A GEOMETRIC INTRODUCTION TO K-THEORY

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Preface

I started assembling the material for this book fourteen years ago, when I taught a course on some of these topics at University of Oregon. My aim was to tell the story of topological and algebraic K-theory together, particularly focusing on the geometric aspects. The connections between algebra, geometry, and topology are what especially excited me when I first encountered this material as a student, and are also the aspects that I found the hardest to learn. The existing texts often seemed to focused on just one perspective, and as a student I remember hitting mysterious spots where I thought "Would this make more sense if I better understood the connections to X, Y, or Z?". I have tried to design this book with the goal of helping students make those connections.

I first learned Serre's definition of intersection multiplicity from Mel Hochster, and I was immediately intrigued by what was—for me—a surprising connection between homological algebra and geometry. I was later led to K-theory, and algebraic topology, because of the work of Roberts [Roberts2, Roberts5] and Gillet-Soulé [GS]. Although there were other factors, it is not far from the truth to say that I became a topologist in order to better understand Serre's definition.

As it has always been for me when learning mathematics, I wanted to know how I could have guessed this definition for myself—what are the underlying principles that *tell us* to go looking in homological algebra for a definition of multiplicity. I still don't have a simple answer to that question, but I do have a complicated answer: it involves the landscape of complex-oriented cohomology theories, K-theory, derived categories of quasi-coherent sheaves, and related machinery. Once one learns to navigate this body of mathematics, Serre's alternating-sum-of-Tors definition almost seems obvious. This book is an attempt to explain what I mean by that.

I once read that certain projects are not so much completed as simply abandoned at some point, and that sentiment very much captures my own experience. In the end this book is not everything I dreamt it would be, and many topics that I planned to cover have out of necessity been left out. The central theme is complex K-theory and its relation with geometry, and although real K-theory and related topics (e.g. Clifford algebras) are certainly discussed the bulk of that material was cut and moved to a different project [D7]. I still hope that, despite its many shortcomings, readers will find this book to be a warm and congenial guide to a strikingly beautiful area of mathematics.

How to use this book. Graduate textbooks do double-duty: as a sourcebook for a course, but also as a reference work. Most of the textbooks that I own are not things that I have read in a linear way. A reader dives in to a part they need, skips back to fill in background, skips ahead to get an explanation of some later connection, etc. But whereas reference works might not be read linearly, *stories* are definitely told linearly. This makes for an inherent tension between different goals.

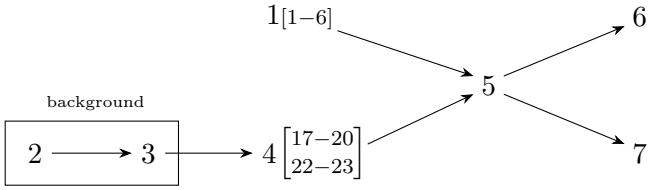
To try to help with this, the book has been structured into "Parts", each of which tells a certain story. Each Part has been divided into Sections, and my original hope was to keep each section under ten pages—something that could be handled in a single reading session. One thing I have learned over the years is that my hopes in writing are never realistic, so in fact many sections extend to 12 or 13 pages, with

a few technical sections being even longer. Still, ten pages was always the goal and I have lingering frustrations about every place I exceeded that.

The Parts are not exactly independent of each other, but are structured with the idea that most readers will jump around. Depending on their background a reader might find it easy to skip whole Parts. To steal a line from [Thomason], an "Honest person" would read the Parts in the order that they appear:

$$1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4 \Longrightarrow 5 \Longrightarrow 6 \Longrightarrow 7.$$

I expect almost no one will actually do this, though. For a "Choose your own adventure" approach, consider the following diagram of dependencies:



Parts 2 and 3 cover background that some readers might already possess (from homotopy theory and vector bundles, respectively), and so could be skipped or skimmed and referred back to as necessary—though at the very least I recommended reading Section 13 on Swan's theorem. The numbers in brackets are the sections that are crucial for Part 5, which is where the real action begins.

The book includes many exercises, and they fall into two basic types. The first type is sprinkled throughout the text and are intended to be (mostly) short supplements to the main discussion. For these the instructions are: "This is something worth thinking about right now, and you should at least read the statement and think for a minute about how you would tackle this."

The second type are exercises given at the end of the sections, and these are problems that would have been a distraction to incorporate in the middle of the discussion. They might involve extended examples that enhance the material from the section, or they might address topics that are important but just didn't naturally fit into the way we told the story.

Some sections have only the first type of problem, some have only the second time, some have both, and a few have neither—the latter mostly happens when the section is particularly technical.

Who this book is for. Throughout the book I assume readers have completed a standard year-long introductory course on algebraic topology, and an introduction to commutative algebra that got as far as heights and Krull dimension. Beyond this I have tried not to assume too much, but inevitably I had to have a certain reader in mind when making expository choices. So in the end I think this is a book on algebraic topology, written for students of that subject, more than a book written for students in algebra or geometry. For example, the book includes several sections of introductory material on algebraic geometry but only a brief section of introductory material from homotopy theory—readers are assumed to have picked up the latter somewhere else. I still hope the book is useful to a wide audience, but if the reader finds that occasionally I seem to be using some homotopy-theoretic technique that is unfamiliar to them, this is the explanation.

AI disclosure. Most of this book was written before modern AI was a significant phenomenon, but in the last year of work I found myself having frequent conversations/arguments with ChatGPT as I was flushing out fine details. The AI was often wrong about things, or required repeated corrections from me before getting things right, but I also found that it knew plenty of things that I didn't—and by using the Socratic method I was often led to the right way of understanding a topic more quickly than would have been possible on my own.

Perhaps most of us have at some point in our careers (often in graduate school) known a mathematician who knew a *ton* of stuff, was exceedingly confident in their knowledge, but where not everything they said was as correct as they believed it to be. This was my experience with ChatGPT. But I did very much find the conversations useful, and I acknowledge them here the same way I would acknowledge helpful conversations with an erudite but overconfident colleague.

None of the writing in this book was done by AI (writing is too much fun!) and every statement in the book is something that I have carefully considered and checked, albeit subject to my usual human fallibility.

Notation and terminology. All rings are commutative unless otherwise noted. When $\mathcal{C}$ is a category we will write $\mathcal{C}(X,Y)$ synonymously with $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. Isomorphisms are typically written $A \cong B$, but in cases where there is a standard and familiar canonical isomorphism we will sometimes write $A = B$ instead, e.g. the isomorphism of sets $X \times Y = Y \times X$. The category of topological spaces is just that—not the category of compactly-generated, weak Hausdorff spaces that is sometimes implicitly assumed by homotopy theorists. At some points I had regrets about this choice, but in the end I still think it is the correct one for an introductory text like this.

An index of symbols is provided at the end. Appendix A covers the background on point-set topology (beyond a first course) that is needed, including some of the issues one inevitably runs into regarding compactly-generated spaces.

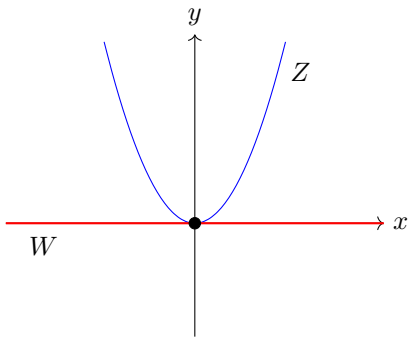
0.1. **Acknowledgments.** I was very fortunate in my student days to have three incredible teachers: Ed Martin, Mel Hochster, and Mike Hopkins. There is a great deal of all of them in this book. While writing a book turned out to be exceedingly difficult and miserable in ways that I had not foreseen, one of the unexpected pleasures was from time to time getting to hear their voices in my head again—remembering the way they would explain something, or the particular way they would tell the story behind some piece of mathematics. If this book passes on even a bit of the beauty of mathematics that they imparted to me, it will have been well worth the effort.

Over the years Nick Addington answered countless questions from me about algebraic geometry, and I am grateful for his time, expertise, and patience. On the algebraic topology side, my go-to source ever since my student days has been Gustavo Granja; I am so appreciative of his help and support over the years, in addition to his insightful knowledge of the field.

Introduction

1. Algebraic intersection multiplicities

Let Z be the parabola $y = x^2$ in $\mathbb{R}^2$, and let W be the tangent line at the vertex: the line $y = 0$. Then Z and W have an isolated point of intersection at $(0, 0)$:



Since high school you have known how to associate a *multiplicity* with this intersection: it is multiplicity 2, essentially because the polynomial x^2 has a double root at $x = 0$. This multiplicity also has a geometric interpretation, coming from intersection theory. If you perturb the intersection a bit, say by moving either Z or W by some small amount, then you get two points of intersection that are near $(0, 0)$—and these points both converge to $(0, 0)$ as the perturbation gets smaller and smaller.

You might object, rightly so, that I am lying to you. If we perturb $y = 0$ to $y = \epsilon$, with $\epsilon > 0$, then indeed we get two points of intersection: $(\sqrt{\epsilon}, \epsilon)$ and $(-\sqrt{\epsilon}, \epsilon)$. And these do indeed converge to $(0, 0)$ as $\epsilon \to 0$. But if we perturb the line in the other direction, by taking ϵ to be negative, then we get no points of intersection at all! To fix this, it is important to work over the complex numbers rather than the reals: the connection between geometry and algebra works out best (and simplest) in this case. If we work over $\mathbb{C}$, then it is indeed true that almost all small perturbations of our equations yield two solutions close to $(0, 0)$.

Our goal will be to vastly generalize the above phenomenom. Let $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$, and let Z be the algebraic variety defined by the vanishing of the f's. We write

$$Z = V(f_1, \ldots, f_k) = \{x \in \mathbb{C}^n \mid f_1(x) = f_2(x) = \cdots = f_k(x) = 0\}.$$

Likewise, let $g_1, \ldots, g_l \in \mathbb{C}[x_1, \ldots, x_n]$ and let $W = V(g_1, \ldots, g_l)$. Assume that P is an isolated point of the intersection $Z \cap W$. Our goal is to determine an algebraic formula, in terms of the f_i's and g_j's, for an intersection multiplicity $i(Z, W; P)$. This multiplicity should have the basic topological property that it coincides with the number of actual intersection points under almost all small deformations of Z and W.

Here are some basic properties, by no means comprehensive, that we would want such a formula to satisfy:

(1) $i(Z, W; P)$ should depend only on local information about Z and W near P.
(2) $i(Z, W; P) \geq 0$ always.

(3) If $\dim Z + \dim W < n$ then $i(Z, W; P) = 0$ (because in this case there is enough room in the ambient space to perturb Z and W so that they don't intersect at all).
(4) If $\dim Z + \dim W = n$ then $i(Z, W; P) > 0$.
(5) If $\dim Z + \dim W = n$ and Z and W meet transversely at P (meaning that $T_P Z \oplus T_P W = \mathbb{C}^n$), then $i(Z, W; P) = 1$.

Note that because of property (1) we can extend the notion of intersection multiplicity to varieties in $\mathbb{C}P^n$, simply by looking locally inside an affine chart for projective space that contains the point P. From now on we will do this without comment. The two statements below are not exactly 'basic properties' along the lines of (1)–(5) above, but they are basic results that any theory of intersection multiplicities should yield as consequences.

(6) Suppose that $X \hookrightarrow \mathbb{C}P^n$ is the vanishing set of a homogeneous polynomial, that is $X = V(f)$. Let L be a projective line in $\mathbb{C}P^n$ that meets X in finitely-many points. Then
$$\sum_{P \in X \cap L} i(X, L; P) = \deg(f).$$
(7) (Bezout's Theorem) Suppose that $X, Y \hookrightarrow \mathbb{C}P^2$ are the vanishing sets of homogeneous polynomials f and g, and that $X \cap Y$ consists of finitely-many points. Then
$$\sum_{P \in X \cap Y} i(X, Y; P) = (\deg f)(\deg g).$$

Note that (6), for the particular case $n = 2$, is a special case of (7).

If you play around with some simple examples, an idea for defining intersection multiplicities comes up naturally. It is

$$i(Z, W; P) = \dim_{\mathbb{C}} \Big[\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k, g_1, \ldots, g_l) \Big]_P. \tag{1.1}$$

Here the subscript P indicates localization of the given ring at the maximal ideal $(x_1 - p_1, \ldots, x_n - p_n)$ where $P = (p_1, \ldots, p_n)$. The localization is necessary because $Z \cap W$ might have points other than P in it, and our definition needs to only depend on what is happening near P.

The best way to get a feeling for the definition in (1.1) is via some easy examples:

Example 1.2. Let $f = y - x^2$ and $g = y$. This is our example of the parabola and the tangent line at its vertex. The point $P = (0, 0)$ is the only intersection point, and our definition tells us to look at the ring
$$\mathbb{C}[x, y]/(y - x^2, y) \cong \mathbb{C}[x]/(x^2).$$
As a vector space over $\mathbb{C}$ this is two-dimensional, with basis 1 and x. So our definition gives $i(Z, W; P) = 2$ as desired. [Note that technically we should localize at the ideal (x, y), which corresponds to localization at (x) in $\mathbb{C}[x]/(x^2)$; however, this ring is already local and so the localization has no effect].

Example 1.3. The above example readily generalizes. If $h(x) \in \mathbb{C}[x]$ then let $f = y - h(x)$ and $g = y$. Factor $h(x) = \prod_i (x - z_i)^{e_i}$ and consider the intersection multiplicity of $V(f)$ and $V(g)$ at the point $(z_1, 0)$. Here we get $\mathbb{C}[x, y]/(f, g) \cong$

$\mathbb{C}[x]/(h(x))$ and after localization at the ideal $(x - z_1)$ the factors $x - z_i$ for $i > 1$ become units so that we have

$$\mathbb{C}[x]_{(x-z_1)}/(h(x)) \cong \mathbb{C}[x]_{(x-z_1)}/(x-z_1)^{e_1} \cong \mathbb{C}[u]_{(u)}/(u^{e_1}).$$

The dimension over $\mathbb{C}$ is then e_1, coinciding with the multiplicity of z_1 as a root of $h(x)$.

Example 1.4. Let $f = y^2 - x^3 - 3x$ and $g = y - \frac{3}{2}x - \frac{1}{2}$. Then $Z = V(f)$ is an elliptic curve, and one can check that $W = V(g)$ is the tangent line at the point $P = (1, 2)$. Let us recall how this works: the gradient vector to the curve is

$$\nabla f = [-3x^2 - 3, 2y]$$

and this is normal to the curve at (x, y). A tangent vector is then $[2y, 3x^2 + 3]$ (since this is orthogonal to ∇f), which means the slope of the curve at (x, y) is $(3x^2 + 3)/2y$. At the point $(1, 2)$ we then get slope $\frac{3}{2}$, and $V(g)$ is the line passing through $(1, 2)$ with this slope.

The line $V(g)$ intersects the curve at one other point, which we find by simultaneously solving $y^2 = x^3 + 3x$ and $y = \frac{3}{2}x + \frac{1}{2}$. This yields the cubic

$$0 = x^3 + 3x - (\tfrac{3}{2}x + \tfrac{1}{2})^2.$$

Since we know that $x = 1$ is a root, we can factor this out and then solve the resulting quadratic. One finds that the cubic factors as

$$0 = (x - 1)^2 \cdot (x - \tfrac{1}{4}).$$

The second point of intersection is found to be $Q = (\frac{1}{4}, \frac{7}{8})$.

Note the appearance of $(x - 1)$ with multiplicity two in the above factorization. The fact that we had a tangent line at $x = 1$ guaranteed that the multiplicity would be strictly larger than one. Likewise, the fact that $(x - \frac{1}{4})$ has multiplicity one tells us that $V(g)$ intersects the curve transversely at the second point. These facts suggest that $i(Z, W; P) = 2$ and $i(Z, W; Q) = 1$. Let us consider these in terms of point-counting under small deformations. We can perturb either Z or W, but it is perhaps easiest to perturb the line W: we can write $\tilde{g} = y - Ax - B$ and then consider what happens for all (A, B) near $(\frac{3}{2}, \frac{1}{2})$. We will need to find the intersection of Z and $\widetilde{W} = V(\tilde{g})$, which as before requires us to solve a cubic. Let us again arrange for there to be a known solution which we can factor out. It is possible to have this solution be either $(1, 2)$ or $(\frac{1}{4}, \frac{7}{8})$. The calculations turn out to be a little easier for the latter, despite the annoying fractions. So we assume $\frac{7}{8} = \frac{A}{4} + B$, or $\tilde{g} = y - A(x - \frac{1}{4}) - \frac{7}{8}$. Since we want to look at A near $\frac{3}{2}$, it is convenient to write $A = \frac{3}{2} + \epsilon$ where ϵ is near zero.

Finding common solutions of $f = 0$ and $\tilde{g} = 0$ yields a cubic with $(x - \frac{1}{4})$ as a factor, and dividing this out we obtain the quadratic

$$0 = x^2 - x(2 + 3\epsilon + \epsilon^2) + (1 - \epsilon + \tfrac{\epsilon^2}{4}).$$

The discriminant of this quadratic is $D = \epsilon(\epsilon^3 + 6\epsilon^2 + 4\epsilon + 16)$, so the quadratic has a double root when $\epsilon = 0$ (as expected) but simple roots for values of ϵ near but not equal to zero. So for these values of ϵ we get two points of intersection of $V(f)$ and $V(\tilde{g})$ near P, and it is easy to see that they converge to P as ϵ approaches zero.

Let us now see what our provisional definition from (1.1) gives. The quotient ring in our definition is

$$\begin{aligned}\mathbb{C}[x,y]/\big(y^2-x^3-3x, y-\tfrac{3}{2}x-\tfrac{1}{2}\big) &\cong \mathbb{C}[x]/\big((\tfrac{3}{2}x+\tfrac{1}{2})^2-x^3-3x\big)\\ &\cong \mathbb{C}[x]/\big((x-1)^2(x-\tfrac{1}{4})\big).\end{aligned}$$

Here we are killing a cubic in $\mathbb{C}[x]$, and so we get a three-dimensional vector space with basis $1, x, x^2$. Note that this is, in some sense, seeing all of the information at P and Q together—this demonstrates the importance of localization. Localization at P corresponds to localizing at $(x-1)$, which turns $(x-\frac{1}{4})$ into a unit. So our localized ring is

$$\mathbb{C}[x]_{(x-1)}/((x-1)^2) \cong \mathbb{C}[t]_{(t)}/(t^2)$$

(where we set $t = x-1$), and this has dimension 2 over $\mathbb{C}$. So $i(Z, W; P) = 2$, as desired.

If we localize at $(x-\frac{1}{4})$ then the $(x-1)^2$ factor becomes a unit, and our localized ring becomes $\mathbb{C}[x]_{(x-\frac{1}{4})}/(x-\frac{1}{4}) \cong \mathbb{C}[t]_{(t)}/(t)$, which is just a copy of $\mathbb{C}$. So $i(Z, W; Q) = 1$.

Note that Examples 1.2 through 1.4 involve a key step where the variable y is eliminated, thus bringing the problem down to the multiplicity of a root in a one-variable polynomial. One cannot always do such an elimination—in fact it happens only rarely. So these examples are very special, although they still serve to give some sense of how things are working.

It turns out that our provisional definition from (1.1) is enough to prove Bezout's Theorem for curves in $\mathbb{C}P^2$. But in some sense one is getting lucky here, and it works only because the dimensions of the varieties are so small. When one starts to look at higher-dimensional varieties it doesn't take long to find examples where the definition clearly gives the wrong answers:

Example 1.5. In $\mathbb{C}^3$ let $X = V(y, z)$ and $Y = V(x, z)$. These are the x-axis and y-axis, respectively, meeting in the origin P. Since $\dim X + \dim Y < 3$ we expect $i(X, Y; P) = 0$. But our provisional definition gives $\dim_{\mathbb{C}} \mathbb{C}[x, y, z]/(x, y, z)$, which is 1.

Perhaps this example starts to give an inkling of what is going wrong in our formula. Between $\mathbb{C}[x, y, z]/(y, z)$ and $\mathbb{C}[x, y, z]/(x, z)$ we are killing z twice, but in the tensor product $\mathbb{C}[x, y, z]/(x, y, z)$ we are only killing it once (killing it twice doesn't really make sense—once it becomes zero, setting it to zero *again* does nothing). If we *could* somehow kill it again, that might bring the answer down to $1 - 1 = 0$, which is what we want.

The next example is much more subtle:

Example 1.6. Let $\mathbb{C}^4$ have coordinates u, v, w, y, and let $X, Y \subseteq \mathbb{C}^4$ be given by

$$X = V(u^3 - v^2, u^2y - vw, uw - vy, w^2 - uy^2), \qquad Y = V(u, y).$$

Note that X is somewhat complicated, but Y is just a plane. If a point (u, v, w, y) is on $X \cap Y$ then $u = y = 0$ and therefore the equations for X say that

$$v^2 = 0, \quad vw = 0, \quad \text{and} \quad w^2 = 0$$

as well. So $X \cap Y$ consists of the unique point $(0, 0, 0, 0)$. Our provisional definition of intersection multiplicities would have us look at the ring

$$\mathbb{C}[u, v, w, y]/(u, y, u^3 - v^2, u^2y - vw, uw - vy, w^2 - uy^2) \cong \mathbb{C}[v, w]/(v^2, vw, w^2)$$

which is three-dimensional over $\mathbb{C}$. If this were the correct answer, then perturbing the plane Y should generically give three points of intersection. However, this is not the case. If we perturb Y to $V(u-\epsilon, y-\delta)$ then the intersection with X is given by the equations

$$u = \epsilon, \quad y = \delta, \quad \epsilon^3 = v^2, \quad \epsilon^2\delta = vw, \quad \epsilon w = v\delta, \quad w^2 = \epsilon\delta^2.$$

As long as $\epsilon \neq 0$ we have two solutions for v, and then the fourth equation determines w completely (the last two equations are redundant). So we only have two points on the intersection, after small perturbations. This is, in fact, the correct answer: $i(Z, W; P) = 2$, and our provisional definition has failed. (A deeper look into this example is given in Section 36.11).

Serre discovered the correct formula for the interesection multiplicity [Serre2]. His formula is as follows. If we set $R = \mathbb{C}[x_1, \ldots, x_n]$ then

$$i(Z, W; P) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}} \Big[\operatorname{Tor}_j^R \Big(R/(f_1, \ldots, f_k), R/(g_1, \ldots, g_l) \Big) \Big]_P. \tag{1.7}$$

There are several things to say here. First, although the sum is written to infinity it turns out that the Tor modules vanish for all $j > n$ (we will prove this later). So it is, in fact, a finite sum. Secondly, the condition that P is an isolated point of intersection forces the $\mathbb{C}$-dimension of all the Tor's to be finite. So the formula does make sense. As to why this gives the "correct" numbers, it will take us a while to explain this. But note that the $j = 0$ term is the dimension of

$$\begin{aligned} \operatorname{Tor}_0(R/(f_1, \ldots, f_k), R/(g_1, \ldots, g_l)) &\cong R/(f_1, \ldots, f_k) \otimes_R R/(g_1, \ldots, g_l) \\ &\cong R/(f_1, \ldots, f_k, g_1, \ldots, g_l). \end{aligned}$$

So our provisional definition from (1.1) is just the $j = 0$ term. One should think of the higher terms as "corrections" to this initial term; in a certain sense these corrections get smaller as j increases (this is not obvious).

Example 1.8. Let us revisit Example 1.5. For $R = \mathbb{C}[x, y, z]$ we can resolve $R/(y, z)$ by the complex

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} -z \\ y \end{smallmatrix}\right]} R^2 \xrightarrow{[\,y\ z\,]} R \longrightarrow R/(y, z) \longrightarrow 0.$$

Tensoring with $R/(x, z)$ gives the complex

$$0 \longrightarrow \mathbb{C}[y] \xrightarrow{\left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right]} \mathbb{C}[y]^2 \xrightarrow{[\,y\ 0\,]} \mathbb{C}[y] \longrightarrow 0,$$

leading to

$$\operatorname{Tor}_i(R/(x, z), R/(y, z)) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

So Serre's formula gives $i(X, Y; P) = 1 - 1 = 0$, which is the desired answer. (From a certain perspective, the Tor_1 term is "remembering" that z is being killed twice when forming the tensor product.)

Exercise 1.9. Let X be the origin in $\mathbb{C}^1$. Determine $i(X, X; X)$ using geometric principles as well as Serre's formula, and show that they match.

An algebraist who looks at (1.7) will immediately notice some possible generalizations. The $R/(\underline{f})$ and $R/(\underline{g})$ terms can be replaced by any finitely-generated module M and N, as long as the $\operatorname{Tor}_j(M,N)$ modules are finite-dimensional over $\mathbb{C}$. For this it turns out to be enough that $M \otimes_R N$ be finite-dimensional over $\mathbb{C}$. Also, we can replace $\mathbb{C}[x_1,\ldots,x_n]$ with any ring having the property that all finitely-generated modules have finite projective dimension—necessary so that the alternating sum of (1.7) is finite. Such rings are called **regular**. Also, instead of localizing the Tor-modules we can just localize the ring R at the very beginning. And finally, in this generality we need to replace $\dim_{\mathbb{C}}$ with a similar invariant: the notion of *length* (meaning the length of a composition series for our module). This leads to the following setup.

Let R be a regular, local ring (all rings are assumed to be commutative and Noetherian unless otherwise noted). Let M and N be finitely-generated modules over R such that $M \otimes_R N$ has finite length. This implies that all the $\operatorname{Tor}_j(M,N)$ modules also have finite length. Define

$$e(M,N) = \sum_{j=0}^{\infty} (-1)^j \ell\big(\operatorname{Tor}_j(M,N)\big) \tag{1.10}$$

and call this the **intersection multiplicty** of the modules M and N.

Based on geometric intuition, Serre made the following conjectures about the above situation:

(1) $\dim M + \dim N \le \dim R$ always
(2) $e(M,N) \ge 0$ always
(3) If $\dim M + \dim N < \dim R$ then $e(M,N) = 0$.
(4) If $\dim M + \dim N = \dim R$ then $e(M,N) > 0$.

In [Serre2] Serre proved all of these in the case that R contains a field, the so-called "geometric case" (some non-geometric examples for R include power series rings over the p-adic integers $\mathbb{Z}_p^{\wedge}$). Serre also proved (1) in general. Conjecture (3) was proven in the mid 80s by Roberts and Gillet-Soulé (independently), using some sophisticated topological ideas that were imported into algebra. Conjecture (2) was proven by Gabber in the mid 90s, using some high-tech algebraic geometry. Conjecture (4) is still open.

Here is an interesting and non-intuitive fact about these alternating-Tor formulas, to whet the reader's appetite. Suppose $Z = V(I)$ and $W = V(J)$ are smooth subvarieties of $\mathbb{C}^n$ having a point P as an isolated point of intersection. Define the *truncated Euler characteristics*

$$d_k = \sum_{i=0}^{k} \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_P$$

for $0 \le k \le n$, where as usual $R = \mathbb{C}[x_1,\ldots,x_n]$. Then d_n is the intersection multiplicity $i(Z,W;P)$, and in some sense the d_k's are approximations to $i(Z,W;P)$. They turn out to satisfy the following two properties:

(i) $d_k \ge i(Z,W;P)$ whenever k is even, and $d_k \le i(Z,W;P)$ whenever k is odd.
(ii) If $d_k = i(Z,W;P)$ then $d_j = i(Z,W;P)$ for all $j \ge k$.

So the d_k's alternate in being above or below $i(Z,W;P)$, and as soon as some d_k actually equals $i(Z,W;P)$ then the sequence stabilizes. This is wildly non-obvious! We will prove it in Section 36.

Another related issue is where the "stablization point" actually occurs. This is the lowest value of k for which $\mathrm{Tor}_k(R/I, R/J)_P$ vanishes (property (ii) above then implies that all of the higher Tor_i also vanish). Under sufficiently "good" conditions this stabilization point can be predicted in terms of the dimensions of all the varieties involved: in the above setting it happens at $1 + \dim(Z \cap W) - [\dim Z + \dim W - n]$. Example 1.8 is a nice demonstration: we have two lines in $\mathbb{C}^3$ intersecting in a point, so the Tor's start to vanish in degree $1 + 0 - [1 + 1 - 3] = 2$. But again, remember this only works when Z and W are sufficiently nice. We discuss the issues at length in Section 36.8.

1.11. **Where we are headed.** Our main goal in these notes is to describe a particular subset of the mathematics surrounding Serre's definition of multiplicity. It is possible to explore this subject purely in algebraic terms, and that is basically what Serre did in his book [Serre2]. In contrast, our main focus will be topological. Although both commutative algebra and algebraic geometry play a large role in our story, we will always adopt a perspective that concentrates on their relations to topology—and in particular, to K-theory.

Here is a brief summary of some of the main points that we will encounter:

(1) There are certain generalized cohomology theories—called *complex-oriented*—which have a close connection to geometry and intersection theory. Any such cohomology can be used to detect intersection multiplicities.
(2) Topological K-theory is a complex-oriented cohomology theory. Elements of the groups $K^*(X)$ are specified by vector bundles on X, or more generally by bounded chain complexes of vector bundles on X. Fundamental classes for complex submanifolds of X are given by *resolutions*.
(3) When X is an algebraic variety there is another version of K-theory called *algebraic K-theory*, which we might denote $K^*_{\mathrm{alg}}(X)$. The analogs of vector bundles are locally free coherent sheaves, or just finitely-generated projective modules when X is affine. Thus, in the affine case elements of $K^*_{\mathrm{alg}}(X)$ can be specified by bounded chain complexes of finitely-generated projective modules. This is the main connection between homological algebra and K-theory.
(4) Serre's definition of intersection multiplicities essentially comes from the intersection product in K-homology, which is the product in K-cohomology translated to homology via Poincaré Duality.

We will spend a large chunk of this book filling in the details behind (1)–(4). But whereas we take our motivation from Serre's definition of multiplicity, that is not the only subject we will cover. Once we have the K-theory apparatus up and running there are lots of neat things to do with it. We have attempted, for the most part, to chose topics that accentuate the relationship between K-theory and geometry in the same way that Serre's definition of multiplicity does.

∘ Exercises ∘

Exercise 1.12. In $\mathbb{C}^3$ let $X = V(z, y)$ and $Y = V(z, y - x^2)$ and let P be the origin. First determine $i(X, Y; P)$ from geometric principles. Compute the algebraic multiplicity both using our provisional definition and then using Serre's formula, comparing the answers. [Hint: Resolve $\mathbb{C}[x, y, z]/(y, z)$ as in Example 1.8.]

Exercise 1.13. Repeat the previous exercise for $X = V(xz, yz)$ and $Y = V(x, y - z)$. Here X is the union of a line and a plane, and Y is a line.

Exercise 1.14. Now repeat the above but with $X = V(x, y, z) = Y$. [If you have not seen a resolution of $\mathbb{C}[x, y, z]/(x, y, z)$, it is not hard to construct one by hand.]

Part 1. K-theory in algebra

In this first part of the book we investigate the K-theory of modules over a commutative ring R. There are two main varieties: one can study the K-theory of all finitely-generated modules, leading to the group $G(R)$, or one can study the K-theory of finitely-generated *projective* modules, leading to the group $K(R)$. In the following sections we get a taste for these groups and the relations between them.

For the duration of the book all rings are commutative with identity unless otherwise stated. Some of the theory we develop works in greater generality, but we will stay focused on the commutative case.

2. A first look at K-theory

Understanding Serre's alternating-sum-of-Tor's formula for intersection multiplicities will be a gradual process. In particular, there is quite a bit of nontrivial commutative algebra that is needed for the story; we will need to develop this as we go along. We will continue to sweep some of these details under the rug for the moment, but let us at least get a couple of things out in the open. To begin with, we will need the following important result:

Theorem 2.1 (Hilbert Syzygy Theorem)**.** *Let k be a field and let R be $k[x_1, \dots, x_n]$ (or any localization of this ring). Then every finitely-generated R-module has a free resolution of length at most n.*

We will prove this theorem in Section 27 below. We mention it here because it implies that $\operatorname{Tor}_j(M,N) = 0$ for $j > n$. Therefore the sum in Serre's formula is actually finite. More generally, a ring is called **regular** if it is Noetherian and every finitely-generated module has a finite projective resolution. It is a theorem that localizations of regular rings are again regular (Exercise 27.28). Hilbert's Syzygy Theorem simply says that polynomial rings over a field are regular. We will find that regular rings are the 'right' context in which to explore Serre's formula.

We will also need the following simple observation. If P is a prime ideal in any commutative ring R, then

$$[\operatorname{Tor}_R(M,N)]_P = \operatorname{Tor}^{R_p}(M_P, N_P).$$

To see this, let $Q_\bullet \to M \to 0$ be an R-free resolution of M. Since localization is exact, $(Q_\bullet)_P$ is an R_P-free resolution of M_P. Hence

$$\begin{aligned}\operatorname{Tor}_j^{R_P}(M_P, N_P) = H_j\big((Q_\bullet)_P \otimes_{R_P} N_P\big) &= H_j\big(Q_\bullet \otimes_R R_P \otimes_{R_P} N \otimes_R R_P\big)\\ &= H_j\big(Q_\bullet \otimes_R N \otimes R_P\big)\\ &= H_j\big(Q_\bullet \otimes_R N\big) \otimes R_P\\ &= \operatorname{Tor}_j^R(M,N) \otimes_R R_P.\end{aligned}$$

The importance of this observation is that it tells us that each Tor in Serre's formula for $i(Z, W; P)$ may be taken over the ring R_P. So we might as well work over this ring from beginning to end. Moreover, without loss of generality we might as well assume that our point of intersection is the origin, which makes the corresponding maximal ideal $(x_1, \dots, x_n)$.

Let $R = \mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$, and let M and N be finitely-generated modules over R. Assume that $\dim_{\mathbb{C}}(M \otimes_R N) < \infty$. It turns out that this implies that

$\dim_{\mathbb{C}} \operatorname{Tor}_j(M,N) < \infty$ for every j, so that we can define

$$e(M,N) = \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{C}} \operatorname{Tor}_j(M,N).$$

The above definition generalizes the notion of intersection multiplicity from pairs $(R/I, R/J)$ to pairs of modules (M,N). The reason for making this generalization might not be clear at first, but the following nice property provides some justification:

Lemma 2.2. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, where $R = \mathbb{C}[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$. Then $e(M,N) = e(M',N) + e(M'',N)$, assuming all three multiplicities are defined (that is, under the assumption that $\dim_{\mathbb{C}}(M \otimes N) < \infty$ and similarly with M replaced by M' and M'').*

Proof. Consider the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_j(M',N) \longrightarrow \operatorname{Tor}_j(M,N) \longrightarrow \operatorname{Tor}_j(M'',N) \longrightarrow \operatorname{Tor}_{j-1}(M',N) \to \cdots$$

This sequence terminates after a finite number of steps, by Hilbert's Syzygy Theorem. By exactness, the alternating sum of the dimensions is zero. This is precisely the desired formula. □

Lemma 2.2 is referred to as the *additivity of intersection multiplicities*. Of course the additivity holds equally well in the second variable, by the same argument.

While exploring ideas in this general area, Grothendieck hit upon the idea of inventing a group that captures **all** the additive invariants of modules. Any invariant such as $e(-,N)$ would then factor through this group. Here is the definition:

Definition 2.3. *Let R be any ring. Let $\mathcal{F}(R)$ be the free abelian group with one generator $[M]$ for every isomorphism class of finitely-generated R-module M. Let $G(R)$ be the quotient of $\mathcal{F}(R)$ by the subgroup generated by all elements $[M]-[M']-[M'']$ for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely-generated R-modules. The group $G(R)$ is called the **Grothendieck group** of finitely-generated R-modules.*

Remark 2.4. It is important in the definition of $G(R)$ that one use only *finitely-generated* R-modules, otherwise the group would be trivial. To see this, if M is any module then let $M^\infty = M \oplus M \oplus M \cdots$. Note that there is a short exact sequence

$$0 \to M \hookrightarrow M^\infty \to M^\infty \to 0$$

where M is included as the first summand. If we had defined $G(R)$ without the finite-generation condition, we would have $[M^\infty] = [M] + [M^\infty]$ and therefore $[M] = 0$. Since this holds for every module M, the group $G(R)$ would be zero. This is called the "Eilenberg Swindle".

Because of the need to focus on finitely-generated modules, and the fact that arguments will often require us to bring in submodules, results from here on out will often assume that R is Noetherian. The first example of this is part (c) of the next result.

The following proposition records some useful ways of obtaining relations in $G(R)$:

Proposition 2.5. *Let R be any ring.*

(a) *If* $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ *is an exact sequence of finitely-generated* R*-modules, then* $\sum(-1)^i[C_i] = 0$ *in* $G(R)$.
(b) *If* $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} = 0$ *is a filtration of* M *by finitely-generated modules, then* $[M] = \sum_i [M_i/M_{i+1}]$ *in* $G(R)$.
(c) *Assume that* R *is Noetherian, and let* $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ *be any chain complex of finitely-generataed* R*-modules. Then* $\sum_i (-1)^i[C_i] = \sum_i (-1)^i[H_i(C)]$ *in* $G(R)$.

Proof. We prove (a) and (c) at the same time. If $C_\bullet$ is a chain complex, note that one has the short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ where Z_i and B_i are the cycles and boundaries in each dimension. One also has $0 \to B_i \hookrightarrow Z_i \to H_i(C) \to 0$. Assuming everything in sight is finitely-generated, one gets a series of relations in $G(R)$ that immediately yield $\sum(-1)^i[C_i] = \sum(-1)^i[H_i(C)]$. So if R is Noetherian we are done, because everything indeed is finitely-generated; this proves (c). In the general case where R is not necessarily Noetherian, we know that each B_i is finitely-generated because it is the image of C_{i+1}. But if $C_\bullet$ is exact then $B_i = Z_i$ and so the Z_i's are also finitely-generated. We have the relations $[C_i] = [Z_i] + [B_{i-1}] = [Z_i] + [Z_{i-1}]$, and from this it is evident that $\sum(-1)^i[C_i] = 0$. This proves (a).

The proof of (b) is similarly easy; one considers the evident exact sequences $0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0$ and the resulting relations in $G(R)$. □

Here are a series of examples:

(1) Suppose $R = F$, a field. Clearly $G(F)$ is generated by $[F]$, since every finitely-generated F-module has the form F^n. If we observe the existence of the group homomorphism $\dim\colon G(F) \to \mathbb{Z}$, which is clearly surjective because it sends $[F]$ to 1, then it follows that $G(F) \cong \mathbb{Z}$.
(2) More generally, suppose that R is a domain. The **torsion-free rank** (often abbreviated to **rank**) of an R-module M is defined to be the dimension of $M \otimes_R QF(R)$ over $QF(R)$, where $QF(R)$ is the quotient field. The rank clearly gives a homomorphism $G(R) \to \mathbb{Z}$, which is surjective because $[R] \mapsto 1$. So $G(R)$ has $\mathbb{Z}$ as a direct summand.
(3) Next consider $R = \mathbb{Z}$. Then $G(\mathbb{Z})$ is generated by the classes $[\mathbb{Z}]$ and $[\mathbb{Z}/n]$ for $n > 1$, by the classification of finitely-generated abelian groups. The short exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \to 0$ shows that $[\mathbb{Z}/n] = 0$ for all n, hence $G(\mathbb{Z})$ is cyclic. Using (b), it follows that $G(\mathbb{Z}) = \mathbb{Z}$. This computation works just as well for any PID.
(4) So far we have only seen cases where $G(R) \cong \mathbb{Z}$. For a case where this is not true, try $R = F \times F$ where F is a field. You should find that $G(R) \cong \mathbb{Z}^2$ here. More generally, the theory of modules over a product ring $R \times S$ yields that $G(R \times S) \cong G(R) \oplus G(S)$ (Exercise 2.24).
(5) The definition of $G(R)$ can also be made for R non-commutative, using left R-modules (one can of course define another group using right R-modules, but that would be $G(R^{op})$). As an example, let G be a finite group and let $R = \mathbb{C}[G]$ be the group algebra. So R-modules are just representations of G on complex vector spaces. The basic theory of such finite-dimensional representations says that each is a direct sum of irreducibles, in an essentially unique way. Moreover, each short exact sequence is split. A little thought shows that this implies that

$G(R)$ is a free abelian group with basis consisting of the isomorphism classes of irreducible representations.

(6) So far all the examples we have computed have $G(R)$ equal to a free abelian group. This is not always the case, although it is difficult to give a counterexample that can be totally explained at this early stage. Just to give some sense of things, let R be the ring of integers in a number field. It turns out that $G(R) \cong \mathbb{Z} \oplus \mathrm{Cl}(R)$, where $\mathrm{Cl}(R)$ is the ideal class group of R. This class group contains sophisticated number-theoretic information about R. It is known to always be torsion, and it is usually nontrivial. We will work out a complete example when we have more tools under our belt: see Section 3.23.

(7) As another simple example, we look at $R = F[t]/(t^2)$ where F is a field. For any module M over R we have the filtration $M \supseteq tM$, and so $[M] = [M/tM]+[tM]$. But both M/tM and tM are killed by t, hence are direct sums of copies of F (where t acts as zero). This shows that $G(R)$ is generated by $[R/tR]$. We also have the function $\dim_F(-)\colon G(R) \to \mathbb{Z}$. Since this function sends $[R/tR]$ to 1, it must be an isomorphism.

(8) The final example we consider here is a variation of the previous one. Let us look at $R = \mathbb{Z}/p^2$. The R-modules are simply abelian groups killed by p^2. Given any such module A one can consider the sequence $0 \to pA \hookrightarrow A \to A/pA \to 0$, and observe that the first and third terms are $\mathbb{Z}/p$-vector spaces. So $[R/p]$ generates $G(R)$. We claim that $G(R) \cong \mathbb{Z}$, and as in the previous example the easiest way to see this is to write down an additive invariant of R-modules taking its values in $\mathbb{Z}$. All finitely-generated R-modules have a finite composition series, and so we can take the Jordan-Hölder length; this is the same as $\ell(A) = \dim_{\mathbb{Z}/p} A/pA + \dim_{\mathbb{Z}/p} pA$. With some trouble one can check that this is indeed an additive invariant (or refer to the Jordan-Hölder theorem), and of course $\ell(\mathbb{Z}/p) = 1$. This completes the calculation.

Exercise 2.6. Prove that $G(R) \cong \mathbb{Z}$ for $R = F[t]/(t^n)$ or $R = \mathbb{Z}/p^n$.

The above examples help establish some basic intuition. In general, though, it can be very hard to compute $G(R)$. In fact, given two modules M and N it can be hard to decide whether or not $[M] = [N]$ in $G(R)$. The following result (taken from [Heller, Lemma 2.1]) at least deconstructs the problem into something concrete:

Proposition 2.7. *Let M and N be finitely-generated R-modules. Then the following are equivalent:*

(1) $[M] = [N]$ in $G(R)$.

(2) There exist two exact sequences of finitely-generated modules $0 \to A \to X \to B \to 0$ and $0 \to A \to Y \to B \to 0$ and a finitely-generated module C such that $X \cong M \oplus C$ and $Y \cong N \oplus C$.

(3) There exist two exact sequences of finitely-generated modules $0 \to A \to X \to B \to 0$ and $0 \to A \to Y \to B \to 0$ such that $M \oplus X \cong N \oplus Y$.

Proof. We will prove (3)⇒(2)⇒(1)⇒(3). For (3)⇒(2) use the sequences

$$0 \to M \oplus N \oplus A \to M \oplus N \oplus Y \to B \to 0$$

and

$$0 \to M \oplus N \oplus A \to M \oplus N \oplus X \to B \to 0$$

with $C = M \oplus X \cong N \oplus Y$. (2)⇒(1) is easy since the hypotheses show that $[M]+[C] = [X] = [A]+[B] = [Y] = [N]+[C]$, and therefore $[M] = [N]$. The real content is therefore (1)⇒(3).

Let $\mathcal{R}el \subseteq \mathcal{F}(R)$ be the subgroup generated by all elements $[J]-[J']-[J'']$ for short exact sequences $0 \to J' \to J \to J'' \to 0$. If $[M]-[N] \in \mathcal{R}el$ then there exist two collections of such sequences $0 \to A_i' \to A_i \to A_i'' \to 0$, $1 \leq i \leq k_1$, and $0 \to B_j' \to B_j \to B_j'' \to 0$, $1 \leq j \leq k_2$, such that

$$[M]-[N] = \sum_{i=1}^{k_1}\big([A_i]-[A_i']-[A_i'']\big) + \sum_{j=1}^{k_2}\big([B_j']+[B_j'']-[B_j]\big)$$

in $\mathcal{F}(R)$. Rearranging, this gives the identity in $\mathcal{F}(R)$

$$[M] + \sum_{i=1}^{k_1}\big([A_i']+[A_i'']\big) + \sum_{j=1}^{k_2}[B_j] = [N] + \sum_{i=1}^{k_1}[A_i] + \sum_{j=1}^{k_2}\big([B_j']+[B_j'']\big).$$

The only way such sums of basis elements can give the same element of $\mathcal{F}(R)$ is if the collection of summands on the two sides are the same up to permutation. But in that case one can write

$$M \oplus \bigoplus_{i=1}^{k_1}(A_i' \oplus A_i'') \oplus \bigoplus_{j=1}^{k_2} B_j \cong N \oplus \bigoplus_{i=1}^{k_1} A_i \oplus \bigoplus_{j=1}^{k_2}(B_j' \oplus B_j'').$$

Finally, consider the evident short exact sequences

$$0 \to \bigoplus_j B_j' \oplus \bigoplus_i A_i' \to \bigoplus_i A_i \oplus \bigoplus_j (B_j' \oplus B_j'') \to \bigoplus_i A_i'' \oplus \bigoplus_j B_j'' \to 0$$

and

$$0 \to \bigoplus_j B_j' \oplus \bigoplus_i A_i' \to \bigoplus_j B_j \oplus \bigoplus_i (A_i' \oplus A_i'') \to \bigoplus_i A_i'' \oplus \bigoplus_j B_j'' \to 0.$$

Adding N to the middle term of the first sequence is isomorphic to the result of adding M to the middle term of the second. □

We can adapt our definition of intersection multiplicity of two modules to define a product on $G(R)$, at least when R is regular. For finitely-generated modules M and N, define

$$[M] \odot [N] = \sum_j (-1)^j [\mathrm{Tor}_j(M,N)].$$

Regularity of R guarantees that this is a finite sum. The long exact sequence for Tor shows that this definition is additive in the two variables, and hence passes to a pairing $G(R) \otimes G(R) \to G(R)$. It is not *at all* clear that this is associative, although we will prove this shortly (Corollary 2.15).

The above product on $G(R)$ is certainly not the first thing one would think of. It is more natural to try to define a product by having $[M]\cdot[N] = [M \otimes_R N]$, but of course this is not additive in the two variables because of the failure of the tensor product to be exact. The higher Tor's are correcting for this. However, we *can* make this naive definition work if we restrict to a certain class of modules. To that end, let us introduce the following definition:

Definition 2.8. *Let R be any ring. Let $\mathcal{F}_K(R)$ be the free abelian group with one generator $[P]$ for every isomorphism class of finitely-generated, projective R-module M. Let $K(R)$ be the quotient of $\mathcal{F}_K(R)$ by the subgroup generated by all elements $[P]-[P']-[P'']$ for every short exact sequence $0 \to P' \to P \to P'' \to 0$ of finitely-generated projectives. The group $K(R)$ is called the* ***Grothendieck group of finitely-generated projective modules****.*

Every short exact sequence of projectives is actually split, so we could also have defined $K(R)$ by imposing the relations $[P \oplus Q] = [P] + [Q]$ for every two finitely-generated projectives P and Q. This makes it a little easier to understand when two modules represent the same class in $K(R)$:

Proposition 2.9. *Let P and Q be finitely-generated projective R-modules. Then $[P] = [Q]$ in $K(R)$ if and only if there exists a finitely-generated projective module W such that $P \oplus W \cong Q \oplus W$. In fact, the same remains true if we require W to be free instead of projective.*

Proof. The first statement is immediate from Proposition 2.7 (which can be reproven verbatim in the present context) using that short exact sequences of projectives always split. For the second claim use that projectives are direct summands of free modules, so that there exists a W' such that $W \oplus W'$ is finitely-generated and free. □

Exercise 2.10. Give a direct proof of Proposition 2.9 along the lines of what we did for Proposition 2.7.

Exercise 2.11. Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$ be an exact sequence of finitely-generated projectives. Prove that $\sum_i (-1)^i [P_i] = 0$ in $K(R)$. [Note that this is almost Proposition 2.5(a) but maybe a tiny bit more thought is required.]

Since projective modules are flat, the product $[P]\cdot[Q] = [P \otimes_R Q]$ is additive and so extends to a product $K(R) \otimes K(R) \to K(R)$. Note that this product is obviously associative, and so makes $K(R)$ into a ring. This is true without any assumptions on R whatsoever (except our standing assumption that R be commutative).

Remark 2.12. Given the motivation of having the tensor product give a ring structure, one might wonder why we used projective modules to define $K(R)$ rather than flat modules. We could have done so, but for finitely-generated modules over commutative, Noetherian rings, being flat and projective are equivalent notions—see [Eisenbud, Corollary 6.6]. For various reasons (which will gradually become clear) it is more common to make the definition using the projective hypothesis.

There is an evident map $\alpha\colon K(R) \to G(R)$ which sends $[P]$ to $[P]$ (note that these two symbols, while they look the same, denote elements of different groups). This brings us to our first important theorem:

Theorem 2.13. *If R is regular then $\alpha\colon K(R) \to G(R)$ is an isomorphism.*

Proof. Surjectivity is easy to see: if M is a finitely-generated module, choose a finite, projective resolution $P_\bullet \to M \to 0$. Then $\sum_j (-1)^j [P_j] = [M]$ in $G(R)$, and this proves that $[M]$ is in the image of α.

Proving injectivity is slightly harder, and it will be most convenient just to define an inverse for α. The above paragraph gives us the definition: for a finitely-generated R-module M, define

$$\beta([M]) = \sum_j (-1)^j [P_j]$$

where $P_\bullet \to M \to 0$ is a finite resolution by finitely-generated projectives. We need to show that this is independent of the choice of P, and that it is additive: these facts will show that β defines a map $G(R) \to K(R)$. It is then obvious that this is a two-sided inverse to α.

Suppose $Q_\bullet \to M \to 0$ is another finite projective resolution of M. Use the Comparison Theorem of homological algebra to produce a map of chain complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Let $T_\bullet$ be the mapping cone of $f\colon P_\bullet \to Q_\bullet$. Recall this means that $T_j = Q_j \oplus P_{j-1}$, with the differential defined by

$$d_T(a,b) = \Big(d_Q(a) + f(b),\ -d_P(b)\Big).$$

There is a short exact sequence of chain complexes

$$0 \to Q \hookrightarrow T \to \Sigma P \to 0$$

where ΣP denotes a copy of P in which everything has been shifted up a dimension (so that $(\Sigma P)_n = P_{n-1}$) and the differential picks up a negative sign ($d_{\Sigma P} = -d_P$). The long exact sequence on homology groups shows readily that T is exact, hence we have $\sum_j (-1)^j [T_j] = 0$ in $K(R)$ by Exercise 2.11. Since $[T_j] = [Q_j] + [P_{j-1}]$ in $K(R)$ this gives that $\sum_j (-1)^j [P_j] = \sum_j (-1)^j [Q_j]$. Hence our definition of β does not depend on the choice of resolution.

A similar argument can be used to show additivity. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, and let $P_\bullet \to M'$ and $Q_\bullet \to M$ be finite projective resolutions. Lift the map $M' \to M$ to a map of complexes $f\colon P_\bullet \to Q_\bullet$, and let $T_\bullet$ be the mapping cone of f. The long exact sequence for homology readily shows that T is a projective resolution of M''. So

$$\beta(M'') = \sum (-1)^j [T_j] = \sum (-1)^j [Q_j] - \sum (-1)^j [P_j] = \beta(M) - \beta(M')$$

and this proves additivity. This completes our proof. □

Using the isomorphism $K(R) \to G(R)$ (when R is regular), we can transplant the ring structure on $K(R)$ to the group $G(R)$. We claim that this gives the product $\odot$ defined via Tor. In the following result, $\beta\colon G(R) \to K(R)$ is the inverse to α defined in the proof of Theorem 2.13.

Proposition 2.14. *Assume that R is regular. Then for any two finitely-generated modules M and N we have*

$$\alpha\Big[\beta([M]) \otimes \beta([N])\Big] = \sum (-1)^j [\mathrm{Tor}_j(M,N)] = [M] \odot [N].$$

Proof. Let $P_\bullet \to M$ and $Q_\bullet \to N$ be finite projective resolutions. Fix j, and consider the complex $P_\bullet \otimes Q_j$. This is a resolution of $M \otimes Q_j$, since Q_j is flat. So $\sum_i (-1)^i [P_i \otimes Q_j] = [M \otimes Q_j]$ in $G(R)$. Using this for each j, we have that

$$\begin{aligned}
\alpha\Big[\beta([M]) \otimes \beta([N])\Big] &= \sum_{i,j} (-1)^{i+j} [P_i \otimes Q_j] \\
&= \sum_j (-1)^j [M \otimes Q_j] \\
&= \sum_j (-1)^j [H_j(M \otimes Q)] \quad \text{using Proposition 2.5(c)} \\
&= \sum_j (-1)^j [\mathrm{Tor}_j(M, N)].
\end{aligned}$$

□

Corollary 2.15. *When R is regular the product $\odot$ on $G(R)$ is associative.*

Proof. This follows immediately from the fact that the tensor product gives an associative multiplication on $K(R)$. □

Let us review the above situation. For any ring R, we have the group $K(R)$ which also comes to us with an easily-defined ring structure $\otimes$. We also have the group $G(R)$—but this does not have any evident ring structure. When R is regular, there is an isomorphism $K(R) \to G(R)$ which allows one to transplant the ring structure from $K(R)$ onto $G(R)$: and this leads us directly to our alternating-sum-of-Tors formula.

This situation is very reminiscent of something you have seen in a basic algebraic topology course. When X is a (compact, oriented) manifold, there were early attempts to put a ring structure on $H_*(X)$ coming from the intersection product. This is technically very difficult. In modern times one avoids these technicalities by instead introducing the cohomology groups $H^*(X)$, and here it is easy to define a ring structure: the cup product. When X is a compact, oriented manifold one has the Poincaré Duality isomorphism $H^*(X) \to H_*(X)$ given by capping with the fundamental class, and this lets one transplant the cup product onto $H_*(X)$. This is the modern approach to intersection theory.

The parallels here are intriguing: $K(R)$ is somehow like $H^*(X)$, and $G(R)$ is somehow like $H_*(X)$. The regularity condition is like being a manifold. We will spend the rest of this course exploring these parallels. [The reader might wonder what happened to the assumptions of compactness and orientability. Neither of these is really needed for Poincaré Duality, as long as one does things correctly. For the version of Poincaré Duality for noncompact manifolds one needs to replace ordinary homology with Borel-Moore homology—this is similar to singular homology, but chains are permitted to have infinitely many terms if they stretch out to infinity. For non-orientable manifolds one needs to use twisted coefficients.]

Exercise 2.16. Check that the tensor product makes $G(R)$ into a left module over $K(R)$ in the evident way. The canonical map $\alpha\colon K(R) \to G(R)$ is just multiplication by the class $[R] \in G(R)$.

Exercise 2.17. Let $f\colon R \to S$ be a map of commutative rings. If P is a finitely-generated projective R-module check that $S \otimes_R P$ is a finitely-generated projective

S-module. Verify that there is an induced map $f_*\colon K(R) \to K(S)$ sending each $[P]$ to $[S \otimes_R P]$, and that f_* is a ring map.

Exercise 2.18. Let $f\colon R \to S$ be a flat map. Prove that there is an induced map of groups $f_!\colon G(R) \to G(S)$ sending each $[M]$ to $[S \otimes_R M]$.

Exercise 2.19. Let $f\colon R \to S$ be a ring map where S is finitely-generated as an R-module. Prove that there is a map of groups $f^*\colon G(S) \to G(R)$ that sends $[{}_SM]$ to $[{}_RM]$ for every finitely-generated S-module M.

Exercise 2.20. In topology $H_*(X)$ is a module over $H^*(X)$ via the cap product. Given $f\colon X \to Y$ there are maps $f^*\colon H^*(Y) \to H^*(X)$ and $f_*\colon H_*(X) \to H_*(Y)$, and f_* is a map of $H^*(Y)$-modules (where the module structure on the domain is via restriction of scalars along f^*): this last statement is the so-called projection formula $f_*(x \cap f^*y) = f_*x \cap y$. If $f\colon R \to S$ is a map where S is module-finite over R, then we have $f_*\colon K(R) \to K(S)$ and $f^*\colon G(S) \to G(R)$. Prove the analogous statement that f^* is a map of $K(R)$-modules. [Note: If the setup here feels backwards of what happens in topology, it's because we are writing things in terms of rings rather than schemes. This will become clearer in the next section.]

2.21. **Some very basic algebraic geometry.** To further develop the analogies between $(K(R), G(R))$ and $(H^*(X), H_*(X))$ we need more of a geometric understanding of the former groups. This starts to require some familiarity with the language of algebraic geometry.

At its most basic level, algebraic geometry attempts to study the geometry of affine n-space $\mathbb{C}^n$ by seeing how it is reflected in the algebra of the ring of polynomial functions $R = \mathbb{C}[x_1, \dots, x_n]$. Hilbert's Nullstellensatz says that points of $\mathbb{C}^n$ are in bijective correspondence with maximal ideals in R: the bijection sends $q = (q_1, \dots, q_n)$ to the maximal ideal $m_q = (x_1 - q_1, \dots, x_n - q_n)$. With a little work one can generalize this bijection. If $S \subseteq \mathbb{C}^n$ is any subset, define $\mathcal{I}(S) = \{f \in R \,|\, f(x) = 0 \text{ for all } x \in S\}$. This is an ideal in R, in fact a radical ideal (meaning that if $f^n \in \mathcal{I}(S)$ then $f \in \mathcal{I}(S)$). In the other direction, if $I \subseteq R$ is any ideal then define $V(I) = \{x \in \mathbb{C}^n \,|\, f(x) = 0 \text{ for all } f \in I\}$. Notice that $V(m_q) = \{q\}$ and $\mathcal{I}(\{q\}) = m_q$.

An **algebraic set** in $\mathbb{C}^n$ is any subset of the form $V(I)$ for some ideal $I \subseteq R$. The algebraic sets form the closed sets for a topology on $\mathbb{C}^n$, called the **Zariski topology**. One form of the Nullstellensatz says that V and $\mathcal{I}$ give a bijection between algebraic sets and radical ideals in R. Under this bijection the prime ideals correspond to **irreducible** algebraic sets—ones that cannot be written as $X \cup Y$ where both X and Y are proper closed subsets. Algebraic sets are also called algebraic **subvarieties**.

The above discussion is summarized in the following table:

Geometry	**Algebra**
$\mathbb{C}^n$ or $\mathbb{A}^n_{\mathbb{C}}$	$\mathbb{C}[x_1, \dots, x_n] = R$
Points $(q_1, \dots, q_n)$	Maximal ideals $(x_1 - q_1, \dots, x_n - q_n)$
Algebraic sets	Radical ideals
Irreducible algebraic sets	Prime ideals

The ring R is best thought of as the set of maps of varieties $\mathbb{A}^n \to \mathbb{A}^1$, with ring operations given by pointwise addition and multiplication. If we restrict to

some irreducible subvariety $X = V(P) \subseteq \mathbb{A}^n$ instead, then the ring of functions $X \to \mathbb{A}^1$ is R/P. This ring of functions is commonly called the **coordinate ring** of X. Much of the dictionary between $\mathbb{A}^n$ and R discussed above adapts verbatim to give a dictionary between X and its coordinate ring:

Geometry	**Algebra**
$X = V(P)$	$\mathbb{C}[x_1, \ldots, x_n]/P = R/P$
Points in X	Maximal ideals in R/P
Algebraic subsets $V(I) \subseteq X$	Radical ideals in R/P
Irreducible algebraic sets $V(Q) \subseteq X$	Prime ideals in R/P.

Note that ideals in R/P correspond bijectively to ideals in R containing P, and likewise for prime (respectively, radical) ideals.

We need one last observation. Passing from $\mathbb{A}^n$ to $\mathbb{A}^{n+1}$ corresponds algebraically to passing from R to $R[t]$. If $X = V(P) \subseteq \mathbb{A}^n$ is an irreducible algebraic set, then $X \times \mathbb{A}^1 \subseteq \mathbb{A}^{n+1}$ is $V(P[t])$ where $P[t] \subseteq R[t]$. That is, the coordinate ring of X is R/P and the coordinate ring of $X \times \mathbb{A}^1$ is $R[t]/P[t] = (R/P)[t]$. We supplement our earlier tables with the following line:

Geometry	**Algebra**
$X \rightsquigarrow X \times \mathbb{A}^1$	$S \rightsquigarrow S[t]$

So far our story involves pairing rings that are finitely-generated over $\mathbb{C}$ with their corresponding geometric objects. With a leap of faith one can extend this to rings that are finitely-generated over an algebraically closed field, and even to finitely-generated rings over any field. One of the great developments in 20th century algebraic geometry is to go all in and extend the theory even further, to *all* commutative rings. To any commutative ring R we attach the geometric object $\operatorname{Spec} R$. As a set, this is the set of prime ideals in R. We equip it with the Zariski topology, where the closed sets are the ones of the form $V(I) = \{P \in \operatorname{Spec}(R) \mid P \supseteq I\}$. As a topological space this is a very primitive object with only bare-bones geometric information. To find more geometry we have to look to the maps between these gadgets.

The object $\operatorname{Spec} \mathbb{Z}[t]$ will be called the **affine line** and denoted $\mathbb{A}^1_{\mathbb{Z}}$. A basic fact to remember is that everything is going to be set up so that $\operatorname{Hom}(\operatorname{Spec} R, \mathbb{A}^1_{\mathbb{Z}}) = R$. That is, R is the ring of functions from $\operatorname{Spec} R$ to the affine line. A map $\operatorname{Spec} R \to \operatorname{Spec} S$ will then give rise to a map of rings $\operatorname{Hom}(\operatorname{Spec} S, \mathbb{A}^1_{\mathbb{Z}}) \to \operatorname{Hom}(\operatorname{Spec} R, \mathbb{A}^1_{\mathbb{Z}})$, i.e. a map of rings $S \to R$. In fact, let us just define the category of affine schemes $\mathcal{A}ff$ to have objects the $\operatorname{Spec} R$ and where maps from $\operatorname{Spec} R$ to $\operatorname{Spec} S$ are the same as ring maps from S to R. That is, $\mathcal{A}ff = (\mathcal{C}omm\mathcal{R}ing)^{op}$.

Often there is a ground field k around, and we want all of our rings to be k-algebras. If $k \to R$ is the structure map then we have $\operatorname{Spec} R \to \operatorname{Spec} k$, and one describes $\operatorname{Spec} R$ as an "affine k-scheme", or a "scheme over $\operatorname{Spec} k$". The category of affine k-schemes is $\mathcal{A}ff_k = (k-\mathrm{alg})^{op}$. The affine line over $\operatorname{Spec} k$ is $\mathbb{A}^1_k = \operatorname{Spec} k[t]$.

One final piece of terminology for now. The phrase "affine scheme" just means one of the objects $\operatorname{Spec} R$. The phrase "affine variety" technically means a $\operatorname{Spec} R$ where R is an integral domain that is finitely-generated over a ground field k. For some reason the word "variety" is the more appealing of the two words, and as a result one sometimes ends up saying "variety" when what one really means is

"scheme". In almost all cases the true intent is clear from context, so we won't worry too much about this distinction.

We have defined $G(-)$ and $K(-)$ as functors taking rings as their inputs, but we could also think of them as taking affine schemes as their inputs. We will write $G(R)$ and $G(\operatorname{Spec} R)$ interchangeably, and similarly for the K-groups. It turns out that the geometric perspective and notation are very useful—many properties of these functors take on a familiar "homological" form when written geometrically. For example, a map $u\colon \operatorname{Spec} R \to \operatorname{Spec} S$ yields a map $u^*\colon K(\operatorname{Spec} S) \to K(\operatorname{Spec} R)$ as one would expect for a cohomology ring.

For the moment we will mostly keep with the algebraic notation, writing $K(R)$ more often than $K(\operatorname{Spec} R)$. But it is good to train oneself to "see" $K(\operatorname{Spec} R)$ even when it is not explicitly written that way.

2.22. **Regular local rings.** We have seen that regularity is an important condition when dealing with K-theory, and so it is worth giving a crash course on the theory of regular local rings. We start with some examples. All of the following local rings are regular:

(1) $k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ where k is a field;
(2) $k[[x_1, \dots, x_n]]$, the ring of formal power series over a field k;
(3) any discrete valuation ring V (equivalently, a local PID), for example $\mathbb{Z}_{(p)}$ or its p-adic completion $\mathbb{Z}_p^\wedge$ or any local ring of the ring of integers in a number field;
(4) $V[[x_1, \dots, x_n]]$ where V is any discrete valuation ring;
(5) $\mathbb{Z}_p^\wedge[[x_1, \dots, x_n]]/(x_1^2 + x_2^2 + \cdots + x_n^2 - p)$.

We next give some general information that will help sort the above examples into classes. Let (R, m, k) be a Noetherian local ring. This means that R is a Noetherian ring with unique maximal ideal m and $k = R/m$. If $\operatorname{char}(R) = \operatorname{char}(k)$ then R is said to be **equicharacteristic**, and otherwise it is **mixed characteristic**. Regular local rings are always integral domains and so the possibilities for the pair $(\operatorname{char}(R), \operatorname{char}(k))$ are only $(0,0)$, (p,p), and $(0,p)$ where p is a prime.

Note that $k[[x_1, \dots, x_n]]$ is equicharacteristic, whereas rings like $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p^\wedge[[x_1, \dots, x_n]]$ are mixed characteristic. The equicharacteristic case has the closer ties to geometry, whereas the mixed characteristic case appears more in number-theoretic situations.

Mixed characteristic local rings (R, m, k) can be further divided into two classes depending on whether or not $p \in m^2$. If $p \in m^2$ one says that R is **ramified**, and otherwise R is **unramified**. Note that $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p^\wedge$ are unramified, whereas the example from (5) above is ramified.

If one takes an algebraic variety over a field k and looks at the local ring at a nonsingular point, one obtains a regular local ring that is most likely not in the list (1)–(5) above. It is far from true, for example, that the local ring of a dimension n nonsingular variety is isomorphic to $k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$. However, it turns out that this is true after completion. The upshot is that there are not very many *complete* regular local rings, and in some sense we know them all. This is part of the Cohen structure theorems for complete local rings. The following theorem brings together several results, but (d) and (e) are the Cohen classification theorems:

Theorem 2.23. *Let (R, m, k) be a local ring and set $n = \dim R$.*

(a) If R is regular then it is a domain.
(b) R is regular if and only if the m-adic completion $\hat{R}$ is regular.
(c) If R is regular and $f \in m - m^2$ then R/f is also regular.
(d) If R is complete and regular and equicharacteristic then $R \cong k[[x_1, \ldots, x_n]]$ for some field k.
(e) Suppose R is complete and regular and mixed characteristic. Then there are two possibilities:
- *If R is unramified ($p \notin m^2$) then $R \cong V[[x_1, \ldots, x_{n-1}]]$ for some complete discrete valuation ring V with maximal ideal (p).*
- *If R is ramified ($p \in m^2$) then $R \cong V[[x_1, \ldots, x_n]]/(p - f)$ for some complete discrete valuation ring V with maximal ideal (p) and some $f \in (p, x_1, \ldots, x_n)^2$ such that $p \nmid f$.*

For parts (a)–(c) see [BH, Propositions 2.2.2–2.2.4]. Parts (d) and (e) are from [CohenI], with [Samuel] another good source. The equicharacteristic case in (d) can also be found in [ZS, Section 12 of Chapter VIII].

Theorem 2.23 gives us a very good handle on complete regular local rings in the equicharacteristic and the unramified mixed characteristic cases. But although the theorem says something concrete and useful about the mixed ramified case, things are much more mysterious here because of how open the choice of f is. Many theorems about regular local rings are known in the equicharacteristic and mixed unramified cases but open in the ramified case.

∘ Exercises ∘

Exercise 2.24. Let R and S be rings. Let $e = (1, 0)$ and $f = (0, 1)$, and recall that every $(R \times S)$-module M decomposes as $M = eM \oplus fM$. Use this to prove that $G(R \times S) \cong G(R) \oplus G(S)$.

Exercise 2.25. Let R be a ring.

(a) Let $Q \subseteq R$ be a prime ideal. Prove that
$$P \mapsto \dim_{QF(R/Q)}(QF(R/Q) \otimes_{R/Q} P/QP)$$
is an additive function on finitely-generated projectives and therefore induces a homomorphism $\operatorname{rank}_Q \colon K(R) \to \mathbb{Z}$.

(b) Let $Q \subseteq R$ be a minimal prime ideal. Then R_Q is Artinian, so every finitely-generated module has finite length. Prove that $M \mapsto \ell_{R_Q}(M_Q)$ is additive and so induces a homomorphism $\ell_Q \colon G(R) \to \mathbb{Z}$.

(c) Prove that in both $K(R)$ and $G(R)$ the class $[R]$ is non-torsion.

Exercise 2.26. Suppose R has the property that every finitely-generated module has a finite resolution by finitely-generated free modules. Prove that $K(R) \cong \mathbb{Z}$ and $G(R) \cong \mathbb{Z}$. Give an example of a Noetherian ring R for which this property fails. Even better, give an example of a regular ring R for which this property fails.

Exercise 2.27. Let $R = k[x]/(x^n)$ where k is a field and $n \geq 2$. Use the classification of modules over $k[x]$ to prove that every finitely-generated projective R-module is free. Verify that $K(R) \cong G(R) \cong \mathbb{Z}$ but that the canonical map $K(R) \to G(R)$ is isomorphic to multiplication-by-n (and in particular, is not an isomorphism).

3. Further properties of $G(R)$

In this section we continue our study of the groups $G(R)$. When R is Noetherian these groups are closely related to the ideal theory of R, and approaching them through that lens turns out to be quite useful.

The first theorem we discuss gives a convenient set of generators for $G(R)$:

Theorem 3.1. *If R is Noetherian then the Grothendieck group $G(R)$ is generated by the set of elements $[R/P]$ where $P \subseteq R$ is prime.*

Before proving this result let us comment on the significance. When X is a topological space, the groups $H_*(X)$ have a geometric presentation in terms of "cycles" and "homologies". The cycles are, of course, generators for the group. The definition of $G(R)$ doesn't look anything like this, but Theorem 3.1 says that the group is indeed generated by classes that have the feeling of "algebraic cycles" on the variety $\operatorname{Spec} R$. One thinks of $G(R)$ as having a generator $[R/P]$ for every irreducible subvariety of R, and then there are some relations amongst these that we don't yet understand. It is worth pointing out that in $H_*(X)$ the cycles are strictly separated by dimension—the dimension i cycles are confined to the single group $H_i(X)$—whereas in $G(R)$ the cycles of different dimensions are all inhabiting the same group. This is one of the main differences between K-theory and singular homology/cohomology.

To prove Theorem 3.1 we first need a lemma from commutative algebra:

Lemma 3.2. *Let R be a Noetherian ring. For any nonzero finitely-generated R-module M there exists a prime ideal $P \subseteq R$ and an embedding $R/P \hookrightarrow M$. Equivalently, there is some $z \in M$ whose annihilator is prime.*

Proof. Consider the set of ideals

$$\mathcal{S} = \{\operatorname{Ann}(m) \,\big|\, m \in M, m \neq 0\}.$$

Equivalently, $\mathcal{S}$ is the set of ideals I such that R/I embeds into M. Note that $S \neq \emptyset$ because $M \neq 0$. Since R is Noetherian $\mathcal{S}$ has a maximal element $I = \operatorname{Ann}(m)$. We will prove that I is prime. Suppose $ab \in I$ and $b \notin I$. Then $bm \neq 0$ so $\operatorname{Ann}(bm) \in \mathcal{S}$ and $\operatorname{Ann}(bm) \supseteq \operatorname{Ann}(m) = I$. By maximality $\operatorname{Ann}(bm) = I$. But $ab \in I$ so $abm = 0$, hence $a \in \operatorname{Ann}(bm)$ and therefore $a \in I$. This completes the proof that I is prime. □

Proof of Theorem 3.1. Let M be a finitely-generated R-module. We will use repeated applications of Lemma 3.2 to construct a so-called **prime filtration** of M. Pick an embedding $R/P_0 \hookrightarrow M$, and let $M_0 = R/P_0$. Next consider M/M_0. If $M/M_0 = 0$, our filtration is complete. If $M/M_0 \neq 0$, then there exists a prime P_1 and an embedding $R/P_1 \hookrightarrow M/M_0$. Let $\pi\colon M \to M/M_0$ denote the projection and define $M_1 = \pi^{-1}(R/P_1)$. Then $\pi\colon M_1 \to R/P_1$ also has kernel M_0; that is, $M_0 \subseteq M_1$ and $M_1/M_0 \cong R/P_1$. Next consider M/M_1 and repeat. This process yields a filtration

$$0 = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

such that $M_i/M_{i-1} \cong R/P_i$. The filtration must be finite since R is Noetherian. By Proposition 2.5(b) we have that $[M] = \sum [M_i/M_{i-1}] = \sum [R/P_i]$, and so the set $\{[R/P] \,|\, P \text{ is prime in } R\}$ generates $G(R)$. □

Remark 3.3. The prime filtrations constructed in the above proof are very useful, and will appear again and again in our arguments. By the "primes appearing in the filtration" we will mean the multiset of primes $\{P_i\}$ such that $\{R/P_i\}$ is the multiset of the filtration quotients. For future use we note that $\operatorname{Ann} M$ kills any subquotient of M, and therefore will be contained in all the primes associated to any prime filtration of M.

3.4. **Finite length modules.** This is a good place to recall the Jordan-Hölder theory of finite-length modules. Recall that a *composition series* for a module is a filtration where the quotients are simple modules. A module is said to have finite length if it has a finite composition series, and the Jordan-Hölder theorem says that any two composition series will have the same length and the same quotients (up to a permutation). The length of M, denoted $\ell(M)$, is the length of any composition series.

Over a commutative ring R, the simple modules are precisely the R/m where m is a maximal ideal. So a module M has finite length if and only if it has a prime filtration where all the primes appearing are maximal.

Exercise 3.5. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. Show that M has finite length if and only if both M' and M'' do. Consequently, if M has a finite filtration where the quotients have finite length, then M also has finite length.

Proposition 3.6. *For a nonzero, finitely-generated module M consider the following statements:*

(1) $\ell(M) < \infty$,
(2) $\operatorname{Ann} M$ *contains a finite product of maximal ideals,*
(3) All minimal primes of $\operatorname{Ann} M$ *are maximal.*

Then (1) $\iff$ *(2)* $\Longrightarrow$ *(3), and if R is Noetherian then (2)* $\iff$ *(3).*

Exercise 3.7. Prove Proposition 3.6.

Now suppose that R is Noetherian and M is a finitely-generated R-module. Let $0 \subseteq F_0 \subseteq \cdots \subseteq F_d = M$ be a prime filtration of M. Then $\operatorname{Ann} M$ is contained in each of the primes appearing in the quotients. Let P be a minimal prime of $\operatorname{Ann} M$ and consider the filtration $(F_\bullet)_P$ of M_P. The new filtration quotients are just the localizations at P of the old fitration quotients. So every filtration quotient that started out as R/P becomes R_P/PR_P, and every filtration quotient that started out as R/Q for $Q \neq P$ becomes zero (since $Q \not\subseteq P$, as $Q \supseteq \operatorname{Ann} M$ and P was minimal over $\operatorname{Ann} M$). So we have a prime filtration of M_P in which only R_P/PR_P appears. Therefore M_P has finite length over R_P, and the number of appearances of R_P/PR_P is exactly $\ell_{R_P}(M_P)$.

We have therefore proven the following:

Proposition 3.8. *Let M be a finitely-generated module over a Noetherian ring R. Let $F_\bullet$ be any prime filtration for M. Then for every minimal prime P over* $\operatorname{Ann} M$*, R/P appears exactly $\ell_{R_P}(M_P)$ times in the filtration quotients of $F_\bullet$.*

Of course non-minimal primes over $\operatorname{Ann} M$ might also appear in the filtration quotients of a prime filtration of M, and the number of such might be different for different filtrations. For example, take $R = \mathbb{Z}$ and $M = R$. Then $0 \subseteq \mathbb{Z}$ is a prime filtration for M, but so are $0 \subseteq p\mathbb{Z} \subseteq \mathbb{Z}$ and $0 \subseteq p^2\mathbb{Z} \subseteq p\mathbb{Z} \subseteq \mathbb{Z}$ for any prime p.

For one more structural result about prime filtrations, see Exercise 3.33.

3.9. **Eagon's presentation for $G(R)$.** Let R be a Noetherian ring. Every finitely-generated R-module M has a prime filtration, and this shows that the classes $[R/P]$ for P a prime generate $G(R)$. So if we let $\mathcal{P}(R)$ be the free abelian group with generators $[R/P]$ for P a prime, we can write $G(R)$ as a quotient of $\mathcal{P}(R)$. Can we describe the relations for this?

If $F_\bullet$ is a prime filtration of a finitely-generated module M, let us write $[F_\bullet] = \sum_i [F_i/F_{i-1}] \in \mathcal{P}(R)$ and $[F_\bullet]_G = \sum_i [F_i/F_{i-1}] \in G(R)$. If $F'_\bullet$ is another prime filtration of M then we have $[F_\bullet]_G = [M] = [F'_\bullet]_G$, so in order to construct $G(R)$ from $\mathcal{P}(R)$ we will need to equate $[F_\bullet]$ and $[F'_\bullet]$. The following result is a triviality:

Proposition 3.10. *Let $S_0 \subseteq \mathcal{P}(R)$ be the subgroup generated by elements $[F_\bullet]-[F'_\bullet]$ for every finitely-generated module M and every two prime filtrations $F_\bullet$ and $F'_\bullet$ of M. Then the canonical map $\mathcal{P}(R)/S_0 \to G(R)$ is an isomorphism.*

Proof. Let $A = \mathcal{P}(R)/S_0$. For M a finitely-generated module, define $\alpha(M) = [F_\bullet] \in A$ for any choice of prime filtration of M. This is well-defined due to the definition of S_0.

Let $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ be a short exact sequence. Choose a prime filtration $F_\bullet$ of M' and a prime filtration $G_\bullet$ of M''. Splicing the filtrations $F_\bullet$ and $p^{-1}G_\bullet$ together gives a prime filtration of M, where the filtration quotients are precisely the concatentation of the ones appearing in $F_\bullet$ and $G_\bullet$. So $\alpha(M) = \alpha(M') + \alpha(M'')$ in A, and hence α gives a group map $G(R) \to A$. This is readily seen to be a two-sided inverse to our projection $A \to G(R)$. □

The above result is not so useful due to the vague description of S_0: we would have to understand all pairs of prime filtrations on *every* module. Eagon obtained better descriptions of S_0 by narrowing the source of the relations [Eagon]. His method starts from the following observation: if P is a prime in R and $x \in R - P$, then there is an exact sequence $0 \longrightarrow R/P \xrightarrow{x} R/P \longrightarrow R/(P,x) \longrightarrow 0$. This shows that $[R/(P,x)] = 0$ in $G(R)$, and therefore if $F_\bullet$ is any prime filtration of $R/(P,x)$ then we must impose the relation $[F_\bullet] = 0$. Define

$$S_{Eagon} = \{[F_\bullet] \,|\, F_\bullet \text{ is a prime filtration of } R/(P,x) \text{ for some prime } P \text{ and some } x \in R - P\} \subseteq \mathcal{P}(R).$$

This is still a large number of relations: one for every choice of P, x, and $F_\bullet$. Still, from a conceptual viewpoint this is more manageable than what we had before.

Theorem 3.11 (Eagon)**.** *The canonical map $\mathcal{P}(R)/S_{Eagon} \to G(R)$ is an isomorphism.*

The main element of the proof is to show that our previous $\alpha(M)$ is still well-defined as an element of $\mathcal{P}(R)/S_{Eagon}$. But unlike before, this is not a triviality and will require a bit of work.

From now on all filtrations are assumed to be finite and all modules are assumed to be finitely-generated.

Let $F_\bullet$ and $G_\bullet$ be two filtrations of M. Say that $G_\bullet$ is a **refinement** of $F_\bullet$, written $G_\bullet \geq F_\bullet$, if every F_j appears somewhere in $G_\bullet$. Loosely speaking, this means that $G_\bullet$ consists of $F_\bullet$ plus extra terms. A basic result in the theory of modules, often called Schreier's Refinement Theorem, says that if $F_\bullet$ and $F'_\bullet$ are two filtrations of a module M then there are refinements $F_\bullet \leq G_\bullet$ and $F'_\bullet \leq G'_\bullet$

such that $G_\bullet$ and $G'_\bullet$ have the same length and the same quotients, the latter up to permutation. In this case we say that $G_\bullet$ and $G'_\bullet$ are **equivalent** filtrations.

Remark 3.12. We will not prove Schreier's Refinement Theorem, as this is something that can be found in basic algebra textbooks (usually in relation to the Jordan-Hölder theorem). But let us at least recall the main idea for why it is true. Suppose $M \supseteq A \supseteq 0$ and $M \supseteq B \supseteq 0$ are two filtrations for M. Consider the refinement of the first given by

$$M \supseteq A + B \supseteq A \supseteq A \cap B \supseteq 0,$$

having quotients $M/(A+B)$, $(A+B)/A \cong B/(A \cap B)$, $A/(A \cap B) \cong (A+B)/B$, and $A \cap B$. Interchanging the roles of A and B gives a similar filtration refining $M \supseteq B \supseteq 0$, having the same set of filtration quotients.

Once one has the above basic idea, it is not hard to extend to longer filtrations.

Lemma 3.13. *Every filtration $F_\bullet$ of a module M can be refined to a prime filtration. If $F_\bullet$ and $F'_\bullet$ are two prime filtrations of a module M, then there are equivalent prime filtrations $G_\bullet$ and $G'_\bullet$ of M that refine $F_\bullet$ and $F'_\bullet$, respectively.*

Proof. For the first statement, the point is to choose a prime filtration for a quotient F_{i+1}/F_i and then intermix this with $F_\bullet$ in the evident way. By doing this for every i, one obtains a prime filtration that refines $F_\bullet$.

For the second statement, Schreier's Theorem says that there are refinements $F_\bullet \leq J_\bullet$ and $F'_\bullet \leq J'_\bullet$ such that $J_\bullet$ and $J'_\bullet$ are equivalent. Fix a choice of pairings of isomorphic filtration quotients of $J_\bullet$ and $J'_\bullet$. Refine $J_\bullet$ to a prime filtration as in the last paragraph. For $J'_\bullet$ do the same, but be careful that for every filtration quotient J'_{i+1}/J'_i one uses the *same* prime filtration as for the paired quotient of $J_\bullet$. This guarantees that the resulting refinements of $J_\bullet$ and $J'_\bullet$ are still equivalent. □

Lemma 3.14. *Let $F_\bullet$ and $F'_\bullet$ be two prime filtrations for a module M. Then $[F_\bullet] - [F'_\bullet] \in S_{Eagon}$.*

Proof. Let $G_\bullet$ be a refinement of $F_\bullet$. Consider one step $F_i \subseteq F_{i+1}$ and let the quotient be R/Q. We know $F_i = G_j$ for some j, and $F_{i+1} = G_k$ for some $j < k$. Then $G_{j+1} \subseteq F_{i+1}$ and so $G_{j+1}/G_j \hookrightarrow F_{i+1}/F_i = R/Q$. The module G_{j+1}/G_j is cyclic and (we can assume) nonzero, so its image in R/Q is generated by some element $x \in R - Q$. Thus $G_{j+1}/G_j \cong (Q,x)/Q \cong R/Q$ (the latter since Q is prime).

The following diagram shows what we currently know:

$$\begin{array}{ccccccc} F_i & & \xrightarrow{\qquad\qquad R/Q \qquad\qquad} & & & & F_{i+1} \\ \| & & & & & & \| \\ G_j & \xrightarrow[R/Q]{} & G_{j+1} & \longrightarrow & \cdots & \longrightarrow & G_k. \end{array}$$

We have $F_{i+1}/G_{j+1} \cong (F_{i+1}/F_i)/(G_{j+1}/G_j) \cong R/(Q,x)$. The filtration $G_k \supseteq G_{k-1} \supseteq \cdots \supseteq G_{j+1}$ induces a prime filtration $J_\bullet$ of F_{i+1}/G_{j+1}, and we see that the $R/Q = F_{i+1}/F_i$ term that appears in $[F_\bullet]$ is replaced by $[R/Q] + [J_\bullet]$ inside of $[G_\bullet]$. Of course $[J_\bullet] \in S_{Eagon}$, by construction.

Repeating this same argument for each link in the F-filtration shows that $[G_\bullet] - [F_\bullet]$ is in S_{Eagon}.

Now return to $F_\bullet$ and $F'_\bullet$. By Lemma 3.13 there are equivalent prime filtrations $G_\bullet$ and $G'_\bullet$ that refine $F_\bullet$ and $F'_\bullet$. By what has already been proven, $[F_\bullet] - [G_\bullet] \in$

S_{Eagon} and $[F'_\bullet]-[G'_\bullet] \in S_{Eagon}$. But $[G_\bullet] = [G'_\bullet]$ since the filtrations are equivalent. Therefore $[F_\bullet] - [F'_\bullet] \in S_{Eagon}$. □

Remark 3.15. In the above proof, observe that every $J_\bullet$ that appears is a prime filtration of $R/(Q,x)$ where R/Q is a subquotient of M and hence contains $\operatorname{Ann} M$. This will be used below to gain additional control over the relations in S_{Eagon}.

Proof of Theorem 3.11. This is now easy. Repeat the argument of Proposition 3.10 to define $\alpha(M) \in \mathcal{P}(R)/S_{Eagon}$, with Lemma 3.14 showing that this is independent of the choice of prime filtration. Everything else is the same. □

Finally, there is one additional step we can take to narrow the defining relations in S. For each prime P and each $x \in R - P$, choose once and for all one prime filtration $F_\bullet^{(P,x)}$ of $R/(P,x)$. Define

$$S'_{Eagon} = \{[F_\bullet^{(P,x)}] \,\big|\, P \subseteq R \text{ prime and } x \in R - P\}.$$

So now we have exactly one relation for every choice of P and x.

Corollary 3.16 (Eagon)**.** *The canonical map $\mathcal{P}(R)/S'_{Eagon} \to G(R)$ is an isomorphism.*

Proof. By definition $S'_{Eagon} \subseteq S_{Eagon}$, and the goal is to prove they are equal. Let T be the set of primes P for which there exists an $x \in R-P$ and a prime filtration $F'_\bullet$ of $R/(P,x)$ such that $[F'_\bullet] \notin S'_{Eagon}$. Since R is Noetherian, if T is nonempty then it has a maximal element P. Choose a corresponding x and $F'_\bullet$ as in the defining conditions.

We have the two prime filtrations $F_\bullet^{(P,x)}$ and $F'_\bullet$ of $R/(P,x)$. We know they can be refined by equivalent prime filtrations $G_\bullet$ and $G'_\bullet$ (respectively). The argument from Lemma 3.14 shows that $[F_\bullet^{(P,x)}]-[G_\bullet]$ equals a sum of terms $[J_\bullet]$ such that $J_\bullet$ is a prime filtration of a module $R/(Q,y)$ where Q is prime, $y \notin Q$, and moreover Q contains $\operatorname{Ann}[R/(P,x)] = (P,x)$ (see Remark 3.15). By maximality of P it follows that each such $J_\bullet$ is in S', and so $[F_\bullet^{(P,x)}] - [G_\bullet] \in S'$.

The same argument shows $[F'_\bullet] - [G'_\bullet] \in S'$, and of course $[G_\bullet] = [G'_\bullet]$ since the two filtrations are equivalent. Therefore $[F'_\bullet] - [F_\bullet^{(P,x)}] \in S'$, but this implies $[F'_\bullet] \in S'$ and that is a contradiction. So $T = \emptyset$ and therefore $S = S'$. □

The proofs of Theorem 3.11 and Corollary 3.16 are from [Eagon].

Exercise 3.17. As an application of Eagon's presentation, prove that if R is a Noetherian ring of dimension 0 then $G(R) \cong \mathbb{Z}^k$ where k is the number of prime ideals of R (this is also the number of connected components of $\operatorname{Spec} R$).

Eagon's presentation is also very useful for one-dimensional rings. Here is the first result in this direction:

Proposition 3.18. *Let (A,m) be a one-dimensional local domain. Then $G(A) \cong \mathbb{Z} \oplus \mathbb{Z}/d$ where the generators of the two summands are $[A]$ and $[A/m]$ and $d = \gcd\{\ell(A/f) \,|\, f \in m - 0\}$.*

Proof. The ring A has only two prime ideals, m and 0. So $[A/m]$ and $[A]$ generate $G(R)$. The map $A \to QF(A)$ induces $G(A) \to G(QF(A)) = \mathbb{Z}$ and this sends $[A]$ to 1 and $[A/m]$ to 0. It follows immediately that $G(A) \cong \mathbb{Z} \oplus \langle [A/m] \rangle$.

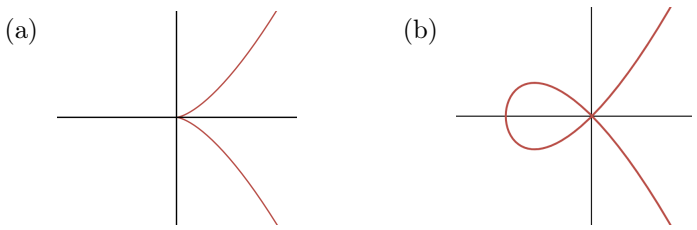
Now we use Eagon's theorem, which says that $G(A)$ is the quotient of $\mathbb{Z}\langle [A], [A/m]\rangle$ by the relations $[F_\bullet] = 0$ for $F_\bullet$ a prime filtration of A/f and f any element in $A - 0$. But the filtration quotients of $F_\bullet$ are all annihilated by f, and so only A/m can appear (not $A/0$). Therefore $F_\bullet$ is a composition series for A/f, and so the number of terms is precisely the length $\ell(A/f)$. Thus, $[F_\bullet] = \ell(A/f)[A/m]$.

The formula for the order of $[A/m]$ in $G(A)$ follows at once. □

Exercise 3.19. Prove that $G(R) \cong \mathbb{Z}$ for each of the following one-dimensional local domains:

(a) $R = \mathbb{C}[x,y]_{(x,y)}/(y^2 - x^3)$ (this is the local ring of a cuspidal cubic at the singular point).

(b) $R = \mathbb{C}[x,y]_{(x,y)}/(y^2 - x^2 - x^3)$ (this is the local ring of a nodal cubic at the singular point).



The above results suggest an interesting variation on $G(R)$. What if, in the relations coming from a prime filtration of M, we only took the pieces coming from minimal primes of $\operatorname{Ann} M$? These are the pieces whose presence and multiplicity is the same in all prime filtrations, and they feel more manageable than the general version. To this end, we proceed as follows.

Let R be a d-dimensional Noetherian domain. Define the **codimension-one Chow homology group**[1] $\mathrm{CH}_{d-1}(R)$ to be the quotient of the free abelian group $\mathbb{Z}\langle [R/P] \,\big|\, P \text{ is a height one prime}\rangle$ by the relations

$$0 = \sum_{P \text{ minimal over } (f)} \ell_{R_P}(R_P/fR_P)[R/P]$$

for every $f \in R - 0$. The sum is over the minimal primes of (f), of which there are only finitely many. Krull's theorem says that such primes have height at most one, and since R is a domain they will have height exactly one.

If $F_\bullet$ is a prime filtration for R/f and P is a minimal prime of (f), then we know by Proposition 3.8 that R/P appears in the quotients exactly $\ell_{R_P}(R_P/fR_P)$ times. Since $0 = [R/f]$ in $G(R)$ (from the exact sequence $0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/f \longrightarrow 0$), we get a relation in $G(R)$ of the form

$$0 = [R/f] = \sum_{P \text{ minimal over } (f)} \ell_{R_P}(R_P/fR_P)[R/P] + (\text{other terms})$$

[1]Be warned that some authors use slightly different definitions. For example, one could replace the $\operatorname{ht}(P) = 1$ criterion with $\dim R/P = 1$, which leads to the same group in most but not all cases. For affine schemes over a field our definition agrees with the one in [Stacks, Chapter III.42].

where the other terms all involve $[R/Q]$ for $Q \supseteq (f)$ but not minimal over (f). In particular, such Q would all have $\operatorname{ht} Q \geq 2$.

Proposition 3.20. *Let R be a d-dimensional Noetherian domain. There is a map $c_1 \colon G(R) \to \mathrm{CH}_{d-1}(R)$ that sends $[R/P]$ to $[R/P]$ when $\operatorname{ht}(P) = 1$, and to 0 for $\operatorname{ht}(P) \neq 1$. This map is called the* ***homological first Chern class****.*

Proof. This follows from Eagon's presentation of $G(R)$. Suppose P is a prime in R, $f \in R - P$, and $F_\bullet$ is a prime filtration for $R/(P, f)$. Consider the defining relation $0 = \sum [F_{i+1}/F_i]$ in Eagon's presentation. If $P = 0$ then the image under c_1 is a defining relation for $\mathrm{CH}_{d-1}(R)$, by our discussion above. If $P \neq 0$ then $\operatorname{ht} P \geq 1$ and so all primes containing (P, f) have height at least two. Therefore the Eagon relation $0 = \sum [F_{i+1}/F_i]$ is mapped to $0 = 0$ under c_1.

Since all of the Eagon relations are preserved by our c_1, it defines a map $G(R) \to \mathrm{CH}_{d-1}(R)$ as claimed. □

Remark 3.21. In algebraic geometry, formal sums of codimension-one subvarieties are called *(Weil) divisors*, and $\mathrm{CH}_{d-1}(R)$ is called the *(Weil) divisor class group* or just the *class group*. It is often denoted $\mathrm{Cl}(R)$, particularly in algebraic number theory where R is a Dedekind domain.
[Aside: There is a related notion of *Cartier divisors*, and a corresponding Cartier divisor class group. Those are related to cocycles and cohomology, whereas the Weil versions are related to cycles and homology. For regular rings one gets the expected isomorphisms.]

We have mentioned this before, but it is worth repeating that in $G(R)$ the generators of different dimensions all interact at once. So for example, we end up with relations involving generators of different dimensions. In the Chow group $\mathrm{CH}_{d-1}(R)$, though, we are in a situation where there are *only* the codimension-one pieces. This turns out to be a big difference, and in some ways makes for a more simpler theory.

It is also possible to define Chow homology groups $\mathrm{CH}_{d-i}(R)$ for any i, but we will not pursue this here. The connections between those groups and $G(R)$ is more difficult to explain. See Exercise 3.35 for a taste.

For one-dimensional Noetherian domains R, in $G(R)$ we have the single height 0 prime that does not interact with anything else and so just gives a $\mathbb{Z}$ summand, and then we have all of the height one primes that interact amongst each other. But that's all there is, and so the latter is just $\mathrm{CH}_0(R)$:

Proposition 3.22. *For a one-dimensional Noetherian domain R we have a split-exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow G(R) \xrightarrow{c_1} \mathrm{CH}_0(R) \longrightarrow 0$$

where the left map sends 1 to $[R]$.

Proof. The Eagon presentation gives $G(R) = \mathbb{Z} \oplus \mathrm{CH}_0(R)$ where the first summand is generated by $[R]$ and the projection $G(R) \to \mathrm{CH}_0(R)$ is precisely c_1. □

3.23. **An extended example.** Let $R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[X]/(X^2 + 3)$. Our aim in this section is to compute $G(R)$. This necessarily involves a certain amount of number theory, since that influences the ideal theory of R. Rather than develop that material in depth, we leave many of the steps as extended exercises for the reader.

For every prime ideal Q in R, the intersection $Q \cap \mathbb{Z}$ will be a prime p in $\mathbb{Z}$. This is just the map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$. One says that "$Q$ lies over p".

Exercise 3.24.

(a) Prove that $(2, 1 + X)$ is the unique prime lying over 2. Find primes lying over 3 and 5 and prove they are unique.
(b) Let $p > 3$ be an integral prime. Prove that p has a unique prime lying over it if and only if -3 is not a square modulo p, and otherwise p has exactly two primes lying over it.
(c) For $p > 3$ show that -3 is a square modulo p if and only if $3|p - 1$. [Hint: Use quadratic reciprocity.]

It will be important for us to know which prime ideals are principal. Let $a, b \in \mathbb{Z}$ and consider the multiplication-by-$(a + bX)$ map $R \to R$. As a map of abelian groups this is $\mathbb{Z}^2 \to \mathbb{Z}^2$, given by the matrix $\left[\begin{smallmatrix} a & -3b \\ b & a \end{smallmatrix}\right]$ (taking $1, X$ as our basis of R). The determinant is $a^2 + 3b^2$, and this equals the order of the cokernel (we can see this using Smith normal form, for example). So if prime Q is principal then the order of R/Q must be of the form $a^2 + 3b^2$.

Since $R/(2, 1 + X) = \mathbb{Z}/2$ and 2 cannot be written as $a^2 + 3b^2$, it follows that $(2, 1 + X)$ is not principal.

Exercise 3.25. Let $Q \subseteq R$ be a prime.

(a) If Q lies over a prime $p > 2$ such that $3 \nmid p - 1$, prove that Q is principal.
(b) Suppose $p > 3$ and $3|p - 1$. It is a standard (but nontrivial) fact that such primes can be written as $p = a^2 + 3b^2$ for some $a, b \in \mathbb{Z}$. Then $(\frac{a}{b})^2 = -3$ in $\mathbb{F}_p$, so the two primes lying over p are $(p, bX - a)$ and $(p, bX + a)$. Prove that each of these is principal.

Altogether this shows that all primes in R except $(2, 1 + X)$ are principal.

Next we turn to the module theory over R.

Exercise 3.26.

(a) Construct a free resolution of $R/(2, 1 + X)$ and prove that $\operatorname{Tor}_i(R/(2, 1 + X), R/(2, 1 + X))$ is nonzero for all i. Deduce that R is not regular. [Hint: The resolution can be made to exhibit a nice periodicity.]
(b) Explain why R_P is regular for every prime P except for $(2, 1 + X)$. Said differently, $(2, 1 + X)$ is the only singular point of $\operatorname{Spec} R$.
(c) Construct an exact sequence $0 \to R/(2, 1 + X) \to R/(2) \to R/(2, 1 + X) \to 0$.
(d) Using the previous parts deduce that $G(R)$ is generated by $[R]$ and $[R/(2, 1+X)]$ and is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, depending on whether $[R/(2, 1 + X)]$ is zero or nonzero.

Let $m = (2, 1+X)$ and consider the map $R \to R_m$. Since this is flat, we have the associated map $G(R) \to G(R_m)$. The ring R_m is a one-dimensional local domain, so we can use Proposition 3.18 to compute the order of $[R/m]$. If we find that $[R/m] \neq 0$ in $G(R_m)$ then this shows it is also nonzero in $G(R)$, and will complete our calculation.

Exercise 3.27. Let $A = R_m$. Let $d = \gcd\{\ell(A/f) \mid f \in A - 0\}$.

(a) Prove that if $f, g \in A - 0$ then there is a short exact sequence $0 \to A/f \to A/fg \to A/g \to 0$ and therefore $\ell(A/fg) = \ell(A/f) + \ell(A/g)$.

(b) Explain why in the definition of d we can replace "$f \in A-0$" with "$f = a+bX \in m-0$ with $a, b \in \mathbb{Z}$". Show that the latter condition implies $a \equiv b \pmod 2$.
(c) Show that $\ell(A/2) = 2$, and deduce that if $f = a + bX \in m - 0$ with $a, b \in 2\mathbb{Z}$ then $\ell(A/f)$ is even.
(d) Suppose $f = a + bX \in m - 0$ with a, b odd integers. Prove that $a^2 + 3b^2 \in fA$, a^2+3b^2 is an odd multiple of 4, and conclude $4 \in fA$. Let u be a multiplicative inverse for b modulo 4. Explain why

$$A/fA = \mathbb{Z}/4[X]/(X^2 - 1, ua + X) = (\mathbb{Z}/4)/(u^2a^2 - 1) = \mathbb{Z}/4.$$

Deduce that $\ell(A/f) = 2$ in this case.
(e) Conclude that $\ell(A/f)$ is even for all $f \in A - 0$, and therefore that $[A/m] \neq 0$ in $G(A)$. So $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

Remark 3.28. We should explain that the ring $\mathbb{Z}[\sqrt{-3}]$ is a particularly easy example. As we saw above, it is remarkably close to being a PID without actually being one. As soon as one moves to $\mathbb{Z}[\sqrt{-5}]$ one is in a situation where there are infinitely many primes that are not principal, and the calculation is trickier. The group $G(\mathbb{Z}[\sqrt{-5}])$ can still be computed with the same techniques, but it takes more work. We will return to this in Section 5 when we have additional tools under our belt that will help simplify matters.

3.29. **The homotopy invariance property.** If M is an R-module write $M[t]$ for the $R[t]$-module $M \otimes_R R[t]$. The functor $M \mapsto M[t]$ is exact because $R[t]$ is flat (in fact, free) over R. So we have an induced map $\alpha\colon G(R) \to G(R[t])$ given by $[M] \mapsto [M[t]]$.

Theorem 3.30 (Homotopy invariance)**.** *The map $\alpha\colon G(R) \to G(R[t])$ is an isomorphism when R is Noetherian.*

We comment on the name "homotopy invariance" for the above result. If $X = \operatorname{Spec} R$ then $\operatorname{Spec} R[t] = X \times \mathbb{A}^1$, so the result says that $G(-)$ gives the same values on X and $X \times \mathbb{A}^1$. This is reminiscent of a functor on topological spaces giving the same values on X and $X \times I$.

Before proving the result, let us help orient the reader a bit. If we think topologically, then we would take the maps $X \xrightarrow{i_0} X \times \mathbb{A}^1 \xrightarrow{\pi} X$ and get induced maps $G(X) \xrightarrow{(i_0)_*} G(X \times \mathbb{A}^1) \xrightarrow{\pi_*} G(X)$ whose composite is the identity. So we get for free that $(i_0)_*$ is injective, and the challenge is to prove surjectivity. The group $G(X \times \mathbb{A}^1)$ is generated by "cycles" on $X \times \mathbb{A}^1$, and we need to show that there are enough relations to move any cycle so that it lies in $X \times \{0\}$. This can be approached by an induction on the dimension. This is Grothendieck's argument, given in [BS, Proposition 8]. It is perhaps most understandable when written in the language of algebraic geometry, but below we give a purely algebraic version due to [Eagon].

Proof of Theorem 3.30. We will first construct a left inverse $\beta\colon G(R[t]) \to G(R)$. A naive possibility for the map β is $J \mapsto J/tJ = J \otimes_{R[t]} R[t]/(t)$, but this doesn't preserve short exact sequences in general. So we correct this using Tor, and instead define

$$\beta([J]) = [\operatorname{Tor}_0^{R[t]}(J, R[t]/(t))] - [\operatorname{Tor}_1^{R[t]}(J, R[t]/(t))].$$

Before checking that this is well-defined, let us analyze the two Tor-groups. Recall that we can calculate Tor by taking an $R[t]$-resolution of either variable. In this

case, it is easier to resolve $R[t]/(t)$:

$$0 \to R[t] \xrightarrow{t} R[t] \to R[t]/(t) \to 0.$$

Tensoring with J yields $0 \to J \xrightarrow{t} J \to 0$, so that $\mathrm{Tor}_0^{R[t]}(J, R[t]/(t)) = J/tJ$ and $\mathrm{Tor}_1^{R[t]}(J, R[t]/(t)) = \mathrm{Ann}_J(t)$. Notice also that $\mathrm{Tor}_i^{R[t]}(J, R[t]/(t)) = 0$ for $i > 1$. We have

$$\beta([J]) = [J/tJ] - [\mathrm{Ann}_J(t)] = \sum_{i=0}^{\infty} (-1)^i [\mathrm{Tor}_i^{R[t]}(J, R[t]/(t))].$$

The fact that β is a well-defined group homomorphism now follows by the usual argument: a short exact sequence of modules induces a long exact sequence of Tor groups, and the alternating sum of these is zero in $G(R)$. It is immediate that $\beta\alpha = Id$: this follows from the fact that for any R-module M one has $M[t]/tM[t] \cong M$ and $\mathrm{Ann}_{M[t]}(t) = 0$. Consequently, α is injective.

The difficult part of the proof is showing that α is surjective. We will use the fact, from Theorem 3.1, that $G(R[t])$ is generated by elements of the form $[R[t]/Q]$ for primes $Q \subseteq R[t]$. It suffices to show that each $[R[t]/Q]$ is in the image of α, and we approach this by Noetherian induction. Define

$$T = \{Q \cap R \mid Q \subseteq R[t] \text{ is prime and } [R[t]/Q] \notin \mathrm{im}(\alpha)\}.$$

Our goal is to show that T must be empty.

Since R is Noetherian, if $T \neq \emptyset$ then it has a maximal element $P = Q \cap R$ for some prime $Q \subseteq R[t]$ such that $[R[t]/Q] \notin \mathrm{im}(\alpha)$. Two observations to start with:

(1) If $I \subseteq R$ is any ideal then the expansion $IR[t]$ equals $I[t]$, the set of polynomials with coefficients in I. One has $R[t]/I[t] \cong (R/I)[t] = \alpha(R/I)$.
(2) Any $R[t]$-module M that is killed by $P + u$ for some $u \in R - P$ must lie in $\mathrm{im}(\alpha)$. This is because for each prime Q_i appearing in a prime filtration of M we have $Q_i \supseteq \mathrm{Ann}_{R[t]}(M) \supseteq P + u$. In particular, none of the $Q_i \cap R$ can be in T since P was chosen to be maximal. So $[R[t]/Q_i] \in \mathrm{im}(\alpha)$ for all these Q_i, and hence $[M] \in \mathrm{im}(\alpha)$ as well.

We know $Q \supseteq P[t]$, and they are not equal because $R[t]/P[t] \in \mathrm{im}(\alpha)$. Pick an $f \in Q - P[t]$ of smallest degree and let u be the leading coefficient. We must have $u \in R - P$, otherwise the leading term is in $P[t]$ and we could subtract it off to reduce the degree of f. If we pass to $R[t]/P[t]$ and also invert u, then Q equals (f) by the division algorithm. So some power of u kills $Q/(P[t], f)$. By observation (2) above, $[Q/(P[t], f)] \in \mathrm{im}(\alpha)$.

Now consider the two exact sequences

$$0 \longrightarrow Q/(P[t], f) \longrightarrow R[t]/(P[t], f) \longrightarrow R[t]/Q \longrightarrow 0,$$

$$0 \longrightarrow R[t]/P[t] \xrightarrow{f} R[t]/P[t] \longrightarrow R[t]/(P[t], f) \longrightarrow 0.$$

Together these show $[R[t]/Q] = -[Q/(P[t], f)] \in G(R[t])$. Hence $[R[t]/Q] \in \mathrm{im}(\alpha)$, and this is a contradiction. □

Here is an interesting consequence of homotopy invariance:

Corollary 3.31. *Let F be a field. Then $K(F[x_1, \ldots, x_n]) \cong \mathbb{Z}$.*

Proof. We have $K(F[x_1,\dots,x_n])\cong G(F[x_1\dots,x_n])$ by Theorem 2.13, since the ring $F[x_1,\dots,x_n]$ is regular by Hilbert's Syzyzy Theorem. We also have $G(F[x_1,\dots,x_n])\cong G(F)$ by homotopy invariance, and $G(F)\cong\mathbb{Z}$ via the dimension map. □

In the next section we will see what Corollary 3.31 says about projectives over $F[x_1,\dots,x_n]$. See Proposition 4.7.

∘ Exercises ∘

Exercise 3.32. Let R be a Noetherian ring. Prove that if R/Q has a finite projective resolution for every prime Q, then R is regular. [Hint: Use prime filtrations and the Horseshoe Lemma.]

Exercise 3.33. Let M be a module over a ring R. A prime P is called an **associated prime** of the module M if there is an embedding $R/P\hookrightarrow M$—or equivalently, if $P=\operatorname{Ann} x$ for some $x\in M$. The set of associated primes of M is denoted $\operatorname{Ass} M$.

