\mathbb{R}_+(16\|16)$	$\overline{\mathbb{C}}_+(16\|16)$	$\mathbb{H}_-(16\|16)$
$\mathbb{H}(1\|1)$	$\mathbb{H}_+(1\|1)$	$\overline{\mathbb{C}}_-(2\|2)$	$\mathbb{R}_-(4\|4)$	$\mathbb{R}(8\|8)$	$\mathbb{R}_+(8\|8)$	$\overline{\mathbb{C}}_+(8\|8)$	$\mathbb{H}_-(8\|8)$	$\mathbb{H}(16\|16)$
$\mathbb{H}_+$	$\overline{\mathbb{C}}_-(1\|1)$	$\mathbb{R}_-(2\|2)$	$\mathbb{R}(4\|4)$	$\mathbb{R}_+(4\|4)$	$\overline{\mathbb{C}}_+(4\|4)$	$\mathbb{H}_-(4\|4)$	$\mathbb{H}(8\|8)$	$\mathbb{H}_+(8\|8)$
$\overline{\mathbb{C}}_-$	$\mathbb{R}_-(1\|1)$	$\mathbb{R}(2\|2)$	$\mathbb{R}_+(2\|2)$	$\overline{\mathbb{C}}_+(2\|2)$	$\mathbb{H}_-(2\|2)$	$\mathbb{H}(4\|4)$	$\mathbb{H}_+(4\|4)$	$\overline{\mathbb{C}}_-(8\|8)$
$\mathbb{R}_-$	$\mathbb{R}(1\|1)$	$\mathbb{R}_+(1\|1)$	$\overline{\mathbb{C}}_+(1\|1)$	$\mathbb{H}_-(1\|1)$	$\mathbb{H}(2\|2)$	$\mathbb{H}_+(2\|2)$	$\overline{\mathbb{C}}_-(4\|4)$	$\mathbb{R}_-(8\|8)$
$\mathbb{R}$	$\mathbb{R}_+$	$\overline{\mathbb{C}}_+$	$\mathbb{H}_-$	$\mathbb{H}(1\|1)$	$\mathbb{H}_+(1\|1)$	$\overline{\mathbb{C}}_-(2\|2)$	$\mathbb{R}_-(4\|4)$	$\mathbb{R}(8\|8)$
$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$

Graded Clifford algebras $\mathrm{Cl}^g_{p,q}$

$\mathbb{R}((8))$	$\mathbb{R}_+((8))$	$\overline{\mathbb{C}}_+((8))$	$\mathbb{H}_-((8))$	$\mathbb{H}((16))$	$\mathbb{H}_+((16))$	$\overline{\mathbb{C}}_-((32))$	$\mathbb{R}_-((64))$	$\mathbb{R}((128))$
$\mathbb{R}_+((4))$	$\overline{\mathbb{C}}_+((4))$	$\mathbb{H}_-((4))$	$\mathbb{H}((8))$	$\mathbb{H}_+((8))$	$\overline{\mathbb{C}}_-((16))$	$\mathbb{R}_-((32))$	$\mathbb{R}((64))$	$\mathbb{R}_+((64))$
$\overline{\mathbb{C}}_+((2))$	$\mathbb{H}_-((2))$	$\mathbb{H}((4))$	$\mathbb{H}_+((4))$	$\overline{\mathbb{C}}_-((8))$	$\mathbb{R}_-((16))$	$\mathbb{R}((32))$	$\mathbb{R}_+((32))$	$\overline{\mathbb{C}}_+((32))$
$\mathbb{H}_-((1))$	$\mathbb{H}((2))$	$\mathbb{H}_+((2))$	$\overline{\mathbb{C}}_-((4))$	$\mathbb{R}_-((8))$	$\mathbb{R}((16))$	$\mathbb{R}_+((16))$	$\overline{\mathbb{C}}_+((16))$	$\mathbb{H}_-((16))$
$\mathbb{H}((1))$	$\mathbb{H}_+((1))$	$\overline{\mathbb{C}}_-((2))$	$\mathbb{R}_-((4))$	$\mathbb{R}((8))$	$\mathbb{R}_+((8))$	$\overline{\mathbb{C}}_+((8))$	$\mathbb{H}_-((8))$	$\mathbb{H}((16))$
$\mathbb{H}_+$	$\overline{\mathbb{C}}_-((1))$	$\mathbb{R}_-((2))$	$\mathbb{R}((4))$	$\mathbb{R}_+((4))$	$\overline{\mathbb{C}}_+((4))$	$\mathbb{H}_-((4))$	$\mathbb{H}((8))$	$\mathbb{H}_+((8))$
$\overline{\mathbb{C}}_-$	$\mathbb{R}_-((1))$	$\mathbb{R}((2))$	$\mathbb{R}_+((2))$	$\overline{\mathbb{C}}_+((2))$	$\mathbb{H}_-((2))$	$\mathbb{H}((4))$	$\mathbb{H}_+((4))$	$\overline{\mathbb{C}}_-((8))$
$\mathbb{R}_-$	$\mathbb{R}((1))$	$\mathbb{R}_+((1))$	$\overline{\mathbb{C}}_+((1))$	$\mathbb{H}_-((1))$	$\mathbb{H}((2))$	$\mathbb{H}_+((2))$	$\overline{\mathbb{C}}_-((4))$	$\mathbb{R}_-((8))$
$\mathbb{R}$	$\mathbb{R}_+$	$\overline{\mathbb{C}}_+$	$\mathbb{H}_-$	$\mathbb{H}((1))$	$\mathbb{H}_+((1))$	$\overline{\mathbb{C}}_-((2))$	$\mathbb{R}_-((4))$	$\mathbb{R}((8))$
$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$

Part 7. What is K-theory?

Of course there is no single answer to this question, and also no easy answer. It is a bit like asking "What is physics?" or "What is a cow?" What constitutes a satisfying answer all depends on one's perspective. A reporter once asked Wiliiam Faulkner what he was trying to say in one of his novels, and the great writer's response was something like "If I could tell you in two minutes I wouldn't have had to write the whole book!" I feel very much the same way about our journey through K-theory over the past 400+ pages. All of the material we have covered has been trying to unveil different aspects of what K-theory "is", but the reader might understandably be feeling that we have, in the end, raised more questions than answers. There are too many places where the question of "what is really going on here?" seems to call out for a deeper explanation.

In this final section I want to look the question "What is K-theory?" straight in the eye and try to sketch a certain kind of answer. But this will involve very different techniques than in the rest of the book. Here I want to mostly leave geometry behind and focus instead on the world of homotopy theory. Unfortunately, that opens up a whole Pandora's Box of evils that could easily take us another 400 pages to unravel. And at some point we all need to retire for the night and get some sleep.

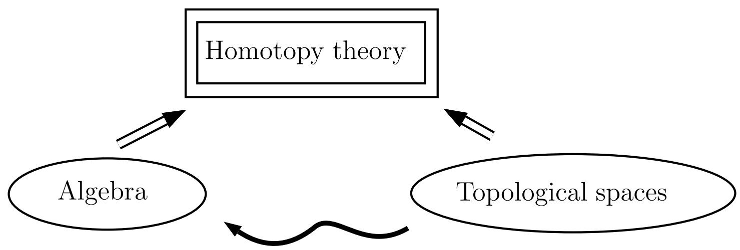
So here is what is going to happen. Sit back, get comfortable, take your feet off the ground. Peter Pan and Tinkerbell are going to take us sailing into the heavens, on a whirlwind tour, and we are going to see some strange lands. Not everything we see is going to make sense to us tonight, and that's okay. We will take in what we can, not ask too many questions, and then come back home and go to sleep before the parents catch on. Deal?

47. The second star to the right

Warning: This section is more about painting a picture than being perfectly rigorous, and so are going to be cavalier about including the occasional "mathematical white lie". The reader should proceed beyond this point at their own risk.

47.1. **What is homotopy theory?** Since homotopy theory is going to be the main backdrop for this journey, we had better get some things straight about it up front.

In an introductory algebraic topology course one gets the idea right away that there are deep and interesting processes for passing from topological information to algebraic information. As researchers delved into this subject and found themselves creating ever more sophisticated machinery to accomplish this transfer, it gradually became apparent that it was best to think of this technology as operating inside a certain realm that had previously been hidden. What came into view was a world lying in between topology and algebra, sharing characteristics with both: this is the world of homotopy theory. More precisely, I should call this the "homotopy theory of topological spaces"; but this is kind of a mouthful and it gives the erroneous impression that the theory is mainly about topology. Here let me just use "homotopy theory" as an abbreviation. The following crude diagram captures the situation somewhat:



When students first encounter homotopy theory it is usually from the perspective that the whole point is to study topological spaces, and as a result topology feels inherently inseparable from what homotopy theory *is*. But that viewpoint is an anchor that keeps one from ascending into the heavens, and it is important to somehow let it go. It is best to view homotopy theory as its own thing, a wonderland that was waiting to be discovered. Topological spaces were our first way into it, but are by no means the only way in.

It is true that the homotopical world can at times feel very geometric. Objects like spheres and manifolds, Lie groups and their homogeneous spaces, cycles and intersections—the list goes on and on—all play a big role. But at other times the homotopical world feels like some kind of highly-sophisticated algebra: it has localizations, completions, bar constructions, resolutions, etc. Every topological space yields a corresponding object in the homotopical world, and in fact every object comes from a topological space—but the realm of spaces is often not the best way to think about what is happening. For example, it turns out that every chain complex of abelian groups also specifies an object in the homotopical world; yet it is far from evident what topological space that should correspond to.

This strange blending of worlds can make it hard for homotopy theorists to talk to mathematicians in other fields. For a homotopy theorist, topological spaces and chain complexes live in the same playpen and get manipulated according to the same rules. This idea can seem very disorienting to algebraists and geometers! The intrepid homotopy theorists have developed all kinds of techniques for working in—and thinking about—the homotopical world, and it can takes years to master. We can't cover all of that in one night, but fasten your seatbelts and we will do a quick flyover.

I need to start assigning some names to things, so let me call this mystery world HS—for the Homotopy theory of Spaces. As the name suggests, the objects in this world are typically called "spaces", but this is hopelessly confusing when one is first getting the lay of the land. Inevitably one starts to confuse the objects with topological spaces, which they are not. Let me instead call the objects "h-sets", short for "homotopical sets". This is not a standard name, but will do for the time being.

If you have been brought up on 20th century mathematics you would undoubtedly expect me to describe HS to you as a category: that is, I should give you precise definitions of the objects and the morphisms. Somewhat surprisingly, this has never quite worked. One can indeed describe certain categories that serve as gateways into HS, but in each case one has to specify extra rules that govern when various constructions in the category match HS and when they don't. Categories with such rules go under the name *model categories*.

The inability to describe HS in terms of ordinary category theory is one of the first oddities one must adapt to. In modern language one might say that HS has to be described as an ∞-*category*. We won't go into the technology of either model categories or ∞-categories, and instead will just keep most of that structure hidden in the background. But it is important to know that it is there.

After going on at length about the difficulties in describing HS, we still need to describe it somehow. To get a sense of the problem, we will make an attempt to give an "algebraic" description of the objects. Remember that these are supposed to model homotopy types, and we know something about those. It seems clear enough that an object X should come with a set of points X_0, and that we should be able to talk about paths—or maybe "edges" is a better term—connecting two points. That seems simple enough, but now things get bumpy. We should be able to concatenate paths (when they line up appropriately) to make new paths, and we should be able to talk about deformations—or homotopies—between paths. Then we are going to need homotopies between homotopies, and so on ad infinitum. Here it gradually becomes clear that there are probably lots and lots of ways to encode all of this stuff: one could use cubes, or simplices, or any other form of higher-dimensional "blobs", and then there is a combinatorial challenge of how to make this stuff all fit together. None of the choices seem particularly canonical, and they all seem fairly complicated. Blech.

Of the many possible approaches there are two that are most commonly used:

(a) Simplicial sets. Here we use simplices as our model for "higher homotopies". The definitions of the objects are not too bad, and one gets a completely combinatorial world in which to work. There are issues, though. One cannot compose edges in a simplicial set, and so one has to find ways around that and related issues. Also, there is certainly nothing canonical about simplices—one might have used cubes just as well, and some problems seem to call out for cubes or even more complicated "parameter spaces". As a result, certain constructions end up seeming unnecessarily complicated when done in the simplicial world.

(b) Topological spaces. This is a kind of Gordian-knot style solution. Observe that topological spaces already have all of the structure we are trying to model, the only problem is that they have too *much* structure. If we only care about the homotopies and higher homotopies inside a space then we don't care about the open sets or other homeomorphism-type information, and we have to find a way to systematically forget that data.

We are not going to introduce the language of model categories here, but the point of that technology is that it offers a systematic approach to "doing homotopy theory" that applies in both of the above worlds (as well as many others). We have two model categories $s\mathcal{S}et$ and $\mathcal{T}op$ as well as adjoint functors $s\mathcal{S}et \rightleftarrows \mathcal{T}op$ allowing us to pass back and forth between them.

Via these models every topological space yields an object in HS, but we also obtain other sources for such objects. Every small category $\mathcal{C}$ has a nerve $N\mathcal{C} \in s\mathcal{S}et$—with the n-simplices being strings of n-composable maps—and the associated object of HS is denoted $B\mathcal{C}$ and called the *classifying space* of $\mathcal{C}$ (perhaps we should say "classifying h-set", but we are going to start reverting to more standard terminology). Likewise, there is the Dold-Kan equivalence of categories $\mathcal{C}h_{\geq 0}(\mathbb{Z}) \cong s\mathcal{A}b$ and via this every non-negatively-graded chain complex of abelian groups yields an associated simplicial set—and hence an object of HS.

47.2. **Homotopical monoids.** Now that we have the notion of an h-set (or at least, the vague idea that there is a world of such things), perhaps an algebraic-minded person would next ask: what is an *h-monoid*? For historical reasons these usually go under the name **A_∞-spaces**: such a thing is an h-set X together with a distinguished point $1 \in X$ and a map $\mu\colon X \times X \to X$ that is "associative and unital up to infinite homotopies". As one would expect, it takes some serious work to make sense of this. We sail past this point without much comment.

Let us note that an A_∞-space is more than just the triple $(X, \mu, 1)$: it is also the infinite collection of "higher homotopies" alluded to in our vague definition. So there can exist multiple A_∞-structures with the same underlying triple $(X, \mu, 1)$. The convention is to be somewhat sloppy with language and to say things like "X is an A_∞-space" in place of "X, with a bunch of hidden data that you might guess but we are not exactly going to spell out, is an A_∞-space". Such is life.

Although we will not need them, let us mention that an A_n-space is like an A_∞-space but where one only has higher homotopies up through level n, with level 2 just consisting of the product $X \times X \to X$ itself.

Here are several examples of A_∞-spaces:

(1) Any monoid in sets, or more generally any topological space with a continuous multiplication that is associative and unital on the nose.
(2) Any loop space ΩX.
(3) Any (homologically graded) differential graded algebra concentrated in non-negative degrees (i.e. a monoid in $\mathcal{C}h_{\geq 0}(\mathbb{Z})$).
(4) If $\mathcal{C}$ is a small monoidal category then the monoidal structure induces a product on $B\mathcal{C}$ making this into an A_∞-space.
(5) For any space X we have the monoid $\mathrm{Vect}(X) = \coprod_n \mathrm{Vect}_n(X)$ of complex vector bundles under direct sum. The sets $\mathrm{Vect}^n(X)$ are represented by $BU(n)$ (or $BGL_n(\mathbb{C})$ if you prefer), and direct sum is represented by the maps of spaces $BU(k) \times BU(n) \to BU(k+n)$ induced by the block sum homomorphism maps $U(k) \times U(n) \to U(k+n)$, i.e.
$$(A, B) \mapsto \begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$
These maps give a multiplication on the space
$$\coprod_n BU(n)$$
that makes it into an A_∞-space

(6) In analogy to (5), if R is any discrete ring then $\coprod_n BGL_n(R)$ has the structure of an A_∞-space. We can also consider
$$\coprod_P B\,\mathrm{Aut}(P)$$
where the coproduct is over the finitely-generated R-projectives. Here one has to be careful as to how to get an indexing *set* for the coproduct. Dodging this point, this is another example of an A_∞-space. Another approach to the same (up to homotopy) space is to take the nerve of the category of fininitely-generated R-projectives with isomorphisms, with the monoidal structure of direct sum (here again one should technically restrict to a small skeletal subcategory).

(7) We can replace $U(n)$ or $GL_n(R)$ with the symmetric groups Σ_n, again using the map $B\Sigma_k \times B\Sigma_n \to B\Sigma_{k+n}$ induced by block sum of permutations. The space $\coprod_n B\Sigma_n$ is again A_∞.

(8) If X is a pointed topological space then the James construction $J(X)$ is the free monoid generated by the points of X with the basepoint as unit, suitably topologized. As $J(X)$ is a topological monoid, it is an A_∞-space.

(9) If X is an A_∞-space then any mapping space $\mathrm{Map}(W, X)$ will inherit an A_∞-structure. In particular, applying this to example (5) shows that if W is path-connected then
$$\coprod_n \mathrm{Map}(W, BU(n))$$
is A_∞.

Every monoid M can be regarded as a category with one object (and endomorphism monoid M), and as such it has a classifying space BM. If we model BM as the geometric realization of the nerve, there is an evident map $M \to \Omega(BM)$ and this is compatible with multiplication (up to homotopy). This generalizes: every A_∞-space X has a classifying space BX, which comes with a map of A_∞-spaces $X \to \Omega BX$. (We have of course not explained what a map of A_∞-spaces is, but so it goes).

47.3. **Homotopical groups and abelian groups.** Now that we know about h-monoids, how about *h-groups*? We might as well keep this party going. Here the answer turns out to be pretty easy, though: an h-group is just an h-monoid X with the property that $\pi_0(X)$ is a group. One can imagine other, more sophisticated, definitions, but they all turn out to be equivalent to this more simplistic one.

Note that every loop space ΩX will be an h-group. It turns out these are all of them!