(a) If P is prime then show $\operatorname{Ass}(R/P)=\{P\}$.
(b) Suppose $0\to M'\to M\to M''\to 0$ is an exact sequence. Prove that $\operatorname{Ass} M\subseteq \operatorname{Ass} M'\cup\operatorname{Ass} M''$. [Hint: If $\operatorname{Ann} x$ is prime and $rx\neq 0$ then $\operatorname{Ann}(rx)=\operatorname{Ann} x$.]
(c) If $F_\bullet$ is a prime filtration of M prove that $\operatorname{Ass} M$ is contained in the primes appearing in $F_\bullet$ (in particular, this proves that $\operatorname{Ass} M$ is finite).
(d) If (R,m) is local and M is a finite length module prove that $\operatorname{Ass} M=\{m\}$. [Hint for $\supseteq$: Start with the largest j such that $m^jM\neq 0$.]
(e) Prove that $\operatorname{Ann}_{S^{-1}M}(x)=S^{-1}(\operatorname{Ann}_M x)$, and use this with (d) to show that every minimal prime of $\operatorname{Ann} M$ belongs to $\operatorname{Ass} M$. [The non-minimal elements of $\operatorname{Ass} M$ are called **embedded primes**.]
(f) Let $R=k[x,y]$ (k a field) and $M=R/(x^2,xy)$. Show that (x,y) is an embedded prime of M, and moreover that $\operatorname{Ass} M=\{(x),(x,y)\}$.
(g) For the above R and M, show that for any $n\geq 1$ there exists a prime filtration of M in which $R/(x,y)$ appears exactly n times in the filtration quotients. Also show there exists a prime filtration in which $R/(x,y-1)$ appears as a quotient.

Exercise 3.34. Fix a prime p and set $R=\mathbb{Z}[x]/(x^2-px)$.

(a) Determine all the prime ideals in R and verify that, as a topological space, $\operatorname{Spec} R$ is two copies of $\operatorname{Spec}\mathbb{Z}$ glued together at the prime (p).
(b) Suppose Q is a prime in R other than (x) and $(x-p)$. Find a short exact sequence $0\to M\to M\to R/Q\to 0$ and conclude $[R/Q]=0$ in $G(R)$. So $G(R)$ is generated by $[R/xR]$ and $[R/(x-p)R]$. Also prove that $[R]=[R/xR]+[R/(x-p)R]$.
(c) Prove that $G(R)\cong\mathbb{Z}^2$. [One way is to use Exercise 2.25(b)].
(d) If Q is a prime other than (x), $(x-p)$, and (x,p) prove that R/Q has a free resolution of length 4.
(e) Find a free resolution of R/xR and use this to prove that $\operatorname{Tor}_*(R/xR,R/(x,p))$ is nonzero in all degrees. Conclude that R is not regular.
[Note: We will investigate $K(R)$ in Exercise 4.17].

Exercise 3.35. Let R be a Noetherian domain of dimension d. Let $\mathcal{P}_k(R)$ be the free abelian group on generators $[R/P]$ where P is a prime such that $\operatorname{ht} P=d-k$.

Define $\mathrm{CH}_k(R)$ to be the quotient of $\mathcal{P}_k(R)$ by the relations

$$0 = \sum_{\substack{P \text{ minimal over } (Q,f) \\ \text{and ht } P = d-k}} \ell_{R_P}(R_P/(Q,f)R_P)[R/P]$$

for every prime Q such that ht $Q = d-k-1$ and and every $f \in R - Q$.

Let $G(R)_k \subseteq G(R)$ be the subgroup generated by the $[R/P]$ where ht $P \geq d-k$. Then $G(R) = G(R)_d \supseteq G(R)_{d-1} \supseteq G(R)_{d-2} \supseteq \cdots$. Prove that there are canonical surjections $\mathrm{CH}_k(R) \to G(R)_k/G(R)_{k-1}$. [This will make more sense when we finally discuss the Atiyah-Hirzebruch spectral sequence, but these surjections are the edge homomorphisms in an algebraic version of that.]

Exercise 3.36. Let $S = \mathbb{Z}[w]/(w^2+w+1)$ and let $R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[X]/(X^2+3)$. It will be convenient to regard $R \subseteq S$ via $X = 2w+1$ (if we regard these rings as inside of $\mathbb{C}$ then $X = \sqrt{3}i$ and $w = \frac{-1+\sqrt{3}i}{2}$). Note that w is integral over $\mathbb{Z}$; it turns out that S is the integral closure of R. The ring S is the ring of integers in the number field $\mathbb{Q}(\sqrt{-3})$.

(a) Check that the unique prime in S lying over $(2, 1+X)$ is (2), and in particular is principal. Note that $R/(2,1+X) \to S/(2)$ is the field extension $\mathbb{F}_2 \hookrightarrow \mathbb{F}_4$.

(b) Observe that $2S \subseteq R$, and that for p an odd prime the map $R/pR \to S/pS$ is an isomorphism. Use these facts to prove that the map $\operatorname{Spec} S \to \operatorname{Spec} R$ is a bijection, with the inverse given by sending a prime $Q \subseteq R$ to QS. [The proofs are different depending on whether the ideal contains 2 or an odd prime.]

(c) Use what we know about R to prove that S is a PID, and conclude $K(S) \cong G(S) \cong \mathbb{Z}$.
[Discussion: What is happening geometrically here is that $\operatorname{Spec} S \to \operatorname{Spec} R$ resolves the singularity in $\operatorname{Spec} R$. The point $(2, 1+X) \in \operatorname{Spec} R$, with residue field $\mathbb{F}_2$, is being "blown up" into the point $(2) \in \operatorname{Spec} S$ with residue field $\mathbb{F}_4$. Away from these points the map $\operatorname{Spec} S \to \operatorname{Spec} R$ is an isomorphism.]

Exercise 3.37. Similarly to $\mathbb{Z}[\sqrt{-3}]$, the ring $R = \mathbb{Z}[\sqrt{-7}]$ has the property that $(2, 1+\sqrt{-7})$ is the unique prime that is not principal. Taking that as a fact, prove that $G(R) \cong \mathbb{Z}$.

Exercise 3.38. Let $R = \mathbb{R}[x,y]_{(x,y)}/(y^2+x^2-x^3)$. This is a one-dimensional local domain. Prove that $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. [Hint: Up to multiplication by units, every $f \in R-0$ may be written as $ax+by+$(higher order terms that all contain x) where $a, b \in \mathbb{R}$. Analyze the cases (i) $a \neq 0$, $b = 0$, (ii) $a = b = 0$. In the final case $b \neq 0$, eliminate y and find that $R/f \cong \mathbb{R}[x]_{(x)}/(x^2 \cdot \text{unit})$.]

Exercise 3.39. Let R be a Noetherian domain of dimension d.

(a) Suppose P and Q are prime ideals and $f, g \in R - 0$ are such that $fP = gQ$. Prove that $[R/P] = [R/Q]$ in $G(R)$.

(b) For $f \in R - \{0\}$ write $\operatorname{div}(f) = \sum_P \ell_{R_P}(R_P/fR_P)[R/P]$ where the sum is over all minimal primes of (f). Prove that $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$.

(c) For $f, g \in R - 0$ define $\operatorname{div}(\frac{f}{g}) = \operatorname{div}(f) - \operatorname{div}(g)$. Prove that this gives a well-defined homomorphism $QF(R)^* \to \mathcal{P}_{d-1}(R)$ (target defined as in Exercise 3.35), and that the cokernel is $\mathrm{CH}_{d-1}(R)$.

(d) For the rest of this problem assume that R is normal. Then for every height one prime P the ring R_P is normal and one-dimensional, hence a DVR. So the

maximal ideal PR_P is principal, and if $x \in PR_P$ is a generator and $f \in R_P - 0$ then $\ell_{R_P}(R_P/fR_P) = n$ if and only if $f = ux^n$ for some unit $u \in R_P$.

(e) Normal domains satisfy $R = \bigcap_{\text{ht } Q=1} R_Q$, where the intersection takes place inside the quotient field of R. Using this, prove that
   (i) For any $f \in R$ one has $(f) = \bigcap_{\text{ht } Q=1}(f)R_Q$.
   (ii) For any height one prime P one has $P = \bigcap_{\text{ht } Q=1} PR_Q$.

(f) Use the following steps to prove that if $[R/P] = 0$ in $\mathrm{CH}_{d-1}(R)$ then P is principal:
   (i) The assumption implies that $\mathrm{div}(g)+[R/P] = \mathrm{div}(f)$ for some $f, g \in R-0$.
   (ii) Argue that $\frac{f}{g} \in R_Q$ for every height one prime Q, hence $\frac{f}{g} \in R$.
   (iii) Conclude $[R/P] = \mathrm{div}(h)$ for some $h \in R - 0$.
   (iv) Prove that $P = (h)$ using (e).

(g) It is a theorem of commutative algebra that a domain is a UFD if and only if every height one prime is principal. Conclude that a normal domain R is a UFD precisely when $\mathrm{CH}_{d-1}(R) = 0$.

Exercise 3.40. Let R be a Noetherian domain. Define a **fractional ideal** to be a finitely-generated R-submodule of $QF(R)$. If I and J are fractional ideals, define $I \sim J$ if there exist $a, b \in R - 0$ such that $aI = bJ$. Equivalence classes of nonzero fractional ideals form a monoid under product, called the **ideal class monoid** and denoted here $\mathrm{Cl}(R)$. The identity element is $[R]$. The group of invertible elements of $\mathrm{Cl}(R)$ is called the **Picard group** of R and denoted $\mathrm{Pic}(R)$.

(a) Prove that every fractional ideal is equivalent to an actual ideal of R.

(b) For a nonzero fractional ideal I define $I^{-1} = \{x \in QF(R) \,|\, xI \subseteq R\}$. Observe that $I \cdot I^{-1} \subseteq R$ always, and check that if I is cyclic then $I \cdot I^{-1} = R$. Then prove that for general I the following are equivalent:
   (1) $I \cdot I^{-1} = R$,
   (2) I is invertible in $\mathrm{Cl}(R)$,
   (3) I_Q is a cyclic R_Q-module for every prime ideal Q.
   An ideal satisfying any of these conditions is called **invertible**.

For the rest of this exercise assume that R is a Dedekind domain—a normal Noetherian domain of dimension 1, or equivalently a Noetherian domain that is locally a DVR.

(c) Prove that every nonzero ideal $I \subseteq R$ is invertible, and conclude that $\mathrm{Cl}(R)$ is a group (so $\mathrm{Cl}(R) = \mathrm{Pic}(R)$ here). Also, prove that $\mathrm{Cl}(R)$ is the group completion of the monoid of equivalence classes of nonzero ideals $I \subseteq R$, where the monoid operation is multiplication of ideals.

(d) Prove that every ideal $I \subseteq R$ is a finite product of prime ideals. In fact, if P is a maximal ideal set $\nu_P(I) = \ell_{R_P}(R_P/IR_P) = \max\{k \,|\, IR_P \subseteq (PR_P)^k\}$. Prove that

$$I = \prod_{\text{ht }(P)=1} P^{\nu_I(P)}$$

   (equality of ideals), and that this is the unique way of writing I as a product of prime ideals.

(e) Show that there is a group homomorphism $\mathrm{CH}_0(R) \to \mathrm{Cl}(R)$ that sends $[R/P]$ to P.

(f) Prove that the map in (f) is an isomorphism. [Hint: The previous parts give a clue of how to define an inverse.]

[Note: It turns out that the Dedekind domains are *precisely* the Noetherian domains having the property that every fractional ideal is invertible.]

4. A closer look at projectives

Recall that a module is projective if and only if it is a direct summand of a free module (there is also a description in terms of lifting criteria, of course). So free modules are projective, and for almost all applications in homological algebra one can get by with using *only* free modules. Consequently, it is common for students not to know many examples of non-free projectives. One of our goals in this section is to remedy this.

It turns out very useful to be able to think about projectives geometrically. Projectives over a commutative ring R correspond to vector bundles over $\operatorname{Spec} R$. This is not at all obvious, and we won't fully understand it until Section 13 after we have developed more tools from the theory of vector bundles. But in this section we will start to see these ideas play out as we import some of the geometric language and intuition into our algebraic discussion.

Before diving into some examples we need a couple of small tools. Just as vector bundles are "locally trivial", projectives are locally free. This is a fun piece of local algebra:

Proposition 4.1. *Over a local ring all projectives are free.*

Proof. Let (R, m, k) be a local ring. A set of generators for an R-module M is minimal (in the sense of no proper subset also generating M) if and only if its image in M/mM is an R/m-basis. So all minimal generating sets for a module have the same size. (Of course this is wildly false when R is not local). Every relation amongst the elements of a minimal set of generators must have all coefficients belonging to m, otherwise there would be a unit coefficient and we would contradict minimality.

Pick a generating set for M and map a free module $F_0 \twoheadrightarrow M$ by sending the free basis to these generators. Then take the kernel K and repeat, mapping a free module $F_1 \twoheadrightarrow K$ by sending the free basis to a minimal generating set. Note that these generating sets might be infinite here. Continuing in this way we build a so-called 'minimal resolution' $F_\bullet \to M \to 0$.

The fact that we picked minimal generating sets at each stage implies that $F_\bullet \otimes R/m$ has vanishing differentials, since all relations amongst the minimal generators have coefficients in m. So $\operatorname{rank}(F_i) = \dim_k \operatorname{Tor}_i(M, k)$.

If M is projective then it is also flat, and hence all of the higher Tor's must vanish. Therefore $F_1 = 0$ and the resolution $0 \to F_0 \to M \to 0$ shows that M is free. □

Let R be a commutative ring, P a finitely-generated projective over R, and $q \subseteq R$ a prime ideal. Then P_q is free and finitely-generated over R_q, so let us define

$$\operatorname{rank}_q(P) = \operatorname{rank}_{R_q}(P_q).$$

Note that rank_q is additive in the sense that $\operatorname{rank}_q(P \oplus Q) = \operatorname{rank}_q(P) \oplus \operatorname{rank}_q(Q)$. If $q \subseteq q'$ then $P_q = (P_{q'})_q$ and so $\operatorname{rank}_q(P) = \operatorname{rank}_{q'}(P)$. Therefore $\operatorname{rank}_?(P)$ is constant on chains of primes. If R is a domain, for example, then all primes can be connected by a chain to (0) and so $\operatorname{rank}_?(P)$ is constant. In this context we will

just talk about the rank of P. (An example of a projective with different ranks at different primes is given in (1) below).

Note that when m is a maximal ideal we can also write

$$\operatorname{rank}_m(P) = \operatorname{rank}_{R_m}(P_m) = \dim_{R_m/mR_m}(P_m/mP_m) = \dim_{R/m}(P/mP).$$

Recall that maximal ideals are the closed points in $\operatorname{Spec} R$. The R/m-vector space P/mP plays the role of the "fiber" of the projective over our closed point.

The following result is quite useful:

Proposition 4.2. *Let R be a ring and let $f\colon J \to W$ be a map of finitely-generated R-projectives of the same rank, meaning that $\operatorname{rank}_m(J) = \operatorname{rank}_m(W)$ for all maximal (or prime) ideals m. If f is a surjection then it is an isomorphism.*

Proof. First recall that over any ring, a surjection of free modules $R^n \twoheadrightarrow R^n$ is necessarily an isomorphism: if A is a matrix representing the map then surjectivity means we can write $I_n = A \cdot B$ for some B, and then taking determinants gives that $\det(A)$ is invertible. Now use the standard formula that constructs A^{-1} in terms of the adjugate matrix.

Returning to the setting of the proposition, the map f is an isomorphism if and only if $f_P\colon J_P \to W_P$ is an isomorphism for all prime ideals P. But J_P and W_P are free over R_P of the same rank, so the preceding paragraph finishes the proof. □

Remark 4.3 (The geometry of local rings)**.** In view of the above discussion this might be a good time to clarify some mysteries about local rings. If q is a prime in R then $\operatorname{Spec} R_q$ may be identified with the subset of $\operatorname{Spec} R$ consisting of all primes contained in q. This subset is usually neither open nor closed in $\operatorname{Spec} R$. What we have instead is this:

$$\operatorname{Spec} R_q = \bigcap_{q \in U^{\text{open}} \subseteq \operatorname{Spec} R} U.$$

That is, $\operatorname{Spec} R_q$ is the intersection of all Zariski open neighborhoods of q in $\operatorname{Spec} R$. For Hausdorff topological spaces such an intersection would always just be the point itself, and therefore non-interesting. The analog here is that $\operatorname{Spec} R_q$ has only one *closed* point, but it nevertheless has a significant amount of information lurking in the non-closed points.

For a typical space that appears in algebraic topology—a manifold or CW-complex, for example—while the intersection of open neighborhoods of a point is just the point itself one can nevertheless sense that the open neighborhoods get similar as they get smaller and smaller. In some sense there is some limiting information there that is not captured at just the set level. This is really what is happening in algebraic geometry. Our $\operatorname{Spec} R_q$ behaves like an "infinitesimal" Zariski neighborhood of q. This idea takes some getting used to, but it is important for understanding how information passes back and forth between algebra and geometry. For additional discussion see Section 28.9.

Now let us turn to some examples of interesting projectives:

(1) Let $R = \mathbb{Z}/6$. Since $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$, both $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective R-modules—and they are clearly not free. More generally, if $R = S \times T$ then we can make S into an R module via $(s,t).u = su$ where $s, u \in S$ and $t \in T$. Then S is projective but not free.

Geometrically, what is happening here is that $\operatorname{Spec} S \times T = \operatorname{Spec} S \amalg \operatorname{Spec} T$. When we make S into an $(S \times T)$-module as above we are constructing the vector bundle that is free of rank one on the $\operatorname{Spec} S$ component but zero on the $\operatorname{Spec} T$ component. Check algebraically that $\operatorname{rank}_m(S)$ is either 0 or 1 and depends on the choice of m.

(2) Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1 + \sqrt{-5}) = (2, 1 - \sqrt{-5})$. For convenience let us write $\mu = \sqrt{-5}$. A standard tool for dealing with this ring is the norm map $N(a + b\mu) = (a + b\mu)(a - b\mu) = a^2 + 5b^2$. This is multiplicative, i.e. $N(xy) = N(x)N(y)$. Using this one argues that if I were generated by a single element $a + b\mu$ then 2 would be a multiple of $a^2 + 5b^2$, and one quickly obtains a contradiction. So I is not principal.

Let K be the kernel of the map $R^2 \to I$ sending e_1 to 2 and e_2 to $1 + \mu$. A little work shows that K is spanned by $(1 + \mu, -2)$ and $(-3, 1 - \mu)$. If one defines $\chi\colon R^2 \to K$ by

$$\chi(e_1) = (3, -1 + \mu), \quad \chi(e_2) = (1 + \mu, -2),$$

it is readily verified that χ is a splitting for the sequence $0 \to K \to R^2 \to I \to 0$. So $K \oplus I \cong R^2$, and hence both K and I are projective.

The inclusion $I \subseteq R$ becomes an isomorphism after tensoring with $QF(R)$, so I has rank one. If I were free then we would have $I \cong R$. However, this would contradict I being non-principal. So I is a non-free projective.

This example generalizes: if D is a Dedekind domain (such as the ring of integers in an algebraic number field) then every ideal $I \subseteq D$ is projective. Non-principal ideals are never free.

Exercise 4.4. Verify that K is generated by $(1 + \mu, -2)$ and $(-3, 1 - \mu)$. Note that the second coordinates of elements of K therefore all belong to I. Prove that the composition $K \hookrightarrow R \xrightarrow{\pi_2} I$ is an isomorphism. Deduce that $I \oplus I \cong R \oplus R$.

(3) Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$. If $C(S^2)$ denotes the ring of continuous functions $S^2 \to \mathbb{R}$, note that we may regard R as sitting inside of $C(S^2)$: it is the subring of polynomial functions on the 2-sphere. The connections with the topology of the 2-sphere will be important below.

Let $\pi\colon R^3 \to R$ be the map $\pi(f, g, h) = xf + yg + zh$. That is, π is left-multiplication by the matrix $\begin{bmatrix} x & y & z \end{bmatrix}$. Let T be the kernel of π:

$$0 \longrightarrow T \rightarrowtail R^3 \xrightarrow{\pi} R \longrightarrow 0.$$

The map π is split via $\chi\colon R \to R^3$ sending $1 \mapsto (x, y, z)$. We conclude that $T \oplus R \cong R^3$, so T is projective.

We claim that T is not free. Suppose, towards a contradiction, that T is free. Since $T \oplus R \cong R^3$ we have $\operatorname{rank}(T) = 2$, and so $T \cong R^2$. Choose an isomorphism $R^2 \to T$, let e_1 and e_2 be the standard basis for R^2, and let the image of e_1 under our isomorphism be (f, g, h). So f, g, and h are polynomial functions on S^2 and

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} f(p) \\ g(p) \\ h(p) \end{bmatrix} = 0$$

for all $p = (p_1, p_2, p_3) \in S^2$. So $p \mapsto (f(p), g(p), h(p))$ is a tangent vector field on S^2. By the Hairy Ball Theorem we can find a point $q = (q_1, q_2, q_3) \in S^2$

such that $f(q) = g(q) = h(q) = 0$. Let $m = (x - q_1, y - q_2, z - q_3) \subseteq R$ and consider the commutative diagram

$$\begin{array}{ccccc} R^2 & \xrightarrow{\cong} & T & \rightarrowtail & R^3 \\ \downarrow & & \downarrow & & \downarrow \\ (R/mR)^2 & \xrightarrow[\cong]{} & T/mT & \rightarrowtail & (R/mR)^3. \end{array}$$

The lower right map is an injection because the upper right map is a split injection. Note that $R/mR \cong \mathbb{R}$ via $F \mapsto F(q)$. Start with e_1 in the upper left corner and compute its image in $(R/mR)^3 \cong \mathbb{R}^3$ under the two outer ways of tracking around the diagram. Along the top route e_1 maps to $(f(q), g(q), h(q))$ which is just $(0, 0, 0)$. On the other hand, along the bottom route e_1 first maps to $(1, 0) \in \mathbb{R}^2$ and then the bottom composite is an injection—so the image in $\mathbb{R}^3$ is nonzero. This is a contradiction, so we conclude that T is not free. (In fact, we have proven more: we have proven that T does not contain R as a direct summand).

Note that T is an algebraic analog of the tangent bundle of S^2. As remarked at the beginning of the section, these parallels between projective modules and vector bundles are very important. We will see much more about them in Section 13.

(4) Let us do one more example where we use topology to produce an example of a non-free projective. This example is based on the Möbius bundle over S^1. Let

$$S = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$$

and let $R \subseteq S$ be the span of the even degree monomials. One should regard S as the ring of polynomial functions on the circle, and R is the ring of polynomial functions $f(x, y)$ satisfying $f(x, y) = f(-x, -y)$. So R is trying to be the ring of polynomial functions on $\mathbb{R}P^1$ (which happens to be homeomorphic to S^1).

Let $P \subseteq S$ be the $\mathbb{R}$-linear span of the homogeneous polynomials with odd total degree. Observe that P is a finitely generated R-module and we have $\pi\colon R^2 \twoheadrightarrow P$ via $\pi(e_1) = x$ and $\pi(e_2) = y$. Define $\chi\colon P \to R^2$ via

$$h \mapsto \chi(h) = \begin{bmatrix} xh \\ yh \end{bmatrix}.$$

One checks that $\pi \circ \chi = id$, so P is projective. We leave it as an exercise for the reader to show that P is not free.

Exercise 4.5. Complete example (4) above by showing that P has rank one and that it is not free. [This is not hard to do algebraically, but it is informative to also explore a topological proof: if P is free, show how to get a nonzero section of the tautological line bundle over $\mathbb{R}P^1$.]

The topological examples (3) and (4), as well as many similar ones, can be found in the lovely paper [Swan]. See also Section 13 below.

Example 4.6. Here is an example of an algebraic problem whose solution involves non-free projectives coming from topology. If A and B are commutative rings such that $A[X] \cong B[X]$, does it follow that $A \cong B$? This is a natural question that once upon a time had people stumped. The first counterexample was due to Hochster [Hochster3] and is closely related to our current discussion.

Let $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ as in (3) above, and let T also be as above. Recall that T is not free but that there is an isomorphism $R^3 \cong T \oplus R$. Apply the symmetric algebra construction $\mathrm{Sym}_R(-)$ to obtain

$$\begin{aligned} R[U,V,W] \cong \mathrm{Sym}_R(R^3) \cong \mathrm{Sym}_R(T \oplus R) &\cong \mathrm{Sym}_R(T) \otimes_R \mathrm{Sym}_R(R) \\ &= \mathrm{Sym}_R(T) \otimes_R R[W] \\ &= \mathrm{Sym}_R(T)[W]. \end{aligned}$$

We take $A = R[U,V]$ and $B = \mathrm{Sym}_R(T)$, and the above gives a ring isomorphism $A[W] \cong B[W]$.

It requires a little work to prove that $A \ncong B$, though it is not too bad. We only give a sktech. One first argues that any isomorphism must be $\mathbb{R}$-linear, then that the isomorphism must map R onto itself. By composing with the inverse to this automorphism of R, one can assume that the isomorphism $A \to B$ is R-linear. But if there is such an R-algebra isomorphism then B is generated as an R-algebra by two elements, and one readily proves that the degree 1 homogeneous components of those two pieces generate T as an R-module. But the arguments from (3) above show that T cannot be generated by two elements, and this is the contradiction. We refer to [Hochster3] for more details.

The original question about A and B, together with some variants, are often called the *Zariski Cancellation Problem*. This is an active area of research. For example, one open question is whether or not there exist counterexamples where A is a polynomial ring over a field of characteristic zero (such counterexamples *are* known in positive characteristic, by recent work of Gupta [Gupta1]). We mention the survey paper [Gupta2] as just one point of entry into this subject.

A projective module P is called **stably free** if there exists a free module F such that $P \oplus F$ is free. The example in (3) gives a projective that is stably-free but not free. It turns out that $K(R)$ can be used to tell us whether such modules exist or not. To see this, recall that if $m \subseteq R$ is a maximal ideal then $\mathrm{rank}_m(-)$ is an additive function on finitely-generated, projective modules. So it induces a map $\mathrm{rank}_m(-)\colon K(R) \to \mathbb{Z}$, which is evidently surjective because $\mathrm{rank}_m(R) = 1$. This shows that $K(R)$ always contains $\mathbb{Z}$ as a direct summand.

Define the **reduced Grothendieck group of projectives** to be

$$\widetilde{K}(R) = K(R)/\langle [R] \rangle.$$

Here is another way to define this group. Take the set of isomorphism classes of finitely-generated projectives and impose the equivalence relation $(P) \sim (P \oplus R)$ for every P. Such equivalences classes are called **stable projectives**. Define a monoid structure on this set by $(P)+(Q) = (P \oplus Q)$, and note that $(0) = (R)$ is the identity. If P is any projective then there exists a Q such that $P \oplus Q$ is free, and therefore $(P)+(Q) = 0$ in this monoid; hence, we have a group. This is called the **Grothendieck group of stable projectives**. One readily checks that this group is isomorphic to $\widetilde{K}(R)$, with the equivalence class (P) corresponding to the element $[P] \in \widetilde{K}(R)$.

Proposition 4.7. *Let R be a commutative ring. The following are equivalent:*
(1) $K(R) \cong \mathbb{Z}$
(2) $\widetilde{K}(R) = 0$

(3) Every finitely-generated, projective R-module is stably-free.

Proof. Immediate. □

Example 4.8. Recall from Corollary 3.31 that if F is a field then $K(F[x_1,\dots,x_n]) = \mathbb{Z}$. Thus, every finitely-generated, projective $F[x_1,\dots,x_n]$-module is stably-free.

In the 1950s, Serre conjectured that every finitely-generated projective over $F[x_1,\dots,x_n]$ is actually free. As we will see later (Remark 14.6 below), the motivation for this conjecture is inspired by topology and the connection between vector bundles and projective modules. Quillen [Quillen6] and Suslin [Suslin] independently proved Serre's conjecture in the 1970s. It is now known as the *Quillen-Suslin theorem*.

Example 4.9. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let I be the ideal $(2, 1+\sqrt{-5})$. We saw in example (2) from the beginning of this section that I is a rank one projective that is not free. Could I be stably free? If it were, then we would have $I \oplus R^k \cong R^{k+1}$, for some k. Apply the exterior product $\Lambda^{k+1}(-)$ to deduce that

$$R \cong \Lambda^{k+1}(R^{k+1}) \cong \Lambda^{k+1}(I \oplus R^k) \cong \Lambda^1(I) \otimes \Lambda^k(R^k) \cong I \otimes R \cong I$$

(in the third isomorphism we have used the formula for the exterior product of a direct sum, together with the general fact that $\Lambda^j(P) = 0$ for $j > \operatorname{rank}(P)$). However, this is a contradiction as we have already seen that I is not free. Hence, I is not stably free and so $[I]$ determines a nonzero class in $\tilde{K}(R)$. By Exercise 4.4 we know $I \oplus I \cong R^2$ and so $2[I] = 0$ in $\tilde{K}(R)$, therefore we have a nonzero 2-torsion class.

Again, this example generalizes to any Dedekind domain D. If $I \subseteq D$ is a non-principal ideal then I is a rank one projective that is not stably free. So a Dedekind domain has $K(D) \cong \mathbb{Z}$ if and only if D is a PID. As another consequence, we observe that over any commutative ring a rank one projective P cannot be stably free unless it is actually free.

4.10. **The Milnor patching theorem.** One can sometimes build projectives over a ring by taking projectives over smaller rings and gluing them together. This is best demonstrated with an example, which quickly leads to a general theorm along these lines.

Exercise 4.11. Let $R = \mathbb{Z}[x]/(x^2 - px)$ as in Exercise 3.34. Recall that geometrically $\operatorname{Spec} R$ is two copies of $\operatorname{Spec}\mathbb{Z}$ that are glued together at the point (p). The theme of this exercise is that projectives over R are obtained by taking free modules of the same rank on the two copies of $\operatorname{Spec}\mathbb{Z}$ and then gluing them together via an isomorphism on their fibers over (p).

(a) For $u \in (\mathbb{Z}/p)^*$ set $J_u = \{(f,g) \in \mathbb{Z}^2 \mid uf \equiv g \pmod{p}\}$. Make J_u into an R-module by $x.(f,g) = (0,pg)$. Prove that J_u is generated over R by $(1,u)$ and $(0,p)$ and that the surjection $R^2 \to J_u$ sending e_1 and e_2 to these two generators has an R-linear splitting. So J_u is projective. Prove that the rank is 1.

(b) Construct an isomorphism $J_u \otimes J_v \to J_{uv}$, for any $u, v \in (\mathbb{Z}/p)^*$. [Hint: Use Proposition 4.2.]

(c) Prove that $J_u \cong R$ if and only if $u = \pm 1$, and more generally $J_u \cong J_v \iff u = \pm v$.

(d) Suppose that P is a rank one projective over R. Then P/xP is a rank one projective over the PID R/xR, so we can choose an R-linear isomorphism $\phi_1\colon P/xP \to R/xR$. Likewise, we can choose an R-linear isomorphism $\phi_2\colon P/(x-p)P \to R/(x-p)R$. We then obtain induced isomorphisms $\bar{\phi}_1, \bar{\phi}_2\colon P/(x,x-p)P \to R/(x,x-p)R$, so that the composite $\bar{\phi}_2\bar{\phi}_1{}^{-1}$ is an R-linear automorphism of $R/(x,x-p)R$ and therefore multiplication by a unit u of this quotient ring. Use these ideas to prove that $P \cong J_u$.

(e) The set of isomorphism classes of rank one projectives becomes a group under the tensor product; this is called the **Picard group** of R and denoted $\mathrm{Pic}(R)$ (see also Exercise 4.18 below). Putting the previous parts together, prove that $\mathrm{Pic}(R) \cong (\mathbb{Z}/p)^*/\pm 1$.

(f) For $A \in GL_n(\mathbb{Z}/p)$ define $J_A = \{(f,g) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid Af = g \pmod p\}$. Extend the above ideas to prove that J_A can be generated by $2n$ elements over R and that J_A is projective of rank n. Prove that $J_{A\bar{P}} \cong J_A \cong J_{\bar{P}A}$ for any $P \in GL_n(\mathbb{Z})$ (where $\bar{P}$ is the mod p reduction), so that $A \mapsto J_A$ gives a map

$$GL_n(\mathbb{Z})\backslash GL_n(\mathbb{Z}/p)/GL_n(\mathbb{Z}) \longrightarrow \big\{\text{iso. classes of rank } n \text{ projectives over } R\big\}.$$

Verify that this is a bijection.

[Discussion: Geometrically, what is happening here is that we are building vector bundles on $\operatorname{Spec} R$ by taking two trivial bundles on $\operatorname{Spec}\mathbb{Z}$ and then gluing together their fibers over p. The "gluing map" is the matrix in $GL_n(\mathbb{Z}/p)$, and the left and right multiplications by $GL_n(\mathbb{Z})$ correspond to change-of-bases in the two factors.]

In the exercises at the end of the section we will use the above classification of R-projectives to compute $K(R)$, but let us pause here and consider more carefully what elements led to this classification. The key ideas come down to the fact that we have the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{x\to p} & \mathbb{Z} \\ {\scriptstyle x\to 0}\downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/p, \end{array}$$

and moreover R can be identified with the pullback: the map $R \to \mathbb{Z}\times_{\mathbb{Z}/p}\mathbb{Z}$ given by $f(x) \mapsto (f(0), f(p))$ is an isomorphism of rings (this is related to the identification $R \cong J_1$ from the exercise). The exercise shows that projectives over R can all be built by taking projectives over the two $\mathbb{Z}$'s (which are necessarily free) and "patching" them with an isomorphism over the $\mathbb{Z}/p$ term.

The above ideas are readily generalized. Suppose given a pullback diagram of rings

$$\begin{array}{ccc} T & \longrightarrow & R \\ \downarrow & & \downarrow \\ S & \longrightarrow & R/I. \end{array} \tag{4.12}$$

Consider the category $\mathcal{PD}$ of "patching data" whose objects are triples (P, Q, ϕ) where P is a finitely-generated projective over S, Q is a finitely-generated projective over R, and $\phi\colon R/I \otimes_S P \to R/I \otimes_R Q$ is an R/I-linear isomorphism. A map

$(P, Q, \phi) \to (P', Q', \phi')$ is a pair of maps $P \to P'$ and $Q \to Q'$ making the evident square commute.

There is a functor T-proj $\to \mathcal{PD}$ that sends a finitely-generated T-projective M to $(S \otimes_T M, R \otimes_T M, \mathrm{id}\colon R/I \otimes_T M \to R/I \otimes_T M)$. So far none of this uses that we had a pullback diagram, only that we had a commutative square. But we will use the pullback hypothesis to construct an inverse for this functor.

Given an object (P, Q, ϕ) in $\mathcal{PD}$, define

$$J_{(P,Q,\phi)} = \{(f, g) \,\big|\, f \in P, g \in Q, \text{ and } \phi(1 \otimes f) = 1 \otimes g \text{ in } R/I \otimes_R Q\}.$$

This becomes a module over T in the evident way: if we represent elements of T as pairs $(s, r) \in S \times_{R/I} R$ then $(s, r).(f, g) = (sf, rg)$.

Theorem 4.13 (Milnor patching theorem)**.** *The module $J_{(P,Q,\phi)}$ is a finitely-generated projective over T. The canonical functor T-proj $\to \mathcal{PD}$ is an equivalence of categories, with inverse equal to J.*

Proof. The tricky part is proving that $J_{(P,Q,\phi)}$ is projective. Once that is done, the equivalence of categories is fairly straightforward and will be left as an exercise.

Choose complements $P \oplus P' \cong S^n$ and $Q \oplus Q' \cong R^k$. Set $P'' = P' \oplus S^k$ and $Q'' = Q' \oplus R^n$, so that $P \oplus P'' \cong S^{n+k}$ and $Q \oplus Q'' \cong R^{n+k}$.

To ease notation set $\tilde{R} = R/I$. Observe the evident isomorphisms

$$\tilde{R} \otimes_S P'' = \tilde{R} \otimes_S (P' \oplus S^k) = (\tilde{R} \otimes_S P') \oplus (\tilde{R})^k = (\tilde{R} \otimes_S P') \oplus (\tilde{R} \otimes_R Q) \oplus (\tilde{R} \otimes_R Q')$$

$$\tilde{R} \otimes_R Q'' = \tilde{R} \otimes_R (Q' \oplus R^n) = (\tilde{R} \otimes_R Q') \oplus (\tilde{R})^n = (\tilde{R} \otimes_R Q') \oplus (\tilde{R} \otimes_R P) \oplus (\tilde{R} \otimes_R P').$$

Let α be the isomorphism from the first to the second which is the identity on the $\tilde{R} \otimes_R P'$ and $\tilde{R} \otimes_R Q'$ summands and $\phi^{-1}\colon \tilde{R} \otimes_R Q \to \tilde{R} \otimes_R P$ on the remaining piece.

It is easy to check that $J_{(P,Q,\phi)} \oplus J_{(P'',Q'',\alpha)} = J_{(S^{n+k}, R^{n+k}, \phi \oplus \alpha)}$. So if we can show that the module on the right is projective, so is $J_{(P,Q,\phi)}$. Thus, we reduce to the case where P and Q are free.

So now we focus on $J_{(S^n, R^n, \phi)}$ where $\phi \in GL_n(R/I)$. It is easy to see that $J_{(S^n,R^n,\phi)} \cong J_{(S^n, R^n, \bar{A}\phi\bar{B})}$ for any $A \in GL_n(R)$ and $B \in GL_n(S)$, where the bars represent passing to coefficients in R/I. Since any elementary matrix in $GL_n(R/I)$ lifts to an elementary matrix in $GL_n(R)$, we see that we are able to multiply ϕ on the right by any elementary matrix without changing the isomorphism class of $J_{(S^n,R^n,\phi)}$.

Now we come to Milnor's trick. Consider $J_{(S^n,R^n,\phi)} \oplus J_{(S^n,R^n,\phi^{-1})} = J_{(S^{2n},R^{2n},F)}$ where $F = \left[\begin{smallmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{smallmatrix}\right]$. If we can show that F is a product of elementary matrices then $J_{(S^{2n},R^{2n},F)} \cong J_{(S^{2n},R^{2n},\mathrm{id})}$, which is free. This will imply that $J_{(S^n,R^n,\phi)}$ is projective, as desired.

To conclude, just observe the product decomposition

$$\begin{bmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{bmatrix} = \begin{bmatrix} I_n & \phi \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -\phi^{-1} & I_n \end{bmatrix} \begin{bmatrix} I_n & \phi \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Each factor is a product of elementary matrices because it can be row/column-reduced to the identity using only transvections (operations where one adds a multiple of one row (or column) to another row (or column)). □

For more experience with the kinds of row reduction arguments used in the above result, see Section 8.2 and especially Lemma 8.10.

Exercise 4.14. Prove the equivalence of categories in the Milnor patching theorem. [Hint: This is almost like a diagram chase, in that at each juncture in the proof there ends up being an evident move to make. Proposition 4.2 is useful in several places.]

The Milnor patching theorem and its proof are from [Milnor7, Section 2]. In the exercises at the end we will explore various applications, but let us at least observe how easy it makes Exercise 4.11. Finitely-generated projectives over $\mathbb{Z}$ are free, so patching data looks like $(\mathbb{Z}^n, \mathbb{Z}^n, \phi)$ where $\phi \in GL_n(\mathbb{Z}/p)$. An isomorphism between patching data for ϕ and ϕ' is a pair of matrices $A, B \in GL_n(\mathbb{Z})$ such that $\phi' = \bar{A}\phi\bar{B}$. So the set of isomorphism classes of rank n projectives is precisely the double quotient space $GL_n(\mathbb{Z})\backslash GL_n(\mathbb{Z}/p)/GL_n(\mathbb{Z})$, as we saw before.

Remark 4.15. Applying Spec to the square (4.12) gives a pushout diagram of schemes; see [Stacks, II.37.14]. Exercise 4.21 below shows a piece of this. For an analogous patching theorem for vector bundles on topological vector spaces, see Corollary 11.7.

Exercise 4.16. Suppose given the Milnor patching square (4.12). Use the Milnor patching theorem for rank 1 projectives to prove that there is a "Mayer-Vietoris" exact sequence

$$0 \to T^* \to S^* \oplus R^* \to (R/I)^* \to \mathrm{Pic}(T) \to \mathrm{Pic}(S) \oplus \mathrm{Pic}(R) \to \mathrm{Pic}(R/I)$$

where all of the maps are the evident ones. Use this to re-compute the Picard group from Exercise 4.11.

∘ Exercises ∘

Exercise 4.17. Fix a prime p and let $R = \mathbb{Z}[x]/(x^2 - px)$. For the following, use the classification of projectives over R from Exercise 4.11 (particularly part (f)).

(a) Prove that $J_u \oplus J_v \cong R \oplus J_{uv}$, and then prove that every finitely-generated projective decomposes as a direct sum of a rank one projective and a free module. Furthermore, show that $R^r \oplus J_u$ is free if and only if $u = \pm 1$. [Hint: Suppose $A, A' \in GL_n(\mathbb{Z}/p)$ and one can obtain A' from A by a row (or column) operation that adds a multiple of one row (or column) to another. Prove that $A \sim A'$ in the orbit space from Exercise 4.11(f). Then use this technique in the first two proofs. For the third, use the determinant.]

(b) Define a map $\mathbb{Z} \oplus [(\mathbb{Z}/p)^*/\pm 1] \to K(R)$ by $(r, u) \mapsto (r-1)[R] + [J_u]$. Prove that this is an isomorphism, so that

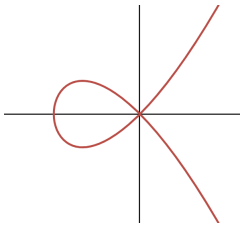
$$K(R) \cong \begin{cases} \mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z} \oplus \mathbb{Z}/(\frac{p-1}{2}) & \text{if } p > 2. \end{cases}$$

Observe that when $p > 3$ the canonical map $K(R) \to G(R)$ is neither injective nor surjective.

Exercise 4.18. Let R be a ring. By a *rank one projective* we mean a finitely-generated projective J such that $\mathrm{rank}_m J = 1$ for all maximal ideals m. This is equivalent to saying that J_Q is free of rank 1 for all primes Q.

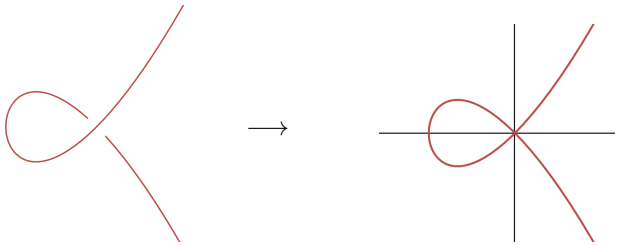
(a) If J is a rank one projective show that the same is true for $\mathrm{Hom}(J, R)$. Prove that the evaluation map $J \otimes \mathrm{Hom}(J, R) \to R$ is an isomorphism. [Hint: Work locally.]
(b) If J and J' are rank one projectives, prove that so is $J \otimes J'$. Let $\mathrm{Pic}(R)$ be the monoid of isomorphism classes of rank one projectives under the tensor product. By (a) this is a group.
(c) Suppose that R is a domain. If J is a rank one projective then $J \otimes_R QF(R) \cong QF(R)$ (non-canonically). But J is a summand of a free module and hence torsion-free, so $J \to J \otimes_R QF(R)$ is an injection. Hence J is isomorphic to a fractional ideal of R—see Exercise 3.40—which by part (b) of that exercise is invertible. Prove that the Picard group as defined there in terms of invertible fractional ideals is isomorphic to the Picard group defined here in terms of rank one projectives.

Exercise 4.19. Let k be a field not of characteristic 2 and let $R = k[x, y]/(y^2 - x^2(x+1))$. This is the coordinate ring of a nodal cubic:



In particular, R is not regular. In this exercise we will explore projectives over R.

Set $t = \frac{y}{x}$ and note that $t^2 = x + 1$, hence t is integral over R. It turns out that the integral closure of R is $\tilde{R} = k[t]$, with $R \hookrightarrow \tilde{R}$ sending $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. Geometrically, the map $\operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ looks as follows:



The curve in the left is just the affine line $\operatorname{Spec} k[t]$, but it has been depicted in a way that is compatible with the map to $\operatorname{Spec} R$. The points $t = 1$ and $t = -1$ (shown in the left picture lying above each other) are sent to $(0, 0)$, and away from these points the map is an isomorphism. Note that $R \subseteq k[t]$ may be regarded as the subring of polynomials f such that $f(1) = f(-1)$.

All vector bundles on the affine line $\operatorname{Spec} \tilde{R}$ are trivial (e.g. use the classification of modules over a PID). We can make bundles on $\operatorname{Spec} R$ by taking a trivial rank n bundle on $\operatorname{Spec} \tilde{R}$ and gluing the two fibers at $t = \pm 1$ together via a fixed

isomorphism $A \in GL_n(k)$. We can interpret this as an example of Milnor patching, using the square

$$\begin{array}{ccc} R & \longrightarrow & k[t] \\ {\scriptstyle \substack{x\to 0\\ y\to 0}}\downarrow & & \downarrow{\scriptstyle f\mapsto(f(1),f(-1))} \\ k & \xrightarrow{\Delta} & k\times k \end{array} .$$

(a) Confirm that the above is a pullback square, and use it to establish an equivalence of categories between R-proj and the category of pairs (F, ϕ) where F is a free $k[t]$-module and $\phi\colon F/(t-1)F \to F/(t+1)F$ is a k-linear isomorphism.
(b) Explain how to get a bijection between isomorphism classes of R-projectives of rank n and the set $GL_n(k)/\sim$ where the equivalence relation is generated by $A \sim B(-1)^{-1}AB(1)$ for any $B \in GL_n(k[t])$.
(c) For any elementary matrix $E \in GL_n(k)$, prove that there is a $B \in GL_n(k[t])$ such that $B(1) = I_n$ and $B(-1) = E$, and likewise a $B' \in GL_n(k[t])$ such that $B'(1) = E$ and $B'(-1) = I_n$. Conclude that for the equivalence relation from (b) every matrix in $GL_n(k)$ is equivalent to one of the form $\mathrm{diag}(d, 1, 1, \ldots, 1)$. Finally, prove that if $A \sim A'$ then $\det A = \det A'$.
(d) For $u \in k^*$ let J_u be the rank 1 projective corresponding to $u \in GL_1(k)$ under the above bijection. Prove that $J_u \oplus J_v \cong R \oplus J_{uv}$, and that $J_u \oplus R^r \cong J_v \oplus R^r$ if and only if $u = v$. Prove that every rank n projective is of the form $J_u \oplus R^{n-1}$ for some $u \in k^*$.
(e) Prove that $\mathrm{Pic}(R) \cong k^*$ and $K(R) \cong \mathbb{Z} \oplus k^*$.

Exercise 4.20. Let $R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[X]/(X^2+3)$ and $S = \mathbb{Z}[w]/(w^2+w+1)$, with $R \subseteq S$ via $X = 2w-1$. Recall from Exercise 3.36 that S is a PID, whereas in R all primes are principal except for $(2, 1+X)$. We identify $\mathbb{F}_4 = \mathbb{Z}/2[w]/(w^2+w+1) = S/2$ and $R/(2, 1+X) = \mathbb{F}_2$.

(a) Check that the diagram

$$\begin{array}{ccc} R & \rightarrowtail & S \\ \downarrow & & \downarrow \\ \mathbb{F}_2 & \longrightarrow & \mathbb{F}_4 \end{array}$$

is a pullback.
(b) Explain why knowing that $GL_n(S) \to GL_n(\mathbb{F}_4)$ is surjective would imply that all finitely-generated R-projectives of rank n are free.
(c) Prove that $GL_n(S) \to GL_n(\mathbb{F}_4)$ is surjective, for all $n \geq 1$. [Hint: Row/column-reduction tells us that $GL_n(\mathbb{F}_4)$ is generated by elementary matrices together with diagonal matrices $\mathrm{diag}(u, 1, 1, \ldots, 1)$ where u is a unit.]

Exercise 4.21. Suppose given a Milnor patching diagram as in (4.12). Prove that the induced map $\mathrm{Spec}\, S \amalg_{\mathrm{Spec}\, R/I} \mathrm{Spec}\, R \to \mathrm{Spec}\, T$ is a bijection. The map is actually a homeomorphism, but proving that is more challenging. [Hint: Identify T with a subset of $S \times R$. Let $J = \{(0, i) \mid i \in I\} \subseteq T$ and let $K = \{(s, t) \in T \mid t = 0\}$. These are ideals in T and $J \cdot K = 0$. Prove that every prime ideal of T containing J is contracted from a prime in S. The slightly tricky part is proving that a prime P of T that does not contain J is contracted from R. If $i \in I$ and $(0, i) \notin P$, let $Q = \{u \in R \mid i^N u \in \pi_2(P) \text{ for some } N \geq 0\}$. Prove that Q is prime and contracts to P.]

Exercise 4.22. Let R be a commutative ring and let G be a finite group acting on R via automorphisms. For $U \subseteq R$ let U^G denote the set of G-invariant elements of U. In particular, $R^G \subseteq R$ is the subring of all G-invariants.

The theme of this exercise is that R^G-projectives should correspond to R-projectives with some extra structure. The goal is to develop a precise statement along these lines.

This exercise needs some commutative algebra that we haven't encountered yet. For an integral extension $A \hookrightarrow B$, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ has special properties governed by the Going-Up property and Incomparability property (these can be found in almost any commutative algebra text), and these imply (in particular) that $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed map.

(a) Prove that $R^G \hookrightarrow R$ is an integral extension. [Hint: For $r \in R$ consider the polynomial $\prod_{g\in G}(X - gr)$.]

(b) If P and Q are primes in R, prove that

$$(P^G \subseteq Q) \Rightarrow \Big(P \subseteq \bigcup_{g\in G} gQ\Big) \Rightarrow (P \subseteq gQ \ \exists g \in G).$$

(c) Prove that if $P^G = Q^G$ then P and Q are conjugate under G. [Hint: Use the incomparability property of integral extensions.]

(d) Prove that $\operatorname{Spec} R \to \operatorname{Spec} R^G$ is a quotient map of topological spaces, and that the fibers are the G-orbits. That is, prove that $\operatorname{Spec} R^G = (\operatorname{Spec} R)/G$ as topological spaces.

(e) Define an **equivariant R-module** to be an R-module M together with a G-action on (the underlying abelian group of) M having the property that $g(r.m) = (gr).(gm)$ for any $g \in G$, $r \in R$, and $m \in M$. Say that the module is **well-generated** if there is a collection of module generators each of which is fixed by the G-action. Note that R with its inherent G-action is an equivariant G-module that is well-generated.

Suppose that $|G|$ is invertible in R and let M be an equivariant R-module. Prove that M is well-generated and R-projective if and only if M is a direct summand (as a module with G-action) of a free module R^α. [Hint: Choose a surjection of R-modules $R^\alpha \twoheadrightarrow M$ sending the free basis to the generators fixed by the G-action. Choose an R-module splitting χ, then define a new splitting $\tilde{\chi}$ via a formula like $\tilde{\chi}(m) = \frac{1}{|G|}\sum_{g\in G}(\text{fill-this-in})$.]

(f) Let $G = \mathbb{Z}/2$ and let $R = \mathbb{Z}$ with the trivial G-action. Find an equivariant R-module whose underlying R-module is projective but which is not well-generated.

(g) Define an **equivariant R-projective** to be an equivariant R-module that is a direct summand, as a module with G-action, of a free module R^α. This implies being an equivariant module that is well-generated and whose underlying R-module is projective, but it is potentially stronger when $|G|$ is not invertible. Prove that

  (i) If E is a projective R^G-module then $R \otimes_{R^G} E$ is an equivariant R-projective.

  (ii) If M is an equivariant R-projective then M^G is an R^G-projective.

  (iii) There is an equivalence of categories

$$((R^G - \text{projectives})) \rightleftarrows ((\text{equivariant } R\text{-projectives}))$$

where the two maps are from (i) and (ii).

5. A brief tour of localization and dévissage

It would be nice if we could compute the K-groups of more rings. For example, we haven't even computed $K(R)$ for a basic ring like $R = \mathbb{Z}[\sqrt{-5}]$. But so far we don't have many techniques to tackle such a computation. An obvious thing to try is to relate the K-groups of R to those of simpler rings made from R, for example quotient rings R/I and localizations $S^{-1}R$. We will start to explore these ideas in the present section. For the moment it will be easier to do this for G-theory, though, rather than K-theory. Note that $R = \mathbb{Z}[\sqrt{-5}]$ is a regular ring, and so $K(R) \cong G(R)$ by Theorem 2.13; hence, the focus on G-groups still gets us what we want in this case.

Let R be a commutative ring and let $f \in R$. Consider the maps

$$G(R/f) \xrightarrow{d_1} G(R) \xrightarrow{d_0} G(f^{-1}R)$$

where $d_1([M]) = [M]$ and $d_0([W]) = f^{-1}W$. Clearly $d_0 \circ d_1 = 0$. We claim that d_0 is also surjective. To see this, let Z be an $f^{-1}R$-module with generators $z_1, \ldots, z_n$. Let $W = R\langle z_1, \ldots, z_n\rangle \subseteq Z$ be the R-submodule generated by the z_i's. Then $f^{-1}W \cong Z$, and so d_0 is surjective.

Theorem 5.1. *When R is Noetherian the sequence*

$$G(R/f) \xrightarrow{d_1} G(R) \xrightarrow{d_0} G(f^{-1}R) \longrightarrow 0$$

is exact.

We will delay the proof of this theorem for the moment, as it is somewhat involved. Let us first look at an example.

Example 5.2. Let $R = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[x]/(x^2+5)$ and $f = (2)$. Note that R is not a PID but $f^{-1}R$ is. Thus $G(f^{-1}R) \cong \mathbb{Z}$. Now we compute

$$R/f = \mathbb{Z}/2[x]/(x^2+5) = \mathbb{Z}/2[x]/(x^2+1) = \mathbb{Z}/2[x]/((x+1)^2) \cong \mathbb{Z}/2[t]/(t^2).$$

We calculated in example (7) from Section 2 that $G(\mathbb{Z}/2[t]/(t^2)) \cong \mathbb{Z}$ and is generated by the module $\mathbb{Z}/2$ with t acting as zero. Translated into the present situation, we are saying $G(R/f) \cong \mathbb{Z}$ with the group being generated by $R/(2, x+1) = R/(2, 1+\sqrt{-5})$.

We have computed that the exact sequence from Theorem 5.1 has the form

$$\mathbb{Z} \xrightarrow{d_1} G(R) \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

where $d_1(1) = [R/(2, 1+\sqrt{-5})]$ and $d_0([R]) = 1$. Let $I = (2, 1+\sqrt{-5})$ and notice that $G(R)$ is generated by $[R]$ and $[R/I]$.

Now look at the short exact sequence $0 \longrightarrow K \longrightarrow R^2 \xrightarrow{\phi} I \longrightarrow 0$ where $\phi(e_1) = 2$, $\phi(e_2) = 1+\sqrt{-5}$, and $K = ker(\phi) = \{(x, y) \,|\, 2x + (1+\sqrt{-5})y = 0\}$. In example (2) from Section 4 we saw that $K \cong I$. So we have $[I] + [I] = [R^2]$ in $G(R)$, or $2([R] - [I]) = 0$. But $[R] - [I] = [R/I]$, hence $2[R/I] = 0$. It follows that $G(R)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, depending on whether the class $[R/I] = [R] - [I]$ is zero or not.

Now use that R is regular, so that $G(R) \cong K(R)$. Recall that we saw in Example 4.9 that $\tilde{K}(R) \neq 0$, or equivalently $K(R) \neq \mathbb{Z}$. In fact we saw precisely that $[R] - [I]$ is not zero in $K(R)$. We conclude that $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, with generators $[R]$ and $[R/I]$ for each of the two summands.

Remark 5.3. Theorem 5.1 gives another parallel between $G(-)$ and singular homology. If $X = \operatorname{Spec} R$ then $A = \operatorname{Spec} R/f$ is a closed subscheme, and $\operatorname{Spec} f^{-1}R = X - A$ is the open complement. So the sequence in Theorem 5.1 can be written as

$$G(A) \to G(X) \to G(X - A) \to 0.$$

This is somewhat reminiscent of the long exact sequence in singular homology $\cdots \to H_*(A) \to H_*(X) \to H_*(X, A) \to \cdots$ but with some important differences. One obvious difference is that our sequence does not yet extend to the left to give a long exact sequence, but that turns out to be just a lack of knowledge on our part: we will eventually see that there are 'higher G-groups' completing the picture. The other evident difference is the presence of $G(X - A)$ as the 'third term' in the long exact sequence, rather than a relative group $G(X, A)$. There are several things to say about this that would be a distraction to delve into at the moment, but perhaps the most relevant is that $H_*(-)$ is really the wrong analogy to be looking at. If we instead consider Borel-Moore homology, then there are indeed long exact sequences that look like $\cdots \to H_*^{BM}(A) \to H_*^{BM}(X) \to H_*^{BM}(X - A) \to \cdots$

Remark 5.4. It is important in Theorem 5.1 that we are using G-theory rather than K-theory. In K-theory we have maps $K(R) \to K(R/f)$ and $K(R) \to K(f^{-1}R)$, both given by tensoring, but in neither case do we have an evident 'third group' that might form an exact sequence. In essence this is because we need relative K-groups; we will start to encounter these in the next section.

We will now work towards proving Theorem 5.1. The proof is somewhat involved, and the result is actually not going to be used much in the rest of the notes. But the proof is very interesting, as it demonstrates many general issues that arise in the subject of K-theory. So it is worth spending time on this.

The proof comes in two parts. For the first part, let us introduce the multiplicative system $S = \{1, f, f^2, f^3, \dots\}$. Write

$$G(M \,|\, S^{-1}M = 0)$$

for the Grothendieck group of all finitely-generated R-modules M such that $S^{-1}M = 0$. The notation is a little slack, but it is very convenient. There are evident maps

$$G(M \,|\, S^{-1}M = 0) \to G(R) \to G(S^{-1}R) \to 0,$$

and we will prove that this is exact for *any* multiplicative system S. This is called the **localization sequence** for G-theory.

The second step is to notice that if M is an R/f-module then as an R-module it has the property that $S^{-1}M = 0$. So we have a map

$$G(R/f) \to G(M|S^{-1}M = 0). \tag{5.5}$$

If M is an arbitrary finitely-generated R-module, the condition $S^{-1}M = 0$ just says that M is killed by a power of f. So we would have a filtration

$$M \supseteq fM \supseteq f^2M \supseteq \cdots \supseteq f^NM = 0$$

where the factors are all R/f-modules. This shows that the map in (5.5) is surjective, and in fact these ideas allow one to define an inverse. The fact that

$$G(R/f) \cong G(M|S^{-1}M = 0)$$

is an example of a general principle known as **dévissage**. When we come to prove this in a moment we will develop the generalization and get a better understanding of what is going on here.

So those are the two pieces for the proof of Theorem 5.1: a general localization sequence where the third term is something we had not considered before—in essence, a relative G-group—and a dévissage theorem identifying that third term with something more familiar.

5.6. **The localization sequence.** To begin with we will need some basic facts about the localization functor $\gamma\colon R - \mathrm{Mod} \to S^{-1}R - \mathrm{Mod}$. Note that if M and N are R-modules then the map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ factors through the S-localization to give

$$S^{-1}\,\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N). \tag{5.7}$$

By extension the same is true for the maps $\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N)$, giving us

$$S^{-1}\,\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N). \tag{5.8}$$

Proposition 5.9. *Let R be a commutative ring and let M and N be R-modules. If M is finitely-presented then the map from (5.7) is an isomorphism. If M has a resolution by finitely-generated projective modules then the map from (5.8) is an isomorphism.*

Proof. First observe that (5.7) is readily checked to be an isomorphism when M is free and finitely-generated. If we have a finite presentation $F_1 \to F_0 \to M \to 0$ then consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & S^{-1}\,\mathrm{Hom}_R(M,N) & \longrightarrow & S^{-1}\,\mathrm{Hom}_R(F_0,N) & \longrightarrow & S^{-1}\,\mathrm{Hom}_R(F_1,N) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}_T(S^{-1}M, S^{-1}N) & \to & \mathrm{Hom}_T(S^{-1}F_0, S^{-1}N) & \to & \mathrm{Hom}_T(S^{-1}F_1, S^{-1}N)
\end{array}$$

where we write $T = S^{-1}R$ for typographical reasons and where each vertical map is an instance of (5.7). The top row is exact by the left exactness of $\mathrm{Hom}_R(-,N)$ together with the exactness of S-localization. The bottom row is exact by the exactness of S-localization and then the left exactness of $\mathrm{Hom}_T(-, S^{-1}N)$. The two vertical maps on the right are isomorphisms because the F_i are free and finitely-generated. So the left vertical map is also an isomorphism.

Note that the map from (5.7) is also an isomorphism when M is a finitely-generated projective, either using the fact that projectives are retracts of free modules or by the fact that finitely-generated projectives are automatically finitely-presented.

For the Ext-statement, choose a projective resolution $P_* \to M$ where the P_i are all finitely-generated. Then $S^{-1}P_* \to S^{-1}M$ is a free resolution over $S^{-1}R$, and so

$$\begin{aligned}\operatorname{Ext}^*_{S^{-1}R}(S^{-1}M, S^{-1}N) &= H^*(\operatorname{Hom}_{S^{-1}R}(S^{-1}P, S^{-1}N)) \\ &= H^*(S^{-1}\operatorname{Hom}_R(P,N)) \\ &= S^{-1}H^*(\operatorname{Hom}_R(P,N)) = S^{-1}\operatorname{Ext}^*_R(M,N).\end{aligned}$$

Here the equalities are really canonical isomorphisms, and the second equality is the Hom-isomorphism we have already proven. □

Exercise 5.10. For the verification that (5.7) is an isomorphism when M is finitely-generated and free, think through where the finite-generation hypothesis is needed.

Corollary 5.11. *Let R be Noetherian and let $S \subseteq R$ be a multiplicative system. In each of the parts below the modules are always assumed to be finitely-generated.*

(a) For any $S^{-1}R$-module W there exists an R-module A and an isomorphism $S^{-1}A \cong W$.

(b) For any R-modules A_1 and A_2 and map of $S^{-1}R$-modules $f\colon S^{-1}A_1 \to S^{-1}A_2$, there exists a map of R-modules $g\colon A_1 \to A_2$ and a diagram of $S^{-1}R$-modules

$$\begin{array}{ccc} S^{-1}A_1 & \xrightarrow{S^{-1}g} & S^{-1}A_2 \\ \| & & \downarrow\cong \\ S^{-1}A_1 & \xrightarrow{f} & S^{-1}A_2. \end{array}$$

Note that the right vertical map need not be the identity.

(c) For any short exact sequence of $S^{-1}R$-modules $0 \to W_1 \to W_2 \to W_3 \to 0$, there exists a short exact sequence of R-modules $0 \to A_1 \to A_2 \to A_3 \to 0$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{-1}A_1 & \longrightarrow & S^{-1}A_2 & \longrightarrow & S^{-1}A_3 & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & 0. \end{array}$$

Proof. We saw the technique for (a) already, in the paragraph prior to the statement of Theorem 5.1: pick a set of generators $w_1, \ldots, w_k$ for W as an $S^{-1}R$-module, and let A be the R-linear span of those generators. The inclusion $A \hookrightarrow W$ gives an inclusion $S^{-1}A \hookrightarrow S^{-1}W$ which is surjective, hence an isomorphism.

Parts (b) and (c) are direct consequences of the surjectivity of the Hom- and Ext^1-maps from Proposition 5.9. □

Remark 5.12. The proofs of Corollary 5.11(b) and (c) are very slick, and sometimes it can be good to also think through more concretes approaches. To this end, here is a sketch of an alternative proof.

For (b), choose presentations $F_1 \to F_0 \to A_1 \to 0$ and $G_1 \to G_0 \to A_2 \to 0$ where the F_i and G_i are finitely-generated and free. The map f lifts to a map of complexes $\tilde{f}\colon S^{-1}F \to S^{-1}G$. Represent the maps $\tilde{f}_0$ and $\tilde{f}_1$ by matrices, and choose $t \in S$ large enough to clear the denominators for all entries at once. Since

the matrices $t\tilde{f}_i$ have entries in R we may regard them as maps $F_i \to G_i$. Consider the square

$$\begin{array}{ccc} F_1 & \xrightarrow{t\tilde{f}_1} & G_1 \\ \downarrow & & \downarrow \\ F_0 & \xrightarrow{t\tilde{f}_0} & G_0. \end{array}$$

This square need not commute, but it commutes after S-localization: so there exists some $u \in S$ such that multiplying the two horizontal maps by u makes the square commutative. Now redefine t to be tu, so that the above square truly does commute. Taking the induced map of cokernels now gives a map $g\colon A_1 \to A_2$. In the diagram

$$\begin{array}{ccccc} S^{-1}F_1 & \xrightarrow{t\tilde{f}_1} & S^{-1}G_1 & \xrightarrow{t^{-1}} & S^{-1}G_1 \\ \downarrow & & \downarrow & & \downarrow \\ S^{-1}F_0 & \xrightarrow{t\tilde{f}_0} & S^{-1}G_0 & \xrightarrow{t^{-1}} & S^{-1}G_0 \\ \downarrow & & \downarrow & & \downarrow \\ S^{-1}A_1 & \xrightarrow{S^{-1}g} & S^{-1}A_2 & \xrightarrow[\cong]{t^{-1}} & S^{-1}A_2 \end{array}$$

the composition of the first two rows are $\tilde{f}_1$ and $\tilde{f}_0$, so the composition of the bottom row is f. This completes the proof for (b).

The proof of (c) is along similar lines. Choose A_1 and A_3 such that $S^{-1}A_1 \cong W_1$ and $S^{-1}A_3 \cong W_3$. Then choose free presentations $F_1 \to F_0 \to A_1$ and $G_1 \to G_0 \to A_3$. Use the Horseshoe Lemma from homological algebra [Weibel1, 2.2.8] to create a free presentation for W_2 of the form $S^{-1}(F_1 \oplus G_1) \to S^{-1}(F_0 \oplus G_0)$, sitting in the middle of a short exact sequence with the previous two. Play the same game as in (b) with clearing denominators on the maps in this presentation, so that one can lift to a diagram of R-modules. There are a couple of slightly tricky points to think through, but one can in this way create the desired short exact sequence of R-modules (though not necessarily with the originally chosen A_1). We leave the reader to ponder the details here.

With these basic results about localization out of the way, we can now derive some consequences for K-theory.

Corollary 5.13. *When R is Noetherian the following subgroups of $G(R)$ are all equal:*
(1) $\big\langle [A]-[B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle$,
(2) $\big\langle [A]-[B] \,\big|\, \text{there exists a map } f\colon A \to B \text{ such that } S^{-1}f \text{ is an isomorphism} \big\rangle$,
(3) $\big\langle [J] \,\big|\, S^{-1}J = 0 \big\rangle$.

Proof. Let H_1, H_2, and H_3 be the subgroups listed in (1)–(3). Clearly $H_1 \supseteq H_2 \supseteq H_3$. The opposite subset $H_1 \subseteq H_2$ follows directly from Corollary 5.11(b). To prove $H_2 \subseteq H_3$, let $f\colon A \to B$ be a map of finitely-generated R-modules such that $S^{-1}f$ is an isomorphism. Consider the short exact sequence

$$0 \to \ker f \to A \to B \to \operatorname{coker} f \to 0,$$

and note that our hypothesis implies that $S^{-1}(\ker f) = 0 = S^{-1}(\operatorname{coker} f)$. But $[A]-[B] = [\ker f] - [\operatorname{coker} f]$ in $G(R)$, so we have that $[A]-[B] \in H_3$. $\square$

Proposition 5.14. *Let R be Noetherian and let $S \subseteq R$ be a multiplicative system. The sequence*

$$G(M|S^{-1}M = 0) \xrightarrow{a} G(R) \xrightarrow{b} G(S^{-1}R) \to 0$$

is exact, where a and b are the evident maps.

Proof. Part (a) of Corollary 5.11 gives surjectivity of b. The somewhat tricky thing is to get the exactness in the middle. Let $\mathcal{F}(R)$ denote the free abelian group on isomorphism classes of finitely-generated R-modules, and let $\mathcal{R}el(R) \subseteq \mathcal{F}(R)$ denote the subgroup generated by elements $[M_i'] + [M_i''] - [M_i]$ for short exact sequences $0 \to M_i' \to M_i \to M_i'' \to 0$. Note that $[0] \neq 0$ in $\mathcal{F}(R)$; we could have imposed this as an extra condition, but it is slightly more convenient to not do so. Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}el(R) & \longrightarrow & \mathcal{F}(R) & \longrightarrow & G(R) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi|_{\mathcal{R}el}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & \mathcal{R}el(S^{-1}R) & \longrightarrow & \mathcal{F}(S^{-1}R) & \longrightarrow & G(S^{-1}R) & \longrightarrow & 0,
\end{array}$$

which we wish to regard as a short exact sequence of chain complexes (the columns become chain complexes by adding zeros above and below). Corollary 5.11(a) gives surjectivity of π, and Corollary 5.11(c) gives surjectivity of $\pi|_{\mathcal{R}el}$. The long exact sequence in homology then becomes

$$0 \to \ker(\pi|_{\mathcal{R}el}) \to \ker(\pi) \to \ker b \to 0. \tag{5.15}$$

We next analyze the kernel of π.

Assume that $x \in \ker(\pi)$. One can write x in the form

$$x = \Big([M_1] + [M_2] + \cdots + [M_k]\Big) - \Big([J_1] + \cdots + [J_l]\Big)$$

for some modules $M_1, \ldots, M_k, J_1, \ldots, J_l$. We then have

$$0 = \pi(x) = \Big([S^{-1}M_1] + [S^{-1}M_2] + \cdots + [S^{-1}M_k]\Big) - \Big([S^{-1}J_1] + \cdots + [S^{-1}J_l]\Big)$$

in $\mathcal{F}(S^{-1}R)$. How can this happen? It can only be that $k = l$ and that for each module $S^{-1}M_j$ there is some i for which $S^{-1}M_j \cong S^{-1}J_i$. By pairing the terms up two by two we find that

$$x \in \big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle \subseteq \mathcal{F}(R).$$

So $\ker \pi = \big\langle [A] - [B] \,\big|\, S^{-1}A \cong S^{-1}B \big\rangle$. It then follows from (5.15) and Corollary 5.13 that

$$\ker b = \langle [J] \,|\, S^{-1}J = 0 \rangle \subseteq G(R).$$

This is what we wanted to prove. □

5.16. **Dévissage.** Now we move to the second stage of the proof of Theorem 5.1. We can rephrase what needs to be shown as saying that the map

$$G(M|M \text{ is killed by } f) \to G(M|M \text{ is killed by a power of } f)$$

is an isomorphism. We have seen a baby version of this argument before, namely back in Section 2 when we showed that

$$G(\mathbb{Z}/p) \to G(\mathbb{Z}/p^2) \quad \text{and} \quad G(F) \to G(F[t]/(t^2))$$

are both isomorphisms. These are both maps of the form

$$G(M|M \text{ is killed by } f) \to G(M|M \text{ is killed by } f^2),$$

for the rings $R = \mathbb{Z}$ and $R = F[t]$, respectively. Iterating the same idea we used to prove these—filter by powers of f—allows one to prove the required generalization. But while we're at it, let us generalize even further. The following result is stated in the context of abelian categories; for a review of those, see Appendix D. It is fine to just restrict to categories of modules, though.

Theorem 5.17 (Dévissage)**.** *Let $\mathcal{B}$ be an abelian category, and let $\mathcal{A} \hookrightarrow \mathcal{B}$ be an abelian subcategory that is closed under taking subobjects and quotients in $\mathcal{B}$. Suppose that every object in $\mathcal{B}$ has a finite filtration whose factors are in $\mathcal{A}$. Then $G(\mathcal{A}) \to G(\mathcal{B})$ is an isomorphism.*

Proof. The inclusion $i\colon \mathcal{A} \to \mathcal{B}$ induces a map $\alpha\colon G(\mathcal{A}) \to G(\mathcal{B})$, and we want to define an inverse $\beta\colon G(\mathcal{B}) \to G(\mathcal{A})$. To do so, for $M \in \mathcal{B}$ choose a filtration

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0,$$

whose quotients M_i/M_{i+1} are in $\mathcal{A}$, and define

$$\beta([M]) = \sum [M_i/M_{i+1}].$$

We must check that β is well-defined, because it seems to depend on the choice of filtration. First observe that our formula gives the same class in $G(\mathcal{A})$ if we refine the filtration: if we replace one of the links $M_i \supseteq M_{i+1}$ with a longer chain $M_i \supseteq M_i^1 \supseteq \cdots \supseteq M_i^r = M_{i+1}$, then we get the same class in $G(\mathcal{A})$ by Proposition 2.5(b) (or really, its analog in the present setting).

Now recall Schreier's Refinement Theorem, which we discussed prior to Remark 3.12. This says that given any two filtrations of M we can refine each one so that the two refinements have the same quotients up to permutation. Together with what has already been shown, this implies that $\beta([M])$ does not depend on the choice of filtration. It is a simple exercise to prove that β is additive, which we leave to the reader.

At this point we have the map $\beta\colon G(\mathcal{B}) \to G(\mathcal{A})$. It is immediate that $\beta\alpha = \mathrm{id}$ and $\alpha\beta = \mathrm{id}$. □

Sometimes in mathematics the hard work goes into showing that something is well-defined, and afterwards the rest is easy. This was the case for the Dévissage Theorem, where all the work went into constructing the map β.

5.18. **Recap and summary.** We embarked on the above journey in order to prove Theorem 5.1, so let us now come back to that.

Proof of Theorem 5.1. Recall that R is a Noetherian ring and $f \in R$. Let S be the multiplicative system $\{f^i \mid i \geq 0\}$. By Proposition 5.14 we have an exact sequence

$$G(M \mid S^{-1}M = 0) \longrightarrow G(R) \longrightarrow G(S^{-1}R) \longrightarrow 0.$$

The inclusion of finitely-generated R/fR-modules into the category of finitely-generated modules M such that $S^{-1}M = 0$ satisfies the hypotheses of the dévissage theorem (Theorem 5.17), by looking at the filtration $M \supseteq fM \supseteq f^2M \supseteq \cdots$. The conditions that $S^{-1}M = 0$ and M is finitely-generated guarantee that $f^kM = 0$

for some k, so that this filtration is finite. Therefore we have an isomorphism $G(R/fR) \cong G(M \mid S^{-1}M = 0)$, allowing us to write our exact sequence as

$$G(R/fR) \longrightarrow G(R) \longrightarrow G(S^{-1}R) \longrightarrow 0.$$

The composite $G(R/fR) \to G(R)$ is the evident map that regards an R/fR-module as an R-module. □

◦ Exercises ◦

Exercise 5.19. Let R be Noetherian and let $I \subseteq R$ be a nilpotent ideal. Use dévissage to prove that $G(R/I) \to G(R)$ is an isomorphism.

Exercise 5.20. Let k be a field and let $R = k[x, y, z]/(xy - z^n)$. Use the localization sequence $G(R/xR) \to G(R) \to G(x^{-1}R) \to 0$ to help prove that $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/n$. [The localization sequence will get you two generators. To determine the order of the torsion generator you will have to do some extra legwork, as in Section 3.]

Exercise 5.21. Let k be a field and let $R = k[x, y]/(xy)$. Prove that $G(R) \cong \mathbb{Z}^2$.

Exercise 5.22. Let R be a Noetherian ring. Prove that $G(R[t, t^{-1}]) \cong G(R)$.

Exercise 5.23. Let $M \subseteq \mathbb{Z}^r$ be a submonoid, let k be a field, and consider the monoid ring $k[M] \subseteq k[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$. This is a rich source of interesting examples. These rings are domains, and it is known that they are normal provided that M satisfies the following property: If $\mathbb{Z}M$ denotes the lattice generated by M, then $v \in \mathbb{Z}M$ and $Nv \in M$ for some $N > 0$ implies $v \in M$. See [BH, Theorem 6.1.4].

In this exercise we will take $r = 2$ and $M = \{(u, v) \in \mathbb{N}^2 \mid u + v \equiv 0 \pmod 3\}$. Our ring is $R = k[x^3, x^2y, xy^2, y^3] \subseteq k[x, y]$. This is a 2-dimensional normal domain (the above condition for normality is readily verified), and it can also be written as

$$R = k[a, b, c, d]/(ac - b^2, bc - ad, c^2 - bd).$$

Show that $G(R) \cong \mathbb{Z} \oplus \mathbb{Z}/3$ via the following steps: (1) use localization sequences to find a prime P such that $[R]$ and $[R/P]$ generate the groups. (2) Use prime filtrations to produce the relation $3[R/P] = 0$. (3) Show that $[R/P] \neq 0$ by using $\mathrm{CH}_1(R)$ and showing that P is not principal (see Exercise 3.39(f)). The fact that R is a graded ring might be helpful for some parts. [For further exploration, let us mention that the book [BG] contains a wealth of material on these $k[M]$ rings.]

Exercise 5.24. Read about abelian categories, Serre subcategories, and Serre quotient categories in Appendix D. If $\mathcal{S}$ is a Serre subcategory of an abelian category $\mathcal{A}$, prove that $G(\mathcal{S}) \to G(\mathcal{A}) \to G(\mathcal{A}/\mathcal{S}) \to 0$ is exact.

6. K-theory of complexes and relative K-theory

Recall that there is always a map $K(R) \to G(R)$ sending the K-class of a projective to the G-class of the same projective. We proved in Theorem 2.13 that when R is regular this map is an isomorphism, and we did this by constructing the inverse: it sends a class $[M]$ to $\sum(-1)^i[P_i]$, where $P_\bullet \to M$ is any bounded resolution of M by finitely-generated projectives. If you go back and examine the proof of that result, you might notice that the alternating sums are largely an *annoyance* in the proof—all the key ideas are best expressed without them, and

they are only forced into the proof so that we get actual elements of $K(R)$. If you think about this enough, it might eventually occur to you to try to make a definition of $K(R)$ that uses chain complexes instead of modules, thus eliminating the need for these alternating sums. We will show how to do this in the present section.

The importance of using chain complexes extends much further than simply changing language to simplify a proof. We will see that defining K-theory in terms of complexes allows us to write down natural definitions for relative K-groups as well.

Throughout this section let R be a fixed commutative ring. By a *bounded* chain complex over R we mean a chain complex which is nonzero in only finitely-many degrees. We begin by making the following definition.

Definition 6.1.

$$K^{cplx}(R) = \frac{\mathbb{Z}\langle\, [P_\bullet] \,\big|\, P_\bullet \text{ is a bounded chain complex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[P_\bullet] = [P'_\bullet]$ *if* $P_\bullet$ *and* $P'_\bullet$ *are chain homotopy equivalent,*

(2) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ *if there is a short exact sequence* $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

The second relation is the one that by now we would expect in a K-group, but the first relation is new to us. If one goes back and thinks about the proof of Theorem 2.13, the need for this first relation quickly becomes clear: it guarantees, for instance, that two projective resolutions of a module will represent the same class in the K-group.

Regarding relation (1), let us introduce some common terminology:

Definition 6.2. *A map of chain complexes* $C_\bullet \to D_\bullet$ *is a* ***quasi-isomorphism*** *if the induced maps* $H_i(C_\bullet) \to H_i(D_\bullet)$ *are isomorphisms for all* $i \in \mathbb{Z}$. *Two chain complexes* $C_\bullet$ *and* $D_\bullet$ *are* ***quasi-isomorphic****, written* $C_\bullet \simeq D_\bullet$, *if there is a zig-zag of quasi-isomorphisms*

$$C_\bullet \xrightarrow{\sim} J^1_\bullet \xleftarrow{\sim} J^2_\bullet \xrightarrow{\sim} \cdots \xrightarrow{\sim} J^n_\bullet \xleftarrow{\sim} D_\bullet.$$

The following proposition is basic homological algebra. We omit the proof, but it is very similar to the proof that two projective resolutions of the same module are chain homotopy equivalent.

Proposition 6.3. *If* P *and* Q *are bounded below complexes of projectives, then every quasi-isomorphism* $P \to Q$ *is a chain homotopy equivalence.*

Exercise 6.4. Prove Proposition 6.3, or look up a proof in a book on homological algebra.

Proposition 6.3 lets us replace the words "chain homotopy equivalence" with "quasi-isomorphism" in any statement about bounded, projective complexes. In particular, we do this in relation (1) from the definition of $K^{cplx}(R)$. The advantage of doing this is simply that quasi-isomorphisms are somewhat easier to identify than chain homotopy equivalences.

Here is our main result concerning the K-theory of complexes:

Proposition 6.5. *$K(R) \cong K^{cplx}(R)$ for any commutative ring R.*

Before giving the proof we record two useful results. For any chain complex C, recall that ΣC denotes the chain complex obtained by shifting every module up one degree and adding a sign to all differentials: $(\Sigma C)_n = C_{n-1}$, and $d_{\Sigma C} = -d_C$. Also, if $f\colon A \to B$ is a map of chain complexes then the mapping cone Cf is the chain complex with $(Cf)_n = A_{n-1} \oplus B_n$ and $d_{Cf}(a,b) = (-da + f(b), db)$. Note that ΣA is the mapping cone of $A \to 0$.

Lemma 6.6. *Let P and Q be bounded complexes of finitely-generated projectives.*
(a) $[\Sigma P] = -[P]$ in $K^{cplx}(R)$.
(b) If $f\colon P \to Q$ is a map of complexes then $[Cf] = [Q] - [P]$ in $K^{cplx}(R)$.

Proof. Recall that there is a short exact sequence of complexes

$$0 \to Q \hookrightarrow Cf \to \Sigma P \to 0,$$

which shows immediately that $[Cf] = [Q]+[\Sigma P]$ in $K^{cplx}(R)$. Let T be the mapping cone of the identity $P \to P$. Note that T is exact, hence quasi-isomorphic to the zero complex. So $0 = [T] = [P] + [\Sigma P]$, from which we get $[\Sigma P] = -[P]$. It then follows that $[Cf] = [Q] + [\Sigma P] = [Q] - [P]$. □

Exercise 6.7. Prove that if relation (1) in the definition of $K^{cplx}(R)$ is replaced with

(1') $\quad [P_\bullet] = 0$ for every exact complex $P_\bullet$,

then the resulting quotient group is also equal to $K^{cplx}(R)$.

Exercise 6.8. Sometimes the sign on the differential is omitted in the definition of the suspension ΣC. Check that the chain complexes obtained from these two different conventions are naturally isomorphic. (For this reason, occasionally we will be sloppy with the sign on the differential).

We now have enough tools to prove the main result of this section:

Proof of Proposition 6.5. If P is a projective R-module, let $P[n]$ denote the chain complex that has P in degree n and in all other degrees is equal to 0. There is an obvious map $\alpha\colon K(R) \to K^{cplx}(R)$ defined by

$$[P] \mapsto \big[P[0]\big].$$

It is somewhat less obvious, but one can define a map in the other direction $\beta\colon K^{cplx}(R) \to K(R)$ by

$$\beta\Big([P_\bullet]\Big) = \sum (-1)^i [P_i].$$

To see that this is well-defined we need to check that it respects the two defining relations for $K^{cplx}(R)$. Relation (2) is obvious, but for the other relation it is convenient to use Exercise 6.7 to replace (1) with (1'). The fact that β respects (1') is immediate, being a consequence of Exercise 2.11.

It is clear that $\beta \circ \alpha = \mathrm{id}$, so α is injective and β is surjective. To finish the proof, it is easiest to prove that α is surjective; we will do this in several steps. If P is a finitely-generated projective then $P[0]$ is obviously in the image of α, and we know that $P[n] = (-1)^n[P[0]]$ by iterated application of Lemma 6.6(a). So $P[n] \in \operatorname{im}\alpha$ for all $n \in \mathbb{Z}$. Said differently, any complex of projectives of length 0 belongs to the

image of α. We next extend this to all bounded complexes by an induction on the length.

Let $P_\bullet$ be a bounded complex of finitely-generated projectives, bounded between degrees k and $n+k$, say. Then $P_k[k]$ is a subcomplex of $P_\bullet$, and the quotient $Q_\bullet$ has length at most $n-1$. We have $[P_\bullet] = [P_k[k]] + [Q_\bullet]$, and both $[P_k[k]]$ and $[Q_\bullet]$ belong to $\operatorname{im}\alpha$ by induction. So $[P_\bullet] \in \operatorname{im}\alpha$, and we are done. □

We will use our identification of $K^{cplx}(R)$ and $K(R)$ implicitly from now on. For example, if P is a bounded complex of projectives we will often write $[P]$ to denote the element of $K(R)$ that would more precisely be written as $\beta([P])$.

Exercise 6.9. Assume that R is Noetherian, and let $G_{\mathrm{fpd}}(R)$ denote the Grothendieck group of finitely-generated modules having finite projective dimension. Prove that $K^{cplx}(R) \cong G_{\mathrm{fpd}}(R)$.

Exercise 6.10. If P is a bounded-below complex of projectives then the functor $(-)\otimes P$ preserves short exact sequences of chain complexes. Using this, show that the tensor product of chain complexes gives a ring structure on $K^{cplx}(R)$ and that the isomorphism $K(R) \cong K^{cplx}(R)$ is an isomorphism of rings.

6.11. **G-theory and chain complexes.** One can prove an analog of Proposition 6.5 in which the 'projective' hypothesis is left out everywhere, showing that $G(R)$ is isomorphic to a Grothendieck group made from bounded chain complexes of arbitrary finitely-generated modules. Here the Grothendieck group of complexes must be defined using relation (1') instead of (1), though, because they are no longer equivalent. Other than this small point, all of the arguments are the same.

What is more interesting, however, is a variant that again uses chain complexes of projectives. For this assume R is Noetherian. Consider chain complexes $P_\bullet$ such that

(i) Each P_i is a finitely-generated projective,
(ii) $P_\bullet$ is bounded-below, in the sense that $P_i = 0$ for all $i \ll 0$.
(iii) $P_\bullet$ has bounded homology, in the sense that $H_i(P) \neq 0$ only for finitely many values of i.

Start with the free abelian group on isomorphism classes of such complexes, and define $G^{cplx}(R)$ to be the quotient by the analogs of relations (1) and (2) (or equivalently, (1') and (2)) in the definition of $K^{cplx}(R)$.

Note that one readily obtains maps $\alpha\colon G^{cplx}(R) \to G(R)$ and $\beta\colon G(R) \to G^{cplx}(R)$ by

$$\alpha([P_\bullet]) = \sum_i (-1)^i [H_i(P)] \quad \text{and} \quad \beta([M]) = [Q_\bullet]$$

where $Q_\bullet \to M$ is any resolution by finitely-generated projectives. The fact that α respects the short-exact-sequence relation follows from the Snake Lemma.

Proposition 6.12. *When R is Noetherian the maps α and β give inverse isomorphisms $G^{cplx}(R) \cong G(R)$.*

Proof. It is immediate that $\alpha\beta = \mathrm{id}$, so that α is surjective and β is injective. The proof will be completed by showing that β is surjective. Let $P_\bullet$ be a bounded-below, homologically bounded chain complex of finitely-generated projectives. We will prove by induction on the number of nonzero homology groups of $P_\bullet$ that

$[P_\bullet] \in \operatorname{im}\beta$. The base is trivial, for if all the homology groups are zero then $P_\bullet \simeq 0$ and so $[P_\bullet] = 0$.

Without loss of generality assume that $P_i = 0$ for $i < 0$. Let n be the smallest integer for which $H_n(P) \neq 0$. If $n > 0$ then $P_1 \to P_0$ is surjective, so there exists a splitting. Using this splitting one sees that P is quasi-isomorphic to a chain complex concentrated in degrees strictly larger than zero. Repeating this argument if necessary, one concludes that P is actually quasi-isomorphic to a chain complex (of f.g. projectives) concentrated in degrees n and higher. So we may assume that P has this property, and then by shifting indices we may assume $n = 0$.

Let $Q_\bullet \to H_0(P)$ be a resolution by finitely-generated projectives (this exists because R is Noetherian). Standard homological algebra gives us a map $f\colon P_\bullet \to Q_\bullet$ inducing an isomorphism on H_0. Let Cf be the mapping cone of f. The long exact homology sequence shows that Cf has one fewer non-vanishing homology group than P, and hence we may assume by induction that $[Cf] \in \operatorname{im}\beta$. But we know from Lemma 6.6 (really, its analog for $G^{cplx}(R)$) that $[Cf] = [Q] - [P]$. Since $[Q] \in \operatorname{im}\beta$ by the definition of β, it follows that $[P] \in \operatorname{im}\beta$ as well. □

When we first learned the definitions of $K(R)$ and $G(R)$, the difference seemed to be about projective versus arbitrary modules. When we look at these groups as $K^{cplx}(R)$ and $G^{cplx}(R)$, however, the difference is about bounded versus bounded-below chain complexes.

6.13. **Relative K-theory.** It may seem like we have brought in an unnecessary level of complexity (no pun intended) by introducing $K^{cplx}(R)$. After all, the proof of Proposition 6.5 shows that for any bounded complex P the class $[P]$ is just the alternating sum $\sum(-1)^i[P_i[0]]$. That is, in $K^{cplx}(R)$ we may decompose any complex into its constituent modules; one really only needs modules, not chain complexes. But we will get some mileage out of these ideas by defining similar K-groups but restricting to complexes *subject to certain conditions*. In these cases we might not be able to 'unravel' the complexes anymore. We give a few examples: in each case we describe a type of chain complex, and the associated group is always defined by taking the free abelian group on isomorphism classes and then quotienting by the analogs of relations (1) and (2) used to define $K^{cplx}(R)$.

(i) Let S be a multiplicative system in R. Define $K(R, S)$ using bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that $S^{-1}P_\bullet$ is exact.

(ii) Let $I \subseteq R$ be an ideal. Define $K(R)_I$ using bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that each $H_k(P)$ is annihilated by I.

(iii) Again let $I \subseteq R$ be an ideal. Define $K(R, I)$ using bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that $P_\bullet \otimes_R R/I$ is exact.

(iv) Fix an $n \geq 0$. Define $K(R, \leq n)$ using bounded complexes $P_\bullet$ of finitely-generated projectives *having the property* that each $H_k(P)$ has Krull dimension at most n. (Recall that the dimension of a module M is the dimension of the ring $R/\operatorname{Ann}(M)$).

Exercise 6.14. In analogy to (iv), define a group $K(R, \geq n)$. If $n > \dim R$ observe that $K(R, \geq n) = 0$. If R is a domain and $n \leq \dim R$, prove that $K(R, \geq n) \cong K(R)$. [Note: By convention the dimension of the zero module is $+\infty$.]

Exercise 6.15. Verify that the tensor product of chain complexes gives a ring structure on $K(R, S)$.

Every map $f\colon P \to Q$ of finitely-generated projectives can be regarded as a chain complex concentrated in degrees 0 and 1. From now on we will often make this identification without comment. If $S^{-1}f$ is an isomorphism then we get a corresponding class in $K(R, S)$. The following lemma about these classes will be very useful:

Lemma 6.16. *Let $\alpha\colon P \to Q$ and $\beta\colon Q \to W$ be maps between finitely-generated projectives, and assume both become isomorphisms after localization at S. Then*

$$[P \xrightarrow{\beta\alpha} W] = [P \xrightarrow{\alpha} Q] + [Q \xrightarrow{\beta} W]$$

in $K(R, S)$.

Proof. Use the following short exact sequence of maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{(\alpha,\mathrm{id})} & Q \oplus P & \xrightarrow{(\mathrm{id},-\alpha)} & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_Q \oplus \beta\alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & Q & \xrightarrow{(\mathrm{id},\beta)} & Q \oplus W & \xrightarrow{(\beta,-\mathrm{id})} & W & \longrightarrow & 0.
\end{array}$$

This gives that

$$[Q \xrightarrow{\mathrm{id}} Q] + [P \xrightarrow{\beta\alpha} W] = [P \xrightarrow{\alpha} Q] + [Q \xrightarrow{\beta} W],$$

but of course the first term on the left is zero in $K(R, S)$. □

There is an evident map $K(R, S) \to K(R)$ that sends a class $[P]$ in $K(R, S)$ to the similarly-named (but different) class $[P]$ in $K(R)$ (and recall that we identify $K(R)$ and $K^{cplx}(R)$ without comment from now on). In colloquial terms, the map simply 'forgets' that a complex P is S-exact. The composite $K(R, S) \to K(R) \to K(S^{-1}R)$ is clearly zero.

Proposition 6.17. *For any multiplicative system in a commutative ring R the sequence $K(R, S) \to K(R) \to K(S^{-1}R)$ is exact in the middle.*

Proof. Suppose $x \in K(R)$ is in the kernel of the map to $K(S^{-1}R)$. Every element of $K(R)$ may be written as $x = [P] - [Q]$ for some finitely-generated projectives P and Q. Then $[S^{-1}P] = [S^{-1}Q]$ in $K(S^{-1}R)$, so by Proposition 2.9 there exists an n such that $S^{-1}P \oplus (S^{-1}R)^n \cong S^{-1}Q \oplus (S^{-1}R)^n$. Alternatively, write this as $S^{-1}(P \oplus R^n) \cong S^{-1}(Q \oplus R^n)$. By Corollary 5.11(b) there exists a map of R-modules $Q \oplus R^n \to P \oplus R^n$ that becomes an isomorphism after S-localization. Regarding this map as a chain complex concentrated in degrees 0 and 1, it gives an element in $K(R, S)$. The image of this element under $K(R, S) \to K(R)$ is clearly x. □

The reader might have noticed that in the above proof we didn't encounter any kind of complicated chain complex when trying to construct our preimage in $K(R, S)$; in fact, we accomplished everything with chain complexes of length 1. This is a general phenomenon, similar to the fact that elements of $K^{cplx}(R)$ can all be decomposed into modules. For the relative K-groups $K(R, S)$ one can't quite decompose *that* far, but one can always get down to complexes of length 1. To state a theorem along these lines, consider maps $f\colon P \to Q$ where P and Q are finitely-generated R-projectives and $S^{-1}f$ is an isomorphism (it is convenient to regard

such maps as chain complexes concentrated in degrees 0 and 1). Let $K(R,S)_{\leq 1}$ be the quotient of the free abelian group on such maps by the following relations:

(1) $[f] = 0$ if f is an isomorphism;
(2) $[f] = [f'] + [f'']$ if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0 \end{array}$$

where the rows are exact.

Notice that there is an evident map $K(R,S)_{\leq 1} \to K(R,S)$.

Theorem 6.18. *For any multiplicative system S in a commutative ring R, the map $K(R,S)_{\leq 1} \to K(R,S)$ is an isomorphism.*

The proof of this theorem is a bit difficult, and the techniques are too distant from the topics at hand to merit spending time on them. A proof can be found in [FH], with some simplifications in [D4]. A similar result in the topological setting is proven in Section 24 below.

Remark 6.19. Theorem 6.18 naturally suggests the following question: why use chain complexes at all, for relative K-theory? That is to say, if one can access the same groups using only chain complexes of length one, why complicate things by making the definition using complexes of arbitrary length? There are two answers to this question. The first concerns the ring structure: the tensor product of two complexes is again a complex in a natural way, giving a ring structure on $K(R,S)$. In contrast, there is not a particularly natural way of defining a ring structure on $K(R,S)_{\leq 1}$.

The second answer comes from algebraic geometry. Let X be a scheme and let U be an open subset of X. Then the 'correct' way to define a relative K-theory group $K(X,U)$ is to use bounded chain complexes of algebraic vector bundles on X that are exact on U. When $X = \operatorname{Spec} R$ and $U = \operatorname{Spec} S^{-1}R$ then it happens that one can get the same groups using only complexes of length one—as we saw above. But even for $X = \operatorname{Spec} R$ not every open subset is of this form. A general open subset will have the form $U = (\operatorname{Spec} S_1^{-1}R) \cup (\operatorname{Spec} S_2^{-1}R) \cup \cdots \cup (\operatorname{Spec} S_d^{-1}R)$, and to get the same relative K-group here using complexes with a fixed bound on their length the best one can do is to take that bound to be d. See [FH, "Main Theorem"] and [D4, Theorem 1.4] for the proof in this case.

When R is a regular ring all localizations $S^{-1}R$ are also regular. So the groups $K(R)$ and $K(S^{-1}R)$ can be identified with $G(R)$ and $G(S^{-1}R)$, by Theorem 2.13. Comparing the localization sequence in K-theory from Proposition 6.17 to the one in G-theory from Proposition 5.14 suggests an identification of the relative terms. Indeed, observe that the usual Euler characteristic map $\chi(P_\bullet) = \sum (-1)^i [H_i(P)]$ gives a well-defined map $K(R,S) \to G(M \,|\, S^{-1}M = 0)$. We have the following:

Theorem 6.20. *If R is regular then $\chi\colon K(R,S) \to G(M \,|\, S^{-1}M = 0)$ is an isomorphism.*

Proof. The proof repeats the ideas we have already seen in Theorem 2.13, Proposition 6.5, and Proposition 6.12. Define $\beta\colon G(M \mid S^{-1}M = 0) \to K(R,S)$ by sending $[M]$ to $[P_\bullet]$ for some finite resolution of M by finitely-generated projectives (which exists because R is regular). The exact same steps as in the proof of Theorem 2.13 show that this is well-defined, and it is clear that $\chi \circ \beta = \mathrm{id}$. So β is injective and χ is surjective. We finish the proof by showing that β is surjective.

Let P be a bounded complex of finitely-generated projectives such that $S^{-1}P$ is exact. If all the homology groups of P are zero then $[P] = 0$ in $K(R,S)$ and so $[P] \in \mathrm{im}(\beta)$. We proceed by induction on the number of non-vanishing homology groups.

By suspending or desuspending we can assume that $P_i = 0$ for $i < 0$. If $H_0(P) = 0$ then we can split off an acyclic complex from the bottom of P and reduce to a complex of smaller length. So we can also assume $H_0(P) \neq 0$. Let Q be a bounded resolution of $H_0(P)$ by finitely-generated projectives. Then there is a map of complexes $f\colon P \to Q$ that is an isomorphism on H_0, so that the mapping cone Cf has fewer nonvanishing homology groups than P. Since $S^{-1}P$ and $S^{-1}Q$ are exact, $S^{-1}(Cf)$ is also exact. We have $[Q] = [Cf] + [P]$ in $K(R,S)$, $[Q]$ is in the image of β by construction, and by induction $[Cf]$ is also in the image of β. So $[P]$ is in the image, and we are done. □

6.21. **Relative K-theory and intersection multiplicities.** We now wish to tie several themes together, and use everything we have learned so far to give a complete, K-theoretic perspective on Serre's definition of intersection multiplicity. This perspective is from the paper [GS].

Let R be a Noetherian ring, and let $Z \subseteq \operatorname{Spec} R$ be any subset. An R-module M is said to be **supported** on Z if $M_P = 0$ for all primes $P \notin Z$. One usually defines $\operatorname{Supp} M$, the **support** of M, to be $\{P \in \operatorname{Spec} R \mid M_P \neq 0\}$. This is a Zariski closed subset of $\operatorname{Spec} R$, as it equals $V(\operatorname{Ann} M)$. To say that M is supported on Z is just the requirement that $\operatorname{Supp} M \subseteq Z$.

When M is finitely-generated and $Z = V(I)$ is closed, M is supported on Z if and only if a power of I annihilates M. Likewise, if S is a multiplicative system then M is supported on $\operatorname{Spec} R - \operatorname{Spec} S^{-1}R$ if and only if $S^{-1}M = 0$.

Let $G(R)_Z$ be the Grothendieck group of all finitely-generated R-modules that are supported on Z.

Similarly, if $C_\bullet$ is a chain complex of R-modules then $\operatorname{Supp} C$ is defined to be $\{P \in \operatorname{Spec} R \mid H_*(C_P) \neq 0\}$. We say that $C_\bullet$ is supported on Z if $\operatorname{Supp} C \subseteq Z$, or if C_Q is exact for every $Q \notin Z$. Note that $C_\bullet$ is supported on Z if and only if all the homology modules $H_*(C)$ are supported on Z.

Similar to our definitions of $K^{cplx}(R)$ and $K(R,S)$, define $K(R)_Z$ to be the Grothendieck-style group of bounded complexes $P_\bullet$ of finitely-generated projective R-modules having the property that $\operatorname{Supp} P_\bullet \subseteq Z$. Note that if $Z = \operatorname{Spec} R - \operatorname{Spec} S^{-1}R$ then $K(R)_Z$ is precisely the group $K(R,S)$ previously defined.

The following statements should be easy exercises for the reader:

(1) The Euler characteristic $\chi(P_\bullet) = \sum_i (-1)^i [H_i(P)]$ defines a group homomorphism $K(R)_Z \to G(R)_Z$.
(2) If R is regular then the map $\chi\colon K(R)_Z \to G(R)_Z$ is an isomorphism.
(3) Tensor product of chain complexes gives pairings

$$\otimes\colon K(R)_Z \otimes K(R)_W \to K(R)_{Z \cap W}$$

for all pairs of closed subsets $Z, W \subseteq \operatorname{Spec} R$.

(4) If M and N are R-modules then $\operatorname{Supp}(M \otimes N) = \operatorname{Supp} M \cap \operatorname{Supp} N$.

(5) Assume that R is regular and transplant the tensor product of chain complexes from (3) to a pairing

$$G(R)_Z \otimes G(R)_W \to G(R)_{Z \cap W}.$$

This sends $[M] \otimes [N]$ to $\sum (-1)^i [\operatorname{Tor}_i(M,N)]$. (Note that this makes sense on the level of supports: If $Z = V(I)$ and $W = V(J)$ then $Z \cap W = V(I+J)$. If M is killed by a power of I and N is killed by a power of J, then $M \otimes N$ and all the $\operatorname{Tor}_i(M,N)$ are killed by some $I^r + J^s$ and therefore by a power of $I + J$).

(6) Let $Z = \{m\}$ where m is a maximal ideal of R. If a module is supported on Z then it has finite length, and the assignment $M \mapsto \ell(M)$ gives an isomorphism $G(R)_Z \xrightarrow{\cong} \mathbb{Z}$.

(7) Let M and N be R-modules such that $\operatorname{Supp}(M \otimes N) = \{m\}$ where m is a maximal ideal of R (geometrically, $\operatorname{Supp} M$ and $\operatorname{Supp} N$ have an isolated point of intersection). Then Serre's intersection multiplicity $e(M,N)$ is the image of $[M] \otimes [N]$ under the composite

$$G(R)_Z \otimes G(R)_W \longrightarrow G(R)_{Z \cap W} \xrightarrow{\ell} \mathbb{Z},$$

where we have written $Z = \operatorname{Supp} M$ and $W = \operatorname{Supp} N$ (and the map labelled ℓ is in fact an isomorphism).

Exercise 6.22. Prove (1)–(7) above.

Remark 6.23. We will understand this better after seeing how intersection multiplicities fit into algebraic topology, but it is worth noting that the group $K(R)_Z$ would—from a topological perspective—be better written as $K(X, X - Z)$, where $X = \operatorname{Spec} R$. For comparison, relative products in a cohomology theory would give pairings

$$K(X, X-Z) \otimes K(X, X-W) \to K(X, (X-Z) \cup (X-W)) = K(X, X - (Z \cap W)),$$

which is what we saw above in the form $K(R)_Z \otimes K(R)_W \to K(R)_{Z \cap W}$. See Sections 20 and 24 for more on relative topological K-theory.

∘ Exercises ∘

Exercise 6.24. Let R be a Noetherian ring and let $C_\bullet$ be a bounded-below chain complex of finitely-generated modules. Prove that there is a quasi-isomorphism $P_\bullet \to C_\bullet$ where $P_\bullet$ is a bounded-below complex of finitely-generated projectives. Even more, prove that the quasi-isomorphism can be constructed to be surjective. [Hint: Construct $P_\bullet$ inductively from the bottom up. Each stage will involve killing relations from the stage below (forcing injectivity on homology), as well as adding generators (forcing surjectivity on homology).]

Exercise 6.25. Let R be a Noetherian ring and let $J \subseteq R$ be an ideal. Define $G^{cplx}(R)_J$ to be the Grothendieck group of bounded below chain complexes of finitely-generated modules having bounded homology that is all killed by J. (By

"Grothendieck group" we mean the group defined analogously to the one in Definition 6.1). Prove that the Euler characteristic $\chi\colon G^{cplx}(R)_J \to G(R/J)$ is an isomorphism.

Exercise 6.26. In the setting of the previous exercise, define $G^{cplx}(R)'_J$ in the same way but only using bounded below chain complexes of finitely-generated projective modules. Prove that this group is naturally isomorphic to $G^{cplx}(R)_J$ as defined above.

7. K-theory of exact complexes

We have seen the isomorphism of groups $K(R) \cong K^{cplx}(R)$. If $P_\bullet$ is a bounded, exact complex of projectives then it gives rise to a relation in $K(R)$, and (equivalently) represents the zero object in $K^{cplx}(R)$. Given this, it might seem surprising to learn that there is yet another model for $K(R)$ in which exact complexes can represent nonzero elements—and even more, *all nonzero elements* can be represented this way. The goal of the present section is to explain this model, as well as some variations. This material is adapted from [Grayson3].

Note: The contents of this section are note needed in the remainder of the book. While the material is intriguing, it can certainly be skipped if desired.

Throughout this section R is a fixed commutative ring.

Definition 7.1.

$$K^{exct}(R) = \frac{\mathbb{Z}\langle\, [P_\bullet] \,\big|\, P_\bullet \text{ is a bounded, exact chain complex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ *if there is a short exact sequence* $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$,

(2) $[\Sigma P_\bullet] = -[P_\bullet]$.

If P is a projective module let CP denote the mapping cone of the identity map $P \to P$. Specifically, CP is a chain complex concentrated in dimensions 0 and 1 where the only nonzero differential is the identity map on P. Observe that there is a group homomorphism $K(R) \to K^{exct}(R)$ that sends $[P]$ to $[CP]$.

Proposition 7.2. *The map $K(R) \to K^{exct}(R)$ is an isomorphism. The inverse is denoted $\chi'\colon K^{exct}(R) \to K(R)$ and called the **derived** (or **secondary**) **Euler characteristic**. If $P_\bullet$ is an exact bounded complex of finitely-generated projectives then*

$$\chi'(P_\bullet) = \sum_j (-1)^{j+1} j[P_j] = \sum_j (-1)^{j-1} [\operatorname{im} d_j]$$

where $d_j\colon P_j \to P_{j-1}$.

Technically speaking the second formula given for χ' doesn't make sense unless we know that each $\operatorname{im} d_j$ is a finitely-generated projective module. This is a simple exercise, but let us record it in a lemma.

Lemma 7.3. *Let $P_\bullet$ be a bounded, exact complex of projectives. Then each $\operatorname{im} d_j$ is projective, and is finitely-generated if P_j is. Moreover, $P_\bullet$ is isomorphic to a direct sum of exact complexes of length one: $P_\bullet \cong \bigoplus_i \Sigma^i C(\operatorname{im} d_{i+1})$.*

Proof. Without loss of generality we can assume that $P_\bullet$ has the form $0 \to P_n \to \cdots \to P_0 \to 0$. So $d_1 \colon P_1 \to P_0$ is surjective, hence $\operatorname{im} d_1 = P_0$ and there is nothing to prove here. Exactness gives us short exact sequences $0 \to \operatorname{im} d_{j+1} \to P_j \to \operatorname{im} d_j \to 0$, for each j. We can assume by induction that $\operatorname{im} d_j$ is projective, hence the sequence is split-exact and therefore $\operatorname{im} d_{j+1}$ is also projective.

Since $\operatorname{im} d_j$ is a quotient of P_j, it is finitely-generated if P_j is.

For the final statement, assume without loss of generality that $P_i = 0$ for $i < 0$. Then $P_1 \to P_0$ is surjective, so choose a splitting χ. The module P_1 splits as $(\operatorname{im} d_2) \oplus (\operatorname{im} \chi)$, so that $P_\bullet$ splits off the acylic complex $[\operatorname{im} \chi \xrightarrow{d} P_0]$ at the bottom. Now continue inductively. □

The splitting of $P_\bullet$ in the above lemma is non-canonical, depending on the choices of splitting at each level. For variety we will see a weaker, but more canonical, version of this decomposition in the next proof.