Theorem 47.4. *Every h-group X is equivalent (as an h-group) to ΩY for some space Y. In fact the map $X \to \Omega BX$ is an equivalence.*

What about h-analogs of commutative monoids and abelian groups? The analogs of commutative monoids are called E_∞-spaces: these are A_∞-spaces together with an infinite collection of higher homotopies governing commutativity. An E_n-space is the associated concept where one only has homotopies up through level n. For example, E_1-spaces are just A_∞-spaces. We define E_n-groups to be E_n-spaces where π_0 is a group.

Theorem 47.5. *The E_n-groups are precisely the n-fold loop spaces $\Omega^n X$.*

Again, several examples:

(1) Every topological commutative monoid is an E_∞-space. For example, if X is a pointed space then the infinite symmetric product $\mathrm{Sp}^\infty(X)$ is E_∞.
(2) If $\mathcal{C}$ is a symmetric monoidal category then $B\mathcal{C}$ is an E_∞-space.
(3) The spaces $\coprod_n B\Sigma_n$, $\coprod_n BGL_n(R)$, $\coprod_P B\,\mathrm{Aut}(P)$, and $\coprod_n BU(n)$ are all E_∞-spaces. The first three are the classifying spaces for the categories of finite sets with isomorphisms, finitely-generated free R-modules with isomorphisms, and finitely-generated projective R-modules with isomorphisms. The last is the classifying space for the *topological* category of finitely-generated $\mathbb{C}$-vector spaces with isomorphisms.

There is a subtlety in Theorem 47.5 that requires some explanation. An n-fold loop space is a space $X = X_0$ together with a sequence of deloopings: spaces $X_1, \ldots, X_n$ and weak equivalences $X_i \simeq \Omega X_{i+1}$. But observe that the space X_0 can have different choices for the deloopings: X_1 can be replaced with any of its components, X_2 can be replaced with its universal cover, etc. The information in the original E_n-group really only corresponds to the *connective* deloopings: the choices where each X_i is $(i-1)$-connected.

The notion of E_∞-space is a reasonable candidate for what "h-abelian group" should mean, but it is not the only candidate. Another perspective is that the "h-abelian groups" are spectra. The difference here is not huge, but is worth commenting on; it is related to the concerns of the preceding paragraph. An E_∞-space X is an infinite-loop space, and therefore is $\Omega^\infty(E)$ for some spectrum E. However, the spectrum E is not unique. For example, a spectrum and its connective cover always have the same Ω^∞.

What is true is that there is (essentially) an equivalence between E_∞-spaces and *connective* spectra. It is common to be somewhat sloppy about the distinction between these two, and we will follow that approach in what follows. So the two different perspectives on "h-abelian groups" amount to just the difference between connective and non-connective spectra. We will not need to care about this distinction for our present purposes.

47.6. **Homotopical group completion.** In usual (non-homotopical) algebra, group completion is the left adjoint to the inclusion i of Groups into Monoids:

$$(\text{Monoids}) \underset{i}{\rightleftarrows} (\text{Groups}).$$

If M is a monoid then the group completion is often denoted M^+. It can be constructed as the quotient of the free group on the underlying pointed set of M by the normal subgroup generated by elements $[x][y][xy]^{-1}$ for all $x, y \in M$. Sometimes the group completion is also referred to as the Grothendieck group of the monoid M. Of course $\mathbb{Z}$ is the group completion of $\mathbb{N}$, and when X is compact $K^0(X)$ is the group completion of the monoid of isomorphism classes of vector bundles on X.

In the homotopical setting we can ask for a homotopical adjoint to the inclusion of h-groups into h-monoids. This is "homotopical group completion". If one starts with an E_∞-monoid then homotopical group completion will produce an E_∞-group, i.e. an infinite loop space. Homotopical group completion is usually denoted by $M \mapsto M^+$ just as in the non-homotopical setting, but we will write $M \mapsto M^{+h}$ to avoid confusion.

There is always an A_∞ map $X \to \Omega(BX)$, and the target is an h-group. It is reasonable to expect this target to be the group completion, and it is:

Theorem 47.7. *If X is an A_∞-space then $X^{+h} \simeq \Omega(BX)$.*

If M is a discrete monoid then the simplicial model of BM together with the Van Kampen theorem immediately gives that $\pi_1(BM) = M^+$, so that we conclude $\pi_0(M^{+h}) = \pi_0(\Omega BM) = \pi_1(BM) = M^+$. It is not hard to believe that this holds more generally: if M is an h-monoid then $\pi_0(M^{+h}) \cong \pi_0(M)^+$. But the interesting question is what $(-)^{+h}$ does to the higher portions of the homotopy type. This is one answer to the question of "what is K-theory?"—at some fundamental level it is the study of homotopical group completion. We constructed the group $K^0(X)$ as the ordinary group completion of the monoid of isomorphism classes of vector bundles on X, but what we should have really been doing is taking homotopical group completion of an A_∞-space of of vector bundles on X. We will say more about this in a moment.

One would almost certainly guess that if M is a discrete monoid then M^{+h} is also discrete, and homotopy equivalent to M^+. But this turns out to be far from the case! In Exercise 47.18 below we give an example of a discrete monoid M for which $BM \simeq S^2$, so that $M^{+h} \simeq \Omega S^2$; in particular, this M^{+h} has non-vanishing higher homotopy groups. In fact McDuff proved that *every* h-group can be realized as the classifying space of a discrete monoid [McD2]. The moral is that monoids can be quite complicated. The naive guess about M^{+h} is indeed true, however, when M is sufficiently nice: see (1) and (2) in the list below for some cases of this.

Here are several examples of homotopical group completion:

(1) If M is a cancellative commutative discrete monoid then $M^{+h} \simeq M^+$ (cancellative means that $x + y = x + z$ implies $y = z$).
(2) If M is a free discrete monoid then $M^{+h} \simeq M^+$.
(3) $(\coprod_n BU(n))^{+h} \simeq \mathbb{Z} \times BU$.
(4) $(\coprod_n BGL_n(R))^{+h} \simeq \mathbb{Z} \times K'(R)$, where $K'(R)$ is a connected space whose homotopy groups are the higher K-groups of R.
(5) $(\coprod_P B\mathrm{Aut}(P))^{+h} \simeq K_0(R) \times K'(R) \simeq K(R)$, where $K(R)$ is the Quillen K-theory space of R.
(6) $(\coprod_n B\Sigma_n)^{+h} \simeq Q(S^0) = \Omega^\infty\Sigma^\infty(S^0)$, the 0th space in the Ω-spectrum for the sphere spectrum. Note that $\pi_i(Q(S^0)) = \pi_i(S)$, the ith stable homotopy group of spheres.
(7) If $\mathcal{C}$ is a symmetric monoidal category then $(B\mathcal{C})^{+h}$ is the Quillen-Segal algebraic K-theory space of $\mathcal{C}$ (the 0th space of the algebraic K-theory spectrum).

We have seen that even if M is discrete then M^{+h} can nevertheless be quite complicated, in the sense of having nonvanishing higher homotopy. In examples (4)–(6) above we find examples of A_∞-spaces M having $\pi_i(M) = 0$ for $i > 1$ but where M^{+h} is again quite complex. The algebraic K-groups of a ring are in general unknown, and there are open questions even in the case $R = \mathbb{Z}$. In example (6) we find the stable homotopy groups of spheres in M^{+h}, and of course those are also notoriously complex. Generally speaking, there is no known process for starting with the homotopy groups of M and then cooking up the homotopy groups of M^{+h}.

We will next try to explain some of the common themes one might notice in examples (3)–(6).

For an h-group X, all of the components must have the same homotopy type: any $x \in X$ has an inverse y in $\pi_0(X)$, and then multiplication-by-x and multiplication-by-y give homotopy inverses between the components X_1 and X_x. So for an h-group we will always have $X \simeq \pi_0(X) \times X_1$, where X_1 is the component of the identity. Thus, one thing group completion must do is "equalize" the different components in an h-monoid. In many of our examples we have an h-monoid X with $\pi_0(X) = \mathbb{N}$, so let us focus on that situation for a moment. Choose a point $p_1 \in X_1$ and form the direct limit system

$$X_0 \xrightarrow{\cdot p_1} X_1 \xrightarrow{\cdot p_1} X_2 \xrightarrow{\cdot p_1} \cdots .$$

Let X_∞ denote the (homotopy) colimit. The composites $X_i \to X^{+h} \xrightarrow{p_1^{-i}} X^{+h}$ are compatible with the maps in the direct limit system, and so yield an induced map from X_∞ to the 0-component of X^{+h}. If we are lucky and X_∞ is actually an h-group then this will be the end of the story: we will get $X^{+h} \simeq \pi_0(X)^+ \times X_\infty$. This happens in example (3), for instance.

In example (4) an issue arises, which is that the X_∞ space is $BGL(R)$ and therefore has nonabelian π_1. But any A_∞-monoid will have an abelian π_1, so in this example X_∞ cannot be the 0-component of the group completion. The same problem occurs in example (6), where π_1 is the infinite symmetric group Σ_∞. In a moment we will see how to fix up these examples: the short answer is one can recover the group completion by just altering X_∞ in order to get rid of the bad parts of π_1. But to explain this, we need a brief detour on homology.

Quillen discovered a general formula for the homology of certain "well-behaved" group completions. The idea is essentially as follows. To construct X^{+h} from X one can examine how $\pi_0(X)^+$ is made from $\pi_0(X)$ and then try to mimic that at the homotopical level. The difficulty is that if M is a general monoid then the passage from M to M^+ might do all kinds of strange things. But if M satisfies some mild hypothesis then M^+ can be obtained from M by a colimit procedure: for example, the colimit—in the category of sets—of the diagram

$$\mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \cdots \tag{47.8}$$

can be identified with $\mathbb{Z}$ in an evident manner. This was secretely what was lying behind the $\pi_0(X) = \mathbb{N}$ examples we looked at above; note how we replaced the colimit with a homotopical one in order to get X_∞.

If M is any monoid, let EM be the translation category for M: the object set is M and for every $x, y \in M$ there is a map $\rho_x \colon y \to yx$ with the properties that $\rho_1 = \mathrm{id}$ and $\rho_a\rho_b = \rho_{ba}$. Let $\underline{M}$ be the diagram $EM \to \mathcal{S}et$ that sends every object to M and each map ρ_x to right-multiplication-by-x. Let M_∞ be the colimit in $\mathcal{S}et$ of $\underline{M}$. Note that this has a very simple description: an element in this colimit can be described as a pair (x, y) where y is the object of EM and $x \in \underline{M}(y)$. The relations in the colimit just say that $(x, y) \sim (xz, yz)$ for every $x, y, z \in M$.

There is a map from the diagram $\underline{M}$ to M^+ sending the pair (x, y) to xy^{-1}, and this induces a map of sets $M_\infty \to M^+$. When M is commutative this is an isomorphism: this is the usual construction of the group completion via pairs of elements. The map is even an isomorphism somewhat more generally, whenever the monoid M admits a certain kind of "calculus of fractions". We will leave the reader to dream about the details there.

We now come to the so-called "Group Completion Theorem", due independently to Quillen and to Barratt-Priddy. If X is an A_∞-space and T is a ring then $H_*(X; T)$

becomes a ring under the Pontryagin product, given by

$$H_p(X;T) \otimes H_q(X;T) \longrightarrow H_{p+q}(X \times X;T) \xrightarrow{\mu_*} H_{p+q}(X;T).$$

The degree 0 subring is precisely the monoid ring $T[\pi_0(X)]$, where $\pi_0(X)$ has the monoid structure inherited from the product on X. The following result is from [BPr] and [Q5]:

Theorem 47.9 (Quillen, Barratt-Priddy)**.** *Let T be a ring and let X be an A_∞-space. Suppose that $\pi_0(X)$ is commutative. Then there is an isomorphism*

$$H_*(X^{+h};T) \cong H_*(X;T)[\pi_0(X)^{-1}].$$

(This also holds under somewhat less restrictive hypothesis, where one only assumes that $\pi_0(X)$ admits a certain kind of calculus of fractions).

Now let us return to the example of the A_∞-space $X = \coprod_n BGL_n(R)$. We have seen that the map $X \to X^{+h}$ induces a map $BGL(R) \to (X^{+h})_0$ (the path-component of the identity element $0 \in \pi_0(X) = \mathbb{N}$), but that $BGL(R)$ cannot be $(X^{+h})_0$ because of π_1 issues. However, the group completion theorem shows that $H_*(BGL(R);T) \to H_*((X^{+h})_0;T)$ will be an isomorphism for every ring T: this is because $H_*(X;T)[\pi_0(X)^{-1}]$ is precisely $\mathbb{Z}[t,t^{-1}] \otimes H_*(BGL(R))$. This tells us that $(X^{+h})_0$ is pretty close to being $BGL(R)$, but it is just that the π_1 is getting in the way. So it is reasonable to envision that if we could just carefully kill off some of the π_1 in $BGL(R)$ then we might get X^{+h}.

Quillen invented his "plus construction" to do exactly this. This construction is related to group completion, though not exactly equal to it, and so the terminology "plus construction" is a bit annoying. If W is a space with a normal subgroup $N \leq \pi_1(W)$ that is perfect (meaning $N = [N,N]$) then Quillen defines a new space W^{+Q} (usually just denoted W^+, sadly) by attaching 2-cells to W to kill off N. This space has the property that $\pi_1(W) \to \pi_1(W^{+Q})$ has kernel N and $W \to W^{+Q}$ induces isomorphisms on homology with any coefficients. [For more details on the plus-construction see [Q1, Section 3], [Q2a, Section 12], and [Lo]].

Returning to our space $BGL(R)$, the normal subgroup $[GL(R),GL(R)] \subseteq GL(R) = \pi_1(BGL(R))$ turns out to be perfect. So we can form $BGL(R)^{+Q}$, and the map $BGL(R) \to X^{+h}$ must factor through it:

$$\begin{tikzcd} BGL(R) \arrow[rr] \arrow[dr] & & X^{+h} \\ & BGL(R)^{+Q}. \arrow[ur, dotted] & \end{tikzcd}$$

The two solid-arrow maps are homology isomorphisms (with any coefficients), so the same is true of the dotted arrow map. But for the dotted arrow map the domain and codomain are both simple spaces (π_1 is abelian and the action on the higher homotopy groups is trivial). A homology isomorphism between simple spaces is a weak homotopy equivalence, so we find that $BGL(R)^{+Q} \simeq X_0^{+h}$. That is, the Quillen plus-construction for $BGL(R)$ gives a model for the 0-component of the homotopical group completion of X.

The same exact story plays out for $\coprod_n B\Sigma_n$ to show that $B\Sigma_\infty^{+Q} \simeq (\coprod_n B\Sigma_n)_0^{+h}$.

Finally, let us look briefly at example (5), which was $X = \coprod_P B\mathrm{Aut}(P)$. Here $\pi_0(X)$ is not $\mathbb{N}$ but rather the monoid of isomorphism classes of finitely-generated

R-projectives under direct sum. This is still commutative and so we can form the corresponding directed system and construct our X_∞ as the homotopy colimit. But the free R-modules are cofinal in this system, since every projective is a direct summand of a free. So we find that X_∞ is weakly equivalent to the corresponding homotopy colimit taken over the free R-modules, which is the same space obtained in example (4). The same homology-based arguments as before now show that $(X^{+h})_0 \simeq BGL(R)^{+Q}$.

We could stay on this subject of group completion for a long time, but let us just mention one more theme. Suppose that $\mathcal{C}$ is a symmetric monoidal category, so that $B\mathcal{C}$ is an A_∞-space. Can we do something to $\mathcal{C}$—back in the combinatorial world of categories—that will mirror the passage from $B\mathcal{C}$ to $(B\mathcal{C})^{+h}$? For example, can we construct a category $\mathcal{C}^{+h}$ having the property that $B(\mathcal{C}^{+h}) \simeq (B\mathcal{C})^{+h}$? This was another question considered by Quillen, and the answer—of course—is yes. This is known as Quillen's $S^{-1}S$-construction, though in our case it would technically be $\mathcal{C}^{-1}\mathcal{C}$. We will give a brief overview, but see [Gr0] for details.

Let $(\mathcal{C}, \oplus)$ be a monoidal category where every map is an isomorphism. Define $\mathcal{C}^{-1}\mathcal{C}$ to be the category with object set $\mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{C})$, and where a morphism $(X_1, X_2) \to (Y_1, Y_2)$ is an equivalence class of triples $[c, f\colon c \oplus X_1 \to Y_1, g\colon c \oplus X_2 \to Y_2]$ where the maps f and g are isomorphisms. The equivalence relation has $[c, f, g] \sim [c', f', g']$ if there is an isomorphism $c \to c'$ making the two evident diagrams commute. If $(\mathcal{C}, \oplus)$ is symmetric monoidal then one can define an action of $\mathcal{C}$ on $\mathcal{C}^{-1}\mathcal{C}$ by $c \odot (X_1, X_2) = (X_1, c \oplus X_2)$, and this is readily checked to be an invertible action (the inverse is given by $c \odot (X_1, X_2) = (c \oplus X_1, X_2)$).

Continuing to assume that $\mathcal{C}$ is symmetric monoidal, we get an induced monoidal structure on $\mathcal{C}^{-1}\mathcal{C}$ where $(X_1, X_2) \oplus (Y_1, Y_2) = (X_1 \oplus Y_1, X_2 \oplus Y_2)$. The symmetry isomorphism is needed for the induced behavior on maps. It is easy enough to check that $\pi_0(\mathcal{C}^{-1}\mathcal{C}) \cong (\pi_0\mathcal{C})^+$. In fact we have the analogous statement at the homotopical level:

Theorem 47.10 (Quillen)**.** *Suppose $(\mathcal{C}, \oplus)$ is a symmetric monoidal category where every map is an isomorphism. Then $B(\mathcal{C}^{-1}\mathcal{C}) \simeq (B\mathcal{C})^{+h}$.*

Although Theorem 47.10 is stated for discrete categories, one can play similar games in the generalized setting of topological categories. Here is one example:

Theorem 47.11. *Let X be a compact Hausdorff space, and let $\mathcal{V}_X$ denote the topological category of complex vector bundles on X with the maps being the isomorphisms. Then*

(a) If $\mathcal{V} = \mathcal{V}_{pt}$ then $B\mathcal{V} \simeq \coprod_n BU(n)$ and $B(\mathcal{V}^{-1}\mathcal{V}) \simeq (\coprod_n BU(n))^{+h} \simeq \mathbb{Z} \times BU$.

(b) $B\mathcal{V}_X \simeq \coprod_n \mathrm{Map}(X, BU(n))$ and $B(\mathcal{V}_X^{-1}\mathcal{V}_X) \simeq \mathrm{Map}(X, \mathbb{Z} \times BU)$. In particular, $\pi_p(B\mathcal{V}_X) \cong K^{-p}(X)$.

47.12. **Exact sequences.** As we know from our study of K^0, sometimes the Grothendieck group of interest arises not as the group completion of a monoid but rather as the universal additive group in which exact sequences are forced to split. To incorporate this kind of framework into our picture we should look for constructions having the form

$$\text{(homotopical gadgets involving a notion of exact sequence)} \longrightarrow (E_\infty\text{-spaces}),$$

where historically the domain has been made explicit with progressively more sophisticated notions. Going very quickly through some history here:

(a) Quillen defined his "Q-construction", which takes a category $\mathcal{C}$ with a notion of exact sequence and outputs a spectrum $K(\mathcal{C})$.
(b) Waldhausen developed machinery that takes a so-called "category with a cofibrations" $\mathcal{D}$ (a category with a well-behaved notion of cofiber sequence) and outputs a spectrum $K_{Wald}(\mathcal{D})$. This allows one to talk about the K-theory spectrum for certain categories of chain complexes, with their homotopical notion of cofiber sequence.
(c) After the introduction of ∞-categories, it was realized early on that the Waldhausen machinery could be adapted to take as its input any stable ∞-category. Blumberg-Gepner-Tabuada [BGT] proved that in this setting one can describe Waldhausen K-theory as the universal construction satisfying a short list of familiar axioms (Morita invariance, additivity, and compatibility with colimits, for those who know what all those words mean).

When $\mathcal{C}$ is a symmetric monoidal category equipped with the notion of split-exact sequence, Quillen's Q-construction agrees (up to homotopy) with the output of the $\mathcal{C}^{-1}\mathcal{C}$-construction we saw earlier. Likewise, Waldhausen K-theory agrees with Quillen K-theory in the cases where one would hope for that. For example, if X is a scheme then the Waldhausen K-theory of perfect complexes over X agrees with the Quillen Q-construction applied to the category of locally-free coherent sheaves on X, and if X is affine then this in turn agrees with the $S^{-1}S$-construction applied to the category of finitely-generated projective R-modules where $R = \mathcal{O}(X)$, which in turn agrees with $K_0(R) \times BGL(R)^{+Q} \simeq (\coprod_P B\mathrm{Aut}(P))^{+h}$.

47.13. **This is all very interesting, but what is the punchline?**

In all of our discussion of K-theory throughout this book, the groups $K^i(X)$ for $i \neq 0$ have always been a bit nebulous. All too often we end up getting our hands on them by using Bott periodicity to shift them to a K^0, which often feels unsatisfying. Even the relative groups $K^0(X, A)$ are a bit intimidating; here we saw different geometric models (e.g. Atiyah's difference bundles, or chain complexes of bundles, or the $\mathcal{L}_\infty$ model from ???) but a preferred model never seemed to rise to the top.

Keeping the above comments in mind, let us now return to the beginnings—but from a somewhat different perspective—and see if we can come to terms with some of this. If X is a compact Hausdorff topological space let $\underline{\mathrm{Vect}}(X)$ be the *space* of vector bundles on X, regarded as an A_∞-space via direct sum. The homotopy type is

$$\underline{\mathrm{Vect}}(X) \simeq \coprod_{n\geq 0} \mathrm{Map}(X, BU(n)).$$

K-theory (and really we should say *connective* K-theory) is the study of the homotopical group completion $\underline{\mathrm{Vect}}(X)^{+h}$, but as we have discussed this is *a priori* a somewhat nebulous object. There is no known general procedure for understanding the homotopy groups of M^{+h} in terms of the homotopy groups of the original A_∞-space M.

One of the "themes" of K-theory is finding models for $\underline{\mathrm{Vect}}(X)^{+h}$ in terms of the geometry of X, and perhaps it is not surprising to discover that there are many such models. Some of these are ????. A recent line of research involves describing a model in terms of certain kind of topological field theories on X: see [ST], [HST], [H], and [U] as just a few places to get started in that area.

47.14. **Exercises.** The exercises below concern discrete monoids M together with their group completions M^+ and homotopical group completions M^{+h}.

Exercise 47.15. Let $M = \{1, h\}$ where $h^2 = h$. Check that M is a monoid and $M^+ = 1$. For each $n \in \mathbb{Z}_+$ find a monoid generated by one element whose group completion is $\mathbb{Z}/n$. In particular, these are examples where $M \to M^+$ is not injective.

Exercise 47.16. Let M be a monoid. Let $M \triangleright h$ denote the monoid whose elements are words involving h and elements of M, subject to the relations that $h^2 = h$ and $mh = h$ for all $m \in M$. Prove that $(M \triangleright h)^+ = 1$ and that $B(M \triangleright h) \simeq *$ (give a contracting homotopy for the simplicial set $B.(M \triangleright h)$). The monoid $M \triangleright h$ acts like a cone on M.

Exercise 47.17. Using the simplicial model for $B.M$ one sees immediately that $H_*(BM) = \mathrm{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z})$. And more generally, if J is an M-module on which all of the generators act as isomorphisms then J is a $\pi_1(BM)$-module and $H_*(BM; J) \cong \mathrm{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, J)$.

(a) Let $M = \langle f \rangle$, the free monoid on one generator f. Let $\alpha\colon S^1 \to BM$ be the evident map sending the 1-simplex of S^1 to f. Prove that α induces an isomorphism on π_1 and on $H_*(-; J)$ for every local coefficient system, and is therefore a weak equivalence.

(b) Generalize part (a) to show that if M is any free discrete monoid then BM is weakly equivalent to a wedge of circles. Deduce that $M^{+h} \simeq M^+$.

Exercise 47.18. Following [F] we will produce a 5-element monoid M such that $BM \simeq S^2$. Then $M^{+h} \simeq \Omega BM \simeq \Omega S^2$, which has nonvanishing higher homotopy groups. So this will be an example where $M^{+h} \not\simeq M^+$.

Let $M = \{1, x_{ij}\}_{1 \le i,j \le 2}$ with the product given by $x_{ij} x_{mn} = x_{in}$.

(a) Verify that M is associative and that $M^+ = 1$.

(b) Let $P_1 = \{x_{11}, x_{12}\}$ and $P_2 = \{x_{21}, x_{22}\}$. As subsets of M these are closed under right multiplication by M. Prove that the map of right $\mathbb{Z}[M]$-modules $\mathbb{Z}[M] \to \mathbb{Z}\langle P_1 \rangle$ sending $1 \mapsto x_{11}$ is split surjective, proving that $\mathbb{Z}\langle P_1 \rangle$ is projective as a right $\mathbb{Z}[M]$-module. Repeat for P_2.

(c) Show that $\mathbb{Z}$ has a projective resolution as right $\mathbb{Z}[M]$-module taking the form

$$0 \longrightarrow \mathbb{Z}\langle P_1 \rangle \oplus \mathbb{Z}\langle P_2 \rangle \longrightarrow \mathbb{Z}[M] \longrightarrow \mathbb{Z}\langle P_1 \rangle \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Use this to prove that $\mathrm{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z}) \cong H_*(S^2)$.

(d) $\pi_1(BM) \cong M^+ = 1$, so by the Hurewicz theorem the map $\pi_2(BM) \to H_2(BM)$ is an isomorphism. Verify that if $f\colon S^2 \to BM$ is a generator for π_2 then f is a homotopy equivalence.

Exercise 47.19. Here we will give another example of a monoid M such that $BM \simeq S^2$. This monoid will not be finite, but the construction will suggest some generalizations.

Let $L = \langle f \rangle$ be the free monoid on one generator. Let M consists of words in the symbols f, h_1, and h_2 subject to the relations that $fh_1 = h_1$, $fh_2 = h_2$, $h_1^2 = h_1$, and $h_2^2 = h_2$. Note that M is the pushout $(L \triangleright h_1) *_L (L \triangleright h_2)$ in the category of monoids, so one can think of it as being the monoid L with two cones attached. We claim that $BM \simeq S^2$.

(a) As a plausibility check, try to compute $H_*(BM)$ algebraically as $\mathrm{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z},\mathbb{Z})$. If we set
$$R = \mathbb{Z}[M] = \mathbb{Z}[f,h_1,h_2]/(fh_1 = h_1, fh_2 = h_2, h_1^2 = h_1, h_2^2 = h_2)$$
then try to show that $\mathbb{Z} = R/(f-1, h_1-1, h_2-1)$ has a free resolution that starts out as
$$\cdots \to R^4 \to R^3 \to R \to \mathbb{Z} \to 0$$
and find a nonzero class in $\mathrm{Tor}_2^R(\mathbb{Z},\mathbb{Z})$.

(b) Here is a better way to compute $H_*(BM) = \mathrm{Tor}_*^{\mathbb{Z}[M]}(\mathbb{Z},\mathbb{Z})$. Set $N_i = L \triangleright h_i \subseteq M$. Before diving in, note that we have the inclusion
$$j\colon BN_1 \cup_{BL} BN_2 \hookrightarrow BM$$
and that the domain of j, being the union of two contractible "cones" over BL, has the homotopy type of $\Sigma(BL) \simeq S^2$. The idea of our computation will be to break the Tor into pieces analgous to the decomposition of S^2 into two hemispheres. Start by proving that M is the disjoint union of left cosets sL where s ranges over the set consisting of 1 and all words in M ending in either h_1 or h_2. Consequently, $\mathbb{Z}[M]$ is free as a right $\mathbb{Z}[L]$-module, on the corresponding set of generators. Do something similar to analyze $\mathbb{Z}[M]$ as a right $\mathbb{Z}[N_i]$ module for $i = 1,2$, in particular showing that it is free.

(c) Use the generators from the previous part to prove that there is a short exact sequence
$$0 \longrightarrow \mathbb{Z}[M] \otimes_{\mathbb{Z}[L]} \mathbb{Z} \longrightarrow (\mathbb{Z}[M] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}) \oplus (\mathbb{Z}[M] \otimes_{\mathbb{Z}[N_2]} \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$
Then use the long exact sequence for $\mathrm{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z},-)$ to prove that $\mathrm{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z},\mathbb{Z}) \cong \mathrm{Tor}_{i-1}^{\mathbb{Z}[L]}(\mathbb{Z},\mathbb{Z})$ for $i \geq 1$ (use the change-of-ring isomorphisms $\mathrm{Tor}_i^S(M, S \otimes_R N) \cong \mathrm{Tor}_i^R(M,N)$ that hold when $R \to S$ is flat). Deduce that $H_*(BM) \cong H_*(S^2)$.

(d) Prove that j is an isomorphism on π_1 and on $H_*(-;\mathbb{Z})$, and so obtain that j is a weak homotopy equivalence.

Exercise 47.19 is a particular case of the analysis done in [McD2]. In fact it is not a huge leap to go from the above ideas to seeing how to construct monoids whose classifying spaces achieve any desired homotopy type.

Exercise 47.20. In Exercise 47.18 we saw a 5-element monoid where $\Omega(BM) \not\simeq M^+$. Are there smaller examples of this?

(a) Prove that up to isomorphism there are exactly two monoids with two elements: one has $BM \simeq \mathbb{R}P^\infty$ and one has $BM \simeq *$. Both have $\Omega(BM) \simeq M^+$.

(b) Prove that up to isomorphism there are exactly seven monoids with three elements, and construct them all. One has $M^+ = \mathbb{Z}/3$, one has $M^+ = \mathbb{Z}/2$, and all of the others have $M^+ = 1$. Prove that in the first case $BM \simeq B\mathbb{Z}/3$, in the second $BM \simeq \mathbb{R}P^\infty$, and in all of the other cases BM is contractible. So all of these monoids have $\Omega(BM) \simeq M^+$. [Hint for the middle case: prove that the map $BM \to B\mathbb{Z}/2$ induced by the monoid map $M \twoheadrightarrow \mathbb{Z}/2$ is an isomorphism on homology with arbitrary coefficients.]

(c) Apparently there are 35 monoids with four elements. I do not know if they all satisfy $\Omega(BM) \simeq M^+$. Explore this!

Part 8. Appendices

Appendix A. Some point-set topology

Exercise A.1. Let $X: I \to \mathcal{T}op$ be a diagram of topological spaces and let Z be locally compact and Hausdorff. Prove that the natural map $\operatorname{colim}_i(X_i \times Z) \to (\operatorname{colim}_i X_i) \times Z$ is a homeomorphism. (Hint: Use [Theorem 46.11][Mu].)

Exercise A.2. Consider a diagram of spaces

$$\begin{tikzcd} X_0 \arrow[r, tail] \arrow[d, two heads] & X_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \end{tikzcd}$$

where the top horizontal maps are injections and the vertical maps are quotient maps. Prove that $\operatorname{colim}_s X_s \to \operatorname{colim}_s Y_s$ is a quotient map.

A.3. Nets and sequences.

Exercise A.4 (Crash course on nets)**.** Let X be a space. Recall that a net in X is a directed set I together with a function $x\colon \operatorname{ob}(I) \to X$. The net is said to have limit $L \in X$ if for every open set U containing L there is an $i \in I$ such that $x_j \in U$ for all $j \geq i$.

(a) Check that if $f\colon X \to Y$ is continuous and x is a net with limit L then $f \circ x$ is a net with limit $f(L)$. (Briefly, we say that continuous functions preserve convergent nets).
(b) Let $x \in X$ and consider the category of open neighborhoods of X where the maps are opposite arrows to inclusions. Check that this is directed.
(c) Let $A \subseteq X$. Prove that $u \in \bar{A}$ if and only if there is a net in A whose limit in X is u. That is, $\bar{A}$ is the set of limit points of nets in X that lie in A.
(d) Prove that a map $f\colon X \to Y$ is continuous if and only if it preserves convergent nets (as in (a)).
(e) If I is a directed set, define a topological space $\hat{I} = I \cup \{\infty\}$ by taking as basis all the singletons $\{i\}$ as well as the sets $[i, \infty) = \{j \in \hat{I} \,|\, i \leq j\}$, for $i \in I$. Note that a subset $S \subseteq \hat{I}$ containing ∞ is open if and only if it intersects I and is closed under inclusion of larger elements. Prove that a net $x\colon I \to X$ has limit L if and only if the function $\hat{x}\colon \hat{I} \to X$ that extends x and has $\hat{x}(\infty) = L$ is continuous.

A map between directed sets $f\colon J \to I$ is a function with the property that whenever $j_1 \leq j_2$ one has $f(j_1) \leq f(j_2)$. A map f is **cofinal** if for every $i \in I$ there exists $j \in J$ such that $i \leq f(j)$. Given a net $x\colon I \to X$, a **refinement** of x is a net of the form $x \circ f$ where J is a directed set and $f\colon J \to I$ is a cofinal map (refinements are also called **subnets** in the literarure, though it is important to note that the map f need not be injective).

Given a net $x\colon I \to X$ and a subset $S \subseteq X$, say that the net **consistently returns to** S if $\{i \in I \,|\, x_i \in S\}$ is cofinal in I—or equivalently, for every $i \in I$ there is a $j \geq i$ such that $x_j \in S$. Given a net $x\colon I \to X$, a point $w \in X$ is an **accumulation point** of x if the net consistently returns to every neighborhood of w.

Exercise A.5 (Crash course on nets, part 2)**.** Given a net $x\colon I \to X$, prove that w is an accumulation point of x if and only if there is a refinement of x that converges to w.

Exercise A.6. Consider the following net in $\mathbb{R}^\omega$ with the box topology:

$$0,\ e_1,\ \tfrac{1}{2}e_2,\ \tfrac{1}{4}e_3,\ \ldots,\ 0,\ \tfrac{1}{2}e_1,\ \tfrac{1}{4}e_2,\ \tfrac{1}{8}e_3,\ \ldots,\ 0,\ \tfrac{1}{4}e_1,\ \tfrac{1}{8}e_2,\ \tfrac{1}{16}e_3,\ \ldots$$

This net is indexed on the ordinal $\omega\cdot\omega$. Prove that this net does not converge to 0.

Let X be a space. We will say "$(x; L)$ is a convergent sequence in X" as a synonym for x being a convergent sequence in X with limit L.

Say that a subset $A \subseteq X$ is **sequentially open** if every convergent sequence $(x; L)$ with $L \in A$ is eventually in A: that is, there exists an $n \geq 1$ such that $x_k \in A$ for all $k \geq n$. Say that $A \subseteq X$ is **sequentially closed** if every convergent sequence $(x; L)$ with values in A has $L \in A$. Note that every open subset is sequentially open, and every closed subset is sequentially closed.

Say that a function $f\colon X \to Y$ is **sequentially continuous** is whenever $(x; L)$ is a convergent sequence in X then $(f(x); f(L))$ is a convergent sequence in Y. Every continuous function is sequentially continuous.

Exercise A.7. Let X be a space.

(a) Check that a subset of X is sequentially open if and only if its complement is sequentially closed. Prove that the sequentially open sets define a topology on X; we will denote this X_{seq}. The identity is a continuous map $X_{seq} \to X$.

(b) Prove that a function $f\colon X \to Y$ is sequentially continuous if and only if $f\colon X_{seq} \to Y$ is continuous.

(c) Prove that the following conditions on X are equivalent:
 (i) Every sequentially open subset is open.
 (ii) Every sequentially closed subset is closed.
 (iii) For every space Y, every sequentially continuous map $f\colon X \to Y$ is continuous.