Proof of Proposition 7.2. Let α denote the map $K(R) \to K^{exct}(R)$ sending $[P] \mapsto [CP]$. It is easy to see that α is surjective, because if $P_\bullet$ is a bounded, exact complex of finitely-generated projectives then it is isomorphic to the direct sum $\bigoplus_j \Sigma^{j-1} C(\operatorname{im} d_j)$. So $[P_\bullet] = \sum_j (-1)^{j-1} [C(\operatorname{im} d_j)]$, and this belongs to $\operatorname{im} \alpha$.

We aim to construct an inverse to α. From the preceding paragraph it is clear that if such an inverse exists it must send $[P_\bullet]$ to $\sum_j (-1)^{j-1} [\operatorname{im} d_j]$. It is only left to check that this formula does indeed define a map $K^{exct}(R) \to K(R)$. To do this we will show that the alternating sum can be rewritten in terms of only the P_i's.

Assume that the smallest degree of $P_\bullet$ containing a nonzero module is degree n. Write $I_j = \operatorname{im} d_j$. Since $\cdots \to P_{j+1} \to P_j \to I_j \to 0$ is exact, we have that $[I_j] = \sum_{k \geq j} (-1)^{k-j} [P_k]$ in $K(R)$. So in $K(R)$ we have

$$\begin{aligned}
\sum_{j \geq n} (-1)^{j-1} [I_j] = \sum_{j \geq n} \sum_{k \geq j} (-1)^{j-1} (-1)^{k-j} [P_k] &= \sum_k (-1)^{k-1} \sum_{n \leq j \leq k} [P_k] \\
&= \sum_k (-1)^{k-1} (k - n + 1) [P_k] \\
&= \sum_k (-1)^{k-1} k [P_k] + (n-1) \chi(P_\bullet) \\
&= \sum_k (-1)^{k-1} k [P_k].
\end{aligned}$$

In the last equality we have used that $\chi(P_\bullet) = 0$ since $P_\bullet$ is exact.

Define $\chi'(P_\bullet) = \sum_k (-1)^{k-1} k [P_k]$. One easily checks that this satisfies relations (1) and (2) in the definition of $K^{exct}(R)$, and hence defines a map $\chi' \colon K^{exct}(R) \to K(R)$. It is trivial to check that $\chi' \circ \alpha = \mathrm{id}$. Therefore α is injective, and since we already proved surjectivity it is an isomorphism and χ' is its inverse. □

7.4. **Derived Euler characteristics.** Now that we have encountered the derived Euler characteristic it seems worthwhile to take a moment and place it into a broader context. Consider the definition

$$\chi_t(P_\bullet) = \sum t^j [P_j] \in K(R)[t, t^{-1}].$$

This function is additive, and in fact one can see that it is the universal additive invariant for bounded complexes of finitely-generated projectives. The usual Euler characteristic is $\chi(P_\bullet) = \chi_t(P_\bullet)|_{t=-1}$. Of course we do not have $\chi_t(\Sigma P_\bullet) = -\chi_t(P_\bullet)$,

this only becomes true after the substitution $t = -1$; what we have instead is the identity

$$\chi_t(\Sigma P_\bullet) = t \cdot \chi_t(P_\bullet). \tag{7.5}$$

If we differentiate χ_t with respect to t then we obtain $\chi'_t(P_\bullet) = \sum jt^{j-1}[P_j]$. Clearly this is also an additive invariant of complexes. The invariant we called χ' is just $\chi'_t(P_\bullet)|_{t=-1}$. Differentiating (7.5) yields the formula

$$\chi'_t(\Sigma P_\bullet) = \chi_t(P_\bullet) + t \cdot \chi'_t(P_\bullet), \tag{7.6}$$

and consequently $\chi'(\Sigma P_\bullet) = \chi(P_\bullet) - \chi'(P_\bullet)$. This is not the kind of behavior we are used to, but notice that if we restrict to complexes $P_\bullet$ with $\chi(P_\bullet) = 0$ then we get the nicer behavior $\chi'(\Sigma P_\bullet) = -\chi'(P_\bullet)$.

One can, of course, iterate this procedure. Let $\chi_t^{(n)}(P_\bullet)$ denote the nth derivative of $\chi_t(P_\bullet)$, and write $\chi^{(n)}(P_\bullet) = \chi_t^{(n)}(P_\bullet)|_{t=-1}$. Call this the ***n*th derived Euler characteristic**. It is an additive function, and if one restricts to complexes such that $0 = \chi^{(n-1)}(P_\bullet)$ then it satifies $\chi^{(n)}(\Sigma P_\bullet) = -\chi^{(n)}(P_\bullet)$.

7.7. **Doubly-exact complexes.** A bicomplex $C_{\bullet,\bullet}$ will be called **bounded** if the modules $C_{i,j}$ are nonzero for only finitely many values of (i, j). The bicomplex will be called **doubly-exact** if every row and every column is exact. By abuse of terminology an ordinary chain complex $D_\bullet$ will be called doubly-exact if it is isomorphic to the total complex of a bounded, doubly-exact bicomplex. Doubly exact complexes all represent zero in $K^{exct}(R)$:

Proposition 7.8. *If $P_\bullet$ is a bounded, doubly-exact complex of finitely-generated projectives then $[P_\bullet] = 0$ in $K^{exct}(R)$.*

Proof. Let $M_{\bullet,\bullet}$ be a doubly-exact bicomplex of finitely-generated projectives. For some n and k, $M_{i,j}$ is zero outside of the rectangle $0 \le i \le n$ and $0 \le j \le k$. Write $M_{i,*}$ for the ordinary complex whose jth term is $M_{i,j}$, and write $M_{\le i,*}$ for the sub-bicomplex of $M_{\bullet,\bullet}$ consisting of all $M_{a,j}$ for $a \le i$. Observe that there are short exact sequences

$$0 \to \mathrm{Tot}(M_{\le(i-1),*}) \hookrightarrow \mathrm{Tot}(M_{\le i,*}) \twoheadrightarrow \Sigma^i M_{i,*} \to 0,$$

for all i. Induction shows that each $\mathrm{Tot}(M_{\le i,*})$ is exact, and therefore in $K^{exct}(R)$ we have

$$[\mathrm{Tot}\, M_{\bullet,\bullet}] = \sum_i [\Sigma^i M_{i,*}] = \sum_i (-1)^i [M_{i,*}]$$

(using the analog of Proposition 2.5(b)). But $M_{\bullet,\bullet}$ may be regarded as an exact sequence of chain complexes

$$0 \to M_{n,*} \to M_{n-1,*} \to \cdots \to M_{1,*} \to M_{0,*} \to 0.$$

The image of each map in this sequence is a chain complex of finitely-generated projectives (using Lemma 7.3), and we have the short exact sequences of chain complexes

$$0 \longrightarrow \mathrm{im}(M_{i+1,*}) \longrightarrow M_{i,*} \longrightarrow \mathrm{im}(M_{i,*}) \longrightarrow 0.$$

By a straightforward induction, each of these image complexes is exact. Each of these short exact sequences gives a relation in $K^{exct}(R)$, and taking their alternating sum shows that $\sum_i (-1)^i [M_{i,*}] = 0$. We have therefore shown that $[\mathrm{Tot}\, M_{\bullet,\bullet}] = 0$ in $K^{exct}(R)$. □

The reader will notice the beginnings of a pattern here. Exact complexes $P_\bullet$ represent zero in $K^{cplx}(R)$, but then we produced a new model for this same group where the exact complexes are the generators. In this new group $K^{exct}(R)$ the doubly-exact complexes represent zero. It is natural, then, to wonder if there is yet another model for this group where the doubly-exact complexes are the generators. Indeed, this works out in what is now a completely straightforward manner, and can be repeated ad infinitum.

Let us use the term *multicomplex* for the evident generalization of bicomplexes to n dimensions. We will denote a multicomplex by $M_\star$, where the symbol $\star$ stands for an n-tuple of integers. Say that the multicomplex is **n-exact** if every linear 'row' (obtained by fixing $n-1$ of the indices) is exact.

Definition 7.9.

$$K^{n-exct}(R) = \frac{\mathbb{Z}\langle\, [M_\star] \,\big|\, M_\star \text{ is a bounded, } n\text{-exact multicomplex of f.g. projectives}\rangle}{\langle\ \textbf{Relation 1}, \textbf{Relation 2}\ \rangle}$$

where the relations are

(1) $[M_\star] = [M'_\star] + [M''_\star]$ *if there is an exact sequence* $0 \to M'_\star \to M_\star \to M''_\star \to 0$,

(2) $[\Sigma M_\star] = -[M_\star]$, *where* Σ *stands for any of the* n *suspension operators on* n*-multicomplexes.*

Given an $(n+1)$-multicomplex $M_\star$ there are $\binom{n+1}{2}$ ways to totalize it to get an n-multicomplex—one needs to choose two of the $n+1$ directions to combine. One can follow the proof of Proposition 7.8 to show that if $M_\star$ is $(n+1)$-exact then each of these totalizations represents zero in $K^{n-exct}(R)$.

If $M_\star$ is an n-multicomplex then let $CM_\star$ denote the cone on the identity map $M_\star \to M_\star$. This is an $(n+1)$-multicomplex, defined in the evident manner. This cone construction induces a group homomorphism $K^{n-exct}(R) \to K^{(n+1)-exct}(R)$.

Proposition 7.10. *The map* $K^{n-exct}(R) \to K^{(n+1)-exct}(R)$ *is an isomorphism, with inverse given by*

$$\chi'(M_\star) = \sum (-1)^{j+1} j [M_{j,\star}]$$

where the symbols $M_{j,\star}$ *represent the various slices of* $M_\star$ *in any fixed direction.*

Proof. Follow the proof of Proposition 7.2 almost verbatim, but where each P_i represents an n-exact multicomplex rather than an R-module. □

We have the sequence of isomorphisms

$$K(R) \to K^{exct}(R) \to K^{2-exct}(R) \to \cdots$$

The composite map $K(R) \to K^{n-exct}(R)$ sends $[P]$ to the n-dimensional cube consisting of P's and identity maps. The composite of the χ' maps in the other direction yields the map $K^{n-exct}(R) \to K(R)$ given by

$$M_\star \mapsto \sum_{j_1,\ldots,j_n} (-1)^{j_1+\cdots+j_n+n} j_1 \cdots j_n [M_{j_1,\ldots,j_n}]. \tag{7.11}$$

If one considers the formal Laurent polynomial

$$\chi_{t_1,\ldots,t_n}(M) = \sum_{j_1,\ldots,j_n} t_1^{j_1} \cdots t_n^{j_n} [M_{j_1,\ldots,j_n}]$$

then the expression (7.11) is the nth order partial derivative $\partial_{t_1} \cdots \partial_{t_n} \chi_{t_1,\ldots,t_n}(M)$ evaluated at $t_1 = t_2 = \cdots = t_n = -1$.

Remark 7.12. Grayson [Grayson3] suggests a perspective where exact complexes are analogous to the formal infinitesimals from nonstandard analysis. Doubly-exact complexes are analogues of products of infinitesimals, and so forth.

8. A taste of K_1

Note: The material in this section will not be needed for most of what follows. We include it for general interest, and because the material fits naturally here. But this section can safely be skipped.

Given a commutative ring R and a multiplicative system $S \subseteq R$, we have seen the exact sequences

$$G(M \mid S^{-1}M = 0) \to G(R) \to G(S^{-1}R) \to 0$$

and

$$K(R, S) \to K(R) \to K(S^{-1}R).$$

It is natural to wonder if these extend to long exact sequences, and the answer is that they do: in the first case there is an extension to the left, and in the latter case there is an extension in both directions. These extensions are not easy to produce, however—they are the subject of 'higher algebraic K-theory', a field that involves some very deep and difficult mathematics. Our aim here is not to start a long journey into that subject, but rather to just give some indications of the very beginnings.

Remark 8.1. From now on the groups $K(R)$ and $G(R)$ will be written $K_0(R)$ and $G_0(R)$.

8.2. **The basic theory of $K_1(R)$.** Let us take the perspective that $K_0(R)$ is, in essence, constructed with the goal of generalizing the familiar notion of dimension for vector spaces. The key property of dimension is additivity for short exact sequences, so consequently one forms the universal group with that property. The group $K_1(R)$ is obtained similarly but with the goal of generalizing the *determinant*.

Determinants are invariants of self-maps—maps with the same domain and target—and we need some language for dealing with such things. Given two self-maps $f\colon A \to A$ and $g\colon B \to B$, we define a map from f to g to be a map $u\colon A \to B$ giving a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{u} & B. \end{array}$$

Likewise, an exact sequence of self-maps is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \xrightarrow{u_0} & P & \xrightarrow{u_1} & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & P' & \xrightarrow{u_0} & P & \xrightarrow{u_1} & P'' & \longrightarrow & 0 \end{array} \tag{8.3}$$

in which the rows are short exact sequences of modules.

Definition 8.4. *Form the free abelian group generated by isomorphism classes of maps $[P \xrightarrow{\alpha} P]$ where P is a finitely-generated projective and α is an isomorphism. Let $K_1(R)$ be the quotient of this group by the following relations:*

(a) $[P \xrightarrow{\alpha} P] = [P' \xrightarrow{\alpha'} P'] + [P'' \xrightarrow{\alpha''} P'']$ *whenever there is a short exact sequence as in (8.3);*

(b) $[P \xrightarrow{\alpha\beta} P] = [P \xrightarrow{\alpha} P] + [P \xrightarrow{\beta} P]$ *for all self-maps $\alpha, \beta\colon P \to P$.*

As a consequence of relation (b) one has that $[P \xrightarrow{\mathrm{id}} P] = [P \xrightarrow{\mathrm{id}} P] + [P \xrightarrow{\mathrm{id}} P]$, and so $[P \xrightarrow{\mathrm{id}} P] = 0$ for any finitely-generated projective P. Note also that if $\alpha\colon P \to P$ and $\beta\colon Q \to Q$ are automorphsms then

$$[P \oplus Q \xrightarrow{\alpha\oplus\beta} P \oplus Q] = [P \xrightarrow{\alpha} P] + [Q \xrightarrow{\beta} Q], \tag{8.5}$$

as a consequence of relation (a).

The use of projective modules in the definition of $K_1(R)$ turns out to be unnecessarily complicated—one can get the same group by only using automorphisms of free modules. Even more, the use of short exact sequences in relation (a) is unnecessarily complicated; one can get the same group by only imposing the weaker relation from (8.5). We will prove both of these claims in just a moment.

Observe that there is a map of groups $GL_n(R) \to K_1(R)$ that sends a matrix A to the class $[R^n \xrightarrow{A} R^n]$ (left-multiplication-by-A). Relation (b) guarantees that this is indeed a group homomorphism. If we let $j\colon GL_n(R) \hookrightarrow GL_{n+1}(R)$ be the usual inclusion, obtained by adding an additional row and column and a 1 along the diagonal, then it is clear that $[R^{n+1} \xrightarrow{j(A)} R^{n+1}] = [R^n \xrightarrow{A} R^n]$. This follows from (8.5) and the fact that $[R \xrightarrow{\mathrm{id}} R] = 0$. Let $GL(R)$ denote the colimit

$$GL(R) = \mathrm{colim}[GL_1(R) \to GL_2(R) \to GL_3(R) \to \cdots],$$

and call this the **infinite general linear group** of R. We have obtained a map $GL(R) \to K_1(R)$, and of course this will factor through the abelianization to give

$$GL(R)_{ab} = GL(R)/[GL(R), GL(R)] \to K_1(R).$$

Exercise 8.6. We abelianized after taking the colimit, but could have just as well done it the other way around. Verify that if $G_1 \to G_2 \to \cdots$ is any sequence of group homomorphisms then $[\mathrm{colim}_n G_n]_{ab} = \mathrm{colim}_n[(G_n)_{ab}]$.

The most fundamental result in the theory of K_1 is the following:

Theorem 8.7. *The map $GL(R)_{ab} \to K_1(R)$ is an isomorphism.*

It will be convenient to prove this at the same time that we give other descriptions for $K_1(R)$. In particular, we make the following definitions:

(1) $K_1^{fr}(R)$ is the group defined similarly to $K_1(R)$ but changing all occurrences of 'projective' to 'free'.
(2) $K_1^{sp}(R)$ is the group defined similarly to $K_1(R)$ but replacing relation (a) by the direct sum relation of (8.5). The "sp" stands for "split".
(3) $K_1^{sp,fr}(R)$ is the group defined by making both the changes indicated in (1) and (2).

One obtains a large diagram as follows:

$$\begin{array}{ccccc} & & \operatorname{colim}_P \operatorname{Aut}(P)_{ab} & \twoheadrightarrow & K_1^{sp}(R) \twoheadrightarrow K_1(R) \\ & & \uparrow & & \uparrow \quad \uparrow \\ GL(R)_{ab} & = & \operatorname{colim}_n GL_n(R)_{ab} & \twoheadrightarrow & K_1^{sp,fr}(R) \twoheadrightarrow K_1^{fr}(R). \end{array} \tag{8.8}$$

The maps labelled as surjections are obviously so. Let us explain the colimit over projectives P. Let $\mathcal{M}$ denote the monoid of isomorphism classes of finitely-generated projectives, with the operation of $\oplus$. The *translation category* $T(\mathcal{M})$ of this monoid has object set equal to $\mathcal{M}$, and the maps from A to B are the elements $C \in \mathcal{M}$ such that $A + C = B$; composition is given by the multiplication in $\mathcal{M}$. This is the indexing category for our colimit. Given an isomorphism $f\colon P \to Q$, there is an induced map of groups $\operatorname{Aut}(P) \to \operatorname{Aut}(Q)$ sending α to $f\alpha f^{-1}$. Changing f gives a different induced map, but it gives the same induced map on $\operatorname{Aut}(P)_{ab} \to \operatorname{Aut}(Q)_{ab}$: this is a consequence of the formula

$$f\alpha f^{-1} = (fg^{-1})(g\alpha g^{-1})(gf^{-1}).$$

We can therefore construct a functor $T(\mathcal{M}) \to \mathcal{A}b$ sending each $[P]$ to $\operatorname{Aut}(P)_{ab}$. If $[Q]$ is a map from $[P]$ to $[J]$ then we choose an isomorphism $f\colon P \oplus Q \to J$ and have T send the map $[Q]$ to the composite $\operatorname{Aut}(P)_{ab} \hookrightarrow \operatorname{Aut}(P \oplus Q)_{ab} \to \operatorname{Aut}(J)_{ab}$. The first map is direct sum with id_Q and the second map is independent of the choice of f, so this is well-defined. The upper left term in our diagram is the colimit of the functor T. The map from this colimit to $K_1^{sp}(R)$ is induced by the one sending an element $\alpha \in \operatorname{Aut}(P)$ to the class $[P \xrightarrow{\alpha} P]$.

Theorem 8.7 will follow as an immediate consequence of the following stronger result:

Theorem 8.9. *All of the maps in (8.8) are isomorphisms.*

We are almost ready to prove this theorem, but we will need one key result first. Let $E(R) \subseteq GL(R)$ be the subgroup generated by the elementary matrices—matrices that have ones along the diagonal and a single nonzero, off-diagonal entry. We likewise define $E_n(R) \subseteq GL_n(R)$, and observe that $E(R) = \operatorname{colim}_n E_n(R)$. We will implicitly identify matrices in $E_n(R)$ with their image in $E(R)$; note that this involves adding trailing ones down the diagonal of the matrix.

Recall that right multiplication by an elementary matrix amounts to performing a column operation where a multiple of one column is added to another; similarly, left multiplication amounts to performing the analogous row operation. These types of operations are called **transvections**. One very useful way to recognize a matrix as belonging to $E(R)$ is to observe that it can be obtained from the identity matrix by using these transvective row and column operations. More generally, we will sometimes use the adjective "transvective" to describe any operation that is a composite of transvections.

Lemma 8.10.

(a) For any $X \in M_n(R)$ the matrix $\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ and its transpose belong to $E(R)$.

(b) If $A \in GL_n(R)$ then $\left[\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right] \in E(R)$.

(c) Let A be a matrix obtained from the identity by switching two columns and multiplying one of the switched columns by -1. Then $A \in E(R)$, and similarly for the transpose of A.

Proof. For part (a) just note that $\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ can be obtained from the identity matrix by a sequence of transvective column operations. Same for the transpose.

For (b) consider the following chain of matrices:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} I & A \\ A^{-2}-A^{-1} & A^{-1} \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ A^{-2}-A^{-1} & A^{-1} \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.$$

Passage from each matrix to the next can be done by transvective row and column operations. Alternatively, each matrix can be obtained from its predecessor by left multiplication by a matrix of the type considered in (a): use the matrices $\left[\begin{smallmatrix} I & A \\ 0 & I \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} I & 0 \\ A^{-2}-A^{-1} & I \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} I & -A^2 \\ 0 & I \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} I & 0 \\ A^{-2}-A^{-3} & I \end{smallmatrix}\right]$.

Finally, for (c) we argue directly in terms of column operations. If v and w are two columns consider the following chain

$$v, w \quad \mapsto \quad v, w-v \quad \mapsto \quad w, w-v \quad \mapsto \quad w, -v.$$

Each link involves adding a multiple of one column to another, and is therefore transvective; therefore the composite operation is an element of $E(R)$. The argument is similar for $v, w \mapsto -w, v$, or one could use the fact that $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in E(R)$ by part (b). □

The following is the key lemma that we will need in our proof of Theorem 8.9:

Lemma 8.11 (Whitehead Lemma)**.** $E(R) = [GL(R), GL(R)]$

Proof. For the $\subseteq$ direction we consider three columns u, v, w, and the following chain of operations (where $r, s \in R$):

$$u, v, w \xrightarrow{1} u, v+ru, w \xrightarrow{2} u, v+ru, w+sv+sru \xrightarrow{3} u, v, w+sv+sru \xrightarrow{4} u, v, w+sru.$$

It should be clear what column operation is being used in each step. Note that the third and fourth operations are the inverses of the first and second, so the composite it a commutator. This shows that any column operation of the type "add a multiple of one column to another" is a commutator, and therefore $E(R) \subseteq [GL(R), GL(R)]$. (We have actually shown $E_n(R) \subseteq [GL_n(R), GL_n(R)]$ for $n \geq 3$).

For the other subset direction, let $A, B \in GL_n(R)$. Consider the following identity:

$$\begin{bmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{bmatrix}.$$

The first matrix is identified with the commutator of A and B inside $GL(R)$, and all of the other matrices are in $E(R)$ by Lemma 8.10(b). So $[A, B] \in E(R)$ as well. □

Corollary 8.12. *For any $A \in GL_n(R)$, $B \in GL_k(R)$, and $X \in M_{n\times k}(R)$, $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ in $GL(R)_{ab}$. If $n = k$ then this matrix also equals $\left[\begin{smallmatrix} AB & 0 \\ 0 & I \end{smallmatrix}\right]$ in $GL(R)_{ab}$.*

Proof. For the first claim simply observe that

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} I & A^{-1}X \\ 0 & I \end{bmatrix}.$$

The second matrix in the product is in $E(R)$ by Lemma 8.10(a), and hence in $[GL(R), GL(R)]$ by the Whitehead Lemma.

For the second claim notice that $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} B & 0 \\ 0 & B^{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} AB & 0 \\ 0 & I \end{smallmatrix}\right]$ and use Lemma 8.10(b) together with the Whitehead Lemma. □

We are now ready to prove that all of our descriptions of $K_1(R)$ give the same group:

Proof of Theorem 8.9. Let $\alpha\colon P \to P$ be an automorphism of a finitely-generated projective, and let Q be a free complement to P: that is, $P \oplus Q \cong R^n$ for some n. Then

$$[P \xrightarrow{\alpha} P] = [P \oplus Q \xrightarrow{\alpha \oplus \mathrm{id}_Q} P \oplus Q]$$

in $K_1^{sp}(R)$, which shows that $K_1^{sp,fr}(R) \to K_1^{sp}(R)$ is surjective. The same proof works for all of the vertical maps in diagram (8.8).

The fact that $\operatorname{colim}_n GL_n(R)_{ab} \to \operatorname{colim}_P \operatorname{Aut}(P)_{ab}$ is an isomorphism is very easy: it is just because the subcategory of $T(\mathcal{M})$ consisting of the free modules is final in $T(\mathcal{M})$ (see [Mac Lane, IX.3] for the notion of final functors). The key observation is that if P is a finitely-generated projective and Q and Q' are such that $P \oplus Q \cong R^n$ and $P \oplus Q' \cong R^k$ then we can form the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{Q'} & R^k \\ {\scriptstyle Q}\downarrow & \searrow^{Q \oplus P \oplus Q'} & \downarrow{\scriptstyle R^n} \\ R^n & \xrightarrow{R^k} & R^{n+k} \end{array}$$

in $T(\mathcal{M})$. This verifies that the undercategory $P \downarrow T(\mathcal{M})$ is connected.

Define a map $K_1^{fr}(R) \to GL(R)_{ab}$ by sending $[R^n \xrightarrow{A} R^n]$ to the matrix A. To see that this is well-defined we need to verify that it respects relations (a) and (b) from Definition 8.4. Relation (b) is self-evident. For (a), suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array}$$

is a short exact sequence of automorphisms between free modules. Then there is a basis for F with respect to which the matrix for α has the form $\left[\begin{smallmatrix} \alpha' & * \\ 0 & \alpha'' \end{smallmatrix}\right]$. Corollary 8.12 verifies that this matrix equals $\left[\begin{smallmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{smallmatrix}\right]$ in $GL(R)_{ab}$.

Now that we have the map $K_1^{fr}(R) \to GL(R)_{ab}$, it is trivial to check that this is a two-sided inverse for the map from (8.8). It follows that all the maps in the bottom row of that diagram are isomorphisms.

The proof for the maps along the top row proceeds in a similar manner. Define a map $K_1(R) \to \operatorname{colim}_P \operatorname{Aut}(P)_{ab}$ by sending $[P \xrightarrow{\alpha} P]$ to the element $\alpha \in \operatorname{Aut}(P)_{ab}$.

One has to check that this respects relations (a) and (b) in the definition of $K_1(R)$, and relation (b) is again trivial. Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \end{array}$$

is a short exact sequence of automorphisms between finitely-generated projectives. Choose free complements Q' for P', and Q'' for P''. Consider the new short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' \oplus Q' & \longrightarrow & P \oplus Q' \oplus Q'' & \longrightarrow & P'' \oplus Q'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha' \oplus \mathrm{id}_{Q'}} & & \downarrow{\scriptstyle \alpha \oplus \mathrm{id}_{Q'} \oplus \mathrm{id}_{Q''}} & & \downarrow{\scriptstyle \alpha'' \oplus \mathrm{id}_{Q''}} & & \\ 0 & \longrightarrow & P' \oplus Q' & \longrightarrow & P \oplus Q' \oplus Q'' & \longrightarrow & P'' \oplus Q'' & \longrightarrow & 0. \end{array}$$

All of the modules in this diagram are free (recall that $P \cong P' \oplus P''$), and so this diagram gives a relation in $K_1^{fr}(R)$. Using the map $K_1^{fr}(R) \to GL(R)_{ab}$ already constructed, we find that

$$\alpha \oplus \mathrm{id}_{Q'} \oplus \mathrm{id}_{Q''} = \big(\alpha' \oplus \mathrm{id}_{Q'}\big) + \big(\alpha'' \oplus \mathrm{id}_{Q''}\big)$$

in $GL(R)_{ab}$ and hence also in $\mathrm{colim}_P \mathrm{Aut}(P)_{ab}$. But this says precisely that $\alpha = \alpha' + \alpha''$ as elements in $\mathrm{colim}_P \mathrm{Aut}(P)_{ab}$, and this is what we needed to check. We have now constructed our map $K_1(R) \to \mathrm{colim}_P \mathrm{Aut}(P)_{ab}$, and it readily follows that it is an inverse for the map in the other direction from (8.8). So all the maps in the top horizontal row of (8.8) are isomorphisms.

We have shown that all horizontal maps in (8.8) are isomorphisms, and that the left vertical map is an isomorphism. So all the maps are isomorphisms. □

We now work towards computing a few examples of $K_1(R)$ in some easy cases. The defining relations we used to construct $K_1(R)$ captured familiar properties of the determinant, so it should not be surprising that the determinant plays a large role here.

Observe that $\det\colon GL(R) \to R^*$ factors through the abelianization and therefore yields an induced map $\det\colon K_1(R) \to R^*$. This map is split, since we can send any $r \in R^*$ to the class of the automorphism $R \xrightarrow{r} R$ (this is a group homomorphism using relation (b) of Definition 8.4). So we always have $K_1(R) \cong R^* \oplus (???)$. The mystery factor is usually called $SK_1(R)$, in analogy with the "S" that appears when one passes from GL_n to SL_n.

We will not calculate K_1 for many rings, but in the easiest examples $SK_1(R)$ always vanishes. We explain this next.

Lemma 8.13. *If A is a diagonal matrix of determinant 1 then A lies in $E(R)$.*

Proof. This can be proven directly by using row and column operations, but the following argument is a bit easier to write. We use that $GL(R)/E(R) \cong K_1(R)$. Let $d_1, \ldots, d_n$ be the diagonal entries of A. Working in $K_1(R)$ we write

$$[R^n \xrightarrow{A} R^n] = [R \xrightarrow{d_1} R] + \cdots + [R \xrightarrow{d_n} R] = [R \xrightarrow{d_1 \cdots d_n} R] = [R \xrightarrow{1} R] = 0$$

where the first equality is by relation (a) in Definition 8.4 and the second equality is by relation (b). □

Proposition 8.14. *If F is a field then $K_1(F) = F^*$.*

Proof. One must show that if $A \in GL(F)$ satisfies $\det(A) = 1$ then $A \in [GL(F), GL(F)] = E(F)$. Lemma 8.13 verifies this in the case where A is diagonal. The proof proceeds by using row and column operations to reduce to this case.

We will use two types of column and row operations: adding a multiple of one column/row to another, and switching two columns (or rows) together with a sign change of one of them. Both types of operation are transvective, the latter by Lemma 8.10(c). Pick any nonzero entry in the matrix, move it into the $(1,1)$ position, and then use it to clear out the rest of its row and column. Proceeding inductively, this transforms A into a diagonal matrix. That is, there exist matrices $E_1, E_2 \in E(F)$ such that E_1AE_2 is diagonal. But by the preceding paragraph we then have $E_1AE_2 \in E(F)$, and so $A \in E(F)$. □

Essentially the same proof as above also shows the following:

Proposition 8.15. *Let R be a Euclidean domain. Then* $\det\colon K_1(R) \to R^*$ *is an isomorphism. In particular, $K_1(\mathbb{Z}) = \{1, -1\} \cong \mathbb{Z}/2$ and when F is a field $K_1(F[t]) \cong F^*$. The conclusion also holds when R is a local ring.*

Proof. We must again show that if $A \in GL_n(R)$ has $\det(A) = 1$ then $A \in E(R)$. For any fixed row of A, the ideal generated by the elements in that row contains $\det(A)$ and is therefore the unit ideal. Pick an element x of smallest degree in this row and then use column operations (and the Euclidean division property) to arrange all other elements in this row to be either zero or have degree smaller than x. By repeating this process, eventually the row will contain a unit and all other entries will be zero. Do a signed transposition to switch this unit into position $(1,1)$, and then do row operations to clear out all other terms in the first column. Repeat this process for the submatrix obtained by deleting the first row and column, and so forth. Eventually the matrix will be reduced to a diagonal matrix, necessarily of determinant 1. Such a diagonal matrix lies in $E(R)$ by Lemma 8.13, so this proves A also lies in $E(R)$.

Essentially the same proof works for local rings, by finding units in the matrix and then using them to clear out their row and column. □

This is a good place to pause for a few exercises that help accentuate the above techniques.

Exercise 8.16. Let R be a commutative ring. Prove that if D is an invertible diagonal matrix and N is strictly lower triangular then $[D + N] = [D]$ in $K_1(R)$.

Exercise 8.17. Prove that the following are equivalent:

(1) $SK_1(R) = 0$,
(2) $K_1(R)$ is generated by the classes $[R \xrightarrow{r} R]$ for $r \in R^*$,
(3) Every invertible matrix over R can be transformed via transvective column operations into a diagonal matrix,
(4) Every invertible matrix over R can be transformed via transvective column operations into a diagonal matrix with exactly one entry that is not a 1.

Exercise 8.18. Let R be a Euclidean domain.

(a) Prove that every invertible matrix can be transformed to a lower triangular matrix using transvective column operations. Use this together with the previous two exercises to give a (slightly) different proof that $SK_1(R) = 0$.
(b) For every $f \in R - \{0\}$ prove that $SK_1(f^{-1}R) = 0$.

Exercise 8.19. Prove that $GL_n(R \times S) = GL_n(R) \times GL_n(S)$ and that $K_1(R \times S) = K_1(R) \oplus K_1(S)$.

Exercise 8.20. Calculate $K_1(\mathbb{Z}/360)$.

It is somewhat of a challenge to come up with an easy ring R for which one can prove by elementary means that $SK_1(R) \neq 0$, or equivalently that $\det\colon K_1(R) \to R^*$ is not an isomorphism. We have just seen that there are no examples amongst fields or Euclidean domains or local rings, and Milnor [Milnor7, Corollary 16.3] proves that one also cannot find examples amongst rings of integers in number fields. There do exist examples where R is a PID, but they are a bit exotic—see Example 8.22 below. In some sense the "easiest" example of such a ring (though not a PID) involves some topology. We outline this in the next example, but skim over the details.

Example 8.21. Let R be a commutative ring.

(a) If $a, b \in R$ are such that $(a, b)R = R$, choose $c, d \in R$ such that $ad - bc = 1$. The matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ lies in $SL_2(R)$ and therefore gives us an element in $K_1(R)$ via the inclusions $SL_2(R) \hookrightarrow SL(R) \hookrightarrow GL(R)$. It turns out this element does not depend on the choice of c and d, for if $aq - bp = 1$ one computes that
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ p & q \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} q & -b \\ -p & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ?? & 1 \end{bmatrix} \in E(R).$$
Write $[a, b]$ for the element in $K_1(R)$ represented in this way, and note that in fact we have $[a, b] \in SK_1(R)$. This is called the **Mennicke symbol** represented by a and b.
(b) With some trouble one can prove that for any $\lambda \in R$ one has
$$[a, b] = [a + \lambda b, b] = [a, b + \lambda a] \qquad \text{and} \qquad [a, b] = [b, a]$$
(see [Milnor7, Lemma 13.2]). We will not need these facts, we just list them to spark the reader's interest.
(c) Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, which we regard as the ring of polynomial functions on the circle S^1. We will argue that the Mennicke symbol $[x, y]$ is nonzero, so that $SK_1(R) \neq 0$.

Note that we get maps $GL_n(R) \to [S^1, GL_n(\mathbb{R})]$ (unpointed homotopy classes) in the evident way: if $A \in GL_n(R)$ then we get a map $S^1 \to GL_n(\mathbb{R})$ by sending the point $(a, b) \in S^1$ to the matrix obtained by plugging in $x = a$ and $y = b$ into A. This is a map of groups when we give $[S^1, GL_n(\mathbb{R})]$ the operation coming from pointwise multiplication. Elementary matrices all go to the identity under this map: if $E = I + N$ where N has a single nonzero entry off the diagonal, then $t \mapsto I + tN$ gives a homotopy (of course elements of $E(R)$ might be products of such matrices, but then one does the homotopy in each factor simultaneously). In this way one gets a map $K_1(R) \to [S^1, GL(\mathbb{R})]$, and likewise a map $SK_1(R) \to [S^1, SL(\mathbb{R})]$. Since $SL(\mathbb{R})$ is a path-connected topological group, unpointed homotopy classes of maps from S^1 agree with the pointed version; that is to say, $[S^1, SL(\mathbb{R})] \cong \pi_1(SL(\mathbb{R}))) \cong \pi_1(SO)$ (the latter

because $SO_n \hookrightarrow SL_n(\mathbb{R})$ is a deformation retract for all n). It is known that this homotopy group is $\mathbb{Z}/2$ and is generated by the image of the standard generator in $\pi_1 SO(2)$ (see Section 15.8 below for this fact).

The Mennicke symbol $[x, y]$ denotes the matrix $\left[\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right] \in SL_2(R)$, and therefore gives the map $S^1 \to SO(2)$ sending $(a, b) \in S^1$ to $\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$. Said differently, this sends $e^{i\theta} \in S^1$ to the 2×2 matrix for (clockwise) rotation by θ, and so is a generator for $\pi_1 SO_2$. In particular, this proves that $[x, y] \neq 0$.

(d) In fact Bass [Bass1, XIII.2, p. 714] proves that $SK_1(R) \cong \mathbb{Z}/2$, but we will not do this much. Let us at least prove that $SK_1(R)$ is 2-torsion, as this is easy. Let $T = \mathbb{C} \otimes_{\mathbb{R}} R = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ and note that as an R-module T is just R^2. So there is an induced map $\pi\colon K_1(T) \to K_1(R)$ sending $[T^n \xrightarrow{f} T^n] \in K_1(T)$ to $[T^n \xrightarrow{f} T^n] \in K_1(R)$ where in the latter case we regard T^n as an R-module. Consider the diagram

$$\begin{array}{ccccc} K_1(R) & \xrightarrow{i} & K_1(T) & \xrightarrow{\pi} & K_1(R) \\ \uparrow & & \uparrow & & \\ SK_1(R) & \xrightarrow{i} & SK_1(T) & & \end{array}$$

where i is induced by the inclusion $R \hookrightarrow T$. Then $\pi \circ i$ sends $[R^n \xrightarrow{g} R^n]$ to $[R^n \oplus R^n \xrightarrow{g \oplus g} R^n \oplus R^n]$ and is therefore multiplication by 2. But if we set $z = x + iy$ and $w = x - iy$ then $T \cong \mathbb{C}[z, w]/(zw - 1) = \mathbb{C}[z, z^{-1}]$. This is a localization of the Euclidean domain $\mathbb{C}[z]$, and essentially the same proof as for Proposition 8.15 shows that $SK_1(T) = 0$ (see also Exercise 8.18 below). Hence, $SK_1(R)$ is 2-torsion.

Example 8.22. Here we describe an example due to Grayson [Grayson2] of a PID R with $SK_1(R) \neq 0$. Let $T = \mathbb{Z}[x]$ and let S be the multiplicative system generated by x and all polynomials $x^m - 1$ where $m \geq 1$. Let $R = S^{-1}T$. Observe that $\dim T = 2$ and that T is a UFD. The latter implies that all height one primes are principal, and the former gives that the height two primes are maximal. But quotienting out by a maximal ideal will yield a finite field, which means that x will map to either zero or a root of unity. Consequently, every maximal ideal of T must intersect S. Since the primes in R are all extended from the primes in T that do not intersect S, we conclude that all of the primes in R are principal. So R is a PID.

The computation of $SK_1(R)$ is a combination of results from [Grayson2], [Lenstra], and [Schoof]. This computation is far too complex for us to include here, but the upshot is that

$$SK_1(R) \cong \bigoplus_{n \geq 2} \mathbb{Z}/n\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \bigoplus_{i \geq 1} (\mathbb{Z}/p^i\mathbb{Z})^{\infty}.$$

So this group is quite big. See [Grayson2] for details.

Other examples of PIDs with $SK_1 \neq 0$ had previously been given by Bass [Bass2] and Ischebeck [Ischebeck].

Exercise 8.23. Attempt to prove that if R is a PID then $SK_1(R) = 0$ and get a sense of what goes wrong.

8.24. **Longer localization sequences.** We next work on extending the localization sequence from Proposition 6.17 to the left using K_1 terms. See Theorem 8.27 below.

Proposition 8.25. *Let $S \subseteq R$ be a multiplicative system.*

(a) The group $K_1(S^{-1}R)$ is generated by classes $[S^{-1}R^n \xrightarrow{S^{-1}\alpha} S^{-1}R^n]$ where $\alpha\colon R^n \to R^n$ is such that $S^{-1}\alpha$ is an isomorphism.

(b) There is a unique map $\partial\colon K_1(S^{-1}R) \to K_0(R,S)$ having the property that if $\alpha\colon R^n \to R^n$ is such that $S^{-1}\alpha$ is an isomorphism, then ∂ sends $[S^{-1}R^n \xrightarrow{S^{-1}\alpha} S^{-1}R^n]$ to the class of the chain complex $0 \to R^n \xrightarrow{\alpha} R^n \to 0$ (concentrated in degrees 0 and 1).

Proof. First let $\beta\colon (S^{-1}R)^n \to (S^{-1}R)^n$ be an automorphism. Let A be the matrix for β with respect to the standard basis, and let $u \in S$ be an element such that uA has entries in R (e.g., take u to be the product of all the denominators of the entries in A). Then uA represents a map $\beta'\colon R^n \to R^n$, and we have the commutative diagram

$$\begin{array}{ccccc} (S^{-1}R)^n & \xrightarrow{\beta} & (S^{-1}R)^n & \xrightarrow{uI_n} & (S^{-1}R)^n \\ \uparrow & & & & \uparrow \\ R^n & & \xrightarrow{\qquad\beta'\qquad} & & R^n \end{array}$$

where the vertical maps are localization. This diagram gives $uI_n \circ \beta = S^{-1}\beta'$, and so $[uI_n] + [\beta] = [S^{-1}\beta']$ in $K_1(S^{-1}R)$. Note that $[uI_n] = n[uI_1]$, and uI_1 is itself the localization of the multiplication-by-u map on R; so we can write

$$[\beta] = [S^{-1}\beta'] - n[S^{-1}u]. \tag{8.26}$$

This shows that $K_1(S^{-1}R)$ is generated by classes $[S^{-1}\alpha]$ for $\alpha\colon R^n \to R^n$, and we have thereby proven (a) and the uniqueness part of (b).

For the existence part of (b), we will define a map $\partial\colon K_1^{sp,fr}(S^{-1}R) \to K_0(R,S)$ and then appeal to Theorem 8.9. Given an automorphism $\beta\colon (S^{-1}R)^n \to (S^{-1}R)^n$, choose a $u \in S$ such that the standard matrix representing $u\beta$ has entries in R. Consider the assignment

$$\beta \mapsto F(\beta,u) = [R^n \xrightarrow{u\beta} R^n] - n[R \xrightarrow{u} R] \in K_0(R,S).$$

Note that this expression doesn't come out of thin air: the expected homomorphism ∂, if it exists, must have this form by (8.26). It remains to show that the above formula does indeed define a homomorphism.

We first show that $F(\beta,u)$ does not depend on the choice of u. It suffices to prove that $F(\beta,tu) = F(\beta,u)$ for any $t \in S$; for if u' is another choice for u then we would have $F(\beta,u) = F(\beta,u'u) = F(\beta,u')$. But now we just compute that

$$\begin{aligned} F(\beta,tu) &= [R^n \xrightarrow{tu\beta} R^n] - n[R \xrightarrow{tu} R] \\ &= [R^n \xrightarrow{t} R^n] + [R^n \xrightarrow{u\beta} R^n] - n\Big[[R \xrightarrow{t} R] + [R \xrightarrow{u} R]\Big] \\ &= [R^n \xrightarrow{u\beta} R^n] - n[R \xrightarrow{u} R] \end{aligned}$$

(the second equality is by Lemma 6.16, applied twice).

Let us now write $F(\beta)$ instead of $F(\beta, u)$. The last things that must be checked are that $F(\beta \oplus \beta') = F(\beta) + F(\beta')$ and $F(\beta\gamma) = F(\beta)F(\gamma)$, but these are both immediate (the latter using Lemma 6.16). So we have established the existence of $\partial\colon K_1(S^{-1}R) \to K_0(R, S)$ having the desired properties. □

Theorem 8.27 (Localization sequence for K-theory)**.** *Let R be a commutative ring and $S \subseteq R$ a multiplicative system. The following sequence is exact:*

$$K_1(R) \longrightarrow K_1(S^{-1}R) \xrightarrow{\partial} K_0(R, S) \longrightarrow K_0(R) \longrightarrow K_0(S^{-1}R).$$

Proof. We will not prove exactness at $K_1(S^{-1}R)$, as this is a bit lengthy and would take us too far afield; a proof can be found in [Weibel2, Theorem III.3.2]. Exactness at $K_0(R)$ was already proven in Proposition 6.17, so it only remains to verify exactness at $K_0(R, S)$.

Let $x \in K_0(R, S)$. We know by Theorem 6.18 that x can be written in the form $x = [P_1 \to P_0] - [Q_1 \to Q_0]$ for finitely-generated projectives P_0, P_1, Q_0, Q_1 over R and maps $\alpha\colon P_1 \to P_0$, $\beta\colon Q_1 \to Q_0$ that become isomorphisms after S-localization. Consider the isomorphism $S^{-1}Q_0 \to S^{-1}Q_1$ that is the inverse to $S^{-1}\beta$. By Corollary 5.11(b) there is a map $\gamma\colon Q_0 \to Q_1$ whose localization is isomorphic to this map. Notice that

$$\begin{aligned} x &= [P_1 \xrightarrow{\alpha} P_0] + [Q_0 \xrightarrow{\gamma} Q_1] - \big([Q_0 \xrightarrow{\gamma} Q_1] + [Q_1 \xrightarrow{\beta} Q_0]\big) \\ &= [P_1 \oplus Q_0 \xrightarrow{\alpha\oplus\gamma} P_0 \oplus Q_1] - [Q_0 \oplus Q_1 \xrightarrow{\gamma\oplus\beta} Q_1 \oplus Q_0]. \end{aligned}$$

So by replacing our original P's and Q's we can assume that $Q_0 = Q_1$.

Let G be a finitely-generated projective such that $Q_0 \oplus G$ is free, and observe that

$$x = x - [G \xrightarrow{\mathrm{id}} G] = [P_1 \to P_0] - [Q_0 \oplus G \to Q_0 \oplus G].$$

So again, by replacing our chosen $Q_0 = Q_1$ we can actually assume that $Q_0 = Q_1$ is free. That is, $x = [P_1 \xrightarrow{\alpha} P_0] - [R^n \xrightarrow{\beta} R^n]$.

Now assume that x maps to zero in $K_0(R)$. This just says that $[P_0] = [P_1]$ in $K_0(R)$, and so there exists a free module G such that $P_0 \oplus G \cong P_1 \oplus G$ (Proposition 2.9). Since $x = x + [G \xrightarrow{id} G]$ we see that we can write x as

$$x = [R^k \xrightarrow{\alpha} R^k] - [R^n \xrightarrow{\beta} R^n]$$

where α and β become isomorphisms after S-localization. It is now immediate that x is in the image of ∂; to be completely specific,

$$x = \partial\Big([S^{-1}R^k \xrightarrow{S^{-1}\alpha} S^{-1}R^k] - [S^{-1}R^n \xrightarrow{S^{-1}\beta} S^{-1}R^n]\Big).$$

□

Example 8.28. This example will be a "reality check". We won't learn anything new, but we will see that the localization sequence is doing something sensible. Let R be a discrete valuation ring (a regular local ring whose maximal ideal is principal), and let F be the quotient field. Let x be a generator for the maximal ideal, and let $S = \{1, x, x^2, \ldots\}$. Note that $S^{-1}R = F$. The localization sequence for K-theory takes on the form

$$R^* \to F^* \xrightarrow{\partial} K_0(R, S) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$$

where we are using $K_1(R) \cong R^*$, $K_1(F) \cong F^*$, $K_0(R) \cong \mathbb{Z}$ (because R is a PID), and the map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ sends $[R]$ to $[F]$ and is therefore an isomorphism. So the localization sequence distills into a single isomorphism

$$F^*/R^* \xrightarrow{\partial} K_0(R,S).$$

The group F^*/R^* is readily checked to be $\mathbb{Z}$, where the isomorphism $\mathbb{Z} \cong F^*/R^*$ sends n to $[x^n]$. On the other hand, we also know by Theorem 6.20 that $K_0(R,S) \cong G(M \,|\, S^{-1}M = 0)$. A finitely-generated module M satisfies $S^{-1}M = 0$ if and only if M is killed by a power of x, or equivalently if M has finite length over R. The length map $\ell\colon G(M \,|\, S^{-1}M = 0) \to \mathbb{Z}$ is easily checked to be an isomorphism.

Finally, let us analyze the map ∂. Given an element $a \in F^*$, we write $a = r/x^n$ for some $r \in R^*$ and $n \geq 0$. The description of ∂ given in the proof of Proposition 8.25 shows that

$$\partial(a) = [R \xrightarrow{r} R] - [R \xrightarrow{x^n} R] = [R \xrightarrow{r} R] - n[R \xrightarrow{x} R].$$

The isomorphism $K_0(R,S) \to G(M \,|\, S^{-1}M = 0)$ sends a complex $P_\bullet$ to the alternating sum of its homology modules, so under this isomorphism we would write

$$\partial(a) = [R/rR] - n[R/xR].$$

Note that $\ell(R/xR) = 1$. We can write $r = ux^k$ for some unit $u \in R$ and $k \geq 0$, in which case $R/rR \cong R/x^kR$ and so $\ell(R/rR) = k$. It follows that the composite

$$F^* \xrightarrow{\partial} K_0(R,S) \xrightarrow{\cong} G(M \,|\, S^{-1}M = 0) \xrightarrow{\cong} \mathbb{Z}$$

sends $\frac{ux^k}{x^n}$ to $k - n$ and so is just the usual x-adic valuation on F^*.

Exercise 8.29. Let $R = \mathbb{Z}$ and $S = \{2^i \,|\, i \geq 0\}$. Compute all the groups and maps in the K-theory localization sequence, and also compare with the isomorphism $K(R,S) \cong G(M \,|\, S^{-1}M = 0)$.

The following example generalizes the previous one, but is a bit more interesting.

Example 8.30. Let D be a Dedekind domain—a regular ring of dimension one. In such a ring all nonzero primes are maximal ideals. Let $S = D - \{0\}$ and let $F = S^{-1}D$ be the quotient field. Our localization sequence looks like

$$K_1(D) \to F^* \to K_0(D,S) \to K_0(D) \to \mathbb{Z}.$$

Although we have not calculated $K_1(D)$, the commutative diagram

$$\begin{array}{ccc} K_1(D) & \longrightarrow & K_1(F) \\ {\scriptstyle \det}\downarrow & & {\scriptstyle \det}\downarrow \cong \\ D^* & \rightarrowtail & F^* \end{array}$$

shows that the image of $K_1(D)$ in F^* is just D^*. The map $K_0(D) \to \mathbb{Z}$ is just the usual rank map, so its kernel is $\widetilde{K}_0(D)$. So we get a short exact sequence

$$0 \to F^*/D^* \xrightarrow{\partial} K_0(D,S) \to \widetilde{K}_0(D) \to 0.$$

We know $K_0(D,S) \cong G(M \,|\, S^{-1}M = 0)$ by Theorem 6.20. The condition $S^{-1}M = 0$ just says that M is a torsion module. Consider the evident map

$$j\colon \bigoplus_{P \neq 0} G_0(D/P) \to G(M \,|\, S^{-1}M = 0)$$

where the direct sum is over all nonzero prime ideals and where the map just forgets that a module is defined over D/P and instead regards it as a D-module. This map is clearly surjective: a torsion D-module M will have a prime filtration in which the primes appearing are all maximal, and $[M]$ will be the sum of the corresponding $[D/P]$'s by the usual argument (see Theorem 3.1 and its proof).

Note that each D/P is a field, and so $G_0(D/P) \cong \mathbb{Z}$. If M is a torsion D-module then M_P is a torsion D_P-module. Since D_P is a discrete valuation ring, this means that M_P has finite length. Define

$$\chi\colon G(M \mid S^{-1}M = 0) \to \bigoplus_{P \neq 0} G_0(D/P)$$

by sending $[M]$ to the tuple of integers $\ell_{D_P}(M_P)$, as P runs over all maximal ideals of D (the only ones that give nonzero lengths are the ones containing $\operatorname{Ann} M$, and there are only finitely-many of these since they are precisely the minimal primes of $\operatorname{Ann} M$). It is easy to check that $\chi \circ j = \mathrm{id}$. Since j was already known to be surjective this means they are inverse isomorphisms. So we can rewrite our short exact sequence as

$$0 \to F^*/D^* \xrightarrow{\partial} \bigoplus_{P \neq 0} \mathbb{Z} \to \widetilde{K}_0(D) \to 0.$$

It will be convenient to write e_P for the basis element of the free abelian group in the middle corresponding to the maximal ideal P. Note that these basis elements correspond to the closed points of $\operatorname{Spec} D$, and so we are looking at a group of 0-cycles.

It remains to analyze the map ∂. By Proposition 8.25, if $r \in D - \{0\}$ then $\partial(r) = [D \xrightarrow{r} D] \in K_0(D, S)$. Under the isomorphisms described above this corresponds to the tuple of integers $\ell_{D_P}(D_P/rD_P)$. This is usually called the **divisor class** of r, and written

$$\operatorname{div}(r) = \sum \ell_{D_P}(D_P/rD_P)e_P.$$

It should be thought of as listing all the zeros of the "function" r, together with their orders of vanishing (see below for an example). For a general element $x \in F^*$ we would just write $x = r/s$ for $r, s \in D - \{0\}$, and then $\partial(x) = \operatorname{div}(r) - \operatorname{div}(s)$; this gives the zeros and poles of x, with multiplicities.

The quotient of $\bigoplus_{P \neq 0} \mathbb{Z}$ by the classes $\operatorname{div}(x)$ for $x \in F^*$ is called the **divisor class group** of D; it is isomorphic to the ideal class group from algebraic number theory, and is visibly the same as the homological Chow group $\mathrm{CH}_0(D)$ defined in Section 3. Our short exact sequence shows that $\widetilde{K}_0(D)$ is also isomorphic to this group, which recovers what we knew from combining Proposition 3.22 with the isomorphism $K_0(D) \cong G_0(D)$.

To demonstrate the geometric intuition behind $\operatorname{div}(r)$, consider the case $D = F[t]$ where F is algebraically closed. If $r = p(t)$ is nonzero then the maximal ideals containing r are the ones $(t - a_i)$ where a_i is a root of $p(t)$. If we write $r = u\prod(t - a_j)^{m_j}$ where $u \in F^*$ and we localize at $P = (t - a_i)$, then r becomes a unit multiple of $(t - a_i)^{m_i}$ and the number $\ell_{D_P}(D_P/rD_P)$ is precisely m_i. So

$$\operatorname{div}(r) = \sum_i m_i \cdot e_{(t-a_i)},$$

as expected. Note that the divisor class group is not very interesting in this case: clearly div is surjective, and so the group is zero. We already knew this for another reason, because $\widetilde{K}_0(D) = 0$ whenever D is a PID.

Example 8.31. As one more example, let us return to the ring $R = \mathbb{Z}[\sqrt{-5}]$ and $S = \{2^i \mid i \geq 0\}$. Some of the computations here are a little difficult, but one can determine that $K_1(R) \to R^*$ is an isomorphism because [Milnor7, Corollary 16.3] tells us this holds for all rings of integers in number fields. One can also compute that $R^* = \{1, -1\}$ by using the fact that the norm map $N(a + b\sqrt{-5}) = a^2 + 5b^2$ is multiplicative. The localization sequence takes the following form:

$$\begin{array}{ccccccccc}
K_1(R) & \longrightarrow & K_1(S^{-1}R) & \xrightarrow{\partial} & K_0(R,S) & \longrightarrow & K_0(R) & \longrightarrow & K_0(S^{-1}R) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathbb{Z}/2 & \xrightarrow{c} & SK_1(S^{-1}R) \oplus \mathbb{Z} \oplus \mathbb{Z}/2 & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{b} & \mathbb{Z} \oplus \mathbb{Z}/2 & \xrightarrow{a} & \mathbb{Z}
\end{array}$$

Let us recall how these isomorphisms work, identify generators for the groups, and see what the maps do. Since $S^{-1}R$ is a PID one has that $K_0(S^{-1}R)$ is $\mathbb{Z}$, generated by $[S^{-1}R]$. We have seen in Example 5.2 that $K_0(R)$ is generated by $[R]$ and $[R]-[I]$ where $I = (2, 1 + \sqrt{-5})$ (recall that I is projective), with the former generating the $\mathbb{Z}$ summand and the latter the $\mathbb{Z}/2$. The map a is S-localization and therefore sends $[R]$ to $[S^{-1}R]$ and $[R] - [I]$ to zero.

Since R is regular we have isomorphisms $K_0(R,S) \cong G(M \mid S^{-1}M = 0) \cong G_0(R/(2))$ where the first is the Euler characteristic and the second is dévissage (Theorem 5.17). Note that $R/(2) \cong \mathbb{Z}/2[x]/(x^2+1) \cong \mathbb{Z}/2[x]/((x+1)^2) \cong \mathbb{Z}/2[u]/(u^2)$ and so $G_0(R/(2))$ is $\mathbb{Z}$, generated by $\mathbb{Z}/2[u]/(u) = R/(2, 1+\sqrt{-5}) = R/I$. Tracing this back into $K_0(R,S)$, we see that the group is generated by the chain complex $[0 \to I \hookrightarrow R \to 0]$. The map b sends this to $[R] - [I]$, which is the generator of the $\mathbb{Z}/2$. So it must be that $2[I \hookrightarrow R]$ is in the image of ∂, but let us check this.

Recall that $K_1(S^{-1}R) = SK_1(S^{-1}R) \oplus (S^{-1}R)^*$. The second summand is clearly $\mathbb{Z} \oplus \mathbb{Z}/2$ with generators 2 and -1. Under ∂ these map to $[R \xrightarrow{2} R]$ and $[R \xrightarrow{-1} R]$, the latter of which is zero (being acyclic). Under the isomorphism $K_0(R,S) \cong G_0(R/(2))$ the complex $[R \xrightarrow{2} R]$ maps to $R/(2)$, and this is twice the generator (look at the dimension over $\mathbb{Z}/2$). So $\partial(2) = 2[I \hookrightarrow R]$, as expected. The long exact sequence now shows that $SK_1(S^{-1}R)$ must be zero.

Remark 8.32. The localization sequence of Theorem 8.27 can be extended further to the left, by defininining K-groups $K_n(R)$ and $K_n(R,S)$ for all $n \geq 1$. This is the subject of higher algebraic K-theory, a deep field with intricate connections to many area of mathematics. It does not really do justice to the subject for us to give just a few references, but some places to start learning about it are [Bass1], [Milnor7], [Quillen5], [Rosenberg], and [Weibel2].

Part 2. A brief detour

9. Background from homotopy theory

In the rest of this book we will be dealing with topological spaces and generalized cohomology theories. Most of the time we can get by with only a basic knowledge of algebraic topology, but there are several places where a deeper familiarity with homotopy theory makes arguments simpler or offers a useful perspective. The language of cofibrations and fibrations, homotopy pushouts and pullbacks, mapping telescopes, and Puppe sequences are all basic tools of the trade. Our aim in this section is to give a very quick tour, just to establish the language and ideas. In the later parts of the section we also give a quick introduction to topological spectra, but this material will not be needed until Part 4. The reader is welcome to skip that discussion and refer back at a later time.

9.1. **The homotopy category of spaces.** In naive approaches to algebraic topology one works with a category of spaces where the maps from X to Y are homotopy classes $[X, Y]$. That is fine for getting the basic ideas, and it is fine when the spaces one encounters are always reasonably nice (e.g. CW-complexes), but it turns out to be the wrong thing to do in general. Over time a certain understanding developed of how to adjust the situation to get the "right" answers, and those ideas were eventually codified into the idea of model categories. A model category is a "model" for a homotopy theory. This is not the only approach, but it is commonly used and quite useful once you get the hang of the ideas.

Our intent here is not to give a course in model categories, as that would take us too far afield and the actual content that we will need amounts to very little. Let us instead just call it a quick tour to establish some necessary language.

Recall that a map of spaces $f\colon X \to Y$ is a **weak homotopy equivalence** if $f_*\colon \pi_0 X \to \pi_0 Y$ is a bijection and $f_*\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for every $n \geq 1$ and every $x \in X$. Briefly, one says that a weak homotopy equivalence is a map that induces isomorphisms on all homotopy groups (overlooking the fact that π_0 is not a group). The phrase "weak homotopy equivalence" is usually abbreviated to just "weak equivalence", and maps that are weak equivalences are denoted by $\xrightarrow{\sim}$. Spaces X and Y are said to be **weakly equivalent** if there is a zig-zag (of arbitrary length) of weak equivalences that starts with X and ends with Y.

Let $\mathcal{W}$ denote the class of weak equivalences. The **homotopy category of spaces** is defined to be the localization $\mathcal{W}^{-1}\mathcal{T}op$, i.e. the category one obtains by formally inverting the morphisms in $\mathcal{W}$. There are known set-theoretic issues in constructing such localization categories, though one can usually get around them using Grothendieck universes. But here one can show explicitly, without any set-theoretic shenanigans, that $\mathcal{W}^{-1}\mathcal{T}op$ does exist (this is one of the things model categories do for us). From now on we will write it as $\mathrm{Ho}\,(\mathcal{T}op)$, though often $\mathrm{Ho}\,\mathcal{T}$ for short. Two spaces X and Y are weakly equivalent if and only if they are isomorphic in $\mathrm{Ho}\,\mathcal{T}$.

If X is a space then the projection $X \times I \to X$ is a left inverse for the two inclusions $i_0, i_1\colon X \hookrightarrow X \times I$. In $\mathrm{Ho}\,(\mathcal{T}op)$ all of these maps become isomorphisms, and so i_0 and i_1 must become *equal* there. It follows that homotopic maps become equal in $\mathrm{Ho}\,(\mathcal{T}op)$, and so the canonical maps $\mathcal{T}op(X, Y) \to \mathrm{Ho}\,\mathcal{T}(X, Y)$ induce maps

$[X, Y] \to \mathrm{Ho}\,\mathcal{T}(X, Y)$. These are not always bijections, though they are when X is a CW-complex (more on this in a moment).

The fact that maps in the homotopy category are not homotopy classes of maps takes some getting used to. This is a place where the language is perhaps not perfect. It helps to define the **naive homotopy category** to be the category whose objects are spaces and where the maps are homotopy classes $[X, Y]$. There is a canonical functor from the naive homotopy category to $\mathrm{Ho}\,(\mathcal{T}op)$, but this is not an equivalence. It *is* an equivalence if one restricts to the full subcategories consisting of the CW-complexes.

Remark 9.2. Let R be a ring, $\mathcal{C}\mathrm{h}_R$ the category of chain complexes over R, and $\mathcal{W}$ the class of quasi-isomorphisms (maps that induced isomorphisms on all homology groups). Algebraists call $\mathcal{W}^{-1}\,\mathcal{C}\mathrm{h}_R$ the **derived category** of R, and topologists call it the homotopy category of R-chain complexes. Note the analogy with $\mathrm{Ho}\,(\mathcal{T}op)$.

In early work in homotopy theory and in homological algebra it developed that certain classes of "nice inclusions" and "nice surjections" arose as being of special importance. Model categories are an attempt to axiomatize the properties that were common to those two (and other) situations.

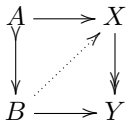
A model category is a category that comes with three classes of maps: the weak equivalences, the cofibrations, and the fibrations. There is a short list of axioms that we will come to in a moment. For $\mathcal{T}op$ the fibrations are the Serre fibrations, and the cofibrations are retracts of generalized cellular inclusions—this means any map that is obtained by iteratively attaching cells (not necessarily in order of their dimension). Any CW-inclusion is a cofibration, and these will be our main examples. Cofibrations are denoted as $\rightarrowtail$ and fibrations as $\twoheadrightarrow$.

Remark 9.3. It should be explained that there were older notions of cofibration/fibration that are in some sense "overwritten" by the model category language, perhaps regretfully. Those older notions are still used by some authors, but here we will call them *Hurewicz cofibrations* and *Hurewicz fibrations*. We mostly will not use these notions, but a Hurewicz cofibration is any map with the homotopy extension property with respect to all spaces, and a Hurewicz fibration is a map with the homotopy lifting property with respect to all spaces. Every Hurewicz fibration is a Serre fibration, and every cofibration is a Hurewicz cofibration.

Here is a short version of the model category axioms. It is intentionally incomplete, but enough to establish the main ideas. We need one more piece of language: maps that are both cofibrations and weak equivalences are called **acyclic cofibrations** (denoted $\xrightarrow{\sim}$), and maps that are both fibrations and weak equivalences are called **acyclic fibrations** (denoted $\xrightarrow{\sim}$).

(1) The class of cofibrations is closed under composition, pushouts, retracts, and transfinite directed composition (that is to say, if $Y_1 \rightarrowtail Y_2 \rightarrowtail \cdots$ is an infinite sequence of cofibrations then the composite $Y_1 \to \operatorname{colim} Y_n$ is a cofibration). Same for the class of acyclic cofibrations.
(2) The class of fibrations is closed under composition, pullbacks, retracts, and transfinite inverse-directed composition. Same for the class of acyclic fibrations.

(3) Any solid-arrow diagram



where either of the vertical maps is a weak equivalence admits a lifting as shown.

(4) Every map $f\colon X \to Y$ can be functorially factored in two ways: as a cofibration followed by an acyclic fibration, or as an acyclic cofibration followed by a fibration. In symbols this looks like



It is a worthwhile exercise to rephrase things like the homotopy extension and homotopy lifting properties in terms of diagrams such as in (3). Property (4) is a "resolution" or "replacement" property: it says (in part) that any map f can be replaced, up to weak equivalence, with one that is a cofibration or a fibration.

If $\emptyset \to X$ is a cofibration then the space X is called **cofibrant**. Every cell complex is cofibrant. In a general model category one could call an object Y **fibrant** if $Y \to *$ is a fibration, but since that is true for all topological spaces Y the term has no point in this context. Any map $\emptyset \to X$ can be factored as $\emptyset \rightarrowtail QX \xrightarrow{\sim}\!\!\!\!\!\rightarrow X$. Any space (such as QX) that is cofibrant and weakly equivalent to X is called a **cofibrant replacement** for X. All such cofibrant replacements are homotopy equivalent. The main example of cofibrant replacements are the the usual CW-approximations.

Remark 9.4. All cofibrant objects are paracompact and Hausdorff, a fact that is worth remembering. To see why, start with the fact that every cofibrant object is a retract of a cell complex. Every cell complex is paracompact and Hausdorff, by the same argument that works for CW-complexes. A retract of a Hausdorff space is necessarily a closed subspace, and closed subspaces of paracompact Hausdorff spaces are paracompact.

The general theory of model categories implies that when X is cofibrant (and Y is fibrant, though that is tautologous for us) the map $[X, Y] \to \mathrm{Ho}\,\mathcal{T}(X, Y)$ is a bijection. Colloquially speaking, when X is cofibrant the naive set of homotopy classes $[X, Y]$ gives the "right" answer.

The category of pointed spaces $\mathcal{T}op_*$ also has an associated model category structure, with the same notions of weak equivalence, cofibration, and fibration as $\mathcal{T}op$ (but where all maps are pointed, of course). In this context, though, the intial object is $*$ (not $\emptyset$) and so a pointed space X is cofibrant if the map $* \to X$ is a cofibration. The terminology "X is well-pointed" is also used for this.

The natural map $[X, Y] \to \mathrm{Ho}\,\mathcal{T}(X, Y)$ is a bijection when X is cofibrant, but it turns out to be a bijection for other spaces X as well. Let us say that X is **semi-cofibrant** if this map is a bijection for all spaces Y. This terminology is not standard, but it is very useful. It is easy to see that a space is semi-cofibrant if

and only if it is homotopy equivalent to a CW-complex (see the following exercise). In older literature one often finds the hypothesis "Let X be a space homotopy equivalent to a CW-complex", and it is much more pleasant just to say "Let X be semi-cofibrant."

Exercise 9.5. Prove that a weak equivalence between semi-cofibrant spaces is a homotopy equivalence. Deduce that every semi-cofibrant space is homotopy equivalent to a CW-complex, and prove the converse as well.

Most spaces one naturally encounters are semi-cofibrant. Milnor showed that all ANR spaces (see Section A.11) are semi-cofibrant. The following result, which we prove in Section A.11, gives some handy sufficient conditions. Say that a space X is **semi-locally contractible** if for every $x \in X$ and every open set $x \in U$ there is an open set $x \in V \subseteq U$ such that the inclusion $V \hookrightarrow U$ is null-homotopic. (Note: This notion is called "locally contractible" in much of the general topology literature.)

Proposition 9.6. *Let X be a topological space. If any of the following conditions are satisfied then X is an ANR and hence semi-cofibrant:*

(1) X is metrizable and semi-locally contractible and has finite covering dimension (note that any subspace of $\mathbb{R}^n$ satisfies the last condition).
(2) X is an open subset of a locally-finite CW-complex (this is just a convenient subset of (1)).
(3) X is paracompact Hausdorff and every point $x \in X$ has an open neighborhood that is homeomorphic to a space of type (1) or (2).

For us the main point is that every locally contractible subset of $\mathbb{R}^n$ is semi-cofibrant, and the same for any paracompact Hausdorff space that looks locally like such things. This covers all complex algebraic varieties, for example.

The standard example of a space that is not semi-cofibrant is the Warsaw circle W. This space is weakly contractible, so $\mathrm{Ho}\,\mathcal{T}(W, W) = *$. Thus if W were semi-cofibrant then $[W, W] = *$ and so the identity map would have to be null-homotopic—i.e., W would be contractible. But W has nontrivial Čech cohomology in degree 1 (with constant $\mathbb{Z}$ coefficients), and Čech cohomology is a homotopy invariant. So W cannot be contractible, and is therefore not semi-cofibrant.

Note also that spaces like $\mathbb{R}^\infty$ and $\mathbb{R}P^\infty$, while cofibrant (hence semi-cofibrant), are not ANRs. The inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ and $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ are cofibrations, so $\emptyset \to \mathbb{R}^\infty$ and $\emptyset \to \mathbb{R}P^\infty$ are as well. But $\mathbb{R}^\infty$ and $\mathbb{R}P^\infty$ are not first-countable (see Exercise A.38), therefore not metrizable, and therefore not ANRs. Proposition 9.6 is useful for its applications to geometric situations, but is not a universal panacea for identifying common semi-cofibrant spaces.

Our discussion here has attempted to give readers a general introduction to the language of model categories with a focus on how it plays out in $\mathcal{T}op$, so that when it occassionally arises it won't seem frightening. What has not come across in this treatment are the beautiful ways that model categories bring together various themes from homotopy theory and homological algebra that once upon a time seemed quite disparate. For example, there is a model category structure on $\mathcal{C}\mathrm{h}_R$ where the weak equivalences are the quasi-isomorphisms, the fibrations are the surjections, and the cofibrant objects are the degreewise-projective complexes. For more information about model categories we recommend [DS] for a nice general introduction, as well as [Hovey] and [Hirschhorn] for more in-depth treatments.

9.7. **Homotopy colimits and limits.** When A is a subspace of X one expects a long exact sequence in cohomology $\cdots \to H_*(A) \to H_*(X) \to \tilde{H}_*(X/A) \to \cdots$ only if $A \hookrightarrow X$ is nice enough, e.g. a cellular inclusion. This is because the space X/A can have the "wrong" homotopy type if the inclusion $A \hookrightarrow X$ is bad. To be more precise, given two inclusions $A \hookrightarrow X$ and $A' \hookrightarrow X'$ and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \simeq\downarrow & & \simeq\downarrow \\ A' & \longrightarrow & X' \end{array}$$

where the vertical maps are weak equivalences, there is no guarantee that the induced map $X/A \to X'/A'$ is a weak equivalence. However this *does* work if $A \hookrightarrow X$ and $A' \hookrightarrow X'$ are cofibrations.

There are two classical ways to get around this problem. One is to factor $A \to X$ as $A \rightarrowtail X' \xrightarrow{\sim}\!\!\!\!\!\rightarrow X$ and then use X'/A instead. In words, we have replaced $A \to X$ with a weakly equivalent cofibration. The other way around the problem is to replace X/A with the mapping cone. In the above situation, it turns out that the map $\mathrm{Cone}(A \to X) \to \mathrm{Cone}(A' \to X')$ is always a weak equivalence (see Exercise 9.21). The space $\mathrm{Cone}(A \to X)$ is a kind of "homotopical quotient"—rather than literally squash A to a point, we have instead added a homotopy that contracts A to a point.

The quotient space X/A is the pushout of $* \leftarrow A \to X$, whereas the mapping cone is called the **homotopy pushout** (equivalently, the **homotopy cofiber** of the inclusion $A \hookrightarrow X$). Note that there is a canonical map $\mathrm{Cone}(A \to X) \to X/A$ that collapses the cone on A to a point.

More generally, for any diagram of spaces $D\colon I \to \mathcal{T}op$ there is a construction called the homotopy colimit, denoted $\mathrm{hocolim}_I D$, which comes with a canonical map $\mathrm{hocolim}_I D \to \mathrm{colim}_I D$. The homotopy colimit has the property that given a map of diagrams $D \to D'$ that is a weak equivalence on each object, the induced map $\mathrm{hocolim}_I D \to \mathrm{hocolim}_I D'$ is a weak equivalence. In addition, if the diagram D is "cofibrant enough" in a certain technical sense (requiring certain maps to be cofibrations) then the canonical map $\mathrm{hocolim}_I D \to \mathrm{colim}_I D$ is a weak equivalence.

There is a dual version of the story for homotopy limits. For example, the homotopy pullback of a diagram $* \to B \xleftarrow{p} E$ (also called the **homotopy fiber** of p) is the pullback of $PB \to B \leftarrow E$ where PB is the path space of B (all paths that start at the basepoint): a point in the homotopy pullback is a point $e \in E$ and a path from $p(e)$ to the basepoint.

There is a very well-developed theory of homotopy colimits and limits in model categories. Modern references are [Hovey] and [Hirschhorn], or see [D6] for an introductory account. But aside from homotopy pushouts and pullbacks, there is really only one other case that we will need in this book. Consider a sequence of maps $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ The homotopy colimit of this diagram can be constructed as the **mapping telescope**:

$$\mathrm{hocolim}\, X_\bullet = \Big(\coprod_{n\geq 1} (X_n \times [0,1])\Big)/\sim$$

where the equivalence relation has $(x,1) \sim (f_n(x),0)$ for $x \in X_n$. See Figure 9.7 for a picture.

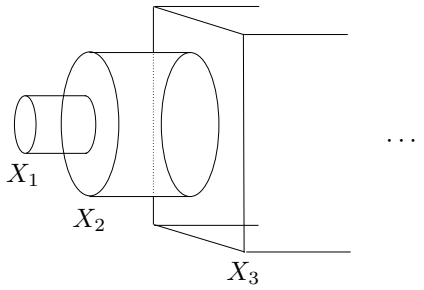


FIGURE 9.7.1. The mapping telescope

Proposition 9.8. *Suppose that all of the X_i are Hausdorff.*

(a) Suppose for convenience that all of the X_i are path-connected, pick a basepoint $* \in X_1$, and let the image of this in each X_i be the basepoint for X_i. Then for every $k \geq 0$ there is a natural map

$$\operatorname*{colim}_n \pi_k(X_n, *) \to \pi_k(\operatorname{hocolim} X_\bullet),$$

and this is an isomorphism.

(b) Suppose that each f_i is a closed inclusion. Then the natural map $\operatorname{hocolim} X_\bullet \to \operatorname{colim} X_\bullet$ *is a weak equivalence.*

Sketch. The point in both parts if that given a sequence $Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots$ of closed inclusions of Hausdorff spaces with colimit Y, the induced map $\operatorname{colim}_n \pi_k(Y_n, *) \to \pi_k(Y, *)$ is an isomorphism. This follows at once from the fact that compact spaces are small with respect to closed inclusions of Hausdorff spaces (Exercise A.6).

For (a), let $T = \operatorname{hocolim} X_\bullet$. Let T_n be the portion of the telescope that runs from $X_1 \times \{0\}$ through $X_n \times \{0\}$. The canonical inclusion $X_n \times \{0\} \hookrightarrow T_n$ is a homotopy equivalence, with homotopy inverse the evident squash map $T_n \to X_n \times \{0\}$. The maps

$$T_0 \hookrightarrow T_1 \hookrightarrow T_2 \hookrightarrow \cdots$$

are closed inclusions and their colimit is T. Let the basepoint for $X_0 = T_0$ be the basepoint for each T_n. Then we have

$$\operatorname*{colim}_n \pi_k(T_n, *) \xrightarrow{\cong} \pi_k(T, *)$$

as well as isomorphisms $\pi_k(T_n, *) \to \pi_k(X_n, *)$ induced by the squash map $T_n \to X_n$. This gives us what we wanted.

Part (b) follows easily from (a) and the similar formula for $\pi_n(\operatorname{colim} X_\bullet)$. □

9.9. **Long exact sequences via homotopy theory.** Let X and Z be pointed spaces. Then $[X, Z]_*$ is just a pointed set, but if we suspend the space in the domain then we get a bit more structure: $[\Sigma X, Z]_*$ is a group, where ΣX is the reduced suspension of X. One way to see this is to collapse the equatorial copy of X in ΣX, to get $\Sigma X \vee \Sigma X$; write this collapse map as

$$\nabla \colon \Sigma X \to \Sigma X \vee \Sigma X.$$

The operation on $[\Sigma X, Z]_*$ is defined by precomposing the wedge of two homotopy classes with ∇. With some trouble one checks that ΣX is a cogroup object in the homotopy category of pointed spaces, which yields that $[\Sigma X, Z]_*$ is a group.

Here is another way to think about this, which relates it to something we already know. Let $F(X, Z)$ be the set of functions from X to Z, equipped with the compact-open topology. We can write

$$[\Sigma X, Z]_* = [S^1, F(X, Z)]_* = \pi_1(F(X, Z))$$

where the basepoint of $F(X, Z)$ is the map sending all of X to the basepoint of Z. Now just use that $\pi_1(F(X, Z))$ is a group.

When $k \geq 2$ we have $[\Sigma^k X, Z]_* = \pi_k(F(X, Z))$ by a similar argument, and so $[\Sigma^k X, Z]_*$ is an abelian group. Alternatively, one proves that now $\Sigma^k X$ is a *cocommutative* cogroup object in the homotopy category.

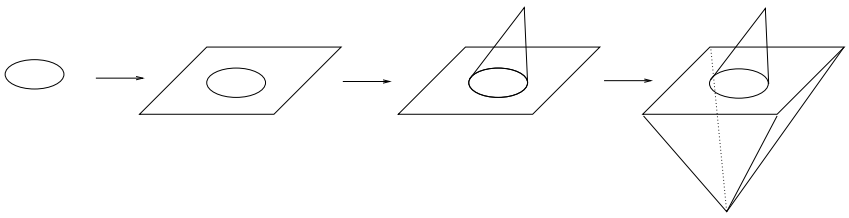
Similar results are obtained by putting conditions on Z rather than X. If Z is a loop space, say $Z \simeq \Omega Z_1$, then $[X, Z]_* \cong [X, \Omega Z_1]_* \cong [\Sigma X, Z_1]_*$, and this is a group by the above arguments. Similarly, if Z is a k-fold loop space for $k \geq 2$, say $Z \simeq \Omega^k Z_1$, then $[X, Z]_* \cong [\Sigma^k, Z_1]_*$ and this is an abelian group.

Homotopy classes of maps into a fixed space Z always give rise to exact sequences:

Proposition 9.10. *Let X, Y be pointed spaces, and let $f\colon X \to Y$ be a pointed map. Consider the mapping cone Cf and the natural map $p\colon Y \to Cf$. For any pointed space Z, the sequence of pointed sets $[X, Z]_* \leftarrow [Y, Z]_* \leftarrow [Cf, Z]_*$ is exact in the middle (meaning that anything in $[Y, Z]_*$ which is sent to the basepoint is in the image of the previous map).*

Proof. Let $h\colon Y \to Z$ and suppose $h \circ f$ is homotopic to the constant map. Choose a pointed homotopy $H\colon X \times I \to Z$ so that $H(X \times \{1\}) = *$. Then H induces a map $CX \to Z$. Let $g\colon Cf \to Z$ be given by H on CX and h on Y. Then clearly $g \circ p = h$. □

Given $f\colon X \to Y$ we form the mapping cone Cf, which comes to us with an inclusion $j_0\colon Y \hookrightarrow Cf$. Next form the mapping cone on j_0, which comes with an inclusion $j_1\colon Cf \hookrightarrow Cj_0$. Keep doing this forever to get the sequence of spaces $X \to Y \to Cf \to Cj_0 \to Cj_1 \to \cdots$ depicted below:



Note that $Cj_0 \simeq \Sigma X$ and $Cj_1 \simeq \Sigma Y$ (this is clear from the pictures). Up to sign the map $Cj_0 \to Cj_1$ is just Σf, so that the sequence of spaces becomes periodic:

$$X \to Y \to Cf \to \Sigma X \to \Sigma Y \to \Sigma(Cf) \to \Sigma^2 X \to \dots$$

This is called the **Puppe sequence**. Note that the composition of two subsequent maps is null-homotopic, and that every three successive terms form a cofiber sequence.

Now let Z be a fixed space and apply $[-, Z]_*$ to the Puppe sequence. We obtain the sequence of pointed sets

$$[X, Z]_* \leftarrow [Y, Z]_* \leftarrow [Cf, Z]_* \leftarrow [\Sigma X, Z]_* \leftarrow [\Sigma Y, Z]_* \leftarrow [\Sigma(Cf), Z]_* \leftarrow \dots$$

By Proposition 9.10 this sequence is exact at every spot where this makes sense (everywhere except at $[X, Z]_*$). At the left end this is just an exact sequence of pointed sets, but as one moves to the right at some point it becomes an exact sequence of groups (namely, at $[\Sigma Y, Z]_*$). As one moves further to the right, it becomes an exact sequence of *abelian* groups by the time one gets to $[\Sigma^2 Y, Z]_*$.

If $Z \simeq \Omega Z_1$ then we can extend the above sequence a little further to the left, by noticing that the sequences for $[-, Z_1]_*$ and $[-, Z]_*$ mesh together:

$$\begin{array}{ccccccccc}
[X, Z_1]_* & \leftarrow & [Y, Z_1]_* & \leftarrow & [Cf, Z_1]_* & \leftarrow & [\Sigma X, Z_1]_* & \leftarrow & [\Sigma Y, Z_1]_* & \leftarrow & [\Sigma(Cf), Z_1] & \leftarrow \\
 & & & & & & \| & & \| & & \| & \\
 & & & & & & [X, \Omega Z_1]_* & \leftarrow & [Y, \Omega Z_1]_* & \leftarrow & [Cf, \Omega Z_1] & \leftarrow \\
 & & & & & & \| & & \| & & \| & \\
 & & & & & & [X, Z]_* & \leftarrow & [Y, Z]_* & \leftarrow & [Cf, Z] & \leftarrow
\end{array}$$

Note that the leftmost cycle of the original sequence, which we had thought consisted just of pointed sets, in fact consisted of groups! If in turn we have $Z_1 \simeq \Omega Z_2$ then we can play this game again and extend the sequence one more cycle to the left, and so forth. If we are *really* lucky then we can do this forever:

Definition 9.11. *An **infinite loop space** is a space Z_0 together with spaces $Z_1, Z_2, Z_3, \dots$ and weak homotopy equivalences $Z_n \simeq \Omega Z_{n+1}$ for all $n \geq 0$.*

Note that if Z is an infinite loop space then we really do get a long exact sequence—infinite in both directions—consisting entirely of abelian groups, having the form

$$\cdots \leftarrow [Cf, Z_{i+1}]_* \leftarrow [X, Z_i]_* \leftarrow [Y, Z_i]_* \leftarrow [Cf, Z_i]_* \leftarrow [X, Z_{i-1}]_* \leftarrow \cdots$$

where it is convenient to use the indexing convention $Z_{-n} = \Omega^n Z$ for $n > 0$.

This situation is very reminiscent of a long exact sequence in cohomology, so let us adopt the following notation: write

$$E_Z^i(X) = [X_+, Z_i]_* = \begin{cases} [X_+, Z_i]_* & i \geq 0, \\ [\Sigma^{-i}(X_+), Z_0]_* & i < 0. \end{cases}$$

For an inclusion of subspaces $j\colon A \hookrightarrow X$ write

$$E_Z^i(X, A) = [Cj, Z_i]_* = \begin{cases} [Cj, Z_i]_* & i \geq 0, \\ [\Sigma^i(Cj), Z_0]_* & i < 0. \end{cases}$$

This is *almost* a generalized cohomology theory, but not quite. There is a small problem with excision. To explain this, suppose we have a triple $U \subseteq A \subseteq X$ where $\bar{U} \subseteq \text{int}(A)$. We get an induced map $\text{Cone}(X - U, A - U) \to \text{Cone}(X, A)$. A little work shows that this is a weak homotopy equivalence—see Exercise 9.23. But without additional assumptions on the spaces it does not seem to be a homotopy equivalence. So applying $[-, Z_i]_*$ does not necessarily give isomorphisms.

The fix is easy enough: we just need to be working in Ho $(\mathcal{T}op_*)$ rather than in the naive homotopy category. If we go back and replace homotopy classes everywhere with maps in Ho $(\mathcal{T}op_*)$, everything works the same except that now excision is also satisfied.

The moral of the story is that every infinite loop space gives rise to a generalized cohomology theory. It also works the other way around: every generalized cohomology theory gives rise to an infinite loop space. But for that part of the story it is best to use the language of spectra.

9.12. **Spectra.** The topological notion of spectra can feel intimidating when one first encounters it. We will need very little of the technical details in the rest of the book, but occasionally it is very helpful to use the basic language. The reader is encouraged to just skim this section for the main ideas.

If X and Y are pointed, finite CW-complexes then the Freudenthal suspension theorem says that the sequence

$$[X, Y]_* \to [\Sigma X, \Sigma Y]_* \to [\Sigma^2 X, \Sigma^2 Y]_* \to \cdots$$

eventually stabilizes. Experience shows that the stable values (which are abelian groups) are better behaved and more accessible than the previous terms in the sequence, and so one might envision a version of the homotopy category where those are the sets of morphisms. Even if X and Y are not finite complexes, one could still take the colimit of the above sequence and hope that it is the right object to study.

Associated to this, one might like to introduce formal desuspensions $\Sigma^{-k}X$ of pointed spaces X. We know what $[\Sigma^{-k}X, \Sigma^{-s}Y]_*$ should be, because it should be the $\operatorname{colim}_n [\Sigma^{n-k}X, \Sigma^{n-s}Y]_*$ and for large enough n we are back in spaces and know how to interpret everything. We might regard objects like $\Sigma^{-k}X$ as "spaces" that have both positive- and negative-dimensional cells.

Finally, if we are going to be extremely optimistic we might envision working with these "spaces" just like we do ordinary spaces. We would like to form homotopy colimits and limits, smash products, function objects, and so forth—all of the usual homotopical constructions that we do for spaces. If all goes well, there will be a model category structure around.

It turns out that one can make sense of all this, and that is what spectra are. You can think about them as spaces with negative cells, and you can think of them as the result of inverting the suspension functor Σ on $\mathcal{T}op$.

We are not going to recount here the machinery that is needed to make all of this work. See [Adams5] for a classical introduction, or [D5] for a survey of the modern approaches via model categories. The paper [Adams4] (especially Chapter 3) gives an interesting survey of some of the early issues people encountered in setting up a stable homotopy category, and includes Adams's much-remembered analogy of the tortoise and the hare.

While our aim is not to give a technical introduction to spectra, we want to recount the most basic definitions and guide the reader in navigating hthis subject.

Definition 9.13. *A* ***spectrum*** *is a sequence of pointed spaces W_n for $n \geq 0$ together with maps (known as "structure maps") $\Sigma W_n \to W_{n+1}$ for every n. A map of spectra $W \to Z$ is a collection of maps $W_n \to Z_n$ making the evident squares commute. Write* ***Spectra*** *for the resulting category.*

In a very formal sense one should think of the spectrum W as the colimit of a sequence of spaces (or perhaps "spaces")

$$W_0 \to \Sigma^{-1}W_1 \to \Sigma^{-2}W_2 \to \cdots$$

where the maps are desuspensions of the structure maps. Of course nothing about this diagram makes sense in $\mathcal{T}op$, but it will turn out that it does make sense in $\mathcal{S}pectra$.

Define the homotopy groups of a spectrum W by

$$\pi_k(W) = \operatorname*{colim}_n \pi_{k+n}(W_n)$$

where the maps in the colimit system are induced by the structure maps: $\pi_{k+n}(W_n) \xrightarrow{\Sigma} \pi_{k+n+1}(\Sigma W_n) \to \pi_{k+n+1}(W_{n+1})$. Note that these are always abelian groups, and also that $\pi_k(W)$ makes sense even for negative k because we only need the groups in the colimit system to be defined for $n \gg 0$. Say that a map of spectra is a weak equivalence if it induces isomorphisms on all homotopy groups. Define the homotopy category $\mathrm{Ho}\,(\mathcal{S}pectra)$ to be the localization of $\mathcal{S}pectra$ at this class of maps.

Every pointed space X gives rise to a **suspension spectrum** $\Sigma^\infty X$: this is the object with $(\Sigma^\infty X)_n = \Sigma^n X$ and the identity maps as structure maps. It is best to think of this as saying that every pointed space "is" a spectrum; the symbol Σ^∞ is just what we write to remind us that we have gone from the world of spaces into the world of spectra. In some cases it is very useful to abbreviate $\Sigma^\infty X$ to just X. Be warned, though, that moving from spaces to spectra loses information: we lose all of the unstable information about a space, keeping only the stable part.

Exercise 9.14. Show that $\pi_k(\Sigma^\infty S^n) = 0$ for $k < n$ and $\pi_n(\Sigma^\infty S^n) \cong \mathbb{Z}$. But convince yourself that $\pi_1(\Sigma^\infty(S^1 \vee S^1)) \cong \mathbb{Z}^2$. Passing from $S^1 \vee S^1$ to $\Sigma^\infty(S^1 \vee S^1)$ has lost the nonabelian information in π_1. In fact, for any connected space show that $\pi_1(\Sigma^\infty X) \cong H_1(X)$.

If W is a spectrum define ΣW to be the result of shifting all of the spaces (and maps) in W down one index: $(\Sigma W)_n = W_{n+1}$ (note that W_0 is discarded in this process). Moving in the other direction, define $\Sigma^{-1}W$ to be the result of shifting all of the spaces in W up one index: $(\Sigma^{-1}W)_n = W_{n-1}$, $(\Sigma^{-1}W)_0 = *$. There is a canonical isomorphism $\Sigma(\Sigma^{-1}W) \cong W$, but in the other direction we only have a canonical map $\Sigma^{-1}(\Sigma W) \to W$ (the spectra are different in degree 0). However, this canonical map is a weak equivalence. So the suspension functor on $\mathcal{S}pectra$ is invertible, at least up to homotopy.

Observe that $\Sigma(\Sigma^\infty X) = \Sigma^\infty(\Sigma X)$. Also, if W is a spectrum we have canonical maps $\Sigma^\infty W_n \to \Sigma^{-1}(\Sigma^\infty W_{n+1})$ coming from the structure maps $\Sigma W_n \to W_{n+1}$. So we can make sense of the diagram

$$\Sigma^\infty W_0 \to \Sigma^{-1}(\Sigma^\infty W_1) \to \Sigma^{-2}(\Sigma^\infty W_2) \to \cdots$$

and W is precisely the degreewise colimit—i.e., the colimit in $\mathcal{S}pectra$. The slogan to keep in mind is that every spectrum is built as a colimit of desuspensions of spaces.

In general, describing $\mathrm{Ho}\,(\mathcal{S}pectra)(A, E)$ in terms of the spaces A_i and E_j can be a difficult problem. However, when A is a suspension spectrum there is a nice

formula. For any $t \in \mathbb{Z}$ one has

$$\mathrm{Ho}\,(\mathcal{S}pectra)(\Sigma^\infty X, \Sigma^t E) \cong \operatorname*{colim}_n \mathrm{Ho}\,(\mathcal{T}op_*)(\Sigma^n X, E_{t+n})$$

where the maps in the colimit system are induced by the structure maps in E. The colimit on the right might feel familiar, as it is similar to the one we used to define the homotopy groups of a spectrum. Indeed, for the special case $X = S^k$ we have $\mathrm{Ho}\,(\mathcal{S}pectra)(\Sigma^\infty S^k, E) \cong \pi_k(E)$, just as it should be.

Every infinite loop space $\{Z_n\}$ gives a spectrum with the same objects: for the structure map $\Sigma Z_n \to Z_{n+1}$ take the adjoint of the map $Z_n \to \Omega Z_{n+1}$. Spectra like this, for which the adjoints of the structure maps are weak equivalences, are known as Ω-spectra (read as "omega spectra"). So Ω-spectra and infinite loops spaces are essentially the same thing.

One can also take a spectrum W—any spectrum—and extract an infinite loop space. Define $\Omega^\infty W$ to be the homotopy colimit of

$$W_0 \to \Omega W_1 \to \Omega^2 W_2 \to \cdots$$

where we have taken the adjoints of the structure maps. More generally, define $\Omega^{\infty-n} W$ to be the homotopy colimit of

$$W_n \to \Omega W_{n+1} \to \Omega^2 W_{n+2} \to \cdots$$

The sequence of spaces $Z_n = \Omega^{\infty-n} W$ is an infinite loop space in a canonical way (some care is needed here, but we skip the technical details).

Going back and forth between spectra and infinite loop spaces doesn't lose any homotopy information at the level of objects (the maps are a little more subtle). If we take a spectrum, extract the associated infinite loop space, and then turn it back into a spectrum, what we get is weakly equivalent to what we started with. And the same in the other direction.

If E is a spectrum there is an associated cohomology theory $E^*(-)$. One can make this by forming the associated infinite loop space and then repeating what we did in the last section, but one can also just write down the groups directly:

$$E^n(X) = \mathrm{Ho}\,(\mathcal{S}pectra)(\Sigma^\infty X_+, \Sigma^n E),$$
$$E^n(X, A) = \mathrm{Ho}\,(\mathcal{S}pectra(\Sigma^\infty C(A \hookrightarrow X), \Sigma^n E).$$

We can also unravel these definitions to use the spaces $\{E_i\}$ making up the spectrum E:

$$E^n(X) = \operatorname*{colim}_k \mathrm{Ho}\,\mathcal{T}_*(\Sigma^k X_+, E_{k+n}),$$
$$E^n(X, A) = \operatorname*{colim}_k \mathrm{Ho}\,\mathcal{T}_*(\Sigma^k C(A \hookrightarrow X), E_{k+n})$$

(the maps in the colimit system are induced by the structure maps of E). Finally, when E is an Ω-spectrum all of the maps in the colimit system are isomorphisms and the formulas reduce even further to

$$E^n(X) = \mathrm{Ho}\,\mathcal{T}_*(X_+, E_n) \quad \text{if } n \geq 0,$$
$$E^n(X) = \mathrm{Ho}\,\mathcal{T}_*(X_+, \Omega^{-n} E_0) \quad \text{if } n < 0,$$

and likewise with $E^n(X, A)$ but replacing X_+ with $C(A \hookrightarrow X)$ in the formulas.

The Brown Representability Theorem says that, conversely, every cohomology theory arises in this way from a spectrum.

Exercise 9.15. Let B be an abelian group. The spaces $K(B,n)$ form an infinite loop space: we use any map $K(B,n) \to \Omega K(B,n+1)$ that induces the identity on π_n. The associated spectrum is written HB and called the **Eilenberg-MacLane spectrum** for B. Check that $\pi_k(HB) = 0$ for $k \neq 0$ and $\pi_0(HB) = B$. Because HB is an Ω-spectrum we have $HB^n(X) = \operatorname{Ho}\mathcal{T}(X, K(B,n))$ for $n \geq 0$ (check that $HB^n(X) = 0$ for $n < 0$), and so $HB^n(X) = H^n(X;B)$.

Exercise 9.16. The spectrum $\Sigma^\infty(S^0)$ is called the **sphere spectrum**, and typically just written as S. Check that $\pi_k(S)$ is the kth stable homotopy group of spheres.

To get a sense of the power of this machinery, write $H = H\mathbb{Z}/2$. The Steenrod operation Sq^2 is a natural transformation of cohomology theories $H^*(X) \to H^{*+2}(X)$. The Yoneda lemma says that this corresponds to a map $H \to \Sigma^2 H$ in $\operatorname{Ho}(\mathit{Spectra})$. Let E be the homotopy cofiber of that map. Then the associated cohomology theory E^* comes with a natural long exact sequence

$$\cdots \longrightarrow H^i(X) \xrightarrow{\mathrm{Sq}^2} H^{i+2}(X) \longrightarrow E^i(X) \longrightarrow H^{i+1}(X) \longrightarrow \cdots$$

Constructing the theory E^* by going through $\mathit{Spectra}$ in this way is trivial, whereas constructing it via a bare-hands approach would be fairly involved.

As soon as one starts working with different cohomology theories and studying their relations to each other, the category of $\mathit{Spectra}$ becomes indispensible.

9.17. **The Milnor exact sequence.** Let $X_1 \to X_2 \to X_3 \to \cdots$ be maps of spaces and write X_∞ for the homotopy colimit (mapping telescope). There are canonical homotopy classes of maps $X_i \to X_\infty$, and these are compatible as i varies. So for any space Y we get a map $\operatorname{Ho}\mathcal{T}(X_\infty, Y) \to \lim_n \operatorname{Ho}\mathcal{T}(X_n, Y)$. It is an easy exercise to see that this is a surjection.

To go further, assume that Y is a double loop space. Then $\operatorname{Ho}\mathcal{T}(Z,Y)$ is an abelian group for any space Z. In this case the kernel of the above map has a nice description in terms of the derived functor of inverse limit:

Proposition 9.18 (The Milnor sequence)**.** *In the above situation there is a natural short exact sequence*

$$0 \to \lim\nolimits_n^1 \mathit{Ho}\mathcal{T}(\Sigma X_n, Y) \to \mathit{Ho}\mathcal{T}(X_\infty, Y) \to \lim_n \mathit{Ho}\mathcal{T}(X_n, Y) \to 0.$$

Here $\lim^1$ is the first derived functor of lim. For a tower of abelian groups $\cdots \xrightarrow{f} A_2 \xrightarrow{f} A_1 \xrightarrow{f} A_0$ one can describe it as

$$\lim\nolimits_n^1 A_n = \operatorname{coker}\Big(\prod_n A_n \to \prod_n A_n\Big)$$

where the map sends a sequence $(a_0, a_1, \cdots)$ to $(a_0, a_1, \ldots) - (fa_1, fa_2, \ldots)$.

See [D6] for discussion of $\lim^1$ and a derivation of the Milnor sequence. But even without working through the technical details, it is easy to see that something like this should hold. Suppose for convenience that all the X_n are cofibrant and that $f\colon X_\infty \to Y$ maps to zero in each $[X_n, Y]$. We can choose null-homotopies $CX_n \to Y$ for each n, but there is no guarantee that this can be done in a compatible way—i.e., the null-homotopy at level n need not have any relation to the one at level $n+1$. If they *could* be made compatible, they could be pasted together to give a null-homotopy for f. Take $CX_{n+1} \to Y$ and restrict it to CX_n, and glue

this to our other map $CX_n \to Y$ to get a map $\Sigma X_n \to Y$. The collection of all of these determines an element in our $\lim^1$ group. The Milnor sequence says that this element is zero precisely when the null-homotopies can be modified so as to become compatible.

The $\lim^1$ groups can be quite strange and difficult to compute, but here is one useful result:

Proposition 9.19 (Mittag-Leffler vanishing)**.** *For an inverse system of abelian groups $A_\bullet$, for each n consider the descending chain*

$$A_n \supseteq f(A_{n+1}) \supseteq f^2(A_{n+2}) \supseteq \cdots$$

Suppose that for each $n \geq 0$ this chain of subgroups stabilizes (the stabilization point can be different for different n). Then $\lim_n^1 A_n = 0$. In particular, if all of the A_n are finite then the stabilization condition is automatically satisfied.

If one only ever studied $[X, Y]$ where X is a finite cell complex then one would probably never encounter the Milnor sequence. But as soon as one tries working with infinite complexes (which are all homotopy colimits of their finite skeleta) then it becomes an important tool. In many cases of interest the $\lim^1$ terms vanish, but one has to be careful about this.

Since cohomology groups $E^*(X)$ are equal to maps in the homotopy category of spectra, the analog of the Milnor sequence in that context gives the following:

Proposition 9.20 (Milnor sequence for cohomology)**.** *Suppose that E is a cohomology theory and $X_1 \to X_2 \to \cdots$ is a sequence of spaces (or spectra). Then for each $k \in \mathbb{Z}$ there is a natural short exact sequence*

$$0 \to \lim_n{}^1 E^{k-1}(X_n) \to E^k(\operatorname*{hocolim}_n X_n) \to \lim_n E^k(X_n) \to 0.$$

See Exercise 9.25 for a simple example where this sequence is nontrivial.

◦ Exercises ◦

Here is a useful fact. Suppose $f\colon X \to Y$ is a map, $\{U, V\}$ is an open cover of Y, and the three maps $f^{-1}(U) \to U$, $f^{-1}(V) \to V$, and $f^{-1}(U \cap V) \to U \cap V$ are all weak equivalences. Then f is itself a weak equivalence. For a proof see [Gray, 16.24]. Let us call this the **Open Decomposition Theorem**.

Exercise 9.21. Suppose given a map of pairs $(X, A) \to (X', A')$ where the maps $A \to A'$ and $X \to X'$ are both weak equivalences. Prove that $\mathrm{Cone}(A \hookrightarrow X) \to \mathrm{Cone}(A' \hookrightarrow X')$ is a weak equivalence.

Exercise 9.22. Let $\{U, V\}$ be an open cover of a space X. Let $j_U\colon U \cap V \hookrightarrow U$ and $j_V\colon U \cap V \hookrightarrow V$ be the inclusions. Let P be the homotopy pushout of $U \longleftarrow U \cap V \longrightarrow V$, i.e. $P = [U \amalg ((U \cap V) \times I) \amalg V] \sim$ with $(x, 0) \sim j_U(x)$ and $(x, 1) \sim j_V(x)$. Prove that the canonical collapse map $P \to X$ is a weak equivalence.

Exercise 9.23. Let X be a space with subspaces $U \subseteq A \subseteq X$ such that $\bar{U} \subseteq \operatorname{int} A$. Let P be the homotopy pushout of the diagram $X - U \xleftarrow{i} A - U \xrightarrow{j} A$ where i and j are the evident inclusions. That is, $P = [(X - U) \amalg ((A - U) \times I) \amalg A]/\sim$ where the equivalence relation is $(a, 0) \sim i(a)$ and $(a, 1) \sim j(a)$ for all $a \in A - U$.

(a) Show that the map $p\colon P \to X$ that squashes the cylinder is continuous.
(b) Use the open cover $\{X - \bar{U}, \operatorname{int} A\}$ of X and the Open Decomposition Theorem to show that p is a weak equivalence.
(c) Compare the inclusion $A \hookrightarrow P$ to $A \hookrightarrow X$ and show that there is a weak equivalence of mapping cones.
(d) Show that the mapping cones of $A \hookrightarrow P$ and $A - U \hookrightarrow X - U$ are weakly equivalent. [Hint: It is easy enough to construct a map. One approach is to then use the Open Decomposition Theorem again.]
(e) Conclude that $\operatorname{Cone}(A - U \hookrightarrow X - U)$ and $\operatorname{Cone}(A \hookrightarrow X)$ are weakly equivalent.

Exercise 9.24. If $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a levelwise exact sequence of inverse systems, then there is a long exact sequence $0 \to \lim A \to \lim B \to \lim C \to \lim^1 A \to \lim^1 B \to \lim^1 C \to 0$. Fix a prime p and consider the three inverse systems

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} \\
 & & \downarrow{\scriptstyle p^3} & & \downarrow{\scriptstyle p^2} & & \downarrow{\scriptstyle p} \\
\cdots & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathbb{Z}/p^3 & \longrightarrow & \mathbb{Z}/p^2 & \longrightarrow & \mathbb{Z}/p
\end{array}$$

Recall that the inverse limit of the bottom row is $\mathbb{Z}_p^\wedge$, the p-adic integers. Show that $\lim^1$ of the top sequence is $\mathbb{Z}_p^\wedge/\mathbb{Z}$, and for that matter identify lim and $\lim^1$ for each row.

Exercise 9.25. Fix an $n \geq 1$ and a prime p. Let X be the homotopy colimit of

$$S^n \xrightarrow{p} S^n \xrightarrow{p} S^n \xrightarrow{p} \cdots$$

This is sometimes called the "p-inverted n-sphere".
(a) Show that $\pi_*(X) \cong \pi_*(S^n) \otimes \mathbb{Z}[\frac{1}{p}]$.
(b) Use the Milnor exact sequence to show that

$$H^*(X) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}_p^\wedge/\mathbb{Z} & \text{if } * = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 9.26. Let E be a cohomology theory and let $k \in \mathbb{Z}$. For any maps of pointed spaces $X_1 \to X_2 \to \cdots$ prove that $\lim_n^1 E^k(X_n)$ and $\lim_n^1 \tilde{E}^k(X_n)$ are canonically isomorphic. For this reason, the Milnor sequence for cohomology is sometimes written with reduced cohomology in the $\lim^1$-term.

Part 3. Bundle theory

Let us take as our starting point that we understand finite-dimensional linear algebra extremely well. There aren't that many isomorphism types of objects (one for each dimension), and we have a decent understanding of the maps between them. Our goal over the next few sections is to explore the idea of doing linear algebra *continuously* over a fixed parameter space X. What this means is that rather than have only one vector space we will have a continuously varying family of vector spaces, parameterized by the points of X.

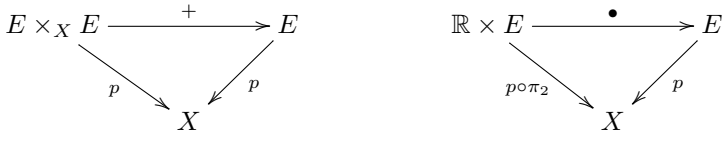
The way to talk about such "continuously varying families" is to bundle the objects together into a single topological space E together with a map $p\colon E \to X$, so that the members of our family appear as the fibers of p. A map from the family $p\colon E \to X$ to the family $p'\colon E' \to X$ will then be a continuous map $F\colon E \to E'$, commuting with the maps down to X, such that F is a linear transformation on each fiber. It turns out that much of linear algebra carries over easily to this enhanced setting. But there are more isomorphism types of objects here, because the topology of X allows for some twisting in the vector space structure of the fibers. The surprise is that studying these 'twisted vector spaces' over a base space X quickly leads to interesting homotopy invariants of X! From a topological viewpoint, K-theory is a cohomology theory that arises out of this study of fiberwise linear algebra.

10. Vector bundles

The point of the next three sections is to establish all of the foundational results we will require for working with vector bundles. Unfortunately, going through all of this carefully ends up being somewhat tedious. The reader might do well to skim these three sections for the basic ideas but not get bogged down in details, referring back for those only as needed.

A (real) vector space is a set V together with operations $+\colon V \times V \to V$ and $\cdot\colon \mathbb{R} \times V \to V$ satisfying a familiar (but long) list of properties. If X is a topological space, a *family of vector spaces* over X will be a continuous map $p\colon E \to X$ together with extra data making each fiber $p^{-1}(x)$ into a vector space, with the operations varying in a continuous manner. The easiest way to say this is as follows:

Definition 10.1. *A **family of (real) vector spaces** is a map $p\colon E \to X$ together with operations $+\colon E \times_X E \to E$ and $\bullet\colon \mathbb{R} \times E \to E$ making the two diagrams*



commute, together with a map $\zeta\colon X \to E$ called the "zero section", such that the operations make each fiber $p^{-1}(x)$ into a real vector space with zero element $\zeta(x)$.

One could write down the above definition completely category-theoretically, in terms of maps and commutative diagrams. Essentially one is defining a "vector

space object" in the category of topological spaces over X. In the case where X is a point, E is simply called a **topological vector space**.

The space X is called the **base** of the family. If $x \in X$ we will write E_x for the fiber $p^{-1}(x)$ regarded with its vector space structure. The dimension of E_x is called the **rank** of the family at x, and denoted $\mathrm{rank}_x(E)$. The rank of E is defined to be

$$\mathrm{rank}(E) = \sup\{\mathrm{rank}_x(E) \,|\, x \in X\},$$

where we include the possibility that $\mathrm{rank}(E)$ is infinite (though we will need this case only rarely).

A **section** of $p\colon E \to X$ is a map $s\colon X \to E$ such that $ps = \mathrm{id}_X$. The condition is equivalent to saying $s(x) \in E_x$ for each $x \in X$. So a section is a continuous choice of element in each fiber. This explains the term "zero-section" for ζ. By abuse of terminology the image of ζ is also sometimes referred to as the zero-section.

Remark 10.2. The additive inverse map $E \to E$ is continuous, as it can be expressed as the composite $E = \{-1\} \times E \hookrightarrow \mathbb{R} \times E \xrightarrow{\cdot} E$. Similarly, the zero-section can be recoved set-theoretically from the scalar multiplication as the image of

$$\{0\} \times E \hookrightarrow \mathbb{R} \times E \xrightarrow{\cdot} E,$$

and this determines a set-theoretic section $X \to E$. However, continuity of this section is not automatic from the other axioms; this is why it is included as part of Definition 10.1.

Definition 10.3. *Given two familes of vector spaces $p\colon E \to X$ and $p'\colon E' \to X$, a **map of families** is a continuous map $f\colon E \to E'$ such that $p'f = p$ and such that f restricts to a linear map on each fiber. Write $\mathsf{FamVS}(X)$ for the category of families of vector spaces over X.*

*A **subfamily** of E is a topological subspace $J \hookrightarrow E$ that contains the zero section and is closed under the operations of addition and scalar multiplication.*

Remark 10.4. Although most often we will work with families over a common base, on occasion we will want to talk about a map between families over two different bases. If $p\colon E \to X$ and $p'\colon E' \to X'$ are families of vector spaces, a map from the first to the second is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{array}$$

such that for any $x \in X$ the induced map $F_x\colon E_x \to E'_{f(x)}$ is linear.

Given a map of families $f\colon E \to E'$ over X, the usual image of f (denoted $\mathrm{im}\, f$) is a subfamily of E'. Let us define the **kernel** of f, denoted $\ker f$, to be the subspace of E consisting of all elements mapped to zero in E'. Equivalently, $\ker f$ is defined to be the pullback

$$\begin{array}{ccc} \ker f & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\zeta'} & E'. \end{array}$$

Exercise 10.5. Review the categorical notions of zero objects, kernels, and cokernels from Appendix D. Verify that the family $X \xrightarrow{\mathrm{id}} X$ is the zero object of the category $\mathsf{FamVS}(X)$ and that the kernel of a map, as defined above, is a kernel in the categorical sense.

Note that kernels in $\mathsf{FamVS}(X)$ are also fiberwise kernels, in the sense that $(\ker f)_x = \ker(f_x)$ for all $x \in X$. Be warned that this property will turn out not to hold for cokernels, though—see Exercise 10.10.

We next explore a few examples of these basic concepts:

Example 10.6.

(a) The simplest example of a family of vector spaces is $E = X \times \mathbb{R}^n$, with the projection map $X \times \mathbb{R}^n \to X$ (here $\mathbb{R}^n$ is equipped with the standard topology). This is called the **trivial family of rank** n, and it is often denoted simply by $\underline{n}_X$. It is also denoted by just $\underline{n}$, with the space X understood. Note that $\underline{0}_X$ is the family $X \xrightarrow{\mathrm{id}} X$. If V is any topological vector space then $\underline{V}_X$ denotes the family $X \times V \to X$, called the trivial family with fiber V.

(b) Let $E = \{(x, v) \,|\, x \in \mathbb{R}^2, v \in \mathbb{R}.\langle x\rangle\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, and let $p\colon E \to \mathbb{R}^2$ be projection onto the first coordinate. Define $(x, v) + (x, v') = (x, v + v')$ and $r.(x, v) = (x, rv)$. This makes $E \to \mathbb{R}^2$ into a family of vector spaces, in fact a subfamily of the trivial family $\underline{2}$. Notice that the fiber $p^{-1}(x)$ is one-dimensional for $x \neq 0$, but 0-dimensional when $x = 0$.

(c) Let $X = \mathbb{R}$ and $E = X \times \mathbb{R}$ be the trivial family of rank 1. Consider the map of familes $E \to E$ given by $(t, v) \mapsto (t, tv)$. This is multiplication-by-t on the fiber over t. The kernel of this map is the subfamily $K \hookrightarrow E$ given by $K = (X \times \{0\}) \cup (\{0\} \times \mathbb{R})$. Note that most of the fibers are 0-dimensional, but the fiber over 0 is one-dimensional.

(d) Let $X = \mathbb{R}$. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$. Let $E \subseteq X \times \mathbb{R}^2$ be the union of $\{(x, re_1) \,|\, x \in \mathbb{Q}, r \in \mathbb{R}\}$ and $\{(x, re_2) \,|\, x \in X\backslash\mathbb{Q}, r \in \mathbb{R}\}$. Then E is a subfamily of the trivial family $\underline{2}_X$.

Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the linear transformation such that $\pi(e_1) = \pi(e_2) = 1$. Define a map $E \to \underline{1}_X$ by $(x, v) \mapsto (x, \pi(v))$. This is a map of families that is an isomorphism on each fiber, but is not an isomorphism of families.

(e) Let $\mathbb{R}^n_{ind}$ denote the vector space $\mathbb{R}^n$ but with the indiscrete topology, so that the only open sets are $\emptyset$ and $\mathbb{R}^n$. Check that $\mathbb{R}^n_{ind}$ is a topological vector space, i.e. that $\mathbb{R}^n_{ind} \to *$ is a family of vector spaces.

Part (d) of the previous example shows that the concept of "family of vector spaces" admits some unpleasant pathology. Even the examples in (b) and (c) show that families can have jumps in the fibers, which doesn't give the feeling of a "continuously varying" family. Part (e) gives the warning that the topology on the fibers might not be what one expects. Very shortly we will start imposing some conditions that eliminate these kind of phenomena.

Remark 10.7. If $E \to X$ is a family of vector spaces, an isomorphism of families $E \cong \underline{V}$ is called a **trivialization**. A family is called **trivializable** if there exists a trivialization, and a **trivial family** is a family together with a chosen trivialization. In practice, though, the adjectives "trivial" and "trivializable" are often used interchangeably, with the precise meaning discerned from context.

If $p\colon E \to X$ is a family of vector spaces and $A \hookrightarrow X$ is a subspace, then $p^{-1}(A) \to A$ is also a family of vector spaces. We will usually write this restriction as $E|_A$.

More generally, suppose that $p\colon E \to X$ is a family of vector spaces and $f\colon Y \to X$ is a map. One may form the pullback $Y \times_X E$, more commonly denoted f^*E in this context:

$$\begin{array}{ccccc} f^*E & = & Y \times_X E & \longrightarrow & E \\ & & \downarrow & & \downarrow{\scriptstyle p} \\ & & Y & \xrightarrow{f} & X. \end{array}$$

A point in f^*E is a pair (y, e) such that $f(y) = p(e)$, and one defines addition and scalar multiplication on f^*E by $(y, e) + (y, e') = (y, e + e')$ and $r \cdot (y, e) = (y, re)$. This makes $f^*E \to Y$ into a family of vector spaces, called the **pullback family**. If $y \in Y$ then there is an evident map of vector spaces $(f^*E)_y \to E_{f(y)}$ which one readily checks is an isomorphism. Note that if f is a subspace inclusion then $f^*E = E|_Y$.

Exercise 10.8. Check carefully that f^*E is a family of vector spaces.

We will need to develop some tools for dealing with maps in $\mathsf{FamVS}(X)$. The following proposition is easy but will be used very often:

Proposition 10.9. *Giving a map of trivial families $X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ is equivalent to giving a continuous map $X \to \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n) \cong \mathbb{R}^{kn}$, where the target has the standard Euclidean topology.*

Proof. Let $F\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ be a map of families. Define $F_i\colon X \to \mathbb{R}^n$ by $F_i(x) = \pi_2 F(x, e_i)$. These maps are continuous, and so they induce a continuous map $\phi\colon X \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k factors) by $\phi(x) = (F_1(x), \ldots, F_k(x))$.

In the other direction, given $\phi\colon X \to \mathbb{R}^{kn} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k factors) define $\tilde{F}$ to be the composite

$$X \times \mathbb{R}^k \xrightarrow{\phi \times \mathrm{id}} \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^k \xrightarrow{LC} \mathbb{R}^n$$

where LC is the "linear combination map" sending $(v_1, \ldots, v_k, \underline{r}) \mapsto r_1 v_1 + \cdots + r_k v_k$. Both maps are continuous, so their composite is as well. Then define $F\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ by $F(x, \underline{r}) = (x, \tilde{F}(x, \underline{r}))$.

It is now routine to check that the two assignments given above are inverses to each other. □

Here is one more example showing the oddities of the category $\mathsf{FamVS}(X)$:

Exercise 10.10. Let $X = [-1, 1]$ and let $f\colon \underline{1}_X \to \underline{1}_X$ be the map that is multiplication-by-t on the fiber over t. Prove that the zero bundle $\underline{0}_X$ is the cokernel of f (see Appendix D for the definition). In particular, note that in this case the cokernel is not the fiberwise cokernel.

Sometimes we will have to deal with families of vector spaces where the fibers are infinite-dimensional. Mostly we will only need trivial families, but even here there are some subtleties. We describe these next.

If S is any set, write $\mathbb{R}\langle S\rangle$ for the vector space with basis S (occasionally we will abbreviate this to $\mathbb{R}S$). There is a natural isomorphism of vector spaces

$$\operatorname*{colim}_{F_{finite} \subseteq S} \mathbb{R}\langle F\rangle \to \mathbb{R}\langle S\rangle$$

where the colimit is over the finite subsets of S. If we give each $\mathbb{R}\langle F\rangle$ the standard Euclidean topology then the colimit inherits an induced topology, giving a topology on $\mathbb{R}\langle S\rangle$. This is called the "finite" topology in some of the literature on topological vector spaces, but we will call it the **colimit topology**. When S is infinite this topology is *different* than the subspace topology induced from the evident embedding $\mathbb{R}\langle S\rangle \subseteq \prod_S \mathbb{R}$, where the product is given the product topology. In fact, the colimit topology is the subspace topology induced by the box topology. See Appendix A.34 for a complete discussion.

A theorem of Kakutani-Klee [KK] says that when $\mathbb{R}\langle S\rangle$ is given the colimit topology, addition is continuous only when S is countable. We will only ever work with $\mathbb{R}\langle S\rangle$ in this case, and from now on we only ever consider families of vector spaces where the rank is countable.

Note that there are at least three distinct topologies that make $\mathbb{R}^\infty$ into a Hausdorff topological vector space: these are the colimit, product, and metric topologies, described in detail in Appendix A.34. So when talking about trivial families with fiber $\mathbb{R}^\infty$ one needs to be careful about specifying which topology is being used. The product and metric topologies have certain algebraic deficiencies—e.g., linear functionals are not always continuous (see Appendix B)—and so we will only use $\mathbb{R}^\infty$ with the colimit topology in our applications.

The background category for families of vector spaces is the overcategory $\mathcal{T}op \downarrow X$ of spaces over X. The terminal object of this category is X itself (or more precisely, the identity map $X \to X$). A "vector" in a family of vector spaces E should be a map in $\mathcal{T}op \downarrow X$ from X to E, which is the same thing as a section of $E \to X$. Given a collection of such "vectors" $s_\alpha \colon X \to E$, for α in an indexing set A, say that these are **linearly independent** (resp. **spanning**) if for every $x \in X$ the vectors $\{s_\alpha(x)\}_{\alpha\in A}$ are linearly independent (resp. spanning) in E_x. Say that the collection is a **weak basis** for E if it is both linearly independent and spanning, i.e. for every $x \in X$ the collection $\{s_\alpha(x)\}_{\alpha\in A}$ is a basis for the vector space E_x. When A is finite a weak basis gives a map of families $X \times \mathbb{R}\langle A\rangle \to E$ which is a bijection (when A is infinite this is more delicate, see Exercise 10.24). However, it need not be an isomorphism! The inverse map need not be continuous: as an example consider a basis $b_1, \ldots, b_n$ in $\mathbb{R}^n_{ind}$ and the associated map $\mathbb{R}^n \to \mathbb{R}^n_{ind}$ sending $e_i \mapsto b_i$.

Define a **strong basis** for a family E to be a collection of sections $\{s_\alpha\}_{\alpha\in A}$ with the property that the induced map $X \times \mathbb{R}\langle A\rangle \to E$ is an isomorphism. So a family is trivializable if and only if it has a strong basis.

Remark 10.11. If $s_1, \ldots, s_n$ is a weak basis for E, then let $\phi_i \colon E \to X \times \mathbb{R}$ be the ith coordinate function: it is the set map that sends a vector $v \in E_x$ to the pair (x, t) where t is the ith coordinate of v with respect to the basis $s_1(x), \ldots, s_n(x)$. The condition that $s_1, \ldots, s_n$ be a strong basis is equivalent to the coordinate functions $\phi_1, \ldots, \phi_n$ all being continuous. (Again, be warned that this doesn't work in the case of an infinite basis, essentially because of the difference between the colimit and product topologies on $\mathbb{R}^\infty$).

Most families of vector spaces will have neither a weak nor a strong basis, as the topology of X gets in the way. The families in Example 10.6(b,c,d) do not have any nonvanishing sections at all! However, demanding that we have strong bases *locally* turns out to be a reasonable thing to require. To this end, if $p \colon E \to X$ is a

family of vector spaces and $x \in X$ define a **local weak (resp. strong) basis** at x to be a neighborhood U of x together with a weak (resp. strong) basis for $E|_U$.

Definition 10.12. *A **vector bundle** is a family of finite rank vector spaces $p\colon E \to X$ that has a local strong basis at every point of X. In other words, for each $x \in X$ there is a neighborhood $x \in U \subseteq X$, an $n \in \mathbb{Z}_{\geq 0}$, and an isomorphism of families of vector spaces*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times \mathbb{R}^n \\ & \searrow \quad \swarrow & \\ & U. & \end{array}$$

The isomorphism in the above diagram is called a "local trivialization". Usually one simply says that a vector bundle is a family of finite rank vector spaces that is locally trivial.

A map of vector bundles is just a map of the underlying families, and $\mathsf{Vect}(X)$ will denote the category of vector bundles over X.

Remark 10.13. Note that the n appearing in Definition 10.12 is $\mathrm{rank}_x(E)$, and it depends on the point x. It is easy to prove, though, that this number is constant on each connected component of X. Vector bundles of constant rank 1 are called **line bundles**, and bundles of constant rank 2 are called **plane bundles**.

Remark 10.14. Definition 10.12 defines *real* vector bundles, but one could do the same with $\mathbb{C}$ or $\mathbb{H}$ replacing $\mathbb{R}$ to define complex and quaternionic bundles.

Remark 10.15. Some authors allow vector bundles to have countably-infinite rank. There is nothing wrong with that, though one has to specify the topology one uses for $\mathbb{R}^\infty$. But if we take this approach then we end up having to add "finite rank" hypotheses in an annoying number of places. In our approach, we will on occasion abuse terminology and use the phrase "vector bundle of infinite rank" for this concept, even though such an object is not a vector bundle according to our definition.

Exercise 10.16. Let $f\colon E_1 \to E_2$ be a map of vector bundles on X that is an isomorphism on each fiber. Prove that f is an isomorphism in $\mathsf{Vect}(X)$.

Of the families of vector spaces we considered in Example 10.6, only the trivial family from (a) is a vector bundle. Before discussing more interesting examples, though, we need some general remarks. Establishing that a given family of vector spaces is a vector bundle requires producing weak local bases *and* proving that the associated coordinate functions are continuous—oftentimes one forgets the latter part, but it is important. This second part can be somewhat annoying, though, so it is useful to know that in many situations one can avoid it. In order to discuss these situations, we start with the following definition:

Definition 10.17. *A family of vector spaces $E \to X$ is said to be **tame** if for every point $x \in X$ and every local weak basis $\{s_\alpha\}$ defined on an open neigborhood U of x, there is an open set $x \in V \subseteq U$ such that the set $\{s_\alpha|_V\}$ is a local strong basis. That is to say, every local weak basis near a point can be restricted to a local strong basis.*

Proposition 10.18. *Let $E \to X$ be a tame family of vector spaces. Then E is a vector bundle if and only if for every $x \in X$ there exists a local weak basis on a neighborhood of x.*

Proof. Immediate. □

Of course every vector bundle is tame. The families from Example 10.6(b,c,d) are also tame, essentially because these families do not have *any* local weak bases at the "exotic" points. The family $\mathbb{R}^n_{ind}$ from Example 10.6(e) is not tame. The concept of tameness is not particularly natural or important, but it is useful to us because of the following result which guarantees tameness in many common situations:

Proposition 10.19. *Let $E \to X$ be a family of vector spaces of finite rank. Then E is tame if either of the following conditions is satisfied:*

(a) E is a subfamily of a trivial family (perhaps of countably infinite rank).
(b) X is locally compact and E is Hausdorff.

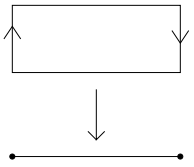
Recall that a CW-complex is locally compact if and only if it is *locally finite*, meaning that every open cell intersects only finitely-many closed cells [FP, Proposition 1.5.10]. So Proposition 10.19(b) applies in a large percentage of the cases one naturally encounters. But note that the CW-complexes S^∞, $\mathbb{R}P^\infty$, and $\mathbb{C}P^\infty$ are *not* locally compact, so Proposition 10.19(b) is not a universal panacea.

The proof of Proposition 10.19 is technical and a distraction from our goals at the moment; it can be found in Appendix B. However, from now on we will use Proposition 10.19 often and—except for the first few times—mostly without comment.

Example 10.20.

(a) Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector space isomorphism. Let $E' = [0,1] \times \mathbb{R}^n$ and let E be the quotient of E' by the relation $(0, v) \sim (1, \phi(v))$. Identifying S^1 with the quotient of $[0,1]$ by $0 \sim 1$, we obtain a map $E \to S^1$ that is clearly a family of vector spaces. We claim this is a vector bundle. If $x \in (0,1)$ then it is evident that E is locally trivial at x, so the only point of concern is $x = 0 = 1 \in S^1$. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$, and let $s_i\colon [0, \frac{1}{4}) \to E'$ be the constant section whose value is e_i. Likewise, let $s'_i\colon (\frac{3}{4}, 1] \to E'$ be the constant section whose value is $\phi(e_i)$. Projecting into E we obtain $s_i(0) = s'_i(1)$, and so the sections s_i and s'_i patch together to give a section $S_i\colon U \to E$, where $U = [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$. The sections $S_1, \ldots, S_n$ are independent. It can be checked directly that these are a local trivialization, or it follows from Propositions 10.18 and 10.19 and the fact that S^1 is locally compact.

When $n = 1$ and $\phi(x) = -x$ the resulting bundle is the *Möbius bundle* M, depicted below:



We further discuss the case of general n and ϕ in Example 14.3.

(b) Let $X = \mathbb{R}P^n$, regarded as the space of lines in $\mathbb{R}^{n+1}$. Let $L \subseteq X \times \mathbb{R}^{n+1}$ be the set

$$L = \{(l, v) \,|\, l \in \mathbb{R}P^n,\ v \in l\}.$$

Then L is a subfamily of the trivial family, and we claim that it is a line bundle over X. To see this, for any $l \in X$ we must produce a local trivialization. By Propositions 10.19(a) and 10.18 it suffices to just produce a local weak basis at every point. By symmetry it suffices to do this at the point $l = \langle e_0 \rangle$. Let $U \subseteq \mathbb{R}P^n$ be the set of lines whose orthogonal projection to $\langle e_0 \rangle$ is nonzero. Then $\mathbb{R}^n \cong U$ via the homeomorphism $\underline{x} \mapsto \langle e_0 + \underline{x} \rangle$, where here we regard $\mathbb{R}^n$ as having basis $e_1, \ldots, e_n$. Define $s\colon U \to L$ by sending l to $(l, e_0 + u)$ where $e_0 + u$ is the unique point on l with $u \in \mathbb{R}^n$. Via the homeomorphism $\mathbb{R}^n \cong U$ we verify at once that s is continuous. This section is clearly nonzero everywhere, so it gives the desired weak basis of $L|_U$. Thus, L is a vector bundle.

To be clear, it is not hard to prove that s induces a local trivialization of L without referencing Propositions 10.18 and 10.19. But those results allow us to avoid the extra steps that would be involved for that. Also note that this argument works equally well for $n = \infty$, since we are dealing with a subfamily of a trivial bundle.

The bundle L is called the **tautological line bundle** over $\mathbb{R}P^n$. Be careful not to ever call this the *canonical line bundle*, as that phrase has a specialized meaning in algebraic geometry that we will see later. Note that when $n = 1$ the bundle L is isomorphic to the Möbius bundle on S^1. (Exercise: Check this!)

(c) One may generalize the previous example as follows. Let V be a vector space and fix an integer $k > 0$. Consider the Grassmannian $\mathrm{Gr}_k(V)$ of k-planes in V (see Exercise 10.26 for a brief review of these spaces). Let

$$\gamma = \{(W, x) \,|\, W \in \mathrm{Gr}_k(V), x \in W\}.$$

Projection to the first coordinate $\pi\colon \gamma \to \mathrm{Gr}_k(V)$ makes γ into a rank k vector bundle, called the **tautological bundle** over $\mathrm{Gr}_k(V)$. To see that it is indeed a bundle, let $W \in \mathrm{Gr}_k(V)$ be an arbitrary k-plane. By choosing an appropriate basis for V we can just assume $W = \langle e_1, \ldots, e_k \rangle$. Equip V with the standard dot product with respect to the e-basis, and let $U \subseteq \mathrm{Gr}_k(V)$ be the collection of all k-planes whose orthogonal projection onto W is surjective (equivalently, an isomorphism). One readily checks that this is an open set of W. For each $J \in U$ let $s_1(J), \ldots, s_k(J)$ be the unique vectors in J that orthogonally project onto $e_1, \ldots, e_k$. One checks that these are continuous sections of $\gamma|_U$, and of course they are clearly independent and hence give a local trivialization.

(d) Let M be a smooth manifold, and let $TM \to M$ be its tangent bundle. So the fiber over each $x \in M$ is the tangent space at x. Let $x \in M$ and let U be a local coordinate patch about x. Let $x_1, \ldots, x_n$ be local coordinates in U, and let $\partial_1, \ldots, \partial_n$ be the associated vector fields (giving the tangent vectors to the coordinate curves in this system). Then $\partial_1, \ldots, \partial_n$ are independent sections of TM, and hence give a local trivialization.

Note that if $f\colon E \to F$ is a map of vector bundles over X then neither $\ker f$ nor $\mathrm{coker}\, f$ will necessarily be a vector bundle. For an example in the case of kernels, let $X = [-1, 1]$ and let $E = \underline{1}$. Define $f\colon E \to E$ by letting it be multiplication-by-t on the fiber over $t \in X$. This does not give an example for cokernels, though, because $\mathrm{coker}\, f$ happens to be a vector bundle (it is the zero bundle). Instead, choose a

continuous function $u\colon X \to \mathbb{R}$ that is nonzero on $(-\frac{1}{2}, \frac{1}{2})$ and zero everywhere else, then define $g\colon E \to E$ as multiplication by $u(t)$ on the fiber over t.

We will give a thorough discussion of kernels and cokernels in Section 12.

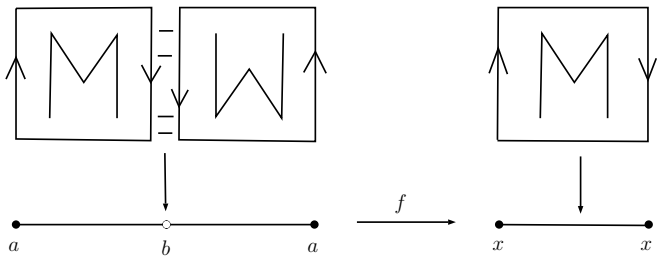
10.21. **Pullback bundles.** If $E \to X$ is a vector bundle and $f\colon Y \to X$ then it is easy to check that f^*E is also a vector bundle. Given composable maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, there is an evident natural isomorphism $(fg)^*E \cong g^*(f^*E)$. For each topological space X and each integer $k \geq 0$, let $\mathrm{Vect}_k(X)$ denote the set of isomorphism classes of vector bundles of rank k on X. The pullback construction then makes $\mathrm{Vect}_k(-)$ into a contravariant functor from $\mathcal{T}op$ into $\mathcal{S}et$.

Example 10.22. Pullback bundles can be slightly non-intuitive. Let $M \to S^1$ be the Möbius bundle, and let $f\colon S^1 \to S^1$ be the map $z \mapsto z^2$. We claim that $f^*M \cong \underline{1}$. This is easiest to see if one uses the following model for M:

$$M = \left\{ \left(e^{i\theta}, re^{i\frac{\theta}{2}}\right) \,\middle|\, \theta \in [0, 2\pi], r \in \mathbb{R} \right\}.$$

The bundle map is projection onto the first coordinate $\pi\colon M \to S^1$. Then $f^*M = \{(e^{i\theta}, re^{i\theta}) \,|\, \theta \in [0, 2\pi], r \in \mathbb{R}\}$. This is clearly isomorphic to $\underline{1}_{S^1}$, via the map $S^1 \times \mathbb{R} \to f^*M$ given by $(e^{i\theta}, r) \mapsto (e^{i\theta}, re^{i\theta})$.

We can also demonstrate the isomorphism $f^*M \cong \underline{1}$ by the following picture:



Here f is the evident map that wraps the circle around itself twice, so that $f^{-1}(x) = \{a, b\}$. We see that f^*M can be thought of as two copies of M that are cut open and then sewn together as shown, thereby producing a cylinder.

◦ Exercises ◦

Exercise 10.23. Let $E_A \to A$, $E_X \to X$, and $E_Y \to Y$ be families of vector spaces and assume given maps of families

$$E_X \longleftarrow E_A \longrightarrow E_Y$$

that lie over subset inclusions $X \supseteq A \subseteq Y$. Assume either that

(i) A is open in X and Y, and the maps $E_A \to E_X$ and $E_A \to E_Y$ are open injections, or

(ii) A is closed in X and Y, and the maps $E_A \to E_X$ and $E_A \to E_Y$ are closed injections.

Prove that the pushout $E = E_X \amalg_{E_A} E_Y$ is a family of vector spaces over $\Omega = X \amalg_A Y$.

[Hint: The difficult part is producing the addition $E \times_\Omega E \to E$. Show that the domain is covered by $E_X \times_X E_X$ and $E_Y \times_Y E_Y$ and that these are open (resp. closed).]

Exercise 10.24. Let $E \to X$ be a family of vector spaces and let $s_1, s_2, \ldots$ be a weak basis for E. If X is locally compact and Hausdorff prove that there is an associated map of families $X \times \mathbb{R}^\infty \to E$ that is a bijection, where $\mathbb{R}^\infty$ is given the colimit topology. [Hint: See Exercise A.4.]

Exercise 10.25. Let $E \to X$ be a rank n vector bundle and $Y \subseteq X$ a closed set. Suppose given a trivialization $u\colon E|_Y \xrightarrow{\cong} \underline{n}_Y$. Define E' to be the pushout of the top row in this diagram:

$$\begin{array}{ccccc} \mathbb{R}^n & \xleftarrow{\pi_2 \circ u} & E|_Y & \rightarrowtail & E \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & Y & \rightarrowtail & X. \end{array}$$

In words, E' is made by gluing all of the fibers of $E|_Y$ together according to the isomorphism u. If there exists an open subset $Y \subseteq U$ and an extension of u to an isomorphism $E|_U \cong \underline{n}_U$, prove that $E' \to X/Y$ is a vector bundle.

Exercise 10.26 (More on Grassmannians)**.** Let us represent k-planes in $\mathbb{R}^n$ as the span of the rows in a $k \times n$ matrix. The rows are independent if and only is some $k \times k$ minor is nonzero. For any choice of k columns $I = \{i_1, \ldots, i_k\}$, whether or not the corresponding minor is nonzero is invariant under row operations and therefore only depends on the k-plane and not on the chosen basis. Define $U_I \subseteq \mathrm{Gr}_k(\mathbb{R}^n)$ to be the subset of k-planes M having a basis where the minor corresponding to I is nonzero. By row-reduction, every M in U_I corresponds to a unique $k \times n$ matrix having the identity submatrix in the columns picked out by I. In this way one gets a homeomorphism $U_I \cong \mathbb{R}^{k(n-k)}$ and proves that $\mathrm{Gr}_k(\mathbb{R}^n)$ is a manifold. This is a standard approach to these spaces. Generalize this story in the following way:

(a) Let W be a fixed k-plane in $\mathbb{R}^n$, and let $\pi_W\colon \mathbb{R}^n \to W$ be orthogonal projection onto W. Let U_W be the collection of all k-planes M for which $\pi_W(M) = W$. Prove that this is an open set in $\mathrm{Gr}_k(\mathbb{R}^n)$.

(b) Given a linear map $L\colon W \to W^\perp$, let $\Gamma(L)$ be its graph: $\Gamma(L) = \{(w, L(w)) \,|\, w \in W\} \subseteq W \oplus W^\perp = \mathbb{R}^n$. Prove that $L \mapsto \Gamma(L)$ gives a homeomorphism $\mathrm{Hom}(W, W^\perp) \cong U_W$.

(c) Explain how parts (a) and (b) connect to the coordinate-specific approach that was explained above.

(d) Use the previous parts to produce k independent sections of $\gamma|_{U_W}$, where γ is the tautological family of vector spaces defined in Example 10.20.

11. Working with vector bundles

This section continues our development of the basis techniques in the study of vector bundles. Here we focus mostly on the theme of building vector bundles, through a variety of methods.

11.1. **Constructing new vector bundles out of old ones.** Let $p\colon E \to X$ and $p'\colon E' \to X$ be two vector bundles. We may form a new bundle $E \oplus E'$, whose underlying topological space is just the pullback $E \times_X E'$. So a point in

$E \oplus E'$ is a pair (e, e') where $p(e) = p'(e')$. The rules for vector addition and scalar multiplication are the evident ones. Note that the fiber of $E \oplus E'$ over a point x is simply $E_x \oplus E'_x$. The local trivializations of E and E' combine to give local trivializations of $E \oplus E'$, showing that this is indeed a vector bundle.

More generally, any canonical construction that one can apply to vector spaces may be extended to apply to vector bundles. So one can talk about the bundle $E \otimes E'$, the dual bundle E^*, the hom-bundle $\underline{\mathrm{Hom}}(E, E')$, the exterior product bundle $\bigwedge^i E$, and so on. We will only carefully define $E \otimes E'$, and leave the other definitions to the reader.

Set-theoretically define

$$E \otimes E' = \{(x, v) \,|\, x \in X, v \in E_x \otimes E'_x\}.$$

This is clear enough, and it is clear how to define addition and scalar multiplication in the fibers. The only thing that takes thought is how to define the topology on $E \otimes E'$, and to check that the operations are continuous. But it is enough to define the topology *locally*, and to check continuity locally. If $x \in X$, let U be a neighborhood of x over with both E and E' are trivializable. Choose isomorphisms $\phi\colon U \times \mathbb{R}^k \to E|_U$ and $\phi'\colon U \times \mathbb{R}^l \to E'|_U$. Then one gets a bijection of sets $U \times (\mathbb{R}^k \otimes \mathbb{R}^l) \to (E \otimes E')|_U$ which is a linear isomorphism on each fiber: one sends $(u, v \otimes w)$ to $(u, \phi(u, v) \otimes \phi'(u, w))$ and then extends linearly. Finally, one uses this bijection to transplant the topology from $U \times (\mathbb{R}^k \otimes \mathbb{R}^l)$ to $(E \otimes E')|_U$. We leave the reader to fill in all the details here.

A brief summary of this technique is "define the new bundle set-theoretically and then use the local trivializations to induce the topology". Technically one should check that different choices of local trivialization yield the same topology, but this is usually routine and left implicit.

Remark 11.2 (External sums and products)**.** Let $E \to X$ and $F \to Y$ be two vector bundles, but this time over possibly different base spaces. One may construct an external direct sum $E \hat{\oplus} F \to X \times Y$ whose fiber over (x, y) is $E_x \oplus F_y$. The underlying topological space of $E \hat{\oplus} F$ is just $E \times F$, and it has the evident operations. Note that $E \hat{\oplus} F$ can also be constructed as $\pi_1^*(E) \oplus \pi_2^*(F)$, where π_1 and π_2 are the projections from $X \times Y$ onto the two factors.

In the case $X = Y$ we can construct the (internal) direct sum from the external one: namely $E \oplus F = \Delta^*(E \hat{\oplus} F)$ where $\Delta\colon X \to X \times X$ is the diagonal map. Thus, the internal and external direct sums determine each other.

One can tell a similar story about external tensor products, or external hom-bundles.

Exercise 11.3. Given two vector bundles $E_1 \to X$ and $E_2 \to X$ construct the bundle $\underline{\mathrm{Hom}}(E_1, E_2)$ whose fiber over $x \in X$ is $\mathrm{Hom}((E_1)_x, (E_2)_x)$. Prove an adjointness formula

$$\mathsf{Vect}(X)(E_1 \otimes E_2, E_3) \cong \mathsf{Vect}(X)(E_1, \underline{\mathrm{Hom}}(E_2, E_3)).$$

Even better, establish an isomorphism of bundles

$$\underline{\mathrm{Hom}}(E_1 \otimes E_2, E_3) \cong \underline{\mathrm{Hom}}(E_1, \underline{\mathrm{Hom}}(E_2, E_3)).$$

11.4. **Constructing vector bundles by patching.** Let X be a space and let A and B be subspaces such that $A \cup B = X$. Recall that if $f_A\colon A \to Y$ and $f_B\colon B \to Y$ are continuous maps that agree on $A \cap B$ then we may patch these

together to get a continuous map $f\colon X \to Y$ provided that either (i) A and B are both closed, or (ii) A and B are both open. This is a basic fact about topological spaces. The analogous facts for vector bundles are very similar in the case of an open cover, but a little more subtle for closed covers.

Proposition 11.5. *Let $E \to X$ be a family of vector spaces.*

(a) If $\{U_\alpha\}$ is an open cover of X and each $E|_{U_\alpha}$ is a vector bundle, then E is a vector bundle.

(b) Let $\{A, B\}$ be a cover of X by closed subspaces. Suppose that either

(i) X is paracompact Hausdorff, or

(ii) B is regular and has a countable basis, or

(iii) For every $x \in A \cap B$ and every neighborhood $x \in U \subseteq X$ there exists a neighborhood $x \in V \subseteq U$ such that $V \cap A \cap B \hookrightarrow V \cap B$ has a retraction.

Then if $E|_A$ and $E|_B$ are both vector bundles, so is E.

Proof. Part (a) is trivial, so we focus on (b). The main issue is producing local trivializations around points $x \in A \cap B$: one can do so in the "A-part" of a neighborhood and in the "B-part" of the neighborhood, but then some care is required in doing the two simultaneously in a compatible way.

Let $x \in X$, with the goal of producing a local trivialization around x. There are three cases: $x \in X - A$, $x \in X - B$, and $x \in A \cap B$. If $x \in X - B$ then we have $x \in (X-B)^{open} \subseteq A$. Since $E|_A$ is a vector bundle there is a subset $x \in U \subseteq X - B$ such that U is open in A and $E|_U$ is trivializable. But then U is open in $X - B$ and hence also open in X, so $E \to X$ has a local trivialization at x. A similar argument works if $x \in X - A$.

We have left to analyze the case $x \in A \cap B$. The fact that $E|_A$ is a vector bundle implies that there exists an open set $x \in U_1 \subseteq X$ such that E is trivializable over $U_1 \cap A$. Similarly, there exists an open set $x \in U_2 \subseteq X$ such that E is trivializable over $U_2 \cap B$. Let $U = U_1 \cap U_2$. Then $E|_{U\cap A}$ and $E|_{U\cap B}$ is a closed cover of $E|_U$, and so $E|_U$ is the pushout of the diagram

$$E|_{U\cap A} \longleftarrow E|_{U\cap A\cap B} \longrightarrow E|_{U\cap B}.$$

Choose a trivialization $E|_{U\cap A} \cong (U \cap A) \times \mathbb{R}^n$, which induces a trivialization of $E|_{U\cap A\cap B}$. So we have the following diagram

$$\begin{array}{ccccc} E|_{U\cap A} & \longleftarrow & E|_{U\cap A\cap B} & \longrightarrow & E|_{U\cap B} \\ {\scriptstyle f_A}\uparrow{\scriptstyle \cong} & & {\scriptstyle f_{A\cap B}}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle f_B} \\ (U \cap A) \times \mathbb{R}^n & \longleftarrow & (U \cap A \cap B) \times \mathbb{R}^n & \longrightarrow & (U \cap B) \times \mathbb{R}^n. \end{array}$$

and the question is whether we can find an isomorphism f_B that makes the diagram commute.

Let j denote the various horizontal maps in the above diagram, which are all subspace inclusions. Choose an isomorphism $h\colon E|_{U\cap B} \to (U \cap B) \times \mathbb{R}^n$. Then the composite $hjf_{A\cap B}$ may be represented by a map $U \cap A \cap B \to GL_n(\mathbb{R})$ as in Proposition 10.9. The construction of f_B is the question of extending this over $U \cap B$.

Under hypothesis (iii), by passing to a smaller neighborhood $x \in U' \subseteq U$ we can find a retraction $U' \cap B \to U' \cap A \cap B$, in which case the required extension is evident.

Properties (i) and (ii) each imply the following: for every Under hypothesis (ii) we know that all subspaces of B are normal [Munkres, Theorems 31.2 and 32.1], and hence we can apply the Tietze extension theorem: the composite map $U \cap A \cap B \to GL_n(\mathbb{R}) \hookrightarrow M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ extends to $U \cap B$. By continuity we can land in the open set $GL_n(\mathbb{R})$ after passing to a smaller neighborhood $x \in U' \subseteq U$.

The argument under hypothesis (i) is similar to the one for (ii) but with slightly different details. Since X is paracompact Hausdorff it is normal, hence regular. So there exists a neighborhood $x \in V \subseteq U$ such that $\bar{V} \subseteq U$, by [Munkres, Lemma 31.1]. Then $\bar{V}$ is a closed subspace of a paracompact Hausdorff space, hence also paracompact Hausdorff, hence normal. By the Tietze extension theorem we can extend $\bar{V} \cap A \cap B \to M_n(\mathbb{R})$ to $\bar{V}$, and then (just as before) by continuity we can arrange for the image to land in $GL_n(\mathbb{R})$ after passing to a smaller neighborhood of x.

Thus, in all cases we have produced the local trivialization of E at x. □

Remark 11.6. A good example of Proposition 11.5(b) is the covering of a sphere S^n by its upper and lower hemispheres, intersecting in the equator. In this situation *all* of (i)—(iii) happen to be satisfied.

Corollary 11.7 (Patching vector bundles)**.** *Let $\{A, B\}$ be a cover of the space X. Suppose given vector bundles $E_A \to A$ and $E_B \to B$, together with a vector bundle isomorphism $\phi\colon E_A|_{A\cap B} \to E_B|_{A\cap B}$. Then there exists a vector bundle $E \to X$ such that $E|_A$ is isomorphic to E_A and $E|_B$ is isomorphic to E_B provided that one of the following conditions holds:*

- *A and B are both open, or*
- *A and B is a closed cover satisfying any of the hypotheses in part (b) of Proposition 11.5.*

Proof. Define E to be the pushout of the following diagram:

$$\begin{array}{ccccc} E_A|_{A\cap B} & \xrightarrow{\phi} & E_B|_{A\cap B} & \rightarrowtail & E_B \\ \downarrow & & & & \vdots \\ E_A & & \cdots\cdots\cdots\cdots\cdots\cdots & \rightarrowtail & E. \end{array}$$

The composite maps $E_A \to A \to X$ and $E_B \to B \to X$ yield a map $E \to X$, and one readily checks that this inherits the structure of a family of vector spaces (see Exercise 10.23). It is also evident that $E|_A \cong E_A$ and $E|_B \cong E_B$. It only remains to verify that E is a vector bundle, and this is a direct application of Proposition 11.5. □

Corollary 11.7(a) admits a generalization to arbitrary open coverings. Suppose $\{U_\alpha\}$ is an open cover of X, and assume given a collection of vector bundles $E_\alpha \to U_\alpha$. For each α and β further assume given an isomorphism

$$\phi_{\beta,\alpha}\colon E_\alpha|_{U_\alpha\cap U_\beta} \xrightarrow{\cong} E_\beta|_{U_\alpha\cap U_\beta}.$$

Let E be the quotient of $\amalg_\alpha E_\alpha$ by the equivalence relation generated by saying $(\alpha, v_\alpha) \sim (\beta, \phi_{\beta,\alpha}(v_\alpha))$ for every α, β, and $v_\alpha \in E_\alpha|_{U_\alpha\cap U_\beta}$. Here we are writing (α, v_α) for the element v_α in $\amalg_\gamma E_\gamma$ that lies in the summand indexed by α.

It is easy to see that in this generality E is a family of vector spaces. It is not necessarily the case, however, that $E|_{U_\alpha} \cong E_\alpha$. If this were true for all α then of

course E would be a vector bundle and we would be done. Here is the trouble, though. Suppose $\alpha_0, \alpha_1, \ldots, \alpha_n$ are a sequence of indices such that $\alpha_0 = \alpha_n = \alpha$. If $v \in E_\alpha$ then we identify v with $\phi_{\alpha_1,\alpha_0}(v)$, which is in turn identified with $\phi_{\alpha_2,\alpha_1}(\phi_{\alpha_1,\alpha_0}(v))$, and so forth—so that v ends up being identified with

$$\Big(\phi_{\alpha_n,\alpha_{n-1}} \circ \phi_{\alpha_{n-1},\alpha_{n-2}} \circ \cdots \circ \phi_{\alpha_1,\alpha_0}\Big)(v). \tag{11.8}$$

Note that, like v, this expression is an element of E_α. So identifications are possibly being made within individual summands of $\amalg_\alpha E_\alpha$, rather than just between different summands. The fibers of $E|_{U_\alpha}$ are quotients of those in E_α, but they might not be identical. To prohibit this from happening we impose some extra conditions: for any indices α, β, γ we require that

(i) $\phi_{\alpha,\alpha} = \mathrm{id}$,
(ii) The two isomorphisms $\phi_{\gamma,\alpha}$ and $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}$ agree on their common domain of definition, which is $E_\alpha|_{U_\alpha \cap U_\beta \cap U_\gamma}$.

We leave it to the reader to check that these conditions force any expression as in (11.8), with $\alpha_0 = \alpha_n$, to just be equal to v (in particular, note that they force $\phi_{\alpha,\beta} = \phi_{\beta,\alpha}^{-1}$). So the fibers of E coincide with the fibers of the E_α's, we get isomorphisms $E|_{U_\alpha} \cong E_\alpha$, and hence E is a vector bundle.

Condition (ii) above is usually called the **cocycle condition**. To see why, consider the case where all of the E_α's are trivial bundles of rank n. Then the data in the $\phi_{\alpha,\beta}$ maps is really just the data of a map $g_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. These $g_{\alpha,\beta}$ maps are called **transition functions**. Condition (ii) is the requirement that the transition functions assemble to give a Čech 1-cocycle with values in the group $GL_n(\mathbb{R})$. Condition (i) is just a normalization condition, so that we are dealing with 'normalized' Čech 1-cocycles. Elements of the (continuous) Čech cohomology group $\check{H}^1_{cts}(U_\bullet; GL_n(\mathbb{R}))$ can be seen to be in bijective correspondence with isomorphism classes of vector bundles on X that are trivializable over the U_α's; if we take the direct limit over all open coverings then we obtain a bijection between isomorphism classes of vector bundles on X and elements of the Čech cohomology group $\check{H}^1_{cts}(X; GL_n(\mathbb{R}))$. But we are getting ahead of ourselves here; see Section 16 for related discussion.

11.9. **Dual bundles.** Let $E \to X$ be a vector bundle of rank n. Using the method of Section 11.1 we can define the dual bundle E^*, which set-theoretically is $\{(x, v) \,|\, x \in X, v \in E_x^*\}$. One can examine this construction in terms of patching trivial bundles. Choose an open cover $\{U_\alpha\}$ of X with respect to which E is trivializable; a choice of trivialization over each U_α then yields a collection of gluing maps $\phi_{\alpha,\beta}$. We think of E as being built from the trivial bundles $E_\alpha = U_\alpha \times \mathbb{R}^n$ via these gluing maps. Then the dual bundle E^* is built from the trivial bundles $U_\alpha \times (\mathbb{R}^n)^*$ via the duals of the gluing maps: that is, $(\phi^{E^*})_{\beta,\alpha} = \big(\phi^E_{\alpha,\beta}\big)^*$.

We will see in a moment (Corollary 11.15) that for real vector bundles over paracompact Hausdorff spaces one always has $E \cong E^*$, although the isomorphism is not canonical. This is not true for complex bundles, however (see Example 11.18).

Let $L \to \mathbb{C}P^n$ be the tautological complex line bundle over $\mathbb{C}P^n$. Its (complex) dual L^* is called the **dual tautological line bundle** over $\mathbb{C}P^n$, or in geometry the **hyperplane line bundle**. Whereas from a topological standpoint neither L nor L^* holds a preferential position over the other, in algebraic geometry there is an important difference between the two. The difference comes from the fact

that L^* has certain naturally defined sections, whereas L does not. For a point $z = [z_0 : \cdots : z_n] \in \mathbb{C}P^n$, L_z is the complex line in $\mathbb{C}^{n+1}$ spanned by $(z_0, \ldots, z_n)$. Given only $z \in \mathbb{C}P^n$ there is no evident way of writing down a point on L_z, without making some kind of arbitrary choice; said differently, the bundle L does not have any easily-described sections. In contrast, it is much easier to write down a *functional* on L_z. For example, let ϕ_i be the unique functional on L_z that sends the point $(z_0, \ldots, z_n)$ to z_i. Notice that this description depends only on $z \in \mathbb{C}P^n$, not the point $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ that represents it; that is, the functional sending $(\lambda z_0, \ldots, \lambda z_n)$ to λz_i is the same as ϕ_i. In this way we obtain an entire $\mathbb{C}^{n+1}$'s worth of sections for L^*, by taking linear combinations of the ϕ_i's. The name "hyperplane line bundle" comes from the fact that each (nonzero) algebraic section of L^* vanishes on a hyperplane.

To be clear, it is important to realize that L has plenty of sections—it is just that one cannot describe them by simple formulas. The slogan to remember is that the bundle L^* has *algebraic* sections, whereas L does not. In algebraic geometry the bundle L^* is usually denoted $\mathcal{O}(1)$, whereas L is denoted $\mathcal{O}(-1)$. More generally, $\mathcal{O}(n)$ denotes $(L^*)^{\otimes n}$ when $n \geq 0$ (so that $\mathcal{O}(0)$ is the trivial line bundle), and denotes $L^{\otimes(-n)}$ when $n < 0$.

11.10. **Inner products on bundles.** It is nearly possible to develop everything we need from bundle theory without using inner products, and in the rest of the text we do try to minimize our use of them. But for some results the use of inner products provides significant simplifications of proofs, and so it is good to know about them.

Definition 11.11. *Let $E \to X$ be a real vector bundle. An **inner product** on E is a map of vector bundles $E \otimes E \to \underline{1}_X$ that induces a positive-definite, symmetric, bilinear form on each fiber E_x. A vector bundle with an inner product is usually called an **orthogonal** vector bundle.*

There is a similar notion for Hermitian inner products on complex vector bundles. If E is a complex vector bundle then let $\bar{E}$ denote the *conjugate bundle*, which is a copy of E but with the new scalar multiplication $\lambda \cdot_{new} v = \bar{\lambda} \cdot_{old} v$. A Hermitian inner product on E is a map of vector bundles $\bar{E} \otimes_{\mathbb{C}} E \to \underline{1}$ that induces a Hermitian inner product on each fiber.

The next result is the first of several places where we will need to use partitions of unity, so let us take a moment to review this concept.

Definition 11.12. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a space X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection of continuous functions $\phi_\alpha \colon X \to [0, 1]$ such that*

(1) [Support condition] For each α, $\overline{\operatorname{Supp} \phi_\alpha} \subseteq U_\alpha$ where $\operatorname{Supp} \phi_\alpha = \phi_\alpha^{-1}(\mathbb{R} - 0)$.
(2) [Local finiteness] For each $x \in X$ there is a neighborhood $x \in V$ such that V has nonempty intersection with $\overline{\operatorname{Supp} \phi_\alpha}$ for only finitely many indices α.
(3) [Partition of 1] For each $x \in X$, $\sum_\alpha \phi_\alpha(x) = 1$ (note that this is a finite sum by (2)).

The basic theory of partitions of unity is discussed in [Munkres, Section 41], though note that Munkres's terminology is slightly different from ours: he defines the support of ϕ_α to be the **closure** of $\phi_\alpha^{-1}(\mathbb{R} - 0)$. The main result we will need

is that on a paracompact Hausdorff space any open cover has a partition of unity subordinate to it [Munkres, Theorem 41.7].

Proposition 11.13. *Assume that X is paracompact and Hausdorff. Then any real bundle on X admits an inner product, and any complex bundle on X admits a Hermitian inner product.*

Proof. The idea is to produce the necessary inner products locally, and then use a partition of unity to average the results into a global inner product. We may as well assume X is connected, and then E has a constant rank n.

Let $E \to X$ be a real vector bundle, and let $\{U_\alpha\}$ be an open cover over which the bundle is trivial. Choose bundle isomorphisms $f_\alpha \colon E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$, for each α. Equip $\mathbb{R}^n$ with the standard Euclidean inner product, and let $\langle -, - \rangle_\alpha$ be the inner product on $E|_{U_\alpha}$ obtained by transplanting the Euclidean product across the isomorphisms f_α.

Let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. For $x \in X$ and $v, w \in E_x$ define

$$\langle v, w \rangle = \sum_\alpha \phi_\alpha(x) \cdot \langle v, w \rangle_\alpha.$$

It is clear that this is continuous in v and w, bilinear, symmetric, and positive-definite—these follow from the corresponding properties of the forms $\langle -, - \rangle_\alpha$ (note that continuity uses the local finiteness property of the partition). So this completes the construction.

The proof for Hermitian inner products on a complex bundle is basically identical. □

Remark 11.14. One can consider the *space* of all possible inner products on a given bundle, which by the above result is nonempty when X is paracompact Hausdorff. It is an important fact that the space is actually contractible: see Exercise 11.29. So any construction that makes use of a choice of inner product will be well-defined up to homotopy.

Corollary 11.15. *Let $E \to X$ be a real vector bundle on a paracompact Hausdorff space X. Then E is isomorphic to its dual E^*.*

Proof. Start by equipping E with an inner product $E \otimes E \to \underline{1}$, and note that the fiberwise forms are nondegenerate (since they are positive-definite). The adjoint of the above bundle map is a map $E \to E^*$, and nondegeneracy of the fiberwise forms shows that this is a fiberwise isomorphism. So it is an isomorphism of bundles. □

Exercise 11.16. Here is an illuminating problem to think through. Every complex vector space may be equipped with a nondegenerate, symmetric bilinear form. Check that the proof of Proposition 11.13 does *not* generalize to show that every complex vector bundle may be equipped with a symmetric bilinear form that is non-degenerate on the fibers—in particular, find the point where the proof breaks down. Note that if the proof did generalize, one could show just as in Corollary 11.15 that every complex bundle was isomorphic to its own dual. This is false, as we will see in Example 11.18 below. The complex version of Corollary 11.15 says that if $E \to X$ is a complex bundle over a paracompact Hausdorff space then E^* is isomorphic to the conjugate bundle $\bar{E}$.

Consider a trivial bundle $X \times \mathbb{R}^n \to X$ and equip it with the standard inner product. This bundle may be considered as trivial in two different ways: the vector bundle structure is trivial, and the inner product structure is also trivial. It is not clear *a priori* that the former property implies the latter, but in fact it does:

Proposition 11.17. *Let X be a space and let $n \in \mathbb{Z}_+$. Every inner product on $\underline{n}_X$ is isomorphic (in the category of vector bundles with inner product) to the 'constant' inner product provided by the standard Euclidean metric.*

Proof. Consider $\mathbb{R}^n$ with its standard basis $e_1, \ldots, e_n$. Inner products on $\mathbb{R}^n$ are in bijective correspondence with symmetric, positive-definite matrices $A \in M_{n\times n}(\mathbb{R})$, by sending an inner product $\langle -, - \rangle$ to the matrix $a_{ij} = \langle e_i, e_j \rangle$. Let $M^{sym,+}$ denote the space of such matrices. To give an inner product on the trivial bundle $\underline{n}_X$ is therefore equivalent to giving a map $X \to M^{sym,+}$.

Given an isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ we may transplant an inner product from the target onto the domain; this gives rise to an action of $GL_n(\mathbb{R})$ on the space of inner products. If $P \in GL_n(\mathbb{R})$ and $A \in M^{sym,+}$ then the action is $P.A = P^T AP$. The fact that every inner product on $\mathbb{R}^n$ has an orthonormal basis shows that $M^{sym,+}$ equals the orbit of the identity matrix I_n under this action. The stabilizer of the identity is of course the orthogonal group O_n, and so we obtain the homeomorphism $GL_n(\mathbb{R})/O_n \cong M^{sym,+}$.

Since O_n is a compact Lie group the quotient $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})/O_n$ is a fiber bundle and hence a fibration (cf. [Palais, Section 4.1]). So we have the fibration sequence $O_n \hookrightarrow GL_n(\mathbb{R}) \to M^{sym,+}$, where the projection map sends a matrix P to $P^T I_n P = P^T P$. The inclusion $O_n \hookrightarrow GL_n(\mathbb{R})$ is a homotopy equivalence by Gram-Schmidt, and so $M^{sym,+}$ is weakly contractible. But actually it is not hard to show directly that there is a homeomorphism $M^{sym,+} \cong \mathbb{R}^{\binom{n+1}{2}}$. For this, recall from linear algebra that a real symmetric matrix A is positive definite if and only if the minors $A[1 \cdots k | 1 \cdots k]$ are positive, for every $1 \le k \le n$ (this is called Sylvester's criterion). When $n = 2$, for example, we can use this to write down a homeomorphism

$$(0,\infty) \times \mathbb{R} \times (0,\infty) \xrightarrow{\cong} M_2^{sym,+}, \qquad (a,b,c) \mapsto \begin{bmatrix} a & b \\ b & \frac{b^2}{a} + c \end{bmatrix}$$

So $M_2^{sym,+} \cong \mathbb{R}^3$. For the general case one works inductively to show $M_n^{\mathrm{sym},+} \cong M_{n-1}^{sym,+} \times \mathbb{R}^{n-1} \times (0,\infty)$. The idea is that after specifying the upper $(n-1)\times(n-1)$ submatrix and the top $n-1$ entries of the last column, we rearrange the cofactor expansion of the determinant about the last column to obtain a lower bound for the final diagonal entry.

Since $M_n^{\mathrm{sym},+}$ is a Euclidean space, it can be given the structure of a CW-complex. So the commutative square

$$\begin{array}{ccc} \{I_n\} & \rightarrowtail & GL_n(\mathbb{R}) \\ \simeq \downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle [P \mapsto P^T P]} \\ M_n^{sym,+} & \xrightarrow{\mathrm{id}} & M_n^{sym,+} \end{array}$$

has a lift as indicated.

As we have discussed, our given inner product on $\underline{n}_X$ is represented by a map $X \to M_n^{sym,+}$. Compose with r to obtain $X \to GL_n(\mathbb{R})$. This map specifies

a bundle isomorphism $\underline{n}_X \to \underline{n}_X$. If we equip the domain with our given inner product and the codomain with the standard inner product, this map preserves the inner products and therefore proves the proposition. □

Suppose that $E \to X$ is a rank n real vector bundle with an inner product. Choose a trivializing open cover $\{U_\alpha\}$, and for each α fix an inner-product-preserving trivialization $f_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ where the codomain has the standard inner product (this is possible by Proposition 11.17). The transition functions $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ therefore factor through O_n, as they must preserve the inner product. This process is usually referred to as *reduction of the structure group*.

We may use these ideas to give another proof of Corollary 11.15, one that is perhaps more down-to-earth. Let $E \to X$ be a real vector bundle on a paracompact Hausdorff space. Since the bundle may be equipped with an inner product, we can choose an open cover $\{U_\alpha\}$ with respect to which there exist local trivializations where the transition functions are maps $g_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to O_n$. So we obtain E by gluing together the spaces $E_\alpha = U_\alpha \times \mathbb{R}^n$ via the maps $g_{\alpha,\beta}$. But then we obtain E^* by gluing together the spaces $U_\alpha \times (\mathbb{R}^n)^*$ via the maps

$$h_{\beta,\alpha} = g^*_{\alpha,\beta}.$$

Recall that in terms of matrices the dual is represented by the transpose. Since each $g_{\alpha,\beta}(x)$ is in O_n we can write $h_{\beta,\alpha}(x) = g_{\alpha,\beta}(x)^{-1}$, or

$$h_{\alpha,\beta}(x) = h_{\beta,\alpha}(x)^{-1} = g_{\alpha,\beta}(x).$$

In other words, the transition functions for E and E^* are exactly the same, and that is why the bundles are isomorphic.

We close this section with the promised example of a complex bundle that is not isomorphic to its dual:

Example 11.18. Let D_+ and D_- denote the upper and lower hemispheres of S^2. Let S^1 be the equator, which we identify with the unit complex numbers. Given a map $f\colon S^1 \to GL_n(\mathbb{C})$ we may construct a complex bundle on S^2 by taking two trivial bundles $\underline{n}_{D_+}$ and $\underline{n}_{D_-}$ and gluing them together using the map f: precisely, for $z \in S^1$ an element $v \in \big(\underline{n}_{D_+}\big)_z$ is glued to $f(z)\cdot v \in \big(\underline{n}_{D_-}\big)_z$. Here we are using Corollary 11.7(b). Let $E(f)$ denote the resulting bundle.

Observe that giving an isomorphism $E(f) \to E(g)$ is equivalent to giving two maps $A\colon D_+ \to GL_n(\mathbb{C})$ and $B\colon D_- \to GL_n(\mathbb{C})$ such that $g(z)\cdot A(z) = B(z)\cdot f(z)$ for all $z \in S^1$. Let us rewrite this as $A(z) = g(z)^{-1}B(z)f(z)$. Now, the map $B|_{S^1}\colon S^1 \to GL_n(\mathbb{C})$ is null-homotopic because it extends over D_-; so it is (unbased) homotopic to the constant map at I_n (this uses connectivity of $GL_n(\mathbb{C})$). Therefore the map $z \mapsto g(z)^{-1}B(z)f(z)$ is homotopic to $g(z)^{-1}f(z)$. But $A|_{S^1}$ is also (unbased) homotopic to the constant map at I_n, because it extends over D_+. So we have proven that if $E(f) \cong E(g)$ then $z \mapsto g(z)^{-1}f(z)$ is unbased homotopic to a constant map.

Next, observe that the dual of $E(f)$ is $E(f')$, where $f'(z) = \big[f(z)^T\big]^{-1}$. So if $E(f)$ is isomorphic to its dual then the map $z \mapsto f(z)^T \cdot f(z)$ is null-homotopic.

Consider the case where $n = 1$ and $f(z) = z$. Since $z \mapsto z^2$ is not null-homotopic, we see that $E(f)$ is not isomorphic to its dual.

Exercise 11.19. Check that $E(f)$ from the end of the above example is isomorphic to the tautological line bundle $L \to \mathbb{C}P^1$.

The above example starts to give an indication that the classification of vector bundles on spheres reduces to a problem in homotopy theory. We will explore this in detail in Section 15, and in Section 16 we will see that it is true not only for spheres but for any paracompact Hausdorff base space X.

11.20. **Principal bundles and frame bundles.** Let G be a topological group.

Definition 11.21. *A **principal G-bundle** is a map $p\colon E \to B$ together with a right G-action on E such that $p(e.g) = p(e)$ for all $g \in G$, $e \in E$, having the property that for every $b \in B$ there is a neighborhood $b \in U$ and a G-equivariant homeomorphism $\phi\colon p^{-1}(U) \to U \times G$ making the diagram*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times G \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

commute.

A map of principal G-bundles consists of a map of G-spaces $E \to E'$ making the expected triangle commute. Let $\mathsf{PrBund}_G(B)$ denote the category of principal G-bundles over B.

Clearly every principal G-bundle is a fiber bundle with fiber G, the G-action on E is free, and the induced quotient map $E/G \to B$ is a homeomorphism. These characteristics are not quite equivalent to being a principal G-bundle, but the counterexamples tend to be pathological.

A ***G*-torsor** is a G-space that is G-homeomorphic to G. So the fibers of a principal G-bundle are all G-torsors. Any G-map between G-torsors is necessarily an isomorphism, and this implies that every map of principal G-bundles is necessarily an isomorphism.

Every rank n vector bundle $J \to B$ gives rise to an associated principal $GL_n(\mathbb{R})$-bundle called the **frame bundle** of J. As a set, define

$$\mathrm{Fr}(J) = \{(b, w_1, \ldots, w_n) \,|\, b \in B, w_1, \ldots, w_n \text{ a basis for } J_b\}.$$

Let $p\colon \mathrm{Fr}(J) \to B$ be projection to the first coordinate, and let $GL_n(\mathbb{R})$ act on the right on tuples $\underline{w} = (w_1, \ldots, w_n)$ in the usual way: $[\underline{w}P]_i = \sum_j w_j P_{ji}$. A local trivialization of J gives an associated bijection $\mathrm{Fr}(J)|_U \cong U \times VI_n(\mathbb{R}^n)$ and we take the topology on $\mathrm{Fr}(J)$ for which these are homeomorphisms. From here it is routine to check that $\mathrm{Fr}(J) \to B$ is a principal $GL_n(\mathbb{R})$-bundle.

If one thinks of an n-bundle in terms of a local trivialization and transition functions with values in $GL_n(\mathbb{R})$, the frame bundle replaces each fiber with $GL_n(\mathbb{R})$ but then uses the same transition functions. One can imagine reversing that process. To describe this carefully we need a new construction, which we describe next.

Suppose $p\colon E \to B$ is a principal G-bundle and X is a left G-space. Define $E \times_G X = (E \times X)/\sim$ where the equivalence relation is $(eg, x) \sim (e, gx)$ for any $e \in E$, $g \in G$, and $x \in X$. By abuse we will also use p to denote the map $E \times_G X \to B$ given by $(e, x) \mapsto p(e)$. It is easy to check that $E \times_G X \to B$ is a fiber bundle with fiber X. Again, if one thinks in terms of local trivializations

then passing from E to $E \times_G X$ changes the fibers from G to X but uses the same transition functions.

Proposition 11.22. *Suppose that V is a real vector space, $G \to \mathrm{Aut}(V)$ is a continuous group homomorphism, and $p\colon E \to B$ is a principal G-bundle. Then $E \times_G V \to B$ is a vector bundle.*

Proof. Left to the reader. □

Proposition 11.23. *There is an equivalence of categories $(\mathsf{Vect}_n(B), iso) \cong \mathsf{PrBund}_{GL_n(\mathbb{R})}(B)$ given by the functors $J \mapsto \mathrm{Fr}(J)$ and $E \mapsto E \times_{GL_n(\mathbb{R})} \mathbb{R}^n$.*

Proof. We again leave this as an exercise. □

The frame bundle gives another perspective on constructing new bundles out of old ones. Suppose $J \to B$ is a rank n vector bundle and we want to construct $\Lambda^k J$. One way is via the formula

$$\Lambda^k J = \mathrm{Fr}(J) \times_{GL_n(\mathbb{R})} (\Lambda^k \mathbb{R}^n).$$

If J and J' are vector bundles of ranks n and k, then $\mathrm{Fr}(J) \times_B \mathrm{Fr}(J')$ is a principal $GL_n(\mathbb{R}) \times GL_k(\mathbb{R})$-bundle and so we can form

$$[\mathrm{Fr}(J) \times_B \mathrm{Fr}(J')] \times_{GL_n(\mathbb{R}) \times GL_k(\mathbb{R})} (\mathbb{R}^n \otimes \mathbb{R}^k).$$

This is a description of the tensor product $J \otimes J'$.

If one pictures the theory of vector bundles as having elements of both topology and linear algebra, the passage from J to $\mathrm{Fr}(J)$ can be regarded as throwing away the linear algebra part. The passage from $\mathrm{Fr}(J)$ to vector bundles of the form $\mathrm{Fr}(J) \times_{GL_n(\mathbb{R})} V$ can be thought of as putting the linear algebra back in.

Remark 11.24. The phrase "G-bundle" or "G-vector bundle" is often used for a vector bundle that comes to us in the form $E \times_G V$ where $E \to B$ is a principal G-bundle and V is a representation of G. For example, a real vector bundle with inner product is often described as an $O(n)$-bundle (see Exercise 11.32). Using this language, "rank n vector bundle" is synonymous with $GL_n(\mathbb{R})$-bundle.

There are two ways one must be careful with this language. First, it is sometimes unclear whether being a "G-bundle" includes the data of E and V, just the fact of their existence, or some kind of equivalence class of such data. Second, some authors use "G-bundle" as an abbreviation for principal G-bundle, which is completely different than the usage described here. So caution is advised.

∘ Exercises ∘

Exercise 11.25. Given a vector bundle $E \to X$, explain how to construct an associated projective bundle $\mathbb{P}(E) \to X$ that is a fiber bundle whose fiber over a point $x \in X$ is the projective space $\mathbb{P}(E_x)$. Verify that your construction admits a natural map of fiber bundles $E \to \mathbb{P}(E \oplus \underline{1})$ that sends a point $e \in E_x$ to the line spanned by $(e, 1)$ in $E_x \oplus \mathbb{R}$.

Exercise 11.26. Let $f\colon X \to Y$ be a finite covering space, and let $E \to X$ be a vector bundle. Define

$$f_*E = \Big\{ (y, v) \;\Big|\; y \in Y,\ v \in \bigoplus_{x \in f^{-1}(y)} E_x \Big\}.$$

Show how to topologize f_*E so that it is a vector bundle over Y. This is called the **pushforward bundle**.

Exercise 11.27. Let $E \to X$ be a vector bundle and let $Y \subseteq X$ be a closed subset.

(a) Prove that every section $s\colon Y \to E$ extends to a section defined on all of X in each of the following two cases:
   (i) $Y \hookrightarrow X$ is a cofibration, or
   (ii) X is locally compact and paracompact and Hausdorff.
   [Note: This demonstrates a common theme in bundle theory, which is that certain results work either in the presence of sufficient cofibration hypotheses or in the presence of technical point-set topology hypotheses. For (ii), use the Tietze extension theorem locally and then patch the results together using a partition of unity.]

(b) Again under hypotheses (i) or (ii), prove that a nonvanishing section $s\colon Y \to E$ extends to a nonvanishing section on some neighborhood of Y.

(c) Suppose that $F \to X$ is another bundle and $u\colon E|_Y \to F|_Y$ is an isomorphism. Under hypotheses (i) or (ii) prove that u extends to an isomorphism $E|_U \to F|_U$ for some neighborhood U of Y.

Exercise 11.28. Let $f\colon Y \to X$ and let E and F be vector bundles over X. Prove that there are natural isomorphisms $f^*(E \otimes F) \cong (f^*E) \otimes (f^*F)$ and $f^*\underline{\mathrm{Hom}}(E,F) \cong \underline{\mathrm{Hom}}(f^*E, f^*F)$.

Exercise 11.29. [Spaces of inner products on a bundle.]

(a) Let V be a real vector space, and define $\mathrm{IP}(V) \subseteq (V \otimes V)^*$ to be the space of inner products on V. Prove that $\mathrm{IP}(V)$ is convex, hence contractible.

(b) For a real vector bundle $E \to X$, let $\mathrm{IP}(E) \subseteq (E \otimes E)^*$ consist of the inner products in each fiber. Prove that $\mathrm{IP}(E) \to X$ is a fiber bundle, and hence an acyclic fibration.

(c) An inner product on E is a section of $\mathrm{IP}(E) \to X$. Prove that any two sections are homotopic (as sections).

(d) If X is paracompact and Hausdorff concluce that the space of inner products on E is contractible.

Exercise 11.30. Let $\mathrm{IP}_n = \mathrm{IP}(\mathbb{R}^n)$ be the space of all inner products on $\mathbb{R}^n$. We know from the previous problem that this is contractible, but let us investigate the topological structure in more detail. Note that IP_n may be identified with the space of all positive-definite, symmetric $n \times n$ matrices, which we topologize as a subspace of $M_{n\times n}(\mathbb{R})$.

(a) Prove $\mathrm{IP}_1 \cong \mathbb{R}_{>0}$.

(b) Consider the process of extending an inner product from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$. Let H^n denote the upper half-space $\{x \in \mathbb{R}^n \mid x_n > 0\}$. Prove that $\mathrm{IP}_n \cong \mathrm{IP}_{n-1} \times H^n$.

(c) Use a similar line of argument to analyze the space of Hermitian inner products on $\mathbb{C}^n$. The analog of H^n is the space $(\mathbb{C}^n - \mathbb{C}^{n-1})/S^1$, where S^1 is the group of unit complex numbers acting via scalar multiplication on $\mathbb{C}^n$. Give an explicit proof that this is contractible.

Exercise 11.31. Prove Propositions 11.22 and 11.23.

Exercise 11.32. For a real vector bundle with inner product, define the associated orthonormal frame bundle. Establish an equivalence of categories between rank n bundles with inner product and principal $O(n)$-bundles.

12. Some results from fiberwise linear algebra

Recall that our basic goal is to learn to do linear algebra "over a base space". The fundamental objects in this setting are the vector bundles, and the maps are the bundle maps. This section contains a miscellany of foundational results that are frequently useful. This material can be safely skipped the first time through and referred back to as needed.

Lemma 12.1. *Let X be any space, and let $f\colon \underline{n} \longrightarrow\!\!\!\!\!\rightarrow \underline{k}$ be a surjective map of bundles. Then f has a splitting.*

Note that the result is not immediately obvious. Of course one can choose a splitting in each fiber, but what guarantees that these can be chosen in a continuous manner?

Proof. Let $W = \{A \in M_{k\times n} \mid \operatorname{rank} A = k\}$, which is the space of surjective maps $\mathbb{R}^n \to \mathbb{R}^k$ (our matrices act on the left). Let Z be the space

$$Z = \{(A,B) \mid A \in M_{k\times n},\ B \in M_{n\times k},\ AB = I\},$$

which is the space of surjective maps with a chosen splitting. We claim that the projection map $p_1\colon Z \to W$ is a fiber bundle with fiber $\mathbb{R}^{k(n-k)}$, but defer the proof for just a moment. The fact that the fiber is contractible then shows that p_1 is weak homotopy equivalence.

Consider the diagram

$$\begin{array}{ccc} & & Z \\ & & \Big\downarrow{\scriptstyle p_1}\sim \\ W & = & W. \end{array}$$

The space W is an open set of $M_{k\times n} \cong \mathbb{R}^{kn}$; indeed, it is the union of the $\binom{n}{k}$ open sets defined by one of the $k \times k$ minors being nonzero. Any open set of Euclidean space may be given a CW-structure—this follows from the fact that every smooth manifold is triangulable, but a simple low-tech argument using subdivided cubes is given in [Hatcher1, Proof of Theorem A.7]. So the standard lifting theorems now show that there is a lifting $r\colon W \to Z$ in the above diagram.

Our surjective bundle map $f\colon \underline{n} \to \underline{k}$ is determined by a map $X \to W$. Composing with $W \to Z$, and then projecting to the second coordinate of Z, gives the desired splitting for f.

It remains to prove the claim about p_1 being a fiber bundle. Let $A \in W$. Since $\operatorname{rank}(A) = k$ there is a $k \times k$ minor of A that is nonzero; without loss of generality let us assume that it is the minor made up of the first k columns of the matrix. Let $U \subseteq W$ be the subspace consisting of all matrices where this same minor is nonzero, which is an open neighborhood of A in W. Writing matrices in block form, U consists of matrices $[X|Y]$ where $\det(X) \neq 0$. Then $p_1^{-1}(U)$ consists of pairs

$$\alpha_{X,Y,J,K} = \left(\begin{bmatrix} X & Y \end{bmatrix}, \begin{bmatrix} J \\ K \end{bmatrix} \right)$$

having the property that $\det(X) \neq 0$ and $XJ + YK = I_k$. We obtain a homeomorphism $U \times M_{n-k,k}(\mathbb{R}) \cong p_1^{-1}(U)$ by sending $([X|Y], K)$ to $\alpha_{X,Y,J,K}$ with $J = X^{-1}(I_k - YK)$. □

Note the significance of the map $W \to Z$ that is produced in the above proof. This assigns to every surjection $\mathbb{R}^n \to \mathbb{R}^k$ a splitting, and it does so in a continuous manner. Of course there is no claim that there is a nice formula for how to do this, and in fact there almost certainly is not—but the proof shows that there does exist *some* way of doing so.

The previous result implies that every surjection of vector bundles is locally split. The following is a global version of this:

Proposition 12.2. *Let X be a paracompact Hausdorff space. Then any surjection of bundles $E \twoheadrightarrow F$ has a splitting. Moreover, the space of all splittings (topologized as a subspace of $\mathrm{Hom}(F, E)$) is contractible.*

Proof. Briefly, for existence we choose local splittings and then use a partition of unity to patch them together.

Choose an open cover $\{U_\alpha\}$ such that both E and F are trivializable over each U_α. Lemma 12.1 shows that there are splittings $\chi_\alpha \colon F|_{U_\alpha} \to E|_{U_\alpha}$. Now choose a partition of unity $\{\phi_\alpha\}$ subordinate to our open cover (cf. Definition 11.12). Set $\chi = \sum_\alpha \phi_\alpha \chi_\alpha$. This sum makes sense and is continuous because the partition of unity is locally finite, and one readily checks that it is a splitting for f.

For contractibility of the space of all splittings, observe that if α and β are two splittings then so is $t\alpha + (1-t)\beta$ for any $t \in \mathbb{R}$. Fixing one splitting α, the deformation $\beta, t \mapsto t\alpha + (1-t)\beta$ gives a contracting homotopy. □

Given a map of vector bundles $f \colon E \to F$ over a space X, we would like to construct kernels, images, and cokernels—bundle operations that reflect the usual constructions of linear algebra. But we have already seen a good example that shows the subtleties here: take $X = [-1, 1]$ and let $f \colon \underline{1} \to \underline{1}$ be multiplication by t on the fiber over $t \in X$. Taking fiberwise kernels or images does not give a vector bundle. To avoid these issues we will require that f have constant rank on the fibers (actually locally constant is enough, as this is equivalent to constant rank on each connected component). Under this assumption the construction of kernel and image bundles is fairly straightforward, as these can be realized as subobjects of bundles we already have. The construction of cokernels is a little more challenging, but also works.

We start by looking at kernels and images. If $f \colon E \to F$ is a map of bundles and $x \in X$, write $\mathrm{rank}_x(f)$ for the rank of $f_x \colon E_x \to F_x$. If these fiberwise ranks are independent of x then we will also write $\mathrm{rank}(f)$ for the common value.

Proposition 12.3. *Let X be any space, and let $f \colon E \to F$ be a map of vector bundles over X. If f has constant rank then $\ker f$ and $\mathrm{im}\, f$ are vector bundles.*

Proof. Let $x \in X$, let $n = \mathrm{rank}_x(E)$, let $k = \mathrm{rank}_x(F)$, and let $r = \mathrm{rank}(f)$. It will suffice to produce a neighborhood U of x together with $n - r$ independent sections of $\ker f$ over U and r independent sections of $\mathrm{im}\, f$ over U. In particular, this makes it clear that we might as well assume that E and F are both trivial bundles; in this case f is specified by a map $X \to W_r$ where $W_r = \{A \in M_{k\times n} \mid \mathrm{rank}(A) = r\}$.

Let Z_r be the space

$$Z_r = \{(A, v_1, \ldots, v_{n-r}) \mid A \in W_r \text{ and } v_1, \ldots, v_{n-r} \text{ span the kernel of } A\}.$$

One can check that the projection $Z_r \to W_r$ is a fiber bundle with fiber $GL_{n-r}(\mathbb{R})$, but this is stronger than what we actually need. We only need that the map is

locally split: any point in W_r has a neighborhood over which there exists a section. Given a map $X \to W_r$, it will then follow that every point in x has a neighborhood over which there exists a lifting into Z_r, and this will give the $n - r$ independent local sections of $\ker f$.

So let A be a point in W_r. Since $\operatorname{rank}(A) = r$, some $r \times r$ minor of A is nonzero. Without loss of generality we might as well assume it is the upper left $r \times r$ minor. Since $\operatorname{rank}(A) = r$, then for $j > r$ the jth column of A is a linear combination of the first r columns in a unique way; said differently, there is a unique vector of the form

$$v_j = e_j - s_1 e_1 - s_2 e_2 - \cdots - s_r e_r$$

that is in the kernel of A. Here the s_i's are certain rational expressions in the matrix entries of A that can be determined using Cramer's Rule. These formulas define sections on the neighborhood U of A consisting of all $k \times n$ matrices of rank r whose upper left $r \times r$ minor is nonzero. This finishes the proof of our claim.

We have established that $\ker f$ is a vector bundle. The proof for $\operatorname{im} f$ is entirely similar but a little easier. Let Y_r be the space

$$Y_r = \{(A, v_1, \ldots, v_r) \,|\, A \in W_r \text{ and } v_1, \ldots, v_r \text{ span the image of } A\}.$$

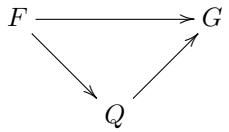
Again, it suffices to show that $Y_r \to W_r$ is locally split. For $A \in W_r$ there is some non-vanishing $r \times r$ minor, and the subset $U \subseteq W_r$ consisting of all matrices with that minor nonzero is a neighborhood of A. We get a section $U \to Y_r$ be sending each $B \in U$ to the pair consisting of B and the r columns of B that are chosen by that minor. □

Next we turn to the construction of cokernels. Let us begin with a precise definition:

Definition 12.4. *Let $f\colon E \to F$ be a map of vector bundles over a space X. A **fiberwise cokernel** for f is a vector bundle Q together with a map of bundles $F \to Q$ with the property that for every $x \in X$ the map $F_x \to Q_x$ makes Q_x a cokernel for f_x.*

The following result says that fiberwise cokernels are also cokernels in the categorical sense (see Appendix D).

Proposition 12.5. *Suppose that Q is a fiberwise cokernel for the bundle map $f\colon E \to F$. Then for any map of bundles $g\colon F \to G$ such that $gf = 0$, there is a unique map of bundles $Q \to G$ making the diagram*



commute.

Proof. Each fiber Q_x is a cokernel for f_x, so there is a unique map of vector spaces $Q_x \to G_x$ making the evident triangle commute. This defines a map of sets $Q \to G$, but we need to check continuity. However, continuity is a local condition and so we can assume that all of the bundles are trivial. In this case the map $p\colon F \to Q$ has a splitting χ by Lemma 12.1. Then $p(\mathrm{id} - \chi p) = 0$, therefore $\mathrm{id} - \chi p$ factors through the kernel of p, which is the image of f. Since $gf = 0$ it follows that $g(\mathrm{id} - \chi p) = 0$

(note that this identity can be checked on fibers). So $g = g\chi p$, and therefore $g\chi$ is the set map $Q \to G$ constructed in the first line. As both g and χ are continuous, we are done. □

Exercise 12.6. Explain why categorical cokernels in the category of vector bundles over X are not always fiberwise cokernels.

Our aim is to prove that fiberwise cokernels for bundle maps $f\colon E \to F$ exist when f has locally constant rank. Let us start with the case where E is a subbundle of F. Define an equivalence relation on F by saying that $v_1 \sim v_2$ if v_1 and v_2 are in the same fiber and $v_1 - v_2 \in E$. Let Q be the quotient space of F under this relation, and let $\pi\colon F \to Q$ be the quotient map. Note that the bundle projection $F \to X$ respects the equivalence relation and so induces $Q \to X$.

Proposition 12.7. *Under the above setup we have*

(a) For the standard inclusion $E = X \times \mathbb{R}^k \hookrightarrow X \times \mathbb{R}^n = F$, then $Q \cong X \times \mathbb{R}^{n-k}$ with the map $F \to Q$ induced by projection onto the last $n-k$ coordinates $\mathbb{R}^n \to \mathbb{R}^{n-k}$.

(b) For any open set $U \subseteq X$, the map $F|_U \to Q|_U$ is a quotient map.

(c) $\pi\colon F \to Q$ is an open map;

(d) The map $F \times_X F \to Q \times_X Q$ is a quotient map;

(e) There is a unique structure of addition and scalar multiplication that makes Q into a family of vector spaces and $F \to Q$ a map of families.

(f) Q is a vector bundle.

Proof. The projection onto the final $n-k$ coordinates $X \times \mathbb{R}^n \to X \times \mathbb{R}^{n-k}$ respects the equivalence relation and so induces a map $Q \to X \times \mathbb{R}^{n-k}$. But we also have the inclusion $\mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^n$ and so can consider the composition $X \times \mathbb{R}^{n-k} \hookrightarrow X \times \mathbb{R}^n \to Q$. One readily verifies that the two maps are inverses, and so $Q \cong X \times \mathbb{R}^{n-k}$.

Part (b) is trivial: for any quotient map $f\colon A \to B$, when $W \subseteq B$ is open then $f^{-1}(W) \to W$ is a quotient map. Apply this to $W = Q|_U$.

By (b) it is enough to prove (c) locally on X, so we immediately reduced to the case where E and F are trivial. The inclusion $f\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^n$ is represented by a map $X \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ where LE denotes the space of linear embeddings. Consider the restriction map $\mathrm{LE}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$. We claim this is locally split. Given a linear embedding $f\colon \mathbb{R}^k \to \mathbb{R}^n$, the associated $n \times k$ matrix has a non-vanishing $k \times k$ minor—without loss of generality assume it is the initial $k \times k$ minor. Then on the open subset of $\mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ where this minor is nonzero, the assignment

$$\begin{bmatrix} A \\ B \end{bmatrix} \mapsto \begin{bmatrix} A & 0 \\ B & I_{n-k} \end{bmatrix}$$

gives a continuous splitting.

Since $\mathrm{LE}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ is locally split, by passing to smaller open sets in X we can assume that $X \to \mathrm{LE}(\mathbb{R}^k, \mathbb{R}^n)$ lifts to $\mathrm{LE}(\mathbb{R}^n, \mathbb{R}^n)$ (via composition with a splitting). This says that there is a bundle isomorphism $X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ making the left square commute in this diagram:

$$\begin{array}{ccccc} X \times \mathbb{R}^k & \xrightarrow{f} & X \times \mathbb{R}^n & \longrightarrow & Q \\ \| & & \cong\downarrow & & \vdots\downarrow \\ X \times \mathbb{R}^k & \xrightarrow{i} & X \times \mathbb{R}^n & \longrightarrow & X \times \mathbb{R}^{n-k}. \end{array}$$

Here i is the standard inclusion. Since Q is defined as a quotient space the diagram shows that we can get an induced homeomorphism $Q \to X \times \mathbb{R}^{n-k}$. So verifying that $X \times \mathbb{R}^n \to Q$ is open is equivalent to the same statement for $X \times \mathbb{R}^n \to X \times \mathbb{R}^{n-k}$, which is true because the standard projection $\mathbb{R}^n \to \mathbb{R}^{n-k}$ is open.

Now we turn to (d). We will use π to denote both the map $F \to Q$ and the associated map $F \times_X F \to Q \times_X Q$. Suppose $U \subseteq Q \times_X Q$ is such that $\pi^{-1}(U)$ is open in $F \times_X F$. Let $(q_1, q_2) \in U$, and choose preimages v_1 for q_1 and v_2 for q_2. Since $\pi^{-1}(U)$ is open in $F \times_X F$, there exist open neighborhoods $v_1 \in V_1 \subseteq F$ and $v_2 \in V_2 \subseteq F$ such that $V_1 \times_X V_2 \subseteq \pi^{-1}(U)$. But then $(q_1, q_2) \in \pi(V_1 \times_X V_2) \subseteq U$. But observe that $\pi(V_1 \times_X V_2) = (\pi V_1) \times_X (\pi V_2)$, and by (c) both πV_1 and πV_2 are open in Q. This proves that U contains a neighborhood of (q_1, q_2).

For (e) we refer to the diagram

$$\begin{array}{ccc} F \times_X F & \xrightarrow{+} & F \\ \downarrow & & \downarrow \\ Q \times_X Q & \dashrightarrow & Q \end{array}$$

and observe that the composition $F \times_X F \xrightarrow{+} F \longrightarrow Q$ respects the equivalence relation on F and so induces the map $Q \times_X Q \to Q$ using (d). A similar argument works for scalar multiplication, using that $\mathbb{R} \times F \to \mathbb{R} \times Q$ is a quotient map [Munkres, Section 46, Exercise 9]. One readily checks that this structure makes Q into a family of vector spaces with the correct properties.

Finally we prove (f). Since we are verifying a local condition, it is enough to do so under the assumption that E and F are trivial. Just as in the proof of (c), we can reduce to the case where $E \hookrightarrow F$ is the standard inclusion $X \times \mathbb{R}^k \hookrightarrow X \times \mathbb{R}^n$, and here we know by (a) that Q is $X \times \mathbb{R}^{n-k}$. □

Corollary 12.8. *Let $f\colon E \to F$ be a map of vector bundles of constant rank. Then a fiberwise cokernel for f exists.*

Proof. By Proposition 12.3 we know that $\operatorname{im} f$ is a vector bundle. Now apply Proposition 12.7 to the inclusion $\operatorname{im} f \hookrightarrow F$. □

From now on, whenever we speak of the cokernel of a map of vector bundles we will implicitly mean the fiberwise cokernel—this conforms with common usage. When these fiberwise cokernels exist they are also categorical cokernels in $\mathsf{FamVS}(X)$; see Exercise 12.20.

The result below is an easy variation on Proposition 12.2; it will be used often, so it is useful to have it stated explicitly.

Corollary 12.9. *Let X be a paracompact Hausdorff space. Then any injection of bundles $E \hookrightarrow F$ has a splitting.*

Proof. We can assume X is connected. Let Q be the cokernel, which is a vector bundle by Corollary 12.8. By Proposition 12.2 the map $F \twoheadrightarrow Q$ has a splitting, which then induces an isomorphism $F \cong E \oplus Q$. The composition $F \xrightarrow{\cong} E \oplus Q \xrightarrow{\pi_1} E$ gives the required splitting of $E \hookrightarrow F$. □

The next result is of a somewhat different nature:

Proposition 12.10. *Suppose that X is compact and Hausdorff. Then every bundle is a subbundle of some finite-rank trivial bundle.*

Proof. Let $\pi\colon E \to X$ be a vector bundle on X. Choose a finite cover $U_1, \ldots, U_s$ over which E is trivializable, which exists because of compactness. For each i choose a trivialization $f_i\colon E|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{R}^n$. Write $F_i = \pi_2 \circ f_i$.

Let $\{\phi_i\}$ be a partition of unity subordinate to the open cover. Define a map

$$\beta\colon E \longrightarrow X \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$$

(where there are s copies of $\mathbb{R}^n$) by the formula

$$(*) \qquad \beta(v) = \Big(\pi v,\, \phi_1(\pi v)F_1(v),\, \ldots,\, \phi_s(\pi v)F_s(v) \Big).$$

We have written πv instead of $\pi(v)$ here, to avoid being overwhelmed by parentheses. Note that if v is not in $E|_{U_i}$ then $F_i(v)$ is undefined, but in this case $\phi_i(\pi v)$ equals 0 and so the formula still makes sense. More rigorously, we can interpret the formula $\phi_i(\pi v)F_i(v)$ as specifying continuous functions on the two closed sets $\pi^{-1}(\overline{\operatorname{Supp}\phi_i})$ (which is contained in $E|_{U_i}$) and on $E\backslash\pi^{-1}(\operatorname{Supp}\phi_i)$ (in this case the zero function), which agree on the overlap—so these patch to give a continuous function on all of X. It is routine to check that formula (*) gives an embedding of bundles. □

We can generalize Proposition 12.10 to paracompact Hausdorff spaces at the expense of passing to trivial bundles of countably-infinite rank. The proof is essentially the same once we have established the following lemma and its corollary. Since we are doing this anyway, it will be convenient to include a refinement for spaces of finite covering dimension (see [Munkres, Section 50] for this notion). Since all topological manifolds have finite covering dimension—as shown in [Munkres]—this is a handy result to have around.

Lemma 12.11. *Let X be a paracompact Hausdorff space. Let $\mathcal{P}$ be a property of open subsets of X such that (1) if U has $\mathcal{P}$ and $V \subseteq U$ then V also has $\mathcal{P}$, (2) if $\{U_\alpha\}$ are disjoint open subsets that all have $\mathcal{P}$ then so does $\bigcup_\alpha U_\alpha$, and (3) X can be covered by open sets with property $\mathcal{P}$. Then X can be covered by countably many open subsets with property $\mathcal{P}$. If in addition X has finite covering dimension, then in fact X can be covered by finitely many open subsets with property $\mathcal{P}$.*

Proof. We take the argument from [MS, Lemma 5.9]. First choose an open cover $\{V_\alpha\}$ by sets with property $\mathcal{P}$, indexed by some set I. Then choose a partition of unity $\{\phi_\alpha\}$ subordinate to this cover. For each nonempty finite subset $S \subseteq I$, define

$$W(S) = \{x \in X \mid \phi_\alpha(x) > \phi_\beta(x) \text{ for all } \alpha \in S \text{ and } \beta \in I - S\}.$$

Said in words, $W(S)$ is the collection of points for which the ϕ's from S are all larger than the ϕ's from outside of S. This is an open set. One sees readily that if $W(S) \cap W(S') \neq \emptyset$ then either $S \subseteq S'$ or $S' \subseteq S$. So if S and S' have the same size then $W(S)$ and $W(S')$ are disjoint.

Let $W_m = \bigcup_{\#S=m} W(S)$. Note that this is a disjoint union. If $x \in X$ then let S_x be the (necessarily finite) collection of all $\alpha \in I$ for which $\phi_\alpha(x)$ is maximal. Then $x \in W(S_x)$, which shows that $\bigcup_m W_m = X$. Finally, for each $S \subseteq I$ note that if $\alpha \in S$ then $W(S) \subseteq \operatorname{Supp}\phi_\alpha \subseteq V_\alpha$. So each $W(S)$ has property $\mathcal{P}$, and therefore each W_m has property $\mathcal{P}$. The set $\{W_m\}_{m>0}$ is the desired countable open cover of X.

For the final statement, assume X has covering dimension n. Then our open cover $\{V_\alpha\}$ has a refinement where no element of X belongs to more than $n+1$ elements. Replace $\{V_\alpha\}$ with this refinement and repeat the above argument. Then $W(S) = \emptyset$ for any S with $\#S > n+1$, hence $W_m = \emptyset$ for $m > n+1$. □

Corollary 12.12. *Let $E \to X$ be a vector bundle, where X is paracompact and Hausdorff. Then there is a countable open cover $\{U_i\}$ for X such that each $E|_{U_i}$ is trivializable. If X has finite covering dimension then the open cover can be chosen to be finite.*

Proof. Apply the lemma, where a subset U has $\mathcal{P}$ if E is trivializable over U. □

Proposition 12.13. *Let X be paracompact and Hausdorff, and let $E \to X$ be a vector bundle. Then there is an embedding of bundles $E \hookrightarrow X \times \mathbb{R}^\infty$, where $\mathbb{R}^\infty$ has the colimit topology (see Appendix A.34). If X has finite covering dimension then E is a subbundle of a finite rank trivial bundle.*

Proof. One chooses a countable trivializing cover as in Corollary 12.12, and then the proof is essentially identical to that for Proposition 12.10 except for one subtlety. The construction from that proof gives a collection of continuous maps $E \to \mathbb{R}^n$ which can be assembled to give an embedding

$$g\colon E \to X \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \cdots \cong X \times \prod_{i=1}^{\infty} \mathbb{R}.$$

The image of each point is a tuple where only finitely many entries are nonzero, so g factors through $X \times \mathbb{R}^\infty$. But the topology on $\mathbb{R}^\infty$ here is the *product topology*—or more precisely, the subspace topology induced by the product topology on $\prod_{i=1}^\infty \mathbb{R}$—and this is different from the colimit topology (see Appendix A.34 for discussion of this). So an extra argument is required to see that the map is still continuous when $\mathbb{R}^\infty$ has the colimit topology.

We need only to consider the map $E \to \mathbb{R}^n \times \mathbb{R}^n \times \cdots$, which we know lands inside the subset $\mathbb{R}^\infty$. For convenience we will still call this g. We need to prove that $g\colon E \to \mathbb{R}^\infty$ is continuous. Since continuity is a local condition, it is sufficient to prove that every $v \in E$ has a neighborhood on which the function is continuous. But by the local finiteness of the partition of unity, there is a neighborhood $\pi(v) \in W$ with the property that W only intersects finitely-many of the $\overline{\operatorname{Supp}\phi_i}$ spaces—$i = 1$ through $i = N$, say. So for all $v' \in \pi^{-1}(W)$ the coordinates of $g(v')$ past the $(N \cdot n)$th all vanish. Then g maps $\pi^{-1}(W)$ into $\mathbb{R}^{Nn}$, and is continuous as such a map. Therefore g is also continuous as a map $g\colon \pi^{-1}(W) \to \mathbb{R}^\infty_{\text{colim}}$. □

Finally, we close this section with a few useful results related to ranks and exactness:

Proposition 12.14. *Let $\alpha\colon E \to F$ be a map of vector bundles over X. Then for any $n \in \mathbb{Z}_{\geq 0}$, the set $\mathcal{R}_n = \{x \in X \mid \operatorname{rank}(\alpha_x) \geq n\}$ is an open subset of X.*

Proof. Let $k = \operatorname{rank}(E)$ and $l = \operatorname{rank}(F)$. Let $x \in \mathcal{R}_n$. We can choose a neighborhood V of x over which both E and F are trivial. The map α is then specified by a continuous function $\alpha\colon V \to \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l) = M_{l\times k}(\mathbb{R})$. Since $\operatorname{rank}(\alpha_x) \geq n$, some $n \times n$ minor of $\alpha(x)$ is nonzero. If $U \subseteq M_{l\times k}(\mathbb{R})$ is the subspace of all matrices for which the corresponding minor is nonzero, this is an open subset of $M_{l\times k}(\mathbb{R})$. Then $\phi^{-1}(U)$ is a neighborhood of x that is completely contained in $\mathcal{R}_n$. □

Remark 12.15. Let $\mathbb{Z}_{+\infty}$ denote the set $\mathbb{Z}$ equipped with the topology where the open sets are the intervals (n, ∞) for $n \in \mathbb{Z}$. Note that since $(n, \infty) = [n+1, \infty)$, the open sets can also be described as half-open intervals. The following two conditions on a function $f\colon X \to \mathbb{Z}$ are readily seen to be equivalent:

(a) f is continuous when $\mathbb{Z}$ is given the topology $\mathbb{Z}_{+\infty}$,
(b) for any convergent net $x\colon I \to X$, there exists $\alpha \in I$ such that $f(\lim_I x) \leq f(x_i)$ for all $i \geq \alpha$.

(See Section A.14 for a quick tour of nets.) When these properties are satisfied the function f is said to be **lower semicontinuous**. Briefly, in a lower semicontinous function the value on the limit of a convergent net can jump down—but not up—from the limiting behavior in the tails of the net. Proposition 12.14 says that the map $x \mapsto \operatorname{rank}(\alpha_x)$ is lower semicontinuous. We leave the reader to think about the inverted notion of upper semicontinuity, and the fact that the map $x \mapsto \operatorname{nullity}(\alpha_x)$ is upper semicontinuous.

Definition 12.16. *Let $E \to F \to G$ be maps of vector bundles whose composite is zero. Say that this is **exact** (in the middle spot) if it is exact on all fibers.*

Proposition 12.17. *Let $E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ be an exact sequence of vector bundles. Then $\operatorname{im}\alpha$ (which equals $\ker\beta$) is a vector bundle.*

Proof. Using Proposition 12.3 it suffices to prove that α has constant rank on each connected component of X. Without loss of generality we can assume that X is connected. Since the question is local on X, we can assume that E, F, and G are all trivial bundles. Let $n = \operatorname{rank}(F)$.

Pick an $x \in X$ and let $p = \operatorname{rank}(\alpha_x)$. Then $\operatorname{rank}(\beta_x) = n - p$ by exactness. Let $U = \{z \in X \mid \operatorname{rank}(\alpha_z) \geq p\}$ and $V = \{z \in X \mid \operatorname{rank}(\beta_z) \geq n - p + 1\}$. Note that exactness implies that $U = X - V$. But both U and V are open by Proposition 12.14, which means they are also both closed. By connectedness, U is either empty or the whole of X. Since $x \in U$, we must have $U = X$.

A similar argument proves that $\{z \in X \mid \operatorname{rank}(\alpha_z) \leq p\} = X$. So for every $z \in X$ we have $p \leq \operatorname{rank}(\alpha_z) \leq p$, and thus the rank of α is constant on X. □

If $E_\bullet$ is a chain complex of vector bundles on X and $x \in X$, write $(E_x)_\bullet$ for the chain complex of vector spaces formed by the fibers over x. Define the **support** of $E_\bullet$, denoted $\operatorname{Supp} E_\bullet$, to be the subspace $\{x \in X \mid (E_x)_\bullet \text{ is not exact}\} \subseteq X$. We will occasionally write $\operatorname{Supp}_i E_\bullet$ for $\{x \in X \mid H_i((E_x)_\bullet) \neq 0\}$. Note that $\operatorname{Supp} E_\bullet = \bigcup_i \operatorname{Supp}_i E_\bullet$.

Proposition 12.18. *Let $E_\bullet$ be a chain complex of vector bundles on X. Then for any $i \in \mathbb{Z}$, the subspace $\operatorname{Supp}_i E_\bullet$ is closed in X. If $E_\bullet$ is a bounded chain complex then $\operatorname{Supp} E_\bullet$ is closed.*

Proof. We will prove that $X - \operatorname{Supp}_i E_\bullet$ is open, so assume x belongs to this set. Write the maps in the chain complex as

$$E_{i+1} \xrightarrow{\alpha} E_i \xrightarrow{\beta} E_{i-1}.$$

Let $n = \operatorname{rank}_x(E_i)$, $a = \operatorname{rank}_x(\alpha)$, and $b = \operatorname{rank}_x(\beta)$. Since the complex is exact at x in the ith spot we have $a + b = n$. By Proposition 12.14 applied twice, there is an open neighborhood U of x such that $\operatorname{rank}_y(\alpha) \geq a$ and $\operatorname{rank}_y(\beta) \geq b$ for all

$y \in U$. Then we can write

$$a \leq \operatorname{rank}_y(\alpha) \leq n - \operatorname{rank}_y(\beta) \leq n - b$$

where the middle inequality follows from the fact that $(E_y)_\bullet$ is a chain complex. Since $a = n - b$ all the inequalities are in fact equalities, and so we have exactness at y for all $y \in U$. That is, $U \subseteq X - \operatorname{Supp}_i E_\bullet$.

The final statement follows from the fact that $\operatorname{Supp} E_\bullet$ is a finite union of the $\operatorname{Supp}_i E_\bullet$ spaces. □

∘ Exercises ∘

Exercise 12.19. In dealing with maps of vector bundles, a condition is said to be local if having it hold at the fiber over x implies that it also holds for all the fibers in some neighborhood of x. Prove that the following conditions are local:

- injectivity,
- surjectivity,
- exactness.

Exercise 12.20. Suppose that $0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0$ is an exact sequence of vector bundles. Prove that in $\mathsf{FamVS}(X)$ the map i is the categorical kernel of p, and that p is the categorical cokernel of i.

13. Swan's Theorem

In this section we explore our first connection between topology and algebra. We will see that vector bundles are closely related to projective modules.

When X is a space let $C(X)$ denote the ring of continuous functions from X to $\mathbb{R}$, where the addition and multiplication are pointwise. If $E \to X$ is a family of vector spaces, let $\Gamma(E)$ denote the vector space of sections under pointwise addition and scalar multiplication. In addition to being a vector space, it is easy to see that this is actually a module over $C(X)$: if $f \in C(X)$ and $s \in \Gamma(E)$ then fs is the section $x \mapsto f(x)s(x)$. The assignment $E \mapsto \Gamma(E)$ gives a functor from the category $\mathsf{FamVS}(X)$ to $C(X)$-modules.

Proposition 13.1. *Suppose that $E' \xrightarrow{i} E \xrightarrow{p} E''$ are maps in $\mathsf{FamVS}(X)$ where i is the kernel of p. Then $0 \to \Gamma(E') \to \Gamma(E) \to \Gamma(E'')$ is exact.*

Proof. We have the pullback diagram

$$\begin{array}{ccc} E' & \xrightarrow{i} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0_X & \xrightarrow{\zeta} & E''. \end{array}$$

This is a pullback both in $\mathsf{FamVS}(X)$ and in $\mathcal{T}op \downarrow X$, since the limits in these categories are the same. Therefore it is also a pullback square in $\mathcal{T}op$. The conclusions about $\Gamma(-)$ follow immediately from categorical diagram chasing. [Alternatively, just prove the result directly by hand.] □

Note in particular that Proposition 13.1 applies when $0 \to E' \to E \to E'' \to 0$ is a fiberwise exact sequence of vector bundles over X, by Exercise 12.20. In fact, with some mild hypotheses on X we can say even more:

Corollary 13.2. *Suppose X is paracompact Hausdorff and $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles. Then $0 \to \Gamma(E') \to \Gamma(E) \to \Gamma(E'') \to 0$ is an exact sequence of $C(X)$-modules, and moreover is split-exact.*

Proof. By Proposition 12.2 the map $E \to E''$ has a splitting, and this gives a splitting for $\Gamma(E) \to \Gamma(E'')$. □

If $E \to X$ is a vector bundle, rather than just a family of vector spaces, then of course the modules of the form $\Gamma(E)$ are not just arbitrary $C(X)$-modules; there is something special about them. It is easiest to say what this is under some assumptions on X:

Proposition 13.3. *If X is compact and Hausdorff, and E is a vector bundle over X, then $\Gamma(E)$ is a finitely-generated, projective module over $C(X)$.*

Proof. By Proposition 12.10 we can embed E into a trivial bundle $\underline{N}$. This embedding has constant rank on each connected component, so by Proposition 12.8 the quotient Q is also a vector bundle. So we have the exact sequence $0 \to E \rightarrowtail \underline{N} \twoheadrightarrow Q \to 0$ of vector bundles on X. By Corollary 13.2 this yields the split-exact sequence $0 \to \Gamma(E) \to \Gamma(\underline{N}) \to \Gamma(Q) \to 0$ of $C(X)$-modules. But $\Gamma(\underline{N}) = C(X)^N$, so $\Gamma(E)$ is a direct summand of a finitely-generated free module. Hence, it is a finitely-generated projective. □

Recall that $\mathsf{Vect}(X)$ denotes the category of vector bundles over X, and let $C(X)$-Mod denote the category of modules over the ring $C(X)$. Let $C(X)$-proj denote the full subcategory of finitely-generated, projective modules. Then when X is compact and Hausdorff Γ is a functor $\mathsf{Vect}(X) \to C(X)$-$\mathsf{proj}$. It is proven in [Swan] that this is actually an equivalence:

Theorem 13.4 (Swan's Theorem)**.** *Let X be a compact, Hausdorff space. Then*

$$\Gamma\colon \mathsf{Vect}(X) \longrightarrow C(X)\text{-}\mathsf{proj}$$

is an equivalence of categories.

Remark 13.5. Theorem 13.4 is also known as the "Serre-Swan theorem", as Serre had earlier proven the analogous result in algebraic geometry. The content of Swan's paper was to demonstrate that the basic framework provided by Serre could also be made to work in the purely topological context. For more on the algebraic geometry side of things, see Section 28 below.

To prove Swan's theorem we need to verify two things:

- Every finitely-generated projective over $C(X)$ is isomorphic to $\Gamma(E)$ for some vector bundle E.
- For every two vector bundles E and F, the induced map
$$\Gamma\colon \mathrm{Hom}_{\mathsf{Vect}(X)}(E, F) \to \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$$
is a bijection.

That is to say, we need to prove that Γ is surjective on isomorphism classes, and is fully faithful. Here is the first part:

Proposition 13.6. *If X is paracompact Hausdorff and P is a finitely-generated projective module over $C(X)$, then $P \cong \Gamma(E)$ for some vector bundle $E \to X$.*

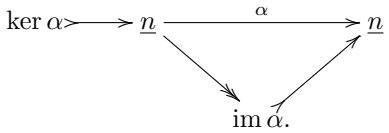
Proof. Choose a surjection $p\colon C(X)^n \twoheadrightarrow P$. Since P is projective, there is a splitting χ. Then $e = \chi p$ satisfies $e^2 = e$, and P is isomorphic to $\mathrm{im}(e)$.

Since $e\colon C(X)^n \to C(X)^n$ we can represent e by an $n \times n$ matrix whose elements are in $C(X)$. Denote the entries of this matrix as e_{ij}. Note that for any $x \in X$ we can evaluate all these functions at x to get an element $e(x) \in M_{n\times n}(\mathbb{R})$. In this way we can regard e as a continuous map $X \to M_{n\times n}(\mathbb{R})$.

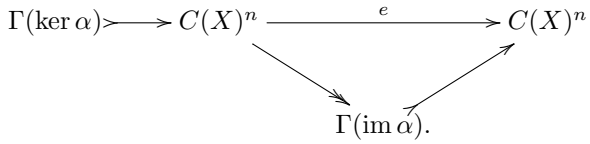
Define a map of vector bundles $\alpha\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ by the formula $\alpha(x, v) = e(x) \cdot v$. Then the sequence

$$\underline{n} \xrightarrow{\alpha} \underline{n} \xrightarrow{1-\alpha} \underline{n}$$

is exact in the middle. Let $E = \mathrm{im}(\alpha)$, which by Proposition 12.17 is a vector bundle on X; the proof of that lemma also shows that α has locally constant rank. We claim that $\Gamma(E) \cong P$. To see this, consider the following diagram of vector bundles:



The map $\underline{n} \to \mathrm{im}\,\alpha$ is split by Proposition 12.2, because X is paracompact. Applying Γ to the above diagram gives



The sequence $0 \to \Gamma(\ker\alpha) \to C(X)^n \to \Gamma(\mathrm{im}\,\alpha) \to 0$ is exact because it was split-exact before applying Γ, and the identification $\Gamma(\ker\alpha) = \ker(\Gamma\alpha)$ shows that $\Gamma(\ker\alpha)$ is the kernel of e. It now follows that $\Gamma(\mathrm{im}\,\alpha)$ is isomorphic to the image of e, which is P. □

Our next goal is to prove that Γ is fully faithful. To do this, it is useful to relate the fibers E_x of our bundle to an algebraic construction based on the module $\Gamma(E)$. For each $x \in X$ consider the evaluation map $\mathrm{ev}_x\colon C(X) \to \mathbb{R}$, and let m_x be the kernel. The ideal $m_x \subseteq C(X)$ is maximal, since the quotient is a field.

Note that we have the evaluation map $\mathrm{ev}_x\colon \Gamma(E) \to E_x$. This map clearly sends the submodule $m_x\Gamma(E)$ to zero so that we get the induced map $\Gamma(E)/m_x\Gamma(E) \to E_x$. In reasonable cases this map is an isomorphism:

Lemma 13.7. *Assume that X is paracompact Hausdorff. Then for any vector bundle $E \to X$ and any $x \in X$, the map $\mathrm{ev}_x\colon \Gamma(E)/m_x\Gamma(E) \xrightarrow{\cong} E_x$ is an isomorphism.*

Proof. We first record the following important fact, which we label (*): if s is a section of E defined on some neighborhood U of x, then there exists a section s' defined on all of X such that s and s' agree on some (potentially smaller) neighborhood of x. To see this, first choose a neighborhood V of x such that $\overline{V} \subseteq U$ (this exists because X is normal, hence regular—see [Munkres, Lemma 31.1]). By

Urysohn's Lemma there is a continuous function $f\colon X \to \mathbb{R}$ such that $f|\overline{V} = 1$ and $f|_{X-U} = 0$. The assignment $z \mapsto f(z) \cdot s(z)$ is readily checked to be a continuous section of E that agrees with s on V.

To prove surjectivity of ev_x, let $v \in E_x$. Since E is locally trivial, one can find a section s defined locally about x such that $s(x) = v$. By principle (*) there is a section s' defined on all of X that agrees with s near x; in particular, $s'(x) = v$.

For injectivity we must work a little harder. Suppose that $s \in \Gamma(E)$ and $s(x) = 0$. We must prove that $s \in m_x\Gamma(E)$. Choose independent sections $e_1, \ldots, e_n$ defined on a neighborhood U of x. Fact (*) says that by replacing U by a smaller neighborhood of x we can assume that the sections are defined on all of X (but only independent on U).