 A topological space X satisfying the above conditions is called **sequentially determined**, or often just **sequential** for brevity.

(d) Prove that an open subset of a sequential space is sequential, and the same for closed subsets.

(e) Prove that any colimit of a diagram of sequential spaces is again sequential. In particular, any quotient space of a sequential space is sequential.

(f) Prove that any first-countable space is sequential (in particular, any metric space is sequential). Conclude that every CW-complex is sequential.

(g) Suppose X has an open cover $\{U_\alpha\}$ where each U_α is sequential. Then X is sequential.

(h) Let $J = \{0\} \cup \{\frac{1}{n} \,|\, n \geq 1\} \subseteq \mathbb{R}$ be equipped with the subspace topology. Check that a function $x\colon J \to X$ is continuous if and only if $x(0)$ is the limit of the sequence $n \mapsto x(\frac{1}{n})$.

(i) For any space X consider the category $J \downarrow X$ whose objects are maps $J \to X$ and whose arrows are commutative triangles that involve an endomorphism of J. There is a canonical functor $(J \downarrow X) \to \mathcal{T}op$ sending each $f\colon J \to X$ to J; denote the colimit by $\operatorname{colim}_{J\to X} J$. There is a canonical map $\theta_X\colon \operatorname{colim}_{J\to X} J \to X$. Prove that X is sequentially determined if and only if θ_X is a homeomorphism.

(j) Suppose that Z is a space that is locally compact Hausdorff and is such that $J \times Z$ is sequentially determined (for example, this holds whenever Z is locally compact Hausdorff and is first-countable). Use the previous parts to show that if X is sequentially determined then so is $X \times Z$, for any space X. (Hint: Use Exercise A.1).

A.8. **Separated maps.**

Exercise A.9.

(a) Let X be a topological space. Show that the following conditions are equivalent:
 (1) X is Hausdorff,
 (2) If a net in X converges to two points x_1 and x_2, then $x_1 = x_2$.
 (3) The diagonal $\Delta \subseteq X \times X$ is closed.

(b) Let $f\colon X \to Y$ be a continuous map. Show that the following conditions are equivalent:
 (1) If a net in X converges to two points x_1 and x_2 and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
 (2) For any directed set I, any diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d, tail] & X \arrow[d, "f"] \\ \hat{I} \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

 has at most one lifting as shown.
 (3) The diagonal $\Delta \subseteq X \times_Y X$ is closed.

 Maps f satisfying these equivalent conditions are called **separated**.

A.10. **Proper maps.** Properness is a relative form of compactness, so let us begin by stating various equivalent conditions for a space to be compact.

Exercise A.11. Let X be a topological space. Prove that the following conditions are equivalent:

(1) X is compact;
(2) Every net $x\colon I \to X$ has a convergent refinement (or equivalently, an accumulation point);
(3) For every space Z the projection $X \times Z \to Z$ is a closed map.

[Hint: (1)⇒(2)⇒(3) are fairly straightforward. For (2)⇒(1), given a cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ with no finite subcover construct a net indexed on the poset of finite subsets of $\mathcal{A}$. For (3)⇒(2), given a net $x\colon I \to X$ take $Z = \hat{I}$ from Exercise A.4.]

Exercise A.12. Let $f\colon X \to Y$ be a map of topological spaces. Prove that the following are equivalent:

(1) Every net $x\colon I \to X$ whose image in Y converges to a point y has a refinement that converges to a point $x \in f^{-1}(y)$; said differently, every diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d, tail] & X \arrow[d, "f"] \\ \hat{I} \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

 has a lifting as shown;
(2) For every space Z and map $Z \to Y$ the pullback $X \times_Y Z \to Z$ is a closed map;

(3) For every space W the map $f \times \mathrm{id}\colon X \times W \to Y \times W$ is a closed map;
(4) f is a closed map and for every compact subset $K \subseteq Y$, the preimage $f^{-1}(K)$ is compact;
(5) f is a closed map and every fiber is compact.

[Outline: Prove that maps satisfying (1) are closed, and also that property (1) is stable under pullbacks. Then prove (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(1). For the last implication, if x does not have an accumulation point in $f^{-1}(y)$ then every $w \in f^{-1}(y)$ has a neighborhood that is not consistently returned to by x. Choose finitely many of these neighborhoods $U_1, \dots, U_r$ that cover $f^{-1}(y)$ and then consider $f(X - (U_1 \cup \cdots \cup U_r))$.]

Definition A.13. *Let $f\colon X \to Y$ be a continuous map.*

*(a) f is **proper** if it satisfies any of the equivalent conditions listed in Exercise A.12.*
*(b) f is **strongly proper** if it is proper and separated, i.e. has the additional property that if a net in X converges to two points x_1 and x_2 and $f(x_1) = f(x_2)$ then $x_1 = x_2$.*
*(c) f is **weakly proper** if the preimage of every compact subset of Y is compact.*

Note that the terms "strongly proper" and "weakly proper" are not standard, and in fact all three notions are called "proper" in various places in the literature. There exist examples showing that the three notions are distinct.

Appendix B. Topological vector spaces and tame families

We start by reviewing some standard material about topological vector spaces over $\mathbb{R}$. There are only two topological vector spaces whose underlying vector space is $\mathbb{R}^n$: $\mathbb{R}^n$ with the standard topology and $\mathbb{R}^n$ with the indiscrete topology. However, when one gets to vector spaces of countably infinite dimension things become more complicated. We begin by reviewing some tools for comparing two topologies on the same set, and then proceed from there to a detailed look at $\mathbb{R}^\infty$.

B.1. **Comparing topologies.** For some reason the language mathematicians use for comparing two topologies on a common set is not universally agreed-upon, and also hard to remember. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set S, the following are equivalent statements:

- $\mathcal{T}_1$ has fewer open sets than $\mathcal{T}_2$, i.e. $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$
- The identity map $S_{\mathcal{T}_2} \to S_{\mathcal{T}_1}$ is continuous.
- $\mathcal{T}_2$ is finer than $\mathcal{T}_1$
- $\mathcal{T}_2$ is a refinement of $\mathcal{T}_1$.

Analysts also use "$\mathcal{T}_2$ is stronger than $\mathcal{T}_1$" or "$\mathcal{T}_1$ is weaker than $\mathcal{T}_2$" as equivalent to the above conditions, but unfortunately topologists often reverse the role of weak/strong here. For example, the "W" in "CW-complex" stands for what topologists call the weak topology on an ascending union, which is finer than other topologies one might consider. So the situation is confusing, and as a result we will try to avoid the weak/strong language. (See [Mu, Section 12] for a similar remark).

If one thinks of a topology as being like a way of distinguishing objects via characteristics—where the open sets are the characteristics—then the fine/coarse terms match up with common usage. A finer topology allows more distinctions, and so has more open sets. It is useful to remember the slogans

$$\text{finer} = \text{more open sets} = \text{easier to map out of.}$$

Finer topologies map to coarser ones. Another way of keeping things straight is to remember that the indiscrete topology is the coarsest topology; one can usually figure out the rest from there.

Example B.2. Recall that if $\{X_\alpha\}$ is a collection of topological spaces then on $\prod_\alpha X_\alpha$ we have the box and product topologies. A basis for the box topology consists of rectangles $\prod U_\alpha$ where each U_α is open in X_α, whereas for the product topology we take rectangles where only finitely many of the U_α differ from X_α. So the box topology has more open sets than the product topology, hence the box topology is the finer one.

Example B.3. More generally, the topology on a limit is the "coarsest topology such that...", whereas the topology on a colimit will be the "finest topology such that...". This can be deduced from the direction of the arrows, remembering always that finer topologies map to coarser ones.

Another way to understand the comparison between two topologies is in terms of the convergent sequences. Continuous maps preserve convergent sequences, so if $\mathrm{id}\colon S_{\mathcal{T}_2} \to S_{\mathcal{T}_1}$ is continuous and a sequence $\{x_i\}$ converges to x_∞ in $\mathcal{T}_2$ then it also converges in $\mathcal{T}_1$. So coarser topologies have more convergent sequences than finer topologies. In the indiscrete topology everything converges to everything.

Convergent sequences do not capture everything about a topology; for that we need to use convergent *nets* instead. Recall that a net in a space X is a directed category I and a function $x\colon I \to X$. The net x converges to a point x_∞ if for every open set U containing x_∞ there exists an $i \in I$ such that $x_j \in U$ for all $j \geq i$. It is a theorem that two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same if and only if they have the same class of convergent nets. Also, a map $f\colon X \to Y$ is continuous if and only if for every convergent map $x\colon I \to X$, $f(\lim x) = \lim f(x)$.

Example B.4. Let $\{X_\alpha\}$ be a collection of topological spaces, and consider the box and product topologies on the product $\prod_\alpha X_\alpha$. Write π_α for the projection onto X_α. Then a net $t\colon I \to \prod_\alpha X_\alpha$ is convergent to T in the product topology if and only if $\pi_\alpha(t)$ converges to $\pi_\alpha(T)$ in X_α, for all α.

It is more complex to analyze the situation in the box topology. The net t is convergent to T in the box topology if for every collection of opens U_α containing $\pi_\alpha(T)$, there exists $i \in I$ such that for all $j \geq i$ we have $\pi_\alpha(t_j) \in U_\alpha$ for all α. This is a very strong condition. As an example, let $\{y_i\}$ be a sequence in $\mathbb{R}$ that converges to a number y. Write $\Delta(y)$ for the diagonal element $(y, y, \ldots)$ in $\prod_{i=1}^\infty \mathbb{R}$. Then $\Delta(y_i)$ converges to $\Delta(y)$ in the box topology means that for every countable collection $U_1, U_2, \ldots$ of neighborhoods of y, there is an N such that for all $k \geq N$ one has $y_k \in \bigcap_i U_i$. Since we can arrange for the infinite intersection to be precisely $\{y\}$, this means that $\Delta(y_i)$ converges to $\Delta(y)$ only when the sequence $\{y_i\}$ is eventually constant. More generally, if all the spaces X_α are Hausdorff and if $\{t_i\}$ is a sequence in $\prod X_\alpha$ that converges to T in the box topology, then one can show that there exists a finite set of indices S and a k such that for all $i \geq k$, t_i agrees with T at all coordinates except possibly the ones in S. In other words, the sequence t is eventually constant in all but finitely-many coordinates. This is a very strong condition on the sequence t.

B.5. $\mathbb{R}^\infty$ **with the colimit topology.** Recall that $\mathbb{R}^\infty$ consists of tuples $(x_1, x_2, \ldots)$ with only finitely many nonzero coordinates. As a set we can write

$$\mathbb{R}^\infty = \operatorname*{colim}_n \mathbb{R}^n$$

where $\mathbb{R}^n$ includes into $\mathbb{R}^{n+1}$ as the subset of tuples whose last coordinate is zero. There are at least four topologies that naturally suggest themselves:

- Give each $\mathbb{R}^n$ its standard topology, and then give $\mathbb{R}^\infty$ the induced topology on the colimit. Denote this as $\mathbb{R}^\infty_{\text{colim}}$.
- Regard $\mathbb{R}^\infty \subseteq \prod_{i=1}^\infty \mathbb{R}$ and give $\mathbb{R}^\infty$ the subspace topology induced from the product topology. Denote this as $\mathbb{R}^\infty_{\text{prod}}$.
- Same as above, but this time give $\mathbb{R}^\infty$ the subspace topology induced from the box topology. Denote this by $\mathbb{R}^\infty_{\text{box}}$.
- The metric topology $\mathbb{R}^\infty_{\text{metric}}$, induced by the usual definition of distance between vectors.

The universal property of the colimit shows that $\mathbb{R}^\infty_{\text{colim}}$ will be finer than the other three topologies. In fact the identity map gives us

$$\mathbb{R}^\infty_{\text{colim}} \longrightarrow \mathbb{R}^\infty_{\text{box}} \longrightarrow \mathbb{R}^\infty_{\text{metric}} \longrightarrow \mathbb{R}^\infty_{\text{prod}}.$$

All of these topologies have the property that if U is open than so is any translate $U + x$ for $x \in \mathbb{R}^\infty$, so we can check continuity by looking at open sets around the origin. For the last map it suffices to observe that an open rectangle $B(0, r_1) \times \cdots \times B(0, r_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$ contains $B(\underline{0}, r)$ where r is the minimum of the r's (or alternatively, the projection maps are continuous for the metric topology). For the second map we observe that $B(\underline{0}, \epsilon)$ contains the product

$$B(0, \tfrac{\epsilon}{2}) \times B(0, \tfrac{\epsilon}{4}) \times B(0, \tfrac{\epsilon}{8}) \cdots$$

by a routine computation.

To see that the last three topologies are all different we can observe that:

- $B(\underline{0}, 1)$ is open in $\mathbb{R}^\infty_{\text{metric}}$ but not in $\mathbb{R}^\infty_{\text{prod}}$;
- $B(0, 1) \times B(0, \frac{1}{2}) \times B(0, \frac{1}{4}) \times \cdots$ is open in $\mathbb{R}^\infty_{\text{box}}$ but not in $\mathbb{R}^\infty_{\text{metric}}$.

Or alternatively,

- The sequence e_1, $\frac{1}{\sqrt{2}}(e_1 + e_2)$, $\frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)$, $\ldots$ converges to $\underline{0}$ in $\mathbb{R}^\infty_{\text{prod}}$ but not in $\mathbb{R}^\infty_{\text{metric}}$;
- The sequence e_1, $\frac{1}{2}(e_1 + e_2)$, $\frac{1}{3}(e_1 + e_2 + e_3)$, $\ldots$ converges to $\underline{0}$ in $\mathbb{R}^\infty_{\text{metric}}$ but not in $\mathbb{R}^\infty_{\text{box}}$.

The reader will have noticed that we have not distinguished the colimit topology from the box topology. That is because they are the same!

Proposition B.6. *The colimit and box topologies on $\mathbb{R}^\infty$ are identical.*

Proof. Our proof requires a sequence of positive numbers $s_1, s_2, \ldots$ having the property that each $s_i < 1$ and $\prod_{i=1}^\infty s_i$ is positive. Many such sequences exist, and the proof below doesn't depend on a specific choice. But one such sequence has terms

$$\tfrac{9}{10}, \ \tfrac{89}{90}, \ \tfrac{889}{890}, \ \tfrac{8889}{8890}, \ldots$$

Note that $\prod_{i=1}^\infty s_i = 0.8888\ldots = \frac{8}{9}$. For a rigorous definition, use $a_1 = 9$, $b_1 = 10$, $b_n = 10a_{n-1}$, and $a_n = b_n - 1$. Set $s_n = \frac{a_n}{b_n}$.

We need to show that every neigbhorhood of $\underline{0}$ in the colimit topology is also open in the box topology. Suppose $\underline{0} \in U \subseteq \mathbb{R}^\infty$ is such a neighborhood. Write $U_n = U \cap \mathbb{R}^n$, which is an open subset of $\mathbb{R}^n$.

There exists an $r_1 > 0$ such that $B(0, r_1) \subseteq U \cap \mathbb{R}^1$. By Lemma B.7 below, there exists $r_2 > 0$ such that $B(0, s_1 r_1) \times B(0, r_2) \subseteq U \cap \mathbb{R}^2$. Applying Lemma B.7 again, there exists $r_3 > 0$ such that $B(0, s_2 s_1 r_1) \times B(0, s_2 r_2) \times B(0, r_3) \subseteq U \cap \mathbb{R}^3$. Continuing in this way, we can choose r_n for all n having the property that

$$B(0, s_{n-1} \cdots s_1 r_1) \times B(0, s_{n-1} \cdots s_2 r_2) \times B(0, s_{n-1} \cdots s_3 r_3) \times \cdots \times B(0, r_n) \subseteq U \cap \mathbb{R}^n.$$

Set $S_k^{n-1} = s_k s_{k+1} \cdots s_{n-1}$, with the convention that $S_n^{n-1} = 1$. So our statement becomes

$$B(0, S_1^{n-1} r_1) \times B(0, S_2^{n-1} r_2) \times \cdots \times B(0, S_n^{n-1} r_n) \subseteq U \cap \mathbb{R}^n.$$

Set $t_k = S_k^\infty = \prod_{i=k}^\infty s_i$. Note that $t_1 > 0$ and then it follows inductively that $t_k > 0$ for all k. Since $s_i < 1$ for all i we also have $t_k < 1$, and even more that $t_k < S_k^{n-1}$ for all n. Consider $W = \mathbb{R}^\infty \cap [B(0, t_1 r_1) \times B(0, t_2 r_2) \times \cdots]$. We claim that $W \subseteq U$, proving that U is open in $\mathbb{R}^\infty_{\text{box}}$.

Let $x \in W$. Then $x \in \mathbb{R}^n$ for some n. Since $t_k < S_k^{n-1}$ we have that

$$x \in B(0, S_1^{n-1} r_1) \times B(0, S_2^{n-1} r_2) \times \cdots \times B(0, S_{n-1}^{n-1} r_{n-1}) \times B(0, r_n).$$

But this product of balls is contained in $U \cap \mathbb{R}^n$, hence $x \in U$. $\square$

Lemma B.7. *Let R be an open rectangle in $\mathbb{R}^{n-1}$ containing $\underline{0}$ and let $U \subseteq \mathbb{R}^n$ be an open set containing $R \times \{0\}$. For $s \in \mathbb{R}$ let $sR = \{sx \mid x \in R\}$. Then for all $0 \leq s < 1$ there exists an $\epsilon > 0$ such that $sR \times (-\epsilon, \epsilon) \subseteq U$.*

Proof. Let S be the set of all $s \in [0, 1)$ such that the conclusion of the lemma holds. It is trivial that $0 \in S$, using that U is an open set containing $\underline{0}$. Note that if $s \in S$ then $[0, s] \subseteq S$, therefore S is an interval. Let z be the least uppper bound of S, and assume that $z \neq 1$.