Using that $e_1(y), \ldots, e_n(y)$ is a basis for E_y when $y \in U$, we can write $s(y) = a_1(y)e_1(y) + \cdots + a_n(y)e_n(y)$ for uniquely defined numbers $a_1(y), \ldots, a_n(y) \in \mathbb{R}$. The functions a_i are continuous (under the local trivialization given by the e_i's the a_i are just compositions of s with projection operators). Regarding the a_i's as local sections of the trivial bundle $X \times \mathbb{R}$, (*) shows we may assume the a_i's are defined on all of X (again, we might need to replace U with a smaller neighborhood here). Since $s(x) = 0$ note that $0 = a_1(x) = a_2(x) = \cdots = a_n(x)$.

Let $t = s - a_1e_1 - \cdots - a_ne_n \in \Gamma E$. Note that t vanishes throughout the neighborhood U of x. Again using the Urysohn Lemma, choose a continuous function $b\colon X \to \mathbb{R}$ such that $b(x) = 0$ and $b|_{X-U} = 1$. So $t = bt$. Then $s = t + \sum a_ie_i = bt + \sum a_ie_i \in m_x\Gamma(E)$, since b and all the a_i are in m_x. □

Remark 13.8. Lemma 13.7 should be thought of as giving an algebraic construction of the geometric fiber E_x. We already encountered this idea back in Section 4 and will develop it in more detail in Sections 28 and 30.

Proposition 13.9. *Assume that X is paracompact Hausdorff. Then for any vector bundles E and F over X, the map $\Gamma\colon \mathrm{Hom}_{\mathrm{Vect}(X)}(E, F) \to \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$ is a bijection.*

Proof. First of all, it is easy to check this when E and F are both trivial. A map of vector bundles $X \times \mathbb{R}^k \to X \times \mathbb{R}^l$ is uniquely specified by a map $X \to M_{l\times k}(\mathbb{R})$, and likewise a map of $C(X)$-modules $C(X)^k \to C(X)^l$ is specified by an $l \times k$ matrix with entries in $C(X)$. One observes that continuous maps $X \to M_{l\times k}(\mathbb{R})$ bijectively correspond with $l \times k$ matrices with entries in $C(X)$.

For the general case, consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Vect}(X)}(E,F) & \xrightarrow{\ \Gamma\ } & \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F) \\ \downarrow & & \downarrow \\ \prod_{x\in X} \mathrm{Hom}(E_x, F_x) & \xrightarrow{\ \cong\ } & \prod_{x\in X} \mathrm{Hom}(\Gamma E/m_x\Gamma E, \Gamma F/m_x\Gamma_F). \end{array}$$

The bottom horizontal map is an isomorphism by Lemma 13.7. The left vertical arrow sends a bundle map $\alpha\colon E \to F$ to the collection of its restrictions to each fiber; surely this map is an injection. It follows at once that Γ is also an injection.

Next we show that the right vertical map is injective. Suppose $\beta \in \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$ is sent to zero. If $\beta \neq 0$ then there is an $s \in \Gamma E$ such that

$\beta(s) \neq 0$. Then $(\beta s)(x) \neq 0$ for some $x \in X$. The square

$$\begin{array}{ccc} \Gamma E & \xrightarrow{\beta} & \Gamma F \\ {\scriptstyle \mathrm{ev}_x}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_x} \\ \Gamma E/m_x\Gamma E & \xrightarrow{\bar{\beta}} & \Gamma F/m_x\Gamma F \end{array}$$

immediately shows that $\bar{\beta}(s(x))$ cannot be zero, which is a contradiction. So indeed the right vertical map is injective.

It remains to show that the top horizontal map (labelled Γ) in our diagram is a surjection, so let $\beta \in \mathrm{Hom}_{C(X)}(\Gamma E, \Gamma F)$. We can apply the right vertical arrow to β, and then find a unique preimage in $\prod_x \mathrm{Hom}(E_x, F_x)$ using that the bottom map is an isomorphism. This gives us a map of sets $\alpha\colon E \to F$, by defining it on each of the fibers. We need to prove that α is continuous. However, this is a local question: so it suffices to do so in the case that E and F are trivial, and this case has already been verified. So we have produced a bundle map $\alpha\colon E \to F$ whose restriction to each fiber agrees with the map β. Then $\Gamma\alpha$ and β are sent to the same object under the right vertical map, therefore they must be equal. □

We have now completed the proof of Swan's Theorem, via Propositions 13.6 and 13.9. Most of the arguments were taken directly from [Swan].

13.10. **Variants of Swan's Theorem.** While Swan's theorem is very pretty, it is unfortunate that the rings $C(X)$ are quite large and unwieldy from an algebraic perspective. For example, these are typically non-Noetherian: choose an infinite descending sequence of sets $X \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and look at the ideals of functions vanishing on each A_i. In many cases, however, a projective coming from Swan's Theorem can be seen to be extended from a projective over a smaller, more manageable ring. As one example, let us consider the ring R of *polynomial* functions on the 2-sphere: $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1) \subseteq C(S^2)$. Let T be the module appearing as the kernel in the following short exact sequence:

$$0 \longrightarrow T \longrightarrow R^3 \xrightarrow{[x\ y\ z]} R \longrightarrow 0.$$

Note that the map $R^3 \to R$ is split by the map $1 \mapsto (x,y,z)$, so T is projective. By applying the functor $(-) \otimes_R C(S^2)$ one sees that $T \otimes_R C(S^2)$ is the $C(S^2)$-module corresponding to the tangent bundle of S^2 under Swan's Theorem.

It is reasonable to regard the finitely-generated projective modules over R as the "algebraic" vector bundles over S^2, and this is exactly what happens in the context of algebraic geometry. Given an affine scheme $X = \mathrm{Spec}\, S$, the category of algebraic vector bundles over X is defined in such a way that it is equivalent (in some sense tautologically) to the category of finitely-generated S-projectives.

As a slight generalization of the previous example, let R be a finitely-generated $\mathbb{C}$-algebra that is an integral domain and let $X = \mathrm{Spec}\, R$. We can consider the topological space $X_{\mathbb{C}}$ of $\mathbb{C}$-valued points in X with its analytic topology, and we get an inclusion $R \hookrightarrow C(X_{\mathbb{C}})$. From this we get a functor $P \mapsto P_{\mathbb{C}} = P \otimes_R C(X_{\mathbb{C}})$ from the category of finitely-generated projective R-modules into the category of finitely-generated projective $C(X_{\mathbb{C}})$-modules, the latter of which is equivalent to the category of complex vector bundles on $X_{\mathbb{C}}$ by Swan's Theorem. For example, when $R = \mathbb{C}[x_1, \ldots, x_n]$ then every finitely-generated R-projective gives us an associated

complex vector bundle on $\mathbb{C}^n$. This technique of passing from algebra into topology will be used often.

Exercise 13.11. Keeping the above discussion in mind, revisit examples (3) and (4) of Section 4 and think about how they fit into this perspective.

∘ Exercises ∘

Exercise 13.12. Check that Swan's Theorem holds for any paracompact Hausdorff space having the property that every vector bundle is trivializable on a finite open cover. (We will use this!)

Exercise 13.13. This exercise is based on an example in [Swan]. Let $L \to \mathbb{R}P^n$ be the tautological line bundle and let $\pi\colon S^n \to \mathbb{R}P^n$ be the projection map.

(a) Check that $\pi^*\colon C(\mathbb{R}P^n) \to C(S^n)$ is an injection and that its image is the set of continuous functions $h\colon S^n \to \mathbb{R}$ such that $h(x) = h(-x)$ for all $x \in S^n$. Let us denote this image as $C_{ev}(S^n)$ and identify it implicitly with $C(\mathbb{R}P^n)$ from now on.

(b) Let $C_{odd}(S^n) \subseteq C(S^n)$ be the subspace of odd functions, namely functions h such that $h(x) = -h(-x)$ for all $x \in S^n$. Check that multiplication of functions makes $C_{odd}(S^n)$ into a module over $C_{ev}(S^n)$.

(c) Verify that $\pi^*L \cong \underline{1}$ and use this to prove that $\Gamma(L) \cong C_{odd}(S^n)$ as $C_{ev}(S^n)$-modules.

(d) Now write $R = C_{ev}(S^n)$ and $P = C_{odd}(S^n)$. Let $x_0, \dots, x_n$ be the usual coordinates on $\mathbb{R}^{n+1}$, regarded as functions on S^n. Produce maps of R-modules $P \to R^{n+1} \to P$ such that the composite is the identity, thereby giving an algebraic proof that P is projective.

(e) Explain the bundle-theoretic analog of the maps in (d).

(f) Let $T = \mathbb{R}[x_0, \dots, x_n]/(\sum x_i^2 - 1)$ and let $T_{ev} \subseteq T$ be the subring generated by the monomials of even degree. Let $T_{odd} \subseteq T$ be the $\mathbb{R}$-subspace spanned by the monomials of odd degree, regarded as a module over T_{ev}. Prove that T_{odd} is a finitely-generated projective T_{ev}-module and that $C(\mathbb{R}P^n) \otimes_{T_{ev}} T_{odd} \cong \Gamma(L)$.

(g) We will eventually prove that $L \oplus (\underline{k-1})$ cannot be embedded as a subbundle of $\underline{n+k-1}$ (this is an easy argument with Stiefel-Whitney classes). Using this, prove that the projective T_{ev}-module $T_{odd} \oplus T_{ev}^{k-1}$ has rank k but cannot be generated by $n+k-1$ elements. (This shows that the number of generators for a projective module can be arbitrarily larger than its rank, even over Noetherian rings.)

14. Homotopy invariance of vector bundles

For a fixed n, let $\mathrm{Vect}_n(X)$ denote the set of isomorphism classes of vector bundles on X. It turns out that when X is a finite complex this set is always countable, and often finite. It actually gives a homotopy invariant of the space X. In this section we prove the homotopy invariance property, and in the next section we will start to compute $\mathrm{Vect}_n(X)$ for some simple spaces X.

Write i_0 and i_1 for the two inclusions $X \hookrightarrow X \times I$ coming from the boundary points of the interval. The key to homotopy invariance is the following result.

Proposition 14.1. *Let X be paracompact Hausdorff, and let $E \to X \times I$ be a vector bundle. Then there is an isomorphism $i_0^*(E) \cong i_1^*(E)$.*

We will delay the proof a bit, but let us give the evident corollaries:

Corollary 14.2. *Let X be paracompact Hausdorff.*

(a) If $f, g\colon X \to Y$ are homotopic then f^* and g^* give the same map $\mathrm{Vect}_n(Y) \to \mathrm{Vect}_n(X)$, for any $n \geq 0$.

(b) If Y is also paracompact Hausdorff and $f\colon X \to Y$ is a homotopy equivalence then $f^*\colon \mathrm{Vect}_n(Y) \to \mathrm{Vect}_n(X)$ is a bijection, for all $n \geq 0$.

(c) If X is contractible then all vector bundles on X are trivializable.

Proof. For (a), let $H\colon X \times I \to Y$ be a homotopy and consider the diagram

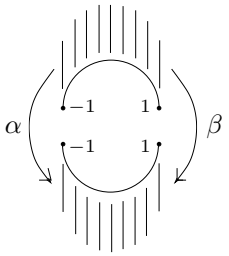
$$\mathrm{Vect}_n(X) \underset{i_1^*}{\overset{i_0^*}{\leftleftarrows}} \mathrm{Vect}_n(X \times I) \xleftarrow{H^*} \mathrm{Vect}_n(Y).$$

One of the compositions is f^*, the other is g^*, and Proposition 14.1 says that the two compositions are the same.

Parts (b) and (c) are simple consequences of (a). □

Example 14.3. To give an idea of how we will apply these results, let us think about vector bundles on S^1. Divide S^1 into an upper hemisphere D_+ and a lower hemisphere D_-, intersecting in two points. Each of D_+ and D_- are contractible, so any vector bundle will be trivializable when restricted to these subspaces.

Given two elements $\alpha, \beta \in GL_n(\mathbb{R})$, let $E_n(\alpha, \beta)$ be the vector bundle on S^1 obtained by taking $\underline{n}_{D_+}$ and $\underline{n}_{D_-}$ and gluing them together via α and β at the two points on the equator. The considerations of the previous paragraph tell us that every vector bundle on S^1 is of this form (up to isomorphism). The following picture depicts the construction of $E_n(\alpha, \beta)$:



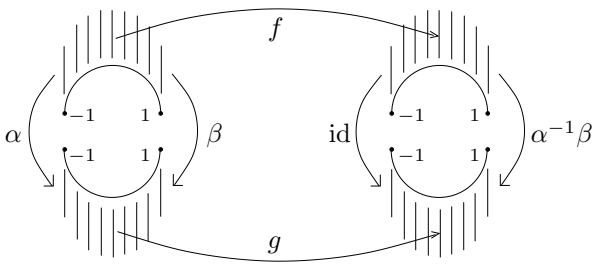
Note that $E_n(\mathrm{id}, \mathrm{id}) = \underline{n}$, and $E_1(\mathrm{id}, -1) = M$ (the Möbius bundle). We will show that:

(1) $E_n(\alpha, \beta) \cong E_n(\mathrm{id}, \alpha^{-1}\beta)$

(2) $E_n(\mathrm{id}, \beta) \cong E_n(\mathrm{id}, \beta')$ if and only if β and β' are in the same path component of $GL_n(\mathbb{R})$ (or equivalently, if $\det(\beta)$ and $\det(\beta')$ have the same sign).

In (2) we have used the fact that $\pi_0(GL_n(\mathbb{R})) = \mathbb{Z}/2$, with the isomorphism being given by the sign of the determinant.

The isomorphism in (1) can be depicted as



Here f and g are maps $D_+ \to GL_n(\mathbb{R})$ and $D_- \to GL_n(\mathbb{R})$ giving the isomorphisms on each fiber; compatibility with the gluing requires that we have $g(-1)\alpha = f(-1)$ and $\alpha^{-1}\beta f(1) = g(1)\beta$. This can be achieved by letting $f(t) = I_n$ and $g(t) = \alpha^{-1}$, for all t.

The proof of (2) is a little more subtle. To give an isomorphism $E(\mathrm{id}, \beta) \cong E(\mathrm{id}, \beta')$ we must again specify maps f and g as above, but this time satisfying $g(-1) = f(-1)$ and $\beta' f(1) = g(1)\beta$. If we paste D_+ and D_- together at -1 and identify the resulting interval with $[0, 1]$, then we are just asking for a map $h\colon [0, 1] \to GL_n(\mathbb{R})$ such that $\beta' h(1) = h(0)\beta$.

If β and β' are in the same path component then choose a path $h\colon I \to GL_n(\mathbb{R})$ such that $h(0) = \beta'$ and $h(1) = \beta$. Since we then have $\beta' h(1) = h(0)\beta$, this yields the desired isomorphism. Conversely, if we have a map h satisfying $\beta' h(1) = h(0)\beta$ then we can rearrange this as $\beta' = h(0)\beta h(1)^{-1}$. The term on the right is path-connected to $h(0)\beta h(0)^{-1}$, using the homotopy $t \mapsto h(0)\beta h(t)^{-1}$. But $h(0)\beta h(0)^{-1}$ has the same determinant as β, so these are also in the same path-component. Hence, β' and β are themselves path-connected and this proves (2).

To summarize, from (1) and (2) it follows that isomorphism types for rank n bundles over S^1 are in bijective correspondence with the path components of $GL_n(\mathbb{R})$. We know that for $n > 0$ there are two such path components, given by the sign of the determinant. They can be represented by the identity matrix and the diagonal matrix J whose diagonal entries are $-1, 1, 1, \ldots, 1$. The corresponding bundles $E_n(\mathrm{id}, \beta)$ are $\underline{n}$ and $M \oplus \underline{(n-1)}$.

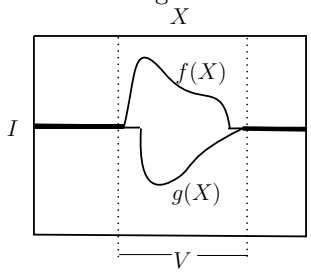
The methods in the above example generalize readily from S^1 to S^k. In fact, the same ideas allow one to get control over vector bundles on any suspension. This will be discussed extensively in Section 15, but see especially Proposition 15.3.

At this point let us now give the proof of Proposition 14.1, starting with an overview of the basic approach. For a bundle on $E \to X \times I$ one can look at the slices $E|_{X\times\{t\}}$ and try to track potential "twisting" that develops as t increases. We need to differentiate between the twisting that was already present in $E|_{X\times\{0\}}$ and twisting that is getting added (or subtracted) as the time variable progresses. To this end, it is useful if we can arrange things so that the new twists only occur in sectors of $E_{X\times\{0\}}$ where the bundle was already trivial. The heart of the proof is reducing to this situation, which will then be handled by the following technical lemma:

Lemma 14.4. *Let X be a Hausdorff space and suppose given maps $f, g\colon X \to X \times I$ that are sections of the projection map $X \times I \to X$. Let $A = \{x \in X \mid f(x) = g(x)\}$. Suppose there exists an open subset V of X such that $\overline{(X - A)} \subseteq V$. Finally,*

assume that $E \to X \times I$ is a vector bundle that is trivializable on $V \times I$. Then $f^*E \cong g^*E$.

The statement of the lemma is a mouthful, so let us explain a bit. The following picture shows the two images $f(X)$ and $g(X)$ inside $X \times I$, with the intersection $f(A) = g(A)$ drawn in bold. The region where the two sections disagree is contained inside of $V \times I$, where we assume the bundle E is trivializable. The proof will construct a fiberwise isomorphism $E|_{f(X)} \to E|_{g(X)}$ that equals the identity on the portion of the bundles over the bold region.



Proof of Lemma 14.4. Let $X_f = f(X)$ and $X_g = g(X)$. The maps $f\colon X \to X_f$ and $X \to X_g$ are homeomorphisms, and we have isomorphisms of bundles $f^*E \cong E|_{X_f}$ and $g^*E \cong E|_{X_g}$. Write $B\colon X_f \to X_g$ for the homeomorphism $g \circ f^{-1}$.

Note that our hypotheses say that the points where X_f and X_g differ are concentrated in the region $V \times I$, where E is trivializable.

The subset A is closed because X is Hausdorff. Let $C = f(A) = g(A)$, $D_f = f\left(\overline{X - A}\right)$, and $D_g = g\left(\overline{X - A}\right)$. We have $X_f = C \cup D_f$ and $X_g = C \cup D_g$, and the subsets appearing in the unions are all closed. Observe that B maps D_f homeomorphically to D_g. Also note that $D_f \cap D_g \subseteq C$.

For $S \subseteq X \times I$ write $E_S = E|_S$ for brevity. Then $E_{X_f} = E_C \cup E_{D_f}$ and $E_{X_g} = E_C \cup E_{D_g}$ are again decompositions into closed sets. We will define a map of bundles $\phi\colon E_{X_f} \to E_{X_g}$ that covers B:

$$\begin{array}{ccc} E_{X_f} & \xrightarrow{\phi} & E_{X_g} \\ \downarrow & & \downarrow \\ X_f & \xrightarrow{B} & X_g. \end{array}$$

Note that $D_f \subseteq V \times I$ and $D_g \subseteq V \times I$. Choose a trivialization $\alpha\colon E_{V\times I} \xrightarrow{\cong} \underline{n}$. This gives the diagram

$$\begin{array}{ccc} E_{D_f} & \overset{\phi_1}{\dashrightarrow} & E_{D_g} \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\alpha} \\ D_f \times \mathbb{R}^n & \xrightarrow[\cong]{B\times \mathrm{id}} & D_g \times \mathbb{R}^n \end{array}$$

and we define ϕ_1 to be the indicated fill-in. This is clearly an isomorphism of bundles. Moreover, for $x \in D_f \cap D_g$ one has $B(x) = x$ and therefore ϕ_1 is the identity on the fiber over x.

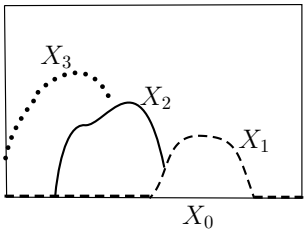
We now have a continuous map $\phi_1\colon E_{D_f} \to E_{D_g} \hookrightarrow E_{X_g}$, and we define $\phi_0\colon E_C \hookrightarrow E_C \cup E_{D_g} = E_{X_g}$ to be the inclusion. We have just seen that ϕ_0

and ϕ_1 agree on the overlap, and so they patch together to define a continuous map $\phi\colon E_{X_f} \to E_{X_g}$. This is an isomorphism on each fiber by construction, and hence an isomorphism of vector bundles. □

Proof of Proposition 14.1. This proof is from [Hatcher2]. Let $E \to X \times I$ be a vector bundle. Pick an $x \in X$. Using the compactness of I and the definition of vector bundle, we may find a neighborhood $U \subseteq X$ of x and values $0 = a_0 < a_1 < ... < a_{n-1} < a_n = 1$ such that E is trivial over each $U \times [a_i, a_{i+1}]$. Patching these together gives a trivialization of the vector bundle over $U \times I$.

Now assume for a moment that X is compact. Then we can cover X by open sets $U_1, \ldots, U_n$ such that E is trivial over each $U_i \times I$. Choose a partition of unity $\phi_1, \ldots, \phi_n$ subordinate to this cover, and set $\beta_0 = 0$, $\beta_i = \phi_1 + \ldots + \phi_i$. Define X_i to be the graph of β_i in $X \times I$. If $f_i\colon X \to X \times I$ is given by $f_i(x) = (x, \beta_i(x))$ then $X_i = f_i(X)$.

Observe that $\beta_n = 1$ and thus $X_n = X \times \{1\}$, $X_0 = X \times \{0\}$. Also, β_{i-1} and β_i agree except on the support of ϕ_i, whose closure is inside of U_i. Because of this, the pair of maps f_{i-1}, f_i satisfies the hypotheses of Lemma 14.4, for each i. The following picture gives an example of the first four sections X_i; each section agrees with the previous one except for a new "bubble" that appears in a region over the set U_i.



Define $B_i\colon X_i \to X_{i-1}$ by $B_i = f_{i-1} \circ f_i^{-1}$. These are homeomorphisms that in terms of the picture can be described as"push down until you hit the next graph".

Lemma 14.4 yields fiberwise isomorphisms ϕ_i making the diagram

$$\begin{array}{ccc} E|_{X_i} & \xrightarrow{\phi_i} & E|_{X_{i-1}} \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{B_i} & X_{i-1} \end{array}$$

commute. Here we have used that X_i and X_{i-1} coincide except over the open set U_i, and that $E|_{U_i \times I}$ is trivial.

Via the identifications $X \cong X_i$, each $E|_{X_i}$ is a vector bundle on X and we have isomorphisms

$$i_1^*(E) = E|_{X_n} \xrightarrow{\cong} E|_{X_{n-1}} \xrightarrow{\cong} E|_{X_{n-2}} \xrightarrow{\cong} \cdots \xrightarrow{\cong} E|_{X_1} \xrightarrow{\cong} E|_{X_0} = i_0^*(E).$$

This gives us what we wanted.

The paracompact case is similar, except for a few details. Let $\mathcal{P}$ be the property that an open subset $U \subseteq X$ is such that E is trivializable on $U \times I$. By Lemma 12.11 there is a countable cover $\{U_i\}$ of X where each U_i has this property. Choose a partition of unity ϕ_i and work as previously to produce fiberwise bundle

isomorphisms

$$\cdots \to E|_{X_n} \longrightarrow E|_{X_{n-1}} \longrightarrow \cdots \longrightarrow E|_{X_1} \longrightarrow E|_{X_0}.$$

For any particular $x \in X$ the maps on the fiber over x are eventually identities as one moves to the left, and after a finite number of steps one has $(E|_{X_n})_x = (E|_{X\times\{1\}})_x$ (equality, not just isomorphism). So we have a fiberwise isomorphism of sets $E|_{X\times\{1\}} \to E|_{X_0}$, and it only remains to check continuity. But continuity is a local condition, and by local finiteness of the partition of unity one knows that each $x \in X$ has a neigborhood where the above sequence stabilizes after a finite number of steps. Continuity is then immediate. □

Remark 14.5. If E is a bundle on $X \times I$ the isomorphism $i_0^*(E) \cong i_1^*(E)$ from the above proof is not canonical. Related to this remark, by Corollary 14.2 one has $\pi^* i_0^* E \cong E$, but the isomorphism is not natural. This is a bit of an annoyance: one might hope that the two functors $i_0^* \colon \mathsf{Vect}(X\times I) \rightleftarrows \mathsf{Vect}(X) \colon \pi^*$ are an equivalence of categories, and indeed there is a natural isomorphism $i_0^*\pi^* \cong \mathrm{id}$, but not in the other direction. Nevertheless, something interesting is happening here. This is a place where one really needs a theory of ∞-categories, and then the above functors form an ∞-categorical equivalence. We will not develop the technology behind this statement, but see Exercise 14.9 for a low-tech foray in that direction.

Remark 14.6. We have seen that all bundles on contractible spaces are trivial, and that there is a close connection between vector bundles and projective modules. Recall that when k is a field then $k[x_1, \ldots, x_n]$ is the algebraic analog of affine space $\mathbb{A}^n$, and that projectives over this ring correspond to algebraic vector bundles. The analogy with topology is what led Serre to conjecture that all finitely-generated projectives over $k[x_1, \ldots, x_n]$ are free, as we discussed in Example 4.8.

We have proven that if E is a vector bundle on $X \times I$ then $i_0^*(E) \cong i_1^*(E)$. It is natural to wonder if this result has a converse, but stating such a thing is somewhat tricky. Here is one possibility: if F and F' are isomorphic vector bundles on X, is there a vector bundle E on $X \times I$ such that $i_0^*(E) \cong F$ and $i_1^*(E) \cong F'$? Unfortunately, this has a trivial answer: yes, just take $E = \pi^*(F)$ where $\pi \colon X\times I \to X$ is the projection. So this phrasing of the question was not very informative.

Here is another possibility: if F and F' are isomorphic vector bundles on X, is there a vector bundle E on $X \times I$ such that $i_0^*(E) = F$ and $i_1^*(E) = F'$? Note the presence of equalities here, as opposed to isomorphisms. This question does not have an obvious answer, but it is also the kind of question that one really doesn't want to be asking: saying that two abstract gadgets are *equal*, rather than just isomorphic, is going to force us down a path that requires us to keep track of too much data.

So we find ourselves in somewhat of a muddle. Perhaps there is an interesting question here, but we don't quite know how to ask it. One approach is to restrict to a class of bundles where "equality" is something we can better control. For example, one can restrict to bundles on X that sit inside of $X \times \mathbb{R}^\infty$. Here, finally, we have an interesting question: if F and F' are two such bundles, which are abstractly isomorphic, is there a bundle E inside of $(X \times I) \times \mathbb{R}^\infty$ that restricts to F and F' at times 0 and 1? The answer is yes, and we will discuss this further in Section 16.

14.7. **Isomorphisms and homotopy invariance.** Let E and F be two vector bundles on a space X, and let $u, v \colon E \to F$ be two bundle maps. Let $\pi \colon X\times I \to X$

be the projection. Define a **homotopy** from u to v to be a bundle map $\pi^*E \to \pi^*F$ that restricts to u on $X \times 0$ and to v on $X \times 1$. Intuitively, this is the same as deforming u to v through bundle maps (we will make this precise in a moment).

For arbitrary bundle maps the set of homotopy classes isn't very interesting, since via scalar multiplication we find that every bundle map is homotopic to the zero map. It is more interesting to put restrictions on the maps, and to require that the homotopies respect these restrictions. For example, we can require u and v to be isomorphisms and then that the homotopy $\pi^*E \to \pi^*F$ also be an isomorphism. When we talk about homotopies between bundle isomorphisms this is always what we will mean.

The following result is a bit silly but will be useful later on. We will also outline a massive generalization in the exercises.

Proposition 14.8. *Let X be a paracompact Hausdorff space, and let E and F be bundles on $X \times I$. Any bundle isomorphism $u_0 \colon E|_{X\times 0} \to F|_{X\times 0}$ may be extended to a bundle isomorphism $u \colon E \to F$, and given any two such extensions their restrictions to $X \times 1$ are homotopic (through isomorphisms).*

Proof. Note that the second statement is trivial, because by definition an extension is a homotopy from $u|_{X\times 1}$ to $u|_{X\times 0}$. Given two such extensions u and u', we simply glue them together at $u|_{X\times 0}$ to obtain a homotopy from $u|_{X\times 1}$ to $u'|_{X\times 1}$.

For the first statement, by Corollary 14.2(b) the map $\pi^* \colon \mathrm{Vect}_n(X) \to \mathrm{Vect}_n(X \times I)$ is a bijection. So there exist bundles E' and F' on X together with isomorphisms $E \cong \pi^*E'$ and $F \cong \pi^*F'$, and therefore it is enough to prove the result for π^*E' and π^*F'. In this case, the first statement simply claims that any bundle isomorphism $E' \to F'$ extends to an isomorphism $\pi^*E' \to \pi^*F'$, which is obvious. □

◦ Exercises ◦

Exercise 14.9. The point of this exercise is to get a taste (but *just* a taste) of the ∞-categorical nature of vector bundles. Let E and F be rank n bundles on X, and recall the bundle $\underline{\mathrm{Hom}}(E,F) \to X$ whose fiber over a point $x \in X$ is $\mathrm{Hom}(E_x, F_x)$. Define $\underline{\mathrm{Iso}}(E,F) \subseteq \underline{\mathrm{Hom}}(E,F)$ to be the subspace whose fibers over x are $\mathrm{Iso}(E_x, F_x)$, the space of vector space isomorphisms from E_x to F_x. Finally, define $\mathrm{Iso}_X(E,F)$ to be the space of sections of $\underline{\mathrm{Iso}}(E,F) \to X$, equipped with the compact-open topology. Said differently, $\mathrm{Iso}_X(E,F)$ is defined by the pullback diagram

$$\begin{array}{ccc} \mathrm{Iso}_X(E,F) & \longrightarrow & \underline{\mathrm{Iso}}(E,F)^X \\ \downarrow & & \downarrow \\ * & \xrightarrow{\mathrm{id}} & X^X \end{array}$$

where the function spaces have the usual compact-open topology.

For the following exercises assume that X is locally compact and Hausdorff, so that one has the natural adjunction isomorphism $\mathcal{T}op(Z, Y^X) \cong \mathcal{T}op(Z \times X, Y)$.

(a) If $E = \underline{n} = F$ verify that $\mathrm{Iso}_X(E,F) \cong GL_n(\mathbb{R})^X$.

(b) Let Z be a space and $\pi \colon Z \times X \to X$ be the projection. Prove that there is an isomorphism of fiber bundles $\underline{\mathrm{Iso}}_{Z\times X}(\pi^*E, \pi^*F) \cong \pi^* \underline{\mathrm{Iso}}_X(E,F)$.

(c) If Z is locally compact and Hausorff verify that $\mathrm{Iso}_X(E,F)^Z \cong \mathrm{Iso}_{Z\times X}(\pi^*E, \pi^*F)$.

(d) Let $u, v\colon E \to F$ be bundle isomorphisms. Prove that u and v are homotopic in the sense defined in Section 14.7 if and only if there is a path $I \to \mathrm{Iso}_X(E,F)$ from u to v.

(e) Given a map $g\colon Y \to X$ where Y is locally compact and Hausdorff explain how to get an induced map of spaces $\mathrm{Iso}_X(E,F) \to \mathrm{Iso}_Y(g^*E, g^*F)$ that has the evident behavior on points. [Hint: One way to approach this is to generalize. If a map p is a pullback of q, show how to get a map from the space of sections of q to the space of sections of p, assuming appropriate spaces are locally compact and Hausdorff.]

(f) Let J and M be bundles on $X\times I$, and assume that X is paracompact Hausdorff. For $i \in \{0,1\}$ prove that the restriction maps

$$\mathrm{res}_i\colon \mathrm{Iso}_{X\times I}(J,M) \to \mathrm{Iso}_{X\times\{i\}}(J|_{X\times\{i\}}, M|_{X\times\{i\}})$$

(induced via (e) by the inclusions $X\times\{i\} \hookrightarrow X\times I$) are acyclic fibrations. [Hint: It suffices to prove this when $J = \pi^*E$ and $M = \pi^*F$ for some bundles E and F on X. Relate the restriction maps to the evaluation maps $\mathrm{ev}_i\colon W^I \to W$ for a certain space W.]

(g) It follows from (f) that the restriction maps are surjective and induce bijections on path components. Relate this statement to Proposition 14.8.

15. Vector bundles on spheres

In this section we explore the set of isomorphism classes $\mathrm{Vect}_n(S^k)$ for various values of k and n. Our goal is to compute as much as we can by elementary means and see how far this gets us.

In the end there are two important points to keep in mind. First, for a fixed k these sets stabilize for $n \gg 0$. Secondly, Bott was able to compute these stable values completely and found an 8-fold periodicity (with respect to k) in the case of real vector bundles, and a 2-fold periodicity in the case of complex bundles. Bott's periodicity theorems are of paramount importance in modern algebraic topology and will be discussed at length later in the book.

15.1. **The clutching construction.** Let X be a pointed space, and let C_+X and C_-X be the positive and negative cones in ΣX. Fix $n \geq 0$. For a map $f\colon X \to GL_n(\mathbb{R})$, let $E_n(f)$ be the vector bundle obtained by gluing $\underline{n}|_{C_+X}$ and $\underline{n}|_{C_-X}$ via the map f (we use Corollary 11.7(b) here). Precisely, if $x \in X$ and v belongs to the fiber of $\underline{n}_{C_+X}$ over x then we glue v to $f(x)\cdot v$ in the fiber of $\underline{n}_{C_-X}$ over x. This procedure for constructing vector bundles on ΣX is called **clutching**, and when X is paracompact Hausdroff every bundle on ΣX is isomorphic to one formed in this way (use Corollary 14.2(c) to see that the bundle is trivial on the two cones). By changing basis in one of the trivial bundles we can always require $f(*) = I_n$; that is, we can require f to be a based map.

Proposition 15.2. *Let X be a paracompact Hausdorff space with a chosen basepoint. If $f, f'\colon X \to GL_n(\mathbb{R})$ are homotopic relative to the basepoint, then $E_n(f) \cong E_n(f')$. Therefore we have a well-defined map $E_n\colon [X, GL_n(\mathbb{R})]_* \to \mathrm{Vect}_n(\Sigma X)$, and in addition this map is surjective.*

Proof. Given a homotopy H between f and f', we use H to make a bundle E on $(\Sigma X)\times I$ by gluing together trivial bundles on $C_+X\times I$ and $C_-X\times I$ via H. Then $E|_{X\times\{0\}}\cong E_n(f)$ and $E|_{X\times\{1\}}\cong E_n(f')$ by construction, so $E_n(f)\cong E_n(f')$ by Proposition 14.1.

To see that the map E_n is surjective, just note that a given vector bundle $\mathcal{E}$ on ΣX can be trivialized on C_+X and C_-X by Corollary 14.2(c). Choose such isomorphisms $\phi_+\colon \mathcal{E}|_{C_+X}\to (C_+X)\times\mathbb{R}^n$ and $\phi_-\colon \mathcal{E}|_{C_-X}\to (C_-X)\times\mathbb{R}^n$, then let F be the composite isomorphism

$$X\times\mathbb{R}^n \xrightarrow{(\phi_+)^{-1}} \mathcal{E}|_X \xrightarrow{\phi_-} X\times\mathbb{R}^n.$$

By Proposition 10.9 this has the form $F(x,v)=(x,f_x(v))$ for a unique map $f\colon X\to GL_n(\mathbb{R})$.

The maps ϕ_+ and ϕ_- can be regarded as vector bundle maps $\mathcal{E}|_{C_+X}\to E_n(f)|_{C_+X}$ and $\mathcal{E}|_{C_-X}\to E_n(g)|_{C_-X}$, and by our construction of f they agree on $\mathcal{E}|_X$. So they induce a vector bundle map $\phi\colon \mathcal{E}\to E_n(f)$ that is a fiberwise isomorphism, and therefore an isomorphism of vector bundles by Exercise 10.16. □

It is natural to guess that $E_n\colon [X,GL_n(\mathbb{R})]_*\to \mathrm{Vect}_n(\Sigma X)$ is an isomorphism, but this is not quite true. It does turn out to be true when $\mathbb{R}$ is replaced by $\mathbb{C}$ and we are dealing with complex vector bundles, but over the reals there is an issue. To see this, observe that $GL_n(\mathbb{R})$ acts on $[X,GL_n(\mathbb{R})]_*$ by pointwise conjugation: if $A\in GL_n(\mathbb{R})$ and $f\colon X\to GL_n(\mathbb{R})$ then define $A.f$ by $(A.f)(x)=A\cdot f(x)\cdot A^{-1}$. If A and B in $GL_n(\mathbb{R})$ are connected by a path then $A.f$ and $B.f$ are based homotopic, and so the action factors through $\pi_0GL_n(\mathbb{R})=\mathbb{Z}/2$. For the analogous story over the complex numbers the action factors through $\pi_0GL_n(\mathbb{C})=*$, and so the action is trivial.

Notice that $E_n(f)\cong E_n(A.f)$ for all A and f; the isomorphism from the former to the latter just consists of left multiplication by A on the trivial bundles $C_+X\times\mathbb{R}^n$ and $C_-X\times\mathbb{R}^n$, which are readily checked to be compatible with the gluing maps. So our E_n map factors as

$$[X,GL_n(\mathbb{R})]_*/(\mathbb{Z}/2)\twoheadrightarrow \mathrm{Vect}_n(\Sigma X)$$

where in the domain we factor out by the group action of $\mathbb{Z}/2$.

Proposition 15.3. *The above maps $E_n^{\mathbb{R}}\colon [X,GL_n(\mathbb{R})]_*/(\mathbb{Z}/2)\to \mathrm{Vect}_n(\Sigma X)$ and $E_n^{\mathbb{C}}\colon [X,GL_n(\mathbb{C})]_*\to \mathrm{Vect}_n^{\mathbb{C}}(\Sigma X)$ are bijections. Moreover, when n is odd the action of $\mathbb{Z}/2$ on $[X,GL_n(\mathbb{R})]_*$ is trivial.*

Proof. Let $f,g\colon X\to GL_n(\mathbb{R})$ be based maps with $E_n(f)\cong E_n(g)$. A choice of isomorphism α amounts to giving maps $\alpha_+\colon C_+X\to GL_n(\mathbb{R})$ and $\alpha_-\colon C_-X\to GL_n(\mathbb{R})$ such that

$$(*)\qquad g\cdot(\alpha_+|_X)=(\alpha_-|_X)\cdot f.$$

Evaluating at the basepoint gives $\alpha_+(*)=\alpha_-(*)$. Let A denote this element of $GL_n(\mathbb{R})$.

Since $\alpha_+|_X$ has an extension to C_+X there is a homotopy relative to the basepoint between $\alpha_+|_X$ and the constant map with value A. The same holds for $\alpha_-|X$. Then (*) gives that $g\cdot A\simeq A\cdot f$, or $g\simeq A\cdot f\cdot A^{-1}$. This verifies that E_n is an injection when we mod out by the $\mathbb{Z}/2$ conjugation action on the domain.

Finally, we need to prove that the $\mathbb{Z}/2$ action is trivial when n is odd. But in this case the two components of $GL_n(\mathbb{R})$ are represented by I and $-I$, and conjugation by both of these elements is trivial. □

Example 15.4. To see the importance of the $\mathbb{Z}/2$-action in the above result, consider the case where $X = S^1$ and $n = 2$. Here we are dealing with

$$E_2 \colon [S^1, GL_2(\mathbb{R})]_* \to \mathrm{Vect}_2(S^2).$$

We can replace $GL_2(\mathbb{R})$ by its homotopy-equivalent subgroup $O(2)$, and then by $SO(2)$ since any pointed map $S^1 \to O(2)$ must land entirely inside $SO(2)$. Let us also identify $S^1 \cong SO(2)$ via $e^{i\theta} \mapsto \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$. We of course know that $[S^1, S^1]_* \cong \mathbb{Z}$ via the degree map. So $[S^1, SO(2)]_* \cong \mathbb{Z}$ with $n \in \mathbb{Z}$ corresponding to the map

$$f_n \colon S^1 \to SO(2), \quad e^{i\theta} \mapsto \left[\begin{smallmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{smallmatrix}\right].$$

The nontrivial path component of $GL_2(\mathbb{R})$ is represented by $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Conjugating f_n by A gives

$$e^{i\theta} \mapsto \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{smallmatrix}\right]$$

and this is precisely f_{-n}. So we see that the $\mathbb{Z}/2$-action is nontrivial here, and the map E_2 is not injective before we quotient out by this action.

Exercise 15.5. Think through the isomorphism $E(f_1) \cong E(f_{-1})$ (or more generally, $E(f_n) \cong E(f_{-n})$) for bundles on S^2 until it feels second nature to you.

Remark 15.6. The need to quotient by the $\mathbb{Z}/2$-action in Proposition 15.3 is related to the difference between vector bundles and *oriented* vector bundles, and isomorphisms versus oriented-isomorphisms. See Exercise 15.26.

Our next goal is to apply Proposition 15.3 when X is a sphere S^{k-1}, in order to obtain a classification of bundles on S^k. For rank 2 bundles on S^2 this was the content of Example 15.4, but we want to see how much further we can push those ideas. The proposition gives a bijection $\mathrm{Vect}_n(S^k) \simeq [\pi_{k-1}GL_n(\mathbb{R})]/(\mathbb{Z}/2)$. It will be convenient to replace $GL_n(\mathbb{R})$ with its subgroup O_n. Recall that $O_n \hookrightarrow GL_n(\mathbb{R})$ is a deformation retract, as a consequence of the Gram-Schmidt process. When $k \geq 2$ any based map $S^{k-1} \to O_n$ must actually factor through the connected component of the identity, which is SO_n. So we have

$$\mathrm{Vect}_n(S^k) \cong [\pi_{k-1}GL_n(\mathbb{R})]/(\mathbb{Z}/2) \cong [\pi_{k-1}O_n]/(\mathbb{Z}/2) \cong [\pi_{k-1}SO_n]/(\mathbb{Z}/2)$$

(where the last isomorphism needs $k \geq 2$).

15.7. **Vector bundles on S^1.** For $k = 1$ and $n > 0$ we need to look at $\pi_0 GL_n(\mathbb{R}) = \mathbb{Z}/2$. The conjugation action is trivial here (it is literally the conjugation on the group $\mathbb{Z}/2$, which is trivial since the group is abelian). So we find that $\mathrm{Vect}_n(S^1) \cong \mathbb{Z}/2$, and we have previously seen in Example 14.3 that the two isomorphism classes are represented by $\underline{n}$ and $M \oplus \underline{(n-1)}$ where M is the Möbius bundle.

15.8. **Vector bundles on S^2.** Here we have $\mathrm{Vect}_n(S^2) \cong [\pi_1 SO_n]/(\mathbb{Z}/2)$. When $n = 1$ we have $SO_1 = \mathbb{Z}/2$ and $\pi_1 SO_1 = *$, so all line bundles on S^2 are trivial. We analyzed $n = 2$ in Example 15.4 and found a bijection $\mathrm{Vect}_2(S^2) \cong \mathbb{Z}_{\geq 0}$. We claim that for $n > 2$ one has $\pi_1 SO_n \cong \mathbb{Z}/2$ and the action of $\mathbb{Z}/2$ is trivial, so that we have the following:

Proposition 15.9. $\mathrm{Vect}_n(S^2) \cong \begin{cases} * & \text{if } n = 1, \\ \mathbb{Z}_{\geq 0} & \text{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \geq 3. \end{cases}$

Proof. For $n = 3$ recall that $SO_3 \cong \mathbb{R}P^3$, so that $\pi_1(SO_3) \cong \mathbb{Z}/2$. To see the homeomorphism use the model $\mathbb{R}P^3 \cong D^3/\sim$ where the equivalence relation has $x \sim -x$ for $x \in \partial D^3$. Map $D^3 \to SO_3$ by sending a vector v to the rotation of $\mathbb{R}^3$ with axis $\langle v \rangle$, through $|v| \cdot \pi$ radians, in the direction given by a right-hand-rule with the thumb pointed along v. Note that this makes sense even for $v = 0$, since the corresponding rotation is through 0 radians. For $x \in \partial D^3$ this map sends x and $-x$ to the same rotation, and so induces a map $\mathbb{R}P^3 \to SO_3$. This is clearly a continuous bijection, and therefore a homeomorphism since the spaces are compact and Hausdorff.

For $n \geq 4$ one can use the long exact sequence associated to the fibration $SO_{n-1} \hookrightarrow SO_n \twoheadrightarrow S^{n-1}$ to deduce that $SO_{n-1} \hookrightarrow SO_n$ induces an isomorphism on π_1.

To deduce that the conjugation action is trivial, we recall that the nonzero element of $\pi_0 O_n$ is represented by any orthogonal matrix of determinant -1. A convenient choice is the diagonal reflection matrix $A_n = \mathrm{diag}(-1, 1, 1, \ldots, 1)$. We know that conjugation by A_n acts trivially on $\pi_1 SO_n$ when $n > 1$ is odd (see the last sentence of Proposition 15.3), and the same statement for even numbers $n \geq 4$ follows from lifting the generator of $\pi_1 SO_n$ back to $\pi_1 SO_{n-1}$. □

Definition 15.10. *Let $\mathcal{O}(n) \in \mathrm{Vect}_2(S^2)$ be the vector bundle E_{f_n} where $f_n \colon S^1 \to SO_2$ is a map of degree n. Note that $\mathcal{O}(0) \cong \underline{2}$.*

The bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}_{\geq 0}$, give a complete list of the rank 2 bundles on S^2. To get to higher ranks we consider the operation of adding on a trivial line bundle, and note that we have commutative diagrams

$$\begin{array}{ccc} \mathrm{Vect}_2(S^2) & \xrightarrow{\oplus 1} & \mathrm{Vect}_3(S^2) \\ \cong\uparrow & & \cong\uparrow \\ \pi_1(SO_2)/(\mathbb{Z}/2) & \xrightarrow{(i_2)_*} & \pi_1(SO_3) \end{array} \qquad \begin{array}{ccc} \mathrm{Vect}_{n-1}(S^2) & \xrightarrow{\oplus 1} & \mathrm{Vect}_n(S^2) \\ \cong\uparrow & & \cong\uparrow \\ \pi_1(SO_{n-1}) & \xrightarrow{(i_{n-1})_*} & \pi_1(SO_n) \end{array}$$

where in the second diagram $n \geq 4$, the bottom maps are induced by the inclusions $i_{n-1} \colon SO_{n-1} \hookrightarrow SO_n$, and we are using the fact that the conjugation action on $\pi_1(SO_n)$ is trivial when $n \geq 3$. We saw in the proof of Proposition 15.9 that $(i_{n-1})_*$ is an isomorphism for $n \geq 4$ and surjective for $n = 3$, and so the same is true for the top maps in the two squares. That is, bundles of rank at least 3 on S^2 all come from bundles of rank 2 by adding on trivial bundles.

The map $\pi_1(SO_2) \to \pi_1(SO_3)$ is readily seen to be the projection $\mathbb{Z} \to \mathbb{Z}/2$ (use the fibration sequence $SO_2 \hookrightarrow SO_3 \to S^2$). This shows that $\mathcal{O}(j) \oplus 1$ is trivial when j is even, and is isomorphic to the nontrivial bundle $\mathcal{O}(1) \oplus 1$ when j is odd.

Putting all of this information together, the following table shows all the vector bundles on S^2:

n	1	2	3	4	5	6
$\mathrm{Vect}_n(S^2)$	$\underline{1}$	$\mathcal{O}(n)$, $n \in \mathbb{Z}_{\geq 0}$	$\underline{3}$, $\mathcal{O}(1) \oplus \underline{1}$	$\underline{4}$, $\mathcal{O}(1) \oplus \underline{2}$	$\underline{5}$, $\mathcal{O}(1) \oplus \underline{3}$	$\cdots$

The operation $(-) \oplus \underline{1}$ moves us from one column of the table to the next, and is completely clear except from column 2 to column 3; as we saw above, there it is given by $\mathcal{O}(j) \oplus \underline{1} \cong \underline{3}$ if j is even, and $\mathcal{O}(j) \oplus \underline{1} \cong \mathcal{O}(1) \oplus \underline{1}$ if j is odd.

To complete our study of these bundles there is one final question that we should answer, namely what happens when one adds two rank 2 bundles (all other sums can be figured out once one knows how to do these):

Proposition 15.11. $\mathcal{O}(j) \oplus \mathcal{O}(k) \cong \begin{cases} \underline{4} & \text{if } j+k \text{ is even,} \\ \mathcal{O}(1) \oplus \underline{2} & \text{if } j+k \text{ is odd.} \end{cases}$

Proof. Let $f_j \colon S^1 \to SO_2$ and $f_k \colon S^1 \to SO_2$ be the clutching functions for $\mathcal{O}(j)$ and $\mathcal{O}(k)$, respectively. The clutching function for the bundle $\mathcal{O}(j) \oplus \mathcal{O}(k)$ is the map $f_j \oplus f_k \colon S^1 \to SO_4$, where $\oplus$ is the (pointwise) block diagonal sum $SO_2 \times SO_2 \to SO_4$, given by

$$(A, B) \mapsto \begin{bmatrix} A & O \\ 0 & B \end{bmatrix}.$$

We can factor $f_j \oplus f_k = (f_j \oplus f_0) \cdot (f_0 \oplus f_k)$ where $\cdot$ is pointwise multiplication and f_0 is the constant map at the identity. It is a standard fact in topology that the group structure on $[S^1, SO_4]_*$ given by pointwise multiplication agrees with the group structure given by concatenation of loops (this is true with SO_4 replaced by any topological group). Note that the homotopy classes of $f_0 \oplus f_k$ and $f_k \oplus f_0$ are the same, since these clutching functions give rise to isomorphic bundles. So we have

$$[f_j \oplus f_k] = [f_j \oplus f_0] + [f_k \oplus f_0]$$

where this is a statement about sums of homotopy classes in $\pi_1(SO_4)$.

But $\pi_1(SO_4) = \mathbb{Z}/2$. The function $f_j \oplus f_0$ is the nontrivial element of $\pi_1 SO_4$ precisely when j is odd, and similarly for $f_k \oplus f_0$. It follows that the sum of these elements is trivial/non-trivial when $j+k$ is even/odd. □

The core argument used in the above proof actually works verbatim in other dimensions, so we record the result below for later use:

Proposition 15.12. *For any n and k, the following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{Vect}_n(S^k) \times \mathrm{Vect}_n(S^k) & & \xrightarrow{\quad\oplus\quad} & & \mathrm{Vect}_{2n}(S^k) \\
\cong\uparrow & & & & \cong\uparrow \\
(\pi_{k-1}SO_n)/(\mathbb{Z}/2) \times (\pi_{k-1}SO_n)/(\mathbb{Z}/2) & & & & (\pi_{k-1}SO_{2n})/(\mathbb{Z}/2) \\
\uparrow\uparrow & & & & \uparrow\uparrow \\
\pi_{k-1}SO_n \times \pi_{k-1}SO_n & \xrightarrow{\quad\mu\quad} & \pi_{k-1}SO_n & \xrightarrow{\quad i_*\quad} & \pi_{k-1}SO_{2n}.
\end{array}$$

Here μ is the group operation on $\pi_{k-1}SO_n$.

Exercise 15.13. Think through the proof of Proposition 15.12.

15.14. **Vector bundles on S^3.** Now we have to calculate $\pi_2 SO_n$. This is trivial for $n \leq 2$ (easy), and for $n = 3$ it also trivial: use $SO_3 \cong \mathbb{R}P^3$ and the fibration sequence $\mathbb{Z}/2 \hookrightarrow S^3 \twoheadrightarrow \mathbb{R}P^3$. Finally, the fibration sequences $SO_{n-1} \hookrightarrow SO_n \twoheadrightarrow S^{n-1}$ now show that $\pi_2 SO_n = 0$ for all n. We have proven

Proposition 15.15. $\mathrm{Vect}_n(S^3) \cong \pi_2(SO_n) \cong 0$. *That is, every vector bundle on S^3 is trivializable.*

15.16. **Vector bundles on S^4.** Once again, the first step is to calculate $\pi_3 SO_n$. Eventually one expects to get stuck here, but so far we have been fortunate. The group is trivial for $n \leq 2$, and for $n = 3$ it is $\mathbb{Z}$ using $SO_3 \cong \mathbb{R}P^3$ and $\mathbb{Z}/2 \to S^3 \twoheadrightarrow \mathbb{R}P^3$. Next look at the long exact homotopy sequence for the fibration $SO_3 \hookrightarrow SO_4 \to S^3$:

$$\cdots \to \mathbb{Z}/2 = \pi_4(S^3) \to \mathbb{Z} \to \pi_3 SO_4 \to \mathbb{Z} \to \pi_2(SO_3) = 0.$$

It follows that $\pi_3 SO_4 \cong \mathbb{Z}^2$. Next do the same thing for $SO_4 \hookrightarrow SO_5 \to S^4$:

$$\cdots \to \mathbb{Z} = \pi_4 S^4 \to \pi_3 SO_4 \to \pi_3 SO_5 \to 0.$$

Unfortunately we cannot go further without calculating the map $\pi_4 S^4 \to \pi_3 SO_4$, which is $\mathbb{Z} \to \mathbb{Z}^2$. So now we are indeed stuck, unless we can resolve this issue. Note, however, that the fibrations $SO_{n-1} \hookrightarrow SO_n \to S^{n-1}$ show that $\pi_3 SO_5 = \pi_3 SO_n$ for $n \geq 5$, so once we've figured this one out we know everything. It will take us a moment, but we will show that the map $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ is a split inclusion. So we get that

Proposition 15.17. $\pi_3(SO_n) \cong \begin{cases} 0 & n \leq 2, \\ \mathbb{Z} & n = 3, \\ \mathbb{Z}^2 & n = 4, \\ \mathbb{Z} & n \geq 5. \end{cases}$

The maps $\pi_3 SO_n \to \pi_3 SO_{n+1}$ are isomorphisms for $n \geq 5$, and a surjection when $n = 4$. Moreover, the $\mathbb{Z}/2$-action is trivial except when $n = 4$, where it is the involution $(k, l) \mapsto (l, k)$.

The way we get lucky here is that we can think of S^3 as the unit quaternions, and then use quaternionic arithmetic to get our hands on both SO_3 and SO_4. For each $q \in S^3$ the conjugation map $\gamma_q \colon \mathbb{H} \to \mathbb{H}$ given by $x \mapsto qx\bar{q}$ is orthogonal and fixes 1. So it induces an orthogonal transformation of $\langle 1 \rangle^\perp = \langle i, j, k \rangle$, which we may identify with an element of SO_3. The resulting map $C_1 \colon S^3 \to SO_3$ given by $q \mapsto \gamma_q|_{\langle i,j,k \rangle}$ has the property that $C_1(q) = C_1(-q)$ for all $q \in S^3$, and the induced map $\mathbb{R}P^3 \to SO_3$ is readily seen to be a homeomorphism. Since elements of $\pi_3 SO_3$ are classified by their degree (use $SO_3 \cong \mathbb{R}P^3$ and $\mathbb{Z}/2 \to S^3 \to \mathbb{R}P^3$), C_1 is a generator. Write C_n for $q \mapsto \gamma_{q^n} = (\gamma_q)^n$, which coincides with the n-fold multiple of C_1 in $\pi_3 SO_3$.

Given a unit quaternion q, let L_q and R_q denote left- and right-multiplication by q on $\mathbb{H}$. These preserve the quaternionic inner product, so taking $1, i, j, k$ as the standard basis we can regard L_q and R_q as elements of SO_4. Since left and right multiplication always commute (for any two elements), we get a continuous group map $S^3 \times S^3 \to SO_4$ via $(a, b) \mapsto L_a R_{\bar{b}}$. The conjugate on the b is just to make the map a group homomorphism.

Proposition 15.18. *The kernel of the above map is $\{(1, 1), (-1, -1)\}$ and the map induces an isomorphism of topological groups $(S^3 \times S^3)/\pm(1, 1) \cong SO_4$.*

Proof. If (a, b) is in the kernel then $x = ax\bar{b}$ for every $x \in \mathbb{H}$, so in particular $1 = a\bar{b}$ and hence $a = b$. Then $xa = ax$ for all x, hence a is in the center of $\mathbb{H}$ which equals $\mathbb{R}$. Thus, $a = \pm 1$ and $(a, b) = \pm(1, 1)$.

At this point we know that we have an injective map of topological groups $(S^3 \times S^3)/\pm(1, 1) \hookrightarrow SO_4$. But both the domain and target are 6-dimensional and connected, so this must be a homeomorphism. □

Proof of Proposition 15.17. The only content we have not discussed is the isomorphism $\pi_3 SO_5 \cong \mathbb{Z}$, the surjection $\pi_3 SO_4 \to \pi_3 SO_5$, and the identification of the $\mathbb{Z}/2$-action on $\pi_3 SO_4$. All will follow as soon as we have a better understanding of $\pi_3 SO_4$.

Since our map $S^3 \times S^3 \to SO_4$ is a covering space it induces isomorphisms on higher homotopy groups. So we find that $\pi_3 SO_4 = \mathbb{Z}^2$, with generators the maps $L, R\colon S^3 \to SO_4$ given by $q \mapsto L_q$ and $q \mapsto R_{\bar{q}}$.

Next consider the $\mathbb{Z}/2$-action on $\pi_3 SO_4$ that comes into our isomorphism $\mathrm{Vect}_4(S^4) \cong (\pi_3 SO_4)/(\mathbb{Z}/2)$. Note the slight subtlety that the $\mathbb{Z}/2$-action preserves the group structure but we are only taking the orbit space *as sets*. For conjugation in SO_4 we use the matrix $B = \mathrm{diag}(1,-1,-1,-1)$, which as a transformation of $\mathbb{H}$ is $B(x) = \bar{x}$ (we could use $A = \mathrm{diag}(-1,1,1,1)$ instead to match our previous work, but that introduces an unwelcome and unhelpful minus sign into the formulas on $\mathbb{H}$). Then we readily compute that BLB^{-1} is the map $q \mapsto \overline{(q\bar{x})} = x\bar{q}$, which is just R. Likewise, $BRB^{-1} = L$. Under the isomorphism $\pi_3 SO_4 = \mathbb{Z}^2$ the $\mathbb{Z}/2$ action is therefore $(k,l) \mapsto (l,k)$, and then $\mathrm{Vect}_4(S^4)$ corresponds to the set quotient $\mathbb{Z}^2/\!\sim$. Knowing this action will actually help us compute the map $\pi_3 SO_4 \to \pi_3 SO_5$, as we explain next.

So now let us turn back to the fiber sequence $SO_4 \to SO_5 \to S^4$ and the homotopy sequence $\mathbb{Z} \to \pi_3 SO_4 \xrightarrow{i_*} \pi_3 SO_5 \to 0$. Since $\pi_3 SO_4 = \mathbb{Z}^2$, the sequence shows that the kernel of i_* has rank at most one. The map i_* respects the $\mathbb{Z}/2$-actions, and the $\mathbb{Z}/2$-action on $\pi_3 SO_5$ is trivial since 5 is odd. The elements $(0,1), (1,0) \in \pi_3 SO_4$ are identified under the action, so their difference must map to zero in $\pi_3 SO_5$. That is, the kernel of i_* contains the element $(1,-1)$. But the kernel has rank at most one, and $\langle (1,-1) \rangle$ is the only rank one subgroup that contains this element. So this is precisely $\ker i_*$, and it follows that $\pi_3 SO_5 \cong \mathbb{Z}$. □

Proposition 15.17 lets us classify all bundles on S^4. The following table gives a partial summary:

n	1	2	3	4	5	6	7
$\mathrm{Vect}_n(S^4)$	$*$	$*$	$\mathbb{Z}$	$\mathbb{Z}^2/\!\sim$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$

To describe the answer more completely we need to give names to some of the bundles. Let $\mathcal{F}_n$ be the rank 3 bundle whose clutching function is C_n. Let $\mathcal{G}_{n,k}$ be the rank 4 bundle whose clutching function is $n(L+R) + kR$, where $n, k \in \mathbb{Z}$ and $k \geq 0$. Geometrically, we are representing orbit classes in $\mathbb{Z}^2/\!\sim$ by the points on and above the diagonal. Note that $\mathcal{G}_{n,0} = \mathcal{F}_n \oplus \underline{1}$: this follows from the fact that C_1 maps to $L + R$ under $\pi_3 SO_3 \to \pi_3 SO_4$.

All of the bundles of rank 5 and higher come from adding trivial bundles to rank 4 bundles, and from rank 5 on this gives bijections. We only need to think a bit about the passage from rank 4 to rank 5. We have seen that $\pi_3 SO_4 \to \pi_3 SO_5$ sends $L - R$ to 0, so it follows that $\mathcal{G}_{a,b} \oplus \underline{1} \cong \mathcal{G}_{a+1,b-2} \oplus \underline{1}$. Therefore, for a complete list of rank 5 bundles we can use $\mathcal{G}_{n,0} \oplus \underline{1} = \mathcal{F}_n \oplus \underline{2}$ and $\mathcal{G}_{n,1} \oplus \underline{1}$, for $n \in \mathbb{Z}$. Using the L, R-basis the map $\pi_3 SO_4 \to \pi_3 SO_5$ is $\mathbb{Z}^2 \to \mathbb{Z}$ given by $aL + bR \mapsto a + b$, and so in our bijection $\mathrm{Vect}_5(S^4) \cong \mathbb{Z}$ we have $\mathcal{F}_n \oplus \underline{2} \mapsto 2n$ (since $nL + nR \mapsto 2n$) and $\mathcal{G}_{n,1} \oplus \underline{1} \mapsto 2n+1$ (since $nL + nR + R \mapsto 2n+1$).

The following table compactly summarizes all of the vector bundles on S^4:

n	1	2	3	4	5	6
$\mathrm{Vect}_n(S^4)$	$*$	$*$	$\mathcal{F}_n$ $(n\in\mathbb{Z})$	$\mathcal{G}_{n,k}$ $(n\in\mathbb{Z},k\in\mathbb{Z}_{\geq 0})$	$\mathcal{F}_n\oplus\underline{2}$, $\mathcal{G}_{n,1}\oplus\underline{1}$ $(n\in\mathbb{Z})$	$\cdots$

The direct sum relations are all deduced from $\mathcal{F}_n\oplus\mathcal{F}_{n'}\cong\underline{3}\oplus\mathcal{F}_{n+n'}$ and $\mathcal{G}_{a,b}\oplus\mathcal{G}_{a',b'}\cong\underline{4}\oplus\mathcal{G}_{a+a',b+b'}$. These follow directly from Proposition 15.12. If we want $\mathcal{F}_n\oplus\mathcal{G}_{a,b}$, which has rank 7, we can instead add $\underline{1}$ and observe that

$$\mathcal{F}_n\oplus\mathcal{G}_{a,b}\oplus\underline{1}\cong\mathcal{G}_{n,0}\oplus\mathcal{G}_{a,b}\cong\underline{4}\oplus\mathcal{G}_{n+a,b}.$$

Since $(-)\oplus\underline{1}$ gives a bijection from rank 7 to rank 8, we have $\mathcal{F}_n\oplus\mathcal{G}_{a,b}\cong\mathcal{G}_{n+a,b}\oplus\underline{3}$.

Remark 15.19. The tangent bundle TS^4 has rank 4 and is therefore isomorphic to $\mathcal{G}_{n,k}$ for a unique n and k in the above indexing scheme. What are these integers? See Exercise 15.27.

15.20. **Vector bundles on S^k.** Although we cannot readily do the calculations for $k>4$, at this point one sees the general pattern. One must calculate $\pi_{k-1}SO_n$ for each n, and these groups vary for a while but eventually stabilize. In fact, $\pi_iSO_n\cong\pi_iSO_{n+1}$ for $n>i+1$. The calculation of these stable groups was an important problem back in the 1950s, eventually solved by Bott. (There is again the conjugation action that must be dealt with, but because of the stabilization isomorphisms this action is trivial when n is large enough.)

Let us phrase things as follows. Consider the inclusions

$$O_1\hookrightarrow O_2\hookrightarrow O_3\hookrightarrow\cdots$$

that send a matrix A to $\begin{bmatrix}A & 0\\ 0 & 1\end{bmatrix}$. The colimit of this sequence is denoted O and called the **infinite orthogonal group**. The homotopy groups of O are the stable values that we encountered above. We computed the first few: $\pi_0O=\mathbb{Z}/2$, $\pi_1O=\mathbb{Z}/2$, $\pi_2O=0$, and $\pi_3O=\mathbb{Z}$. Bott's calculation showed the following:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
π_iO	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The pattern is 8-fold periodic: $\pi_{i+8}O\cong\pi_iO$ for all $i\geq 0$. One is supposed to remember the pattern of groups to the tune of "Twinkle, Twinkle, Little Star":

zee - two - zee - two - ze - ro - zee ze - ro - ze - ro - ze - ro - zee.

At some point it becomes important to understand Bott's computations at a deeper level; in particular, one sometimes needs to get one's hands on explicit generators. See the discussion of Bott periodicity in [D7] for an extensive treatment. But for now we will just accept that the values are as given above.

15.21. **Complex vector bundles on spheres.** One can repeat the above analysis for complex vector bundles on a sphere. One finds that

$$\mathrm{Vect}_n^{\mathbb{C}}(S^k)\cong\pi_{k-1}(GL_n(\mathbb{C}))\cong\pi_{k-1}(U_n),$$

where $U_n\hookrightarrow GL_n(\mathbb{C})$ is the unitary group. Analogously to the real case, one has fiber bundles $U_{n-1}\hookrightarrow U_n\twoheadrightarrow S^{2n-1}$ coming from the fact that when U_n acts on $\mathbb{C}^n$ the orbit of e_1 is S^{2n-1} and the stabilizer is U_{n-1}. Using that $U_1\cong S^1$ one can again compute $\mathrm{Vect}_n^{\mathbb{C}}(S^k)$ for small values of k. Here is what you get:

n	1	2	3	4	5	6	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^1)$	0	0	0	0	0	0	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^3)$	0	0	0	0	0	0	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^4)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
$\mathrm{Vect}_n^{\mathbb{C}}(S^5)$	0	$\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	0 or $\mathbb{Z}/2$	$\cdots$

The stable value in the last row turns out to be 0, although one cannot figure this out without computing a connecting homomorphism in the long exact homotopy sequence.

Exercise 15.22. Verify all of the computations in the above table.

The fiber bundles $U_n \hookrightarrow U_{n+1} \twoheadrightarrow S^{2n+1}$ again imply that $\pi_i U_n$ stabilizes as n grows. In fact, $\pi_i U_n \cong \pi_i U_{n+1}$ for $n > \frac{i}{2}$. We can write the stable value as $\pi_i U$ where U is the **infinite unitary group** defined as the colimit of

$$U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \cdots$$

Bott computed the homotopy groups of U to be 2-fold periodic, with

$$\pi_i U = \begin{cases} \mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Again, for now we will just accept this result; but eventually we will have to understand the computation in more detail. In particular, we will need to get our hands on specific generators.

∘ Exercises ∘

Exercise 15.23. If X is path-connected prove that every real line bundle on ΣX is trivializable.

Exercise 15.24. Consider the tangent bundle TS^2. Fix a basis at the north pole and trivialize the bundle on the upper hemisphere C_+S^1 using parallel transport along great circles (it should be intuitively clear what this means, even if you don't know the precise definition of the words). Likewise, fix a basis at the south pole and trivialize the bundle on C_-S^1 similarly. Work out the clutching function $S^1 \to SO(2)$ and show that it has degree ± 2 (the sign depends on orientation conventions).

Exercise 15.25. Using the methods we have demonstrated for S^n, $n \leq 4$, attempt to understand all vector bundles on S^5. Look up whatever homotopy groups of spheres you need to do the calculations. Establishing a complete classification is within reach based on the techniques we developed in this section. You should be able to show the following:

- All bundles of rank 1 and 2 are trivial.
- There is exactly one nontrivial bundle of rank 3. Describe the clutching function as concretely as you can.
- There are exactly three different isomorphism types of rank 4 bundles.

- All bundles of rank 5 are obtained by adding trivial bundles to rank 4 bundles, and there are exactly two isomorphism types.
- [Accepting the Bott results] All bundles of rank 6 or higher are trivial.

[Hint: Study the technique in the proof of Proposition 15.17 that uses the $\mathbb{Z}/2$-action to produce elements in the kernel of i_*. This should get you over most of the humps.]

Exercise 15.26. Define an **oriented n-bundle** to be a real vector bundle $E \to X$ of rank n together with a homotopy class of nonzero sections of $\Lambda^n E \to X$. Note that we could also describe this as a nonzero section up to multiplication by functions $X \to \mathbb{R}_+$, or in other ways. Intuitively, this is a vector bundle with a consistently chosen orientation on all of the fibers.

Likewise, if E and F are oriented n-bundles say that an isomorphism $E \to F$ is oriented if it is orientation-preserving on all of the fibers. Let $\mathrm{Vect}_n^{or}(X)$ denote the set of oriented-isomorphism classes of oriented n-bundles on X.

For convenience assume that X is path-connected from now on.

(a) Convince yourself that not every bundle is orientable, and that every orientable bundle has exactly two orientations. Prove that an n-bundle is orientable if and only if $\Lambda^n E$ is trivializable.

(b) For an oriented bundle E, let σE denote the same bundle but with the opposite orientation (i.e., the orientation obtained by multiplying the section of $\Lambda^n E$ by -1). This induces a $\mathbb{Z}/2$-action on $\mathrm{Vect}_n^{or}(Y)$ for any space Y. The evident map $\mathrm{Vect}_n^{or}(Y) \to \mathrm{Vect}_n(Y)$ sends E and σE to the same object, and so induces a set map
$$\mathrm{Vect}_n^{or}(Y)/(\mathbb{Z}/2) \to \mathrm{Vect}_n(Y).$$
Prove that this is an injection.

(c) Give a one-line proof that E and σE are oriented-isomorphic whenever the rank is odd.

(d) Prove that the map $\mathrm{Vect}_n^{or}(\Sigma X)/(\mathbb{Z}/2) \to \mathrm{Vect}_n(\Sigma X)$ is a bijection.

(e) Prove that there is a natural isomorphism $\mathrm{Vect}_n^{or}(\Sigma X) \cong [X, GL_n^+(\mathbb{R})]_*$, where $GL_n^+(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ is the subgroup of matrices with positive determinant. Explain why the same statement is true with $SO_n(\mathbb{R})$ replacing $GL_n^+(\mathbb{R})$. Finally, explain why the $\mathbb{Z}/2$-action on the left side of the above isomorphism lines up with the one on the right that we introduced in the text.

(f) Think about the perspective the previous parts offer on Proposition 15.3.

Exercise 15.27. Show that the tangent bundle TS^4 is isomorphic to $\mathcal{G}_{-1,2}$ in our classification of bundles on S^4. One way is to follow the steps below:

(1) First check that $TS^4 \oplus \underline{1}$ is trivial, then deduce that $TS^4 \cong \mathcal{G}_{-a,2a}$ for some $a \geq 0$.

(2) Regard S^4 as the sphere inside $\mathbb{R}\langle e_N \rangle \oplus \mathbb{H}$, where e_N is the "north pole". The equator of S^4 is then the unit quaternions, and the tangent space of S^4 at e_N is identified with $\mathbb{H}$ by projection. Orient S^4 by having $1, i, j, k$ be an oriented basis of $T_{e_N} S^4$, and then verify that $-1, i, j, k$ is an oriented basis of $T_{-e_N} S^4$.

(3) Let $\langle x, q \rangle = \mathrm{Re}(x\bar{q})$ be the standard inner product on $\mathbb{H}$. For each $q \in S^3$ consider the transformation $R_{\theta,q}$ of $\mathbb{R} \oplus \mathbb{H}$ that rotates the plane $\mathrm{Span}\{e_N, q\}$ by θ radians and is the identity on $\mathrm{Span}\{e_N, q\}^\perp$. We rotate in the direction such that when $\theta = \frac{\pi}{2}$ the vector e_N rotates to q, and q to $-e_N$. Prove that

for $x \in \mathbb{H} = T_{e_N} S^4$ we have

$$R_{\theta,q}(x) = -(\sin\theta)\langle x, q\rangle e_N + (\cos\theta - 1)\langle x, q\rangle q + x.$$

Conclude that $R_{\pi,q}(x) = x - 2\langle x, q\rangle q$, which is the reflection of x in the hyperplane $\langle q\rangle^{\perp}$ of $\mathbb{H}$. [Note: The rotations $R_{\theta,q}$ are doing the parallel transport of tangent vectors as we follow the great circle from the north pole, down through q, and on towards the south pole.]

(4) Verify that to get the clutching function for TS^4 we can use $q \mapsto R_{\pi,q}$ but with one caveat. The way we have set things up, this maps S^3 into the determinant -1 component of O_4. To instead get into SO_4 we need to compose this with a fixed orthogonal transformation of determinant -1; choose the transformation $A(x) = -\bar{x}$. In this way we get the clutching function $h\colon S^3 \to SO_4$ given by

$$q \mapsto [x \mapsto -(\overline{x - 2\langle x, q\rangle q})].$$

(5) We know that $h = -a(L + R) + 2aR$ in our L, R-basis for $\pi_3 SO_4 = \mathbb{Z}^2$, as this was the definition of $\mathcal{G}_{-a,2a}$. To determine a it suffices to compute the image of h under the map $(p_1)_*\colon \pi_3 SO_4 \to \pi_3 S^3$, where $p_1\colon SO_4 \to S^3$ is projection onto the first column. This is because $(p_1)_*$ takes L to 1 and R to -1, and so it will take h to $-2a$. But using our formula for h from the previous part, $p_1 \circ h$ is the map

$$q \mapsto h(q)(1) = 2\langle 1, q\rangle \bar{q} - 1.$$

Verify that this map has degree -2 as a map $S^3 \to S^3$, for example by using the local degree theorem.

16. Classifying spaces

Consider the functor sending a paracompact Hausdorff space X to the set $\mathrm{Vect}_n(X)$. This is contravariant and homotopy invariant, so it has the feel of a functor $[-, Y]$ for some space Y—that is, a functor that is representable in the homotopy category. In this section we prove that the functor $\mathrm{Vect}_n(-)$ is indeed representable. A representing space Y is called a **classifying space** for rank n vector bundles. There are different methods for constructing such a space, some more homotopical and some more geometric.

16.1. **A view of the landscape.** Before embarking on the construction, a few words are in order about how to orient yourself towards this material. As we have discussed, a vector bundle may be regarded as a family of vector spaces indexed by the base space. In general, it is often useful to view a map of spaces $E \to B$ as a family of ____ if each fiber is a ____ (as another example, a principal G-bundle is a family of G-torsors). Taking a cue from the subject of moduli spaces, one could naively hope that families over B of some mathematical object are in bijection with maps from B to a certain "moduli space" for that mathematical object. With this naive idea, families over $*$ would be in bijective correspondence with points of our moduli space. But for rank n vector bundles this would mean that the moduli space has only one point, since there is only one real vector space of rank n. So this can't work.

The error in the above paragraph, which might already be apparent to the reader, is that there is *not* only one real vector space of rank n. There is only one isomorphism class, but that is not the same thing. So one starts to wonder how to expand the simple idea of moduli spaces to incorporate isomorphisms. What turns

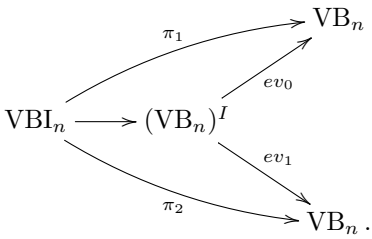
out to work is a homotopical version of the theory, where rank n vector bundles over B are in bijective correspondence with *homotopy classes of maps* from B to a 'homotopical moduli space'. This will be our desired classifying space.

In the geometric approach to these classifying spaces, instead of just one rank n vector space we look at all of the rank n subspaces of some fixed "big" vector space. It turns out that $\mathbb{R}^\infty$ is big enough. This gets somewhat technical because of the ins-and-outs of dealing with the topology on $\mathbb{R}^\infty$, which requires some care. But this takes us to the study of Grassmannians, and in particular to the spaces $\mathrm{Gr}_n(\mathbb{R}^\infty)$.

We close this overview with a slightly deeper look at some of the key ideas. Suppose given a space VB_n and a rank n vector bundle $\gamma_n \to \mathrm{VB}_n$. To every map $f\colon X \to \mathrm{VB}_n$ we can associate the pullback bundle $f^*\gamma_n$, and by Corollary 14.2(a) this gives a natural transformation $[X, \mathrm{VB}_n] \to \mathrm{Vect}_n(X)$ for paracompact Hausdorff spaces X. It will then be a matter of proving surjectivity and injectivity of this natural map, and our construction of VB_n will give surjectivity almost automatically: that is, for any vector bundle $E \to X$ there is a map $f\colon X \to \mathrm{VB}_n$ and an isomorphism $E \cong f^*\gamma_n$. The bundle $\gamma_n \to \mathrm{VB}_n$ is called a "universal bundle", since every other rank n bundle is pulled back from it. Injectivity of our natural transformation is the slightly subtle part.

For injectivity we will produce an analogous classifying space for isomorphisms between bundles of rank n. This will be a space VBI_n together with two canonical maps $\pi_1, \pi_2\colon \mathrm{VBI}_n \to \mathrm{VB}_n$ and an isomorphism $\phi\colon \pi_1^*\gamma_n \to \pi_2^*\gamma_n$. For any two maps $f, g\colon X \to \mathrm{VB}_n$ and an isomorphism $f^*\gamma_n \cong g^*\gamma_n$ there will exist a map $u\colon X \to \mathrm{VBI}_n$ such that $\pi_1 \circ u \simeq f$ and $\pi_2 \circ u \simeq g$. This is the analog of the surjectivity part of the previous paragraph, and is again fairly easy.

Finally, we will construct a map $\mathrm{VBI}_n \to (\mathrm{VB}_n)^I$ sitting in a homotopy-commutative diagram



As an exercise, check that injectivity of $[X, \mathrm{VB}_n] \to \mathrm{Vect}_n(X)$ follows formally from these considerations. So the work ahead of us comes down to constructing VB_n and VBI_n with the properties outlined above.

16.2. **Grassmannians.** Now we dive into the basic theory of Grassmannians for topological vector spaces. In practice the vector space will always be $\mathbb{R}^n$ or $\mathbb{R}^\infty$ with the colimit topology, so the reader is free to think only about those cases. See Appendix B for background material on topological vector spaces.

Let W be a topological vector space. Let $VI_n(W) \subseteq W^n$ be the subset of tuples $(w_1, \ldots, w_n)$ that are linearly independent, equipped with the subspace topology (the 'V' comes from the usual notation for Stiefel manifolds, and here 'I' stands for 'independent'). Define $\mathrm{Gr}_n(W)$ to be the quotient space of $VI_n(W)$ with respect

to the relation where $(w_1, \ldots, w_n) \sim (w'_1, \ldots, w'_n)$ if the vectors span the same subspace. As usual, we identify points of $\mathrm{Gr}_n(W)$ with n-dimensional subspaces of W.

From another perspective, change-of-basis gives a continuous group action of $GL_n(\mathbb{R})$ on $VI_n(W)$, and $VI_n(W) \to \mathrm{Gr}_n(W)$ is just quotienting by this action. Let us construct the action carefully. If $\underline{w} = (w_1, \ldots, w_n)$ is a tuple of vectors in W and $P \in GL_n(\mathbb{R})$, define $[\underline{w} \cdot P]_j = \sum_i w_i P_{ij}$. If $W = \mathbb{R}^k$ and we regard each w_i as a column vector then $\underline{w}$ is a $k \times n$ matrix and this definition is just matrix multiplication. Our definition clearly gives a map of sets $VI_n(W) \times GL_n(\mathbb{R}) \to VI_n(W)$. To see continuity start with the "linear combination map" $LC\colon W^n \times \mathbb{R}^n \to W$, $(\underline{w}, \underline{r}) \mapsto \sum_i r_i w_i$. This is continuous because W is a topological vector space. Next consider the diagram

$$\begin{array}{ccc} VI_n(W) \times GL_n(\mathbb{R}) & \dashrightarrow & VI_n(W) \\ \downarrow & & \downarrow \\ W^n \times GL_n(\mathbb{R}) & & \\ \downarrow & & \\ W^n \times (\mathbb{R}^n)^n & \longrightarrow & W^n. \end{array}$$

The map $GL_n(\mathbb{R}) \hookrightarrow (\mathbb{R}^n)^n$ sends a matrix to the n-tuple of its columns, and the bottom horizontal map is a product of n copies of LC and so is continuous. The down-bottom composite factors through the subspace $VI_n(W)$, thereby giving the upper horizontal map.

Let W' be another topological vector space. If $f\colon W \to W'$ is a continuous linear embedding then the map $W^n \to (W')^n$ restricts to $VI_n(W) \to VI_n(W')$ and therefore induces a map $\mathrm{Gr}_n(W) \to \mathrm{Gr}_n(W')$. We will generally denote all of these maps by f as well. The following result just says that homotopies among the maps $W \to W'$ descend to homotopies among the induced maps of Grassmannians.

Lemma 16.3. *Suppose that $H\colon W \times I \to W'$ is a homotopy such that each H_t is a linear embedding. Then there is a homotopy $\mathrm{Gr}_n(W) \times I \to \mathrm{Gr}_n(W')$ that at each time t is the map induced by H_t.*

Proof. Let $\tilde{H}$ be the composite

$$W^n \times I \xrightarrow{\mathrm{id} \times \Delta} W^n \times I^n \xrightarrow{H^n} (W')^n$$

which sends $(w_1, \ldots, w_n, t) \mapsto (H_t(w_1), \ldots, H_t(w_n))$. This is continuous by construction. Since each H_t is a linear embedding the composite restricts to give

$$\tilde{H}\colon VI_n(W) \times I \longrightarrow VI_n(W').$$

In the diagram

$$\begin{array}{ccccc} VI_n(W) \times I & \longrightarrow & VI_n(W') & \longrightarrow & \mathrm{Gr}_n(W') \\ \downarrow & & & \nearrow & \\ \mathrm{Gr}_n(W) \times I & & & & \end{array}$$

the projection $VI_n(W) \times I \to \mathrm{Gr}_n(W) \times I$ is a quotient map by [Munkres, Chapter 29, Exercise 11], and the quotient relations are respected by the horizontal map. So the indicated extension exists. □

Corollary 16.4. *If $k < N \leq \infty$ then all linear embeddings $\mathbb{R}^k \hookrightarrow \mathbb{R}^N$ induce homotopic maps on Grassmannians.*

Proof. First assume $N < \infty$. Topologize the space of linear maps $\mathbb{R}^k \to \mathbb{R}^N$ as a subspace of the function space $(\mathbb{R}^N)^{(\mathbb{R}^k)}$ with the compact-open topology. By Exercise B.11 evaluating at any basis gives a homeomorphism from the space of linear maps to $(\mathbb{R}^N)^k$. The subspace of linear embeddings then maps homeomorphically to $VI_k(\mathbb{R}^N)$.

The usual Stiefel manifold $V_k(\mathbb{R}^N)$ of orthornormal frames sits inside $VI_k(\mathbb{R}^N)$ as a deformation retract, by Gram-Schmidt. But the space $V_k(\mathbb{R}^N)$ is connected, for example by the usual inductive argument using the fibrations $V_{k-1}(\mathbb{R}^{N-1}) \to V_k(\mathbb{R}^N) \to V_1(\mathbb{R}^N)$ (this is where we need $k < N$). So any two points are connected by a path, i.e. any two linear embeddings are connected by a homotopy through linear embeddings.

For the case $N = \infty$ just use that any linear map $\mathbb{R}^k \to \mathbb{R}^\infty$ factors through a finite $\mathbb{R}^s$, and then appeal to what has already been proven. □

Here is a useful property of the VI-spaces:

Proposition 16.5. *There is a natural bijection between $\mathcal{T}op(X, VI_n(W))$ and the set of fiberwise injections of families of vector spaces $\underline{n}_X \hookrightarrow X \times W$.*

Proof. Start with the linear combination map $LC\colon VI_n(W) \times \mathbb{R}^n \to W$. Given a map $f\colon X \to VI_n(W)$, let ϕ_f be the composite

$$X \times \mathbb{R}^n \xrightarrow{f \times \mathrm{id}} VI_n(W) \times \mathbb{R}^n \xrightarrow{LC} W.$$

This is evidentally continuous. We then get $(\pi_1, \phi_f)\colon X \times \mathbb{R}^n \to X \times W$ and one readily checks that this is a fiberwise injective map of families.

In the other direction, suppose given a fiberwise injective map of families $j\colon X \times \mathbb{R}^n \hookrightarrow X \times W$. For each i let α_i be the composite

$$X \times \{e_i\} \hookrightarrow X \times \mathbb{R}^n \xrightarrow{j} X \times W \xrightarrow{\pi_2} W,$$

which is clearly continuous. These induce $\alpha\colon X \to W^n$, and the image lies in the subspace $VI_n(W)$. So we get a continuous map $X \to VI_n(W)$. This construction is readily checked to be a two-sided inverse for the construction of the previous paragraph. □

To prove more about $VI_n(W)$ and $\mathrm{Gr}_n(W)$ we will have to assume that W is a reasonable topological vector space in the sense of Definition B.2. This ensures that finite-dimensionsal subspaces of W are homeomorphic to $\mathbb{R}^n$, and that when $W = F + W'$ is a direct sum decomposition with F finite-dimensionsal then the corresponding projections from W to F and to W' are continuous. The main non-trivial example to keep in mind is $W = \mathbb{R}^\infty$ (with the colimit topology), though of course the finite-dimensional spaces $\mathbb{R}^n$ are also examples. Note that $\mathbb{R}^\infty$ in the metric or product topologies are not reasonable topological vector spaces in this sense (see Appendix B).

Proposition 16.6. *Let W be a reasonable topological vector space and let $p\colon VI_n(W) \to \mathrm{Gr}_n(W)$ be the standard quotient map. Let $F \subseteq W$ be a subspace of dimension n and let W' be a fixed complement. Write π_F and $\pi_{W'}$ for the associated projection maps (note that despite the notation the maps depend on both F and W'). Let $U_{F,W'} = \{J \in \mathrm{Gr}_n(W) \,|\, \pi_F(J) = F\} = \{J \in \mathrm{Gr}_n(W) \,|\, J \cap W' = 0\}$.*

(a) $U_{F,W'}$ is an open set in $\mathrm{Gr}_n(W)$.

(b) If $b_1, \ldots, b_n \in F$ is a basis then the map $(W')^n \to U_{F,W'}$ that sends $(w_1', \ldots, w_n') \to \mathrm{Span}\{b_1 + w_1', \ldots, b_n + w_n'\}$ is a homeomorphism.

(c) For any linear map $L\colon F \to W'$ let $\Gamma(L) = \{v + L(v) \,|\, v \in F\}$ be the graph of L. Then $L \mapsto \Gamma(L)$ gives a homeomorphism $\mathrm{Hom}(F, W') \cong U_{F,W'}$.

(d) There is a $GL_n(\mathbb{R})$-equivariant homeomorphism $p^{-1}(U_{F,W'}) \cong U_{F,W'} \times GL_n(\mathbb{R})$ that commutes with the projection maps down to $U_{F,W'}$.

(e) p is a principal $GL_n(\mathbb{R})$-bundle.

Proof. First note that π_F and $\pi_{W'}$ are continuous because W is reasonable (Proposition B.4). Also, write $U = U_{F,W'}$ for the remainder of the proof.

Observe that

$$p^{-1}(U) = \{(w_1, \ldots, w_n) \in W^n \,|\, \pi_F(w_1), \ldots, \pi_F(w_n) \text{ is a basis for } F\}.$$

The set $p^{-1}(U)$ is open in $VI_n(W)$ because it sits in the pullback diagram

$$\begin{array}{ccccc} VI_n(W) & \hookrightarrow & W^n & \xrightarrow{\pi_F} & F^n \\ \uparrow & & & & \uparrow \\ p^{-1}(U) & & \longrightarrow & & VI_n(F) \end{array}$$

and $VI_n(F) \hookrightarrow F^n$ is open. The last statement is because $F \cong \mathbb{R}^n$ (this uses that W is reasonable) and for n vectors in $\mathbb{R}^n$ independence is determined by the nonvanishing of the determinant. Also, note that we are using continuity of π_F.

Since $p^{-1}(U)$ is open in $VI_n(W)$, we have that U is open in $\mathrm{Gr}_n(W)$.

Fix a basis $b_1, \ldots, b_n$ of F. Consider the map $(W')^n \to VI_n(W)$ given by $(w_1', \ldots, w_n') \to (b_1 + w_1', \ldots, b_n + w_n')$. This is continuous because it is the composite

$$(W')^n \times \{(b_1, \ldots, b_n)\} \hookrightarrow (W')^n \times W^n = (W' \times W)^n \hookrightarrow (W \times W)^n \xrightarrow{+} W^n.$$

The image lies in $p^{-1}(U)$, so composing with p gives a map $f\colon (W')^n \to U$. This is clearly a bijection, but we need to check that it is a homeomorphism. For this, start with the continuous map

$$\alpha\colon p^{-1}(U) \to (W')^n \times VI_n(F),\ \underline{w} \mapsto (\pi_{W'}(w_1), \ldots, \pi_{W'}(w_n), \pi_F(w_1), \ldots, \pi_F(w_n)).$$

Then consider the composite

$$\begin{array}{ccccccccc} p^{-1}(U) & & & & & & & & (W')^n \\ \downarrow \alpha & & & & & & & & \uparrow c \\ (W')^n \times VI_n(F) & \xrightarrow{a} & (W')^n \times GL_n(\mathbb{R}) & \xrightarrow{\mathrm{id} \times \mathrm{inv}} & (W')^n \times GL_n(\mathbb{R}) & \xrightarrow{b} & (W')^n \times (\mathbb{R}^n)^n & & \end{array}$$

The map a comes from the identification $F \cong \mathbb{R}^n$ provided by the chosen basis; the coordinates of the ith element become the ith column of a matrix. The map inv sends a matrix to its inverse. The map b sends a matrix in $GL_n(\mathbb{R})$ to its tuple of columns, and the map c consists of several instances of the linear combination map $(W')^n \times \mathbb{R}^n \to W'$ given by $(\underline{(w)}, \underline{(r)}) \mapsto \sum r_i w_i$. All of these maps are

continuous. The composite takes a tuple of the form $(v_1 + w_1', \dots, v_n + w_n')$ where $(v_1, \dots, v_n) \in VI_n(F)$ and each $w_i' \in W'$, does a change of basis to bring it into the form $(b_1 + w_1'', \dots, b_n + w_n'')$ with $w_i'' \in W'$ for all i, and then sends it to the tuple $(w_1'', \dots, w_n'')$. This map respects the quotient relation for $p^{-1}(U) \to U$ and so descends to give a map $g\colon U \to (W')^n$. This is the inverse to f, thereby proving that we had a homeomorphism.

The composite $\mathrm{Hom}(F, W') \xrightarrow{\Gamma} U \xrightarrow{g} (W')^n$ is precisely the map that evaluates at the basis $b_1, \dots, b_n$. This composite is a homeomorphism by Exercise B.11 and g is also a homeomorphism, hence so is Γ.

It only remains to prove (d), since (e) follows immediately. Observe that we have already produced maps $U \xrightarrow{g} (W')^n \to p^{-1}(U)$ whose composite is a section of p. Call this composite s. We get an associated map $U \times GL_n(\mathbb{R}) \to p^{-1}(U)$ by $(M, P) \mapsto s(M) \cdot P$, using the right $GL_n(\mathbb{R})$-action on $VI_n(W)$. The inverse is given by p in the first coordinate and $\pi_2 a \alpha$ in the second coordinate (see the previous diagram). Note that $\pi_2 a \alpha$ takes a tuple $(f_1 + w_1', \dots, f_n + w_n')$ with $f_i \in F$ and $w_i' \in W'$ and sends it to the unique matrix $P \in GL_n(\mathbb{R})$ for which $(f_1, \dots, f_n) = (b_1, \dots, b_n) \cdot P$. □

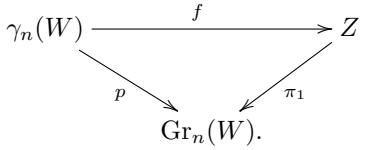
Define $\gamma_n(W)$ to be the vector bundle associated to the principal $GL_n(\mathbb{R})$-bundle $VI_n(W) \to \mathrm{Gr}_n(W)$ (Section 11.20). That is,

$$\gamma_n(W) = VI_n(W) \times_{GL_n(\mathbb{R})} \mathbb{R}^n.$$

There is a continuous map $\phi\colon \gamma_n(W) \to W$ defined by $((w_1, \dots, w_n), (r_1, \dots, r_n)) \mapsto r_1 w_1 + \cdots + r_n w_n$; for $P \in GL_n(\mathbb{R})$ note that $(\underline{w}P, \underline{r})$ and $(\underline{w}, P\underline{r})$ both map to $\underline{w}P\underline{r}$, where $\underline{w}$ is regarded as a row and $\underline{r}$ as a column. From ϕ we can construct $\gamma_n(W) \to \mathrm{Gr}_n(W) \times W$ which is the projection in the first coordinate and ϕ in the second. This is a map of vector bundles over $\mathrm{Gr}_n(W)$ and is clearly a fiberwise injection.

Proposition 16.7. *The canonical map $\gamma_n(W) \to \mathrm{Gr}_n(W) \times W$ is a homeomorphism onto its image (in the subspace topology).*

Proof. Let f denote the canonical map in question, and let Z denote the image of f with the subspace topology. We have the triangle



The map $Z \to \mathrm{Gr}_n(W)$ is a rank n subfamily of the trivial family of vector spaces $\mathrm{Gr}_n(W) \times W$, and f is an isomorphism on each fiber. If we show that $Z \to \mathrm{Gr}_n(W)$ is a vector bundle it will then follow that f is an isomorphism of vector bundles and hence a homeomorphism. Since Z is a subfamily of a trivial family, Propositions 10.18 and 10.19 imply that it is enough to locally produce n independent sections near every point.

For each n-dimensional subspace $F \subseteq W$ with chosen complement W' let $U_{F,W'} \subseteq \mathrm{Gr}_n(W)$ be as in Proposition 16.6. This is open and $F \in U_{F,W'}$. Proposition 16.6 also showed that if $b_1, \dots, b_n$ is a basis for F then the map $\phi_{\underline{b}}\colon (W')^n \to U_{F,W'}$ given by $(w_1', \dots, w_n') \mapsto \mathrm{Span}(b_1 + w_1', \dots, b_n + w_n')$ is a

homeomorphism. Now simply write down the sections $M \mapsto (M, b_i + (\phi_{\underline{b}}^{-1}(M))_i)$, for $1 \leq i \leq n$. For $M \in U_{F,W'}$ this simply picks out the unique basis of the form $b_1 + w'_1, \ldots, b_n + w'_n$ and then chooses the ith element. □

The above result says that we can reinterpret $\gamma_n(W)$ as the space

$$\gamma_n(W) = \{(M, x) \,|\, M \in \mathrm{Gr}_n(W),\ x \in M\} \subseteq \mathrm{Gr}_n(W) \times W.$$

The vector bundle $\gamma_n(W) \to \mathrm{Gr}_n(W)$ is called the **tautological bundle** over the Grassmannian. It is the "evident" subbundle of $\underline{W}$ that one can write down.

Remark 16.8. It is very common in algebraic topology to use γ for the tautological vector bundles on Grassmannians, as we have done here. This goes back at least to [MS]. However, in K-theory one also has the γ-operations, which are something completely different. This conflict in notation leads to the tautological bundles often being denoted by η in K-theory contexts. We will mostly stick to γ, but will sometimes use η—one has to be light on one's feet in this business.

The bundles $\gamma_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ are also referred to as **universal bundles** because of the role they play in the classifying space story that we will develop shortly. It will be important that these bundles can also be described as colimits of the $\gamma_n(\mathbb{R}^s)$ bundles. The standard inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2} \hookrightarrow \cdots$ give an induced system of maps

$$\mathrm{Gr}_n(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{n+1}) \hookrightarrow \cdots \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$$

and similarly for the VI_n and γ_n constructions.

Proposition 16.9. *The canonical maps* $\mathrm{colim}_s\, VI_n(\mathbb{R}^s) \to VI_n(\mathbb{R}^\infty)$, $\mathrm{colim}_s\, \mathrm{Gr}_n(\mathbb{R}^s) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, *and* $\mathrm{colim}_s\, \gamma_n(\mathbb{R}^s) \to \gamma_n(\mathbb{R}^\infty)$ *are all homeomorphisms.*

Proof. Consider the following three directed systems and their abutments:

$$\begin{array}{ccccccccc}
(\mathbb{R}^n)^n & \longrightarrow & (\mathbb{R}^{n+1})^n & \longrightarrow & (\mathbb{R}^{n+2})^n & \longrightarrow & \cdots & \longrightarrow & (\mathbb{R}^\infty)^n \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
VI_n(\mathbb{R}^n) & \longrightarrow & VI_n(\mathbb{R}^{n+1}) & \longrightarrow & VI_n(\mathbb{R}^{n+2}) & \longrightarrow & \cdots & \longrightarrow & VI_n(\mathbb{R}^\infty) \\
\downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle p_{n+2}} & & & & \downarrow{\scriptstyle p} \\
\mathrm{Gr}_n(\mathbb{R}^n) & \longrightarrow & \mathrm{Gr}_n(\mathbb{R}^{n+1}) & \longrightarrow & \mathrm{Gr}_n(\mathbb{R}^{n+2}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Gr}_n(\mathbb{R}^\infty).
\end{array}$$

It is trivial that in each case the map from the colimit to the rightmost entry is a continuous bijection, but a little work is required to check that these are homeomorphisms. We will give two different proofs, and the reader can choose the technique they like better.

All of the spaces in the above diagram are sequentially determined. This follows from Exercise A.19 and Proposition B.8. So the colimits in question are also sequentially determined, and to establish the required homeomorphisms we just need to establish that the convergent sequences are the same for the two spaces. But the horizontal maps are all closed inclusions between Hausdorff spaces, so by Exercise A.20 in each of the top two rows the convergent sequences are precisely the ones that lie in some finite stage and are convergent there. This takes care of

the top two rows. The bottom row then follows from Exercise A.2, which says that a colimit of quotient maps is a quotient map.

For $\operatorname{colim}_s \gamma_n(\mathbb{R}^s)$ and $\gamma_n(\mathbb{R}^\infty)$, observe the latter is sequentially determined because it is a quotient of $VI_n(\mathbb{R}^\infty) \times \mathbb{R}^n$: we have already shown that $VI_n(\mathbb{R}^\infty)$ is sequentially determined, so the product is by Exercise A.19(l). Using that $\gamma_n(\mathbb{R}^\infty)$ is a subspace of $\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$, the convergent sequences are precisely the ones that land in some finite stage and are convergent there; these are the same as in $\operatorname{colim}_s \gamma_k(\mathbb{R}^s)$.

Despite the number of words in the above explanation, the actual content is very small and with practice one learns to think through the arguments quickly.

Next we give alternative proofs that avoid the use of sequentially determined spaces. By Corollary B.9 we have that $(\mathbb{R}^\infty)^n$ is the colimit of the top directed system. The intersection of $VI_n(\mathbb{R}^\infty)$ with $(\mathbb{R}^k)^n$ is $VI_n(\mathbb{R}^k)$, and this is open in $(\mathbb{R}^k)^n$ because it is the set of $n \times k$ matrices with at least one non-vanishing $n \times n$ minor. So $VI_n(\mathbb{R}^\infty)$ is open in $(\mathbb{R}^\infty)^n$ and is the colimit of the middle system. Finally, suppose $U \subseteq \mathrm{Gr}_n(\mathbb{R}^\infty)$ is such that its intersection with each $\mathrm{Gr}_n(\mathbb{R}^s)$ is open in $\mathrm{Gr}_n(\mathbb{R}^s)$. Then $p_s^{-1}(U \cap \mathrm{Gr}_n(\mathbb{R}^s))$ is open in $VI_n(\mathbb{R}^s)$, but this set is also $p^{-1}(U) \cap VI_n(\mathbb{R}^s)$. Therefore $p^{-1}(U)$ is open in $VI_n(\mathbb{R}^\infty)$, and hence U is open. This confirms that $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is the colimit of the bottom system.

Finally, we turn to the map $\operatorname{colim}_s \gamma_n(\mathbb{R}^s) \to \gamma_n(\mathbb{R}^\infty)$. For this consider the two directed systems

$$\begin{array}{ccccccc} VI_n(\mathbb{R}^n) \times \mathbb{R}^n & \rightarrowtail & VI_n(\mathbb{R}^{n+1}) \times \mathbb{R}^n & \rightarrowtail & \cdots & \rightarrowtail & VI_n(\mathbb{R}^\infty) \times \mathbb{R}^n \\ \downarrow & & \downarrow & & & & \downarrow \\ \gamma_n(\mathbb{R}^n) & \longrightarrow & \gamma_n(\mathbb{R}^{n+1}) & \longrightarrow & \cdots & \longrightarrow & \gamma_n(\mathbb{R}^\infty) \end{array}$$

Since $\mathbb{R}^n$ is locally compact and Hausdorff the colimit of the top row is the space on the right; this uses Exercise A.4 together with what we have already proven about VI_n. It then follows formally that the induced map $VI_n(\mathbb{R}^\infty) \times \mathbb{R}^n \to \operatorname{colim}_s \gamma_n(\mathbb{R}^s)$ is a quotient map (Exercise A.2). The quotient relations are the same as the ones that define $\gamma_n(\mathbb{R}^\infty)$, so we deduce that $\operatorname{colim}_s \gamma_n(\mathbb{R}^s) \to \gamma_n(\mathbb{R}^\infty)$ is a homeomorphism. □

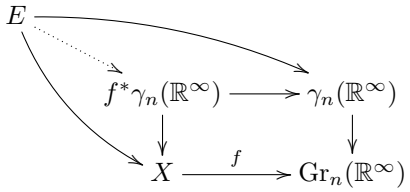
16.10. **Grassmannians and bundle theory.** The bundle $\gamma_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ allows us to construct the natural transformation $\Theta_X \colon [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n(X)$ sending $f \mapsto f^*\gamma_n$ (for X paracompact Hausdorff).

Proposition 16.11. *For X paracompact Hausdorff the map Θ_X is a surjection.*

Proof. Let $\pi \colon E \to X$ be a rank n vector bundle. By Proposition 12.13 we know that there is a fiberwise embedding of bundles $j \colon E \hookrightarrow X \times \mathbb{R}^\infty$. Define the function $f \colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by $f(x) = \pi_2(j(E_x))$. We need to check that f is continuous. This can be done locally on X, and so we can assume E is trivial. In that case Proposition 16.5 says that the map j is determined by a continuous map $X \to VI_n(\mathbb{R}^\infty)$. The composite of this map with the projection to $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is precisely f, and this verifies continuity.

Consider the map $(f\pi, \pi_2 j) \colon E \to \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$. This is clearly continuous, and its image lands in the subspace $\gamma_n(\mathbb{R}^\infty)$ (this implicitly uses Proposition 16.7).

The diagram



gives us an induced map $E \to f^*\gamma_n$ which is readily seen to be an isomorphism on each fiber and hence an isomorphism of bundles. So the map f is a preimage for E under Θ_X. □

Remark 16.12. Note in the above proof that if X were actually compact then the bundle embedding could be chosen to be $E \hookrightarrow X \times \mathbb{R}^s$ for some $s < \infty$, and then f maps to the finite Grassmannian $\mathrm{Gr}_n(\mathbb{R}^s)$.

Proposition 16.13. *Let $j^{ev}, j^{odd}\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be given by $j^{ev}(x_1, x_2, ...) = (0, x_1, 0, x_2, ...)$ and $j^{odd}(x_1, x_2, ...) = (x_1, 0, x_2, 0, ...)$. Then $j^{ev} \simeq \mathrm{id}$ and $j^{odd} \simeq \mathrm{id}$, via homotopies H having the property that each H_t is a linear embedding. Consequently, the maps $j^{ev}, j^{odd}\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ are homotopic.*

Proof. We prove the claim for j^{ev}; the proof for j^{odd} is analogous. Define a homotopy $H\colon \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ by $H(x,t) = tj^{ev}(x) + (1-t)x$. This is continuous using that $\mathbb{R}^\infty \times I$ is the colimit of the $\mathbb{R}^k \times I$ (Exercise A.4). This is clearly a homotopy between j^{ev} and id. It remains to show that this is a homotopy through linear embeddings. Let $t \in (0,1)$ and suppose that $H(x,t) = 0$. We need to show that $x = 0$. Our assumption yields $0 = ((1-t)x_1, tx_1 + (1-t)x_2, (1-t)x_3, tx_2 + (1-t)x_4, ...)$. Therefore $(1-t)x_i = 0$ for all odd i; but since $t \neq 1$, this means that $x_i = 0$ for all odd i. Likewise, observe that $tx_n + (1-t)x_{2n} = 0$ for all $n \in \mathbb{N}$. So $x_n = 0$ implies $x_{2n} = 0$. Since we have $x_i = 0$ for all odd i and every natural number n can be written in the form $n = 2^e \cdot (\text{odd})$, it follows that $x = 0$.

The last statement in the proposition follows from Lemma 16.3 □

Corollary 16.14. *Any two linear embeddings $f, g\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ are homotopic through linear embeddings, and therefore induce homotopic maps on $\mathrm{Gr}_n(\mathbb{R}^\infty)$.*

Proof. By Proposition 16.13 the map f is homotopic to $j_{ev} \circ f$, and the map g is homotopic to $j_{odd} \circ g$. So we may assume that $\mathrm{im}(f) \subseteq \mathbb{R}^\infty_{ev}$ and $\mathrm{im}(g) \subseteq \mathbb{R}^\infty_{odd}$. Now consider the map $H\colon \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ given by $H(x,t) = tf(x) + (1-t)g(x)$. □

Remark 16.15. There is an important technique demonstrated in the above corollary and its proof, and this will be used frequently in all of our future work with the spaces $\mathrm{Gr}_n(\mathbb{R}^\infty)$. In any particular situation, up to homotopy one is free to push things into convenient infinite-dimensional subspaces of $\mathbb{R}^\infty$. Often this dramatically simplifies some issues.

Theorem 16.16. *The map $\phi\colon [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n(X)$ is a bijection when X is paracompact and Hausdorff.*

Proof. Surjectivity was established in Proposition 16.11, so only injectivity remains to be shown. Assume $f, g\colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ are such that $f^*(\gamma_n) \cong g^*(\gamma_n)$ as vector bundles over X. We will show that f is homotopic to g. By Proposition 16.13, we may replace f by $j^{ev} \circ f$ and g by $j^{odd} \circ g$. In doing so, we are effectively assuming that $f(x) \subseteq \mathbb{R}^\infty_{ev}$ and $g(x) \subseteq \mathbb{R}^\infty_{odd}$, for each $x \in X$.

Let $\alpha\colon f^*\gamma_n \to g^*\gamma_n$ be a choice of isomorphism. Define a map $H\colon X \times I \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ by the formula

$$H(x,t) = \{tv + (1-t)\alpha(v) \,|\, v \in f(x)\}.$$

It is easy to see that $H(x,t)$ is indeed an n-dimensional subspace of $\mathbb{R}^\infty$, using that $f(x) \subseteq \mathbb{R}^\infty_{ev}$ and $g(x) \subseteq \mathbb{R}^\infty_{odd}$. Also $H(x,0) = g$ and $H(x,1) = f$ by definition. It only remains to check continuity of H. This is a local issue, so we can restrict to an open set $U \subseteq X$ on which $f^*\gamma_n$ is trivializable.

Choose an isomorphism $A\colon \underline{n}_U \to f^*\gamma_n|_U$ and let $B = \alpha \circ A$. The composite $U \times \mathbb{R}^n \xrightarrow{A} (f^*\gamma_n)|_U \hookrightarrow U \times \mathbb{R}^\infty$ is a fiberwise embedding and so corresponds to a map $\tilde{f}\colon U \to VI_n(\mathbb{R}^\infty)$ by Proposition 16.5, and the image of $\tilde{f}$ actually lies in $VI_n(\mathbb{R}^\infty_{ev})$. Likewise, the composite $U \times \mathbb{R}^n \xrightarrow{B} (g^*\gamma_n)|_U \hookrightarrow U \times \mathbb{R}^\infty$ corresponds to a map $\tilde{g}\colon U \to VI_n(\mathbb{R}^\infty)$ that factors through $VI_n(\mathbb{R}^\infty_{odd})$.