For a rectangle $B(x_1, r_1) \times \cdots \times B(x_a, r_a)$, let us refer to r_i as the ith cross-sectional radius.

Consider $\partial(zR)$, which is a closed bounded subset of $\mathbb{R}^n$ and hence compact. Since $z < 1$ we have $\partial(zR) \subseteq R$. For each point $x \in \partial(zR)$ there is an open rectangle around $(x, 0)$ that is contained in U. Such open rectangles cover $\partial(zR)$, so by compactness we can cover $\partial(zR)$ by finitely many of these. Let $u > 0$ be smaller than any of the cross-sectional radii that appear in these finitely many rectangles, and also smaller than z.

Let $s = z - \frac{u}{2}$. Since $s < z$ we have $s \in S$, so there exists an $\epsilon > 0$ such that $sR \times (-\epsilon, \epsilon) \subseteq U$. Let σ be smaller than both u and ϵ. Then the union of $sR \times (-\epsilon, \epsilon)$ with our chosen open rectangles contains $sR \times (-\sigma, \sigma)$ for $s = z$ but also for s slightly larger than z. This contradicts z being the least upper bound of S.

We conclude that $z = 1$, hence $S = [0, 1)$. This completes the proof. $\square$

Corollary B.8. *If a sequence $x \colon \mathbb{N} \to \mathbb{R}^\infty_{\text{colim}}$ is convergent with limit L then there exists an $N \geq 1$ and a $k \geq 1$ such that $L \in \mathbb{R}^k$, $x_i \in \mathbb{R}^k$ for all $i \geq N$, and the limit of $x|_{\geq N}$ in $\mathbb{R}^k$ equals L.*

Proof. This follows by the remarks in Example B.4, but we also include a detailed proof. We know $L \in \mathbb{R}^M$ for some M. Assume that the sequence x is not eventually contained in some $\mathbb{R}^n$. Then there exist $i_1 < i_2 < \cdots$ and $M < n_1 < n_2 < \cdots$ such that $(x_{i_r})_{n_r} \neq 0$ for every r. If $j = n_r$ let $U_j = B(0, |(x_{i_r})_{n_r}|)$, and if $j \notin \{n_1, n_2, \ldots\}$ set $U_j = \mathbb{R}$. Let $U = \prod_j U_j$. Then by Proposition B.6 U is an open neighborhood of $\underline{0}$, hence $L + U$ is a neighborhood of L. Observe that we have constructed things so that $x_{i_r} \notin L + U$, for every r. This contradicts the assumption that x converges to L.

We have now proven that the sequence is eventually contained in some $\mathbb{R}^k$, and we can take k large enough so that the limit L is also in $\mathbb{R}^k$. Now just apply the projection map $\mathbb{R}^\infty \to \mathbb{R}^k$ (which is continuous) to the sequence $x|_{\geq N}$ to see that the limit in $\mathbb{R}^k$ is also equal to L. $\square$

It is now easy to check that the vector sum and scalar multiplication maps $\mathbb{R}^\infty_{\text{colim}} \times \mathbb{R}^\infty_{\text{colim}} \xrightarrow{+} \mathbb{R}^\infty_{\text{colim}}$ and $\mathbb{R} \times \mathbb{R}^\infty_{\text{colim}} \to \mathbb{R}^\infty_{\text{colim}}$ are continuous, using the box topology in place of the colimit topology. These maps make $\mathbb{R}^\infty_{\text{colim}}$ into a topological vector space.

Remark B.9. The metric and product topologies also make $\mathbb{R}^\infty$ into a topological vector space. To check that the vector sum is continuous, check that if $x, y\colon I \to \mathbb{R}^\infty$ are nets that converges to a and b (with either the metric or product topology on $\mathbb{R}^\infty$) then the net $i \mapsto x_i + y_i$ converges to $a + b$. Similar for scalar multiplication.

Next we start to place conditions on topological spaces so that they are reasonably well-behaved from an algebraic perspective.

Definition B.10. *Let V be a topological vector space. Say that V is **reasonable** if*

(1) For every indepedent set $v_1, \ldots, v_n \in V$ the induced map $\mathbb{R}^n \to \langle v_1, \ldots, v_n\rangle$ sending $e_i \mapsto v_i$ is a homeomorphism, where the target is given the subspace topology.

(2) Every linear functional $\phi\colon V \to \mathbb{R}$ is continuous.

(In both (1) and (2) $\mathbb{R}$ has the standard Euclidean topology).

Not every topological vector space is reasonable! The space $\mathbb{R}^n_{ind}$ does not satisfy property (1), and neither $\mathbb{R}^\infty_{metric}$ nor $\mathbb{R}^\infty_{prod}$ satisfies (2). To see the latter, consider the linear functional $\phi\colon \mathbb{R}^\infty \to \mathbb{R}$ given by $\underline{x} \mapsto x_1 + x_2 + x_3 + \cdots$. The sequence $s_i = \frac{1}{n}\sum_{i=1}^n e_i$ converges to 0 in both $\mathbb{R}^\infty_{metric}$ and $\mathbb{R}^\infty_{prod}$ but its image under ϕ does not converge to 0.

Recall that Tychonoff's theorem says that every finite-dimensional topological vector space is isomorphic to $\mathbb{R}^n$ with either the Euclidean or indiscrete topology. So condition (1) is equivalent to saying that each $\langle v_1, \ldots, v_n\rangle$ with the subspace topology is Hausdorff.

Proposition B.11. *Let V be a reasonable topological vector space. Let $F \subseteq V$ be a finite-dimensional subspace with a chosen complement W. Then the associated projection maps $\pi_F\colon V \to F$ and $\pi_W\colon V \to W$ are continuous.*

Proof. By condition (1) of being reasonable, F is isomorphic to $\mathbb{R}^n$ with the product topology. Composing with this isomorphism, π_F becomes a linear map $V \to \mathbb{R}^n$. Every component of this map is a linear functional, hence continuous by condition (2). So π_F is continuous.

If $j\colon F \hookrightarrow V$ is the inclusion then the map π_W is $\mathrm{id} - j\pi_F$, hence continuous. $\square$

Let V be a topological vector space over $\mathbb{R}$. The finite-dimensional subspaces of V, with inclusions, make up a directed category and we have a canonical map

$$\operatorname*{colim}_{J^{fin.dim} \subseteq V} J \longrightarrow V$$

where the colimit is taken in the category of sets. This is a bijection. If we equip each J with the standard topology and the map is a homeomorphism, we refer to V as a **standard** topological vector space.

Example B.12. $\mathbb{R}^n$ with the usual topology is a standard topological vector space, as is $\mathbb{R}^\infty$ with the colimit topology. For the latter, consider the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{J^{fin.dim} \subseteq \mathbb{R}^\infty} J \arrow[r] & \mathbb{R}^\infty_{\text{colim}}. \\
\operatorname*{colim}_n \mathbb{R}^n \arrow[u] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The subspaces $\mathbb{R}^n$ are cofinal in the category of all finite-dimensional subspaces of $\mathbb{R}^\infty$, and so the vertical map is a homeomorphism. Hence the horizontal map is a homeomorphism as well.

By the same line of reasoning, $\mathbb{R}^\infty$ with the metric or product topologies is not a standard topological vector space.

Proposition B.13. *Let $b_1, b_2, \dots$ be a basis for $\mathbb{R}^\infty$. Then the linear map $L\colon \mathbb{R}^\infty_{\text{colim}} \to \mathbb{R}^\infty_{\text{colim}}$ sending e_i to b_i is a homeomorphism.*

Proof. The restriction of L to $\mathbb{R}^n$ factors through some $\mathbb{R}^k$, and the map $\mathbb{R}^n \to \mathbb{R}^k$ is continous, so $L|_{\mathbb{R}^n}$ is continuous as well. Now pass to the colimit to obtain the continuity of L.

Let $s_i = L^{-1}(e_i)$. Since L is an isomorphism of vector spaces, $\{s_1, s_2, \dots,\}$ is a basis for $\mathbb{R}^\infty$. So just as in the last paragraph, the linear map $\mathbb{R}^\infty_{\text{colim}} \to \mathbb{R}^\infty_{\text{colim}}$ sending $e_i \mapsto s_i$ is continuous. But this is precisely L^{-1}, therefore L is a homeomorphism. $\square$

Corollary B.14. *Let $b_1, b_2, \dots$ be a basis for $\mathbb{R}^\infty$, and let $\hat{b}_1, \hat{b}_2, \dots$ be the dual linear functionals given by $\hat{b}_j(v) = a_j$ if $v = \sum a_j b_j$. Then each $\hat{b}_j$ is continuous as a map $\mathbb{R}^\infty_{\text{colim}} \to \mathbb{R}$.*

Proof. Let $L\colon \mathbb{R}^\infty_{\text{colim}} \to \mathbb{R}^\infty_{\text{colim}}$ send e_i to b_i. Then $\hat{b}_j = \pi_j \circ L^{-1}$, but both π_j and L^{-1} are continuous. $\square$

B.15. **Topological vector spaces.**

Proposition B.16. *Let $N \leq \infty$. Every linear transformation $\mathbb{R}^\infty \to \mathbb{R}^N$ is continuous, and every linear isomorphism $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is a homeomorphism.*

Proof. The second statement follows directly from the first. For the first, observe that when $k < \infty$ every linear map $\mathbb{R}^k \to \mathbb{R}^\infty$ factors through some $\mathbb{R}^s$. Since $\mathbb{R}^k \to \mathbb{R}^s$ is continuous, so is $\mathbb{R}^k \to \mathbb{R}^\infty$.

So every linear map $\mathbb{R}^k \to \mathbb{R}^N$ is contininuous. Now use that $\mathbb{R}^\infty = \operatorname{colim}_k \mathbb{R}^k$ to conclude that every linear map $\mathbb{R}^\infty \to \mathbb{R}^N$ is continuous. $\square$

Proposition B.17. *For every $n \geq 1$ the topological vector space $(\mathbb{R}^\infty)^n$ is isomorphic (as topological vector spaces) to $\mathbb{R}^\infty$. In particular, the underlying topological spaces are homeomorphic.*

Proof. It suffices to prove this when $n = 2$. Let $\phi\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the map $(\underline{a}, \underline{b}) \mapsto (a_1, b_1, a_2, b_2, \ldots)$. It is immediate that this is an isomorphism of vector spaces, but we need to think about the topology aspect. Consider the maps $f\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $e_i \mapsto e_{2i-1}$ and $f\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $e_i \mapsto e_{2i}$. These are continuous by Proposition B.16. The map ϕ is the composite

$$\mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{f\times g} \mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{+} \mathbb{R}^\infty.$$

so ϕ is continuous.

The inverse ϕ^{-1} is a map $\mathbb{R}^\infty \to \mathbb{R}^\infty \times \mathbb{R}^\infty$. The projections to the two factors are continuous by Proposition B.16, therefore ϕ^{-1} is continous. $\square$

Corollary B.18. *The evident map* $\operatorname{colim}_k(\mathbb{R}^k)^n \to (\mathbb{R}^\infty)^n$ *is a homeomorphism.*

Proof. The map is evidently a continuous bijection, but it is not immediate that the inverse is continuous. This does not seem to follow purely from point-set topology, due to the usual incompatibilities between the product and colimits.

However, Proposition B.17 implies that $(\mathbb{R}^\infty)^n$ is the colimit of its finite-dimensional subspaces. The family of subspaces $(\mathbb{R}^k)^n$ is cofinal in these, and to the result follows. $\square$

B.19. **Tame families.** In this appendix we prove Proposition 8.17. If $E \to X$ is a family of vector spaces of finite rank, this says that E is tame if either (a) E is a subfamily of a trivial family, or (b) X is locally compact and E is Hausdorff. The proofs in the two cases are quite different.

Lemma B.20. *Every trivial family of vector spaces of finite rank is tame.*

Proof. Let X be a space, and let $s_1, \ldots, s_n$ be a local weak basis of $E = X \times \mathbb{R}^n$ defined on an open neighborhood U of a point $x \in X$. Write S_i for the composite

$$U \xrightarrow{s_i} E|_U \xrightarrow{\pi_2} \mathbb{R}^n.$$

We can regard $S_1, \ldots, S_n$ as giving a map $F\colon U \to M_{n\times n}(\mathbb{R})$, where the columns are the S_i's. Since $S_1(y), \ldots, S_n(y)$ is a basis for $\mathbb{R}^n$, for every $y \in U$, the image of F actually lies in $GL_n(\mathbb{R})$. The inverse map $J\colon GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ sending $A \to A^{-1}$ is continuous (a formula is given using determinants), so we can consider the composite

$$U \times \mathbb{R}^n \xrightarrow{S} E|_U \xrightarrow{\mathrm{id}\times JF} U \times \mathbb{R}^n$$

where the second map is $(y, v) \mapsto (y, JF(y)(v))$. This composite is the identity. It follows that the map S is a homeomorphism, which is what we wanted to prove. $\square$

Proof of Proposition 8.17 in case (a). Assume that E is a subfamily of $X \times V$ where V is either $\mathbb{R}^n$ or $\mathbb{R}^\infty_{\text{colim}}$. Let $x \in X$ and let $s_1, \ldots, s_n$ be a local weak basis defined on an open neighborhood U of x. Let S_i be the composite $U \xrightarrow{s_i} E \hookrightarrow X \times V \xrightarrow{\pi_2} V$. The vectors $S_1(x), \ldots, S_n(x)$ are linearly independent, so we can choose a basis $\{f_\alpha\}_{\alpha\in A}$ for V containing these vectors. Let $\beta\colon V \to \mathbb{R}^n$ be the linear map sending each $S_i(x)$ to e_i and all the other f_α to 0. If V is finite-dimensional then β is obviously continuous, whereas if $V = \mathbb{R}^\infty_{\text{colim}}$ the continuity follows from Corollary B.14. We will also write β for the associated map $X \times V \to X \times \mathbb{R}^n$.

Consider now the composites $U \xrightarrow{S_i} U \times V \xrightarrow{\beta} U \times \mathbb{R}^n \xrightarrow{\pi_2} \mathbb{R}^n$. Taken together, they are the columns of a map $F\colon U \to M_{n\times n}(\mathbb{R})$, and we can further consider the composite

$$U \to M_{n\times n}(\mathbb{R}) \xrightarrow{\det} \mathbb{R}.$$

Let $W \subseteq U$ be the preimage of $\mathbb{R} - 0$, which is open and contains x. As in Lemma B.20 consider the composite

$$W \times \mathbb{R}^n \xrightarrow{S} E|_W \hookrightarrow W \times V \xrightarrow{\beta} W \times \mathbb{R}^n \xrightarrow{\mathrm{id}\times JF} W \times \mathbb{R}^n$$

where $J\colon GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is the inverse map. The composition is the identity, therefore S is a homeomorphism, and so $s_1, \ldots, s_n$ is a local strong basis on W. $\square$

The proof for case (b) requires some technical preliminary work. A classical result from the theory of topological vector spaces, originally due to Tychonoff, says that if an n-dimensional topological vector space V is T_1 (points are closed) then $V \cong \mathbb{R}^n$ as topological vector spaces. We take the techniques from that proof and generalize them to families of vector spaces.

Let $E \to X$ be a family of vector spaces and let $W \subseteq E$ be any subset. If $S \subseteq \mathbb{R}$ then write SW for the image of W under the composite $S \times E \hookrightarrow \mathbb{R} \times E \longrightarrow E$. When $S = \{\lambda\}$ we just write λW for $\{\lambda\}W$.

Note that if $\lambda \neq 0$ then the multiplication-by-λ map $E \to E$ is continuous, as is multiplication by λ^{-1}. So these are in fact homeomorphisms, and therefore λW is open whenever W is. In a similar vein, if $s\colon X \to E$ is a section then we can consider the map $+s\colon E \to E$. This is continuous, being the composite

$$E = E \times_X X \xrightarrow{\mathrm{id}\times s} E \times_X E \xrightarrow{+} E.$$

But $-s\colon E \to E$ is also continuous, and is the inverse to $+s$; so again, these are both homeomorphisms. Thus, if W is open in E then so is the set

$$s + W = \{s(x) + w_x \,|\, x \in X, w_x \in E_x \cap W\}.$$

We let $z\colon X \to E$ denote the zero-section. Note that this is a continous map, by the definition of family of vector spaces.

Definition B.21. *Let $E \to X$ be a family of vector spaces. A subset $W \subseteq E$ is called **balanced** if $[-1,1]W \subseteq W$.*

Proposition B.22. *Let $E \to X$ be a family of vector spaces. Let U be an open set containing the point $z(x)$, for some $x \in X$. Then there is a balanced open set V such that $z(x) \in V \subseteq U$.*

Proof. Scalar multiplication $\mathbb{R} \times E \to E$ is continuous and $(0, z(x)) \mapsto z(x)$. So there is an $\epsilon > 0$ and an open set W containing $z(x)$ such that $(-\epsilon, \epsilon)W \subseteq U$. Let J be the set of all points v in U having the property that $[-1,1]v \subseteq U$. Then $\frac{\epsilon}{2}W \subseteq J \subseteq U$. Since $\frac{\epsilon}{2}W$ is open and contains $z(x)$, this shows that J contains a neighborhood of $z(x)$. Note also that $[-1,1]J \subseteq J$ by virtue of the definition of J.

Let M be the interior of J. Then M is an open set containing $z(x)$, and $M \subseteq J \subseteq U$. If $0 < |\lambda| \leq 1$ then $\lambda M \subseteq \lambda J \subseteq J$. But λM is open, so in fact $\lambda M \subseteq M$. Note that we only have this for $\lambda \neq 0$, though, so we cannot yet conclude that M is balanced.