The assignment $(u,t) \mapsto t\tilde{f}(u) + (1-t)\tilde{g}(u)$ gives a continuous map $J\colon U \times I \to VI_n(\mathbb{R}^\infty)$. To check continuity just refer to the diagram

$$\begin{array}{ccc} VI_n(\mathbb{R}^\infty_{ev}) \times VI_n(\mathbb{R}^\infty_{odd}) \times I & \dashrightarrow & VI_n(\mathbb{R}^\infty) \\ \downarrow & & \downarrow \\ (\mathbb{R}^\infty)^n \times (\mathbb{R}^\infty)^n \times I & \longrightarrow & (\mathbb{R}^\infty)^n \end{array}$$

where the bottom map is $(\underline{a}, \underline{b}, t) \mapsto t\underline{a} + (1-t)\underline{b}$ and is cleary continuous. The image of the top left corner in $(\mathbb{R}^\infty)^n$ lands in $VI_n(\mathbb{R}^\infty)$ by algebra, and so we get the induced dotted arrow. The map J is obtained just by precomposing the dotted arrow with $(\tilde{f}, \tilde{g})$.

The projection of J down to $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is exactly $H|_{U\times I}$. This verifies continuity of H and completes the proof. □

The above is the standard proof for injectivity, and while it is not hard the reader may well be left wondering what exactly happened. It is the kind of proof where the core idea is almost obscured by its apparent simplicity. Let us go back to the paradigm discussed in Section 16.1 and see how it fits here.

Define $\mathrm{VB}_n = \mathrm{Gr}_n(\mathbb{R}^\infty)$, which is our proposed classifying space for n-bundles. Pulling back the universal bundle gives our natural map $[X, VB_n] \to \mathrm{Vect}_n(X)$, which we know is surjective. Consider the tautological bundles $\gamma_n^{ev} \to \mathrm{Gr}_n(\mathbb{R}^\infty_{ev})$ and $\gamma_n^{odd} \to \mathrm{Gr}_n(\mathbb{R}^\infty_{odd})$ and let π_1, π_2 be the projections from $\mathrm{Gr}_n(\mathbb{R}^\infty_{ev}) \times \mathrm{Gr}_n(\mathbb{R}^\infty_{odd})$ to its two factors. Define

$$\mathrm{VBI}_n = \underline{\mathrm{Iso}}\big(\ \pi_1^*\gamma_n^{ev},\ \pi_2^*\gamma_n^{odd}\ \big)$$

(see Exercise 14.9 for the definition of $\underline{\mathrm{Iso}}$). This is a fiber bundle over the space $\mathrm{Gr}_n(\mathbb{R}^\infty_{ev}) \times \mathrm{Gr}_n(\mathbb{R}^\infty_{odd})$. The "even" and "odd" are just for extra clarity and can of course be removed everywhere. The space VBI_n comes equipped with two special bundles—the pullbacks of γ_n^{ev} and γ_n^{odd}—and a fixed isomorphism between them. It is established implicitly in the above proof that given two bundles E and J on X together with an isomorphism $E \cong J$, there is a map $X \to \mathrm{VBI}_n$ classifying that data.

The second element of the above proof is the construction of a map $\mathrm{VBI}_n \to (\mathrm{VB}_n)^I$. This is the part that takes an isomorphism between a subspace of $\mathbb{R}^\infty_{ev}$ and

a subspace of $\mathbb{R}^\infty_{odd}$ and uses it to construct a straight-line path between them in the Grassmannian.

From these two basic components one obtains the diagram

$$\begin{array}{ccc}
\mathcal{T}op(X, \mathrm{VBI}_n) & \longrightarrow & \mathcal{T}op(X, \mathrm{VB}_n^I) \\
\Big\downarrow\Big\downarrow & & \overset{ev_0}{\Big\downarrow}\Big\downarrow\overset{ev_1}{} \\
\mathcal{T}op(X, \mathrm{VB}_n) & = & \mathcal{T}op(X, \mathrm{VB}_n) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Vect}_n(X) & \dashrightarrow & [X, \mathrm{VB}_n]
\end{array}$$

where the left column is a coequalizer. The induced map $\mathrm{Vect}_n(X) \to [X, \mathrm{VB}_n]$ is the inverse for the map we had at the beginning.

Exercise 16.17. Flesh out the details of the above outline. In particular, if E and J are vector bundles on X with an isomorphism $E \to J$ produce a classifying map $X \to \mathrm{VBI}_n$. Also show that the left column is a coequalizer.

16.18. **Representing operations on vector bundles.** If $\mathcal{C}$ is a category then a map $f\colon W \to Z$ in $\mathcal{C}$ induces a map of representable functors $f_*\colon \mathcal{C}(-, W) \to \mathcal{C}(-, Z)$. The Yoneda lemma says this in fact gives a bijection between $\mathcal{C}(W, Z)$ and the set of natural transformations $\mathcal{C}(-, W) \to \mathcal{C}(-, Z)$. In particular, this implies that natural transformations $\mathrm{Vect}_n(X) \to \mathrm{Vect}_r(X)$ correspond to homotopy classes of maps $\mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_r(\mathbb{R}^\infty)$.

A simple example of this is the assignment $E \mapsto E \oplus 1$, regarded as a natural transformation $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$. The corresponding map $P_1\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$ (really, just a homotopy class) can be described as the composite

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_{n+1}(\mathbb{R} \oplus \mathbb{R}^\infty) \xrightarrow{\cong} \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$$

where the first map sends an n-plane W to $\langle e_0 \rangle \oplus W$, with e_0 the standard basis for the added copy of $\mathbb{R}$. The second map is the one induced by any linear homeomorphism $\mathbb{R} \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ (they all induce homotopic maps on the Grassmannisn by Corollary 16.14). For specificity let us choose the linear homeomorphism that sends $e_i \mapsto e_{i+1}$ for all i. To check that P_1 really does induce the "$\oplus\underline{1}$" natural transformation, just verify that $P_1^*\gamma_{n+1} \cong \gamma_n \oplus 1$; this is routine.

As another example consider the direct sum operation $\mathrm{Vect}_n(X) \times \mathrm{Vect}_k(X) \to \mathrm{Vect}_{n+k}(X)$, sending $(E, E') \mapsto E \oplus E'$. This will be induced by a homotopy class of maps $S\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+k}(\mathbb{R}^\infty)$. Such a map can be constructed as

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty) \xrightarrow{\oplus} \mathrm{Gr}_{n+k}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \xrightarrow{\cong} \mathrm{Gr}_{n+k}(\mathbb{R}^\infty)$$

where the first map sends a pair of subspaces (J_1, J_2) to $J_1 \oplus J_2$ and the second is induced by any linear homeomorphism $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ (again, by Corollary 16.14 the exact choice doesn't matter). For example, choose any bijection $\alpha\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ and then use the linear map that sends $e_i \oplus e_j \mapsto e_{\alpha(i,j)}$. Again, to justify that S does represent the direct sum map one verifies that $S^*\gamma_{n+k} \cong \gamma_n \hat{\oplus} \gamma_k$, which is an easy exercise.

Continuity of the $\oplus$ map requires some comment. If the product $\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty)$ is given the compactly-generated topology then it may be identified with $\mathrm{colim}_{s,t}[\mathrm{Gr}_n(\mathbb{R}^s) \times \mathrm{Gr}_k(\mathbb{R}^t)]$ and then continuity of the map is immediate. However,

one can also verify continuity when the domain has the product topology. Start with the composite

$$VI_n(\mathbb{R}^\infty) \times VI_k(\mathbb{R}^\infty) \rightarrowtail (\mathbb{R}^\infty)^n \times (\mathbb{R}^\infty)^k = (\mathbb{R}^\infty \times \{0\})^n \times (\{0\} \times \mathbb{R}^\infty)^k \downarrow (\mathbb{R}^\infty \oplus \mathbb{R}^\infty)^{n+k}$$

and observe that the image lands inside $VI_{n+k}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty)$. Consider the diagram

$$\begin{array}{ccccc} VI_n(\mathbb{R}^\infty) \times VI_k(\mathbb{R}^\infty) & \longrightarrow & VI_{n+k}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) & \longrightarrow & \mathrm{Gr}_{n+k}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \\ \downarrow & & & \nearrow & \\ \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty). & & & & \end{array}$$

The maps $VI_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ and $VI_k(\mathbb{R}^\infty) \to \mathrm{Gr}_k(\mathbb{R}^\infty)$ are fiber bundles and therefore so is their product. The product is therefore a quotient map. The quotient relations for $VI_n(\mathbb{R}^\infty) \times VI_k(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty)$ are satisfied by the composite across the top of the diagram, so this yields the dotted arrow. This map is readily verified to be $\oplus$.

For a third method of verifying the continuity of $\oplus$, see Exercise16.21.

There are many examples similar to the previous two where one constructs explicit maps on classifying spaces representing some algebraic construction. Let us do just one more. Given any vector space V we know how to construct the exterior product $\Lambda^k V$ and this is functorial. An n-plane $J \subseteq \mathbb{R}^\infty$ therefore gives an $\binom{n}{k}$-plane $\Lambda^k J \subseteq \Lambda^k \mathbb{R}^\infty$. Note that we topologize $\Lambda^k \mathbb{R}^\infty$ as the colimit of its finite-dimensional subspaces. Choosing a linear homeomorphism $\Lambda^k \mathbb{R}^\infty \cong \mathbb{R}^\infty$ therefore lets us form the composite map

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \xrightarrow{\Lambda^k} \mathrm{Gr}_{\binom{n}{k}}(\Lambda^k \mathbb{R}^\infty) \xrightarrow{\cong} \mathrm{Gr}_{\binom{n}{k}}(\mathbb{R}^\infty)$$

which we temporarily just call L. Given a rank n bundle $E \to X$ we could define its kth exterior product to be the bundle represented by $L \circ f$, where f is any representing map for E.

Exercise 16.19. Verify continuity of the Λ^k map in the above composite. [Hint: One way is to lift the map to $VI_n(\mathbb{R}^\infty) \to VI_{\binom{n}{k}}(\Lambda^k \mathbb{R}^\infty)$. Another way is to show that the map preserves convergent sequences and use Exercise A.20.]

Exercise 16.20. Think about some of the pros and cons of constructing $\Lambda^k E$ in the above manner, compared to the construction that chooses a local trivialization of the bundle and applies Λ^k to each patch, or to the construction via frame bundles.

Exercise 16.21. Prove that $\mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty)$ is sequentially determined, and then prove that the map $\oplus\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+k}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty)$ is continuous by proving that it preserves convergent sequences. For a bonus, prove that the natural map $\mathrm{colim}_{s,t}[\mathrm{Gr}_n(\mathbb{R}^s) \times \mathrm{Gr}_k(\mathbb{R}^t)] \to \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty)$ is a homeomorphism. [Hint: Consult Proposition 16.6, Exercise A.19(g), and Exercise A.20.]

16.22. **Stabilization of vector bundles.** Here is a simple application of classifying spaces that we will occasionally find useful. Fix a space X. If $E \to X$ is a vector bundle of rank n, then of course $E \oplus \underline{1}$ is a vector bundle of rank $n+1$. We

get a sequence of maps

$$\mathrm{Vect}_0(X) \xrightarrow{\oplus 1} \mathrm{Vect}_1(X) \xrightarrow{\oplus 1} \mathrm{Vect}_2(X) \xrightarrow{\oplus 1} \cdots$$

Are these maps injective? Surjective? Are there more and more isomorphism classes of vector bundles as one goes up in rank, or is it the case that all "large" rank vector bundles actually come from smaller ones via addition of a trivial bundle? A homotopical analysis of classifying spaces yields some partial answers here. We handle both the case of real and complex bundles:

Proposition 16.23. *Let X be a finite-dimensional CW-complex. For real vector bundles, $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ is a bijection for $n \geq \dim X + 1$ and a surjection for $n = \dim X$. For complex bundles, $\mathrm{Vect}^{\mathbb{C}}_n(X) \to \mathrm{Vect}^{\mathbb{C}}_{n+1}(X)$ is a bijection for $n \geq \frac{1}{2}\dim X$ and a surjection for $n \geq \frac{1}{2}(\dim X - 1)$.*

Proof. The map $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ is represented by the map of spaces $P_1 \colon \mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$ discussed above. To establish the proposition we must analyze when $[X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \xrightarrow{(P_1)_*} [X, \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)]$ is injective/surjective.

The key fact is that $\mathrm{Gr}_n(\mathbb{R}^\infty) \to \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$ is n-connected. This can be argued in different ways: one way is to examine the Schubert cell decompositions of each space and observe that they are identical until one reaches dimension $n+1$—see Exercise 18.9 (and especially part (d)) for an introduction to Schubert cells if they are unfamiliar. Example 16.28 gives a different proof of this fact using classifying spaces. This connectivity result implies that $[X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \to [X, \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)]$ is bijective for CW-complexes with $\dim X \leq n-1$, and surjective for CW-complexes with $\dim X = n$.

For the complex case, $\mathrm{Gr}_n(\mathbb{C}^\infty) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{C}^\infty)$ is now $(2n+1)$-connected (by similar arguments). So we get the analogous bijection for CW-complexes X of dimension at most $2n$, and the surjection when $\dim X = 2n+1$. □

16.24. **Classifying spaces for principal bundles.** We have previously seen that the category of n-bundles and isomorphisms is equivalent to the category of principal $GL_n(\mathbb{R})$-bundles. The classifying space story for n-bundles is really part of a larger story about classifying spaces for principal bundles. We give a brief tour, but see [Steenrod1] or [Husemoller] for details.

Fix a topological group G. A **universal G-bundle** is a principal G-bundle $\gamma_G \colon EG \to BG$ where EG is weakly contractible. It turns out that such bundles always exist and are unique up to a certain notion of homotopy equivalence. Given a universal bundle, the associated map $[X, BG] \to \mathrm{PrBund}_G(X)$ sending $f \mapsto f^*\gamma_G$ is a bijection.

The space BG is called a **classifying space** for G. Technically it is only defined up to weak equivalence. There are constructions for EG, BG, and γ_G that are functorial in G. These constructions are essentially homotopical in nature. The original construction of [Milnor2] uses an infinite join, whereas most modern references use the simplicial bar complex [Segal1], [May3]. Here is an almost purely homotopical version, just to get the flavor: let $G\mathsf{Tors}$ be the topological category of right G-torsors and G-maps, let $\Gamma \colon G\mathsf{Tors} \to \mathcal{T}op$ be the (enriched) functor that sends every G-torsor to its underlying space, and define EG to be the (enriched) homotopy colimit of Γ. Then EG has a free G-action and turns out to be weakly contractible,

and when constructed carefully enough the map $EG \to (EG)/G$ is a principal G-bundle. We mention this only to help orient the reader, as the infinite-join and bar complex versions are really just models for this homotopy colimit.

Because $G \to EG \to BG$ is a fiber bundle and EG is weakly contractible, we get a canonical (up to homotopy) weak equivalence $G \xrightarrow{\simeq} \Omega BG$.

Example 16.25.

(a) If G is discrete then the fiber sequence $G \to EG \to BG$ shows that $\pi_1(BG) \cong G$ and $\pi_i(BG) = 0$ for $i \neq 1$. So BG is a $K(G, 1)$-space.
(b) Consider $\mathbb{Z}/2$ acting on $S^\infty \subseteq \mathbb{R}^\infty$ by the antipodal map. The quotient map $S^\infty \to \mathbb{R}P^\infty$ is a principal $\mathbb{Z}/2$-bundle and S^∞ is contractible, so it is the univeral bundle. Therefore $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$. Of course we also knew this from (a), since $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$.
(c) Identify S^1 with the unit complex numbers and consider the action on $S^\infty \subseteq \mathbb{C}^\infty$. The quotient $S^\infty \to \mathbb{C}P^\infty$ is a universal S^1-bundle, so that $BS^1 \simeq \mathbb{C}P^\infty$.
(d) Consider $VI_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$. This is a principal $GL_n(\mathbb{R})$-bundle, and the total space is contractible. So $BGL_n(\mathbb{R}) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$.
(e) Equip $\mathbb{R}^\infty$ with the Euclidean inner product and consider the space $V_n(\mathbb{R}^\infty)$ of orthonormal n-frames. The projection $V_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ is a universal $O(n)$-bundle, therefore $BO(n) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$.

It should not be a surprise that $BO(n) \simeq BGL_n(\mathbb{R})$, since in fact $O(n) \simeq GL_n(\mathbb{R})$. Moving forward, any homotopical statement involving $BGL_n(\mathbb{R})$ can just as well be phrased in terms of $BO(n)$. The standard practice in homotopy theory is to use the latter, partly because it is somewhat shorter and partly because the compact Lie groups $O(n)$ are slightly more appealing than their noncompact analogs $GL_n(\mathbb{R})$.

Exercise 16.26. Use the fibration sequences $V_{n-1}(\mathbb{R}^{s-1}) \to V_n(\mathbb{R}^s) \to V_1(\mathbb{R}^s)$ to show that $V_n(\mathbb{R}^s)$ (and hence also $VI_n(\mathbb{R}^s)$) is $(s-n-1)$-connected. Explain why $VI_n(\mathbb{R}^\infty)$ is weakly contractible.

Often we want to compare the classifying space of a group to that of a subgroup:

Proposition 16.27. *Let G be a Lie group and $H \leq G$ a closed subgroup. Then there is a homotopy fiber sequence $G/H \to BH \to BG$.*

Proof. [Sketch] The best proofs are homotopy-theoretic and require more machinery than we are willing to develop, so instead we give a rough sketch that omits some details. Start with $EG \to BG$. Take the right G-action on EG and restrict it to an H-action, then form EG/H. Then $EG \to EG/H$ is a principal H-bundle with contractible total space, so EG/H is a model for BH. Then consider the fiber sequence $G/H \to EG/H \to EG/G$, which can be rewritten as $G/H \to BH \to BG$.

This argument used that $EG \to EG/H$ was a principal H-bundle and that $EG/H \to EG/G$ is a fibration. These statements are not automatic and can fail without some niceness hypotheses. It is usually enough to have $H \to G \to G/H$ a fiber bundle, which is true in the Lie case by [Palais, Section 4.1].

See [May3, Section 8] for more detailed information, though note that there "G/H" is defined via a bar complex and only coincides with the geometric G/H under sufficient niceness conditions. □

Example 16.28. Consider the homotopy fiber sequence $O_{n+1}/O_n \to BO_n \to BO_{n+1}$. We know that $O_{n+1}/O_n \cong S^n$, so $BO_n \to BO_{n+1}$ is n-connected. The similar sequence $U_{n+1}/U_n \to BU_n \to BU_{n+1}$ shows that $BU_n \to BU_{n+1}$ is $(2n+1)$-connected.

Remark 16.29. The bijection $[X, BG] \cong \mathrm{PrBund}_G(X)$ should really be regarded as the π_0-version of a space-level equivalence. Regard $\mathsf{PrBund}_G(X)$ as a topological category by equipping the set of morphisms between two principal G-bundles with the evident topology. One can take the nerve of this topological category—or, in modern language, regard it as an ∞-category—and then there is a weak equivalence $\mathrm{Map}(X, BG) \simeq \mathsf{PrBund}_G(X)$. We will not develop this further, but it is good to be aware that these higher equivalences are operating in the background. The analogs also hold for the vector bundle versions, so that $\mathrm{Map}(X, BO_n)$ is weakly equivalent to the ∞-groupoid of rank n vector bundles on X.

∘ Exercises ∘

Exercise 16.30. Consider the natural transformation $\mathrm{Vect}_1(X) \times \mathrm{Vect}_1(X) \to \mathrm{Vect}_1(X)$ given by $(L_1, L_2) \mapsto L_1 \otimes L_2$. This is represented by a map $m\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$. Prove that the induced map m^* on $H^*(-; \mathbb{Z}/2)$ sends the generator $x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ to $x \otimes 1 + 1 \otimes x$. [Note that since $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$ this characterization completely determines the homotopy class of m.]

Exercise 16.31. Let X be a paracompact Hausdorff space. If $L \to X$ is a real line bundle then $L \otimes L$ is trivial. Give three proofs of this fact, using the following strategies:

(i) Explain why the fibers of $L \otimes L$ have a well-defined notion of positive and negative. Construct a nonzero section of $L \otimes L$ by patching local positive sections together.

(ii) Prove that if L is any line bundle then the evaluation homomorphism $L \otimes L^* \to \underline{1}$ is an isomorphism, and then use that $L \cong L^*$.

(iii) Write down a model for the map $f\colon \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ that represents the natural transformation $\mathrm{Vect}_1(X) \to \mathrm{Vect}_1(X)$ given by $L \to L \otimes L$, and prove that f is null homotopic.

Exercise 16.32. Let X be a paracompact Hausdorff space. Consider the following sets of isomorphism classes:

- Rank n real vector bundles on X.
- Rank n real vector bundles on X with inner products.
- Principal $GL_n(\mathbb{R})$-bundles on X.
- Principal O_n-bundles on X.

All of these sets are in bijective correspondence with each other. Think through the bijections from a variety of different perspectives.

Exercise 16.33. Define the **oriented Grassmannian** $\mathrm{Gr}_n^{or}(\mathbb{R}^\infty)$ to be the set of pairs (M, ω) where $M \in \mathrm{Gr}_n(\mathbb{R}^\infty)$ and ω is an orientation of M.

(a) Show that $\mathrm{Gr}_n^{or}(\mathbb{R}^\infty)$ can be topologized in a unique way such that the projection $\mathrm{Gr}_n^{or}(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ is a covering space (of degree 2).

(b) Show that there are natural bijections $[X, \mathrm{Gr}_n^{or}(\mathbb{R}^\infty)] \cong \mathrm{Vect}_n^{or}(X) \cong \mathrm{PrBund}_{SO(n)}(X)$.
(c) Explain why there is a weak equivalence $BSO(n) \simeq \mathrm{Gr}_n^{or}(\mathbb{R}^\infty)$.
(d) Suppose that X is path-connected and $\pi_1(X)$ has no subgroup of index 2. Prove that every vector bundle on X is orientable. [Hint: Use covering space theory.]
(e) Explain the isomorphisms $\mathrm{Vect}_1(X) \cong H^1(X; \mathbb{Z}/2)$ and $\mathrm{Vect}_2^{or}(X) \cong H^2(X; \mathbb{Z})$.
(f) Let X and Z be pointed spaces. Recall that there is a natural action of $\pi_1(Z)$ on $[X, Z]_*$, and the evident map $[X, Z]_* \to [X, Z]$ yields a bijection $[X, Z]_*/\pi_1(Z) \to [X, Z]$ [Hatcher1, Proposition 4.A.2]. Prove that $\pi_1 BO_n = \mathbb{Z}/2$ and explain how the diagram

$$\begin{array}{ccccc} [\Sigma X, BO_n]_* & \xrightarrow{\cong} & [X, \Omega BO_n]_* & \xrightarrow{\cong} & [X, O_n]_* \\ \downarrow & & & & \\ [\Sigma X, BO_n] & & & & \end{array}$$

connects to Proposition 15.3 and Exercise 15.26.
(g) Explain why $H^1(BO_n; \mathbb{Z}/2) = \mathbb{Z}/2$. The unique nonzero class is called the **first Stiefel-Whitney class** and denoted w_1. It can be regarded as a map $w_1 \colon BO_n \to K(\mathbb{Z}/2, 1)$. Explain why the maps $BSO_n \to BO_n \xrightarrow{w_1} K(\mathbb{Z}/2, 1)$ give a homotopy fiber sequence.
(h) If $E \to X$ is an n-bundle then there is a representing map $f \colon X \to BO_n$, and the composite $X \to BO_n \xrightarrow{w_1} K(\mathbb{Z}/2, 1)$ gives an element $w_1(E) \in H^1(X; \mathbb{Z}/2)$. This only depends on the isomorphism class of E. Explain the following fact: E is orientable if and only if $w_1(E) = 0$.

Exercise 16.34. Back in Section 15 we showed that all real bundles on S^3 are trivializable. Reprove this using the isomorphisms $\mathrm{Vect}_n(S^3) \cong [S^3, BO_n]$ and facts you know about the spaces BO_n for n small. Exercise 16.33 might also be helpful.

Exercise 16.35. Suppose $f \colon G \to H$ is a group homomorphism. If $E \to X$ is a principal G-bundle prove that $EG \times_G H \to X$ is a principal H-bundle. There is an induced map of spaces $Bf \colon BG \to BH$ making the diagram

$$\begin{array}{ccc} [X, BG] & \xrightarrow{\cong} & \mathrm{PrBund}_G(X) \\ \downarrow & & \downarrow \\ [X, BH] & \xrightarrow{\cong} & \mathrm{PrBund}_H(X) \end{array}$$

commute. For any $g \in G$, if $c_g \colon G \to G$ denotes the conjugation automorphism prove that $Bc_g \simeq \mathrm{id}$.

Exercise 16.36. Define a map $O_k \times O_m \to O_{k+m}$ by $(A, B) \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. This is a group homomorphism, so it induces $BO_k \times BO_m = B(O_k \times O_m) \to BO_{k+m}$. Explain why

$$\begin{array}{ccc} [X, BO_k] \times [X, BO_m] & \longrightarrow & [X, BO_{k+m}] \\ \cong \downarrow & & \cong \downarrow \\ \mathrm{Vect}_k(X) \times \mathrm{Vect}_m(X) & \xrightarrow{\oplus} & \mathrm{Vect}_{k+m}(X) \end{array}$$

is commuative.

Part 4. K-theory in topology

At this point in our story it is easy enough to write down the topological K-groups: for X compact and Hausdorff $KO(X)$ is simply the Grothendieck of real vector bundles on X, and likewise $KU(X)$ (more commonly just written $K(X)$) is the Grothendieck group of complex vector bundles on X. These groups would probably not be so interesting except for the fact that they extend to full-fledged generalized cohomology theories. That fact leads to them being extremely computable, and—this is less direct—they have interesting connections to geometry. In the next several sections we will develop these cohomology theories, explore their basic properties, and start learning to compute with them.

17. Topological K-theory

For a compact and Hausdorff space X, let $KO(X)$ denote the Grothendieck group of (finite rank) real vector bundles over X. Swan's Theorem gives that $KO(X) \cong K_{\text{alg}}(C(X))$, where the latter denotes the Grothendieck group of finitely-generated projectives. We can repeat this definition for both complex and quaternionic bundles, to define groups $KU(X)$ and $KSp(X)$, respectively. The group $KU(X)$ is most commonly just written as $K(X)$. In this section we start to develop the general theory of these groups, mostly concentrating on $KO(X)$ because the story is very analogous in the three cases.

The main point is that $KO(X)$ is the 0th group in a generalized cohomology theory, and likewise for $K(X)$ and $KSp(X)$. We will explain the construction of these theories, though some key elements will be postponed until later.

17.1. **Initial observations on KO.** Until we explicitly mention otherwise, all spaces in this section are assumed to be compact and Hausdorff.

Observe that $KO(-)$ is a contravariant functor: if $f\colon X \to Y$ then $f^*\colon KO(Y) \to KO(X)$ sends $[E]$ to $[f^*E]$. In particular, the squash map $p\colon X \to *$ yields a split-inclusion $p^*\colon KO(*) \to KO(X)$, where the splitting is induced by any choice of basepoint in X. One has $KO(*) \cong \mathbb{Z}$, so $\mathbb{Z}$ is a direct summand of $KO(X)$. To analyze the complement we can take two different approaches:

Definition 17.2. *For $x \in X$ let $\widetilde{KO}(X,x) = \ker[KO(X) \xrightarrow{i^*} KO(x)]$ where $i\colon \{x\} \hookrightarrow X$. Further, define $KO^{st}(X) = KO(X)/p^*KO(*)$.*

The group $\widetilde{KO}(X,x)$ is called the **reduced KO-group** of the pointed space X. We call $KO^{st}(X)$ the **Grothendieck group of stable vector bundles on X**. The reason for the latter terminology will be clear momentarily. These two groups are isomorphic; algebraically, this is coming from the split-exact sequence

$$0 \longrightarrow KO(*) \xrightarrow{p^*} KO(X) \longrightarrow KO^{st}(X) \longrightarrow 0.$$

If $i\colon \{x\} \hookrightarrow X$ is the inclusion then i^* is a splitting for the first map in the sequence. One gets an isomorphism between $KO^{st}(X)$ and $\ker i^*$ in the evident way, by sending a class $[E]$ to $[E] - p^*i^*[E]$. This isomorphism is used so frequently that it is worth recording more visibly:

$$KO^{st}(X) \cong \widetilde{KO}(X,x) \text{ via } [E] \mapsto [E] - [\underline{\text{rank}_x(E)}]. \tag{17.3}$$

Remark 17.4. Both $KO^{st}(X)$ and $\widetilde{KO}(X,x)$ appear often in algebraic topology, and topologists are somewhat cavalier about mixing them up. We give here one example where this can cause confusion.

Tensor product of bundles makes $KO(X)$ into a ring, via the formula $[E]\cdot[F] = [E\otimes F]$ and extending linearly. Then $\widetilde{KO}(X,x)$ is an ideal of this ring. Therefore $KO^{st}(X)$ may be given a product via the above isomorphism, but this product is *not* $[E]\cdot[F] = [E\otimes F]$. Indeed, it is clear that this definition would not be invariant under $E \mapsto E\oplus 1$. The product on $KO^{st}(X)$ is instead $[E]\cdot[F] = [E\otimes F] - (\operatorname{rank} E)[F] - (\operatorname{rank} F)[E] + (\operatorname{rank} E)(\operatorname{rank} F)$.

We offer the following alternative description of $KO^{st}(X)$. Let $\mathrm{Vect}(X)$ be the set of isomorphism classes of vector bundles on X, and impose the equivalence relation $E \simeq E\oplus \underline{1}$ for every vector bundle E. The set of equivalence classes is obviously a monoid under direct sum, but it is actually more than a monoid: it is a group. To see this, recall that if E is any vector bundle over X then there exists an embedding $E \hookrightarrow \underline{N}$ for sufficiently large N (Proposition 12.10). If Q is the quotient then we have the exact sequence $0\to E\to \underline{N}\to Q\to 0$, which is split by Proposition 12.2. So $E\oplus Q \cong \underline{N}$. Yet $\underline{N} = 0$ under our equivalence relation, and so E has an additive inverse. It is easy to see that $KO^{st}(X)$ is precisely this group of equivalence classes.

Finally, here is a third description of $KO^{st}(X)$. Consider the chain of maps

$$\mathrm{Vect}_0(X) \xrightarrow{\oplus 1} \mathrm{Vect}_1(X) \xrightarrow{\oplus 1} \mathrm{Vect}_2(X) \xrightarrow{\oplus 1} \cdots$$

When X is path-connected the colimit is the set of equivalence classes described in the preceding paragraph, and therefore coincides with $KO^{st}(X)$. Note that if X were not path connected then we would only be getting the monoid of vector bundles of constant rank on X. Recall from Section 16 that $\mathrm{Vect}_n(X) = [X, \mathrm{Gr}_n(\mathbb{R}^\infty)]$, and that the $\oplus 1$ map is represented by the map of spaces

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_{n+1}(\mathbb{R}\oplus\mathbb{R}^\infty) = \mathrm{Gr}_{n+1}(\mathbb{R}^\infty)$$

that sends a subspace $W\subseteq\mathbb{R}^\infty$ to $\mathbb{R}\oplus W\subseteq \mathbb{R}\oplus\mathbb{R}^\infty$. Let $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ denote the colimit of these maps

$$\mathrm{Gr}_1(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \mathrm{Gr}_2(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \mathrm{Gr}_3(\mathbb{R}^\infty) \xrightarrow{\oplus 1} \cdots$$

(in some sense we really want the homotopy colimit, but in this case the colimit has the same homotopy type and is good enough). Recall from Section 11.20 that $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is a model for BO_n. It follows that likewise $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ is a model for BO, where O is the infinite orthogonal group $O = \operatorname{colim}_n O_n$. In fact, the above direct limit system of Grassmannians is (up to homotopy) the same as the system

$$BO_1 \to BO_2 \to BO_3 \to \cdots$$

where each stage is the map induced by inclusion $O_n \hookrightarrow O_{n+1}$. We will go back and forth between writing $\mathrm{Gr}_\infty(\mathbb{R}^\infty)$ and BO, almost indiscriminately.

Then for compact spaces X we have a bijection

$$\operatorname*{colim}_n [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \longrightarrow [X, \mathrm{Gr}_\infty(\mathbb{R}^\infty)].$$

So we have learned that $KO^{st}(X) \simeq [X, BO]$ (set bijection) when X is compact, Hausdorff, and path-connected. The bijection sends a class represented by a rank n bundle E to $X \xrightarrow{f_E} BO_n \to BO$ where the f_E is a classifying map for E.

In fact, BO has an H-space structure which makes $[X, BO]$ into a monoid and the above bijection into an isomorphism of monoids. Elements of O can be thought of as infinite matrices that are equal to the identity in the tails. Consider the map $O \times O \to O$ that sends (A, B) to the matrix that has A placed in all the even coordinates and B placed in all the odd coordinates. This is a group homomorphism, so it induces $BO \times BO \to BO$. The map $KO^{st}(X) \to [X, BO]$ is then a map of monoids, using Exercise 16.36.

If X has a basepoint then we can consider $[X, BO]_*$ instead of $[X, BO]$. There is the evident map $[X, BO]_* \to [X, BO]$. Typically there would be no reason for this to be a bijection, but BO is a path-connected H-space: and in that setting the map *is* a bijection. So in fact we can write our group isomorphism as

$$KO^{st}(X) \cong [X, BO]_* \qquad (X \text{ path-connected}).$$

Applying this in particular to $X = S^k$ we have that for $k \geq 1$

$$\begin{aligned}\widetilde{KO}(S^k) \cong KO^{st}(S^k) \cong [S^k, BO] \cong [S^k, BO]_* = \pi_k(BO) &= \pi_{k-1}(\Omega BO) \\ &= \pi_{k-1}(O).\end{aligned}$$

Bott's calculations of $\pi_* O$ (Section 15) therefore give us the values of $\widetilde{KO}(S^k)$ when $k \geq 1$. For $k = 0$ observe that $KO(S^0) = KO(* \sqcup *) \cong \mathbb{Z} \oplus \mathbb{Z}$, so we have $\widetilde{KO}(S^0) \cong \mathbb{Z}$. This lets us fill out the table:

TABLE 17.4. Reduced KO-theory of spheres

k	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\widetilde{KO}(S^k)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\cdots$

Exercise 17.5. In $KO^{st}(X)$ we have thrown away the rank invariant of bundles, since we identify E with $E \oplus 1$. To put that invariant back in we can write our isomorphism for path-connected X as

$$KO(X) \xrightarrow{\cong} [X, \mathbb{Z} \times BO] = [X_+, \mathbb{Z} \times BO]_*, \qquad E \mapsto (\operatorname{rank} E, [E]^{st}).$$

That is, $[E] \in KO(X)$ is sent to the map $X \xrightarrow{f_E} BO = \{\operatorname{rank} E\} \times BO \hookrightarrow \mathbb{Z} \times BO$. Check that this isomorphism (suitably interpreted) works just as well for spaces X that are not path-connected. Here $[X, \mathbb{Z} \times BO]$ is a group by extending the H-space structure on BO to one on $\mathbb{Z} \times BO$ in the evident way, so that $(\{a\} \times BO) \times (\{b\} \times BO)$ maps to $\{a + b\} \times BO$.

Now we come to the miracle in the story. Bott's calculations didn't just show that $\pi_* O$ was 8-fold periodic, they gave an explicit weak equivalence $O \simeq \Omega^8 O$. Even more, they showed that $\Omega^7 O \simeq \mathbb{Z} \times BO$. Putting these two together, the most convenient version of the result is the following:

Theorem 17.6 (Bott Periodicity, Strong Version)**.** *There is map of H-spaces $\mathbb{Z} \times BO \to \Omega^8(\mathbb{Z} \times BO)$ that is a weak homotopy equivalence (the H-space structure on the target is the loop-space structure).*

If we now write down the periodic sequence of spaces

$$\mathbb{Z} \times BO,\ \Omega^7(\mathbb{Z} \times BO),\ \Omega^6(\mathbb{Z} \times BO)\ \ldots, \mathbb{Z} \times BO,\ \Omega^7(\mathbb{Z} \times BO),\ \ldots$$

this is an infinite loop space. The resulting spectrum is called the KO-spectrum, and this leads to the definition

$$KO^0(X) = \mathrm{Ho}\,\mathcal{T}_*(X_+, \mathbb{Z} \times BO) = \mathrm{Ho}\,\mathcal{T}(X, \mathbb{Z} \times BO),$$

for arbitrary (not necessarily compact) spaces X. For a pointed space X we have $\widetilde{KO}^0(X) = \mathrm{Ho}\,\mathcal{T}_*(X, \mathbb{Z} \times BO)$. We of course also have groups $KO^k(X)$ for all $k \in \mathbb{Z}$, defined in terms of maps to the above list of spaces. Precisely, $KO^k(X) = \mathrm{Ho}\,\mathcal{T}_*(X, \Omega^r(\mathbb{Z} \times BO))$ where $r \in \{0, \ldots, 7\}$ is chosen so that 8 divides $k + r$.

Note that the periodicity in our infinite loop space gives the periodicity formula

$$KO^{k+8}(X) \cong KO^k(X)$$

for all $k \in \mathbb{Z}$ and all spaces X.

Proposition 17.7. *Suppose that X is compact, Hausdorff, and semi-cofibrant. Then there is an isomorphism of groups $KO^0(X) \cong KO(X)$*

Proof. Since X is semi-cofibrant we have $KO^0(X) = \mathrm{Ho}\,\mathcal{T}(X, \mathbb{Z} \times BO) = [X, \mathbb{Z} \times BO]$, and we saw in Exercise 17.5 that for X compact Hausdorff this is isomorphic to $KO(X)$. There is one subtlety here: the group structure on $KO^0(X)$ is the one that comes from the loop-space structure on $\mathbb{Z} \times BO$, whereas for Exercise 17.5 we want the one coming from the H-space structure on $\mathbb{Z} \times BO$ that corresponds to direct sum of bundles. The two coincide because the Bott equivalence $\mathbb{Z} \times BO \to \Omega^8(\mathbb{Z} \times BO)$ is an H-space map. □

17.8. **The complex and quaternionic cases.** In the complex case, Bott obtained the two-fold periodicity $\pi_* U \cong \pi_{*+2} U$, coming from an equivalence $\Omega^2 U \simeq U$. He also proved that $\Omega U \simeq \mathbb{Z} \times BU$, so that complex Bott periodicity is ofted stated as

Theorem 17.9 (Complex Bott Periodicity, Strong Version)**.** *There is a map of H-spaces $\mathbb{Z} \times BU \to \Omega^2(\mathbb{Z} \times BU)$ that is a weak homotopy equivalence.*

Note that $\Omega(\mathbb{Z} \times BU) = \Omega BU \simeq U$, so the above Ω^2 statement is really *equivalent* to having a weak equivalence of H-spaces $\Omega U \simeq \mathbb{Z} \times BU$.

Complex Bott periodicity lets us write down the infinite loop space

$$\mathbb{Z} \times BU,\ U,\ \mathbb{Z} \times BU,\ U\ \ \ldots$$

The resulting spectrum is denoted K (occasionally KU) and called the **complex K-theory spectrum**. The associated cohomology theory is given by

$$K^{2n}(X) = \mathrm{Ho}\,\mathcal{T}_*(X_+, \mathbb{Z} \times BU) = \mathrm{Ho}\,\mathcal{T}(X, \mathbb{Z} \times BU),$$
$$K^{2n+1}(X) = \mathrm{Ho}\,\mathcal{T}_*(X_+, U) = \mathrm{Ho}\,\mathcal{T}(X, U).$$

Here periodicity is $K^n(X) \cong K^{n+2}(X)$, for all $n \in \mathbb{Z}$. When X is compact, Hausdorff, and semi-cofibrant we have $K^0(X) \cong K(X)$.

Finally, we can tell a similar story for quaternionic K-theory based on the symplectic group Sp in place of the groups O and U. But it turns out that this is actually the same as the real story. Bott proved that

$$\mathbb{Z} \times BSp \simeq \Omega^4(\mathbb{Z} \times BO)$$

and so the loop spaces of $\mathbb{Z} \times BSp$ are the same as the loop spaces of $\mathbb{Z} \times BO$ but with a 4-fold shift in the indexing. If one writes down the symplectic K-theory spectrum KSp in the evident way, it just turns out that $KSp \simeq \Sigma^4 KO$. Or in terms of cohomology theories,

$$KSp^*(X) \cong KO^{*+4}(X).$$

So we actually don't need symplectic K-theory, since the information is already there in KO.

17.10. **A first computation.** The point of this section was to construct the cohomology theories KO and K, having the properties that when X is compact Hausdorff and semi-cofibrant the groups $KO^0(X)$ and $K^0(X)$ coincide with the Grothendieck groups of real and complex vector bundles over X. We have now accomplished this! We will spend the rest of these notes exploring what one can do with such cohomology theories, i.e., what they are good for. We have already said that one thing they are good for is calculation. So let us give a first example demonstrating the benefits and limitations here.

Let us try to compute $KO(\mathbb{R}P^2)$. Start with the ubiquitous decomposition $KO(\mathbb{R}P^2) = \mathbb{Z} \oplus \widetilde{KO}(\mathbb{R}P^2) = \mathbb{Z} \oplus KO^{st}(\mathbb{R}P^2)$. Next use the fact that $\mathbb{R}P^2$ can be built by attaching a 2-cell to $\mathbb{R}P^1 = S^1$, where the attaching map wraps S^1 around itself twice. That is, $\mathbb{R}P^2$ is the mapping cone for $S^1 \xrightarrow{2} S^1$. The Puppe sequence for this map looks like

$$S^1 \xrightarrow{2} S^1 \longrightarrow \mathbb{R}P^2 \longrightarrow S^2 \xrightarrow{2} S^2 \longrightarrow \cdots$$

hence we have an exact sequence

$$\leftarrow \cdots \leftarrow \widetilde{KO}(S^1) \leftarrow \widetilde{KO}(S^1) \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \widetilde{KO}(S^2) \leftarrow \widetilde{KO}(S^2) \leftarrow \cdots$$

Note that this is just the long exact sequence for the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$ in $\widetilde{KO}$-cohomology, where we are using the identification $\widetilde{KO}(S^2) = \widetilde{KO}^0(S^2) = \widetilde{KO}^{-1}(S^1)$.

We know $\widetilde{KO}(S^k)$ for all $k \geq 0$, so the above sequence becomes

$$\mathbb{Z}/2 \leftarrow \mathbb{Z}/2 \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \mathbb{Z}/2 \leftarrow \mathbb{Z}/2$$

where both maps $\mathbb{Z}/2 \leftarrow \mathbb{Z}/2$ are multiplication by 2, i.e. the 0 map. Hence we have a short exact sequence

$$0 \leftarrow \mathbb{Z}/2 \leftarrow \widetilde{KO}(\mathbb{R}P^2) \leftarrow \mathbb{Z}/2 \leftarrow 0, \tag{17.11}$$

and so $\widetilde{KO}(\mathbb{R}P^2)$ is either $(\mathbb{Z}/2)^2$ or $\mathbb{Z}/4$. It remains to decide which one.

The short exact sequence in (17.11) is really

$$0 \longleftarrow \widetilde{KO}(S^1) \xleftarrow{i^*} \widetilde{KO}(\mathbb{R}P^2) \xleftarrow{p^*} \widetilde{KO}(S^2) \longleftarrow 0$$

where $i\colon S^1 \hookrightarrow \mathbb{R}P^2$ is the inclusion of the 1-skeleton and $p\colon \mathbb{R}P^2 \to S^2$ is the map that squashes the 1-skeleton to a point. We have previously seen that the generator of $\widetilde{KO}^0(S^1) = KO^{st}(S^1)$ corresponds to the Mobius bundle $[M]$, and the generator of $\widetilde{KO}(S^2) = KO^{st}(S^2)$ is $[\mathcal{O}(1)]$, the rank 2 bundle whose clutching map is the isomomorphism $S^1 \to SO(2)$. The image of $[\mathcal{O}(1)]$ in $\widetilde{KO}^0(\mathbb{R}P^2)$ is $p^*\mathcal{O}(1)$.

We happen to know one bundle on $\mathbb{R}P^2$, the tautological line bundle γ. When we restrict γ to $\mathbb{R}P^1$ we get M, and so $[\gamma]$ is a preimage for $[M]$ under i^*. We need

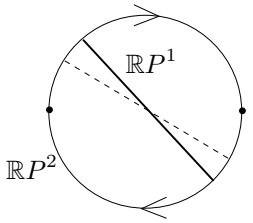
to decide if $2[\gamma] = 0$ in $KO^{st}(\mathbb{R}P^2)$; if it is, then $\widetilde{KO}(\mathbb{R}P^2) \cong (\mathbb{Z}/2)^2$ and if it is not then $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$. So the question becomes: is $\gamma \oplus \gamma$ stably trivial?

The answer turns out to be that $\gamma \oplus \gamma$ is *not* stably trivial; this is an elementary exercise using characteristic classes (Stiefel-Whitney classes), but we have not discussed such techniques yet—see Section 38.10 below for complete details. For now we will just accept this fact, and conclude that $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$. Note that this calculation demonstrates an important principle to keep in mind: often the machinery of cohomology theories get you a long way, but not quite to the end, and one has to do some geometry to complete the calculation.

There is a better way to think about this calculation, and we can't resist pointing it out even though it won't make complete sense yet. But it ties in to intersection theory, which is our overarching theme in these notes. In our discussion above we used $KO^{st}(\mathbb{R}P^2)$ as our model for $\widetilde{KO}(\mathbb{R}P^2)$, but let us change perspective and use the model that is the kernel of $KO(\mathbb{R}P^2) \to KO(*)$, for some chosen basepoint. Recall that $[E]$ in $KO^{st}(\mathbb{R}P^2)$ corresponds to $[E] - \mathrm{rank}(E)$ in $\widetilde{KO}(\mathbb{R}P^2)$; so the class we wrote as $[\gamma]$ is $[\gamma] - 1$ in the shifted perspective, and we need to decide if $2([\gamma] - 1) = 0$ in $KO(\mathbb{R}P^2)$. The element $1 - [\gamma]$ should be thought of as corresponding to a chain complex of vector bundles

$$0 \to \gamma \to 1 \to 0,$$

and thinking of it this way one finds that it plays the role of the K-theoretic fundamental class of the submanifold $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$ (we will discuss these connections extensively in Part 5). Then $(1 - [\gamma])^2$ represents the self-intersection product of $\mathbb{R}P^1$ inside $\mathbb{R}P^2$, which we know is a point by the standard geometric argument (shown in the picture below, depicting an $\mathbb{R}P^1$ and a small perturbation of it)):



In particular, the self-intersection is not empty. This translates to the statement that $(1 - [\gamma])^2 \neq 0$. But

$$(1 - [\gamma])^2 = 1 - 2[\gamma] + [\gamma^2] = 1 - 2[\gamma] + 1 = 2(1 - [\gamma])$$

where we have used that $\gamma \otimes \gamma \cong \underline{1}$ (this holds for any real line bundle, see Exercise 16.31). So this explains why $2(1 - [\gamma]) \neq 0$. Again, we understand this argument doesn't make much sense yet. We will come back to arguments like this in Part 5. For the moment just get the idea that it is the intersection theory of submanifolds in $\mathbb{R}P^2$ that is ultimately forcing $\widetilde{KO}(\mathbb{R}P^2)$ to be $\mathbb{Z}/4$ rather than $(\mathbb{Z}/2)^2$. [This is a little bit of a lie. The connection to intersection theory works great for K^* and complex submanifolds, but for KO and real submanifolds it is more subtle than we are indicating here.]

Remark 17.12. It seems worth pointing out that in fact for every n one has $\widetilde{KO}(\mathbb{R}P^n) \cong \mathbb{Z}/2^k$ for a certain value k depending on n. See [D7] for this calculation.

Exercise 17.13. It is a good idea for the reader to try their hand at similar calculations, to see how the machinery is working. Try calculating some of the groups below, at least for small values of n:

- $K(\mathbb{C}P^n)$ (reasonably easy)
- $KO(\mathbb{C}P^n)$ (a little harder)
- $K(\mathbb{R}P^n)$ (even harder)
- $KO(\mathbb{R}P^n)$ (hardest).

Don't worry if you can't completely determine some of the groups; just see how far the machinery takes you.

17.14. ***K*-theory for non-compact spaces.** The reader will perhaps have noticed that for arbitrary spaces X we now have two competing definitions:

$$\begin{aligned} KO_{Grt}(X) &= \text{Grothendieck group of vector bundles over } X \\ KO^0(X) &= \mathrm{Ho}\,\mathcal{T}(X, \mathbb{Z} \times BO). \end{aligned}$$

In older literature the second group is sometimes called **representable *KO*-theory**. When X is compact, Hausdorff, and semi-cofibrant we have seen that these groups are isomorphic. What about more generally?

For X paracompact and Hausdorff there is a natural map $\Phi\colon KO_{Grt}(X) \to KO^0(X)$. To explain this we might as well assume that X is path-connected as well. Recall from Proposition 12.2 that all short exact sequences of vector bundles are split. From this it follows that $KO_{Grt}(X)$ is the group completion of the monoid $\mathrm{Vect}(X)$ of isomorphism classes of vector bundles. To a rank n vector bundle $E \to X$ we assign the pair $(n, j \circ f)$ where f is a classifying map $X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ and $j\colon \mathrm{Gr}_n(\mathbb{R}^\infty) \hookrightarrow \mathrm{Gr}_\infty(\mathbb{R}^\infty)$ is the standard inclusion. The map f involves a choice, but it is well-defined up to homotopy. So this assignment gives a map $\mathrm{Vect}(X) \to \mathrm{Ho}\,\mathcal{T}(X, \mathbb{Z} \times BO)$, which is readily seen to be a map of monoids. Since the target is a group it induces $KO_{Grt}(X) \to KO^0(X)$.

The natural transformation Φ extends to all spaces as follows. For any space X, choose a cofibrant replacement $QX \to X$ and define Φ_X to be the unique map that makes the following diagram commute:

$$\begin{array}{ccc} KO_{Grt}(X) & \overset{\Phi_X}{\dashrightarrow} & KO^0(X) \\ \downarrow & & \downarrow \cong \\ KO_{Grt}(QX) & \xrightarrow{\Phi_{QX}} & KO^0(QX). \end{array}$$

Note that Φ_{QX} was already defined because QX is paracompact and Hausdorff.

Exercise 17.15. Check that the definition of Φ_X does not depend on the choice of QX. (Hint: Use a factorization $\emptyset \rightarrowtail \tilde{X} \xrightarrow{\sim} X$).

We know that Φ is an isomorphism when X is compact Hausdorff and semi-cofibrant. We now endeavor to generalize this somewhat. The following definition

encapsulates what we need for a space to have a bundle theory that behaves like that of a compact Hausdorff space.

Definition 17.16. *A space is* ***bund-compact*** *if it is paracompact, Hausdorff, and homotopy equivalent to a compact Hausdorff space.*

Proposition 17.17. *Let X be a bund-compact space. Then the following are true:*

(a) Every vector bundle on X can be embedded as a subbundle of a finite rank trivial bundle.
(b) If $E \to X$ is a vector bundle then there exists a vector bundle $Q \to X$ such that $E \oplus Q$ is trivial.
(c) If E and F are vector bundles on X and $[E] = [F]$ in $K^0(X)$, then there is a vector bundle $Q \to X$ such that $E \oplus Q \cong F \oplus Q$. In fact, one can even insist that Q be trivial.
(d) If X is semi-cofibrant then the map $\Phi_X \colon KO_{Grt}(X) \to KO^0(X)$ is an isomorphism.

Proof. We leave (a)–(c) as an exercise for the reader. For (d), the hypothesis implies there is a compact Hausdorff space W and a homotopy equivalence $f \colon W \to X$. Since X is semi-cofibrant, W is also semi-cofibrant. Consider the square

$$\begin{array}{ccc} KO_{Grt}(X) & \xrightarrow{\Phi_X} & KO^0(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ KO_{Grt}(W) & \xrightarrow{\Phi_W} & KO^0(W). \end{array}$$

By Corollary 14.2(b) the map $f^* \colon \mathrm{Vect}_n(X) \to \mathrm{Vect}_n(W)$ is a bijection for every n. Since f^* preserves direct sums it therefore induces an isomorphism of monoids $\mathrm{Vect}(X) \to \mathrm{Vect}(W)$, and therefore an isomorphism on the group completions $KO_{Grt}(X) \to KO_{Grt}(W)$. The right f^* is an isomorphism by definition of KO^0, since f is a weak equivalence. The map Φ_W is an isomorphism, therefore Φ_X is an isomorphism. □

Exercise 17.18. Prove (a)–(c) of the above proposition.

The notion of bund-compactness is still a little too strong to be all that useful, mostly because it involves homotopy equivalence rather than weak equivalence. The following is a different notion that is a little more robust:

Definition 17.19. *A space is* ***homotopically compact*** *if it is weakly equivalent to a finite cell complex. A pair (X, A) is* ***homotopically compact*** *is there is a finite CW-complex Y and subcomplex B for which there is a zig-zag of weak equivalences between (X, A) and (Y, B).*

As a few examples, $\mathbb{R}^n$ is homotopically compact but not compact. The space $\prod_{i=1}^{\infty} S^2$ is compact Hausdorff (by the Tychonoff theorem), hence also bund-compact, but not homotopically compact—this space has $\prod_{i=1}^{n} S^2$ as a retract for every n, and hence must have nonzero homology groups in infinitely many degrees. The following result shows that examples of homotopically compact spaces that are not bund-compact are mostly pathological (though they do exist, e.g., the long line).

Proposition 17.20. *Suppose X is paracompact, Hausdorff, semi-cofibrant, and homotopically compact. Then X is bund-compact.*

Proof. Since X is homotopically compact, X is weakly equivalent to a finite CW-complex W. This implies there is a map $W \to X$ that is a weak homotopy equivalence. Since X is semi-cofibrant, this is in fact a homotopy equivalence. □

The map $\Phi_X \colon KO_{Grt}(X) \to KO^0(X)$ is in general neither injective nor surjective. We will explain an example where the latter fails. For this example it will be convenient to use K rather than KO, and we will have to use some calculations that have not been explained yet. But the example is still informative, even with not all details justified.

Example 17.21. We claim that the map $K_{Grt}(\mathbb{R}P^\infty) \to K^0(\mathbb{R}P^\infty)$ is not surjective. Let $L \to \mathbb{R}P^\infty$ be the tautological bundle, and let $J = cL$ be its complexification (Exercise 17.25). With some trouble one can completely analyze $\mathrm{Vect}_n^{\mathbb{C}}(\mathbb{R}P^\infty) = [\mathbb{R}P^\infty, BU(n)]$, and one finds that it consists of the bundles $rJ \oplus (\underline{n-r})$ for $0 \leq r \leq n$ (we will explain this below). Consequently, $K_{Grt}(\mathbb{R}P^\infty) = \mathbb{Z} \oplus \mathbb{Z}$, with the summands generated by $\underline{1}$ and $[J]$. In contrast, we claim that $K^0(\mathbb{R}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}_2^\wedge$ where the second summand denotes the 2-adic integers. The completion is a phenomenon that often arises when dealing with homotopy classes of maps out of infinite complexes, because of the Milnor sequence. Using the standard skeletal filtration of $\mathbb{R}P^\infty$ by finite projective spaces, one has the associated Milnor exact sequence

$$0 \longrightarrow \lim_n{}^1 K^{-1}(\mathbb{R}P^n) \longrightarrow \tilde{K}^0(\mathbb{R}P^\infty) \longrightarrow \lim_n \tilde{K}^0(\mathbb{R}P^n) \longrightarrow 0$$

(see Section 9.17). One can compute that $K^{-1}(\mathbb{R}P^n)$ is 0 when n is even and $\mathbb{Z}$ when n is odd (see [D7, Section 6]). The Mittag-Leffler condition then shows that the $\lim^1$ vanishes.

In fact the standard computations (again see [D7]) also yield that the $\widetilde{K}^0(\mathbb{R}P^n)$ groups are of the form $\mathbb{Z}/(2^{??})$ with the exponents increasing with n, and that as n varies the induced maps are surjections. So one obtains $\mathbb{Z}_2^\wedge$ for the inverse limit. Therefore $K^0(\mathbb{R}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}_2^\wedge$, as claimed. So the map $K_{Grt}(\mathbb{R}P^\infty) \to K^0(\mathbb{R}P^\infty)$ cannot be surjective (in fact the domain is countable and the target is uncountable).

Elements of this example can be generalized to BG, where G is a finite group. Let $\mathrm{Rep}_n(G)$ denote the set of isomorphism classes of n-dimensional complex representations of G, which by elementary algebra is in bijective correspondence with the set of isomorphism classes of n-dimensional unitary representations of G. Let $\mathrm{Rep}(G)$ be the complex representation ring of G, which is the group completion of the commutative monoid $(\amalg_{n \geq 0} \mathrm{Rep}_n(G), \oplus)$.

The set of group homomorphisms $\mathrm{Hom}(G, U_n)$ has an action of U_n by conjugation, and we have a natural bijection $\mathrm{Hom}(G, U_n)/\mathrm{conjug.} \xrightarrow{\cong} \mathrm{Rep}_n(G)$. Quotienting by conjugacy is tantamount to taking isomorphism classes on the right. Every homomorphism $G \to U_n$ also yields an induced map $BG \to BU_n$, and conjugate homomorphisms induced homotopic maps by Exercise 16.35. So we obtain

the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(G, U_n)/\text{conjug.} & \xrightarrow{\cong} & \mathrm{Rep}_n(G) \\ \downarrow & & \downarrow \\ [BG, BU_n] & \xrightarrow{\cong} & \mathrm{Vect}_n(BG) \end{array}$$

where the right vertical map sends a representation V to the vector bundle $EG \times_G V \to BG$. The maps on the right assemble to induced a map of abelian groups (in fact, rings) $\mathrm{Rep}(G) \to K_{Grt}(BG)$.

Dwyer-Zabrodsky [DZ, Theorem 1.1] show that the left vertical map is an isomorphism whenever G is a p-group (in fact they prove this result with U_n replaced by any compact connected Lie group). Thus, the right vertical map is an isomorphism as well. Since this holds for all n, we also get that $\mathrm{Rep}(G) \to K_{Grt}(BG)$ is an isomorphism.

In particular, when $G = \mathbb{Z}/2$ there are exactly two irreducible complex representations—the trivial and sign representations on $\mathbb{C}$—and we find that $\mathrm{Vect}_n(\mathbb{R}P^\infty) \cong \mathrm{Rep}_n(\mathbb{Z}/2) = \{(n-k)[\mathbb{C}_{triv}] + k[\mathbb{C}_{sign}] \,|\, 0 \le k \le n\}$. And $K_{Grt}(\mathbb{R}P^\infty) \cong \mathrm{Rep}(\mathbb{Z}/2) \cong \mathbb{Z}^2$.

Let $I(G)$ be the kernel of the dimension map $\dim\colon \mathrm{Rep}(G) \to \mathbb{Z}$, which is typically called the **augmentation ideal**. The Atiyah-Segal completion theorem [ASe] says that $K^0(BG) = \mathrm{Rep}(G)^\wedge_I$. We get the diagram

$$\begin{array}{ccc} \mathrm{Rep}(G) & \longrightarrow & \mathrm{Rep}(G)^\wedge_I \\ \downarrow & & \downarrow \cong \\ K_{Grt}(BG) & \xrightarrow{\Phi} & K^0(BG). \end{array}$$

When G is a p-group the left vertical map is an isomorphism by [DZ], the top horizontal map is injective [Atiyah1, Lemma 6.9], and so the bottom horizontal map is injective as well (this argument is from [JO, Corollary 1.9]).

Let us explore what the above says when $G = \mathbb{Z}/2$. As we have mentioned, here there are two irreducible representations, namely the trivial and sign representations on $\mathbb{C}$. If we denote the latter as x then $\mathrm{Rep}(G) = \mathbb{Z}[x]/(x^2-1)$. The augmentation ideal is $I = (x-1)$. If we set $R = \mathrm{Rep}(G)$ then $\mathrm{Rep}(G)^\wedge_I$ is the inverse limit of

$$\cdots \twoheadrightarrow R/I^3 \twoheadrightarrow R/I^2 \twoheadrightarrow R/I.$$

One quickly finds that $R/I = \mathbb{Z}$ (generated by 1), $R/I^2 = \mathbb{Z} \oplus \mathbb{Z}/2$ where the first summand is generated by 1 and the second by $x-1$, and $R/I^3 = \mathbb{Z} \oplus \mathbb{Z}/4$ with the same generators. To identify $R/I^n = \mathbb{Z}[x]/(x^2-1, (x-1)^n)$ we use the division algorithm in $\mathbb{Z}[x]$ to write

$$(x-1)^n = (x^2-1)f(x) + (\text{linear polynomial}).$$

Since $x-1$ divides the first two terms the linear polynomial must have the form $k(x-1)$ for some $k \in \mathbb{Z}$. Divide by $x-1$ to get

$$(x-1)^{n-1} = (x+1)f(x) + k,$$

and then obtain k by plugging in $x = -1$: so $k = (-2)^{n-1}$. Thus one finds that $R/I^n = \mathbb{Z}[x]/(x^2-1, 2^{n-1}(x-1)) \cong \mathbb{Z}[x]/(x-1) \oplus \mathbb{Z}[x]/(x+1, 2^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^{n-1}$,

with the first summand generated by 1 and the second by $x-1$. The completion $R_I^\wedge$ is therefore $\mathbb{Z} \oplus \mathbb{Z}_2^\wedge$, as expected.

17.22. **Connective K-theory.** If X is a path-connected space recall the nth Postnikov section P_nX, obtained from X by inductively attaching cells to kill off all homotopy groups in degrees larger than n. This comes with a map $X \to P_nX$, and the space P_nX is well-defined up to weak equiavlence. Let W_nX be the homotopy fiber of $X \to P_nX$. This is called the ***n*-connected cover** of X. It comes with a map $W_nX \to X$ that is an isomorphism on π_i for $i > n$, and $\pi_i(W_nX) = 0$ for $i \leq n$. Note that W_1X is just the universal covering space of X (up to homotopy). By convention we set $W_{-1}X = X$. If X is not path-connected we let W_0X be the component of X containing a fixed basepoint and $W_n(X) = W_n(W_0X)$ for $n > 0$.

In $\mathcal{S}pectra$ we can make similar constructions, but now for any $n \in \mathbb{Z}$. The spectrum $W_{-1}E$ is called the **connective cover** of E. The map $W_{-1}E \to E$ is an isomorphism on non-negative homotopy groups, and $W_{-1}E$ has vanishing homotopy groups in negative degrees.

Definition 17.23. *The **connective K-theory spectra** are $ko = W_{-1}KO$ and $ku = W_{-1}K$ (some authors write bo and bu instead of ko and ku).*

If one prefers to think of these in terms of space-level constructions, that is easy enough. There are canonical (up to homotopy) weak equivalences $W_{n-1}(\Omega X) \simeq \Omega(W_nX)$. So if $\{Z_i\}$ is an infinite loop space then $\{W_{i-1}Z_i\}$ is also an infinite loop-space, since we have

$$W_{k-1}Z_k \xrightarrow{\simeq} W_{k-1}(\Omega Z_{k+1}) \simeq \Omega(W_kZ_{k+1}).$$

This is the "connective cover" of our infinite loop space. Note that the 0th space is still Z_0, but that the nth space in the connective cover is $(n-1)$-connected.

Exercise 17.24. Let E be a spectrum and let $J \to E$ be its connective cover. Explain why the maps $J^k(X) \to E^k(X)$ are isomorphisms for any space X and any $k \leq 0$. If X is a finite-dimensional CW-complex, prove that $J^k(X) = 0$ for $k > \dim X$.

Whereas the spaces in the Ω-spectra for KO and K are (by now) familiar friends, the spaces in the Ω-spectra for ko and k are difficult to describe other than via the abstract constructions above—and they are not periodic, since they become more and more highly connected. So from this perspective the connective covers seem much less natural than KO and K. Why do we care about them?

In some sense KO and K were discovered before they had a right to be, due to the miracle of Bott periodicity. The constructions we have given feel jury-rigged, and can be frustrating to work with (see for example the discussion of products in Section 19). One of the goals of the field of algebraic K-theory has been to find more natural constructions, but it turns out that these algebraic constructions tend to produce ko and k rather than their periodic analogs. Perhaps this is not so surprising, as one would expect to discover the ring $\mathbb{Z}[x]$ before $\mathbb{Z}[x, x^{-1}]$. In modern approaches KO and K are usually constructed by first starting with ko and k and then inverting the Bott elements.

Another reason homotopy theorists care about ko and k is that in some situations they have computational advantages over the periodic versions. We will not pursue this, though.

In this book we will not need connective K-theory, but it is good to know that it exists and to understand the basic idea of what it is.

◦ Exercises ◦

Exercise 17.25.

(a) For any real vector space V, the complexification is defined to be $c(V) = \mathbb{C}\otimes_{\mathbb{R}} V$. Explain how to construct the complexification of a real vector bundle.
(b) Explain how to construct maps $\mathrm{Gr}_k(\mathbb{R}^N) \to \mathrm{Gr}_k(\mathbb{C}^N)$ so that, upon taking $N = \infty$, one obtains a map representing complexification $c\colon \mathrm{Vect}_k(X) \to \mathrm{Vect}_k^{\mathbb{C}}(X)$.
(c) Given a rank k complex vector bundle E, forgetting the complex structure yields a real vector bundle of rank $2k$ denoted $r_{\mathbb{R}}(E)$. Explain how to construct a map of Grassmannians that represents $r_{\mathbb{R}}\colon \mathrm{Vect}_k^{\mathbb{C}}(X) \to \mathrm{Vect}_{2k}(X)$.
(d) Explain the identities $r_{\mathbb{R}}c(E) \cong E \oplus E$ and $cr_{\mathbb{R}}(E) \cong E \oplus \bar{E}$, where $\bar{E}$ is the conjugate bundle.
(e) Produce group homomorphisms $O_k \to U_k$ and $U_k \to O_{2k}$ such that the associated maps of classifying spaces represent c and $r_{\mathbb{R}}$.
(f) Explain why there are induced maps $c\colon KO_{Grt}(X) \to K_{Grt}(X)$ and $r_{\mathbb{R}}\colon K_{Grt}(X) \to KO_{Grt}(X)$, that $r_{\mathbb{R}}c$ is multiplication by 2, and explain why c is a ring map.
(g) For arbitary X explain why we have maps c and $r_{\mathbb{R}}$ as in the previous part but with KO^0 and K^0, and why $r_{\mathbb{R}}c$ is multiplication by 2. (The fact that c is a ring homomorphism in this generality will have to wait for our discussion of products in Section 19.)

18. The space $\mathbb{Z} \times BU$

Understanding the spaces $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$ is absolutely fundamental to working with topological K-theory. The goal of this section is to help the reader become familiar with different geometric models of these spaces, and how to move back and forth between them. We do this through a series of guided exercises. For convenience we concentrate on the complex case, but the real case is identical.

Exercise 18.1. Let $\mathbb{C}_+^p$ denote the vector space with basis $e_1^+, \dots, e_p^+$, and $\mathbb{C}_-^q$ the vector space with basis $e_1^-, \dots, e_q^-$. Write $\mathbb{C}^{p,q} = \mathbb{C}_+^p \oplus \mathbb{C}_-^q$. Note the evident inclusions $\mathbb{C}^{p,q} \hookrightarrow \mathbb{C}^{p+1,q}$ and $\mathbb{C}^{p,q} \hookrightarrow \mathbb{C}^{p,q+1}$. At times it will be convenient to regard $\mathbb{C}^{p,q}$ as equipped with the Hermitian metric where $e_1^+, \dots, e_p^+, e_1^-, \dots, e_q^-$ is an orthonormal basis.

For any vector space V write $\mathrm{Gr}_\bullet(V) = \coprod_{k\geq 0}\mathrm{Gr}_k(V)$. Then for W finite-dimensional, consider the doubly-infinite array of inclusions

$$\begin{array}{ccccccc}
\mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{0,0}) & \overset{1}{\rightarrowtail} & \mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{1,0}) & \overset{1}{\rightarrowtail} & \mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{2,0}) & \overset{1}{\rightarrowtail} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{0,1}) & \overset{1}{\rightarrowtail} & \mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{1,1}) & \overset{1}{\rightarrowtail} & \mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{2,1}) & \overset{1}{\rightarrowtail} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{0,2}) & \overset{1}{\rightarrowtail} & \mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{1,2}) & \overset{1}{\rightarrowtail} & \mathrm{Gr}_\bullet(W\oplus\mathbb{C}^{2,2}) & \overset{1}{\rightarrowtail} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The vertical maps are induced by the inclusions $\mathbb{C}^{p,q}\hookrightarrow\mathbb{C}^{p,q+1}$ and the associated maps $\mathrm{Gr}_k(W\oplus\mathbb{C}^{p,q})\hookrightarrow\mathrm{Gr}_k(W\oplus\mathbb{C}^{p,q+1})$. The horizontal maps are a little different: they are induced by the inclusions $\mathbb{C}^{p,q}\hookrightarrow\mathbb{C}^{p+1,q}$ and the associated maps $\mathrm{Gr}_k(W\oplus\mathbb{C}^{p,q})\hookrightarrow\mathrm{Gr}_{k+1}(W\oplus\mathbb{C}^{p+1,q})$ that send a subspace N to $N\oplus\langle e^+_{p+1}\rangle$. The horizontal maps in the diagram are broken with the label "1" as a reminder that the index on the path components is shifted by 1.

So the vertical maps do not alter the subspace, but add a negative basis element to the ambient space. The horizontal maps add a positive basis element to both the ambient space and the subspace. Looked at a different way, the horizontal maps add a new positive basis element to a subspace W, whereas the vertical maps add a new negative basis element to the complement $W^\perp$.

Let $\mathcal{Z}(W)$ denote the colimit of the above diagram. We will mostly be interested in $\mathcal{Z}(0)$, but it is useful to consider the extra generality. By category theory, the colimit can be computed by first taking colimits along the columns and then taking the colimit of the resulting row. This identifies $\mathcal{Z}(W)$ with the colimit of

(18.2)
$$\mathrm{Gr}_\bullet(W\oplus\mathbb{C}^\infty_-)\overset{1}{\rightarrowtail}\mathrm{Gr}_\bullet(W\oplus\mathbb{C}^1_+\oplus\mathbb{C}^\infty_-)\overset{1}{\rightarrowtail}\mathrm{Gr}_\bullet(W\oplus\mathbb{C}^2_+\oplus\mathbb{C}^\infty_-)\overset{1}{\rightarrowtail}\cdots$$

(a) Think through the above for $\mathcal{Z}(0)$ and show that $\mathcal{Z}(0)$ is precisely the space $\mathbb{Z}\times\mathrm{Gr}_\infty(\mathbb{C}^\infty)$ introduced in Section 17.

(b) Check that assigning $V\subseteq\mathbb{C}^{p,q}$ the integer $\dim V - p$ gives a bijection $\pi_0(\mathcal{Z}(0))\cong\mathbb{Z}$. From now on we use this as our standard labeling of the path components.

(c) By choosing an isomorphism $W\cong\mathbb{C}^{n,0}$, identify the bigraded array for W with a cofinal subarray inside of the one for 0. Deduce that $\mathcal{Z}(W)\cong\mathcal{Z}(0)$.

(d) When $W\subseteq W'$ there is an evident map $\mathcal{Z}(W)\hookrightarrow\mathcal{Z}(W')$. Prove that this is a weak homotopy equivalence, for example via the following steps:
   (i) Prove it for $\mathcal{Z}(0)\hookrightarrow\mathcal{Z}(W)$ by using (18.2) and studying the inclusions $\mathrm{Gr}_k(\mathbb{C}^\infty)\hookrightarrow\mathrm{Gr}_k(W\oplus\mathbb{C}^\infty)$ (Corollary 16.14 is helpful).
   (ii) Deduce the general case by the two-out-of-three property.

   [Note: We will see in Exercise 18.9 below that $\mathcal{Z}(0)$ has the structure of a CW-complex, so in fact the maps $\mathcal{Z}(W)\hookrightarrow\mathcal{Z}(W')$ are homotopy equivalences not just weak homotopy equivalences.]

(e) Let $V_0 \subsetneq V_1 \subsetneq \cdots \subseteq \mathbb{C}^\infty$ be an infinite sequence of proper subspaces, and suppose given a colimit diagram

$$\mathrm{Gr}_{k_0}(V_0) \to \mathrm{Gr}_{k_1}(V_1) \to \mathrm{Gr}_{k_2}(V_2) \to \cdots$$

where each inclusion $\mathrm{Gr}_{k_j}(V_j) \hookrightarrow \mathrm{Gr}_{k_{j+1}}(V_{j+1})$ is induced by adding $k_{j+1} - k_j$ prescribed elements to the subspace $N \subseteq V_j$. Prove that if $\lim_j k_j = \infty$ and $\lim_j(\dim V_j - k_j) = \infty$ then the colimit is isomorphic to $\mathrm{Gr}_\infty(\mathbb{C}^\infty)$. [Hint: First do a change of basis to make the V_j's and the prescribed elements as nice as you can. Then try to find the sequence of Grassmannians as a cofinal subsequence of our infinite array.]

(f) For a finite-dimensional W prove that $\mathcal{Z}(W)$ is homeomorphic to the colimit

$$\mathrm{Gr}_\bullet(W) \hookrightarrow \mathrm{Gr}_\bullet(W \oplus \mathbb{C}^{1,1}) \hookrightarrow \mathrm{Gr}_\bullet(W \oplus \mathbb{C}^{2,2}) \hookrightarrow \cdots$$

where in each case the map sends a subspace $V \subseteq W \oplus \mathbb{C}^{n,n}$ to $V \oplus \langle e^+_{n+1} \rangle$. Check that assigning to $V \subseteq \mathbb{C}^{n,n}$ the number $\dim V + \dim W - n$ distinguishes the different path components of $\mathcal{Z}(W)$.

Remark 18.3. In the future we will write $\mathbb{Z} \times BU_{gr} = \mathcal{Z}(0)$ and $(\mathbb{Z} \times BU_{gr})(W) = \mathcal{Z}(W)$ for the spaces from the previous exercise. When we write $\mathbb{Z} \times BU$ the reader will usually be flexible to pick whatever model they want, or else a specific model should be clear from context. But typically the model $\mathbb{Z} \times BU_{gr}$ is the most convenient one to have in mind.

Exercise 18.4. Let X be compact Hausdorff, and (for convenience) connected. Then $[X, \mathbb{Z} \times BU_{gr}] = \mathrm{colim}_{p,q}[X, \mathrm{Gr}_\bullet(\mathbb{C}^p_+ \oplus \mathbb{C}^q_-)]$.

(a) Show that the assignment sending $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^p_+ \oplus \mathbb{C}^q_-)$ to $f^*\gamma_k - p \in K^0_{Grt}(X)$ is well-defined and gives an isomorphism $[X, \mathbb{Z} \times BU_{gr}] \to K^0_{Grt}(X)$. *Note: There is a sign choice in the definition of this map, as we could have just as well sent f to $p - f^*\gamma_k$. There are arguments in favor of each convention, but the one we have chosen seems to be a little more satisfying in the long run.*

(b) Define $[X, \mathbb{Z} \times BU_{gr}(W)] \to K^0_{Grt}(X)$ by sending $f\colon X \to \mathrm{Gr}_k(W \oplus \mathbb{C}^p_+ \oplus \mathbb{C}^q_-)$ to $f^*\gamma_k - p$. Prove that this is also an isomorphism, and that the following two diagrams commute:

$$\begin{array}{ccc} [X, \mathbb{Z} \times BU_{gr}(0)] & \longrightarrow & K^0_{Grt}(X) \\ \downarrow{\scriptstyle i_*} & & \| \\ [X, \mathbb{Z} \times BU_{gr}(W)] & \longrightarrow & K^0_{Grt}(X) \end{array} \qquad \begin{array}{ccc} [X, \mathbb{Z} \times BU_{gr}(0)] & \longrightarrow & K^0_{Grt}(X) \\ \downarrow{\scriptstyle +W} & & \downarrow{\scriptstyle +\underline{W}} \\ [X, \mathbb{Z} \times BU_{gr}(W)] & \longrightarrow & K^0_{Grt}(X). \end{array}$$

Here $i\colon \mathbb{Z} \times BU_{gr}(0) \to \mathbb{Z} \times BU_{gr}(W)$ is induced by the inclusions $\mathrm{Gr}_k(\mathbb{C}^{p,q}) \hookrightarrow \mathrm{Gr}_k(W \oplus \mathbb{C}^{p,q})$ whereas $+W\colon \mathbb{Z} \times BU_{gr}(0) \to \mathbb{Z} \times BU_{gr}(W)$ is induced by $\mathrm{Gr}_k(\mathbb{C}^{p,q}) \to \mathrm{Gr}_{k+|W|}(W \oplus \mathbb{C}^{p,q})$, $N \mapsto W \oplus N$.

Exercise 18.5. Be warned that the space BU_{gr} is not first-countable. Write $\mathrm{Gr}_n = \mathrm{Gr}_n(\mathbb{C}^{n,n})$ so that $BU_{gr} = \mathrm{colim}_n \mathrm{Gr}_n$ where the inclusion $\mathrm{Gr}_n \hookrightarrow \mathrm{Gr}_{n+1}$ sends a subspace W to $W \oplus \langle e^+_{n+1} \rangle$.

(a) Check that the image of $\mathrm{Gr}_n \hookrightarrow \mathrm{Gr}_{n+1}$ has empty interior. So a nonempty open subset of BU_{gr} cannot be contained in any Gr_n.

(b) Let $b \in BU$ be the point represented by $\langle e^+_1, \ldots, e^+_n \rangle$ (these are the same point, for any n). Suppose $T_1, T_2, \ldots$ are a countable sequence of neighborhoods of

b. Show there exist $n_1 < n_2 < \cdots$ and $x_i \in (\mathrm{Gr}_{n_i} - \mathrm{Gr}_{n_{i-1}}) \cap T_i$. Prove that $BU - \{x_1, x_2, \ldots\}$ is a neighborhood of b and does not contain any T_i.

(c) Conclude that BU is not first-countable.

Not being first-countable, BU_{gr} is not metrizable and cannot be given the structure of a locally-finite CW-complex. It does have a CW-structure with countably many cells, as we will see in Exercise 18.9 below.

The infinite colimit model of $\mathbb{Z} \times BU_{gr}$ suggests that we can regard the points as certain infinite-dimensional subspaces of $\mathbb{C}^{\infty,\infty} = \mathbb{C}^{\infty}_{+} \oplus \mathbb{C}^{\infty}_{-}$. The next few exercises develop this perspective.

Exercise 18.6. Let $H = \mathbb{C}\langle e_1, e_2, \ldots\rangle$ with the Hermitian inner product for which $\{e_i\}$ is an orthonormal basis. Let X_m be the collection of subspaces $W \subseteq H$ such that $W \supseteq \langle e_{m+1}, e_{m+2}, \ldots\rangle$, and let $X = \bigcup_m X_m$. Topologize X_m so that the bijection $X_m \to \mathrm{Gr}_{\bullet}(\mathbb{C}^m)$ given by $W \mapsto W/\langle e_{m+1}, e_{m+2}, \ldots\rangle$ is a homeomorphism, and give X the colimit topology. Prove that $X \cong \coprod_{m\geq 0} \mathrm{Gr}_m(\mathbb{C}^{\infty})$ via the assignment $W \mapsto W^{\perp}$.

We can 'fix' the model from the last exercise by including positive and negative copies of H and being careful with the kinds of subspaces we consider:

Exercise 18.7. Let $\mathcal{H} = \mathbb{C}^{\infty,\infty} = \mathrm{colim}_n\, \mathbb{C}^{n,n}$. Write $\mathcal{H}_m = \mathbb{C}^{m,m}$, $\mathcal{H}_{>m} = \mathcal{H}_m^{\perp}$, $\mathcal{H}^{+} = \mathrm{Span}\{e_j^{+} \mid j \geq 1\}$, and $\mathcal{H}^{+}_{>m} = \mathcal{H}^{+} \cap \mathcal{H}_{>m}$.

Say that a subspace $V \subseteq \mathcal{H}$ is **finite-type** if there exists $m \geq 0$ such that $\mathcal{H}_m + \mathcal{H}^{+}_{>m} \supseteq V \supseteq \mathcal{H}^{+}_{>m}$. Loosely speaking, a finite-type subspace is one that differs from $\mathcal{H}^{+}$ in finitely-many spots. Let $\mathrm{Gr}_{ft}(\mathcal{H})$ be the set of finite-type subspaces.

(a) For each $m \geq 0$, let $\mathrm{Gr}(\mathcal{H}; m) \subseteq \mathrm{Gr}_{ft}(\mathcal{H})$ be the subset of those V for which $\mathcal{H}_m + \mathcal{H}^{+}_{>m} \supseteq V \supseteq \mathcal{H}^{+}_{>m}$. Check that the quotient map $V \mapsto V/\mathcal{H}^{+}_{>m}$ yields a bijection $\mathrm{Gr}(\mathcal{H}; m) \to \mathrm{Gr}_{\bullet}(\mathcal{H}_m)$.

(b) Give $\mathrm{Gr}(\mathcal{H}; m)$ the topology induced from $\mathrm{Gr}_{\bullet}(\mathcal{H}_m)$ by the bijection from (a), and then topologize $\mathrm{Gr}_{ft}(\mathcal{H})$ as $\bigcup_m \mathrm{Gr}(\mathcal{H}; m)$. Show that the maps $\mathrm{Gr}_{\bullet}(\mathcal{H}^m) \to \mathrm{Gr}_{ft}(\mathcal{H})$ given by $W \mapsto W + \mathcal{H}^{+}_{>m}$ yield a continuous map $\mathcal{Z}(0) \to \mathrm{Gr}_{ft}(\mathcal{H})$ that is a homeomorphism.

(c) Let W be a finite-dimensional vector space. Define a subspace $V \subseteq W \oplus \mathcal{H}$ to be finite-type if there exists $m \geq 0$ such that $W + \mathcal{H}_m + \mathcal{H}^{+}_{>m} \supseteq V \supseteq \mathcal{H}^{+}_{>m}$. Write $\mathrm{Gr}_{ft}(W \oplus \mathcal{H})$ for the set of finite-type subspaces. Repeat the above parts to topologize this space and prove that it is homeomorphic to $\mathcal{Z}(W)$.

Note: The model in the next two exercises is really just $\mathrm{Gr}_{ft}(\mathbb{C} \oplus \mathcal{H})$ from part (c) of the previous one. Nevertheless, the indexing is done slightly differently and certain aspects of this are useful on occasion. The point of the exercises is to learn to navigate this particular setup. Our discussion will largely follow that in [PrSe].

Exercise 18.8. Let H be a complex inner product space with countable orthonormal basis $\{e_k \mid k \in \mathbb{Z}\}$, topologized as the colimit of its finite-dimensioal subspaces. Let $H_{\geq m} = \mathbb{C}\langle e_k \mid k \geq m\rangle$ and let $H_{-} = \mathbb{C}\langle e_k \mid k < 0\rangle$. Say that a subspace $W \subseteq H$ is **finite-type** if $H_{\geq k} \supseteq W \supseteq H_{\geq m}$ for some $k \leq m$. Define $\mathrm{Gr}_{ft}(H)$ to be the set of finite-type subspaces of H, and let $\mathrm{Gr}(H; m) \subseteq \mathrm{Gr}_{ft}(H)$ be those W for which $H_{\geq(-m)} \supseteq W \supseteq H_{\geq m}$. The assignment $\mathrm{Gr}(H; m) \to \mathrm{Gr}_{\bullet}(H_{\geq(-m)}/H_{\geq m}) \cong \mathrm{Gr}_{\bullet}(\mathbb{C}^{2m})$ is a bijection and we equip $\mathrm{Gr}(H; m)$ with the topology that makes this a homeomorphism. Then we give $\mathrm{Gr}_{ft}(H) = \bigcup_m \mathrm{Gr}(H; m)$ the colimit topology. We know from the previous exercise that $\mathrm{Gr}_{ft}(H) \cong \mathbb{Z} \times BU_{gr}$.

Let $\mathbb{Z}_- \subseteq \mathbb{Z}$ be the set of negative numbers. For $m \in \mathbb{Z}$ let $\overrightarrow{m} = \{k \in \mathbb{Z} \,|\, m \leq k\}$. Say that a subset $S \subseteq \mathbb{Z}$ is **finite-type** if $S \cap \mathbb{Z}_-$ is finite and $\overrightarrow{m} \subseteq S$ for some $m \in \mathbb{Z}$. For such an S, define the **stabilization point** to be the smallest m such that $\overrightarrow{m} \subseteq S$. Define the **virtual cardinality** to be $\mathrm{vcard}(S) = \#(S \cap \mathbb{Z}_-) - \#(\mathbb{N} \setminus S)$. Colloquially speaking, if we call the members of S "elements" and the members of $\mathbb{Z} \setminus S$ "holes", then $\mathrm{vcard}(S)$ is the number of negative elements minus the number of non-negative holes.

For a nonzero vector $v \in H$, define its **indicator** $\mathrm{indc}(v)$ to be the smallest k for which $v_k \neq 0$. The set $\mathcal{U}_k = \{u \in H \,|\, u_k \neq 0\}$ is then an open neighborhood of v for which $\mathrm{indc}(u) \leq \mathrm{indc}(v)$ for all $u \in \mathcal{U}_k$. So the map $\mathrm{indc}\colon H - 0 \to \mathbb{Z}$ is upper semicontinuous.

For a subspace $W \subseteq H$, let $S_W = \{\mathrm{indc}(w) \,|\, w \in W\} \subseteq \mathbb{Z}$. Call this the **indicator set** of W. If a subspace W is finite-type then its indicator set is finite-type (be warned that the converse is not true); in this case define its **virtual dimension** to be $\mathrm{v.dim}(W) = \mathrm{vcard}(\mathrm{indc}(W))$. Note that $\mathrm{v.dim}(H_{\geq 0}) = 0$.

(a) If $W \in \mathrm{Gr}(H; m)$ prove that $\mathrm{v.dim}(W) = \dim(W/H_{\geq m}) - m$.
(b) Given a path in $\mathrm{Gr}_{ft}(H)$, the indicator sets along the path can change by having a negative element transform into filling a non-negative hole, or vice versa. So the virtual dimension remains constant. Give a rigorous proof of this fact and show that v. dim enumerates the path components of $\mathrm{Gr}_{ft}(H)$.
(c) Let $T\colon H \to H$ be the "shift" transformation that sends each e_i to e_{i+1}. Check that $W \subseteq H$ is finite-type if and only if $T(W)$ is finite-type. Also check that $\mathrm{v.dim}\, T(W) = \mathrm{v.dim}(W) - 1$, and the induced function $T\colon \mathrm{Gr}_{ft}(H) \to \mathrm{Gr}_{ft}(H)$ is a homeomorphism.
(d) Define $G_m \subseteq \mathrm{Gr}_{ft}(H)$ to consist of all finite-type $W \subseteq H$ such that $H_{\geq m} \subseteq W$ and $\dim W/H_{\geq m} = m$. Prove that $G_m \subseteq G_{m+1}$ and $\bigcup_m G_m$ is the path-component of $\mathrm{Gr}_{ft}(H)$ where the virtual dimension is zero.
(e) Prove that there are homeomorphisms $\mathrm{Gr}_m(\mathbb{C}^\infty) \to G_m$ making the diagram

$$\begin{array}{ccccc} \mathrm{Gr}_1(\mathbb{C}^\infty) & \xrightarrow{\oplus 1} & \mathrm{Gr}_2(\mathbb{C}^\infty) & \xrightarrow{\oplus 1} & \cdots \\ \downarrow & & \downarrow & & \\ G_1 & \rightarrowtail & G_2 & \rightarrowtail & \cdots \end{array}$$

commute and thereby inducing $\mathrm{Gr}_\infty(\mathbb{C}^\infty) \xrightarrow{\cong} \mathrm{Gr}_{ft}(H)_{\mathrm{v.dim}=0}$.

Exercise 18.9 (Schubert cell structure)**.** This problem uses the model $\mathrm{Gr}_{ft}(H)$ from the previous exercise. Fix a finite-type subset $S \subseteq \mathbb{Z}$, and set $d = \mathrm{vcard}(S)$. Let m be the stabilization point of S. The number of elements in $S - \overrightarrow{m}$ is $d + m$. Set $s_i = i$ for $i \geq m$, and let $s_{-d}, s_{-d+1}, \ldots, s_{m-1}$ be the elements of $S - \overrightarrow{m}$ listed in increasing order. We call s the **standard sequence associated to S**. Note that $s_{m-1} < m - 1$, or else $m - 1$ would be the stabilization point; by a reverse induction we get $s_i < i$ for all $i < m$.

Define $\ell(S) = \sum_{k \geq -d}(k - s_k)$. This is a finite sum, since for $k \geq m$ we have $k = s_k$. Since $k - s_k \geq 1$ for $k < m$ we have $\ell(S) \geq d + m$.

Let $C_S \subseteq \mathrm{Gr}_{ft}(H)$ denote the subset of all W whose indicator set equals S. This is an open (see below) cell of complex dimension $\ell(S)$: the point is that every

element $W \in C_S$ has a canonical basis taking the form of the rows of the matrix

$$\begin{bmatrix} 1 & * & * & 0 & * & \cdots \\ 0 & 0 & 0 & 1 & * & \cdots \\ \vdots & & & & & \end{bmatrix} \tag{18.10}$$

where the columns are indexed by the basis elements e_i for $i \geq s_{-d}$ and the leading 1's appear in the columns corresponding to each e_{s_k}. The $*$'s can be any complex number, and this is why C_S is a complex affine space. The number of $*$'s in the first row is $-d - s_{-d}$, in the second row is $(-d+1) - s_{-d+1}$, and so forth, so that the total number is $\ell(S)$ (so the real dimension of the cell is $2\ell(S)$).

We call C_S the **Schubert cell** associated to S, and the closure $\overline{C_S}$ is called the corresponding **Schubert variety** (or sometimes just "closed Schubert cell").

(a) Explicity identify all of the 0-cells and 2-cells.
(b) Prove that C_S is an open subset of $\mathrm{Gr}_{ft}(H)$. [Hint: Reduce to a finite Grassmannian and describe C_S in terms of the nonvanishing of a $k \times k$ minor.]
(c) Convince yourself that taking F_n to be the union of all the Schubert cells of dimension $\leq n$ gives a CW-structure on $\mathrm{Gr}_{ft}(H)$.
(d) Observe that the spaces G_m from the previous exercise are subcomplexes of $\mathrm{Gr}_{ft}(H)$ under this cell structure. Prove that all of the cells in the complement $G_m - G_{m-1}$ have dimension at least $2m$.
(e) Show that the number of S having $\ell(S) = k$ and $\mathrm{vcard}(S) = 0$ equals the number of partitions of k. (This takes some inspiration).
(f) Since BU_{gr} is not first-countable (Exercise 18.5), it cannot have the structure of a locally-finite CW-complex. Explicitly demonstrate that the Schubert cell structure is not locally-finite.

19. Products in K-theory

The story of products in K-groups seems to be inherently complicated. Every approach I know requires genuine hard work somewhere along the way. The beginnings are easy enough: tensor product of bundles makes $KO_{Grt}(X)$ and $K_{Grt}(X)$ into rings. So when X is compact Hausdorff and semi-cofibrant this gives a ring structure on $KO^0(X)$ and $K^0(X)$. But the next questions are:

(1) Do the ring structures extend to $KO^0(X)$ and $K^0(X)$ for general spaces X?
(2) Do the above extend to products on pairs, e.g. $KO^0(X, A) \otimes KO^0(Y, B) \to KO^0(X \times Y, X \times B \cup A \times Y)$?
(3) Do these extend to graded ring structures on $KO^*(X)$ and $K^*(X)$, as well as external pairings, products on pairs, and so forth?

The answers are essentially all yes (with an important caveat for the second item), though none of the solutions comes easily.

The goal of this section is to give a brief tour of some of the issues involved, point the reader to places where the questions are completely settled, but not to provide all of the difficult details.

From a very modern perspective, there are ways of setting up a category of spectra so that it comes equipped with a symmetric monoidal product denoted $\wedge$, which is the analog of the smash product of pointed spaces. A *ring spectrum* is then simply a monoid in this category. Having that technology around makes arguments much clearer and simpler than older approaches. Be warned that older literature

(e.g. [Switzer]) used the term "ring spectrum" for a monoid in the homotopy category of spectra, which is different. Sometimes "structured ring spectrum" is used to accentuate the modern notions. See [BGH] for a general introduction to modern approaches to spectra.

It is a fact, though a difficult one, that KO and K can be constructed as ring spectra. This—and similar problems—was the motivation for much work on so-called *infinite loop-space machines*, carried out largely by Peter May and his collaborators. The category of real (or complex) vector spaces has two symmetric monoidal products, $\oplus$ and $\otimes$, and the idea is to feed this structure into a certain "machine" and have the ring spectra KO and K come out. An outline of this idea is already in [Segal4, Section 5], and for May's work in this arena see (as just two examples) [May4] and [May5]. A modern work that develops the state of the art and also fills some previous gaps is [EM]. We also mention the approach of [Lurie] that uses ∞-categories.

There is another approach, due to Joachim, that avoids infinite loop-space machinery, though still making use of a certain modern category of spectra with a well-behaved smash product. In technical lingo, [Joachim] constructs KO and K as *symmetric spectra*. Joachim makes essential use of the fact that K-theory can be represented by spaces of Fredholm operators. We have not discussed this yet, but see the treatment in [D7]. This involves less technology than a generic infinite loop-space machine, but at the expense of using a more complicated model of K-theory.

Here we are going to avoid the modern machinery and give a low-tech (though necessary incomplete and unsatisfying) discussion. To first approximation, giving a ring structure on a spectrum E amounts to giving pairings $E_k \wedge E_n \to E_{k+n}$ for all $k, n \geq 0$ that are associative (in some sense) and compatible with the structure maps in E. There is more to it, but this gives the essence. When $E = KO$ the case $k = n = 0$ amounts to a pairing $(\mathbb{Z} \times BO) \wedge (\mathbb{Z} \times BO) \to \mathbb{Z} \times BO$, but even producing just *this* is nontrivial. This is what is required for the answer to question (1). For higher k and n one can use an induced map

$$\Omega^i(\mathbb{Z} \times BO) \wedge \Omega^j(\mathbb{Z} \times BO) \to \Omega^{i+j}((\mathbb{Z} \times BO) \wedge (\mathbb{Z} \times BO)) \to \Omega^{i+j}(\mathbb{Z} \times BO)$$

and then use the Bott periodicity maps to adjust the $i + j$ index in the target by multiples of 8 if necessary. This basically works, though it is a little clunky. It still leaves the problem of the pairing on $\mathbb{Z} \times BO$, though.

Giving any kind of complete account of constructing KO and K as ring spectra would take us too far afield. Readers can take it as given that this works and then some day undertake a study of the details when they are needed. In this section we instead want to focus on two basic questions. Assuming it all works, what pairings on cohomology groups does one actually obtain (this is (2) and (3) above)? And secondly, why is the basic pairing $(\mathbb{Z} \times BO) \wedge (\mathbb{Z} \times BO) \to \mathbb{Z} \times BO$ difficult to produce? Along the way we will produce the pairing on $\mathbb{Z} \times BU$, as that case is significantly easier (though still nontrivial).

19.1. **Issues with products.** As soon as one starts seriously dealing with products, a few technical issues become important. First, if X and Y are CW-complexes then there is the usual product cell stucture on $X \times Y$. But the weak (or colimit) topology associated with this cell structure is not always the same as the product

topology. If X and Y are countable CW-complexes, the two topologies are the same by [Milnor1, Lemma 2.1].

If X and Y are pointed spaces then the smash product is $X \wedge Y = (X \times Y)/(X \times pt \cup pt \times Y)$. Because of the quotienting, weak equivalences $\tilde{X} \to X$ and $\tilde{Y} \to Y$ do not necessarily induce weak equivalences $\tilde{X} \wedge \tilde{Y} \to X \wedge Y$. If all of the spaces are cofibrant (as pointed spaces) then the weak equivalences are homotopy equivalences, and then all is fine. So to get the "correct" smash product of X and Y one should really take cofibrant-replacements $QX \to X$ and $QY \to Y$ and form $QX \wedge QY$. This yields the so-called "derived" smash product on $\mathrm{Ho}\,\mathcal{T}_*$. We will not have to use the derived smash product often, but when we do we will denote it as $\underline{\wedge}$. Note that $X_+ \underline{\wedge} Y_+ = (QX)_+ \wedge (QY)_+ = QX \times QY \simeq X \times Y = X_+ \wedge Y_+$, so in this special case there is no difference (up to homotopy) between the derived and underived smash products.

Finally, any serious work with spectra and smash products leads one to really want to be working in a category of spaces where the product (and smash product) commute with colimits in each variable and have well-behaved adjunction properties. The category $\mathcal{T}op$ of all topological spaces is simply not sufficient for this. Standard solutions are to instead work with the category of k-spaces (also called compactly-generated spaces), or perhaps with the category of compactly-generated weak Hausdorff spaces. After [Steenrod2] topologists often call these "convenient categories of topological spaces".

In such categories it follows formally that if $A \rightarrowtail B$ and $C \rightarrowtail D$ are cofibrations then so is $(A \times D) \amalg_{A \times C} (B \times C) \to B \times D$, and similarly for smash products in the pointed setting. In particular, if A is cofibrant and $C \rightarrowtail D$ is a cofibration then so it $A \times C \rightarrowtail A \times D$. As an even more special case, if A and B are cofibrant then so is $A \times B$.

Unfortunately, in these enhanced settings "subspace" does not always coincide with the usual subspace topology, and "product" does not always coincide with the usual product topology. In most cases of interest they do coincide, but one has to be careful about this issue. As one benefit, though, in these settings the topology on a product of CW-complexes is exactly the one that is needed for the CW-structure—so the previous issue of this working only for countable CW-complexes goes away (this is tied to the fact mentioned in the last paragraph that a product of two cofibrant objects is again cofibrant).

19.2. **Products on pairs.** Let $E = \{E_i\}_{i \geq 0}$ be an infinite loop space, and suppose given additional pairings $\mu_{i,j} \colon E_i \wedge E_j \to E_{i+j}$. Given pointed spaces W and Z we get

$$\mathrm{Ho}\,\mathcal{T}_*(W, E_i) \otimes \mathrm{Ho}\,\mathcal{T}_*(Z, E_j) \longrightarrow \mathrm{Ho}\,\mathcal{T}_*(W \underline{\wedge} Z, E_i \underline{\wedge} E_j) \longrightarrow \mathrm{Ho}\,\mathcal{T}_*(W \underline{\wedge} Z, E_{i+j}),$$

where the first map sends $f \otimes g$ to $f \wedge g$ and the second is post-composition with μ. Taking $W = X_+$ and $Z = Y_+$ this gives the pairing $E^i(X) \otimes E^j(Y) \to E^{i+j}(X \times Y)$. When $X = Y$ we can pull back along the diagonal map to get $E^i(X) \otimes E^j(X) \to E^{i+j}(X)$.

Remark 19.3. There is a small white lie in the above paragraph regarding bilinearity. If W is a suspension then we can use the operation on $\mathrm{Ho}\,\mathcal{T}_*(W, E_i)$ induced by the cogroup structure on W (the usual pinch map), and by the usual Eckmann-Hilton argument this is the same as the group structure induced by the loop-space structure on E_i. Using the structure coming from W, linearity in the first variable

is easy—and similarly for the second variable, assuming Z is a suspension. This is where bilinearity is simple and natural.

If one works entirely in the category of spectra, then every object is a suspension and the same argument goes through. If one insists on working at the level of spaces, then $\mathrm{Ho}\,\mathcal{T}_*(W, E_i) \cong \mathrm{Ho}\,\mathcal{T}_*(W, \Omega E_{i+1}) \cong \mathrm{Ho}\,\mathcal{T}_*(\Sigma W, E_{i+1})$ and so one can shift the domain to being a suspension at the expense of passing to E_{i+1}. We would then have to use $\mu_{i+1,j+1}$ rather than our pairing $\mu_{i,j}$. Some basic compatibility between the μ-maps is required to ensure that the products one gets from the two methods are the same.

The upshot is that bilinearity really *is* "easy", as long as one has enough structure around to make the above arguments work.

So we have seen how to use pairings among the $\{E_i\}$ to get induced pairings on the non-relative E-cohomology groups. The more challenging question is to see what happens on pairs. Recall that if (X, A) is a pair then $E^*(X, A)$ is defined in terms of the mapping cone $C(X, A)$. The fundamental issue is that given pairs (X, A) and (Y, B) there is not a simple way to compare $C(X, A) \wedge C(Y, B)$ to $C(X \times Y, X \times B \cup A \times Y)$.

Definition 19.4. *Let (X, A) and (Y, B) be two pairs. Let $P(X, Y, A, B) \subseteq X \times Y$ be the subspace of pairs (x, y) where either $x \in A$ or $y \in B$ (we will often denote this space as P for short). We have a commutative diagram*

$$\begin{array}{ccc} A \times B & \longrightarrow & A \times Y \\ \downarrow & & \downarrow \\ X \times B & \longrightarrow & P. \end{array} \tag{19.5}$$

Let $HP = HP(X, Y, A, B)$ be the homotopy pushout of $X \times B \longleftarrow A \times B \longrightarrow A \times Y$. Say that (X, A) and (Y, B) are ***mateable*** *if the canonical collapse map $HP \to P$ is a weak equivalence.*

Remark 19.6. In $\mathcal{T}op$, the space $P(X, Y, A, B)$ is more commonly (and helpfully) written as $(X \times B) \cup (A \times Y)$. If one is instead working in a category such as k-spaces, the product $X \times B$ might not be a subspace of $X \times Y$—one only has a canonical map. This is why one must be careful with the language.

In what follows, we will often be lazy and write $X \times B$ both for the product and for its image in $X \times Y$. When working with categories other than $\mathcal{T}op$ just be warned that there is a subtle distinction that requires extra thought. This distinction disappears if A and B are both open or both closed.

Proposition 19.7. *The pairs (X, A) and (Y, B) are mateable in each of the following circumstances:*

(1) A is open in X and B is open in Y.
(2) (X, A) and (Y, B) are CW-pairs where X and Y are countable CW-complexes.
(3) We are working in k-spaces or a similar convenient category of topological spaces and $A \hookrightarrow X$ is a cofibration and B is cofibrant (or alternatively, A is cofibrant and $B \hookrightarrow Y$ is a cofibration).

Proof. In case (1), $X \times B$ and $A \times Y$ give an open covering for P and so diagram (19.5) is a pushout. The fact that $HP \to P$ is a weak equivalence is by Exercise 9.22.

In case (2), the natural product cell structure gives a CW-structure on the Cartesian product $X \times Y$ (this does not work for uncountable CW-complexes). Then $X \times B$ and $A \times Y$ are subcomplexes, so P is the pushout of (19.5). The maps $A \times B \to X \times B$ and $A \times B \to A \times Y$ are cofibrations (being subcomplexes), hence the map from the homotopy pushout to the pushout is a weak equivalence.

For (3) the hypotheses imply that $A \times B \to X \times B$ is a cofibration (see Section 19.1). It follows from [Hirschhorn, Theorem 13.1.10 and Proposition 13.5.4] that the canonical map from a homotopy pushout to the pushout is a weak equivalence as soon as we know that at least one of the maps is a cofibration, and our assumptions guarantee this. □

We claim that there is a distinguished map

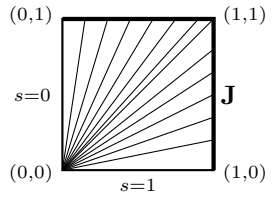
$$\theta\colon C(X \times Y, HP) \to C(X, A) \wedge C(Y, B).$$

Defining this map is unpleasant, but the intuition is easy enough. The point is that in $C(X, A) \wedge C(Y, B)$ the subspace $A \times B \subseteq X \times Y$ can be coned off in two different ways, one through the first factor and one through the second. But using the canonical map $CA \times CB \to C(X, A) \wedge C(Y, B)$, inside the product $CA \times CB$ we have a homotopy between the two ways. Constructing θ is just a matter of writing down this homotopy.

To describe this rigorously, let $J \subseteq I^2$ be the subspace of points (s, t) where either $s = 1$ or $t = 1$. Fix once and for all a map $H\colon I^2 \to I^2$ with the following properties:

$$H(0,t) = (0,t), \quad H(1,t) = (t,0), \quad H(s,0) = (0,0), \quad H(s,1) \in J$$

for all $s, t \in I$. All such maps are homotopic through maps with these properties, so the exact choice does not matter. One possible choice is depicted below:



The rays are indexed by $s \in [0, 1]$ and the parameter t represents the location along each ray, with $t = 0$ at the origin and $t = 1$ on J. Write $H(s, t) = (H_1(s, t), H_2(s, t))$.

Denote points in X by x, points in A by a, and so forth. If we want to regard $a \in A$ as belonging to X we will write a_X. Using this notation, the space HP consists of points (x, b), $((a, b), s)$, and (a, y) with the identifications $((a, b), 0) = (a_X, b)$ and $((a, b), 1) = (a, b_Y)$. The space $C(X \times Y, HP)$ is the quotient space $[(X \times Y) \amalg (HP \times I)]/\sim$ and so consists of points (x, y), $((x, b), t)$, $((a, y), t)$, and $((a, b), s, t)$ with the identifications

$$\begin{aligned}
&((x,b),0) = (x,b), \quad ((a,y),0) = (a,y), \quad ((a,b),s),0) = (a,b)\\
&((a,b),0,t) = (a_X,b,t), \quad ((a,b),1,t) = (a,b_Y,t)\\
&((x,b),1) = ((a,y),1) = ((a,b),s,1).
\end{aligned}$$

Define $\theta\colon C(X \times Y, HP) \to C(X, A) \wedge C(Y, B)$ by the following formulas:

- $\theta(x, y) = (x, y)$,
- $\theta((x, b), t) = (x, (b, t))$,

- $\theta((a,y),t) = ((a,t),y)$,
- $\theta((a,b),s,t) = ((a,H_1(s,t)),(b,H_2(s,t))$.

The formulas make this look worse than it is.

Exercise 19.8. Check that the formulas defining θ respect the identifications and so really do determine a continuous map as claimed.

The map θ allows us to construct our pairings, at least under some assumptions:

Proposition 19.9. *There are natural pairings*

$$E^i(X,A) \otimes E^j(Y,B) \to E^{i+j}(X \times Y, P(X,Y,A,B))$$

defined whenever (X,A) and (Y,B) are mateable, extending the pairings previously defined for $A = B = \emptyset$.

Proof. Let QX, QA, QY, and QB be cofibrant-replacements for X, A, Y, and B. Elements of $f \in E^i(X,A)$ can be represented by maps $f\colon C(QX,QA) \to E_i$, and likewise for elements $g \in E^j(Y,B)$.

Write P_Q and HP_Q for the pushout and homotopy pushout of $QX \times QB \longleftarrow QA \times QB \longrightarrow QA \times QY$. Consider the diagram

$$\begin{array}{ccccc}
C(QX \times QY, P_Q) & \longleftarrow & C(QX \times QY, HP_Q) & \xrightarrow{\theta} & C(QX,QA) \wedge C(QY,QB) \\
\downarrow & & \downarrow \simeq & & \downarrow f \wedge g \\
C(X \times Y, P) & \xleftarrow[\simeq]{} & C(X \times Y, HP) & & E_i \wedge E_j \\
 & & & & \downarrow \\
 & & & & E_{i+j}.
\end{array}$$

The assumption that (X,A) and (Y,B) are mateable says that $HP \to P$ is a weak equivalence. The map $HP_Q \to HP$ is a weak equivalence because it comes from a map of pushout diagrams that is a weak equivalence in every spot. Therefore the two indicated morphisms of mapping cones are also weak equivalences.

The evident zig-zag in the diagram has therefore produced an element of $\mathrm{Ho}\,\mathcal{T}_*(C(X \times Y, P), E_{i+j})$, which is precisely $E^{i+j}(X \times Y, P)$. This is our product of f and g. □

With some trouble one can work out in detail the compatibility properties one wants between the above pairings and other elements of the generalized cohomology theory E^*. The main ones are encapsulated in the following definition:

Definition 19.10. *A **multiplicative generalized cohomology theory** is a cohomology theory $\mathcal{E}$ equipped with product maps*

$$\mu\colon \mathcal{E}^p(X,A) \otimes \mathcal{E}^q(Y,B) \longrightarrow \mathcal{E}^{p+q}(X \times Y, X \times B \cup A \times Y)$$

for mateable pairs (X,A) and (Y,B) (see Definition 19.4), with the product maps satisfying the following requirements:

(1) Naturality in both (X,A) and (Y,B);
(2) There exists a two-sided unit $1 \in \mathcal{E}^0(pt,\emptyset) = \mathcal{E}^0(pt)$;
(3) The pairings are associative;

(4) *The pairings are compatible with the connecting homomorphisms, in the sense that the following two diagrams commute:*

$$\begin{array}{ccc}
\mathcal{E}^p(A)\otimes\mathcal{E}^q(Y,B) & \xrightarrow{\mu} & \mathcal{E}^{p+q}(A\times Y, A\times B) \\
{\scriptstyle \delta\otimes\mathrm{id}}\downarrow & & \uparrow{\scriptstyle \cong} \\
\mathcal{E}^{p+1}(X,A)\otimes\mathcal{E}^q(Y,B) & & \mathcal{E}^{p+q}(X\times B\cup A\times Y, X\times B) \\
& {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\delta} \\
& & \mathcal{E}^{p+q+1}(X\times Y, X\times B\cup A\times Y)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{E}^p(X,A)\otimes\mathcal{E}^q(B) & \xrightarrow{\mu} & \mathcal{E}^{p+q}(X\times B, A\times B) \\
{\scriptstyle (-1)^p\,\mathrm{id}\otimes\delta}\downarrow & & \uparrow{\scriptstyle \cong} \\
\mathcal{E}^p(X,A)\otimes\mathcal{E}^{q+1}(Y,B) & & \mathcal{E}^{p+q}(X\times B\cup A\times Y, A\times Y) \\
& {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\delta} \\
& & \mathcal{E}^{p+q+1}(X\times Y, X\times B\cup A\times Y)
\end{array}$$

(the vertical δ maps are the connecting homomorphisms for the evident triples, and the indicated maps are isomorphisms precisely because (X,A) and (Y,B) are mateable).

See [Switzer, 13.56] for the above commutative diagrams (and others).

Remark 19.11. The above definition is not the best one, but it is a version that is not too frightening to the uninitiated. As we have mentioned, for serious work one needs not just a cohomology theory but the corresponding spectrum, and for a multiplicative cohomology this should be a "structured ring spectrum". It requires a significant amount of language just to say what those are. For at least 95 percent of what we do in this book, the above definition will suffice. Occasionally the only way to make an argument is to use the more structured version, and when the need arises we will alert the reader but not give all details. See Section 27.21 for an example of this.

Theorem 19.12. *KO and K have the structure of multiplicative cohomology theories, where on the degree zero groups for compact Hausdorff, semi-cofibrant spaces the pairing is the tensor product of vector bundles.*

We will not completely prove the above result, but see [Switzer] or the other references mentioned earlier. In the remainder of this section we will show a bit of what is required for a naive, brute-force approach.

Remark 19.13. To simplify the discussion we have not talked about the commutativity of the product, but the fact of the matter is that we have as much commutativity as we have any business expecting. The spectra KO and K can be constructed as commutative (structured) ring spectra, or as E_∞ ring spectra. Again, see the previously-mentioned references for details.

19.14. **The pairing on $\mathbb{Z}\times BU$.** We seek to produce a map $(\mathbb{Z}\times BU)\times(\mathbb{Z}\times BU)\to \mathbb{Z}\times BU$ that represents the tensor product on $K^0(X)$ when X is compact Hausdorff and semi-cofibrant. We know that the product of components indexed by integers a and b will map into the component indexed by ab, so it is really a matter of handling the case $a=b=0$. We can start with the maps

$$\mathrm{Gr}_k(\mathbb{C}^\infty)\times \mathrm{Gr}_n(\mathbb{C}^\infty)\to \mathrm{Gr}_{kn}(\mathbb{C}^\infty\otimes\mathbb{C}^\infty)$$

that classify tensor product of bundles, but these are not compatible with the inclusions $\mathrm{Gr}_k(\mathbb{C}^\infty)\hookrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}\oplus\mathbb{C}^\infty)$. The problem is that the tensor product on stable K-theory $K^{st}(X)$ is given by $E,F\mapsto (E-\operatorname{rank}E)\otimes(F-\operatorname{rank}F)=E\otimes F-(\operatorname{rank}E)F-(\operatorname{rank}F)E+(\operatorname{rank}E)(\operatorname{rank}F)$, and so we need to start with maps that represent this entire expression—not just the tensor product, which is only the first term. However, because of the negative signs it is not at all clear how to construct simple models of such maps.

One way to attempt to get around this issue is to work entirely in the homotopy category, where the negative terms are easy to handle. We have classes $\gamma_k-k\in \widetilde{K}^0_{Grt}(\mathrm{Gr}_k(\mathbb{C}^\infty))$, therefore we have classes

$$\xi_{k,r}=(\gamma_k-k)\otimes(\gamma_r-r)\in \widetilde{K}^0_{Grt}(\mathrm{Gr}_k(\mathbb{C}^\infty)\times \mathrm{Gr}_r(\mathbb{C}^\infty)).$$

Via the natural transformation $\widetilde{K}^0_{Grt}\to\widetilde{K}^0$ we can regard these as elements of $\widetilde{K}^0(\mathrm{Gr}_k(\mathbb{C}^\infty)\times \mathrm{Gr}_r(\mathbb{C}^\infty))$.

Now take the direct limit with respect to the inclusions $\mathrm{Gr}_k(\mathbb{C}^\infty)\hookrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}^\infty)$, and likewise for Gr_r. The associated Milnor sequence is

$$0\longrightarrow \lim{}^1[\Sigma(\mathrm{Gr}_k(\mathbb{C}^\infty)\times \mathrm{Gr}_r(\mathbb{C}^\infty)),\mathbb{Z}\times BU]\longrightarrow \widetilde{K}^0(BU\times BU)$$
$$\lim \widetilde{K}^0(\mathrm{Gr}_k(\mathbb{C}^\infty)\times \mathrm{Gr}_r(\mathbb{C}^\infty))\longrightarrow 0.$$

The $\xi_{k,r}$ classes are compatible as k and r vary, hence define an element ξ in the inverse limit term. The $\lim^1$ term vanishes because all of the groups are actually zero, by the following result:

Lemma 19.15. *Let X be a connected CW-complex having only odd-dimensional cells (other than a single 0-cell). Then $[X,\mathbb{Z}\times BU]=0$.*

Proof. Write X_n for the n-skeleton of X. One proves the result inductively for finite-dimensional X, using the cofiber sequences $X_{2n-1}\hookrightarrow X_{2n+1}\to \vee S^{2n+1}$ and the fact that $K^0(S^{2n+1})=0$.

For infinite-dimensional X we need to consider the Milnor sequence

$$0\to \lim{}^1_n[\Sigma(X_{2n+1}),\mathbb{Z}\times BU]\to K^0(X)\to \lim_n K^0(X_{2n+1})\to 0.$$

The term on the right vanishes by the finite-dimensional case. For the $\lim^1$ term consider the cofiber sequence $\vee S^{2n+1}\to \Sigma X_{2n-1}\to \Sigma X_{2n+1}$, which gives

$$\widetilde{K}^0(S^{2n+1})\longleftarrow \widetilde{K}^0(\Sigma X_{2n-1})\longleftarrow \widetilde{K}^0(\Sigma X_{2n+1}).$$

Since K^0 vanishes on odd spheres, the map $\widetilde{K}^0(\Sigma X_{2n+1})\to\widetilde{K}^0(\Sigma X_{2n-1})$ is surjective. As this is true for all n, the Mittag-Leffler vanishing condition implies that the $\lim^1$-term is zero. □

So we have shown that there is a unique element $\tilde{\xi} \in \widetilde{K}^0(BU \times BU)$ that lifts ξ. This is an element of $[BU \times BU, BU]$. Restricting to either $BU \times *$ or $* \times BU$ gives zero, so we lift to a homotopy class of maps $BU \wedge BU \to BU$ (alternatively, re-do the above arguments with smash products rather than Cartesian products).

A version of the above argument can also be found in [Switzer, 13.90].

Now that we have $BU \wedge BU \to BU$, we extend to $(\mathbb{Z} \times BU) \wedge (\mathbb{Z} \times BU) \to \mathbb{Z} \times BU$ by using the same map for each of the pairings $(\{a\} \times BU) \wedge (\{b\} \times BU) \to \{ab\} \wedge BU$.

Exercise 19.16. The multiplication on BU gives two maps $BU \wedge BU \wedge BU \to BU$ in the usual way ($(ab)c$ versus $a(bc)$). Prove that these are the same up to homotopy by analyzing the Milnor sequence for $\widetilde{K}^0(BU \wedge BU \wedge BU)$. The key ideas can be explained in just a few sentences.

The above method for producing the pairing on $\mathbb{Z} \times BU$ is not very pleasant. It is even harder for $\mathbb{Z} \times BO$, as there one cannot use the "odd-dimensional cells" argument and so more work is required to prove the vanishing of the $\lim^1$ term. Again, there are less computational methods for producing these maps that in some ways are more satisfying but come at the expense of needing to develop much more technology.

19.17. **Summary.** As we warned at the beginning of the section, the story of products in topological K-theory is inherently difficult. When first learning this material it is best to just get the basic ideas, accept that the details can be worked out, and then move on to using the theory. This isn't satisfying, and we're not saying you shouldn't be bothered by this state of affairs. But things often work out this way in homotopy theory, where one is essentially taking some structure that exists in a higher plane of existence and forcing it to fit into ours. See [D7, What is K-theory?] for more discussion of this point.

20. Relative K-theory in degree zero

Given a pair (X, A), we have defined groups $KO^0(X, A)$ and $K^0(X, A)$ in terms of maps in the homotopy category from the mapping cone $C(X, A)$ into the spaces $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$. Just as vector bundles give us a geometric source for (some) classes in $K^0(X)$, it is helpful if we can discover a similar geometric source for classes in $K^0(X, A)$. That is the goal of this section.

When we talked about algebraic K-theory back in Part 1, we defined the relative K-group $K^0(X, A)$ using quasi-isomorphism classes of chain complexes that were exact on A (Section 6.13). We will make a similar construction in the topological case, with some important differences. With some work one can produce a homomorphism from the algebraically-defined group into the topological one. Like our comparison map $K^0_{Grt}(X) \to K^0(X)$, we only expect an isomorphism under certain finiteness (or compactness) hypotheses on the pair (X, A).

In this section we outline the results, but the detailed proofs are difficult and mostly deferred to Section 24. For convenience we state results in this section for just the K^0 groups, but they all hold for KO^0 as well.

20.1. **Grothendieck groups for relative K-theory.** Let $A \hookrightarrow X$ be an inclusion of topological spaces.

Definition 20.2. *Let $\mathcal{F}(X, A)$ be the free abelian group on isomorphism classes of bounded chain complexes of vector bundles $E_\bullet$ on X that are exact on A (meaning*

that for every $x \in X$ the complex of vector spaces $(E_x)_\bullet$ is exact). Define $\mathcal{K}(X, A)$ to be the quotient of $\mathcal{F}(X, A)$ by the following relations:

(1) $[E_\bullet \oplus F_\bullet] = [E_\bullet] + [F_\bullet]$;
(2) $[E_\bullet] = 0$ *whenever* $E_\bullet$ *is exact on all of* X;
(3) If $\mathcal{E}_\bullet$ *is a bounded complex of vector bundles on* $X \times I$ *that is exact on* $A \times I$ *then* $[\mathcal{E}|_{X\times 0}] = [\mathcal{E}|_{X\times 1}]$.

Note that pullback of vector bundles makes $\mathcal{K}(X, A)$ into a contravariant functor.

Remark 20.3. We will eventually produce a comparison map $\mathcal{K}(X, A) \to K^0(X, A)$, but this will be an isomorphism only when the inclusion $A \hookrightarrow X$ is nice enough—for example, if it is a cofibration and both X and A are homotopically compact. One should be warned up front that the groups $\mathcal{K}(X, A)$ might not be the groups you think they are without such restrictions.

Relations (1) and (2) are familiar from Part 1, although the reader might be surprised that (1) only deals with direct sums and not short exact sequences. We will say more about this in a moment. Let us first make some remarks on relation (3), since in our work on algebraic Grothendieck groups we did not encounter relations of this type.

If $d\colon E \to F$ is a map of vector bundles over X there is a clear, intuitive notion of a *deformation* of d. One way to make this rigorous is to consider the subspace $\mathrm{VB}(E, F) \subseteq \underline{\mathcal{T}op}(E, F)$ consisting of the vector bundle maps; then a deformation of d is just a continuous map $I \to \mathrm{VB}(E, F)$ that sends 0 to d. If $\pi\colon X \times I \to X$ is the projection, a little thought shows that when X is locally compact Hausdorff the above notion of deformation is the same as a map of vector bundles $\pi^*E \to \pi^*F$ over $X \times I$ that restricts to d on $X \times \{0\}$. The reason for the hypotheses on X is that one needs to use an adjunction $\underline{\mathcal{T}op}(E, F)^I \cong \underline{\mathcal{T}op}(E \times I, F)$, and this only works if E is locally compact and Hausdorff (which is implied by X having those properties).[1]

Let us call a vector bundle map $\pi^*E \to \pi^*F$ a **product-deformation** of d, and a map $I \to \mathrm{VB}(E, F)$ a **path-deformation**. In general one has a continuous map $\underline{\mathcal{T}op}(E \times I, F) \to \underline{\mathcal{T}op}(E, F)^I$, but it need not be a homeomorphism. This shows that a product-deformation gives a path-deformation, but not necessarily vice versa.

Likewise, a deformation of a chain complex $E_\bullet$ over X can be thought of in two ways. One way involves a collection of path-deformations for all the maps of $E_\bullet$, having the property that at any given time t the maps in the deformation assemble into a chain complex. The other way is simply as a chain complex structure on the set of vector bundles $\{\pi^*E_i\}$ which when restricted to $X \times \{0\}$ is isomorphic to $E_\bullet$. As before, a product-deformation always gives a path-deformation but the two notions are equivalent only when X is locally compact and Hausdorff.

The point here is that if $(E_\bullet, d)$ is a given chain complex and d' is another differential that is related to d by a product-deformation, then relation (3) implies that $[(E_\bullet, d)] = [(E_\bullet, d')]$. (To be precise, there is an exactness condition required for the deformation, namely that at every time t the differential d_t is exact on A). Moreover, if X is paracompact Hausdorff then by Corollary 14.2(b) every bundle on $X \times I$ is isomorphic to the pullback of a bundle from X; it follows that *every*

[1] As always, the hypotheses can be dropped if $\mathcal{T}op$ is replaced by a convenient category of spaces, though at the expense of changing the notion of product.

relation from (3) can be recast in this form. That is to say, for spaces that are paracompact and Hausdorff it is equivalent to replace (3) by

(3') $[(E_\bullet, d)] = [(E_\bullet, d')]$ for any bounded chain complex $(E_\bullet, d)$ that is exact on A and any d' that is related to d by a product-deformation (where each d_t is also required to be exact on A).

If in addition X is locally compact then one can use path deformations instead. But since spaces such as $\mathbb{C}P^\infty$ are not locally compact, one does not want to be bound by that restriction.

Remark 20.4. The notion of path-deformation is in some ways more appealing than a product-deformation, because in practice one wants to communicate the homotopy by writing down the assignment $t \mapsto d_t$. Even though not every path-deformation gives rise to a product deformation, many do. For example, if $d\colon E \to F$ is a bundle map then the path-deformation $t \mapsto t \cdot d$ is readily checked to also be a product deformation, essentially because the scalar product map $\mathbb{R} \times F \to F$ is continuous. Often in proofs we will write down a path-deformation and leave the reader to check that it is actually a product-deformation too.

The following important lemma will help give a feeling for the idea of deforming a chain complex:

Lemma 20.5. *Let $E_\bullet$ be a bounded complex of vector bundles on X that is exact on A. Then $[E_\bullet] = -[\Sigma E_\bullet]$ in $\mathcal{K}(X, A)$, where $\Sigma E_\bullet$ is the shifted complex having E_i in degree $i+1$ and $d_{\Sigma E} = -d_E$.*

Proof. First note that if $V_\bullet$ and $W_\bullet$ are exact complexes of vector spaces and $f\colon V_\bullet \to W_\bullet$ is any map, then the mapping cone Cf is still exact. This follows by the long exact sequence on homology. Consequently, if $E_\bullet \to F_\bullet$ is a map between complexes of vector bundles on X, each of which is exact on A, then the mapping cone is also exact on A.

Let C denote the mapping cone of the identity map $E_\bullet \xrightarrow{\text{id}} E_\bullet$. We depict this complex as follows:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & & \\
 & \nearrow & \oplus & \nearrow_{\text{id}} & \oplus & \nearrow_{\text{id}} & \oplus & \nearrow_{\text{id}} & & & \\
\cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & & & &
\end{array}$$

The arrows depict the various components of the differentials in the mapping cone; recall that $d(a,b) = (da + \text{id}(b), -db)$ for $(a,b) \in E_n \oplus E_{n-1}$, where we have written $\text{id}(b)$ instead of b just to indicate the role of the original chain map.

Consider the deformation of C obtained by putting a t in front of all the diagonal arrows and letting $t \mapsto 0$. That is, $C(t)$ is the mapping cone for $t(\text{id})\colon E_\bullet \to E_\bullet$. Then $C(t)$ is exact on A for every t, and when $t = 0$ we have $C(0) = E_\bullet \oplus \Sigma E_\bullet$. So in $\mathcal{K}(X, A)$ we have $[C] = [C(1)] = [C(0)] = [E_\bullet] + [\Sigma E_\bullet]$.

But C is exact on all of X, being the mapping cone of an identity map. So $[C] = 0$ in $\mathcal{K}(X, A)$, and hence $[E_\bullet] = -[\Sigma E_\bullet]$. □

Exercise 20.6. Let $E_\bullet$ and $F_\bullet$ be complexes of vector bundles on X that are exact on A and let $f\colon E_\bullet \to F_\bullet$ be any map, with Cf denoting the mapping cone. Use the ideas in the above proof to show that $[Cf] = [F_\bullet] - [E_\bullet]$ in $\mathcal{K}(X, A)$.

Our next task is to analyze exact complexes. We will see that just as in homological algebra (for complexes of projectives) they split up into basic pieces.

Definition 20.7. *An **elementary complex** is one of the form*

$$[0 \to \cdots \to 0 \to E \xrightarrow{\mathrm{id}} E \to 0 \to \cdots \to 0]$$

where E is a vector bundle on X and the E's occur in some dimensions i and $i+1$. Denote this complex as $D_i(E)$.

Proposition 20.8. *Let X be a paracompact Hausdorff space. If $E_\bullet$ is a bounded complex of vector bundles on X that is exact, then $E_\bullet$ is a direct sum of elementary complexes.*

Proof. The proof is really the same as in homological algebra (see Lemma 7.3). Assume without loss of generality that $E_i = 0$ for $i < 0$. Then $E_1 \to E_0$ is a surjection, so the kernel K_1 is a vector bundle by Proposition 12.3. By Proposition 12.2 the sequence $0 \to K_1 \to E_1 \to E_0 \to 0$ is split-exact, and a choice of splitting allows us to write $E_1 = K_1 \oplus Q_1$ where the composite $Q_1 \hookrightarrow E_1 \to E_0$ is an isomorphism. Noting that $E_2 \to E_1$ has image contained in K_1, the complex $E_\bullet$ splits as the direct sum of $D_0(E_0)$ and a complex that is zero in dimensions smaller than 1. Now continue inductively, replacing $E_\bullet$ with this smaller factor, until the nonzero degrees of $E_\bullet$ have been exhausted. □

Remark 20.9. Observe now that relation (2) of Definition 20.2 could be replaced with the relation that $[D_i(E)] = 0$ for any vector bundle E on X and any $i \in \mathbb{Z}$. This fact is sometimes useful.

The next result explains why we were able to forego short exact sequences in relation (1) from Definition 20.2.

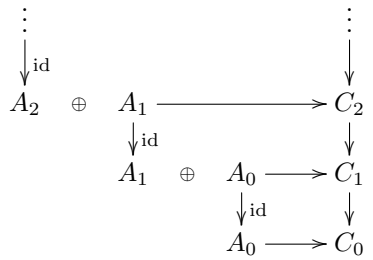
Proposition 20.10. *Let X be paracompact and Hausdorff, and let $A \subseteq X$. Assume given a short exact sequence $0 \to E'_\bullet \to E_\bullet \to E''_\bullet \to 0$ of complexes of vector bundles on X, where each complex is exact on A. Then $[E_\bullet] = [E'_\bullet] + [E''_\bullet]$ in $\mathcal{K}(X, A)$.*

Proof. Let $C_\bullet$ be the mapping cone of $E'_\bullet \hookrightarrow E_\bullet$, and recall that there is a natural surjective map $C_\bullet \twoheadrightarrow E''_\bullet$. Let $K_\bullet$ be the kernel, which is a chain complex of vector bundles by Proposition 12.3. Elementary homological algebra (applied in each fiber) shows that $K_\bullet$ is exact on X. By Lemma 20.11 below the inclusion $K_\bullet \hookrightarrow C_\bullet$ is split, and so $C_\bullet \cong K_\bullet \oplus E''_\bullet$. Then $[C_\bullet] = [K_\bullet] + [E''_\bullet] = [E''_\bullet]$ in $\mathcal{K}(X, A)$. Yet Exercise 20.6 gives $[C_\bullet] = [E_\bullet] - [E'_\bullet]$. □

Lemma 20.11. *Let X be a paracompact Hausdorff space. Let $j\colon K_\bullet \hookrightarrow C_\bullet$ be an inclusion between bounded complexes of vector bundles on X, and assume that $K_\bullet$ is exact. Then the map j admits a splitting $\chi\colon C_\bullet \to K_\bullet$.*

Proof. Without loss of generality assume that $K_i = 0 = C_i$ for $i < 0$. By Proposition 20.8 we can write $K = \oplus_{i=0}^{N} D_i(A_i)$ for some vector bundles $A_0, \ldots, A_N$ on X.

The inclusion j looks as follows:



Starting at the bottom, choose a splitting χ_0 for the inclusion $A_0 \hookrightarrow C_0$, using Corollary 12.9. Likewise, choose a splitting α_1 for the inclusion $A_1 \hookrightarrow C_1/A_0$ (note that C_1/A_0 is a vector bundle by Corollary 12.8). Define $\chi_1 \colon C_1 \to A_1 \oplus A_0$ to be the sum of $C_1 \to C_1/A_0 \xrightarrow{\alpha_1} A_1$ and $C_1 \to C_0 \xrightarrow{\chi_0} A_0$. It is readily checked that χ_1 is a splitting for j_1 and that $d\chi_1 = \chi_0 d$. Continue inductively to define χ at all levels. □

The groups $\mathcal{K}(X, A)$ are readily seen to be homotopy invariant constructions, essentially because this was built into the definition:

Proposition 20.12. *For any map of pairs $f \colon (X, A) \to (Y, B)$ that is part of a homotopy equivalence (of pairs), the induced map $f^* \colon \mathcal{K}(Y, B) \to \mathcal{K}(X, A)$ is an isomorphism.*

Proof. If $j_0, j_1 \colon (X, A) \hookrightarrow (X \times I, A \times I)$ are the evident inclusions then it is clear that $j_0^* = j_1^*$, by relation (3) in Definition 20.2. It then follows by category theory that homotopic maps $(X, A) \to (Y, B)$ induce the same map upon applying $\mathcal{K}(-, -)$. Consequently, if $f \colon (X, A) \to (Y, B)$ is part of a homotopy equivalence of pairs then it induces an isomorphism on $\mathcal{K}$-groups. □

Before finishing with our basic exploration of the group $\mathcal{K}(X, A)$, let us note the following simple result:

Proposition 20.13. *For any paracompact Hausdorff space X there is an isomorphism $\mathcal{K}(X, \emptyset) \to K^0_{Grt}(X)$ given by the formula $[E_\bullet] \to \sum_i (-1)^i [E_i]$, where $K^0_{Grt}(X)$ is the Grothendieck group of vector bundles on X.*

Proof. It is immediate that the indicated formula gives a group homomorphism $\chi \colon \mathcal{K}(X, \emptyset) \to K^0_{Grt}(X)$; the only nontrivial part is verifying relation (3), but here one uses that if F is a vector bundle on $X \times I$ then $F|_{X \times 0} \cong F|_{X \times 1}$ by Proposition 14.1.

There is also the evident map $j \colon K^0_{Grt}(X) \to \mathcal{K}(X, \emptyset)$ sending a vector bundle E to the chain complex $E[0]$ consisting of E in degree 0 and zeros in all other degrees. Certainly $\chi \circ j = \mathrm{id}$.

If $E_\bullet$ is any chain complex of vector bundles on X then we may deform $E_\bullet$ to the complex with zero differentials, by putting a t in front of all the d maps and letting $t \mapsto 0$. So $[E_\bullet] = [(E_\bullet, d = 0)] = \sum_i [\Sigma^i E_i]$ in $\mathcal{K}(X, \emptyset)$. But by Lemma 20.5 we know $[\Sigma^i E_i] = (-1)^i [E_i]$. This proves that $j \circ \chi = \mathrm{id}$. □

Let $E_\bullet$ and $F_\bullet$ be bounded chain complexes of vector bundles on X. Let $E_\bullet \otimes F_\bullet$ denote the usual tensor product of chain complexes, giving another complex of

vector bundles on X. In contrast to this, there is also an *external* tensor product. If $G_\bullet$ is a complex of vector bundles on a space Y, define

$$E_\bullet \hat{\otimes} G_\bullet = \pi_1^*(E_\bullet) \otimes \pi_2^*(G_\bullet)$$

where $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are the two projections. Note that if $\Delta \colon X \to X \times X$ is the diagonal map then $E_\bullet \otimes F_\bullet \cong \Delta^*(E_\bullet \hat{\otimes} F_\bullet)$.

The internal and external tensor products induce pairings on the $\mathcal{K}$-groups defined above, taking the form

$$\otimes \colon \mathcal{K}(X, A) \otimes \mathcal{K}(X, B) \to \mathcal{K}(X, A \cup B)$$

and

$$\hat{\otimes} \colon \mathcal{K}(X, A) \otimes \mathcal{K}(Y, B) \to \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)).$$

The main point is that if $V_\bullet$ and $W_\bullet$ are bounded exact sequences of vector spaces and $V_\bullet$ is exact, then $V_\bullet \otimes W_\bullet$ is exact. It follows that if $E_\bullet$ and $F_\bullet$ are complexes on X, with $E_\bullet$ exact on A and $F_\bullet$ exact on B, then $E_\bullet \otimes F_\bullet$ is exact on $A \cup B$ (with a similar analysis for the external case). Note again that the internal and external tensor products are connected by the formula

$$[E_\bullet] \otimes [F_\bullet] = \Delta^*\big([E_\bullet] \hat{\otimes} [F_\bullet]\big).$$

The following theorem is essentially due to Atiyah, Bott and Shapiro [ABS].

Theorem 20.14. *There is a natural transformation of functors $\chi \colon \mathcal{K}(X, A) \to K^0(X, A)$ such that when $A = \emptyset$ one has $\chi(E_\bullet) = \sum_i (-1)^i [E_i]$, and all such natural transformations agree on pairs (X, A) where both X and A are homotopically compact. Moreover, χ is a natural isomorphism on pairs that are homotopy equivalent to a finite CW-pair. Also, χ is compatible with (external and internal) products in the sense that $\chi(E_\bullet \otimes F_\bullet) = \chi(E_\bullet) \cdot \chi(F_\bullet)$ for mateable pairs (X, A) and (Y, B) where all of X, Y, A, and B are homotopically compact.*

The proof of Theorem 20.14 involves some technicalities that would be a distraction at this particular moment, so we postpone the proof until Section 24 below. See, in particular, Section 24.43 for the final proof.

In light of the above theorem, it is unclear how well-behaved the groups $\mathcal{K}(X, A)$ are for pairs (X, A) that are not homotopy equivalent to a finite CW-pair. This is unfortunate, because we will need to work with groups like $K^0(X, X - Z)$ where Z is a closed subvariety of X. The above result does not allow us to replace these with the analogous $\mathcal{K}$ groups. Still, we do have a map from the latter to the former, and that is often enough for us. Essentially, the $\mathcal{K}$ construction is good for producing elements and relations in the K^0 groups, even when it is not good for computing them explicitly.

From now on we assume that a specific natural transformation χ has been chosen, but the indeterminancy in this choice will not affect any arguments we ever need to make.

20.15. **Relative exact complexes.** There is another model for relative K-groups that works somewhat better for computations, but at the expense of not having a good product structure. Given a pair (X, A), define an ***A*-relative exact complex** on X to be a collection of vector bundles $\{E_i\}_{i \in \mathbb{Z}}$ on X together with a differential on $(E_\bullet)|_A$ that gives an exact chain complex. Note that the differentials are only defined over A, not on all of X. A map $E_\bullet \to E'_\bullet$ of A-relative exact complexes is

a collection of vector bundles maps $\phi_i\colon E_i \to E_i'$ whose restrictions to A commute with the differentials. It is important that the ϕ's are defined on all of X.

Every chain complex of vector bundles on X that is exact over A gives an A-relative exact complex, just by restricting the differentials.

Now we mimic the definition of $\mathcal{K}(X,A)$ but using these A-relative exact complexes:

Definition 20.16. *Let $\mathcal{F}(X,A)_A$ be the free abelian group on isomorphism classes of bounded A-relative exact complexes on X. Define $\mathcal{K}(X,A)_A$ to be the quotient of $\mathcal{F}(X,A)_A$ by the following relations:*

(1) $[E_\bullet \oplus F_\bullet] = [E_\bullet] + [F_\bullet]$;

(2) $[E_\bullet] = 0$ *whenever $E_\bullet$ is the restriction of an exact complex of vector bundles on X;*

(3) If $\mathcal{E}_\bullet$ is a bounded $A \times I$-relative exact complex of vector bundles on $X \times I$ then $[\mathcal{E}|_{X\times 0}] = [\mathcal{E}|_{X\times 1}]$.

There is an evident map of groups $\mathcal{K}(X,A) \to \mathcal{K}(X,A)_A$, but in general there is no reason to believe this is either injective or surjective. But we have the following:

Theorem 20.17. *Suppose that $A \hookrightarrow X$ is a cofibration and X is paracompact Hausdorff. Then the map $\mathcal{K}(X,A) \to \mathcal{K}(X,A)_A$ is an isomorphism.*

We again postpone the proof until Section 24. Combining with Theorem 20.14 this gives a natural transformation $\mathcal{K}(X,A)_A \to K^0(X,A)$ for cofibrant pairs (X,A) that is an isomorphism when X and A are homotopically compact.

Exercise 20.18. Think about what goes wrong when trying to use tensor product of relative exact chain complexes to model the product on K-groups.

We close this section with an extended example. We know from the Bott computations that $\widetilde{KO}^0(S^1) = \mathbb{Z}/2 = \widetilde{KO}^0(S^2)$. The question arises as to whether the square of the nonzero class in $\widetilde{KO}^0(S^1)$ is equal to the nonzero class in $\widetilde{KO}^0(S^2)$. We will prove that this is so by a purely geometric/algebraic analysis.

Let $M \to S^1$ be the Möbius bundle. We know that M generates $KO^{st}(S^1) = \mathbb{Z}/2$, or equivalently that $M - 1$ generates $\widetilde{KO}^0(S^1)$. Likewise, let $E \to S^2$ be the rank 2 bundle whose clutching function is the standard generator of $\pi_1(SO(2))$. The computation $\pi_1 SO = \mathbb{Z}/2$ shows that $\widetilde{KO}^0(S^2) = \mathbb{Z}/2$ is generated by E. Refer to Section 15 for details.

We lift the problem from spheres to disks via the diagram

$$\begin{array}{ccc} KO^0(D^1,\partial D^1) \otimes KO^0(D^1,\partial D^1) & \longrightarrow & KO^0(D^2,\partial D^2) \\ \uparrow \cong & & \uparrow \cong \\ KO^0(S^1,*) \otimes KO^0(S^1,*) & \longrightarrow & KO^0(S^2,*). \end{array}$$

The upper horizontal map is modeled by the tensor product of complexes, by Theorem 20.14. Note that this is *not* true for the bottom map. The bottom map is

$$KO^0(S^1,*) \otimes KO^0(S^1,*) \longrightarrow KO^0(S^1 \times S^1, S^1 \vee S^1) \xleftarrow{\cong} KO^0(S^1 \wedge S^1, *)$$

and only the first map is given by tensor product. This is a standard problem, where there is no geometric way to move from relative classes on a product to classes on the smash product.

Let $K(x)$ be the complex $\underline{1} \xrightarrow{x} \underline{1}$ on D^1, where the differential is multiplication by x over the point $x \in D^1 = [-1, 1]$. This represents the generator of $KO^0(D^1, \partial D^1)$. We know it represents the generator because when restricted to ∂D^1 we get the map $\partial D^1 \to O(1)$ that is the clutching function for our generator M of $KO^{st}(S^1)$. (Note that the 'K' in $K(x)$ is for Koszul, not for a K-group; see Section 27.5 for more on Koszul complexes).

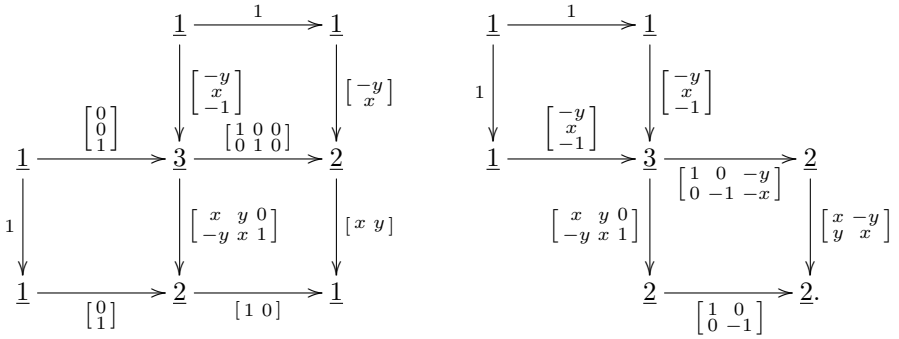
Let E be the complex $\underline{2} \xrightarrow{z} \underline{2}$ on D^2, where we identify $\mathbb{R}^2$ with $\mathbb{C}$ and where the differential is complex multiplication by z on the point over $z \in D^2$. Said differently, over $(x, y) \in D^2$ the differential is left multiplication by $\left[\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right]$. Then E represents the generator of $KO^0(D^2, \partial D^2)$: again, restricting the differential to ∂D^2 gives the map $S^1 \to SO(2)$ which is the clutching function for the generator of $KO^{st}(S^2)$.

Our task is to show that $K(x) \hat{\otimes} K(x)$ equals E as elements of $KO^0(D^2, \partial D^2)$. The former is the complex $K(x, y)$ given by

$$0 \longrightarrow \underline{1} \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} \underline{2} \xrightarrow{\left[\begin{smallmatrix} x & y \end{smallmatrix}\right]} \underline{1} \longrightarrow 0,$$

where again we are indicating the differentials over the point (x, y). So our task is this: using the relations that define $\mathcal{KO}(D^2, \partial D^2)$, find an explicit sequence that connects $K(x, y)$ to E. This is challenging to do directly, but it is much easier if we use $\mathcal{KO}(D^2, \partial D^2)_{\partial D^2}$.

In the following diagram, the complexes are written vertically. These are two exact sequences of ∂D^2-relative exact complexes on D^2:



Note that for the complex in the middle (which is the same in the two cases), the differential only squares to zero when $(x, y) \in \partial D^2$. So this is only a ∂D^2-relative exact complex on D^2.

Exercise 20.19. First check that all the squares in the above diagram commute and that the horizontal sequences are short-exact. Then check that in each case the inclusion is split-exact (as a map of relative exact complexes). Therefore the short exact sequences actually give direct sum decompositions.

Let C be the middle complex, which is the same in the two cases. By the above exercise we have the following relations in $\mathcal{KO}(D^2, \partial D^2)_{\partial D^2}$:

$$C = (\text{elementary complex}) + K(x, y), \qquad C = (\text{elementary complex}) + E.$$

Since elementary complexes are zero in $\mathcal{KO}(D^2, \partial D^2)_{\partial D^2}$, this shows $K(x,y) = C = E$.

The above argument is very elegant, and simple enough that even a linear algebra student could understand it. But the reader is probably wondering where those two exact sequences came from—we just pulled them out of our hat. In Section 24 we will see a general procedure for taking a relative exact complex of length $n \geq 2$ and "folding the top degree down" so as to give an equivalent relative exact complex of length $n-1$, and the above exact sequences come from applying that method to the complex $K(x,y)$. The middle complex C is an example of the $\tilde{\Gamma}$-construction from Section 24.16.

The folding construction can be applied iterative to reduce any relative exact complex to an equivalent complex of length one. Here is one general result along these lines:

Proposition 20.20. *Let (X, A) be a pair. Let $(E_\bullet, d)$ be a bounded A-relative exact complex on X. Suppose given inner products on all of the bundles E_i and let d^* be the adjoints of the d maps. Then*

$$0 \to \bigoplus_{i \text{ odd}} E_i \xrightarrow{d+d^*} \bigoplus_{i \text{ even}} E_i \to 0$$

is an A-relative exact complex that represents the same class as $E_\bullet$ in $\mathcal{KO}(X,A)_A$. [For complex K-theory, do the same thing but with Hermitian inner products.]

For the proof of this result, see Exercise 24.47.

Exercise 20.21. Use Proposition 20.20 to give a much-streamlined proof that $K(x,y)$ and E represent the same element of $KO^0(D^2, \partial D^2)$.

Exercise 20.22. [Challenge] Consider the third tensor power of $K(x)$, which lies in $KO(D^3, \partial D^3)$. It is easy enough to write this complex down explicitly in terms of matrices. On the one hand, we know this class in $KO(D^3, \partial D^3)$ must be zero because the group is $\widetilde{KO}(S^3) = 0$. Find explicit relations showing that the complex is zero in $\mathcal{KO}(D^3, \partial D^3)_{\partial D^3}$. [Note: These are not easy to find, though the final answer isn't incredibly difficult either. The exercise is to try, not necessarily to succeed. For more information see [D7, Section 20].]

21. Complex Bott periodicity via the K-functor

The sections on Bott periodicity in [D7] give a very detailed treatment, from a variety of perspectives. This section, in contrast, is more of a brief introduction. Here we will prove the complex result, in the version $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. Atiyah and Bott [AB] discovered how to give a fairly low-tech proof using the K^0-functor, and we explain their method. This proof is also given in [Atiyah5, Chapter 2.2].

Initial readers might be well-advised to stop after Corollary 21.8 and move on to Section 22. The remainder of the current section is occupied with the proof itself, which involves some knotty technical details. There are good ideas worth learning at some point, but their use is mostly confined to just this one proof—they don't reappear elsewhere in the book.

Before we dive in let us try to give a bit of intuition. Bott periodicity involves two equivalences: $\Omega(\mathbb{Z} \times BU) \simeq U$ and $\Omega U \simeq \mathbb{Z} \times BU$. If we regard $\mathbb{Z} \times BU$ as something

like a moduli space of (stable) subspaces of $\mathbb{C}^\infty$, then the first equivalence says that a loop of subspaces is—in some homotopical sense—the same as a (unitary) automorphism. There is a certain geometric intuition behind this: if one pictures a k-dimensional subspace M of $\mathbb{C}^\infty$ moving around and coming back to itself, one can choose an orthonormal basis for M and watch that move along as well. It comes back to a possibly *different* orthonormal basis for M, and one gets a unitary transformation from M to itself by sending the first basis to the second. Of course this is just path lifting in the fibration sequence $U_k \to V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ and is the 0-level of the equivalence $\Omega\,\mathrm{Gr}_k(\mathbb{C}^\infty) \simeq \Omega BU_k \simeq U_k$.

The *second* equivalence of Bott periodicity is the less obvious one. It says that a loop of (unitary) automorphisms of $\mathbb{C}^\infty$ is—again, in some homotopical sense—the same as a (stable) subspace. Here it is much harder to see how to make a geometric connection between these two; no method for taking a loop of automorphisms and extracting a subspace jumps out. The Atiyah-Bott argument boils down to two ideas. First, for certain basic loops (what we will call *linear loops*) of automorphisms, functional analysis does provide a very nice way to extract a subspace[1]. Second, by using some stabilization techniques that are familiar from playing around with the K^0-functor, one can with only a little trouble reduce to the special case just mentioned. Everything else is just bookkeeping to make this line of argument work.

21.1. **Background material.** For this section we will NOT assume that we have K as a cohomology theory. All we have is the K^0 functor given by $K^0(X) = \mathrm{Ho}\,\mathcal{T}(X, \mathbb{Z} \times BU)$, which for semi-cofibrant compact Hausdorff spaces X is isomorphic to the Grothendieck group of vector bundles. For $A \subseteq X$ we defined $K^0(X, A) = \mathrm{Ho}\,\mathcal{T}_*(Cj, \mathbb{Z} \times BU)$ where Cj is the mapping cone of the inclusion $j\colon A \hookrightarrow X$. When X is pointed we define $\widetilde{K}^0(X) = K^0(X, pt) = \mathrm{Ho}\,\mathcal{T}_*(X, \mathbb{Z} \times BU)$. These definitions gives us half of a long exact sequence:

$$K^0(A) \longleftarrow K^0(X) \longleftarrow K^0(X, A) \longleftarrow \widetilde{K}^0(\Sigma(A_+)) \longleftarrow \widetilde{K}^0(\Sigma(X_+)) \longleftarrow \cdots$$

By Section 19 we have a pairing $(\mathbb{Z} \times BU) \wedge (\mathbb{Z} \times BU) \to \mathbb{Z} \times BU$ that makes $K^0(X)$ into a ring and gives external pairings $K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$, both agreeing with the tensor product of bundles when the spaces involved are compact, Hausdorff, and semi-cofibrant. We also have relative pairings $K^0(X, A) \otimes K^0(Y, B) \to K^0(X \times Y, X \times B \cup A \times Y)$ when the pairs (X, A) and (Y, B) are mateable.

We will need to know $K^0(S^1)$ and $K^0(S^2)$. For the rest of this section let us identify $S^2 \cong \mathbb{C}P^1$, as this gives us the tautological line bundle $L \to \mathbb{C}P^1$ and all of its powers (positive and negative).

Proposition 21.2. *$K^0(S^1) = 0$ and $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$ with generators $\underline{1}$ and L.*

Proof. These follow directly from our analysis of bundles on spheres via clutching functions back in Section 15.21: $\mathrm{Vect}_n^{\mathbb{C}}(S^1) \cong \pi_0(U_n) = \{*\}$ and $\mathrm{Vect}_n^{\mathbb{C}}(S^2) \cong \pi_1(U_n) = \mathbb{Z}$ for all $n \geq 1$. In the second case let us be careful and note the

[1]One key example of this is easy to explain: if $T\colon \mathbb{C}^N \to \mathbb{C}^N$ is a linear transformation with no eigenvalues on the unit circle then $z \mapsto zI - T$ is a linear loop of automorphisms of $\mathbb{C}^N$, and the associated subspace will be related to the eigenspaces of T.

isomorphisms of groups $\widetilde{K}^0(S^2) = [S^2, \mathbb{Z}\times BU]_* \cong [S^1, U]_* = \pi_1(U) \cong \mathbb{Z}$, the second using $\Omega(\mathbb{Z}\times BU) = \Omega BU \simeq U$. Therefore $K^0(S^2) \cong \mathbb{Z}\oplus\mathbb{Z}$.

To write down generators for $K^0(S^2)$ observe that since the inclusions $U_n \hookrightarrow U_{n+1}$ induce isomorphisms on π_1 (use the fiber sequence $U_n \hookrightarrow U_{n+1} \twoheadrightarrow S^{2n+1}$), the $(-)\oplus\underline{1}$ maps $\mathrm{Vect}^{\mathbb{C}}_n(S^2) \to \mathrm{Vect}^{\mathbb{C}}_{n+1}(S^2)$ are bijections for all $n \geq 1$. So we get a generator for $K^{st}(S^2)$ by taking any complex line bundle whose clutching function generates $\pi_1(U_1) = \mathbb{Z}$. The bundle L has clutching function $z \mapsto z^{-1}$, so this does the job. □

At the end of the above proof we could also have chosen L^* as the generator, and it is useful to understand how the two choices relate. The bundle L on S^2 can be obtained by the clutching function $S^1 \to GL_1(\mathbb{C})$ given by $z \mapsto z^{-1}$: indeed, if we take the section $[1:w] \xrightarrow{s_+} (1,w)$ on the upper half of $\mathbb{C}P^1$ (containing the point at infinity $[1:0]$) and the section $[z:1] \xrightarrow{s_-} (z,1)$ on the lower half of $\mathbb{C}P^1$, then on the equator $S^1 = \{z = [z:1]\}$ we have $s_1(z) = \frac{1}{z}\cdot s_2(z)$.

We claim that $2L \cong L^2 \oplus \underline{1}$. One way to see this is via the following sequence of transvective row and column operations:

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}\end{bmatrix} \simeq \begin{bmatrix} z^{-1} & 1 \\ 0 & z^{-1}\end{bmatrix} \simeq \begin{bmatrix} 1 & 1 \\ z^{-1}-z^{-2} & z^{-1}\end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ z^{-1}-z^{-2} & z^{-2}\end{bmatrix}$$
$$\simeq \begin{bmatrix} 1 & 0 \\ 0 & z^{-2}\end{bmatrix}.$$

Each transvective operation yields a corresponding homotopy between clutching functions: take the elementary matrix representing the operation and add a factor of t onto the unique off-diagonal entry. So all of the clutching functions in the above chain yield isomorphic bundles. The leftmost matrix is the clutching function for $2L$, and the rightmost is the one for $\underline{1}\oplus L^2$.

So we have that $L^2 = 2L - 1$ in $K^0(S^2)$. Multiplying through by L^* gives $L = 2 - L^*$, or $L^* = 2 - L$. We will often use this in the version

$$L^* - 1 = 1 - L. \tag{21.3}$$

Observe that $\widetilde{K}^0(S^2) \cong \mathbb{Z}$, generated by $1 - L$. We take this as our preferred generator. Some authors use $-(1-L) = 1 - L^*$, but $1 - L$ is the most standard and works best with regard to issues we will encounter later coming from algebraic geometry.

21.4. **Bott periodicity and the K^0-functor.** The following K^0-version of Bott periodicity is our main goal in this chapter:

Theorem 21.5 (Complex Bott periodicity)**.** *The external tensor product map $K^0(X)\otimes K^0(S^2) \to K^0(X\times S^2)$ is an isomorphism for all spaces X that are compact Hausdorff and semi-cofibrant.*

Since $K^0(S^2) \cong \mathbb{Z}\oplus\mathbb{Z}$, the above result yields that $K^0(X\times S^2) \cong K^0(X)\oplus K^0(X)$. This does not immediately look like a periodicity theorem, but with a little massaging we can turn it into one.

Corollary 21.6. *The external pairing $K^0(X)\otimes\widetilde{K}^0(S^2) \to K^0(X\times S^2, X\times pt)$ is an isomorphism when X is compact Hausdorff and semi-cofibrant. In particular, since $\widetilde{K}^0(S^2) \cong \mathbb{Z}$ and is generated by $1-L$ the map $(1-L)\otimes(-)\colon K^0(X) \longrightarrow K^0(X\times S^2, X\times pt)$ is a natural isomorphism.*

If $(X, *)$ is a finite CW-pair then $\widetilde{K}^0(X) \otimes \widetilde{K}^0(S^2) \to K^0(X \times S^2, X \times pt \cup pt \times S^2)$ is also an isomorphism, and from this one obtains a natural isomorphism $\widetilde{K}^0(X) \cong \widetilde{K}^0(X \wedge S^2)$.

The assumption that $(X, *)$ is a finite CW-pair is stronger than what we actually need for the second part, but it covers all of our applications and avoids some technical issues. The argument actually works as long as $X \vee S^2 \hookrightarrow X \times S^2$ is a cofibration.

Proof. Start with the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0(X) \otimes K^0(S^2, *) & \longrightarrow & K^0(X) \otimes K^0(S^2) & \longrightarrow & K^0(X) \otimes K^0(pt) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & K^0(X \times S^2, X \times *) & \longrightarrow & K^0(X \times S^2) & \longrightarrow & K^0(X) & \longrightarrow & 0. \end{array}$$

The top row is the result of tensoring $K^0(X)$ onto a split-exact sequence, hence is also split-exact. The bottow row is also split-exact. Theorem 21.5 says that the middle vertical map is an isomorphism, so the left map is as well.

For the second part we consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}^0(X) \otimes \widetilde{K}^0(S^2) & \longrightarrow & K^0(X) \otimes \widetilde{K}^0(S^2) & \longrightarrow & K^0(pt) \otimes \widetilde{K}^0(S^2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & K^0(X \times S^2, X \vee S^2) & \longrightarrow & K^0(X \times S^2, X \times *) & \longrightarrow & K^0(S^2, *) & \longrightarrow & 0 \end{array}$$

Again, the top row is split-exact and the right two vertical maps are isomorphisms. So if we can show the bottom row is exact, we are done. This bottom row comes from the sequence of mapping cones along the top row here:

$$\begin{array}{ccccc} C(X \times S^2, X \vee S^2) & \longleftarrow & C(X \times S^2, X \times *) & \longleftarrow & C(S^2, *) \\ \downarrow & & \downarrow & & \downarrow \\ X \wedge S^2 & \longleftarrow & X_+ \wedge S^2 & \longleftarrow & S^2. \end{array}$$

The vertical maps are the canonical projections from the mapping cones down to the quotients. Our assumptions on X imply that these vertical maps are all weak equivalences, and also that the similar projection $C(X_+ \wedge S^2, S^2) \to X \wedge S^2$ is a weak equivalence: this is because, for each pair, the inclusion of the smaller space into the larger one is a cofibration. Applying $\mathrm{Ho}\,\mathcal{T}_*(-, \mathbb{Z} \times BU)$ we find that the sequence we are trying to show is exact coincides with the right three terms of the long exact sequence

$$\cdots \to \widetilde{K}^0(\Sigma(S^2_+)) \longrightarrow K^0(X_+ \wedge S^2, S^2) \longrightarrow \widetilde{K}^0(X_+ \wedge S^2) \longrightarrow \widetilde{K}^0(S^2).$$

The rightmost map, as well as its mate three steps to the left, are both surjective: they are split by the map induced from the projection $X_+ \wedge S^2 \to S^2$. So we have our desired short exact sequence. □

Corollary 21.7. *For $n \geq 1$ we have $\widetilde{K}^0(S^{2n-1}) = 0$ and $\widetilde{K}^0(S^{2n}) \cong \mathbb{Z}$ with generator $(1 - [L])^{\otimes(n)}$, where $L \to \mathbb{C}P^1 = S^2$ is the tautological line bundle. In*

particular, the ring $\bigoplus_{n\geq 0}\widetilde{K}^0(S^n)$ *is the polynomial ring* $\mathbb{Z}[\beta]$ *where* $\beta = 1-[L]\in \widetilde{K}^0(S^2)$.

Proof. This follows immediately from Proposition 21.2 and Corollary 21.6. □

Corollary 21.8. *There is a weak equivalence* $\Omega^2(\mathbb{Z}\times BU)\simeq \mathbb{Z}\times BU$.

Proof. We have the map $(\mathbb{Z}\times BU)\wedge(\mathbb{Z}\times BU)\to \mathbb{Z}\times BU$ that represents external tensor product of bundles. We also have a pointed map $S^2\to \mathbb{Z}\times BU$ representing the element $1-L\in\widetilde{K}^0(S^2)$. Consider the composite

$$S^2\wedge(\mathbb{Z}\times BU)\to(\mathbb{Z}\times BU)\wedge(\mathbb{Z}\times BU)\xrightarrow{\mu}\mathbb{Z}\times BU.$$

The adjoint $\mathbb{Z}\times BU\to\Omega^2(\mathbb{Z}\times BU)$ has the property that for X compact Hausdorff and semi-cofibrant the induced map

$$[X,\mathbb{Z}\times BU]_*\to[X,\Omega^2(\mathbb{Z}\times BU)]_*=[S^2\wedge X,\mathbb{Z}\times BU]_*$$

represents $(1-L)\otimes(-)\colon \widetilde{K}^0(X)\to\widetilde{K}^0(X\wedge S^2)$. Corollary 21.6 implies that for all $n\geq 0$ this induced map is an isomorphism for $X=S^n$. This says precisely that our map of spaces is a weak homotopy equivalence. □

Exercise 21.9. If we make $\mathbb{Z}\times BU$ into a double-loop space via the equivalence from the above proof, explain why the induced group structure on $[X_+,\mathbb{Z}\times BU]_*$ is the one corresponding to direct sum of vector bundles. (This should just be a matter of making some elementary observations, though it does require care.) Even better, prove that the map $\mathbb{Z}\times BU\to\Omega^2(\mathbb{Z}\times BU)$ from the above proof is a map of H-spaces. [Hint: On some level this boils down to the tensor product distributing over direct sum. But to get the actual homotopy-commutative diagram that you want, an argument with the Milnor exact sequence like we saw in Section 19.14 might be necessary.]

21.10. **The essence of periodicity.** It remains to prove Theorem 21.5, and this will occupy us for the rest of this section. From now on X is a fixed compact Hausdorff space. We will need to get our hands on the bundle theory of $X\times S^2$.

If J and J' are bundles on a space W recall the vector bundle $\underline{\mathrm{Hom}}(J,J')\to W$, its space of sections $\mathrm{Hom}(J,J')$, the fiber bundle $\underline{\mathrm{Iso}}_W(J,J')\subseteq\underline{\mathrm{Hom}}(J,J')$, and its space of sections $\mathrm{Iso}_W(J,J')\subseteq\mathrm{Hom}(J,J')$. See Section 11.1 and Exercise 14.9. These will appear often in our discussion. The appearance of W as a subscript is not necessary, but as we will be dealing with multiple base spaces it will often be a convenient reminder.

Let D_+ and D_- denote the upper and lower hemispheres of S^2. Here is one way to make a bundle on $X\times S^2$. Start with a bundle $E\to X$, then form $J_+=\pi_1^*(E)$ on $X\times D_+$ and $J_-=\pi_1^*(E)$ on $X\times D_-$. Pick an isomorphism $f\colon (J_+)|_{X\times S^1}\xrightarrow{\cong}(J_-)|_{X\times S^1}$ and then form a new bundle on $X\times S^2$ using f as a clutching function. Let us denote this new bundle $E(f)$. Note that f can be regarded as an element of $\mathrm{Iso}_{X\times S^1}(\pi^*E,\pi^*E)$, or (equivalently) as a map from S^1 into the space $\mathrm{Iso}_X(E,E)$. We call f an ***E*-clutching function**. Just as we saw in Section 15.1, we can assume that $f\colon S^1\to\mathrm{Iso}_X(E,E)$ sends the basepoint to the identity. If f is homotopic to f' then $E(f)\cong E(f')$. If E_1 and E_2 are two bundles on X and f_1 and f_2 clutching functions associated to each, then

$$(E_1\otimes E_2)(f_1\otimes f_2)\cong E_1(f_1)\otimes E_2(f_2). \tag{21.11}$$

Note as well that $E(\mathrm{id}) = \pi^* E$, where $\pi\colon X \times S^2 \to X$ is the projection. We will usually write $E(1)$ for $E(\mathrm{id})$.

Exercise 21.12. Let f and f' be E-clutching functions. Prove that $E(f) \cong E(f')$ if and only if f and f' are homotopic (as maps $S^1 \to \mathrm{Iso}_X(E,E)$).

Remark 21.13. In a couple of places we will have to deal with two bundles E and E' on X and a clutching function $f \in \mathrm{Iso}_{X\times S^1}(\pi^*(E), \pi^*(E'))$. Obviously this can only exist if $E \cong E'$ and so secretly it is an instance of the above discussion, but sometimes no particular isomorphism $E \to E'$ is in sight. In this case we will write (E, f, E') for the resulting bundle on $X \times S^2$.

Let us write $\mathcal{L} = \underline{1}_X \hat{\otimes} L$ on $X \times S^2$; this can also be described as $\mathcal{L} = \pi^* L$ where $\pi\colon X \times S^2 \to S^2$ is the projection. From now on we identify the equator of S^2 with the standard circle $S^1 \subseteq \mathbb{C}$, and under our identification $S^2 \cong \mathbb{C}P^1$ we let D_+ be the hemisphere containing the point at infinity. We let z denote the inclusion $S^1 \hookrightarrow \mathbb{C}$, regarded as a complex-valued function on the circle. Note that $\underline{1}_X(z) \cong \mathcal{L}^*$, and so by (21.11) we have the equivalent statements

$$E(zf) \cong E(f) \otimes \mathcal{L}^*, \qquad E(f) \cong E(zf) \otimes \mathcal{L}. \tag{21.14}$$

Since all bundles on $X \times D_+$ are (up to isomorphism) pulled back from X, and likewise for $X \times D_-$, we know that every bundle on $X \times S^2$ is isomorphic to $E(f)$ for some bundle $E \to X$ and some E-clutching function f. Our next task will be to understand how to simplify these clutching functions as much as possible.

Given any $a \in \mathrm{Iso}_X(E,E)$ and any $k \in \mathbb{Z}$, the map $z^k a$ is an E-clutching function. If $a_k \in \mathrm{Hom}(E,E)$ then a sum

$$\sum_{k=-n}^{n} z^k a_k$$

may or may not be an E-clutching function, depending on whether or not the values for $z \in S^1$ are always isomorphisms. When this is the case, we call it a **Laurent clutching function**. A Laurent clutching function for which there are no negative powers of z in the sum is called a **polynomial clutching function**.

Choose a metric for the bundle E. The choice will not actually matter in the end, but to talk about Fourier approximations it is convenient to be able to measure how close functions are to each other. The theory of Fourier series tells us that for any E-clutching function f there is a sequence of Laurent clutching functions $\{f_n\}$ that converges to f uniformly on $X \times S^1$ (here it is important that X is compact, of course).

Remark 21.15. Here is a very brief review of the Fourier series material we needed for the above. Let $f\colon S^1 \to V$ be a map into a finite-dimensional complex vector space. For each $k \in \mathbb{Z}$ define

$$a_k = \tfrac{1}{2\pi i} \int_{S^1} f(z) z^{-(k+1)}\, dz \in V.$$

Define $S_n(z) = \sum_{k=-n}^{n} a_k z^k$, the nth Fourier approximation to f. The sequence $\{S_n\}$ does not necessarily converge uniformly to f. To correct this issue one instead looks at the so-called *Cesaro means* $f_n = \frac{1}{n+1}\sum_{k=0}^{n} S_n$. These functions *do*

converge uniformly to f, and f_n is still a Laurent polynomial function of degree at most n.

All of this carries over to the bundle setting. If $E \to X$ is a bundle and $f\colon S^1 \to \mathrm{Hom}(E,E)$ then performing the constructions fiberwise yields $a_k \in \mathrm{Hom}(E,E)$ and so forth.

To aid our discussion and help isolate what we really care about, let us make the following definitions. If $J \to W$ is a vector bundle and $U \subseteq J$, say that U is **fiberwise convex** if $U \cap J_w$ is convex (as a subset of J_w) for every $w \in W$. This will be useful to us because any two sections of J that land inside U are homotopic via the straight-line homotopy.

Proposition 21.16. *Let $p\colon J \to W$ be a vector bundle over a compact space W, and let $s\colon W \to J$ be a section. Let $U \subseteq W$ be an open subset that contains $s(W)$. Then there exists an open set V such that $s(W) \subseteq V \subseteq U$ and such that V is fiberwise convex.*

Proof. Choose a metric for the bundle J. For $\epsilon > 0$ define $B_\epsilon(s) = \{v \in J \mid d_{p(v)}(v, sp(v)) < \epsilon\}$. In words, this is a small tube of fiberwise-radius ϵ around the image of s. It is an open subset of J containing $s(W)$, and it is fiberwise convex.

If $U \subseteq J$ is any open set containing $s(W)$ and W is compact, there exists an ϵ such that $B_\epsilon(s) \subseteq U$—this is a slight generalization of the usual "tube lemma". One readily produces such ϵ's locally, and then by compactness finds that a single ϵ works for the whole space. □

Now return to the setting where f is an E-clutching function. Then f is a section of $\underline{\mathrm{Hom}}_{X\times S^1}(\pi^*E, \pi^*E)$ that lands inside the open subset $\underline{\mathrm{Iso}}_{X\times S^1}(\pi^*E, \pi^*E)$. Say that a collection of E-clutching functions $\{f_\alpha\}$ is **ultra-close** to f if their images all lie in some fibrewise-convex neighborhood V of the image of f such that $V \subseteq \underline{\mathrm{Iso}}_{X\times S^1}(\pi^*E, \pi^*E)$. This implies that the straight-line homotopy between any two of the sections is a homotopy of E-clutching functions.

Returning to our discussion of Fourier series, uniform convergence of $\{f_n\}$ implies that for some $N \geq 0$ the collection $\{f_n\}_{n\geq N}$ is ultra-close to f (in fact it tells us that for any $\epsilon > 0$ the graphs of f_n land inside $B_\epsilon(f)$ for $n \gg 0$). Therefore the clutching functions f_n and f are homotopic, so we then have $E(f) \cong E(f_n)$. This proves that all bundles on $X \times S^2$ can be obtained via a Laurent clutching function.

Replacing f_n by $z^n f_n$ gets us a polynomial clutching function, but at the expense of tensoring the bundle with $\mathcal{L}^{-n}$. Said differently, $E(f_n) \cong E(z^n f_n) \otimes \mathcal{L}^n$. So up to tensoring with powers of $\mathcal{L}$, all bundles on $X \times S^2$ can be obtained by polynomial clutching functions.

The next reduction is to *linear* clutching functions. Suppose $p(z) = \sum_{k=0}^n A_k z^k$ is a polynomial E-clutching function, so that each $A_k \in \mathrm{Hom}(E,E)$. We can produce a linear function $\mathrm{Lin}_{n+1}(p)$ on $E^{\oplus(n+1)}$ via the block matrix formula

$$\mathrm{Lin}_{n+1}(p) = \begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_n \\ -zI & I & 0 & \cdots & 0 \\ 0 & -zI & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & -zI & I \end{bmatrix}. \tag{21.17}$$

We will often abbreviate $\mathrm{Lin}_{n+1}(p) = \mathrm{Lin}(p)$.

Remark 21.18 (Important)**.** For the rest of this section we will abbreviate $E^{\oplus(s)}$ to $E^{(s)}$. The parentheses serve to distinguish the sum from a tensor product.

The set $R = \mathrm{Hom}_{X\times S^1}(\pi^*E, \pi^*E)$ is a ring, from which we obtain the group $GL_{n+1}(R)$. This group acts on $(\pi^*E)^{(n+1)}$, and therefore on the set of $E^{(n+1)}$-clutching functions. This action can be regarded as row and column operations on block matrices as above. Use of transvective row and column operations readily shows that the matrix in (21.17) is equivalent to

$$\begin{bmatrix} p(z) & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \tag{21.19}$$

(inductively reduce the A_i blocks to zero, starting with A_n and working backwards). In particular, the original matrix is nonsingular. Hence $\mathrm{Lin}(p)$ is an $E^{(n+1)}$-clutching function.

For any transvective row or column operation there is a homotopy connecting the initial clutching function to the ending one: take the elementary matrix representing the operation and put a t in front of the unique off-diagonal entry. Therefore the two clutching functions in (21.17) and (21.19) are homotopic and we have $E^{(n+1)}(\mathrm{Lin}(p)) \cong E(p) \oplus nE(1)$.

Proposition 21.20. *Let $E \to X$ be a vector bundle and let p be a polynomial E-clutching function of degree $\le n$. Then*
(a) $E^{(n+1)}(\mathrm{Lin}(p)) \cong E(p) \oplus nE(1)$.
(b) $E^{(n+2)}(\mathrm{Lin}_{n+2}(p)) \cong E^{(n+1)}(\mathrm{Lin}_{n+1}(p)) \oplus E(1)$.
(c) $E^{(n+2)}(\mathrm{Lin}_{n+2}(zp)) \cong E(z) \oplus E^{(n+1)}(\mathrm{Lin}_{n+1}(p))$.

Proof. Part (a) was explained above. For part (b), put a t in from of the $-zI$ in the bottom row of $\mathrm{Lin}_{n+2}(p)$. This gives a homotopy of $E^{(n+2)}$-clutching functions $\mathrm{Lin}_{n+2}(p) \simeq \mathrm{Lin}_{n+1}(p) \oplus I$.

For (c) use the series of row and column operations

$$\begin{bmatrix} 0 & A_0 & A_1 & \cdots \\ -zI & I & 0 & \cdots \\ \vdots & & & \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & A_0 & A_1 & \cdots \\ -zI & 0 & 0 & \cdots \\ \vdots & & & \end{bmatrix} \rightsquigarrow \begin{bmatrix} zI & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & \cdots \\ \vdots & & & \end{bmatrix}.$$

The first multiplies column-block one by z^{-1} and adds it to column-block two, which is a series of transvections. The second exchanges the first two row-blocks and negates the first. Recall that the "transpose two rows and negate one of them" operation can be accomplished by a sequence of transvections (see Lemma 8.10). Since transvections can be translated into straight-line homotopies, we therefore can homotop the $\mathrm{Lin}_{n+2}(zp)$ clutching function to $zI \oplus \mathrm{Lin}_{n+1}(p)$. □

21.21. **Linear S^1-families of isomorphisms.** Fix $r \ge 1$. Bott's original arguments produced an explicit map $\beta_r\colon \mathrm{Gr}_\bullet(\mathbb{C}^r) \to \Omega U(r)$ and showed that this becomes more and more highly connected as $r \mapsto \infty$ (see the discussion in [D7]). The stabilization is a little tricky, but this gives the equivalence $\mathbb{Z}\times BU \to \Omega U$. A homotopy inverse to β_r—if it existed—would take a loop in $U(r)$ and produce an

explicit vector space in $\mathbb{C}^r$. Such an assignment is hard to produce in this generality (and maybe even impossible), but Atiyah and Bott realized that they could produce such a thing in an explicit but important case. For this discussion it will be convenient to replace $U(r)$ with $GL_r(\mathbb{C})$. Define a **linear loop** in $GL_r(\mathbb{C})$ to be a function $S^1 \to GL_r(\mathbb{C})$ of the form $z \mapsto A + Bz$ where $A, B \in M_{r\times r}(\mathbb{C})$ are fixed. We will also call these **linear S^1-families of isomorphisms**. We will explain how to assign an explicit vector space to such a loop, in a way that varies continuously with the loop.

To get us started, let $T\colon \mathbb{C}^r \to \mathbb{C}^r$ be linear and suppose that T has no eigenvalues of norm 1. Then $z \mapsto zI - T$ is a linear loop in $GL_r(\mathbb{C})$. The Jordan normal form of T can be used to obtain a decomposition $\mathbb{C}^r = V_{in} \oplus V_{out}$ into T-invariant subspaces where all eigenvalues of T on V_{in} lie in the interior of the unit disk, and all eigenvalues of T on V_{out} lie outside the unit disk. Just set

$$V_{in} = \sum_{|a|<1} \ker(aI - T)^r, \qquad V_{out} = \sum_{|a|>1} \ker(aI - T)^r.$$

We are literally taking the decomposition of $\mathbb{C}^r$ that gives the Jordan normal form of T and grouping the blocks into two classes, depending on whether the relevant eigenvalue is inside or outside the unit circle. If we now imagine that T is allowed to vary in a continuous family (maintaining always that no eigenvalues have norm 1), then the V_{in} and V_{out} subspaces will also vary continuously. This is intuitively obvious, but a concrete way to see it is to realize that

$$Q_{in} = \tfrac{1}{2\pi i} \int_{S^1} (zI - T)^{-1}\, dz \tag{21.22}$$

is the projection operator on $\mathbb{C}^r$ that projects onto the V_{in} subspace and sends V_{out} to zero. This shows Q_{in}, and hence V_{in}, to be a continuous invariant of T. To justify this integral formula, by placing T into Jordan normal form we can reduce to the case of a single Jordan block $T = aI + N$ where N is nilpotent. Then

$$(zI-T)^{-1} = ((z-a)I-N)^{-1} = \tfrac{1}{z-a}\cdot(I-\tfrac{N}{z-a})^{-1} = \tfrac{1}{z-a}\cdot\left(I+\tfrac{N}{z-a}+\tfrac{N^2}{(z-a)^2}+\cdots\right)$$

(the final sum is finite). Appling $\frac{1}{2\pi i}\int_{S^1}$ and using the Cauchy integral formula gives

$$\tfrac{1}{2\pi i}\int_{S^1} (zI-T)^{-1}\, dz = \begin{cases} I & \text{if } |a|<1, \\ 0 & \text{if } |a|>1. \end{cases}$$

The fact that Q_{in} is the claimed projection operator now follows from putting all the blocks of T together.

The integral formula for Q_{in} is a basic tool in functional analysis, where Q_{in} is called the **Riesz projection**. In general, when one has a simple closed contour Γ that does not pass through any points of the spectrum $\sigma(T)$ of an operator T, then the Riesz projection isolates a T-invariant subspace where the spectrum of the restricted operator is precisely the portion of $\sigma(T)$ enclosed by Γ.

Atiyah and Bott realized that the above story extends to any linear S^1-family of isomorphisms of $\mathbb{C}^r$. If V is a finite-dimensional complex vector space and $A, B \in \mathrm{End}(V)$, consider the function $p\colon \mathbb{C} \to \mathrm{End}(V)$ given by $p(z) = A + Bz$. If $p(z)$ is an isomorphism for every $z \in S^1$ we say this is a "linear S^1-family of isomorphisms of V". If T is an endomorphism of $\mathbb{C}^r$ then the S^1-family $zI - T$ is a family of isomorphisms if and only if T has no eigenvalues on S^1.

Proposition 21.23. *Let $A, B \in M_{r\times r}(\mathbb{C})$ and suppose that $p(z) = A + Bz$ is nonsingular for every $z \in S^1$. Set $Q_{in} = \frac{1}{2\pi i}\int_{S^1} p(z)^{-1}dp(z)$ and $Q'_{in} = \frac{1}{2\pi i}\int_{S^1} dp(z)p(z)^{-1}$.*

(a) Q_{in} and Q'_{in} are projection operators on $\mathbb{C}^r$.

(b) Set $V_{in} = \mathrm{im}(Q_{in})$ and $V_{out} = \mathrm{im}(1 - Q_{in})$, as well as $V'_{in} = \mathrm{im}(Q'_{in})$ and $V'_{out} = \mathrm{im}(1 - Q'_{in})$. Then each $p(z)$ maps V_{in} into V'_{in} and V_{out} into V'_{out}. Furthermore, p decomposes as $p = p_{in} \oplus p_{out}$ where p_{in} (resp., p_{out}) is a linear family of isomorphisms $V_{in} \to V'_{in}$ (resp., $V_{out} \to V'_{out}$).

(c) $\dim V_{in}$ is equal to the number of roots of the polynomial $\det p(z)$ inside the unit circle, counted with multiplicity. Likewise, $\dim V_{out}$ is equal to the number of roots of $\det p(z)$ outside the unit circle, counted with multiplicity.

(d) $p_{in}(z)$ is an isomorphism for all $|z| \geq 1$ and $p_{out}(z)$ is an isomorphism for all $|z| \leq 1$. (In particular, taking $z = 0$ shows that A_{out} is an isomorphism.)

(e) Finally, if we write $p_{in}(z) = A_{in} + B_{in}z$ then B_{in} is an isomorphism (this is related in spirit to taking $z = \infty$ in the previous part).

Proof. The special case $p(z) = zI - T$ was dealt with above. Note that in this case $Q_{in} = Q'_{in}$, $V_{in} = V'_{in}$, etc. If $p(z) = A + Bz$ where B is invertible then we can factor out B and reduce to the previous case. Atiyah-Bott found that a clever change of coordinates allows us to reduce the general case to the one where B is invertible.

Consider a conformal change of coordinate of the form $z = \frac{aw+b}{cw+d}$ with $a, c, b, d \in \mathbb{C}$. We want this to preserve the unit circle, as well as the inside of the unit circle. Up to rotations we might as well assume $1 \mapsto 1$, and to make things easier let us also see what happens if we require $-1 \mapsto -1$. The general formula for a transformation with these properties is $z = \frac{aw+1}{w+a}$. To force it to preserve the circle we need that the image of i has norm 1, or that $|i + a| = |ia + 1| = |a - i|$. This forces a to be real. As the map sends $0 \mapsto a^{-1}$ it will preserve the inside of the circle if $|a| > 1$.

Under the change of coordinate we have

$$p(z) = A + Bz = A + B(\tfrac{aw+1}{w+a}) = \tfrac{(Aa+B)+(A+Ba)w}{w+a}.$$

Now we pause and consider the choice of a. If $A + Ba$ is invertible then we can factor it out of the numerator and reduce to an expression that looks like $wI - T$. This is easy enough to arrange. Since $A + Bz$ is invertible at $z = 1$, it is also invertible in a neighborhood of 1. So we can choose a real number $a > 1$ such that $A + Ba$ is invertible. Let us set $q(z) = (A + Ba)^{-1}p(z) = p(a)^{-1}p(z)$, which is another linear S^1-family of isomorphisms. Moreover, $q^{-1}dq = p^{-1}dp$ and $dq \cdot q^{-1} = p(a)^{-1} \cdot dp \cdot p^{-1} \cdot p(a)$. If we set $T = -(A + Ba)^{-1}(Aa + B)$ then

$$q(z) = \tfrac{wI-T}{w+a}.$$

Notice that as long as $w \neq -a$ then $wI - T$ is invertible if and only if $q(z)$ is invertible, which in turn is equivalent to $p(z)$ being invertible. Since $w \in S^1$ if and only if $z \in S^1$, we have that $wI - T$ is invertible for all $w \in S^1$.

We next compute that

$$\begin{aligned} q^{-1}\,dq = (w + a)(wI - T)^{-1} \cdot \tfrac{(w+a)I-(wI-T)}{(w+a)^2}\,dw &= (wI - T)^{-1}dw - (w + a)^{-1}I\,dw \\ &= dq \cdot q^{-1}. \end{aligned}$$

Therefore

$$Q_{in} = \tfrac{1}{2\pi i}\int_{S^1} p^{-1}dp = \tfrac{1}{2\pi i}\int_{S^1} q^{-1}dq = \tfrac{1}{2\pi i}\int_{S^1} (wI-T)^{-1}\,dw$$

since $\frac{1}{w+a}$ has no poles inside the unit disk. Likewise,

$$Q'_{in} = \tfrac{1}{2\pi i}\int_{S^1} dp\cdot p^{-1} = \tfrac{1}{2\pi i}\int_{S^1} p(a)\cdot dq\cdot q^{-1}p(a)^{-1} = p(a)\tfrac{1}{2\pi i}\int_{S^1} dq\cdot q^{-1}\cdot p(a)^{-1}$$
$$= p(a)\cdot Q_{in}\cdot p(a)^{-1}.$$

We therefore see that Q_{in} and $Q_{out} = 1 - Q_{in}$ are the projection operators corresponding to the complementary subspaces V_{in} and V_{out} which form the Jordan block components of T determined by the eigenvalues having norm less than 1 (for V_{in}) and norm greater than 1 (for V_{out}). The operators Q'_{in} and $1 - Q'_{in}$ are the projections for the complementary pair $p(a)(V_{in})$ and $p(a)(V_{out})$. We call these V'_{in} and V'_{out} from now on.

Since T preserves the subspaces V_{in} and V_{out}, so does $wI - T$, and therefore so does $q(z)$. Since $q(z) = p(a)^{-1}p(z)$ this implies that $p(z)$ maps V_{in} into $p(a)(V_{in}) = V'_{in}$ and V_{out} into V'_{out}.

Using the decomposition $\mathbb{C}^r = V_{in}\oplus V_{out}$ in the domain and $\mathbb{C}^r = V'_{in}\oplus V'_{out}$ in the codomain, the matrices A and B may be written in block diagonal form as

$$A = \begin{bmatrix} A_{ii} & A_{io} \\ A_{oi} & A_{oo}\end{bmatrix}, \qquad B = \begin{bmatrix} B_{ii} & B_{io} \\ B_{oi} & B_{oo}\end{bmatrix}.$$

The fact that $A + Bz$ maps V_{in} into V'_{in} and V_{out} into V'_{out} for all $z \in S^1$ implies that $A_{oi} + B_{oi}z$ and $A_{io} + B_{io}z$ have all entries equal to zero, for all $z \in S^1$. A degree 1 polynomial that vanishes on all of S^1 must be the zero polynomial, so we conclude that $A_{oi} = B_{oi} = A_{io} = B_{io} = 0$. Let us now write $A_{in} = A_{ii}$, $A_{out} = A_{oo}$, and similarly for B. So if we set $p_{in}(z) = A_{in} + zB_{in}$ and $p_{out}(z) = A_{out} + zB_{out}$ these are linear S^1-families of isomorphisms. We have now proven (b).

For (c) we start with the formula

$$\det(A+Ba)^{-1}\det p(z) = \det q(z) = \frac{\det(wI-T)}{(w+a)^r}.$$

Since the $z \mapsto w$ transformation preserves the unit disk, the number of roots of $\det p(z)$ inside the unit circle coincides with the same number for $\det(wI-T)$. But this is precisely the dimension of V_{in}.

It only remains to prove (d) and (e). We know that $wI - T$ is invertible on V_{in} for $|w| \geq 1$, and invertible on V_{out} for $|w| \leq 1$. The latter implies that $q(z)$ is likewise invertible on V_{out} for $|z| \leq 1$, which implies the same for $p(z)$.

For the region $|w| \geq 1$ we must contend with the points where things blow up to infinity. Recall $z = \infty$ when $w = -a$, and $w = \infty$ when $z = a$. Since $wI - T$ is invertible on V_{in} for $|w| > 1$ it certainly follows that $q(z)$—and therefore $p(z)$—are invertible on V_{in} for $|z| > 1$ and $z \neq a$. But a was chosen so that $p(a)$ was invertible, hence $p(z)$ is invertible on V_{in} for all $|z| \geq 1$.

Finally, we return to the decomposition $p_{in}(z) = A_{in} + zB_{in}$. Suppose $B_{in}u = 0$ for some $u \in V_{in}$. Then $Bu = 0$ as well. So

$$Tu = -(A+Ba)^{-1}(Aa+B)u = -(A+Ba)^{-1}Aau = -a(A+Ba)^{-1}(A+Ba)u = -au.$$

But V_{in} only contained eigenvectors of T where the eigenvalue has norm less than 1. Since $a > 1$ this forces $u = 0$. Hence B_{in} is invertible on V_{in}. □

Exercise 21.24. Suppose A is nonsingular and we take $p(z) = A$ (the constant function). Check that $V_{in} = 0$. If B is nonsingular and we take $p(z) = Bz$, check that $V_{in} = \mathbb{C}^r$. Make sure you can also identify V_{out}, V'_{in}, and V'_{out} in these examples.

Finally, check that if $p_0(z) = A_0 + B_0 z$ and $p_1(z) = A_1 + B_1 z$ are linear S^1-families of isomorphisms on $\mathbb{C}^r$ and on $\mathbb{C}^k$, respectively, then $p_0 \oplus p_1$ is a family of isomorphisms on $\mathbb{C}^r \oplus \mathbb{C}^k$ and the resulting V_{in} is equal to $V_{in}(p_0) \oplus V_{in}(p_1)$ (and likewise for V_{out}, V'_{in}, and V'_{out}).

It is worth taking a moment and setting the above result in greater context. Suppose that $f(z) \in \mathbb{C}[z]$ and has no roots on the unit circle. Then

$$\begin{aligned}\int_{S^1} \frac{df}{f} &= \#\text{ of roots of } f \text{ inside the unit circle} \\ &= \text{the winding number } \omega(f) \text{ of } f \text{ as a map } S^1 \to \mathbb{C}^*.\end{aligned}$$

For the first equality, if $f(z) = \lambda \prod_i (z - a_i)$ then $\frac{df}{f} = \sum \frac{dz}{z-a_i}$ and now use the Cauchy integral formula. For the second, write $f(z) = \lambda \prod_{|a_i|<1}(z-a_i) \cdot \prod_{|b_i|>1}(z-b_i)$. We can deform $\lambda \simeq 1$, all of the a's to zero, and all of the b's to 5 (this is just a convenient large number). So $f(z) \simeq z^{n_a}(z-5)^{n_b}$ where n_a is the number of a's and n_b is the number of b's. But for $z \in S^1$ the straight-line homotopy from $z-5$ to -5 will never cross zero, and so $z^{n_a}[t(z-5) + (1-t)(-5)]^{n_b}$ shows that f is homotopic to $z^{n_a} \cdot (-5)^{n_b}$, and homotoping -5 to 1 brings us to z^{n_a}. So $\omega(f)$ is precisely n_a.

Now return to the context of $p(z) = A + Bz$ being a linear S^1-family of isomorphisms of $\mathbb{C}^r$. Then $d(z) = \det p(z)$ is a polynomial that has no roots on the unit circle. The winding number of $d(z)$ is a homotopy invariant of the family, and Proposition 21.23(c) says that this is equal to $\dim V_{in}$. So in some sense we have lifted our homotopy invariant from a simple number to a linear-algebraic object.

21.25. **Back to vector bundles and K-theory.** Now suppose that $E \to X$ is a vector bundle and $p(z) = A + Bz$ is a linear S^1-family of E-clutching functions. For any $x \in X$ we can look at the slice $\{x\} \times S^1 \subseteq X \times S^1$, and the restriction p_x gives a linear S^1-family of isomorphisms of E_x. Proposition 21.23 gives subspaces $(V_{in})_x, (V_{out})_x, (V'_{in})_x, (V'_{out})_x \subseteq E_x$. We can do this for every x simultaneously, and the subspaces assemble into subbundles $V_{in}, V_{out}, V'_{in}, V'_{out} \subseteq E$ such that $E = V_{in} \oplus V_{out}$ and $E = V'_{in} \oplus V'_{out}$: the point here is that the formula for the Riesz projections Q_{in} and Q'_{in} are continuous in the operator, and so give a bundle map $E \to E$ of constant rank. Taking the kernels and images gives the four $V_{in/out}$ bundles.

Exercise 21.26. Explain why Q_{in} and Q'_{in} have constant rank.

The E-clutching function p decomposes into the sum of two linear clutching functions $p_{in} = A_{in} + zB_{in}$ and $p_{out} = A_{out} + zB_{out}$. The first is an isomorphism for all $|z| \geq 1$ and the second for all $|z| \leq 1$.

We will write $V_{in}(E, p)$, or just $V_{in}(p)$, when we need to include the dependence on E and p. Likewise for the other elements of the above decomposition.

The bundle maps $p_{in}(z)\colon V_{in} \to V'_{in}$ and $p_{out}(z)\colon V_{out} \to V'_{out}$ are isomorphisms for any $z \in S^1$, and $V_{in} \oplus V_{out} = E$ by construction. So to understand these bundles as elements of $K(X)$ it suffices to focus on just one of them; we choose V_{in}.

Exercise 21.27. If $A \in \mathrm{Iso}_X(E,E)$ and our linear clutching function is $p(z) = A$ then check that $V_{in} = 0$. If $B \in \mathrm{Iso}_X(E,E)$ and our linear clutching function is $p(z) = Bz$ then check that $V_{in} = E$.

Proposition 21.28. *Suppose p and q are linear S^1-families of E-clutching functions, and that p and q are homotopic through such families. Then $V_{in}(p) \cong V_{in}(q)$ as bundles over X.*

Proof. The homotopy H can be regarded as a linear S^1-family of π^*E-clutching functions, where $\pi\colon X \times I \to X$ is the projection. We obtain a vector bundle $V_{in}(H)$ over $X \times I$ that restricts to $V_{in}(p)$ and $V_{in}(q)$ over $X \times 0$ and $X \times 1$. Therefore $V_{in}(p) \cong V_{in}(q)$. □

When $p(z)$ is a polynomial E-clutching function of degree at most r, then we have the linear clutching function $\mathrm{Lin}(p)$ on $E^{(r+1)}$. We write $V_{in}^{r+1}(\mathrm{Lin}(p))$ as shorthand for $V_{in}(E^{(r+1)}, \mathrm{Lin}(p))$.

Proposition 21.29. *Let $E \to X$ be a vector bundle and let $p(z) = A + Bz$ be a linear S^1-family of E-clutching functions. Then the bundles V_{in} and V_{out} over X satisfy $V_{in} \oplus V_{out} \cong E$ and $E(p) \cong V_{in}(z) \oplus V_{out}(1)$.*

Proof. We have decompositions $E = V_{in} \oplus V_{out}$ and $E = V'_{in} \oplus V'_{out}$ together with $p = p_{in} \oplus p_{out}$ where the latter are linear families of isomorphisms $p_{in} = A_{in} + B_{in}z$ mapping $V_{in} \to V'_{in}$ and $p_{out} = A_{out} + B_{out}z$ mapping $V_{out} \to V'_{out}$. Then $E(p) \cong (V_{in}, p_{in}, V'_{in}) \oplus (V_{out}, p_{out}, V'_{out})$ (see Remark 21.13 for notation).

We also know that B_{in} belongs to $\mathrm{Iso}_X(V_{in}, V'_{in})$ and A_{out} belongs to $\mathrm{Iso}_X(V_{out}, V'_{out})$, that the map $p_{in}(z)$ is an isomorphism for $|z| \geq 1$, and that $p_{out}(z)$ is an isomorphism for $|z| \leq 1$. These are all by Proposition 21.23.

Consider the homotopy $tA_{in} + zB_{in}$ for $0 \leq t \leq 1$, which gives zB_{in} at $t = 0$ and $p_{in}(z)$ at $t = 1$. We claim that this is a homotopy of clutching functions. To see this for $t \in (0,1)$ observe that

$$tA_{in} + zB_{in} = t(A_{in} + t^{-1}zB_{in}) = t \cdot p_{in}(t^{-1}z)$$

and $p_{in}(t^{-1}z)$ is an isomorphism since $|t^{-1}z| > 1$. It follows that $(V_{in}, p_{in}, V'_{in}) \cong (V_{in}, zB_{in}, V'_{in})$.

Likewise, consider the homotopy $A_{out} + tzB_{out} = p_{out}(tz)$ with $0 \leq t \leq 1$. Since $|tz| \leq 1$ this is a clutching function for every value of t, so we have a homotopy of clutching functions from A_{out} to $p_{out}(z)$. Therefore $(V_{out}, p_{out}, V'_{out}) \cong (V_{out}, A_{out}, V'_{out})$.

Finally, use the evident change-of-basis isomorphisms $(V_{in}, zB_{in}, V'_{in}) \cong V_{in}(z)$ and $(V_{out}, A_{out}, V'_{out}) \cong V_{out}(1)$. Putting everything together, we have shown $E(p) \cong V_{in}(z) \oplus V_{out}(1)$. □

Corollary 21.30. *Under the hypotheses of Proposition 21.29 we have the identity $E(p) = V_{in} \cdot (\mathcal{L}^* - 1) \oplus E(1)$ in $K(X \times S^2)$.*

Proof. We just write

$$\begin{aligned} E(p) = V_{in}(z) + V_{out}(1) = V_{in}(z) + (E - V_{in})(1) &= V_{in} \otimes \mathcal{L}^* - V_{in}(1) + E(1) \\ &= V_{in}(\mathcal{L}^* - 1) + E(1). \end{aligned}$$

□

Corollary 21.31. *Let $E \to X$ be a vector bundle and let p be a polynomial clutching function of degree $\leq r$. Then $E(p) = V_{in}(\mathrm{Lin}_{r+1} p) \cdot (\mathcal{L}^* - 1) + E(1)$ in $K(X \times S^2)$.*

Proof. We have $E(p) = E^{(r+1)}(\mathrm{Lin}_{r+1} p) - rE(1)$ by Proposition 21.20(a) and $E^{(r+1)}(\mathrm{Lin}_{r+1} p) = V_{in}(\mathrm{Lin}_{r+1} p) \cdot (\mathcal{L}^* - 1) + E^{(r+1)}(1)$ by Corollary 21.30. Just put these two together, remembering that $E^{(r+1)} = (r+1)E$. □

Corollary 21.32. *The tensor product map $K(X) \otimes K(S^2) \to K(X \times S^2)$ is surjective.*

Proof. Let W denote the image of our map. Since $K(S^2)$ is spanned by 1 and L^*, W is the span of bundles $E(1)$ and $E(z)$ for all bundles $E \to X$. The group W is also closed under tensoring with both positive and negative powers of $\mathcal{L}$.

For any bundle $E \to X$ and any E-clutching function f we need to prove that $E(f)$ is in W. Corollary 21.31 shows that if f is polynomial then $E(f) \in W$. If f is a Laurent clutching function then using $E(f) \cong E(z^r f) \otimes \mathcal{L}^r$ and the fact that $z^r f$ is polynomial for $r \gg 0$, we know $E(f) \in W$.

Finally, if f is an arbitrary clutching function then we know $E(f) \cong E(f')$ for some Laurent approximation f' to f. So $E(f) \in W$. □

We have almost completed the proof that $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism. All that remains is to convince ourselves that we can take the basic technique from Corollary 21.32 and use it to construct an explicit inverse. Before taking this on we state one easy lemma that will be needed:

Lemma 21.33. *Let $E \to X$ be a vector bundle and let $p(z)$ be a polynomial clutching function of degree at most r. Then*
(a) $V_{in}^{r+2}(\mathrm{Lin}(p_r)) \cong V_{in}^{r+1}(\mathrm{Lin}(p_r))$.
(b) $V_{in}^{r+2}(\mathrm{Lin}(zp_r)) \cong V_{in}^{r+1}(\mathrm{Lin}(p_r)) \oplus E$.

Proof. In the proof of Proposition 21.20 we established homotopies of clutching functions $\mathrm{Lin}_{r+2}(p_r) \simeq \mathrm{Lin}_{r+1}(p_r) \oplus I$ and $\mathrm{Lin}_{r+2}(zp_r) \simeq \mathrm{Lin}_{r+1}(p_r) \oplus zI$. Now use Proposition 21.28, Exercise 21.27, and the fact that V_{in} respects direct sum decompositions. □

Proof of Theorem 21.5. First note that the ring structure on $K(S^2)$ makes $K(X) \otimes K(S^2)$ a right module over $K(S^2)$. We can therefore write $E \cdot L$ for $E \otimes L$, and $(E_1 + E_2 L)L$ for $(E_1 \otimes L) + (E_2 \otimes L^2)$, and so forth. Also, let $\Theta\colon K(X) \otimes K(S^2) \to K(X \times S^2)$ denote the external tensor product map.

Let $J \to X \times S^2$ be a vector bundle. Set $E = J|_{X \times pt}$. We know that $J \cong E(f)$ for some E-clutching function $f \in \mathrm{Iso}_{X \times S^1}(\pi^* E, \pi^* E)$, and that f is well-defined up to homotopy of E-clutching functions (see Exercise 21.12).

Pick $r \gg 0$ such that the Laurent approximations f_k of f for $k \geq r$ are ultra-close to f. Let $p_r = z^r f_r$, which is a polynomial clutching function of degree $2r$. Set $V_{in} = V_{in}^{2r+1}(\mathrm{Lin}(p_r))$. Finally, define

$$\Phi(J) = (V_{in} \cdot (L^* - 1) + E) \cdot L^r \in K(X) \otimes K(S^2).$$

Then

$$\Theta\Phi(J) = (V_{in}(\mathcal{L}^* - 1) + E) \cdot \mathcal{L}^r = E(p_r)\mathcal{L}^r = E(z^r f_r)\mathcal{L}^r = E(f_r) = E(f) = J$$

where the second equality is from Corollary 21.31.

We need to check that $\Phi(J)$ is well-defined and additive. If it is, it induces $\Phi\colon K(X \times S^2) \to K(X) \otimes K(S^2)$. Suppose we replace p_r with $p_{r+1} = z^{r+1} f_{r+1}$. The E-clutching functions f_r and f_{r+1} are close enough that the straight-line homotopy between them passes entirely through E-clutching functions, and so $p_{r+1} = z^{r+1} f_{r+1}$ is homotopic to $zp_r = z^{r+1} f_r$. Then by Proposition 21.28 $V_{in}(p_{r+1}) = V_{in}(zp_r) = V_{in}(p_r) + E$, where the second is by Lemma 21.33 and we are writing $V_{in}(p)$ instead of $V_{in}(\operatorname{Lin} p)$ for brevity. But

$$\begin{aligned}(V_{in}(p_{r+1})(L^* - 1) + E)L^{r+1} &= ((V_{in}(p_r) + E)(L^* - 1) + E)L^{r+1} \\ &= (V_{in}(p_r)(L^* - 1) + EL^*)L^{r+1} \\ &= (V_{in}(p_r)(1 - L) + E)L^r \\ &= (V_{in}(p_r)(L^* - 1) + E)L^r\end{aligned}$$

where in the last equality we have used that $1 - L = L^* - 1$ (21.3). This shows that $\Phi(J)$ is the same whether we define V_{in} via p_r or via p_{r+1}. So at this point we know that $\Phi(J)$ does not depend on r—but it might depend on f. Let us temporarily write $\Phi(E, f)$.

Suppose we had chosen a different clutching function f' instead of f. We know that f' and f will be homotopic. A choice of homotopy may be regarded as an $\hat{E}$-clutching function F on $(X \times I) \times S^1$ instead of just $X \times S^1$, where $\hat{E}$ is the pullback of E to $X \times I$. Consider the element $\Phi(\hat{E}, F)$. It is clear that this element of $K(X \times I) \otimes K(S^2)$ pulls back to $\Phi(E, f)$ and $\Phi(E, f')$ along the two inclusions of X into $X \times I$. Since these inclusions induce the same map on K-groups, $\Phi(E, f) = \Phi(E, f')$. So finally we know that $\Phi(J)$ is independent of all choices. The additivity is then clear: $\Phi(J \oplus J') = \Phi(J) + \Phi(J')$. This follows from the additivity of the V_{in} construction.

It only remains to check that $\Phi\Theta = \mathrm{id}$. Equivalently, for every bundle $E \to X$ we need to check that $\Phi(E(1)) = E$ and $\Phi(E(z)) = E \cdot L^*$. But for the linear clutching function $p(z) = \mathrm{id}$ we know that $V_{in} = 0$, and therefore $\Phi(E(1)) = E \cdot L^0 = E$ by definition. Likewise, for the linear clutching function $p(z) = zI$ we know that $V_{in} = E$ and so $\Phi(E(z)) = (E(L^* - 1) + E)L^1 = E$. □

Remark 21.34. I confess that the above proof does not speak to me. Whatever secrets it contains seem closely guarded. It reminds me of proofs of the Pythagorean Theorem where one chases around a big complicated diagram, eventually proving the result but coming away with no intuition about why it is true. That's fine if the goal is just to be sure the result is true so that one can go off to apply it freely. But it can be frustrating if the goal is to understand the deeper reason *why* something is true.

This proof is nice in that it requires fairly little technology and one can check the details in a few pages. It is probably the most "low-tech" proof of complex periodicity. But there are many other proofs of Bott periodicity that contain more insight. See the relevant sections of [D7] for a careful study.

Remark 21.35. It is worth considering whether the above proof can be modified to obtain real Bott periodicity. Unfortunately, the proof depended heavily on several unique aspects of the complex numbers and of the circle S^1: Jordan canonical form, Riesz projections via complex analysis, and Fourier series all played key roles. For 8-fold real periodicity none of this is available, and so one runs into brick walls before even getting started.

Another approach is to get at real periodicity through the relationship between real and quaternionic K-theory. Using $\pi_3 GL_n(\mathbb{H}) \cong \pi_3 GL_1(\mathbb{H}) \cong \pi_3(S^3) \cong \mathbb{Z}$ one finds that $KSp^0(S^4) \cong \mathbb{Z}^2$, generated by 1 and the tautological quaternionic line bundle over $S^4 = \mathbb{H}P^1$. One can try to prove that the external tensor product $KO^0(X) \otimes KSp^0(S^4) \to KSp^0(X \times S^4)$ is an isomorphism. Here one would have to analyze quaternionic clutching functions, and quaternionic linear transformations do admit a theory of Jordan canonical form. But we are still in trouble because of the change from S^1 to S^3, as we don't have the same tools from analysis to feed into the argument. Also, it turns out that $KO^0(X) \otimes KSp^0(S^4) \to KSp^0(X \times S^4)$ is *not* always an isomorphism: $X = S^1$ is a counterexample. The isomorphism does hold in the form $KO^0(X) \otimes \widetilde{KSp^0}(S^4) \to KSp^0(X \times S^4, X \times *)$, but now things are inherently more difficult because of the relative group. So again, we are stuck. I am not aware of any self-contained proof of real periodicity along the lines developed in this section. It doesn't seem hopeless, but some serious effort would be required for developing all of the necessary tools in this context.

As a way of grappling with what is going on behind the scenes here, recall that there are three division algebras over the real numbers:

$$\mathbb{C}, \qquad \mathbb{R}, \qquad \mathbb{H}.$$

We have separated them this way for a reason: grouping them according to their center (which is what the theory of division algebras tells us to do), $\mathbb{R}$ and $\mathbb{H}$ really do go together—they both have center $\mathbb{R}$, whereas $\mathbb{C}$ is its own center. From a certain perspective Bott periodicity seems to be two stories: one for $\mathbb{C}$, and a separate story about $\mathbb{R}$ and $\mathbb{H}$ together. All of the nice tools from this section seem to work only in the former case.

We have the projective spaces $\mathbb{C}P^1$, $\mathbb{R}P^1$, and $\mathbb{H}P^1$, with their tautological line bundles $L_{\mathbb{C}}$, $L_{\mathbb{R}}$, and $L_{\mathbb{H}}$. There are fundamental isomorphisms

$$\begin{aligned} \widetilde{KO}^0(S^1) &\cong \mathbb{Z}/2, \qquad \text{generated by } 1 - L_{\mathbb{R}}, \\ \widetilde{K}^0(S^2) &\cong \mathbb{Z}, \qquad \text{generated by } 1 - L_{\mathbb{C}}, \\ \widetilde{KSp^0}(S^4) &\cong \mathbb{Z}, \qquad \text{generated by } 1 - L_{\mathbb{H}} \end{aligned}$$

induced by the simple calculations $\pi_0 O = \pi_0 O(1) = \mathbb{Z}/2$, $\pi_1 U = \pi_1 U(1) \cong \mathbb{Z}$, and $\pi_3 Sp = \pi_3 Sp(1) \cong \mathbb{Z}$. If one lets $\beta_F = 1 - L_F$ in the three cases, then one might naively expect that $\widetilde{K}^0(S^*) \cong \mathbb{Z}[\beta_{\mathbb{C}}]$, $\widetilde{KO}^0(S^*) \cong \mathbb{Z}[\beta_{\mathbb{R}}]/(2\beta_{\mathbb{R}})$, and $\widetilde{KSp^0}(S^*) \cong \mathbb{Z}[\beta_{\mathbb{H}}]$. Only the first of these is actually correct—in fact, in the third case there is not even an evident ring structure due to the lack of a quaternionic tensor product. For the latter two, what somehow happens is that the two expected answers—because they are forced into being combined—merge to yield the $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$ pattern. It starts out looking like $\mathbb{Z}[\beta_{\mathbb{R}}]/(2\beta_{\mathbb{R}})$, but then the $\mathbb{Z}[\beta_{\mathbb{H}}]$ pattern takes over for a while, before repeating. There are spectral sequences where one can see this merging happening explicitly, though the precise connections to the story we are telling here are a bit nebulous. See Exercise 23.15 for this.

Lastly, there *are* approaches to real Bott periodicity that try to get at it using complex periodicity but incorporating certain $\mathbb{Z}/2$ symmetries. In such an approach one would still be relying mostly on tools for circles and the complex numbers, but then trying to add in an extra layer of symmetries on top of it. See the discussion in [D7].

22. Vector fields on spheres

It is a classical problem to determine how many linearly independent vector fields can be constructed on a given sphere S^n. This problem was heavily studied throughout the 1940s and 1950s, and then finally solved by Adams in 1962 using K-theory. It is one of the great successes of generalized cohomology theories. In this section we discuss some background to the vector field problem. This material will at first seem like a digression from our task of setting up the foundations of K-theory, but soon enough we will see that the techniques we encounter are intimately connected to the heart of the subject.

A discussion of Adams's solution to the vector field problem can be found in [D7].

22.1. **The vector field problem.** Given a nonzero vector $u = (x, y)$ in $\mathbb{R}^2$, there is a formula for producing a (nonzero) vector that is orthogonal to u: namely, $(-y, x)$. However, there is no analog of this that works in $\mathbb{R}^3$. That is, there is no single formula that takes a vector in $\mathbb{R}^3$ and produces a (nonzero) orthogonal vector. If such a formula existed then it would give a nonvanishing vector field on S^2, and we know by elementary topology that such a thing does not exist.

Let us next consider what happens in $\mathbb{R}^4$. Given $u = (x_1, x_2, x_3, x_4)$, we can produce an orthogonal vector via the formula $(-x_2, x_1, -x_4, x_3)$. But of course this is not the only way to accomplish this: we can vary what pairs of coordinates we choose to flip. In fact, if we consider

$$v_1 = \begin{bmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -x_4 \\ -x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

then we find that v_1, v_2, and v_3 are not only orthogonal to u but they are orthogonal to each other as well. In particular, at each point of S^3 we have given an orthogonal basis for the tangent space.

We aim to study this problem for any $\mathbb{R}^n$. What is the maximum k for which there exist formulas for starting with $u \in \mathbb{R}^n - \{0\}$ and producing k orthogonal (nonzero) vectors, with u as the first of the set? The following gives a different phrasing for essentially the same question:

Question 22.2. *On S^n, how many (tangent) vector fields $v_1, v_2, \ldots, v_r$ can we find so that $v_1(x), v_2(x), \ldots, v_r(x)$ are linearly independent for each $x \in S^n$?*

In colloquiual usage we will sometimes drop the phrase "linearly independent" and leave it to be understood. So for example, if we talk about constructing two vector fields on S^5 it is implicit that we mean independent vector fields, as otherwise the problem would be trivial!

Note that by the Gram-Schmidt process we can replace "linearly independent" by "orthonormal." If n is even, the answer is zero because there does not exist even a single nonvanishing vector field on an even sphere. To start to see what happens when n is odd, we look at a couple of more examples.

Let $u \in S^5$ have the standard coordinates. We notice that the vector $v_1 = (-x_2, x_1, -x_4, x_3, -x_6, x_5)$ is orthogonal to u. However, a little legwork shows that no other pattern of switching coordinates will produce a vector that is orthogonal to both u and v_1. Of course this does not mean that there isn't some more elaborate

formula that would do the job, but it shows the limits of what we can do using our naive constructions.

For $u \in S^7$ we can divide the coordinates into the top four and the bottom four. Take the construction that worked for S^3 and repeat it simultaneously in the top and bottom coordinates—this yields a set of three orthonormal vector fields on S^7, given by the formulas

$$\begin{aligned}(-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),\\ (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6),\\ (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5).\end{aligned} \tag{22.2}$$

This idea generalizes at once to prove the following:

Proposition 22.3. *If there exist r (independent) vector fields on S^{n-1}, then there also exist r vector fields on S^{kn-1} for all k.*

For example, since there is one vector field on S^1 we also know that there is at least one vector field on S^{2k-1} for every k. Likewise, since there are three vector fields on S^3 we know that there are at least three vector fields on S^{4k-1} for every k.

We have constructed three vector fields on S^7, but one can actually make seven of them. This can be done via trial-and-error attempts at extending the patterns in (22.2), but there is a slicker way to accomplish this as well. Recall that S^3 is a Lie group, being the unit quaternions inside of $\mathbb{H}$. We can choose an orthonormal frame at the origin and then use the group structure to push this around to any point, thereby obtaining three independent vector fields; in other words, for any point $x \in S^3$ use the derivative of right-multiplication-by-x to transport our vectors in T_1S^3 to T_xS^3. The space S^7 is not quite a Lie group, but it still has a multiplication coming from being the set of unit octonions. The multiplication is not associative, but this is of no matter—the same argument works to construct 7 vector fields on S^7. Note that this immediately gives us 7 vectors fields on S^{15}, S^{23}, etc.

Based on the data so far, one would naturally guess that if $n = 2^r$ then there are $n-1$ vector fields on S^{n-1}. However, this guess turns out to fail already when $n = 16$ (and thereafter). To give a sense of how the numbers grow, we give a chart showing the maximum number of vector fields that exist on low-dimensional spheres:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$n-1$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
vf on S^{n-1}	1	3	1	7	1	3	1	8	1	3	1	7	1	3	1	9

So far we have explained how to construct the listed number of vector fields until we get to S^{15}—there we have shown how to make seven of them, but the claim is that one more can be made. Once we know how to make eight on S^{15} we automatically know how to make eight on S^{31}, but the claim is again that one more can be made beyond this. By the end of this section we will know how to construct all of these.

Now that we have a basic sense of the problem let us explain the numerology behind the answer.

Definition 22.4. *If $n = m \cdot 2^{a+4b}$ where m is odd, then the* ***Hurwitz-Radon number*** *for n is $\rho(n) = 2^a + 8b - 1$.*

Theorem 22.5 (Hurwitz-Radon)**.** *There exist at least $\rho(n)$ independent vector fields on S^{n-1}.*

Consider $n = 32 = 2^5 = 2^{1+4\cdot 1}$. In the terms of Definition 22.4 we have $a = b = 1$, so that $\rho(32) = 2^1 + 8(1) - 1 = 9$. That is, there are at least 9 vector fields on S^{31}. If $n = 1024 = 2^{10} = 2^{2+4\cdot 2}$ then $\rho(n) = 2^2 + 8 \cdot 2 - 1 = 19$; one can make 19 independent vector fields on S^{1023}. One should of course notice that these numbers are not going up very quickly.

We will prove the Hurwitz-Radon theorem by a slick, modern method using Clifford algebras. But it is worth pointing out that the theorem can be proven through very naive methods, too (it was proven in the 1920s). All of the Hurwitz-Radon vector fields follow the general patterns that we have seen, of switching pairs of coordinates and changing signs—one only has to find a way to organize the bookkeeping behind these patterns.

22.6. **Sums-of-squares formulas.** Hurwitz and Radon were not actually thinking about vector fields on spheres. They were instead considering an algebraic question about the existence of certain kinds of "composition formulas" for quadratic forms. For example, the following identity is easily checked:

$$(*) \qquad (x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2.$$

Hurwitz and Radon were looking for more formulas such as this one, for larger numbers of variables:

Definition 22.7. *A* ***sum-of-squares formula*** *of type $[r, s, n]$ is an identity*

$$(x_1^2 + x_2^2 + \ldots + x_r^2)(y_1^2 + y_2^2 + \ldots + y_s^2) = z_1^2 + z_2^2 + \ldots + z_n^2$$

in the polynomial ring $\mathbb{R}[x_1, \ldots, x_r, y_1, \ldots, y_s]$, where each z_i is a bilinear expression in x's and y's.

We will often refer to the above as an "$[r, s, n]$-formula", for brevity. For what values of r, s, and n does such a formula exist? This is currently an open question. There are three formulas that are easily produced, coming from the normed algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The multiplication is a bilinear pairing, and the identity $|xy|^2 = |x|^2|y|^2$ is the required sums-of-squares formula. These algebras give formulas of type $[2, 2, 2]$, $[4, 4, 4]$, and $[8, 8, 8]$. (Check that the $[2, 2, 2]$ formula that comes from $\mathbb{C}$ is exactly formula (*) above). In a theorem from 1898 Hurwitz proved that these are the only normed algebras over the reals, and in doing so ruled out the existence of $[n, n, n]$-formulas for $n \notin \{1, 2, 4, 8\}$. The question remained (and remains) about other types of formulas. See [Shapiro] for a detailed history of this problem.

Exercise 22.8. Use the multiplication table for $\mathbb{H}$ to write down the corresponding $[4, 4, 4]$ formula.