For the final step, let $T = z^{-1}(M)$. Since z is continuous, this is an open subset of X containing x. Consider $N = M \cap p^{-1}(T)$. This is open, contains $z(x)$, and is contained in U. For λ nonzero in $[-1,1]$ we have that $\lambda N \subseteq N$ because the same is true for both M and $p^{-1}(T)$. For $v \in N$ we have that $p(v) \in T$, and so $0 \cdot v = z(p(v)) \in M$. Then $0 \cdot v \in N$, thus $0N \subseteq N$ and so N is balanced. $\square$

Proposition B.23. *Let $E \to X$ be a family of vector spaces of finite rank n. If E is Hausdorff and X is locally compact, then any weak basis is a strong basis.*

Proof. The weak basis gives us a continuous bijection $S\colon X \times \mathbb{R}^n \to E$, and we must show that S^{-1} is continuous. So for every open subset $\Omega \subseteq X \times \mathbb{R}^n$ we must prove that $S(\Omega)$ is open in E. Let $y \in S(\Omega)$, so that $y = S(x,u)$ for some $x \in X$ and $u \in \mathbb{R}^n$ with $(x,u) \in \Omega$. There is an open set $x \in U$ and an $\epsilon > 0$ such that $U \times B(u,\epsilon) \subseteq \Omega$, where $B(u,\epsilon)$ denotes the evident open ball.

We first treat the case where $u = 0$. Since X is locally compact, there is a compact set K that contains an open neighborhood W of x. By replacing W with $W \cap U$ we can assume $W \subseteq U$. Let S_ϵ be the sphere in $\mathbb{R}^n$ of radius ϵ, and consider $C = S(K \times S_\epsilon)$. Since $K \times S_\epsilon$ is compact, C is also compact. Since E is Hausdorff, C is therefore closed in E. Observe that the set $S(W \times B(0,\epsilon))$ does not intersect C, since S is a bijection. In particular, $z(x) = S(x,0) \notin C$.

Now apply Proposition B.22 to $z(x) \in E - C$, noting that $E - C$ is open. So there is a balanced open set V of E that contains $z(x)$ and is such that $V \subseteq E - C$. Let $V' = p^{-1}(W) \cap V$, which is still open and balanced. Since V' is balanced and does not intersect C we must have $V' \subseteq S(W \times B(0,\epsilon))$. Indeed, if $v \in V'$ and $v \notin S(W \times B(0,\epsilon))$ then $v = S(w,t)$ for some $w \in W$ and $t \in \mathbb{R}^n$ with $|t| \geq \epsilon$. Since $V \subseteq E - C$, we in fact have $|t| > \epsilon$. But since V' is balanced we must then have $\frac{\epsilon}{|t|}v \in V'$, which contradicts $V \subseteq E - C$ since this vector is $S(w, \frac{\epsilon}{|t|}t)$.

So we have $z(x) \in V' \subseteq S(W \times B(0,\epsilon)) \subseteq S(U \times B(0,\epsilon)) \subseteq S(\Omega)$. Thus, $S(\Omega)$ contains an open neighborhood of $y = z(x)$.

Now consider the general case where $u \neq 0$. Let $k\colon X \to E$ be the section $k(a) = S(a,u)$ for $a \in X$. Then $-k + S(U \times B(u,\epsilon)) = S(U \times B(0,\epsilon))$ and this set contains $z(x) = s(x,0)$. By the case treated already, there is an open set V of E such that $z(x) \in V \subseteq S(U \times B(0,\epsilon))$. Then $k + V$ is an open set such that $y \in k + V \subseteq S(U \times B(u,\epsilon)) \subseteq S(\Omega)$, and again we have proven that $S(\Omega)$ contains an open neighborhood of y. Since this holds for all y, we have that $S(\Omega)$ is open. $\square$

Proof of Proposition 8.17 in case (b). This follows immediately from Proposition B.23, using that an open subset of a locally compact space is again locally compact. $\square$

Appendix C. Bernoulli numbers

There are different conventions for naming the Bernoulli numbers, especially when one enters the topology literature. We adopt what seems to be the most common definition, which is the following:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \cdot \frac{x^k}{k!}.$$

Expanding the power series yields

$$\begin{aligned}\frac{x}{e^x-1} &= 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots \\ &= 1 - \left(\frac{1}{2}\right)x + \left(\frac{1}{6}\right)\cdot\frac{x^2}{2} - \left(\frac{1}{30}\right)\cdot\frac{x^4}{4!} + \left(\frac{1}{42}\right)\cdot\frac{x^6}{6!} - \left(\frac{1}{30}\right)\cdot\frac{x^8}{8!}\cdots\end{aligned}$$

So we have

k	0	1	2	3	4	5	6	7	8	9
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0

From the table one guesses that $B_{2n+1} = 0$ for $n > 0$. This is easy to prove: if we set $f(x) = \frac{x}{e^x-1}$ then we can isolate the odd powers of x by examining $f(x)-f(-x)$. But algebra yields

$$\left(\frac{x}{e^x-1}\right) - \left(\frac{-x}{e^{-x}-1}\right) = -x.$$

Computing the coefficients of $\frac{x}{e^x-1}$ is not the most efficient way of computing Bernoulli numbers, as one can deduce from the large denominators in the above formula. A better method is via a certain recursive formula, and this is best remembered by a "mnemonic":

$$(B+1)^n = B^n. \tag{C.1}$$

Do not take this formula literally! It is shorthand for the following procedure. First expand the left-hand-side via the Binomial Formula, treating B as a formal variable. Then rewrite the formula by "lowering all indices", meaning changing every B^i to a B_i. This gives the desired recursive formula.

For example: $(B+1)^2 = B^2$ yields $B^2 + 2B + 1 = B^2$, which in turn gives $B_2 + 2B_1 + 1 = B_2$. Cancelling the B_2's we obtain $2B_1 + 1 = 0$, or $B_1 = -\frac{1}{2}$. Likewise, $(B+1)^3 = B^3$ yields $B_3 + 3B_2 + 3B_1 + 1 = B_3$, thereby giving

$$B_2 = -\tfrac{1}{3}(1 + 3B_1) = -\tfrac{1}{3}\cdot -\tfrac{1}{2} = \tfrac{1}{6}.$$

And so on. For the record here are a few more of the Bernoulli numbers, computed in this way:

k	0	1	2	4	6	8	10	12	14	16	18
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$

Note that we have not yet justified the recursive formula (C.1). We will do this after a short interlude.

C.2. **Sums of powers.** The Bernoulli numbers first arose in work of Jakob Bernoulli on computing formulas for the power sums

$$1^t + 2^t + 3^t + \cdots + n^t.$$

Most modern students have seen the formulas

$$1 + 2 + \cdots + n = \tfrac{n(n+1)}{2} \quad \text{and} \quad 1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6}.$$

The Bernoulli formulas generalize these to give

$$1^t + 2^t + \cdots + n^t = P_t(n)$$

where P_t is a degree $t+1$ polynomial in n with rational coefficients. It is somewhat surprising that the formulas for the P_t's can be given using a single set of coefficients, the Bernoulli numbers.

The Bernoulli formulas are most succinctly written using our mnemonic device of lowering indices. We write

$$1^t + 2^t + \cdots + n^t = \tfrac{1}{t+1}\Big[\big(B + (n+1)\big)^{t+1} - B^{t+1}\Big]. \tag{C.3}$$

Let us work through the first few examples of this. For $t = 1$ we have

$$\begin{aligned} 1 + 2 + \cdots + n = \tfrac{1}{2}\Big[(B + (n+1))^2 - B^2\Big] &= \tfrac{1}{2}\Big[B_2 + 2B_1(n+1) + (n+1)^2 - B_2\Big] \\ &= \tfrac{1}{2}\Big[(n+1)^2 - (n+1)] \\ &= \tfrac{1}{2}(n+1)n. \end{aligned}$$

For $t = 2$ we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 &= \tfrac{1}{3}\Big[(B + (n+1))^3 - B^3\Big] \\ &= \tfrac{1}{3}\Big[3B_2(n+1) + 3B_1(n+1)^2 + (n+1)^3\Big] \\ &= \tfrac{1}{3}(n+1)\Big[\tfrac{1}{2} - \tfrac{3}{2}(n+1) + (n+1)^2\Big] \\ &= \tfrac{1}{3}(n+1)\Big[\tfrac{1}{2}n + n^2\Big] \\ &= \tfrac{1}{6}(n+1)n(2n+1). \end{aligned}$$

We leave it to the reader to derive the $t = 3$ formula:

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Proof of the Bernoulli formula (C.3). Start with the identity of power series

$$1 + e^x + e^{2x} + \cdots + e^{nx} = \frac{e^{(n+1)x} - 1}{e^x - 1} = \Big(\frac{x}{e^x - 1}\Big) \cdot \Big(\frac{e^{(n+1)x} - 1}{x}\Big).$$

The coefficient of x^t on the left-hand-side is

$$\frac{1}{t!}(1^t + 2^t + \cdots + n^t).$$

The coefficient of x^t on the right-hand-side is

$$\sum_{k=0}^{t} \frac{B_k}{k!} \cdot \frac{(n+1)^{t+1-k}}{(t+1-k)!}.$$

Equating coefficients and rearranging yields the Bernoulli formula immediately. $\square$

Now let us return to our recursive formula (C.1) for computing the Bernoulli numbers. Note that it is an immediate consequence of (C.3) by taking $n = 0$ to get $0 = (B+1)^{t+1} - B^{t+1}$.

C.4. **Miscellaneous facts.**

Theorem C.5 (Claussen/von Staudt)**.**
(a) $(-1)^n B_{2n} \equiv \sum_p \frac{1}{p}$ *mod* $\mathbb{Z}$*, where the sum is taken over all primes* p *such that* $p-1$ *divides* $2n$*.*
(b) When expressed as a fraction in lowest terms, B_{2n} *has square-free denominator consisting of the product of all primes* p *such that* $p-1$ *divides* $2n$*.*

For example, we can now immediately predict that the denominator of B_{20} will be $2 \cdot 3 \cdot 5 \cdot 11 = 330$. Note that the primes 2 and 3 will always appear in the denominators of Bernoulli numbers.

The following strange fact is relevant to the appearance of Bernoulli numbers in topology:

Proposition C.6. *For any even* n *and any* $k \in \mathbb{Z}$*,* $\frac{k^n(k^n-1)B_n}{n} \in \mathbb{Z}$*.*

Proof. We follow Milnor and Stasheff here [MS]. Write

$$f(x) = 1 + e^x + e^{2x} + \cdots + e^{(k-1)x} = \frac{e^{kx}-1}{e^x-1}.$$

Note that

$$f^{(r)}(0) = 1^r + 2^r + \cdots + (k-1)^r.$$

In particular, the derivatives of f evaluated at 0 are all integers.

Next consider the logarithmic derivative

$$\begin{aligned}
\frac{f'(x)}{f(x)} = D(\log(f(x))) &= \frac{ke^{kx}}{e^{kx}-1} - \frac{e^x}{e^x-1} \\
&= k\Big[\frac{1}{1-e^{-kx}}\Big] - \Big[\frac{1}{1-e^{-x}}\Big] \\
&= \frac{1}{x}\Big[\frac{-kx}{e^{-kx}-1} - \frac{-x}{e^{-x}-1}\Big] \\
&= \frac{1}{x}\Big[\sum \tfrac{B_i}{i!}(-kx)^i - \sum \tfrac{B_i}{i!}(-x)^i\Big] \\
&= \sum_i (-1)^i \tfrac{B_i}{i!}(k^i-1)x^{i-1} \\
&= \tfrac{k-1}{2} + \tfrac{B_2}{2!}(k^2-1)x + \tfrac{B_4}{4!}(k^4-1)x^3 + \cdots
\end{aligned}$$

The $(2t-1)$st derivative of this expression, evaluated at 0, is $\frac{B_{2t}}{2t}(k^{2t}-1)$.

However, iterated use of the quotient rule shows that the $(2t-1)$st derivative of $f'(x)/f(x)$, evaluated at 0, can be written as an integral linear combination of $f(0), f'(0), f''(0), \ldots$ divided by $f(0)^{2t}$. Since $f(0) = k$ and all the deriviates of f have integral values at 0, this gives

$$\frac{B_{2t}}{2t}(k^{2t}-1) \cdot k^{2t} \in \mathbb{Z}.$$

□

Appendix D. The algebra of symmetric functions

Let $S = \mathbb{Z}[x_1, \ldots, x_n]$ be equipped with the evident Σ_n-action that permutes the indices. It is a well-known theorem that the ring of invariants is a polynomial ring on the elementary symmetric functions:

$$S^{\Sigma_n} \cong \mathbb{Z}[\sigma_1, \ldots, \sigma_n].$$

Let $s_k = x_1^k + x_2^k + \cdots + x_n^k$, the **$k$th power sum** of the variables x_i. Since s_k is a Σ_n-invariant we have

$$s_k = S_k(\sigma_1, \ldots, \sigma_n)$$

for a unique polynomial S_k in n variables (with integer coefficients). The polynomial S_k is called the kth **Newton polynomial**.

Let us calculate the simplest examples of the Newton polynomials. Clearly $s_1 = \sigma_1$, and so $S_1(\sigma_1, \ldots, \sigma_n) = \sigma_1$. For s_2 we compute that

$$s_2 = x_1^2 + \cdots + x_n^2 = (x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n) = \sigma_1^2 - 2\sigma_2.$$

These calculations get more difficult as the exponents get larger.

It is useful to adopt the following notation when working with the ring of invariants. If m is a monomial in the x_i's then $[m]$ denotes the sum of all elements in the Σ_n-orbit of m. For example,

$$[x_1x_2] = \sigma_2, \quad [x_1^k] = s_k, \quad \text{and} \quad [x_1^2x_2] = \sum_{i \neq j} x_i^2 x_j.$$

If $H \leq \Sigma_n$ is the stabilizer of m then we can also write

$$[m] = \sum_{g \in \Sigma_n / H} gm.$$

Let us use the above notation to help work out the third Newton polynomial. Elementary algebra easily yields the equation

$$s_3 = [x_1^3] = [x_1]^3 - 3[x_1^2x_2] - 6[x_1x_2x_3].$$

Here one considers the product $(x_1 + \cdots + x_n)^3$ and reasons that a term like $x_1^2x_2$ appears three times in the expansion, and terms like $x_1x_2x_3$ appear six times. Via a similar process we work out that

$$[x_1^2x_2] = [x_1] \cdot [x_1x_2] - 3[x_1x_2x_3].$$

Putting everything together, we have found that

$$s_3 = \sigma_1^3 - 3(\sigma_1\sigma_2 - 3\sigma_3) - 6\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

This final expression is the third Newton polynomial S_3.

Lemma D.1 (The Newton identities)**.** *For $k \geq 2$ one has the identity*

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^k \sigma_{k-1} s_1 + (-1)^{k+1} k\sigma_k.$$

Consequently, there is analogous inductive formula for the Newton polynomials:

$$S_k = \sigma_1 S_{k-1} - \sigma_2 S_{k-2} + \cdots + (-1)^k \sigma_{k-1} S_1 + (-1)^{k+1} k\sigma_k.$$

Proof. The key is the formula

$$[x_1x_2 \cdots x_{j-1}x_j^k] = [x_1 \cdots x_j] \cdot [x_1^{k-1}] - [x_1 \cdots x_j x_{j+1}^{k-1}]$$

which holds for $k > 2$, whereas when $k = 2$ we have

$$[x_1x_2 \cdots x_{j-1}x_j^2] = [x_1 \cdots x_j] \cdot [x_1] - (j+1)[x_1 \cdots x_j x_{j+1}].$$

In the latter case the point is that a term $x_1 \ldots x_{j+1}$ appears $j+1$ times in the product $[x_1 \ldots x_j] \cdot [x_1]$.

When $k = 2$ the identity from the statement of the lemma has already been verified by direct computation. For $k > 2$ start with the simple formula

$$s_k = [x_1^k] = [x_1] \cdot [x_1^{k-1}] - [x_1 x_2^{k-1}] = \sigma_1 s_{k-1} - [x_1 x_2^{k-1}].$$

Next observe that

$$[x_1 x_2^{k-1}] = \begin{cases} [x_1 x_2] \cdot [x_1^{k-2}] - [x_1 x_2 x_3^{k-2}] & \text{if } k > 3, \\ [x_1 x_2] \cdot [x_1^{k-2}] - 3[x_1 x_2 x_3] & \text{if } k = 3. \end{cases}$$

If $k = 3$ we are now done, otherwise repeat the above induction step. The details are left to the reader. □

As an application of Lemma D.1 observe that we have

$$S_3 = \sigma_1 S_2 - \sigma_2 S_1 + 3\sigma_3 = \sigma_1(\sigma_1^2 - 2\sigma_2) - \sigma_2\sigma_1 + 3\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3,$$

agreeing with our earlier calculation. Here is a table showing the first few Newton polynomials:

Table 4.2. Newton polynomials

k	S_k
1	σ_1
2	$\sigma_1^2 - 2\sigma_2$
3	$\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$
4	$\sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4$
5	$\sigma_1^5 - 5\sigma_1^3\sigma_2 + 4\sigma_1^2\sigma_3 + 5\sigma_1\sigma_2^2 - 3\sigma_2\sigma_3 - 5\sigma_1\sigma_4 + 5\sigma_5$

The Newton polynomials also show up in the following:

Proposition D.3. *Let $\alpha = \alpha_1 t + \alpha_2 t^2 + \cdots \in R[[t]]$, where R is a commutative ring. Then*

$$\tfrac{d}{dt}\big(\log(1+\alpha)\big) = \tfrac{\alpha'}{1+\alpha} = \mu_1 + \mu_2 t + \mu_3 t^2 + \cdots$$

where $\mu_k = (-1)^k S_k(\alpha_1, \dots, \alpha_k)$.

Proof. Equate coefficients in the identity

$$\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^3 + \cdots \ = (1 + \alpha_1 t + \alpha_2 t^2 + \cdots) \cdot (1 + \mu_1 t + \mu_2 t^2 + \cdots).$$

This gives a series of identities for each μ_k that parallel the Newton identities. The result then follows by an easy induction. □

Appendix E. Homotopically compact pairs

By a **pair of topological spaces** we mean an ordered pair (X, A) where A is a subspace of A. A map of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$, and such a map is said to be a weak equivalence if both $f\colon X \to Y$ and $f|A\colon A \to B$ are weak equivalences. Two maps $f, g\colon (X, A) \to (Y, B)$ are said to be homotopic if there is a map $H\colon (X \times I, A \times I) \to (Y, B)$ such that $H|_{X\times 0} = f$ and $H|_{X\times 1} = g$.

Define a topological space X to be **homotopically compact** if it is weakly equivalent to a finite CW-complex. Likewise, define a pair of topological spaces (X, A) to be homotopically compact if there exists a finite CW-pair (X', A') and a weak equivalence $(X', A') \to (X, A)$. In this case we call (X', A') a **finite model** for (X, A).

Proposition E.1. *A pair (X, A) is homotopically compact if and only if both X and A are homotopically compact.*

Proof. The "only if" direction is trivial, and the other direction is an immediate consequence of the slightly more general lemma below. □

Lemma E.2. *Let $f\colon A \to X$ be a map, where both A and X are homotopically compact. Let $\gamma_a\colon \widetilde{A} \to A$ be any finite model for A. Then there exists a finite CW-complex $\widetilde{X}$, containing $\widetilde{A}$ as a subcomplex, together with a weak equivalence $\widetilde{X} \to X$ such that the square*

$$\begin{tikzcd} \widetilde{A} \arrow[r, tail] \arrow[d, "\gamma_A"'] & \widetilde{X} \arrow[d, "\gamma_X"] \\ A \arrow[r, "f"] & X \end{tikzcd}$$

commutes.

Proof. Let $\gamma_X : \widetilde{X} \to X$ be any finite model X. Since $[\widetilde{A}, \widetilde{X}] \to [\widetilde{A}, X]$ is a bijection, there is a map $\tilde{f}\colon \widetilde{A} \to \widetilde{X}$ such that $\gamma_X \tilde{f} \simeq f\gamma_A$. By cellular approximation, we may assume that $\tilde{f}$ is cellular. Choose such a homotopy. Let C_X denote the mapping cylinder of $\tilde{f}$, and let $\gamma_C\colon C_X \to X$ be the evident map. Note that γ_C gives a finite model for X, and that the diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r, tail] \arrow[d, "\simeq"] & C_X \arrow[d, "\simeq"] \\ A \arrow[r, "f"] & X \end{tikzcd}$$

commutes. Here $\widetilde{A} \hookrightarrow C_X$ is the canonical inclusion into the top of the mapping cylinder. Since $(C_X, \tilde{A})$ is a finite CW-pair, the lemma is proven. □

Proposition E.3. *Let (X, A) be homotopically compact. If $f_0\colon (X_0, A_0) \to (X, A)$ and $f_1\colon (X_1, A_1) \to (X, A)$ are finite models for (X, A), then there exists a map $(X_0, A_0) \to (X_1, A_1)$ such that the triangle*

$$\begin{tikzcd} (X_0, A_0) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & (X, A) \\ (X_1, A_1) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

commutes up to homotopy. Additionally, there exists a zig-zag of finite models

$$\begin{tikzcd} (Y_0, B_0) \arrow[r] \arrow[drr, "\simeq"'] & (Y_1, B_1) \arrow[dr, "\simeq"] & (Y_2, B_2) \arrow[l] \arrow[r] \arrow[d, "\simeq"] & \cdots & (Y_r, B_r) \arrow[l] \arrow[dll, "\simeq"] \\ & & (X, A) & & \end{tikzcd}$$

such that $(Y_0, B_0) \to (X, A)$ equals $(X_0, A_0) \to (X, A)$ and $(Y_r, B_r) \to (X, A)$ equals $(X_1, A_1) \to (X, A)$. That is, the category of finite models of (X, A) is connected.

Proof. First use that $[A_0, A_1] \to [A_0, A]$ is a bijection to produce a map $g\colon A_0 \to A_1$ whose image under the bijection is $f_0|_A$. We may assume that g is cellular. Choose a homotopy from $f_0|_A$ to $f_1|_A \circ g$, and using the Homotopy Extension Property extend this to a homotopy $H\colon X_0 \times I \to X$ such that $H_0 = f_0$. Let $f' = H_1$. We now have a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & X_1 \arrow[d, "\simeq"] \\ X_0 \arrow[r, "f'"] & X \end{tikzcd}$$

and so by the Relative Homotopy Lifting Property (???) there exists a map $h\colon X_0 \to X_1$ such that the upper triangle commutes and the lower triangle commutes up to a homotopy relative to A_0. And again, we map assume that h is cellular. Putting our two homotopies together, we get the required homotopy-commutative triangle.

For the final statement of the proposition we can use a four-step zig-zag as follows:

$$\begin{tikzcd} (X_0, A_0) \arrow[r, "i_0"] \arrow[dr, "\simeq", "f_0"'] & (X_0 \times I, A_0 \times I) \arrow[d, "J"', "\simeq"] & (X_0, A_0) \arrow[l, "i_1"'] \arrow[dl, "\simeq"', "f_1 h"] \arrow[r, "h"] & (X_1, A_1) \arrow[dll, bend left, "\simeq"', "f_1"] \\ & (X, A). & & \end{tikzcd}$$

The map labelled J is a homotopy for the triangle in the first part of the proposition. We leave the details for the reader. $\square$

For us what is very useful about the class of homotopically compact spaces is that it includes all algebraic varieties:

Theorem E.4. *If X is an algebraic variety over $\mathbb{C}$ then X is homotopically compact.*

Proof. When X is a subvariety of some $\mathbb{C}^n$ this is a consequence of [Hir2, Theorem on page 170 and Remark 1.10]. For the general case we do an induction on the size of an affine cover for X. Suppose that $\{U_1, \ldots, U_n\}$ is an affine cover, and let $A = U_2 \cup \cdots \cup U_n$. Then we have the pushout diagram

$$\begin{tikzcd} U_1 \cap A \arrow[r] \arrow[d] & A \arrow[d] \\ U_1 \arrow[r] & X, \end{tikzcd}$$

which is also a homotopy pushout by [DI, Corollary 1.6]. By induction we know that A is homotopically compact. Moreover, since U_1 is affine it is a subvariety of some $\mathbb{C}^n$, and therefore the same is true of $U_1 \cap A$. So both U_1 and $U_1 \cap A$ are homotopically compact by the base case. The result then follows by Lemma E.5 below. $\square$

Lemma E.5. *Let A, X, and Y be homotopically compact spaces. Then the homotopy pushout of any diagram $X \leftarrow A \to Y$ is also homotopically compact.*

Proof. Let $f\colon A\to X$ and $g\colon A\to Y$ denote the maps, and choose a finite model $\widetilde{A}\to A$. By Lemma E.2 (applied twice) there exists a diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\simeq"'] & \widetilde{A} \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & \widetilde{Y} \arrow[d, "\simeq"] \\ X & A \arrow[l] \arrow[r] & Y. \end{tikzcd}$$

where $(\widetilde{X},\widetilde{A})$ and $(\widetilde{Y},\widetilde{A})$ are finite CW-pairs. The homotopy pushout of $X\leftarrow A\to Y$ is therefore weakly equivalent to that of $\widetilde{X}\leftarrow\widetilde{A}\to\widetilde{Y}$, and the latter clearly has the homotopy type of a finite CW-complex (in fact, in the latter case the pushout is itself a model for the homotopy pushout). □

Corollary E.6. *If (X,A) is a pair of algebraic varieties over $\mathbb{C}$ then (X,A) is homotopically compact.*

Proof. This follows from Theorem E.4 and Proposition E.1. □

Appendix F. Reducing the length of complexes in K-theory

Appendix G. Abelian categories and exact categories

The first part of this book developed K-theory in the context of modules over a ring R. This can be generalized in various ways, e.g. to abelian categories, to exact categories, or (perhaps) to triangulated categories. While not strictly necessary for most of our present purposes, it is good to have a basic sense of how some of these generalizations work.

This section discusses the foundations of abelian categories, Serre quotients, and exact categories. We outline the theory via a series of exercises. Most readers will probably not want to do *all* of these exercises; many of them are things one can just accept and move on. But doing some portion of them gives a nice introduction to the ins and outs of working with these objects.

G.1. **Abelian categories.** A **zero object** in a category $\mathcal{C}$ is an object $*$ that is both initial and terminal. This means that $\mathcal{C}(*,X)$ and $\mathcal{C}(X,*)$ are both singletons, for every object X. Most often we will be in settings where $\mathcal{C}$ is enriched over abelian groups, meaning that the hom-sets have abelian group structures with respect to which composition is bi-additive. In such settings a zero object is usually denoted by 0.

In a category with a zero object a **kernel** of a map $f\colon A\to B$ is a map $K\to A$ that makes a pullback diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ 0 \arrow[r] & B. \end{tikzcd}$$

Dually, a **cokernel** of $A\to B$ is a map $B\to C$ that makes an analogous pushout diagram. Note that by abuse of terminology we often refer to the object K as the kernel of f, but in fact it is really the map $K\to A$ that is the kernel. We write $\ker f$ for the domain of the kernel and $\operatorname{coker} f$ for the codomain of the cokernel. When we say that "the kernel of f is zero" we mean that it is the unique map $0\to A$, and dually for the cokernel.

We will use categorical notions of monomorphisms and epimorphisms. The former will be denoted by $\rightarrowtail$ and the latter by $\twoheadrightarrow$.

Exercise G.2.

(a) Prove that a pullback of a monomorphism is a monomorphism, and dually for epimorphisms. Deduce that a kernel of a map is always a monomorphism and the cokernel of a map is always an epimorphism.

(b) Suppose that $\mathcal{C}$ is enriched over abelian groups and has a zero object. Prove that $f\colon A \to B$ is a monomorphism (resp. epimorphism) if and only if $\mathcal{C}(X, A) \to \mathcal{C}(X, B)$ (resp. $\mathcal{C}(B, X) \to \mathcal{C}(A, X)$) is a monomorphism for every object X.

(c) In the setting of (b) prove that a map is a monomorphism if and only if its kernel is zero, and dually for epimorphisms and cokernels.

Definition G.3. *An* **abelian category** *is a category enriched over abelian groups that has a zero object, finite coproducts and products which are equal, all kernels and cokernels, and where every monomorphism is the kernel of a map and every epimorphism is the cokernel of a map.*

Abelian categories were first introduced in [Bu] under the name "exact category" (a phrase that now means something different). The concept had been independently developed by Grothendieck and was used as the foundation for his treatment of homological algebra in [Gro]. Gabriel later developed the foundations of abelian categories extensively in [Ga], and this remains a very good reference. Note that the definition of abelian category as we have given it above is not identical to the one given in [Gro] and [Ga], but they are equivalent.

Here are a few examples:

- The category of modules over a ring (this is really the canonical example).
- The category of torsion abelian groups. More generally, if R is a commutative ring and S is a multiplicative system then the category of S-torsion modules (modules M where $S^{-1}M = 0$) is an abelian category.
- If R is Noetherian, the category of finitely-generated R-modules is abelian. Note that without the Noetherian hypothesis this category would not be guaranteed to have kernels.
- If $\mathcal{A}$ is an abelian category and I is a small category then the category of functors $\mathrm{Func}(I, \mathcal{A})$ is again an abelian category.
- If X is a topological space then the category of sheaves of abelian groups on X is an abelian category.

Exercise G.4. Let $\mathcal{A}$ be an abelian category.

(a) If $f\colon A \to B$ is a monomorphism prove that f is the kernel of the map $B \to \mathrm{coker}(f)$ ("every monomorphism is the kernel of its cokernel"). Note the dual result for epimorphisms.

(b) Prove that if a map is both a monomorphism and an epimorphism then it is an isomorphism.

(c) Prove that $\mathcal{A}$ has pullbacks and pushouts (hint: construct these as certain kernels and cokernels). If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a pullback square, prove that $\ker f \to \ker g$ is an isomorphism; then prove the dual result about pushout squares and cokernels. Give an example in $\mathcal{A}b$ of a pullback square that is not a pushout, and also vice versa.

Prove that if the above is a pullback square and g is an epimorphism, then it is also a pushout square and f is an epimorphism. Dually, if the diagram is a pushout square and f is a monomorphism, then the diagram is also a pullback square and g is a monomorphism.

(d) Prove that the following conditions are equivalent:
  (i) $0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$ is such that j is the kernel of p and p is the cokernel of j;
  (ii) $0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$ yields left-exact sequences of abelian groups after applying $\mathrm{Hom}(X,-)$ and after applying $\mathrm{Hom}(-,X)$, for every X.

These equivalent conditions provide the notion of "short exact sequence" in an abelian category.

(e) The **image** of a map $f\colon A \to B$ is defined to be the kernel of the map $B \to \mathrm{coker}(f)$. Likewise, the **coimage** of f is the cokernel of $\ker(f) \to A$. There is a canonical map from the coimage to the image; prove that this is an isomorphism. Consequently, the canonical map $A \to \mathrm{im}(f)$ is an epimorphism and $\mathrm{coim}(f) \to B$ is a monomorphism. [Hint: This is a little tricky, so refer to the diagram and outline below:

$$\begin{tikzcd}
 & PB \arrow[r] \arrow[d] \arrow[dl, dotted] & Z \arrow[d, "u"] & & \\
\ker(f) \arrow[r, "i"] & A \arrow[r, two heads] & \mathrm{coker}(i) \arrow[r, equal] \arrow[d, "\alpha"] & \mathrm{coim}(f) & \\
 & & \mathrm{im}(f) \arrow[r, tail] \arrow[d, "v"'] & B \arrow[r] \arrow[d] & \mathrm{coker}(f) \\
 & & W \arrow[r] & PO \arrow[ur, dotted] &
\end{tikzcd}$$

We are trying to show that α is an isomorphism, so first assume $u\colon Z \to \mathrm{coker}(i)$ is such that $\alpha u = 0$. Form the pullback PB and argue that the map $PB \to A$ factors through $\ker(f)$ and hence $PB \to \mathrm{coker}(i)$ is zero. Then conclude $u = 0$ since $PB \to Z$ is an epimorphism. A dual argument shows that if $v\colon \mathrm{im}(f) \to W$ is such that $v\alpha = 0$ then $v = 0$.]

(f) Given a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ such that $gf = 0$ there is a canonical map $\mathrm{im}(f) \to \ker(g)$. Say that the sequence is exact at B if this canonical map is an isomorphism. Prove that a sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact in the sense of (e) if and only if it is exact at each spot. Show that a long exact sequence breaks up into a collection of interlocking short exact sequences as usual.

(g) Prove the Snake Lemma: Given a diagram with exact rows

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

there is an exact sequence

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \xrightarrow{\partial} \mathrm{coker}(a) \longrightarrow \mathrm{coker}(b) \longrightarrow \mathrm{coker}(c)$$

where all of the maps except for ∂ are the evident ones. If $A \to B$ is injective then so is $\ker(a) \to \ker(b)$, and if $B' \to C'$ is surjective then so is $\mathrm{coker}(b) \to \mathrm{coker}(c)$. [Hint: Construct the pullback PB

$$\begin{tikzcd} & PB \arrow[r] \arrow[d] & \ker(c) \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & B \arrow[r] & C \end{tikzcd}$$

and the induced map $A \to PB$, and prove that the cokernel of $A \to PB$ is $\ker(c)$ (use that the pullback of an epimorphism is an epimorphism). The map $PB \to B'$ factors through A', and therefore one gets an induced map $\mathrm{coker}(A \to PB) \to \mathrm{coker}(a)$. This constructs ∂, and everything else is tedious but mostly routine.]

Suppose given an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories. The kernel of F is defined to be the full subcategory of $\mathcal{A}$ consisting of objects X for which $F(X) \cong 0$. Such a subcategory has the property given in the following definition:

Definition G.5. *A full subcategory $\mathcal{S} \subseteq \mathcal{A}$ is called a **Serre subcategory** if it has the property that whenever $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$ then B is in $\mathcal{S}$ if and only if A and C are in $\mathcal{S}$. That is, a full subcategory is Serre if and only if it is closed under subobjects, quotients, and extensions.*

Is every Serre subcategory the kernel of an exact functor? It turns out the answer is yes. One can construct a quotient abelian category $\mathcal{A}/\mathcal{S}$ where the objects in $\mathcal{S}$ are all identified with 0 and where $\mathcal{A}/S$ has the expected universal property. The projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{S}$ is exact and has kernel equal to $\mathcal{S}$.

The next couple of exercises will take the reader through the basic theory behind all of this.

Exercise G.6.

(a) Given a Serre subcategory $\mathcal{S} \subseteq \mathcal{A}$, define a map $f\colon A \to B$ to be an **$\mathcal{S}$-equivalence** if $\ker f$ and $\mathrm{coker}\, f$ are in $\mathcal{S}$. Prove that the $\mathcal{S}$-equivalences have the two-out-of-three property: if $A \xrightarrow{f} B \xrightarrow{g} C$ are composable maps in $\mathcal{A}$ and two of f, g, and gf are $\mathcal{S}$-equivalences, then so is the third. [Hint: Produce a long exact sequence of the form $0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \mathrm{coker}(f) \to \mathrm{coker}(gf) \to \mathrm{coker}(g) \to 0$.]

(b) If f is an S-equivalence and a monomorphism prove that any pushout of f is also those things. Dually, if g is an S-equivalence and an epimorphism prove than any pullback is as well.

(c) Prove that if $f\colon A \to B$ is an S-equivalence and a monomorphism then any pullback of f is also those things, and dually for epimorphisms and pushouts. [This one is tricky, so use the following outline. Starting with the pullback diagram on the left form the pushout diagram on the right:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \oplus C \arrow[r, two heads, "\pi_2"] \arrow[d, "-f \oplus p"'] & C \arrow[d] \\ B \arrow[r] & Q. \end{tikzcd}$$

Prove that $B \to Q$ is an epimorphism and is equal to the cokernel of f, hence Q is in $\mathcal{S}$. Then prove that the square on the right is also a pullback, so that the kernels of the two vertical maps are isomorphic. Use this to prove that the cokernel of $P \to C$ is a suboject of Q and hence lies in $\mathcal{S}$.]

(d) Prove that the S-equivalences are closed under pullbacks and pushouts. [Hint: First show that any S-equivalence can be factored as an epimorphism followed by a monomorphism, both of which are themselves S-equivalences.]

(e) Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Recall that an R-module M is said to be S-torsion if $S^{-1}M = 0$. Check that the S-torsion modules form a Serre subcategory, and that the S-equivalences are the maps $f\colon M \to N$ such that $S^{-1}f$ is an isomorphism.

Exercise G.7. Let $\mathcal{A}$ be an abelian category and $\mathcal{S} \subseteq \mathcal{A}$ a Serre subcategory. Define a new category $\mathcal{A}/\mathcal{S}$ as follows: the objects are the same as those of $\mathcal{A}$, and morphisms from A to B consist of equivalence classes of certain zig-zags:

$$\mathcal{A}/\mathcal{S}(A, B) = \{\ A \xleftarrow[\sim_{\mathcal{S}}]{} A' \longrightarrow B' \xleftarrow[\sim_{\mathcal{S}}]{} B\ \}/\sim .$$

Here the indicated maps are $\mathcal{S}$-equivalences, and two zig-zags are equivalent if there is a commutative diagram

$$\begin{tikzcd} & A' \arrow[r] \arrow[dl, "\sim_{\mathcal{S}}"'] & B' & \\ A & \Gamma \arrow[l, "\sim_{\mathcal{S}}"] \arrow[r] \arrow[u] \arrow[d] & \Theta \arrow[u] \arrow[d] & B \arrow[l, "\sim_{\mathcal{S}}"] \arrow[ul, "\sim_{\mathcal{S}}"'] \arrow[dl, "\sim_{\mathcal{S}}"] \\ & A'' \arrow[r] \arrow[ul, "\sim_{\mathcal{S}}"] & B'' & \end{tikzcd}$$

with the original two zig-zags forming the upper and lower edges. Note that it is not immediately clear that this relation is transitive, and it is also not clear how to define composition in $\mathcal{A}/\mathcal{S}$—these will both be explained shortly. [Warning: If $\mathcal{A}$ is not small then there are potential set-theoretic issues in the above definition, as $\mathcal{A}/\mathcal{S}(A, B)$ is not clearly a set. One can bypass this issue by using Grothendieck universes, but see also the comment in (a) below.]

(a) Prove that given a commutative diagram as on the left (below) there exists a commutative diagram on the right having the same upper and lower edges.

$$\begin{tikzcd} & A' \arrow[r] \arrow[dl, "\sim_{\mathcal{S}}"'] \arrow[d] & B' \arrow[d] & \\ A & \Gamma \arrow[l, "\sim_{\mathcal{S}}"] \arrow[r] & \Theta & B \arrow[l, "\sim_{\mathcal{S}}"] \arrow[ul, "\sim_{\mathcal{S}}"'] \arrow[dl, "\sim_{\mathcal{S}}"] \\ & A'' \arrow[r] \arrow[ul, "\sim_{\mathcal{S}}"] \arrow[u] & B'' \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} & A' \arrow[r] \arrow[dl, "\sim_{\mathcal{S}}"'] & B' & \\ A & \tilde{\Gamma} \arrow[l, "\sim_{\mathcal{S}}"] \arrow[r] \arrow[u] \arrow[d] & \tilde{\Theta} \arrow[u] \arrow[d] & B. \arrow[l, "\sim_{\mathcal{S}}"] \arrow[ul, "\sim_{\mathcal{S}}"'] \arrow[dl, "\sim_{\mathcal{S}}"] \\ & A'' \arrow[r] \arrow[ul, "\sim_{\mathcal{S}}"] & B'' & \end{tikzcd}$$

Use this to prove that the relation used to define $\mathcal{A}/\mathcal{S}(A,B)$ is transitive, hence an equivalence relation. [Hint: Exercise G.6(d).]

(b) Prove that every zig-zag in $\mathcal{A}/\mathcal{S}(A,B)$ is equivalent to a zig-zag where $A' \to A$ is a monomorphism and $B \to B'$ is an epimorphism—call these "special" zig-zags. Prove additionally that if two special zig-zags are equivalent then there is a diagram as above where the middle zig-zag is also special. [Note: This only requires Exercise G.6(c)]. [Discussion: This implies that we can redefine $\mathcal{A}/\mathcal{S}(A,B)$ via the analogous definitions that only involve special zig-zags. In abelian categories where the objects have underlying sets, this provides a constraint that allows us to avoid the set-theoretic issues mentioned above.]

(c) The set $\mathcal{A}/\mathcal{S}(A,B)$ is sometimes described as

$$\operatorname*{colim}_{A' \hookrightarrow A,\, B \twoheadrightarrow B'} \mathrm{Hom}_{\mathcal{A}}(A',B')$$

where $A' \hookrightarrow A$ is a monomorphism that is an $\mathcal{S}$-equivalence and $B \twoheadrightarrow B'$ is an epimorphism that is an $\mathcal{S}$-equivalence. The colimit is taken over a certain category that can be easily reconstructed by looking at the definition of $\mathcal{A}/\mathcal{S}(A,B)$ above. Do this reconstruction and prove that the two descriptions of the morphism sets are the same. [Also, check that when we fix A the category whose objects are monomorphisms $A' \hookrightarrow A$ that are $\mathcal{S}$-equivalences is cofiltered.]

(d) Prove that every zig-zag in $\mathcal{A}/\mathcal{S}(A,B)$ is equivalent to one in which $A' \to A$ is the identity map, and also to one in which $B \to B'$ is the identity map (note that we are not claiming that both maps can be made to be the identity at once).

(e) Define composition in $\mathcal{A}/\mathcal{S}$ as follows. Given classes $U \in \mathcal{A}/\mathcal{S}(A,B)$ and $V \in \mathcal{A}/\mathcal{S}(B,C)$ represent U by a zig-zag where the final map is the identity and represent V by a zig-zag where the initial map is the identity. Then take the evident concatenation of zig-zags. Check that this is well-defined.

(f) Given $U, V \in \mathcal{A}/\mathcal{S}(A,B)$ define $U+V$ as follows. First represent U and V by special zig-zags, then prove that one can in fact represent them by zig-zags where the first and third map in U are equal to the first and third map in V (respectively). Define $U+V$ to be the zig-zag obtained by adding the two middle maps (it might be useful to also think about the description of the hom-sets from (c)). Verify that this is well-defined and makes $\mathcal{A}/\mathcal{S}$ into a category enriched over abelian groups.

(g) Verify that $\mathcal{A}/\mathcal{S}$ is an abelian category (this is somewhat long and tedious).

(h) Prove that all objects in $\mathcal{S}$ are isomorphic to 0 in $\mathcal{A}/\mathcal{S}$.

(i) Let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ be the identity on objects and send a map $f\colon A \to B$ to the zig-zag $A \xleftarrow{\mathrm{id}} A \xrightarrow{f} B \xleftarrow{\mathrm{id}} B$. Prove the following:

  (i) A zig-zag $A \xleftarrow{\sim_{\mathcal{S}}} A' \longrightarrow B' \xleftarrow{\sim_{\mathcal{S}}} B$ represents 0 in $\mathcal{A}/\mathcal{S}(A,B)$ if and only if $A' \to B'$ factors through an object in $\mathcal{S}$. In particular, if $f \in \mathcal{A}(A,B)$ then $\pi(f) = 0$ if and only if f factors through an object in $\mathcal{S}$.

  (ii) π sends monomorphisms to monomorphisms, and (dually) epimorphisms to epimorphisms.

  (iii) π is an exact functor.

  (iv) π sends $\mathcal{S}$-equivalences to isomorphisms.

(j) Given A in $\mathcal{A}$, $\pi(A) \cong 0$ if and only if A is in $\mathcal{S}$. Likewise, given $f\colon A \to B$ in $\mathcal{A}$ one has that $\pi(f)$ is an isomorphism if and only if f is an $\mathcal{S}$-equivalence.

(k) Prove that if $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories and $F(X) \cong 0$ for every X in $\mathcal{S}$, then there is a unique functor $\bar{F}\colon \mathcal{A}/\mathcal{S} \to \mathcal{B}$ making the evident triangle commute.

(l) For every pair of objects X, Y in $\mathcal{A}$ and a map $X \xrightarrow{u} Y$ in $\mathcal{A}/\mathcal{S}$, there exist maps $X \to \hat{Y}$ and $\hat{X} \to Y$ in $\mathcal{A}$ together with commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & \hat{Y} \arrow[d, "\cong"] \\ X \arrow[r, "u"] & Y \end{tikzcd} \qquad \begin{tikzcd} \hat{X} \arrow[r] \arrow[d, "\cong"'] & Y \arrow[d, equal] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

in $\mathcal{A}/\mathcal{S}$.

(m) Given a monomorphism $W \rightarrowtail Z$ in $\mathcal{A}/\mathcal{S}$ there exists a monomorphism $\hat{W} \rightarrowtail \hat{Z}$ in $\mathcal{A}$ and a commutative diagram

$$\begin{tikzcd} \hat{W} \arrow[r, tail] \arrow[d, "\cong"'] & \hat{Z} \arrow[d, "\cong"] \\ W \arrow[r, tail] & Z \end{tikzcd}$$

in $\mathcal{A}/\mathcal{S}$ where the vertical maps are isomorphisms.

(n) Every exact sequence in $\mathcal{A}/\mathcal{S}$ is isomorphic to the image under π of an exact sequence in $\mathcal{A}$.

[Compare parts (l) and (m) to Corollary 4.11. The connection will be clearer after Exercise G.11 below.]

Remark G.8. The category $\mathcal{A}/\mathcal{S}$ from the above exercise is called the **Gabriel quotient** or **Serre quotient** of abelian categories. Often it is just called the **quotient**. It was first constructed by Gabriel [Ga], and in that reference Serre categories were called *épaisse* ("thick") subcategories.

The construction of $\mathcal{A}/\mathcal{S}$ is elegant from a theoretical perspective, but it can be hard to work with in practice. In many nice situations, though, $\mathcal{A}/\mathcal{S}$ can be identified with a subcategory of $\mathcal{A}$. For this to work out, we need to have enough objects in $\mathcal{A}/\mathcal{S}$ that "see" all of the objects in $\mathcal{S}$ as zero and also "see" all of the $\mathcal{S}$-equivalences as if they were isomorphisms. We explore this next.

Exercise G.9. Let $\mathcal{S} \subseteq \mathcal{A}$ be a Serre subcategory.

(a) Say that an object X of $\mathcal{A}$ is **$\mathcal{S}$-null** if $\operatorname{Hom}_{\mathcal{A}}(A, X) = 0$ for all A in $\mathcal{S}$. Likewise, say that X is **$\mathcal{S}$-local** if for every $\mathcal{S}$-equivalence $A \to B$ the induced map $\operatorname{Hom}_{\mathcal{A}}(B, X) \to \operatorname{Hom}_{\mathcal{A}}(A, X)$ is an isomorphism. Prove that if X is $\mathcal{S}$-local then it is $\mathcal{S}$-null, and give a counterexample to the converse.

(b) Verify that if X is $\mathcal{S}$-local then any solid-arrow diagram below has a unique lifting as shown:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\sim_{\mathcal{S}}"'] & X. \\ B \arrow[ur, dotted] & \end{tikzcd}$$

(c) If X and Y are $\mathcal{S}$-local and $f\colon X \to Y$ is an $\mathcal{S}$-equivalence, prove that f is in fact an isomorphism.

(d) Prove that the following are equivalent:

(i) X is $\mathcal{S}$-local,
(ii) $\mathrm{Hom}_{\mathcal{A}}(A, X) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(A, X)$ for all A in $\mathcal{S}$,
(iii) For every A in $\mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(A, X) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(\pi A, \pi X)$ is an isomorphism.

[Hint: For (i)⇒(iii) construct the inverse to the given map.]

(e) Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Let $\mathcal{S}$ be the Serre subcategory of S-torsion modules. Verify that a module M is $\mathcal{S}$-local if and only if it is S-local (meaning that every element of S acts invertibly on M).

The above exercise suggests the idea that we try to embed $\mathcal{A}/\mathcal{S}$ into $\mathcal{A}$ via the $\mathcal{S}$-local objects. So for every object X in $\mathcal{A}/\mathcal{S}$ (equivalently, an object of $\mathcal{A}$) we need to attach a corresponding $\mathcal{S}$-local object. The case of S-torsion R-modules suggests that this sometimes can be done in a universal way, as in the following definition:

Definition G.10. *Let $\mathcal{S} \subseteq \mathcal{A}$ be a Serre subcategory. An **$\mathcal{S}$-localization** of an object X in $\mathcal{A}$ is an object $X_{\mathcal{S}}$ and an $\mathcal{S}$-equivalence $X \to X_S$ satisfying the following universal property:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y^{\mathcal{S}\text{-local}}. \\ X_{\mathcal{S}} \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

*That is, $X \to X_{\mathcal{S}}$ is univeral among maps to $\mathcal{S}$-local objects. The Serre subcategory $\mathcal{S}$ is called **localizing** if every object has an $\mathcal{S}$-localization.*

Exercise G.11. Suppose $\mathcal{S} \subseteq \mathcal{A}$ is a Serre subcategory.

(a) If $\mathcal{S}$ is localizing, show that there exists a functor $T\colon \mathcal{A}/\mathcal{S} \to \mathcal{A}$ sending each X to its $\mathcal{S}$-localization $X_{\mathcal{S}}$ and that this functor is right adjoint to π.

(b) Conversely to the situation in (a), assume given a functor $T\colon \mathcal{A}/\mathcal{S} \to \mathcal{A}$ that is right adjoint to π (but without assuming that $\mathcal{S}$ is localizing). Prove the following:
(i) The image of T lies in the $\mathcal{S}$-local objects.
(ii) The co-unit $\epsilon_{\pi X}\colon \pi T(\pi X) \to \pi X$ is an isomorphism, for every X. Since π is surjective on objects, this implies that every co-unit is an isomorphism (one then says that T is a "section" of π).
(iii) Applying the functor π to the unit $X \to T\pi X$ yields an isomorphism, for every X.
(iv) The unit map $X \to T\pi X$ is an $\mathcal{S}$-localization, for every X. Hence $\mathcal{S}$ is localizing.

[Note that many texts define a localizing subcategory to be a Serre subcategory where π admits a right adjoint. The above shows this definition to be equivalent to ours.]

(c) Continue to assume that T is the right adjoint to π. Prove that T is fully-faithful, thereby identifying $\mathcal{A}/\mathcal{S}$ with the full subcategory of $\mathcal{A}$ consisting of the $\mathcal{S}$-local objects.

(d) Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Prove that the quotient category R-Mod $/\mathcal{S}$ is equivalent to the category of $S^{-1}R$-modules.

G.12. **Exact categories.** The notion of an "exact category" was introduced in [Q3]. It is a weakening of the notion of abelian category in which one does not require the existence of kernels and cokernels for *all* maps, but only for certain select classes. A good example to keep in mind is the category of finitely-generated projectives over a ring R, where one is only guaranteed kernels (resp. cokernels) for *split* surjections (resp. split injections). The basic tools of K-theory can be developed in this weakened context, and this allows (for example) for tools that apply to the groups $K(R)$ and $G(R)$ simultaneously.

Let $\mathcal{A}$ be a category enriched over abelian groups that has finite products and coproducts, which are equal. Suppose given a subclass of monomomorphisms and a subclass of epimorphisms, called "admissible" monomomorphisms and epimorphisms (respectively). In this situation we say that $\mathcal{A}$ is an **exact category** if the following criteria are satisfied:

(1) Admissible monomorphisms are closed under composition and isomorphism, and dually for admissible epimorphisms.
(2) Pullbacks of admissible epimorphisms exist and are again admissible epimorphisms, and dually for pushouts of admissible monomorphisms.
(3) Admissible monomorphisms have cokernels, and those are admissible epimorphisms; moreover, every admissible monomorphism is the kernel of its cokernel. Dually for admissible epimorphisms.
(4) For every pair of objects X and Y, the inclusion $X \hookrightarrow X \oplus Y$ is an admissible monomorphism (the projection $X \oplus Y \to Y$ is its cokernel and therefore an admissible epimorphism).

Given an exact category, define a sequence $A \xrightarrow{i} B \xrightarrow{p} C$ to be **exact** if i is an admissible monomorphism and p is its cokernel (and note that this is equivalent to saying that p is an admissible epimorphism and i is its kernel).

Here are a few examples:

- Every abelian category can be considered an exact category by declaring all monomorphisms and epimorphisms to be admissible.
- Let R be a ring and let $\mathcal{P}(R)$ be the full subcategory of R-Mod consisting of the finitely-generated projectives. Define a monomorphism in $\mathcal{P}(R)$ to be admissible if its cokernel is a projective R-module, and likewise define an epimorphism to be admissible if its kernel is a projective R-module. Then $\mathcal{P}(R)$ is an exact category.
- Let R be a ring and $\mathcal{A} \subseteq R$-Mod any full subcategory that is closed under direct sums. Define an epimorphism $f\colon A \to B$ to be admissible if it is split and $\ker f$ is in $\mathcal{A}$, and dually for admissible epimorphisms. Then $\mathcal{A}$ is an exact category.
- Suppose that $\mathcal{A}$ is an abelian category and $\mathcal{B} \subseteq \mathcal{A}$ is a full subcategory with the property that if $X \in \mathrm{ob}(\mathcal{B})$ and $X \cong Y$ in $\mathcal{A}$ then $Y \in \mathrm{ob}(\mathcal{B})$. Suppose also that $\mathcal{B}$ is closed under extensions: if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{A}$ and $X', X'' \in \mathrm{ob}(\mathcal{B})$ then $X \in \mathrm{ob}(\mathcal{B})$. Define a monomorphism in $\mathcal{B}$ to be admissible if it is a monomorphism in $\mathcal{A}$ and its cokernel (as a map in $\mathcal{A}$) lies in $\mathcal{B}$, and dually for admissible epimorphisms. Then $\mathcal{B}$ is an exact category.

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