Perhaps surprisingly, most of what is known about the non-existence of sums-of-squares formulas comes from topology. To phrase the question differently, we are looking for a function $\phi\colon \mathbb{R}^r \otimes \mathbb{R}^s \to \mathbb{R}^n$ such that $|\phi(x, y)|^2 = |x|^2 \cdot |y|^2$ for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. The bilinear expressions $z_1, \ldots, z_n$ are just the coordinates of $\phi(x, y)$.

Write $z = \phi(x,y) = \sum x_j A_j y$, where the A_j's are $n \times s$ matrices of real numbers. The sum of squares formula says that $z^T z = (x^T x) \cdot (y^T y)$. But $z^T z = \sum_{i,j} (y^T A_j^T x_j)(x_i A_i y)$, hence

$$y^T \left(\sum_{i,j} x_i x_j A_j^T A_i \right) y = z^T z = (x^T x)(y^T y) = y^T \Big((x^T x) I \Big) y$$

for all y. The first and last expressions are quadratic forms in y, and they are equal only if $\sum_{i,j} x_i x_j A_j^T A_i = (x^T x) I = \sum x_i^2 I$. This must hold for all x, so equating coefficients of the monomials in x we find that

- $A_i^T A_i = I$ (that is, $A_i \in O_n$) for every i, and
- $A_j^T A_i + A_i^T A_j = 0$ for every $i \neq j$.

The case $s = n$ turns out to be significantly simpler to address than the general case. If $s = n$ we may set $B_i = A_1^{-1} A_i = A_1^T A_i$. Then $B_1 = Id$ and the conditions to satisfy for $i \geq 2$ become

- $B_i^T = B_i^{-1}$
- $B_i^2 = -I_n$
- $B_i B_j = -B_j B_i$ for all $i \neq j$.

Note that the first two conditions imply $B_i^T = -B_i$, and using this the third condition can be rewritten as $B_j^T B_i + B_i^T B_j = 0$. So by replacing the A's with the B's we have proven the following:

Corollary 22.9. *If an $[r,n,n]$-formula exists, then one exists where $A_1 = I$ and $A_i^T = -A_i$ for $i \geq 2$.*

In the setting of the corollary, the necessary conditions on the matrices $A_2, A_3, \ldots, A_r$ become that $A_i^2 = -I$ and $A_i A_j = -A_j A_i$.

Corollary 22.10. *If an $[r,n,n]$-formula exists, then there exist $r-1$ independent vector fields on S^{n-1}.*

Proof. Assume the A_i are chosen as in Corollary 22.9. If $y \in S^{n-1}$ then $\phi(e_i, y) \in S^{n-1}$ for $i = 1, 2, \ldots, r$: this follows from the identity $|\phi(e_i,y)|^2 = |e_i|^2 \cdot |y|^2$. We also have that $\phi(e_1, y) = y$ since $A_1 = I$. We claim that $\phi(e_i, y) \perp \phi(e_j, y)$ if $i \neq j$. To see this, note that the norm formula gives

$$|\phi(e_i + e_j, y)|^2 = |e_i + e_j|^2 \cdot |y|^2 = 2|y|^2.$$

On the other hand,

$$\begin{aligned} |\phi(e_i + e_j, y)|^2 &= |\phi(e_i, y) + \phi(e_j, y)|^2 \\ &= |\phi(e_i, y)|^2 + |\phi(e_j, y)|^2 + 2\phi(e_i, y) \cdot \phi(e_j, y) \\ &= 2|y|^2 + 2\phi(e_i, y) \cdot \phi(e_j, y). \end{aligned}$$

We conclude $\phi(e_i, y) \cdot \phi(e_j, y) = 0$. Therefore we have established that $\phi(e_2, -)$, $\phi(e_3, -), \ldots, \phi(e_r, -)$ are orthonormal vector fields on S^{n-1}. $\square$

22.11. **Clifford algebras.** We have seen that we get $r-1$ independent vector fields on S^{n-1} if we have a sums-of-squares formula of type $[r,n,n]$. Having such a formula amounts to producing matrices $A_2, A_3, \ldots, A_r \in O_n$ such that $A_i^2 = -I$ and $A_iA_j + A_jA_i = 0$ for $i \neq j$. If we disregard the condition that the matrices be orthogonal, we can encode the latter two conditions by saying that we have a representation of a certain algebra:

Definition 22.12. *The **Clifford algebra** Cl_k is defined to be the quotient of the tensor algebra $\mathbb{R}\langle e_1, \ldots, e_k\rangle$ by the relations $e_i^2 = -1$ and $e_ie_j + e_je_i = 0$ for all $i \neq j$.*

The first few Clifford algebras are familiar: $\mathrm{Cl}_0 = \mathbb{R}$, $\mathrm{Cl}_1 = \mathbb{C}$, and $\mathrm{Cl}_2 = \mathbb{H}$. After this things become less familiar: for example, it turns out that $\mathrm{Cl}_3 = \mathbb{H} \times \mathbb{H}$ (we will see why in just a moment). It is somewhat of a miracle that it is possible to write down a precise description of *all* of the Clifford algebras, and *all* of their modules. Before doing this, let us be clear about *why* we are doing it:

Theorem 22.13. *An $[r,n,n]$-formula exists if and only if there exists a Cl_{r-1}-module structure on $\mathbb{R}^n$. Consequently, if there is a Cl_{r-1}-module structure on $\mathbb{R}^n$ then there are $r-1$ independent vector fields on S^{n-1}.*

Before giving the proof, we need one simple fact. The collection of monomials $e_{i_1}\cdots e_{i_r}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ give a vector space basis for Cl_k, which has size 2^k (note that we include the empty monomial, corresponding to 1, in the basis). This is an easy exercise. This is the "standard basis" for Cl_k.

Proof of Theorem 22.13. The forward direction is trivial: Given an $[r,n,n]$-formula, Corollary 22.9 gives us such a formula with $A_1 = I$, $A_i^2 = -I$, and $A_iA_j = -A_jA_i$ for $i,j \geq 2$. Then define a Cl_{r-1}-module structure on $\mathbb{R}^n$ by letting e_i act as multiplication by A_{i+1}, for $1 \leq i \leq r-1$.

Conversely, assume that Cl_{r-1} acts on $\mathbb{R}^n$. We can almost reverse the procedure of the previous paragraph, except that there is no guarantee that the e_i's act orthogonally on $\mathbb{R}^n$—and we need $A_i \in O_n$ to get an $[r,n,n]$-formula.

Equip $\mathbb{R}^n$ with a positive-definite inner product, denoted $x, y \mapsto x \cdot y$. This inner product probably has no compatibility with the Clifford-module structure. So define a new inner product on $\mathbb{R}^n$ by

$$\langle v, w\rangle = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq r-1} (e_I v) \cdot (e_I w),$$

where $e_I = e_{i_1}e_{i_2}\cdots e_{i_j}$ and the sum runs over all 2^{r-1} elements of the standard basis for Cl_{r-1}. Basically we are averaging out the dot product. Our inner product $\langle v, w\rangle$ is a symmetric bilinear form, and it is positive definite because the dot product is positive definite. It also has the property that it is invariant under the Clifford algebra action: $\langle e_iv, e_iw\rangle = \langle v, w\rangle$ for all i.

Now let $v_1, \ldots, v_n$ be an orthonormal basis for $\mathbb{R}^n$ with respect to our new inner product. Let A_i be the matrix for e_i with respect to this basis. Then the A_i's are orthogonal matrices, and the relations $A_i^2 = -I$ and $A_iA_j + A_jA_i = 0$ are automatic because they are satisfied in Cl_{r-1}. In this way we obtain the desired $[r,n,n]$-formula. □

Remark 22.14. Most modern treatments of vector fields on spheres go straight to Clifford algebras and their modules, without ever talking about sums-of-squares

formulas. But the sums-of-squares material is an interesting part of this whole story, both for historical reasons and for its own sake.

From now on we can focus on the following question: For what values of n do we have a Cl_{r-1}-module structure on $\mathbb{R}^n$? This is one of the most intriguing parts of the story, because on the face of things it doesn't seem like we have accomplished anything by shifting our perspective onto Clifford algebras. We have, after all, just rephrased the basic question. But a miracle now occurs, in that we can analyze all the Clifford algebras by a simple technique.

22.15. **Clifford algebras over general rings.** To make it clear that our analysis does not use anything special about $\mathbb{R}$, let us change our ground ring to any ring R not of characteristic two. Define $\mathrm{Cl}_n(R)$ to be the algebra generated over R by symbols $e_1, \ldots, e_n$ that commute with elements of R and satisfy the relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. It turns out that these algebras show an interesting pattern that depends on the residue of n mod 8, reflecting how the elements e_i interact.

Here $\mathrm{Cl}_n(R)$ is a free R-module of rank 2^n with a basis consisting of the monomials $e_{i_1} \cdots e_{i_k}$ with $i_1 < \cdots < i_k$. Also recall that if S is a ring then $S[e]/(e^2 - 1) \cong S \times S$ via the isomorphism $f(e) \mapsto (f(1), f(-1))$. Our derivation will proceed by finding elements in the Clifford algebras that square to 1 and commute with certain others.

In this discussion it will be convenient to abbreviate $\mathrm{Cl}_n(R)$ as just R_n. Also, if S is a ring then we will abbreviate the matrix algebra $M_{n\times n}(S)$ as just $S(n)$.

Here are a series of observations:

(1) $(e_1 \cdots e_n)^2 = (-1)^{\binom{n+1}{2}}$. So the product squares to 1 when n is 0 or 3 mod 4, and to -1 when n is 1 or 2 mod 4.

(2) $e_1 \cdots e_n$ commutes with all of e_1, e_2, …, e_{n-1} precisely when n is odd.

(3) When $n \equiv 3$ mod 4 then $e_1 \cdots e_n$ squares to 1 and commutes with all elements of R_{n-1} and so we conclude that $R_n \cong R_{n-1} \times R_{n-1}$. Let us restate this as
$$R_{4k+3} \cong R_{4k+2} \times R_{4k+2}.$$

(4) In R_{4k+r} consider the elements $\gamma_i = e_1 \cdots e_{4k} e_{4k+i}$ for $i \leq 1 \leq r$. Then $\gamma_i^2 = -1$, γ_i anticommutes with γ_j for $i \neq j$, and each γ_i commutes with elements of $\mathrm{Cl}_{4k}(R)$. This proves that
$$R_{4k+r} \cong R_{4k} \otimes_R R_r.$$
So once we know R_i for $i \leq 4$ we can derive all of the others.

(5) $R_4 \cong R_2(2)$. We can derive an isomorphism as follows. If we think of the ring $R_3 = R_2 \times R_2$ as sitting inside $R_2(2)$ as the diagonal matrices, then what we are missing are the anti-diagonal matrices. The obvious anti-diagonal matrix that squares to -1 is $f = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Note that f anticommutes with elements $\mathrm{diag}(a, -a)$ for $a \in R_2$. So define $\phi\colon R_4 \to R_2(2)$ by the formulas $e_1 \mapsto \mathrm{diag}(e_1, -e_1)$, $e_2 \mapsto \mathrm{diag}(e_2, -e_2)$, $e_3 \mapsto \mathrm{diag}(e_1 e_2, -e_1 e_2)$, and $e_4 \mapsto f$. The images satisfy the defining relations for the Clifford algebra, so ϕ is an algebra map. One readily checks that ϕ is surjective (this uses that $\mathrm{char}(R) \neq 2$). Then since the domain and codomain are free modules of the same rank it is an isomorphism.

At this point we can make the following partial table of Clifford algebras over R, and could continue on ad infinitum:

n	$\mathrm{Cl}_n(R)$
0	R
1	R_1
2	R_2
3	$R_2 \times R_2$
4	$R_2(2)$
5	$(R_2 \otimes_R R_1)(2)$
6	$(R_2 \otimes_R R_2)(2)$
7	$(R_2 \otimes_R R_2)(2) \times (R_2 \otimes_R R_2)(2)$
8	$(R_2 \otimes_R R_2)(4)$

In fact we can simplify this table further, because we can identify the algebras $R_2 \otimes_R R_1$ and $R_2 \otimes_R R_2$. For this, observe that if A is any R-algebra then there is a natural map $A \otimes_R A^{op} \to \mathrm{End}_R(A)$ that sends $a \otimes b$ to the endomorphism $x \mapsto axb$. This is not an isomorphism in general, but sometimes it is (this is essentially the situation for the so-called *Azumaya algebras*, though we will not need this term). In addition, Clifford algebras are inherently isomorphic to their own opposite algebras and this lets us bypass the "op":

(6) There is an anti-automorphism of $\mathrm{Cl}_n(R)$ that sends $e_{i_1} \cdots e_{i_k} \mapsto e_{i_k} \cdots e_{i_1}$. This is called the **transpose** and written $x \mapsto x^t$. Observe that $(ab)^t = b^t a^t$ for all a and b. Note that the transpose map on R_1 is just the identity.

(7) We have algebra maps $\theta_n \colon \mathrm{Cl}_n(R) \otimes_R \mathrm{Cl}_n(R) \to \mathrm{End}_R(\mathrm{Cl}_n(R))$ induced by $a \otimes b \mapsto [x \mapsto axb^t]$ (it would not be an algebra map without the transpose). The target $\mathrm{End}_R(\mathrm{Cl}_n(R))$ denotes the algebra of R-module endomorphisms, so it is isomorphic to $R(2^n)$. The domain and codomain of θ_n are both free R-modules of rank 2^{2n}. The θ maps are not always isomorphisms, e.g. when $n = 1$ one has $\theta(e_1 \otimes 1) = \theta(1 \otimes e_1)$. But when $n = 2$ luck is on our side, as we are about to see.

Proposition 22.16. *θ_2 is an isomorphism, so $R_2 \otimes_R R_2 \cong R(4)$. If R is commutative then the restriction of θ_2 to $R_2 \otimes_R R_1 \subseteq R_2 \otimes_R R_2$ is an isomorphism $R_2 \otimes_R R_1 \cong R_1(2)$.*

Proof. We prove the second claim first. Let α denote the restricted map $R_2 \otimes_R R_1 \to \mathrm{End}_R(R_2)$, still given by $a \otimes b \mapsto [x \mapsto axb^t]$. Since R is commutative, R_1 is also commutative. It follows that the map $x \mapsto axb^t$ is R_1-linear when we give R_2 the action of R_1 via right multiplication. So in fact α is a map $R_2 \otimes_R R_1 \to \mathrm{End}_{R_1}(R_2)$. Both the domain and codomain are free R_1-modules of rank 4, and α is R_1-linear, so it suffices to prove that α is surjective. Recalling that we treat R_2 as a right R_1-module, and choosing the R_1-basis $1, e_2$, we identify $\mathrm{End}_{R_1}(R_2)$ with 2×2 matrices over R_1: that is, we identify $f \in \mathrm{End}_{R_1}(R_2)$ with the matrix A where $f(1) = a_{11} + e_2 a_{21}$ and $f(e_2) = a_{12} + e_2 a_{22}$. Then we compute by hand that the elements $1 \otimes 1$, $e_1 \otimes 1$, $e_2 \otimes 1$, and $e_1 e_2 \otimes 1$ map under α to the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} e_1 & 0 \\ 0 & -e_1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -e_1 \\ -e_1 & 0 \end{bmatrix}.$$

For example, $e_1 \otimes 1$ becomes the map $1 \mapsto e_1$, $e_2 \mapsto e_1e_2 = -e_2e_1$, giving us the second matrix. As these four matrices are clearly a spanning set for $R_1(2)$, this verifies the surjectivity of α.

To prove that θ_2 is an isomorphism we can use a similar strategy, but here we would need 4×4 matrices over R and it would be preferable not to have to write down 16 of them. We can simplify things somewhat by using the above observation that $R_2 \otimes_R R_1$ maps into the subspace of R_1-linear endomorphisms, as well as the parallel observation that $R_2 \otimes_R (e_2R_1)$ maps into the subspace of R_1-antilinear endomorphisms (those such that $f(xe_1) = -f(x)e_1$). Since $\mathrm{End}_R(R_2)$ is the direct sum of R_1-linear and R_1-antilinear subspaces, we are reduced to checking that $R_2 \otimes_R (e_2R_1) \to \mathrm{End}_{R_1-anti}(R)$ is an isomorphism. Again representing endomorphisms via matrices in the usual way, the images of the basis elements $1 \otimes e_2$, $e_1 \otimes e_2$, $e_2 \otimes e_2$, and $e_1e_2 \otimes e_2$ are

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -e_1 \\ -e_1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -e_1 & 0 \\ 0 & e_1 \end{bmatrix}.$$

Again, since $\mathrm{char}(R) \neq 2$ these are a basis for $M_{2\times 2}(R_1)$ and the proof is complete. □

We can use Proposition 22.16 to simplify our table of Clifford algebras. For convenience we show both the table for general R and also the special case $R = \mathbb{R}$:

TABLE 22.17. Clifford algebras

n	$\mathrm{Cl}_n(R)$
0	R
1	R_1
2	R_2
3	$R_2 \times R_2$
4	$R_2(2)$
5	$R_1(4)$
6	$R(8)$
7	$R(8) \times R(8)$
8	$R(16)$

n	$\mathrm{Cl}_n(\mathbb{R})$
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \times \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \times \mathbb{R}(8)$
8	$\mathbb{R}(16)$

With the appearance of $R(16)$ in spot 8 the rest of the table becomes periodic, using the isomorphisms $R_{4k+r} \cong R_{4k} \otimes_R R_r$. So $R_9 \cong R_1(16)$, $R_{10} \cong R_2(16)$, and so on: we just add "__(16)" to each of the terms in the above table.

So we have calculated all of the Clifford algebras!

Exercise 22.18. For the ground ring $R = \mathbb{C}$ prove that $R_1 \cong \mathbb{C} \times \mathbb{C}$ and $R_2 \cong \mathbb{C}(2)$, and observe that the resulting table of Clifford algebras actually has its 8-fold quasi-periodicity absorbed into a 2-fold quasi-periodicity.

22.19. **Modules over Clifford algebras.** Now that we know all the Clifford algebras, it is actually an easy process to determine all of their finitely-generated modules. We need three facts:

- If A is a division algebra then all finitely-generated modules over A are free;

- By Morita theory, the finitely-generated modules over $A(n)$ are in bijective correspondence with the finitely-generated modules over A. The bijection sends an A-module M to the $A(n)$-module M^n.
- If R and S are algebras then modules over $R \times S$ can all be written as $M \times N$ where M is an R-module and N is an S-module.

In the following table we list each Clifford algebra $\mathrm{Cl}_n = \mathrm{Cl}_n(\mathbb{R})$ and the $\mathbb{R}$-dimension of its smallest nonzero module, and in parentheses next to the dimension we also identify the module.

TABLE 22.20. Dimensions of Clifford modules

n	Cl_n	Smallest dim. of a Cl_n-module
0	$\mathbb{R}$	1 $(\mathbb{R})$
1	$\mathbb{C}$	2 $(\mathbb{C})$
2	$\mathbb{H}$	4 $(\mathbb{H})$
3	$\mathbb{H} \times \mathbb{H}$	4 $(\mathbb{H})$
4	$\mathbb{H}(2)$	8 $(\mathbb{H}^2)$
5	$\mathbb{C}(4)$	8 $(\mathbb{C}^4)$
6	$\mathbb{R}(8)$	8 $(\mathbb{R}^8)$
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	8 $(\mathbb{R}^8)$
8	$\mathbb{R}(16)$	16 $(\mathbb{R}^{16})$
9	$\mathbb{C}(16)$	32 $(\mathbb{C}^{16})$
10	$\mathbb{H}(16)$	64 $(\mathbb{H}^{16})$

Observe that the third column has a quasi-periodicity, where the number in row $k+8$ is 16 times the one in row k (and likewise the module is obtained by applying $(-) \otimes_{\mathbb{R}} \mathbb{R}^{16}$).

After all of this, we are ready to prove the Hurwitz-Radon theorem about constructing vector fields on spheres. Recall that if Cl_{r-1} acts on $\mathbb{R}^n$ then there are $r-1$ independent vector fields on S^{n-1}. Going down the rows of the above table, we make the following deductions:

$$\begin{aligned}
&\mathrm{Cl}_1 \text{ acts on } \mathbb{R}^2, \text{ therefore we have 1 vector field on } S^1\\
&\mathrm{Cl}_2 \text{ acts on } \mathbb{R}^4, \text{ therefore we have 2 vector fields on } S^3\\
&\mathrm{Cl}_3 \text{ acts on } \mathbb{R}^4, \text{ therefore we have 3 vector fields on } S^3\\
&\mathrm{Cl}_4 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 4 vector fields on } S^7\\
&\mathrm{Cl}_5 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 5 vector fields on } S^7\\
&\mathrm{Cl}_6 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 6 vector fields on } S^7\\
&\mathrm{Cl}_7 \text{ acts on } \mathbb{R}^8, \text{ therefore we have 7 vector fields on } S^7\\
&\mathrm{Cl}_8 \text{ acts on } \mathbb{R}^{16}, \text{ therefore we have 8 vector fields on } S^{15}.
\end{aligned}$$

It is not hard to deduce the general pattern here. The key is knowing where the jumps in dimension occur, and then just doing bookkeeping. To this end, note that the smallest dimension of a nonzero module over Cl_r is $2^{\sigma(r)}$ where

$$\sigma(r) = \#\{s : 0 < s \leq r \text{ and } s \equiv 0, 1, 2, \text{ or } 4 \mod (8)\}$$

(the numbers 0,1,2,4 mod 8 are the rows where the jumps occur in the third column of Table 22.19). Our analysis has shown that we can construct r independent vector fields on $S^{2^{\sigma(r)}-1}$.

Proof of Theorem 22.5 (Hurwitz-Radon Theorem). First note that we know much more about Clifford modules than is indicated in Table 22.20. For each Clifford algebra Cl_r we know the complete list of all isomorphism classes of finitely-generated modules, and their dimensions are all *multiples* of the dimension listed in the table. This is important.

Given an $n \geq 1$, our job is to determine the largest r for which Cl_r acts on $\mathbb{R}^n$. We will then know that there are r vector fields on S^{n-1}. If we write $n = 2^u \cdot (\text{odd})$ it is clear from Table 22.20 and the previous paragraph that the only way Cl_r could act on $\mathbb{R}^n$ is if it actually acts on $\mathbb{R}^{2^u}$. Moreover, the quasi-periodicity in the table shows that if we add 4 to u (i.e., we multiply n by 16) then the largest r goes up by 8. It follows at once that if $u = a + 4b$ then the formula for the largest r is $8b$+??? where the missing expression just needs to be something that works for the values $a = 0, 1, 2, 3$. One readily finds that $r = 8b + 2^a - 1$ does the job.

So we know that there are $8b + 2^a - 1$ vector fields on S^{n-1}, when n has the form $(\text{odd}) \cdot 2^{a+4b}$. □

Remark 22.21 (First connection with KO^*)**.** Return to Table 22.20 and look at the column with the smallest dimensions of the modules. As one reads down the column, consider where the jumps in dimensions occur: we have "jump-jump-nothing-jump-nothing-nothing-nothing-jump," which then repeats. This is oddly reminiscent of the periodic sequence

$$\mathbb{Z}_2\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}\ldots$$

At first blush this feels like quite an amazing and unexpected connection! We will eventually see, following [ABS], that there is a very direct connection between the groups KO^* and the module theory of the Clifford algebras. For now we leave it as an intriguing coincidence, but will develop the ideas a bit more in Section 23.

22.22. **Adams's Theorem.** So far we have done all this work just to construct collections of independent vector fields on spheres. The Hurwitz-Radon lower bound is classical, and was well-known in the 1940's. The natural question is, can one do any better? Is there a different construction that would yield *more* vector fields than we have managed to produce, or is the bound provided by the Hurwitz-Radon construction the best possible? People were actively working on this problem throughout the 1950's. Adams finally proved in 1962 [Adams2] that the Hurwitz-Radon bound was maximal, and he did this by using K-theory:

Theorem 22.23 (Adams)**.** *There do not exist $\rho(n) + 1$ independent vector fields on S^{n-1}.*

This is a difficult theorem, and it will be a long while before we are able to prove it. We are introducing it here largely to whet the reader's appetite. Note that it is far from being immediately clear how a cohomology theory would help one prove the result. There are several reductions one must make in the problem, but the first one we can explain without much effort:

Proposition 22.24. *If there are $r-1$ vector fields on S^{n-1} then the projection $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1} \to \mathbb{R}P^{un-1}/\mathbb{R}P^{un-2} \cong S^{un-1}$ has a section in the homotopy category, for every $u > \frac{2r-2}{n}$.*

The existence of a section in the homotopy category is something that can perhaps be contradicted by applying a suitable cohomology theory $E^*(-)$. See Exercise 22.26 below for a simple example.

We close this section by sketching the proof of Proposition 22.24. Recall that the Stiefel manifold of k-frames in $\mathbb{R}^n$ is the space

$$V_k(\mathbb{R}^n) = \{(u_1, \ldots, u_k) \,|\, u_i \in \mathbb{R}^n \text{ and } u_1, \ldots, u_k \text{ are orthonormal}\}.$$

Consider the map $p_1 \colon V_k(\mathbb{R}^n) \to S^{n-1}$ which sends $(u_1, \ldots, u_n) \mapsto u_1$. There exist r vector fields on S^{n-1} if and only if there is a section of $p_1 \colon V_{r+1}(\mathbb{R}^n) \to S^{n-1}$.

We need a fact from basic topology, namely that there is a cell structure on $V_k(\mathbb{R}^n)$ where the cells look like

$$e^{i_1} \times \cdots \times e^{i_s}$$

with $n-k \leq i_1 < i_2 < \cdots < i_s \leq n-1$ and s is arbitrary. We will not prove this here: see Hatcher [Hatcher1, Section 3.D] or Mosher-Tangora [MT, Chapter 5].

The cell structure looks like

$$\big[e^{n-k} \cup e^{n-k+1} \cup \cdots \cup e^{n-1}\big] \cup \big[(e^{n-k} \times e^{n-k+1}) \cup (e^{n-k} \times e^{n-k+2}) \cup \cdots\big] \cup \cdots$$

If $n-1 < (n-k)+(n-k+1)$ (these are the dimensions of the last cell in the first group and the first cell in the second group) then the $(n-1)$-skeleton just consists of the cells e^{n-k} through e^{n-1}. This looks like the top part of the cell structure for $\mathbb{R}P^{n-1}$, and indeed it is. To begin to see this, start with the map $\rho \colon \mathbb{R}P^{n-1} \to O(n)$ that sends a line $\ell \subseteq \mathbb{R}^n$ to the reflection in the hyperplane $\ell^\perp$. Let $f \colon \mathbb{R}P^{n-1} \to V_k(\mathbb{R}^n)$ be the composite

$$\mathbb{R}P^{n-1} \xrightarrow{\rho} O(n) \xrightarrow{p_{\leq k}} V_k(\mathbb{R}^n)$$

where $p_{\leq k}$ sends a matrix $A \in O(n)$ to the tuple of its first k columns. The subspace $\mathbb{R}P^{n-k-1} \subseteq \mathbb{R}P^{n-1}$ consisting of points $[0 : \cdots : 0 : x_{k+1} : \cdots : x_n]$ is all sent to the standard frame $(e_1, \ldots, e_k)$ under f, so we obtain the induced map

$$\tilde{f} \colon \mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \longrightarrow V_k(\mathbb{R}^n).$$

If r is a reflection in a hyperplane $\ell^\perp$ then $r(v) - v$ lies in ℓ for every vector v. So one can recover ℓ from any vector v such that $r(v) \neq v$. Thus, if $(\rho_\ell(e_1), \ldots, \rho_\ell(e_k)) \neq (e_1, \ldots, e_k)$ then ℓ is uniquely determined by the k-tuple on the left, whereas $(\rho_\ell(e_1), \ldots, \rho_\ell(e_k)) = (e_1, \ldots, e_k)$ if and only if $\ell \subseteq \langle e_1, \ldots, e_k\rangle^\perp$. This shows that $\tilde{f}$ is an injection. The cell structure on $V_k(\mathbb{R}^n)$ (which we have kept in a black box) is defined in such a way that the image of $\tilde{f}$ is indeed the $(n-1)$-skeleton when $n+2 > 2k$.

For future reference let us record this point:

Proposition 22.25. *When $n+2 > 2k$ there is a cell structure on $V_k(\mathbb{R}^n)$ where the $(n-1)$-skeleton is $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1}$.*

We will need one more fact about this situation. The composite map $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} \xrightarrow{\tilde{f}} V_k(\mathbb{R}^n) \xrightarrow{p_1} S^{n-1}$ sends the subspace $\mathbb{R}P^{n-2} = \{[0 : x_2 : \cdots : x_n]\}$ to e_1 and so factors through $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2}$. The induced map

$\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \to S^{n-1}$ is a homeomorphism, using the same considerations described above for proving that $\tilde{f}$ is injective. So we have a commutative diagram of the form

$$(*) \qquad \begin{array}{ccc} \mathbb{R}P^{n-1}/\mathbb{R}P^{n-k-1} & \xrightarrow{\tilde{f}} & V_k(\mathbb{R}^n) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\ \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} & \xrightarrow{\cong} & S^{n-1}. \end{array}$$

Proof of Proposition 22.24. If there exist $r-1$ vector fields on S^{n-1} there also exist $r-1$ vector fields on S^{un-1} for any u (see Proposition 22.3). Then $p_1 \colon V_r(\mathbb{R}^{un}) \to S^{un-1}$ has a section s. By the cellular approximation theorem the map s is homotopic to a cellular map s'. So s' factors through the $(un-1)$-skeleton of $V_r(\mathbb{R}^{un})$, which by the above remarks is $\mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1}$ provided u is large enough so that $un+2 > 2r$. From diagram (*) (but with different indices) we deduce that the composition

$$S^{un-1} \xrightarrow{s'} \mathbb{R}P^{un-1}/\mathbb{R}P^{un-r-1} \xrightarrow{\pi} \mathbb{R}P^{un-1}/\mathbb{R}P^{un-2} \cong S^{un-1}.$$

is homotopic to $\pm\,\mathrm{id}$. If it happened to be $-\,\mathrm{id}$, alter s' by precomposing with a degree -1 map to fix this. □

Exercise 22.26. Use singular cohomology to prove that $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-3} \to S^{n-1}$ does not have a section when n is odd. Deduce that an even sphere does not have a non-vanishing vector field (which you already knew).

Exercise 22.27 (For those who know Steenrod operations)**.** Use Steenrod operations to prove that there do not exist two independent vector fields on S^5, by proving that the projection $\pi \colon \mathbb{R}P^5/\mathbb{R}P^2 \to S^5$ does not have a section.

∘ Exercises ∘

The following exercises take us through another approach to computing the Clifford algebras over $\mathbb{R}$. For this we will need a slight generalization, which turns out to be incredibly useful throughout the subject. Given a real vector space V and a quadratic form $q \colon V \to \mathbb{R}$, define

$$\mathrm{Cl}(V,q) = T_{\mathbb{R}}(V)/\langle v \otimes v = q(v)\cdot 1 \,|\, v \in V\rangle.$$

For $\mathbb{R}^k$ with $q(x_1,\dots,x_k) = -(x_1^2+\cdots+x_k^2)$ this recovers the algebra Cl_k. For $q(x_1,\dots,x_k) = x_1^2+\cdots+x_k^2$ this gives a new algebra we will call Cl_k^-. It will be convenient to temporarily rename Cl_k as Cl_k^+. Of course there are other quadratic forms on $\mathbb{R}^k$, but these will be the only two we need for our present purposes.

Exercise 22.28. Prove that there are isomorphisms of algebras $\mathrm{Cl}_k^{\pm} \cong \mathrm{Cl}_2^{\pm} \otimes_{\mathbb{R}} \mathrm{Cl}_{k-2}^{\mp}$. [Hint: Make some suitable guesses for where to send each algebra generator e_i and then just prove that it works.]

Exercise 22.29. We have already remarked that $\mathrm{Cl}_0^+ \cong \mathbb{R}$, $\mathrm{Cl}_1^+ \cong \mathbb{C}$, and $\mathrm{Cl}_2^+ \cong \mathbb{H}$. Prove that $\mathrm{Cl}_0^- \cong \mathbb{R}$, $\mathrm{Cl}_1^- \cong \mathbb{R}\times\mathbb{R}$, and $\mathrm{Cl}_2^- \cong \mathbb{R}(2)$. [To get the last isomorphism, note that Cl_2^- is generated by e_1 and e_2 subject to the relations $e_1^2 = 1$, $e_2^2 = 1$, and $e_1e_2 = -e_2e_1$. The conditions $e_i^2 = 1$ might make you think of reflections, and we

can try to realize the skew-commutativity relation by using two reflections through carefully chosen lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$. Get an algebra homomorphism $\mathrm{Cl}_2^- \to \mathbb{R}(2)$ by sending e_i to the matrix for reflection in ℓ_i, and then prove that this map is an isomorphism.]

Exercise 22.30. Use the previous two exercises to prove that $\mathrm{Cl}_3^+ \cong \mathbb{H} \times \mathbb{H}$, $\mathrm{Cl}_4^+ \cong \mathbb{H}(2)$, $\mathrm{Cl}_3^- \cong \mathbb{C}(2)$, and $\mathrm{Cl}_4^- \cong \mathbb{H}(2)$. Then continue with the same method to show

$$\mathrm{Cl}_5^+ \cong (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})(2),\ \mathrm{Cl}_6^+ \cong (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H})(2),\ \mathrm{Cl}_5^- \cong \mathbb{H}(2) \times \mathbb{H}(2),\ \mathrm{Cl}_6^- \cong \mathbb{H}(4).$$

Miller [Miller] compares this process to lacing up a shoe.

Exercise 22.31. The algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ can be written in a simpler form: in fact we have $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$. Find where we already proved these in this section and write down explicit isomorphisms.

Exercise 22.32. Now continue "lacing up the shoe" and deduce that

$$\mathrm{Cl}_7^+ \cong \mathbb{R}(8) \times \mathbb{R}(8),\ \mathrm{Cl}_8^+ \cong \mathbb{R}(16),\ \mathrm{Cl}_7^- \cong \mathbb{C}(8),\ \mathrm{Cl}_8^- \cong \mathbb{R}(16).$$

Note the synchronicity $\mathrm{Cl}_8^+ \cong \mathrm{Cl}_8^- \cong \mathbb{R}(16)$, and convince yourself that the interlacing process now yields the 8-fold quasi-periodicity.

For future reference we give the entire table of our generalized Clifford algebras, with the quasi-periodicity $\mathrm{Cl}_{r+8}^{\pm} \cong \mathrm{Cl}_r^{\pm}(16)$ implicit on both sides:

TABLE 22.33. The Clifford algebras $\mathrm{Cl}_r^{\pm}$

r	Cl_r^+	Cl_r^-
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \times \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \times \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \times \mathbb{H}(2)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$
7	$\mathbb{R}(8) \times \mathbb{R}(8)$	$\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$

Comparing the two columns reveals some interesting patterns. These phenomena, and others that we have encountered in this section, are perhaps best explained in terms of the super Brauer group of $\mathbb{Z}/2$-graded real division algebras. That story is explored in [D7].

23. The coefficient groups of K and KO

In order to use the cohomology theories KO and K we will need to have a good understanding of their coefficient groups. In the negative range these are the homotopy groups of $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$, and we have already indicated that these were computed by Bott. The groups in the positive range are determined by periodicity. In addition to identifying the groups (and recall that they are all cyclic), Bott's methods can be used to produce very explicit generators and to determine the ring structure. A very efficient way to go about this was provided by Atiyah-Bott-Shapiro [ABS], through the use of Clifford algebras and their modules.

We will delay a complete treatment of these results until [D7], where we do the hard work of proving the Bott periodicity theorems. But the results themselves are necessary foundations for working with K and KO, and we outline them here so that we can freely use them in what follows.

23.1. **Clifford modules and elements of $KO(S^r)$.** Regard the sphere S^r as contained in Cl_r via the standard inclusion $S^r \subseteq \mathbb{R} \oplus \mathbb{R}^r \subseteq \mathrm{Cl}_r$, where the first copy of $\mathbb{R}$ consists of the scalar multiples of the identity element. Note that S^r is actually contained in the units of Cl_r, as this follows from the identity

$$(a_0 + a_1e_1 + \cdots + a_re_r)(a_0 - a_1e_1 - \cdots - a_re_r) = \sum a_i^2.$$

Let M be a Cl_{r-1}-module. Construct a bundle over S^r by clutching: over the upper and lower hemispheres we take $D^r_\pm \times M$, and for $x \in \partial D^r_+$ and $m \in M$ we glue (x, m) to $(x, xm) \in D^r_- \times M$. The gluing maps are all isomorphisms because each x is a unit in Cl_{r-1}. So this constructs a bundle over S^r of rank $\dim_{\mathbb{R}} M$. Call this bundle $\Theta(M)$.

Theorem 23.2 (Atiyah-Bott-Shapiro)**.** *Let $r \geq 1$. If M is an irreducible Cl_{r-1}-module then the group $\widetilde{KO}(S^r)$ is generated by the element $[\Theta(M)] - \underline{\dim M}$. In fact, this construction gives a surjective homomorphism*

$$G(\mathrm{Cl}_{r-1}) \twoheadrightarrow \widetilde{KO}(S^r)$$

where the domain is the Grothendieck group of finitely-generated Cl_{r-1}-modules.

The analogous statements are true over the complex numbers, using $\mathrm{Cl}_{r-1}(\mathbb{C})$ and $\widetilde{K}(S^r)$.

The Clifford algebras are all semisimple: the finitely-generated modules are direct sums of simple modules, in a unique way. This is clear just by looking at the algebras as they appear in Table 22.15. So the Grothendieck groups $G(\mathrm{Cl}_{r-1})$ are all free abelian.

Exercise 23.3. Determine the groups $G(\mathrm{Cl}_{r-1})$ for all $r \geq 1$.

Let M be a Cl_{r-1}-module. It is not hard to see that if M extends to a Cl_r-module then $\Theta(M)$ is trivializable. This is because the composite $S^{r-1} \hookrightarrow \mathrm{Cl}_{r-1} \hookrightarrow \mathrm{Cl}_r$ can be null-homotoped while staying in the subset of invertible elements (in fact, it can be null-homotoped just inside of S^r). So the clutching map that constructs $\Theta(M)$ can be homotoped—through clutching functions—to the constant map at the identity. This is the trivialization.

The inclusion $\mathrm{Cl}_{r-1} \hookrightarrow \mathrm{Cl}_r$ yields the restriction-of-scalars map $\rho_r\colon G(\mathrm{Cl}_r) \to G(\mathrm{Cl}_{r-1})$. By what we have just said, we obtain maps

$$G(\mathrm{Cl}_{r-1})/\operatorname{im}\rho_r \twoheadrightarrow \widetilde{KO}^0(S^r).$$

Amazingly, these turn out to be isomorphisms. In fact the groups in the domain can easily be computed algebraically, are found to exactly match the Bott computations, and so surjectivity of the above map implies injectivity (since all of the groups are cyclic). Things work similarly in the complex case. This story is recounted in detail in [D7].

Exercise 23.4. Use your knowledge of the real Clifford algebras to determine the quotients $G(\mathrm{Cl}_{r-1})/\operatorname{im}\rho_r$ for $r \geq 1$. Then do the same for the complex case.

23.5. **Comparing KO and K.** Recall from Exercise 17.25 that we have natural maps $c\colon KO^0(X) \to K^0(X)$ and $r_{\mathbb{R}}\colon K^0(X) \to KO^0(X)$, and that the former is a ring map. The Atiyah-Bott-Shapiro theorem lets us identify these maps when X is a sphere.

Theorem 23.6. *The complexification map $c\colon \widetilde{KO}^0(S^r) \to \widetilde{K}^0(S^r)$ is*

- *an isomorphism when $r \equiv 0$ (mod 8),*
- *multiplication by two when $r \equiv 4$ (mod 8),*
- *zero otherwise.*

The restriction map $r_{\mathbb{R}}\colon \widetilde{K}^0(S^r) \to \widetilde{KO}^0(S^r)$ is

- *multiplication by two when $r \equiv 0$ (mod 8),*
- *the surjection $\mathbb{Z} \to \mathbb{Z}/2$ when $r \equiv 2$ (mod 8),*
- *an isomorphism when $r \equiv 4$ (mod 8),*
- *zero otherwise.*

This falls out of Theorem 23.2 and we will leave it as an exercise, but let us give some hints on how to proceed. Just as an example, start with $\mathrm{Cl}_3(\mathbb{C}) = \mathbb{C}(2)\times\mathbb{C}(2)$. Let $M = \mathbb{C}^2$ be the irreducible module where the left factor acts nontrivially and the right factor acts trivially. If we restrict scalars and regard M as a $\mathrm{Cl}_3(\mathbb{R})$-module, it must be irreducible because it has real dimension four and this is the smallest dimension for modules over $\mathrm{Cl}_3(\mathbb{R}) = \mathbb{H} \times \mathbb{H}$. It follows from the Atiyah-Bott-Shapiro theorem that $r_{\mathbb{R}}\colon \widetilde{K}^0(S^4) \to \widetilde{KO}^0(S^4)$ is surjective. The relation $r_{\mathbb{R}}c = 2$ then implies that $c\colon \widetilde{KO}^0(S^4) \to \widetilde{K}^0(S^4)$ is multiplication by two.

Exercise 23.7. Use the above ideas, together with your knowledge of the real and complex Clifford algebras, to prove Theorem 23.6. [For a more elementary proof of this result, which avoids Theorem 23.2 but uses other aspects of Bott periodicity, see Exercise 23.14 below.]

23.8. **The coefficient rings.** Define the real Bott element $\beta_{\mathbb{R}} \in \widetilde{KO}^0(S^8)$ to be the unique class having the property that $c(\beta_{\mathbb{R}}) = \beta$. This exists by Theorem 23.6. The KO-version of real Bott periodicity is:

Theorem 23.9. *For any compact Hausdorff and semi-cofibrant space X, the external tensor product map $KO^0(X) \otimes KO^0(S^8) \to KO^0(X \times S^8)$ is an isomorphism. Likewise, the restricted maps $KO^0(X) \otimes \widetilde{KO}^0(S^8) \to KO^0(X \times S^8, X \times *)$ and $\widetilde{KO}^0(X) \otimes \widetilde{KO}^0(S^8) \to \widetilde{KO}^0(X \wedge S^8)$ are isomorphisms, the latter assuming X is pointed semi-cofibrant.*

We will not prove this until [D7]. But assuming it, we can pull everything we know together and state the following:

Theorem 23.10 (The coefficient rings of K-theory)**.**

(a) There is an isomorphism of rings $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$ where $\beta \in K^{-2}(pt)$ corresponds to $1 - L$ under the suspension isomorphism $K^{-2}(pt) \cong \widetilde{K}^0(\mathbb{C}P^1)$.

(b) Likewise, $KO^*(pt) = \mathbb{Z}[x, u, \beta_{\mathbb{R}}, \beta_{\mathbb{R}}^{-1}]/(2x, x^3, xu, u^2 - 4\beta_{\mathbb{R}})$ where $|x| = -1$, $|u| = -4$, $|\beta_{\mathbb{R}}| = -8$, β_R is the unique element of $KO^{-8}(pt)$ such that $c(\beta_{\mathbb{R}}) = \beta^4$, and u is the unique element of $KO^{-4}(pt)$ such that $c(u) = 2\beta^2$.

Proof. Part (a) follows directly from Corollary 21.7.

For (b), we know by Theorem 23.6 that there is a unique class $u \in \widetilde{KO}^0(S^4)$ such that $c(u) = 2\beta^2$. Then $c(u^2) = 4\beta^4 = c(4\beta_{\mathbb{R}})$ and therefore $u^2 = 4\beta_{\mathbb{R}}$, since c is an isomorphism on S^8.

We proved at the end of Section 20.15 that the square of the generator of $\widetilde{KO}^0(S^1) = \mathbb{Z}/2$ equals the generator of $\widetilde{KO}^0(S^2) = \mathbb{Z}/2$ (for a different proof, see Exercise 23.16 below). All of the other elements of the ring structure are now determined by degree considerations, the additive structure of $\pi_*(\mathbb{Z} \times BO)$, and periodicity. □

23.11. **The missing gaps.** During the course of this initial tour of topological K-theory, there are several places where we have stated difficult results without proof. Let us list these missing pieces to be clear about what they are. Readers can find the details in [D7].

(i) The space-level version of real Bott periodicity: the existence of weak equivalences $\Omega^8 O \simeq O$.

(ii) Bott's calculations of $\pi_* O$. [This follows almost immediately from (i) together with geometric models for the spaces $\Omega^i O$, which fall out of the proof of (i).]

(iii) Enhancing (i), a weak equivalence $\mathbb{Z} \times BO \to \Omega^8(\mathbb{Z} \times BO)$ that is a map of H-spaces.

(iv) The product map $(\mathbb{Z} \times BO) \times (\mathbb{Z} \times BO) \to \mathbb{Z} \times BO$. We also claimed that there are pairings of spectra $KO \wedge KO \to KO$ and $K \wedge K \to K$. [We will not prove these; see the references given in Section 19.]

(v) Theorem 23.9, the KO-version of the real periodicity theorem. [This comes from identifying the map in (iii) as the map representing multiplication by the class $\beta_{\mathbb{R}} \in \widetilde{KO}^0(S^8)$.]

(vi) The Atiyah-Bott-Shapiro theorems that $G(\mathrm{Cl}_{r-1})/\operatorname{im}\rho_r \to \widetilde{KO}^0(S^r)$ are isomorphisms, and similarly in the complex case.

(vii) The proof of Theorems 20.14 and 20.17, which give geometric models of the relative groups $KO^0(X, A)$ and $K^0(X, A)$ under some hypotheses. [This gap will be filled in Section 24].

Remark 23.12. Several of the above missing pieces relate to products, especially as to how they interact with real Bott periodicity. This is an inherently difficult subject. As just one example, it is possible to enhance Theorem 23.2 into a map of graded rings, and this is quite helpful, but figuring out the correct multiplication on the domain turns out to be subtle—not hard, in the end, but subtle. The sections on Bott periodicity in[D7] take a detailed look at these (and related) issues regarding products.

◦ Exercises ◦

Exercise 23.13. Suppose that $\mathbb{R}^n$ is a module over Cl_r. Using only first principles, explain how to get an element of $\pi_{r+1}GL_n(\mathbb{R})$ from this data. If the Cl_r-action extends to a Cl_{r+1}-action, prove that this homotopy element vanishes. Explain how this yields a group homomorphism $G(\mathrm{Cl}_r)/\operatorname{im}\rho_{r+1} \to \pi_{r+1}(GL(\mathbb{R}))$.

Exercise 23.14. Theorem 23.6 can also be proven using the more detailed consequences of real Bott periodicity. Bott explicitly described all of the spaces $\Omega^i(BO)$ as homogeneous spaces for the infinite versions of familiar Lie groups.

(a) The inclusions $O_n \hookrightarrow U_n$ induce an inclusion $O \hookrightarrow U$, and there is a homotopy fiber sequence $U/O \to BO \to BU$. Bott proved that $\Omega(U/O) \simeq \mathbb{Z} \times BO$, so we know the homotopy groups of U/O. Use this to prove the statements about c in Theorem 23.6.

(b) Likewise, the inclusions $U_n \hookrightarrow O_{2n}$ induce $U \hookrightarrow O$, and there is a homotopy fiber sequence $O/U \to BU \to BO$. Bott proved that $\Omega O \simeq O/U$, and so again we can deduce the homotopy groups of O/U. Use this knowledge to prove the statements about r in Theorem 23.6.

Exercise 23.15. [For those who are comfortable with spectral sequences.] There turns out to be a multiplicative spectral sequence of the form

$$E_2^{*,*} = \mathbb{Z}[x,y]/(2x), \quad |x| = (1,1), \quad |y| = (4,0)$$

with $d_r\colon E_r^{p,q} \to E_r^{p-1,q+r}$ and $E_\infty^{p,*}$ an associated graded of $\widetilde{KO}^0(S^p)$. There is a unique pattern of differentials that is compatible with the fact that $\widetilde{KO}^0(S^3) = 0$. Find the pattern of differentials and use the spectral sequence to re-compute $\bigoplus_{p\geq 0}\widetilde{KO}^0(S^p)$ as a ring. [This spectral sequence appears in a number of different places. If you invert y it appears as the homotopy-fixed-point spectral sequence for $K^{h\mathbb{Z}/2}$, or (without inverting y) it appears as a truncation of the similar spectral sequence for $(\mathbb{Z}\times BU)^{h\mathbb{Z}/2}$. See [Mitchell4, Example before 5.1] or [HS]. It also appears as a 2-completed form of the motivic spectral sequence for the algebraic K-theory of $\mathbb{R}$, and (in a similar vein) as a spectral sequence for KR-theory in [D3].]

Exercise 23.16. This exercise uses Stiefel-Whitney classes; for readers unfamiliar with these, see Section 38.10. We will give an alternative proof that the square of the generator in $\widetilde{KO}^0(S^1)$ is the generator in $\widetilde{KO}^0(S^2)$.

Let $M \to S^1$ be the Möbius bundle, which generates $KO^{st}(S^1) = \mathbb{Z}/2$; equivalently, $M-1$ generates $\widetilde{KO}^0(S^1)$. Likewise, let $E \to S^2$ be the rank 2 bundle whose clutching function is the standard generator of $\pi_1(SO(2))$. We know that E generates $KO^{st}(S^2) = \mathbb{Z}/2$. We need to prove that $(M-1)\cdot(M-1) = E-2$ as elements of $\widetilde{KO}^0(S^2)$.

Start by considering the cofiber sequence $S^1 \vee S^1 \hookrightarrow S^1 \times S^1 \to S^2$ and the induced split-exact sequence

$$0 \longrightarrow \widetilde{KO}^0(S^2) \longrightarrow KO^0(S^1\times S^1) \longrightarrow \mathbb{Z} \oplus \widetilde{KO}^{st}(S^1) \oplus \widetilde{KO}^{st}(S^1) \longrightarrow 0.$$

(Recall that $X \vee Y \hookrightarrow X \times Y \to X \wedge Y$ always induces a split-exact sequence on any cohomology theory, since the inclusion is split after one suspension by the sum

of the two projections.) In the middle group we have $M\hat{\otimes}1$, $1\hat{\otimes}M$, and $M\hat{\otimes}M$. Identifying the group on the right with $\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$, show that

$$M\hat{\otimes}1\to(1,1,0),\qquad 1\hat{\otimes}M\to(1,0,1),\qquad M\hat{\otimes}M\to(1,1,1),\qquad 1\to(1,0,0).$$

The element $M\hat{\otimes}M-M\hat{\otimes}1-1\hat{\otimes}M+1$ maps to zero, and hence lifts to a unique element of $\widetilde{KO}^0(S^2)$. This is the element $(M-1)\cdot(M-1)$. Since $\widetilde{KO}^0(S^2)=\mathbb{Z}/2$ we only need to show that this product element is nonzero in $KO^0(S^1\times S^1)$. Equivalently, the question is whether the rank 2 bundles $(M\hat{\otimes}M)\oplus 1$ and $(M\hat{\otimes}1)\oplus(1\hat{\otimes}M)=\pi_1^*(M)\oplus\pi_2^*(M)$ are stably isomorphic. Use Stiefel-Whitney classes to prove that they are not.

Remark 23.17. In the above exercise we encountered one of the standard issues in working with these structures: whereas the tensor product $KO^0(X)\otimes KO^0(Y)\to KO^0(X\times Y)$ is very concrete and geometric, the "reduced tensor product" $\widetilde{KO}^0(X)\otimes\widetilde{KO}^0(Y)\to\widetilde{KO}^0(X\wedge Y)$ is less so. The reduced tensor product naturally lives in a relative group $KO^0(X\times Y,X\vee Y)$, and there is no direct method for distilling such classes into linear combinations of vector bundles on $X\wedge Y$. In Section 20.15 we got around this issue by focusing on $KO^0(D^n,S^{n-1})$ rather than $\widetilde{KO}^0(S^n)$ and doing all of our computations with relative classes.

24. More about relative K-theory

Our aim in this section is to revisit the topic of relative K-theory and give the proof of Theorem 20.14. Recall that this is a comparison between the geometric groups $\mathcal{K}(X,A)$, defined in terms of chain complexes of vector bundles that are exact on A, and the group $K^0(X,A)$ that comes to us from topology once we have the K-theory spectrum. The proof is somewhat clunky and frustrating in a way that requires some apology.

This kind of clunkiness often surfaces when one tries to compare the world of geometry to the world of homotopy theory. Somehow these subjects inhabit different universes, and bringing the homotopical world down to earth turns out to be an inherently messy undertaking. While we have this beautiful cohomology theory K^*, at the moment we only have one access to it: that is the connection between vector bundles and $K^0(X)$. Theorem 20.14 is essentially about building another access route, but our only technique is to go through the one we already have. Thus, the clunkiness.

The ideas behind the proof are interesting and have their own intrinsic appeal, but within the context of this book they hardly ever resurface outside the confines of this one argument. Once we have this new access route, we just use it and can largely forget about the trouble that went into constructing it. So this section should be regarded as giving some technical information that is important, but not necessary for anything later in the book.

Throughout this section (X,A) will be a pair of spaces. At various points we will need to assume that the spaces are paracompact Hausdorff, or that $A\hookrightarrow X$ is a cofibration (e.g. a relative cell complex), and sometimes we will assume that (X,A) is a finite CW-pair. We remind the reader that the pair (X,A) is said to be cofibrant if A is cofibrant and $A\hookrightarrow X$ is a cofibration.

Recall the group $\mathcal{K}(X,A)$ introduced in Definition 20.2, made from bounded complexes of vector bundles on X that are exact on A. For specificity we focus

on $\mathbb{C}$-vector bundles, but the same arguments work for the real and quaternionic cases. We take several steps aimed at analyzing these groups:

(1) Whereas $\mathcal{K}(X, A)$ is defined via generators and relations, we know that the additive inverse of a chain complex is just another chain complex (in fact, its suspension). So we can construct an isomorphic group just as certain equivalence classes of non-negatively graded, bounded chain complexes, with the monoid structure induced by direct sum. We call this construction $\mathcal{M}_\infty(X, A)$. It has the advantage over $\mathcal{K}(X, A)$ of the elements being a bit easier to get our hands on.

(2) Rather than consider all bounded complexes, we can consider complexes that are concentrated in degrees between 0 and n, for some fixed $n \geq 1$. This leads to sets (in fact, monoids) $\mathcal{M}_n(X, A)$. We prove that—for nice enough pairs (X, A)—all choices of n give rise to isomorphic monoids; so in some sense the use of chain complexes is overkill, as it suffices to just look at complexes of length 1. (The use of all chain complexes makes for a more natural theory, however—for example, the tensor product of two complexes of length 1 is a complex of length 2, and can only be turned back into a complex of length 1 by a cumbersome process).

(3) Finally, and most importantly, we replace our consideration of chain complexes by that of a related but different construct. Namely, we consider $\mathbb{Z}$-graded collections of vector bundles $\{E_i\}$ together with an exact differential defined only over the set A (that is, a differential on $(E_\bullet)|_A$). Let us call such things "A-relative exact complexes". From these we construct monoids $\mathcal{M}_n(X, A)_A$ for all $n \geq 1$, where as above the integer n bounds the length. When (X, A) is sufficiently nice it turns out that every A-relative exact complex may be extended to give an ordinary chain complex, and the space of all possible extensions is contractible. So homotopically speaking there is no real difference between the theories obtained from the two notions. The A-relative exact complexes turn out to give a theory that is a bit easier to manipulate, essentially because one doesn't have to deal with extraneous data.

What motivates this line of argument? Recall that $K^0(X, A)$ is defined to be $\operatorname{Ho}\mathcal{T}_*(Cj, \mathbb{Z} \times BU)$ where Cj is the mapping cone of the inclusion $j\colon A \hookrightarrow X$. Assuming A and X are cofibrant, elements of this set are represented by maps $X \to \mathbb{Z} \times BU$ together with a choice of null homotopy for the restriction to A. If X and A are homotopically compact then we understand maps into $\mathbb{Z} \times BU$ entirely in terms of bundle theory, and so the above leads one to think about bundles E on X together with a specific trivialization of $E|_A$. This is precisely an A-relative complex of the form $0 \to \underline{n} \to E \to 0$. It turns out that all elements of $\mathcal{M}_1(X, A)_A$ are of this form, since we can always add an elementary complex to make the bundle in degree 1 trivial. So $\mathcal{M}_1(X, A)_A$ is an algebraic construct that has a close connection to the topological definition of $K^0(X, A)$. We relate $\mathcal{K}(X, A)$ to $\mathcal{M}_1(X, A)_A$ by the reductions described above, and then we connect $\mathcal{M}_1(X, A)_A$ to $K^0(X, A)$ directly.

As one final remark before we get started, it is possible to simplify some of the arguments in this section by only working with finite CW-pairs (X, A). This is what is done in [ABS]. The book [Atiyah5] states the results for compact pairs, but in several places it seems to use implicit hypotheses like $A \hookrightarrow X$ being a cofibration. Unfortunately, in applications we will often want to consider $(\mathbb{C}^n, \mathbb{C}^n - 0)$ or $(E, E -$

ζ_E) (E a vector bundle) or $(X, X - Z)$ (X an algebraic variety and Z a subvariety). So we would prefer not to restrict to compact spaces, and would also prefer not to assume that $A \hookrightarrow X$ is a cofibration. We have therefore endeavored to use constructions that will apply to pairs (X, A) in as broad a generality as we could manage without making life too difficult. It remains true, though, that for the strongest forms of the results we need to assume that X and A are homotopically compact and also that $A \hookrightarrow X$ is a cofibration.

In outline the arguments in this section largely follow Atiyah's in [Atiyah5, Section 2.6], which were in turn modeled on the ones in [ABS]. However, there are a few important differences that will be explained along the way.

24.1. **Relative chain complexes.** Let (X, A) be a pair of spaces. Let $\mathcal{C}\mathrm{h}(X, A)$ denote the category whose objects are bounded chain complexes of vector bundles on X that are exact on A. A map in this category is simply a map of chain complexes of vector bundles. In contrast to this, let $\mathcal{C}\mathrm{h}(X, A)_A$ denote the category whose objects are collections $\{E_i\}$ of vector bundles on X, all but finitely-many of which are zero, together with maps $d\colon E_{i+1}|_A \to E_i|_A$ making the restriction $E_\bullet|_A$ into an exact chain complex of vector bundles on A. A map $E \to E'$ in $\mathcal{C}\mathrm{h}(X, A)_A$ is a collection of maps $E_i \to E'_i$ of vector bundles on X that commute with the maps d where defined (i.e., over the set A). An object in $\mathcal{C}\mathrm{h}(X, A)_A$ will be called an ***A*-relative exact complex**.

Note the difference between $\mathcal{C}\mathrm{h}(X, A)$ and $\mathcal{C}\mathrm{h}(X, A)_A$: in the former the differentials are defined on all of X, whereas in the latter they are only defined on A. In both cases the maps between chain complexes are defined on all of X. Observe that there is an evident functor $\mathcal{C}\mathrm{h}(X, A) \to \mathcal{C}\mathrm{h}(X, A)_A$, which we will denote $E \mapsto E(A)$; this just restricts all the differentials to A.

Say that two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$ are **homotopic** if there is an object $\mathcal{E}$ in $\mathcal{C}\mathrm{h}(X \times I, A \times I)$ together with isomorphisms $\mathcal{E}|_{X\times 0} \cong E_\bullet$ and $\mathcal{E}|_{X\times 1} \cong E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$. Likewise, define two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)_A$ to be homotopic if there is an object $\mathcal{E}$ in $\mathcal{C}\mathrm{h}(X \times I, A \times I)_{A\times I}$ together with isomorphisms $\mathcal{E}|_{X\times 0} \cong E_\bullet$ and $\mathcal{E}|_{X\times 1} \cong E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)_A$. In each of these two settings the notion of homotopy is readily seen to be an equivalence relation. Write $\sim_h$ for this relation.

Remark 24.2. Observe that the above notion of homotopy has isomorphisms built into it: we only required that $\mathcal{E}|_{X\times 0}$ was isomorphic to E, not equal to it. So with this definition isomorphic complexes are automatically homotopic, via a constant homotopy. The choice to bundle these two relations together rather than keeping them separate is just for convenience, as it makes some results easier to state and prove. It would have been equivalent for us to define "strict homotopy" (with equalities rather than isomorphisms) and then taken the equivalence relation generated by strict homotopies and isomorphisms.

We will also need a second equivalence relation on chain complexes. Say that two complexes $E_\bullet$ and $E'_\bullet$ in $\mathcal{C}\mathrm{h}(X, A)$ are **stably equivalent** if there exist elementary complexes (see Definition 20.7) $P_1, \dots, P_r, Q_1, \dots, Q_s$ in $\mathcal{C}\mathrm{h}(X, A)$ such that

$$E \oplus P_1 \oplus \cdots \oplus P_r \cong E' \oplus Q_1 \oplus \cdots \oplus Q_s.$$

Write this as $E_\bullet \sim_{st} E'_\bullet$. We define a similar equivalence relation on the objects of $\mathcal{C}\mathrm{h}(X, A)_A$.

Let $\mathcal{C}\mathrm{h}_n(X,A)$ be the full subcategory of $\mathcal{C}\mathrm{h}(X,A)$ consisting of chain complexes $E_\bullet$ such that $E_i = 0$ when $i \notin [0,n]$, and let $\mathcal{C}\mathrm{h}_n(X,A)_A$ be the analogous subcategory of A-relative complexes. It will be convenient to allow the index n to be ∞ here; note that this still corresponds to bounded complexes, but where there is no fixed upper bound on the nonzero region.

Let $\mathcal{M}_n(X,A)$ denote the set of equivalence classes of objects in $\mathcal{C}\mathrm{h}_n(X,A)$ under the equivalence relation generated by the homotopy relation and the stable-equivalence relation. Define $\mathcal{M}_n(X,A)_A$ analogously. The restriction functor $E \mapsto E(A)$ (restrict the maps to A, leave the objects alone) clearly induces a map of sets $\mathcal{M}_n(X,A) \to \mathcal{M}_n(X,A)_A$.

Direct sum of complexes makes $\mathcal{M}_n(X,A)$ into a monoid, and similarly for $\mathcal{M}_n(X,A)_A$. Observe that we have the evident maps $\mathcal{M}_n(X,A) \to \mathcal{M}_{n+1}(X,A)$ obtained by regarding a complex concentrated in degrees $[0,n]$ as also concentrated in degrees $[0,n+1]$, and the direct limit of these maps is $\mathcal{M}_\infty(X,A)$. The same holds for the A-relative version.

Exercise 24.3. Let X be a space. Prove that the map $\mathcal{M}_1(X,\emptyset) \to K^0_{Grt}(X)$ (the Grothendieck group of vector bundles on X) sending $E_\bullet$ to $[E_0] - [E_1]$ is an isomorphism of monoids. (One step in the proof will probably be to show that the domain is actually a group). Repeat for the map $\mathcal{M}_n(X,\emptyset) \to K^0_{Grt}(X)$ that sends $E_\bullet$ to $\sum(-1)^i[E_i]$.

Observe that there is a natural map $\mathcal{M}_\infty(X,A) \to \mathcal{K}(X,A)$ induced by sending a chain complex $E_\bullet$ to its class in $\mathcal{K}(X,A)$, since the definition of $\mathcal{K}(X,A)$ has the $\sim_h$ and $\sim_{st}$ relations built into it. This is a map of monoids, and in fact is an isomorphism under mild hypotheses:

Proposition 24.4. *The monoid $\mathcal{M}_\infty(X,A)$ is always a group, and when X is paracompact Hausdorff the natural map $\mathcal{M}_\infty(X,A) \to \mathcal{K}(X,A)$ is an isomorphism. The same statements hold for $\mathcal{M}_\infty(X,A)_A$ and the map $\mathcal{M}_\infty(X,A)_A \to \mathcal{K}(X,A)_A$.*

Proof. Suppose $E_\bullet$ is a chain complex in $\mathcal{C}\mathrm{h}_\infty(X,A)$ and let $C_\bullet(t)$ be the mapping cone of the multiplication by t map on $E_\bullet$. By the homotopy relation we have $C_\bullet(1) = C_\bullet(0)$ in $\mathcal{M}_\infty(X,A)$. But $C_\bullet(1)$ is exact on X and so is a direct sum of elementary complexes: if X is paracompact Hausdorff one can cite Proposition 20.8 for this, but $C_\bullet(1)$ is nice enough that one can also produce the splitting by hand without any assumptions on X at all (see Exercise 24.5 below). Hence $C_\bullet(1) \sim 0$ in $\mathcal{M}_\infty(X,A)$. Finally, observe that $C_\bullet(0) = E_\bullet \oplus \Sigma E_\bullet$. This shows that $E_\bullet$ has an additive inverse in $\mathcal{M}_\infty(X,A)$, namely $\Sigma E_\bullet$. Thus, $\mathcal{M}_\infty(X,A)$ is a group.

For the second claim, consider the assignment that sends a chain complex $E_\bullet \in \mathcal{C}\mathrm{h}(X,A)$ whose lowest nonzero degree is n ($n \in \mathbb{Z}$) to $(-1)^n[\Sigma^{-n}E_\bullet] \in \mathcal{M}_\infty(X,A)$. We will show that this induces a group homomorphism $\mathcal{K}(X,A) \to \mathcal{M}_\infty(X,A)$ by verifying that it respects the defining relations for $\mathcal{K}(X,A)$. Relation (3) (the homotopy relation) is automatic. Relation (2) says that exact complexes are zero, and here is where we need that X is paracompact Hausdorff: Proposition 20.8 says that in this setting all exact complexes are isomorphic to direct sums of elementary complexes.

Finally to check relation (1) assume that $E_\bullet$ and $J_\bullet$ are two complexes whose lowest degrees are n and k with $n \leq k$. We need to verify that

$$(-1)^n[\Sigma^{-n}(E_\bullet \oplus J_\bullet)] = (-1)^n[\Sigma^{-n}E_\bullet] + (-1)^k[\Sigma^{-k}J_\bullet]$$

in $\mathcal{M}_\infty(X,A)$. But $\Sigma^{-n}J_\bullet = \Sigma^{k-n}\Sigma^{-k}J_\bullet$ and in the first paragraph we showed that $[\Sigma^{k-n}(\Sigma^{-k}J_\bullet)] = (-1)^{k-n}[\Sigma^{-k}J_\bullet]$ in $\mathcal{M}_\infty(X,A)$. This gives what we needed. Consequently, our assignment induces a group map $\mathcal{K}(X,A) \to \mathcal{M}_\infty(X,A)$. This is readily checked to be a two-sided inverse for the map from the statement of the proposition.

The proofs for $\mathcal{M}_\infty(X,A)_A$ and $\mathcal{K}(X,A)_A$ are identical. □

Exercise 24.5. Let $E_\bullet$ be any bounded complex of vector bundles on a space X, and let $C_\bullet$ be the mapping cone of the identity. Prove that $C_\bullet$ is a direct sum of elementary complexes.

Exercise 24.6. Below we will prove that $\mathcal{M}_n(X,A)$ is also a group when $n \geq 1$ and (X,A) is a cofibrant pair. Try to prove this for $n = 1$ assuming only that X is paracompact Hausdorff and see where you get stuck.

All of the constructions we have introduced are contravariant functors of pairs and have homotopy invariance formally built into them:

Proposition 24.7. *Let $f,g\colon (X,A) \to (Y,B)$ be maps of pairs and suppose $H\colon (X\times I, A\times I) \to (Y,B)$ is such that $H_0 = f$ and $H_1 = g$. Then $f^* = g^*$ as maps $\mathcal{K}(Y,B) \to \mathcal{K}(X,A)$, and likewise for the corresponding maps $\mathcal{M}_n(Y,B) \to \mathcal{M}_n(X,A)$ and $\mathcal{M}_n(Y,B)_B \to \mathcal{M}_n(X,A)_A$ for $1 \leq n \leq \infty$. Consequently, if f is part of a homotopy equivalence of pairs then f^* is an isomorphism in all cases.*

Proof. The arguments for the different functors are all the same, so let us just do one: $\mathcal{M}_n(X,A)$. If $i_0, i_1\colon (X,A) \hookrightarrow (X\times I, A\times I)$ are the inclusions then $f^* = (i_0)^*H^*$ and $g^* = (i_1)^*H^*$. So it suffices to prove that $i_0^* = i_1^*$, but this follows from the fact that the homotopy relation was built into the definition of $\mathcal{M}_n(X,A)$. □

24.8. **Strategy of the proof of Theorem 20.14.** Let $\mathcal{L}(X,A)$ and $\mathcal{L}(X,A)_A$ be defined similarly to the $\mathcal{M}$ sets except that we do not include the homotopy relation—so the equivalences are generated by isomorphisms and stable equivalences. There are canonical surjections from the $\mathcal{L}$-constructions to the corresponding $\mathcal{M}$-constructions.

Consider the following diagram:

$$\begin{array}{ccccccc}
 & & \mathcal{M}_1(X,A) & \longrightarrow & \mathcal{M}_n(X,A) & \longrightarrow & \mathcal{M}_\infty(X,A) \xrightarrow{(24.4)} \mathcal{K}(X,A) \\
 & & \downarrow{\scriptstyle (24.12)} & & \downarrow{\scriptstyle (24.12)} & & \downarrow{\scriptstyle (24.12)}\ \textcircled{3} \qquad \downarrow \\
K^0(X,A) & \xleftarrow[(24.34)]{\chi_1} & \mathcal{M}_1(X,A)_A & \xrightarrow[(24.23)]{\textcircled{1}} & \mathcal{M}_n(X,A)_A & \xrightarrow[(24.23)]{\textcircled{2}} & \mathcal{M}_\infty(X,A)_A \xrightarrow{(24.4)} \mathcal{K}(X,A)_A \\
 & & \uparrow{\scriptstyle (24.46)} & & \uparrow{\scriptstyle (24.46)} & & \uparrow{\scriptstyle (24.46)} \\
 & & \mathcal{L}_1(X,A)_A & \longrightarrow & \mathcal{L}_n(X,A)_A & \longrightarrow & \mathcal{L}_\infty(X,A)_A.
\end{array}$$

All of the maps are the evident ones except for χ_1, which will need to be constructed below. By the end of this section we will have proven that they are all bijections when (X,A) is a cofibrant pair, except for χ_1 which also requires that both X and A are homotopically compact. The references in the diagram are to the results that establish the indicated bijections. For convenience here are the actual conditions needed:

- (24.4): X is paracompact and Hausdorff.
- (24.12): $A \hookrightarrow X$ is a cofibration.
- (24.23): A is cofibrant and $A \hookrightarrow X$ is a cofibration.
- (24.46): A is cofibrant and $A \hookrightarrow X$ is a cofibration.
- (24.34): A and X are cofibrant and homotopically-compact.

For cofibrant pairs (X, A) our natural transformation $\mathcal{K}(X, A) \to K^0(X, A)$ will be obtained by zig-zagging through this diagram. We do not really need the third level with the $\mathcal{L}$-sets, but these are used in [ABS] and [Atiyah5] and it is convenient to include them as part of the story. As the $\mathcal{L}$ and $\mathcal{M}$ groups in the diagram end up being isomorphic (though this is not obvious) it is a matter of taste whether one uses $\mathcal{M}_n(X, A)_A$ or $\mathcal{L}_n(X, A)_A$.

For general pairs (X, A) we can still get a natural transformation by one additional step, using a cofibrant approximation $(\tilde{X}, \tilde{A}) \to (X, A)$. This is useful, but in the process we lose control of some of the isomorphisms.

Once we know that the maps labelled ① are isomorphisms for all n, we immediately get the same for ② by passing to the colimit. So the steps we will follow are:

- Show that map ③ is an isomorphism.
- Show that the maps $\mathcal{M}_n(X, A)_A \to \mathcal{M}_{n+1}(X, A)_A$ are isomorphisms for $n \geq 1$. Then ① and ② are isomorphisms as well.
- Construct and analyze the natural transformation $\chi_1 \colon \mathcal{M}_1(X, A)_A \to K^0(X, A)$.

The isomorphisms $\mathcal{L}_n(X, A)_A \to \mathcal{M}_n(X, A)_A$ will fall out of this outline as a simple corollary.

Remark 24.9. A natural question is why we use $\mathcal{K}(X, A)$ or $\mathcal{M}_\infty(X, A)$ at all, when apparently the connection to $K^0(X, A)$ is easier with the A-relative groups $\mathcal{M}(X, A)_A$. So it is worth remarking up front that the tensor product of chain complexes gives $\mathcal{K}(X, A) \otimes \mathcal{K}(Y, B) \to \mathcal{K}(X \times Y, A \times Y \cup X \times B)$ (or the same with $\mathcal{M}_\infty$), whereas one does *not* get pairings like these with the A-relative constructions. If differentials are only defined on A and B then on the external tensor product one only obtains differentials on $A \times B$. So $\mathcal{K}$ and $\mathcal{M}_\infty$ stand out as the spots in the diagram with naturally-defined products. Relating those products to the ones on K^0 will be another challenge, undertaken in Section 24.40.

Remark 24.10. Although not in the above diagram, one can introduce a final set of groups $\mathcal{L}_n(X, A)$ consisting of chain complexes defined on all of X, exact on A, up to isomorphism and stability. However, these are not isomorphic to the above groups, as shown by the following example. Let $X = \mathbb{R}$ and $A = \emptyset$. The chain complexes $\underline{1} \xrightarrow{0} \underline{1}$ and $\underline{1} \xrightarrow{\mathrm{id}} \underline{1}$ are homotopic and so represent the same element in $\mathcal{K}(X, A)$, but they do not become isomorphic after taking direct sums with elementary complexes. In particular, the former complex has homology in the fiber over 0 whereas the latter does not, and these properties will not change upon addition of exact complexes. In constrast, note that when we regard these as objects in $\mathcal{C}\mathrm{h}(X, A)_A$ they are actually isomorphic, and therefore identified in $\mathcal{L}_\infty(X, A)_A$.

24.11. **Comparing complexes to A-relative complexes.** Our first important result is the following:

Proposition 24.12. *Assume that $A \hookrightarrow X$ is a cofibration. Then for any $1 \leq n \leq \infty$ the map $\mathcal{M}_n(X, A) \to \mathcal{M}_n(X, A)_A$ is a bijection.*

We start with a lemma. Recall that $E \mapsto E(A)$ is the functor that takes a chain complex E and sends it to the A-relative chain complex obtained by restricting the differential so that it is only defined on A.

Lemma 24.13. *When $A \hookrightarrow X$ is a cofibration the functor $\mathcal{C}\mathrm{h}(X, A) \to \mathcal{C}\mathrm{h}(X, A)_A$ is surjective on objects. Additionally, if E_1 and E_2 are objects in $\mathcal{C}\mathrm{h}(X, A)$ such that $E_1(A) \cong E_2(A)$ then E_1 and E_2 are homotopic.*

Proof. Given a $\mathbb{Z}$-indexed collection of vector spaces V_i, let Ch-struct$(V) \subseteq \prod_i \mathrm{Hom}(V_{i+1}, V_i)$ denote the collection of sequences $(d_i)_{i\in\mathbb{Z}}$ satisfying $d_i \circ d_{i+1} = 0$ for all i. Regard Ch-struct(V) as a topological space by giving it the subspace topology. Note that if $d \in$ Ch-struct(V) then $t \cdot d \in$ Ch-struct(V) for any $t \in \mathbb{C}$. Picking any path in $\mathbb{C}$ from 0 to 1 (e.g. the standard one along the real line) thereby gives a contracting homotopy showing that Ch-struct(V) is contractible.

Now suppose that $\{E_i\}_{i\in\mathbb{Z}}$ is a collection of vector bundles on X having the property that only finitely many are nonzero. For $x \in X$ write E_x for the collection $\{(E_i)_x\}_{i\in\mathbb{Z}}$. Let Ch-struct$(E)$ be the set of pairs $(x \in X, d \in$ Ch-struct$(E_x))$, topologized so that the map $\pi_1\colon$ Ch-struct$(E) \to X$ is a fiber bundle. The assumption that only finitely many E_i are nonzero implies that each x has a neighborhood over which all the bundles are trivial, and we use such neighborhoods in the usual way to define the topology on Ch-struct(E). As the fibers of Ch-struct$(E) \to X$ are contractible this map is a weak homotopy equivalence.

The fiber bundle Ch-struct$(E) \to X$ can also be regarded as a sub-fiber-bundle of the vector bundle $\bigoplus_i \underline{\mathrm{Hom}}(E_{i+1}, E_i)$. Making $\{E_i\}$ into a chain complex is precisely the same as giving a section of Ch-struct$(E) \to X$. Likewise, equipping $\{E_i\}$ with a chain complex differential *over* A is the same as giving a section defined over A.

Suppose given an object E in $\mathcal{C}\mathrm{h}(X, A)_A$. Consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & \text{Ch-struct}(E) \\ \downarrow & & \downarrow \simeq \\ X & \longrightarrow & X \end{array} \tag{24.14}$$

where the top horizontal map encodes the differentials on E. Since $A \to X$ is a cofibration and Ch-struct$(E) \to X$ is an acyclic fibration, there is a lifting $X \to$ Ch-struct(E). This lifting precisely gives a chain complex structure on $\{E_i\}$, defined on all of X, that extends the one defined over A. This proves that the functor $E \mapsto E(A)$ is surjective on objects.

Next observe that if $f\colon Y \to X$ is any map then Ch-struct(f^*E) is canonically identified with the pullback of $Y \to X \leftarrow$ Ch-struct(E). This is an easy exercise. Suppose that (E, s) and (E, s') are two preimages in $\mathcal{C}\mathrm{h}(X, A)$ for the same object E in $\mathcal{C}\mathrm{h}(X, A)_A$. Then s and s' correspond to two liftings in the square (24.14). Given this data, form the new diagram

$$\begin{array}{ccc} (X \times 0) \amalg_{(A\times I)} (X \times 1) & \longrightarrow & \text{Ch-struct}(E) \\ \downarrow & & \downarrow \simeq \\ X \times I & \xrightarrow{\ \pi\ } & X \end{array}$$

where $\pi\colon X \times I \to X$ is the projection, $A \times I \to$ Ch-struct(E) is the constant homotopy, and the top horizontal map equals s and s' on the two copies of X. The left map is a cofibration so once again the diagram has a lifting. The resulting map $X \times I \to$ Ch-struct(E) corresponds to a section of Ch-struct$(\pi^* E) \to X \times I$, and so specifies a complex of vector bundles on $X \times I$. The differentials are constant with respect to 'time' on $A \times I$, and so in particular the complex is exact on $A \times I$. So it lives in $\mathcal{C}\mathrm{h}(X \times I, A \times I)$. By construction it restricts to the two liftings (E, s) and (E, s') at times 0 and 1. So any two extensions of the same A-relative chain complex are homotopic.

Now suppose that E and F are objects in $\mathcal{C}\mathrm{h}(X, A)$ and that $E(A) \cong F(A)$. So there are isomorphisms $u_i\colon E_i \to F_i$ of bundles over X which, when restricted to A, commute with the differentials d. Let d'_i be the composite

$$E_i \xrightarrow{u_i} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{u_{i-1}^{-1}} E_{i-1}.$$

Note that the d' maps give a chain complex structure on $\{E_i\}$; call this new chain complex E'. We have $E' \cong F$ as objects in $\mathcal{C}\mathrm{h}(X, A)$. Observe that $d'|_A = d|_A$, and so $E(A) = E'(A)$. By what we proved in the last paragraph, E and E' are homotopic in $\mathcal{C}\mathrm{h}(X, A)$. Since E' and F are (trivially) homotopic (see Remark 24.2), transitivity gives that E is homotopic to F. □

Remark 24.15. The above proof was written in part to demonstrate the technique of translating a desired task into a lifting problem. This is a useful technique that we will need again later in this section. However, it is worth pointing out that in the case of relative cell complexes the lifting problems could be solved in a very concrete and simple way. If (Y, B) is a relative cell complex and $E_\bullet$ is an object in $\mathcal{C}\mathrm{h}(Y, B)_B$, we extend the differentials from B to all of Y by an induction over the cells of $Y - B$. If e^n is such a cell, we assume inductively that the differentials have been defined over the boundary. Let $f\colon D^n \to Y$ be the characteristic map of the cell, and consider the pullback $f^* E_\bullet$. Choose trivializations for all of these bundles over the disk. Points in the interior of the disk have the form tx for $x \in \partial D^n$ and $t \in [0, 1)$. Define the differential over tx to be t times the differential over x. This gives the required extension.

Proof of Proposition 24.12. We need to prove that $\mathcal{M}_n(X, A) \to \mathcal{M}_n(X, A)_A$ is bijective, and surjectivity is provided by Lemma 24.13. For injectivity, it will suffice to prove, given two objects E and F in $\mathcal{C}\mathrm{h}_n(X, A)$, that

(1) if $E(A) \sim_h F(A)$ then $E = F$ in $\mathcal{M}_n(X, A)$, and
(2) if $E(A) \sim_{st} F(A)$ then $E = F$ in $\mathcal{M}_n(X, A)$.

For suppose that $E(A)$ and $F(A)$ are identified in $\mathcal{M}_n(X, A)_A$. Then there is a finite chain of objects $\tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_r$ in $\mathcal{C}\mathrm{h}_n(X, A)_A$ such that $\tilde{E}_1 = E(A)$, $\tilde{E}_r = F(A)$, and for each i either $\tilde{E}_i \sim_h \tilde{E}_{i+1}$ or $\tilde{E}_i \sim_{st} \tilde{E}_{i+1}$. By Lemma 24.13 there are chain complexes $E_i \in \mathcal{C}\mathrm{h}_n(X, A)$ such that $E_i(A) = \tilde{E}_i$, for each i. We can choose $E_1 = E$ and $E_r = F$. By iterated applications of (1) and (2) we then know that $E_1, E_2, \ldots, E_r$ are all identified in $\mathcal{M}_n(X, A)$, and in particular E and F are identified. This is what we needed to prove.

We turn to the proofs of (1) and (2). Suppose that $E(A) \sim_h F(A)$. Then there exists an $\tilde{\mathcal{E}} \in \mathcal{C}\mathrm{h}(X \times I, A \times I)_{A \times I}$ together with isomorphisms $\tilde{\mathcal{E}}|_{X \times 0} \cong E(A)$ and $\tilde{\mathcal{E}}|_{X \times 1} \cong F(A)$. By Lemma 24.13 there is an $\mathcal{E} \in \mathcal{C}\mathrm{h}(X \times I, A \times I)$ such that

$\mathcal{E}(A) = \tilde{\mathcal{E}}$. Then $\mathcal{E}|_{X\times 0}(A) \cong E(A)$, and so by Lemma 24.13 $\mathcal{E}|_{X\times 0}$ is homotopic to E. The same reasoning gives that $\mathcal{E}|_{X\times 1}$ is homotopic to F. But $\mathcal{E}|_{X\times 0}$ and $\mathcal{E}|_{X\times 1}$ are homotopic, so by transitivity E is homotopic to F.

Finally, suppose that $E(A) \sim_{st} F(A)$. So there exist elementary complexes $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_s$ in $\mathcal{C}\mathrm{h}_n(X, A)_A$ such that $E(A) \oplus \bigoplus_i P_i \cong F(A) \oplus \bigoplus_j Q_j$. Note that each elementary complex actually lives in $\mathcal{C}\mathrm{h}_n(X, A)$, as its only nonzero differential is the identity and therefore is defined on all of X.

Let $P = \oplus_i P_i$ and $Q = \oplus_j Q_j$. Then $(E \oplus P)(A) \cong (F \oplus Q)(A)$, and so Lemma 24.13 tells us that $E \oplus P$ and $F \oplus Q$ are homotopic. Consequently, $E \oplus P$ is identified with $F \oplus Q$ in $\mathcal{M}_n(X, A)$. But the stability relation identifies E with $E \oplus P$ and F with $F \oplus Q$, hence E and F are themselves identified. □

24.16. **An unusual construction from homological algebra.** Our next aim is to prove that the maps $\mathcal{M}_n(X, A)_A \to \mathcal{M}_{n+1}(X, A)_A$ are bijections when $n \geq 1$. This will be based on a strange technique for folding the top group of an exact complex two terms lower down, to construct a new complex that also happens to be exact. We now describe this unusual construction.

Let V be an exact complex of vector spaces and assume that $V_i = 0$ for $i > n$. (The complex could actually consist of projectives over some ring, but let us stick with the simpler setting). Since the complex is exact there exists a contracting homotopy: maps $e\colon V_i \to V_{i+1}$ such that $de + ed = \mathrm{id}$. Let ΓV be the following chain complex, concentrated in degrees smaller than n and agreeing with V in degrees smaller than $n - 2$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V_{n-1} & \longrightarrow & V_{n-2} \oplus V_n & \longrightarrow & V_{n-3} & \longrightarrow & V_{n-4} & \longrightarrow & \cdots \\ & & x & \longrightarrow & (dx, ex) & & & & & & \\ & & & & (a, b) & \longrightarrow & da. & & & & \end{array}$$

It is an elementary exercise to prove that ΓV is exact, but this will also follow directly from the two decompositions we produce next.

Exercise 24.17. Check that the above construction is a chain complex and prove via naive means that it is exact. Additionally, prove that $V_{n-1} = d(V_n) \oplus ed(V_{n-1})$.

Remark 24.18. We know (e.g. Lemma 7.3) that an exact complex of vector spaces splits (non-canonically) as a direct sum of elementary complexes. The above construction is essentially taking the top piece of such a splitting and shifting it down one degree. The contracting homotopy e encodes the splitting.

For any vector space W and any $k \in \mathbb{Z}$, recall that $D_k(W)$ denotes the elementary chain complex consisting of W in degrees k and $k + 1$, where the differential is the identity. The next exercise develops some basic material about contracting homotopies and how they relate to splittings of a complex into elementary subcomplexes.

Exercise 24.19. Let V be a bounded chain complex of vector spaces and suppose e is a contracting homotopy for V.

(a) Prove that $V_i = e(\ker d_{i-1}) \oplus d(V_{i+1})$ and that $d_i\colon e(\ker d_{i-1}) \to d(V_i)$ is an isomorphism. So the contracting homotopy yields a decomposition of V into a direct sum of elementary complexes.

(b) Define a splitting for V to be a sequence of vector spaces L_i and an isomorphism $\phi\colon \bigoplus_i D_i(L_i) \to V$. Say that two splittings (L,ϕ) and (L',ϕ') are equivalent if there is a collection of isomorphisms $L_i \to L'_i$ making the expected triangle with ϕ and ϕ' commute. Prove that there is a bijection between the equivalence classes of splittings of V and the set of contracting homotopies of V.

(c) Check that the set of contracting homotopies for V has a natural affine structure: if e and f are contracting homotopies then so is $\lambda e + (1-\lambda)f$ for any λ in the ground field. If the ground field is $\mathbb{R}$ or $\mathbb{C}$ then we topologize the set of contracting homotopies as a subspace of $\prod_i \mathrm{Hom}(V_i, V_{i+1})$. Prove that the resulting space is contractible.

(d) Recall the internal Hom-complex $\mathrm{Hom}(V,V)$: in degree k this is $\prod_i \mathrm{Hom}(V_i, V_{i+k})$, and given a collection $\{\alpha_i\colon V_i \to V_{i+k}\}$ the differential is the collection of maps $\{d\circ\alpha_i - (-1)^k \alpha_{i-1}\circ d\}$. Check that a contracting homotopy for V is just an element $e \in \underline{\mathrm{Hom}}(V,V)_1$ satisfying $d(e) = \mathrm{id}$, and if e is a contracting homotopy then the set of all contracting homotopies is $e + d^{-1}(0)$. (This gives another perspective on the affine structure from part (d)).

(e) Suppose (L,ϕ) is a splitting for V. If e is any contracting homotopy for V, prove that the set of all contracting homotopies can be identified with the coset $e + \bigoplus_i \big[\mathrm{Hom}(L_i, L_{i+1}) \oplus \mathrm{Hom}(L_i, L_{i+2})\big]$.

Returning to our chain complex V with contracting homotopy e, note that for $x \in V_n$ one has $ed(x) = x$. Using this, we can write down a natural chain map $V \to D_{n-1}(V_n)$ that equals the identity in degree n and the map $e\colon V_{n-1} \to V_n$ in degree $n-1$. Let $\widetilde{\Gamma}V$ be the desuspension of the mapping cone of this map (or equivalently, the homotopy fiber); specifically, $\widetilde{\Gamma}V$ is the following chain complex, concentrated in dimensions at most n and agreeing with V in dimensions smaller than $n-2$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V_n & \longrightarrow & V_{n-1}\oplus V_n & \longrightarrow & V_{n-2}\oplus V_n & \longrightarrow & V_{n-3} & \longrightarrow & V_{n-4} \longrightarrow \cdots \\ & & x & \longrightarrow & (dx, -x) & & (x,y) & \longrightarrow & dx & & \\ & & & & & & (a,b) & \longrightarrow & (da, ea+b) & & \end{array}$$

There is a canonical inclusion $D_{n-2}(V_n) \hookrightarrow \widetilde{\Gamma}V$, and the quotient is V. Moreover, this inclusion has a canonical splitting χ defined by $\chi(a,b) = ea + b$ for $(a,b) \in V_{n-1}\oplus V_n = (\widetilde{\Gamma}V)_{n-1}$ and $\chi(a,b) = b$ for $(a,b) \in V_{n-2}\oplus V_n = (\widetilde{\Gamma}V)_{n-2}$. This splitting gives an isomorphism $\widetilde{\Gamma}V \cong D_{n-2}(V_n)\oplus V$.

Notice as well that there is an evident map $D_{n-1}(V_n) \hookrightarrow \widetilde{\Gamma}V$: in degree n this equals the identity and in degree $n-1$ it equals the differential of $\widetilde{\Gamma}V$. The cokernel of this inclusion is precisely ΓV. Moreover, there is again a canonical splitting for the inclusion: in degree n it is equal to the identity, and in degree $n-1$ it is the negation of the projection map $V_{n-1}\oplus V_n \to V_n$. This splitting gives $\widetilde{\Gamma}V \cong D_{n-1}(V_n)\oplus \Gamma V$.

Summarizing, we have produced two split-exact sequences

$$0 \to D_{n-2}(V_n) \to \widetilde{\Gamma}V \to V \to 0 \quad \text{and} \quad 0 \to D_{n-1}(V_n) \to \widetilde{\Gamma}V \to \Gamma V \to 0$$

and these induce isomorphisms

$$\widetilde{\Gamma}V \cong D_{n-2}(V_n)\oplus V \quad \text{and} \quad \widetilde{\Gamma}V \cong D_{n-1}(V_n)\oplus \Gamma V. \tag{24.20}$$

An important point is that the maps in these short exact sequences, their splittings, and therefore the induced isomorphisms in (24.20) are all canonical in the pair (V, e).

By the way, notice that it follows immediately from the isomorphisms in (24.20) that the homology groups of ΓV and V coincide; therefore ΓV is exact.

The constructions from above depended on the choice of contracting homotopy e. But recall from Exercise 24.19 that we understand very well the space of all contracting homotopies, and that it is contractible. This will be important for our applications in the next section.

Remark 24.21. The Γ- and $\widetilde{\Gamma}$-constructions used in this section seem to have first appeared in [Dold]. Notice that very little about the contracting homotopy e was ever used—in fact, all we really needed was the component of e in the top dimension, the map $e\colon V_{n-1} \to V_n$. And all that was important about this map was that it was a splitting for the differential $d_n\colon V_n \to V_{n-1}$. Rather than use the space of contracting homotopies as a parameter space, we could have used the (simpler) space of splittings for d_n. The reader can check that this is again an affine space, homeomorphic to the vector space of all maps $f\colon V_{n-1} \to V_n$ such that $fd = 0$; in particular, this parameter space is again contractible.

We have used the space of contracting homotopies because this approach is important in generalizations to algebraic K-theory. See [FH] and [D4] for more information.

24.22. **Back to topology.** Let $E_\bullet$ be a bounded chain complex of vector bundles on a paracompact Hausdorff space Z, and assume that $E_\bullet$ is exact. By Proposition 20.8 $E_\bullet$ has a splitting as a direct sum of elementary complexes, and so just as in Exercise 24.19 there is a contracting homotopy for $E_\bullet$. As in that exercise, the space of contracting homotopies has an affine structure and is therefore contractible. In fact, if one chooses an isomorphism $E_\bullet \cong \bigoplus_i D_i(L_i)$ then the space of contracting homotopies is homeomorphic to the vector space $\bigoplus_i[\mathrm{Hom}(L_i, L_{i+1}) \oplus \mathrm{Hom}(L_i, L_{i+2})]$, using Exercise 24.19(e).

If e is a chosen contracting homotopy for $E_\bullet$ then we can form the associated chain complex ΓE by repeating the construction from Section 24.16 but in the bundle setting. This is a new chain complex of vector bundles that is still exact on Z. This construction of course depends on the choice of contracting homotopy e, and so we should probably write $\Gamma_e E$. But since any two choices for e are homotopic, ΓE is well-defined up to homotopy.

We will use the above construction to prove the following:

Proposition 24.23. *Assume that A is cofibrant and $A \hookrightarrow X$ is a cofibration. Then for any $n \geq 2$ the map $j\colon \mathcal{M}_{n-1}(X, A)_A \to \mathcal{M}_n(X, A)_A$ is a bijection.*

Proof. Let $E_\bullet$ be a chain complex in $\mathcal{C}\mathrm{h}_n(X, A)_A$, where $n \geq 2$. By the preceding considerations, since A is cofibrant there exists a contracting homotopy for $(E_\bullet)|_A$. Using such a contraction e we can form $\Gamma_e E$, which is an object in $\mathcal{M}_{n-1}(X, A)_A$. Different choices for e are homotopic and therefore give rise to homotopic complexes, so we get a well-defined function

$$\Gamma\colon\ \mathcal{C}\mathrm{h}_n(X, A)_A \to \mathcal{M}_{n-1}(X, A)_A.$$

It is an elementary exercise to check that $\Gamma(E \oplus J) = \Gamma(E) + \Gamma(J)$, and it follows from this that if $E \sim_{st} E'$ then $\Gamma E \sim_{st} \Gamma E'$. Likewise, if $E \sim_h E'$ then pick an $\mathcal{E} \in \mathcal{C}\mathrm{h}_n(X \times I, A \times I)_{A\times I}$ with $\mathcal{E}|_{X\times 0} \cong E$ and $\mathcal{E}|_{X\times 1} \cong E'$. Choose a contracting

homotopy e for $\mathcal{E}|_{A\times I}$ and note that $[(\Gamma_e\mathcal{E})|_{X\times 0}] = [\Gamma E]$ and $[(\Gamma_e\mathcal{E})|_{X\times 1}] = [\Gamma E']$. So $[\Gamma E] = [\Gamma E']$. Putting everything together, we have shown that Γ gives a map

$$\Gamma\colon \mathcal{M}_n(X,A)_A \to \mathcal{M}_{n-1}(X,A)_A.$$

It is trivial that $\Gamma j = \mathrm{id}$. We claim that $j\Gamma = \mathrm{id}$ as well, thus establishing that j is a bijection. This claim almost follows directly from the canonical isomorphisms (24.20), except there is an important step that must be filled in. We would like to say that these isomorphisms globalize to give

$$\widetilde{\Gamma} E \cong D_{n-2}(E_n) \oplus E \quad \text{and} \quad \widetilde{\Gamma} E \cong D_{n-1}(E_n) \oplus \Gamma E. \tag{24.24}$$

This is certainly true if we restrict all the chain complexes to the subspace A. However, to give an isomorphism in $\mathcal{C}\mathrm{h}(X,A)_A$ we actually need to give a collection of isomorphisms for bundles **over X** (they are only required to commute with the differentials over A, however). So we must verify that in each degree the isomorphisms from (24.20) globalize not just to A but to X. Those isomorphisms were obtained from split short exact sequences, so we must check that all the maps involved can be extended over X. But we have formulas for all of these maps, and most of them are inclusions of a summand or projections onto a summand—these obviously extend to all of X. The one exception is in one of the splittings, where we used the map $\chi\colon E_{n-1}\oplus E_n \to E_n$ given on fibers by $(a,b)\mapsto ea+b$ where e was part of the given contracting homotopy. Since e is only defined on A, this does not automatically make sense on all of X. However, the particular map e we are using in this formula is a section over A of the bundle $\underline{\mathrm{Hom}}(E_{n-1},E_n)\to X$. The diagram

$$\begin{array}{ccc} A & \longrightarrow & \underline{\mathrm{Hom}}(E_{n-1},E_n) \\ \downarrow & & \downarrow\simeq \\ X & \xrightarrow{\ \mathrm{id}\ } & X \end{array}$$

must have a lifting, and this gives an extension of e to a bundle map $\tilde{e}\colon E_{n-1}\to E_n$ defined on all of X. The formula $(a,b)\mapsto \tilde{e}a+b$ then gives the desired splitting that works on all of X. Note: it is important here that the splitting is only required to commute with the differentials on A, since this is all we have guaranteed.

To summarize, we have indeed justified the isomorphisms in (24.24). These imply that ΓE and E represent the same class in $\mathcal{M}_n(X,A)_A$. In other words, we have proven that $j\circ\Gamma = \mathrm{id}$, and so Γ is a two-sided inverse for j. □

Remark 24.25 (Atiyah's proof)**.** Atiyah proves a version of Proposition 24.23 in [Atiyah5, Chapter 2.6]; the argument orginally comes from [ABS]. We will explain the basic ways his proof differs from ours, and why these differences are important.

Let $E_\bullet$ be a chain complex in $\mathcal{C}\mathrm{h}_n(X,A)_A$. There is a canonical map $D_{n-1}(E_n)\to E_\bullet$ which in degree n equals the identity and in degree $n-1$ equals the differential $E_n\to E_{n-1}$. Let $E'_\bullet = E_\bullet \oplus D_{n-2}(E_n)$ and consider the composition

$$D_{n-1}(E_n)\to E_\bullet \hookrightarrow E'_\bullet.$$

The map in degree $n-1$ is $d\oplus 0\colon E_n\to E_{n-1}\oplus E_n$, which is defined only on A. Atiyah shows via a lifting argument that this can be extended to a monomorphism of bundles on X. Let Q be the quotient, and observe that by Proposition 12.2 the sequence $0\to E_n\to E'_n\to Q\to 0$ is a split-exact sequence of bundles over X. A

choice of splitting $\chi\colon E'_n \to E_n$ then shows that $E'_\bullet$ is the direct sum of $D_{n-1}(E_n)$ and a complex

$$0 \to Q \to E_{n-2} \oplus E_n \to E_{n-3} \to E_{n-4} \to \cdots \tag{24.26}$$

This last complex lies in $\mathcal{C}\mathrm{h}_{n-1}(X,A)_A$, and it represents the same class as E in $\mathcal{M}_n(X,A)_A$. This shows that $j\colon \mathcal{M}_{n-1}(X,A)_A \to \mathcal{M}_n(X,A)_A$ is surjective.

This argument does not give an inverse for j, however. The complex in (24.26) depends on a choice (the extension of a certain map to all of X), and so it is not clear how to use this construction to make an inverse for j.

In our argument we gave a construction ΓE that **did not depend** on choosing any such extensions to X. Such extensions did appear, but only in the isomorphisms showing that our ΓE had the correct properties. By pushing these choices into the maps rather than the objects, we were able to write down an explicit inverse for j.

In Atiyah's case he found a clever way around his problem, by instead constructing a map $\mathcal{M}_n(X,A)_A \to \mathcal{M}_1(X,A)_A$ that is an inverse to the appropriate composition of j's. This is enough to deduce injectivity of all the j maps. Atiyah's construction proceeds by choosing Hermitian inner products on all of the bundles E_i, and then letting $\alpha\colon E_i \to E_{i+1}$ be the adjoint of the differential. The map $d+\alpha\colon \bigoplus_{i \text{ odd}} E_i \to \bigoplus_{i \text{ even}} E_i$ is seen to lie in $\mathcal{C}\mathrm{h}_1(X,A)_A$, and it clearly has the desired properties. This element seems to again depend on choices, namely the choice of inner products; but the space of all such choices is contractible (Exercise 11.29), and so one indeed gets a well-defined element of $\mathcal{M}_1(X,A)_A$. Exercise 24.27 below will give you enough information to fill in the details of this approach.

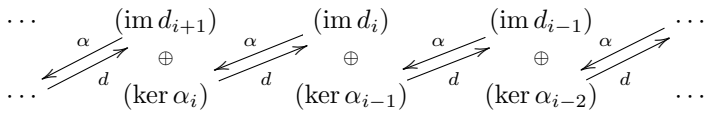
The disappointing aspect of Atiyah's argument is that it does not work in the related context of algebraic K-theory. In that setting one cannot play a corresponding game with inner products. In contrast, our argument with contracting homotopies does generalize. See [FH] and [D4] for more information.

Exercise 24.27. Assume V is an exact chain complex of finite-dimensional real vector spaces, and that $V_i = 0$ for $i < 0$ and for $i > n$. Choose an inner product on each V_i, and let $\alpha_i\colon V_i \to V_{i+1}$ be the adjoint of $d_{i+1}\colon V_{i+1} \to V_i$. That is, for each $x \in V_i$ and $y \in V_{i+1}$ one has $\langle \alpha x, y\rangle = \langle x, dy\rangle$.

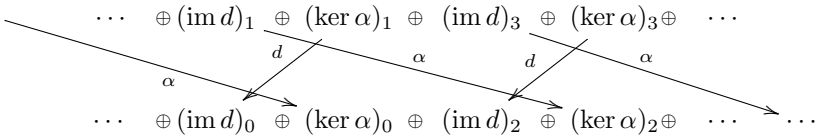
(a) Prove that $\alpha^2 = 0$, and so (V,α) is a cochain complex. Observe that this is isomorphic to the dual complex V^*, and therefore is exact.

(b) For each i prove that $\ker\alpha_i$ is the orthogonal complement of $\operatorname{im} d_{i+1}$ inside of V_i. As a corollary, deduce that d restricts to an isomorphism $\ker\alpha_i \to \operatorname{im} d_i$ and α restricts to an isomorphism $\operatorname{im} d_i \to \ker\alpha_i$. Produce an example when $n = 1$ showing that these isomorphisms need not be inverses.

(c) Observe that $V_i = (\operatorname{im} d_{i+1}) \oplus (\ker\alpha_i)$ for each i, and confirm that the following picture shows V decomposing into a direct sum of acyclic complexes of length one:



(d) Let $V_{odd} = \oplus_{i \text{ odd}} V_i$ and $V_{ev} = \oplus_{i \text{ even}} V_i$. Observe that $d + \alpha \colon V_{odd} \to V_{ev}$ is an isomorphism, by the following diagram:



Note: Here we have written $(\operatorname{im} d)_i$ for the component of $\operatorname{im} d$ contained in degree i; i.e., $(\operatorname{im} d)_i = \operatorname{im} d_{i+1}$, but $(\ker \alpha)_i = \ker \alpha_i$.

(e) Prove an analog of these results for exact complexes of $\mathbb{C}$-vector spaces, in which one chooses Hermitian inner products on all of the V_i's.

24.28. **Another interlude on where we are headed.** To prove Theorem 20.14 we are required to produce a natural map of groups $\mathcal{K}(X, A) \to K^0(X, A)$ such that when $A = \emptyset$ the map sends $E_\bullet$ to $\sum_i (-1)^i [E_i]$. At this point we have established the chain of isomorphisms of monoids

$$\mathcal{M}_1(X, A)_A \xrightarrow{\cong} \mathcal{M}_n(X, A)_A \xrightarrow{\cong} \mathcal{M}_\infty(X, A)_A \xleftarrow{\cong} \mathcal{M}_\infty(X, A) \xrightarrow{\cong} \mathcal{K}(X, A)$$

for cofibrant pairs (X, A) (recall that this hypothesis ensures, in particular, that X and A are paracompact and Hausdorff). So for this situation our problem is equivalent to producing a natural map from any of these gadgets into $K^0(X, A)$ with the desired properties. We will call any such natural map an **Euler characteristic**.

In the next part of the argument we will construct an Euler characteristic on $\mathcal{M}_1(X, A)_A$ and prove that in good cases it is an isomorphism. Later we will see that for pairs (X, A) where X and A are homotopically compact it is also unique.

Let us make one simple but useful observation that falls out of what we have done already. *A priori* the sets $\mathcal{M}_n(X, A)_A$ are only monoids, but the above chain of isomorphisms shows they are actually groups.

Corollary 24.29. *Let (X, A) be a cofibrant pair. Then for all $1 \leq n \leq \infty$ the commutative monoid $\mathcal{M}_n(X, A)_A$ is a group.*

24.30. **The difference bundle construction.** Suppose given a pair (X, A). Our goal in this section is to construct a map

$$\chi_1 \colon \mathcal{M}_1(X, A)_A \to K^0(X, A),$$

natural in (X, A), and prove that it is an isomorphism when X and A are both cofibrant and homotopically compact.

Write $j \colon A \hookrightarrow X$ for the inclusion. Write $\operatorname{Cyl}(j)$ for the mapping cylinder, A_T for the copy of A at the top of the mapping cylinder ('T' is for top), and $\operatorname{Cone}(j) = \operatorname{Cyl}(j)/A_T$ for the mapping cone. Let $D(X, A, X)$ denote the *double mapping cylinder*, namely

$$D(X, A, X) = \operatorname{Cyl}(j) \amalg_{A_T} \operatorname{Cyl}(j).$$

It will be convenient to denote the left copy of $\operatorname{Cyl}(j)$ inside $D(X, A, X)$ as Cyl_0 and the right copy as Cyl_1. Each of these mapping cylinders contains a copy of X, which will likewise be denoted X_0 and X_1. Let $j_i \colon X_i \hookrightarrow D(X, A, X)$ be the two inclusions, and let $\pi \colon D(X, A, X) \to X$ be the evident projection.

In the long exact sequence for the pair $(D(X, A, X), X_0)$ the map j_0^* is split by π^*, hence the sequence breaks up into short exact sequences. At the K^0 level this

is

$$0 \longrightarrow K^0(D(X,A,X),X_0) \longrightarrow K^0(D(X,A,X)) \xrightarrow{j_0^*} K^0(X_0) \longrightarrow 0. \tag{24.31}$$

In addition, we have the sequence of isomorphisms

$$\begin{array}{ccccc} K^0(D(X,A,X),X_0) & \xleftarrow[\cong]{j_0^*} & K^0(D(X,A,X),\mathrm{Cyl}_0) & \xrightarrow[\cong]{j_1^*} & K^0(\mathrm{Cyl}_1, A_T) \\ & & & & \cong\uparrow{\scriptstyle \pi^*} \\ & & & & K^0(X,A). \end{array} \tag{24.32}$$

The first and last of these are isomorphisms by homotopy invariance, whereas for j_1^* this is by excision and homotopy invariance (we can fatten up the two cylinders into open sets if we like, intersecting in $A \times (0,1)$). Note that all of our diagrams are natural in the pair (X,A).

Given an element in $\mathcal{C}\mathrm{h}_1(X,A)_A$ our strategy will be to produce an element in $K^0(D(X,A,X),X_0)$ via a geometric construction and then push it into $K^0(X,A)$ via the above string of isomorphisms.

Suppose $E_1 \xrightarrow{d} E_0$ is an object in $\mathcal{C}\mathrm{h}_1(X,A)_A$, and recall that the map d is only defined over A. Note that d is an isomorphism $(E_1)|_A \to (E_0)|_A$. Construct a bundle $\mathcal{B}(E_0,E_1,d)$ on $D(X,A,X)$ by taking $\pi_0^*(E_0)$ on Cyl_0 and $\pi_1^*(E_1)$ on Cyl_1 and gluing them along the isomorphism

$$\pi_1^*(E_1)|_{A_T} = E_1|_A \xrightarrow{d} E_0|_A = \pi_0^*(E_0)|_{A_T}.$$

Here we are using Corollary 11.7 for the gluing construction, which works because the inclusion $A_T \hookrightarrow D(X,A,X)$ has the required properties. The bundle $\mathcal{B}(E_0,E_1,d)$ is called the **difference bundle** corresponding to $E_1 \xrightarrow{d} E_0$.

Consider the element $\alpha(E_\bullet) = [\mathcal{B}(E_0,E_0,\mathrm{id})] - [\mathcal{B}(E_0,E_1,d)] \in K^0(D(X,A,X))$ (we are implicitly using the Φ functor from Section 17.14). We can also write this as $\alpha(E_\bullet) = [\pi^*E_0] - [\mathcal{B}(E_0,E_1,d)]$. Restricting to X_0 sends this class to $[E_0]-[E_0]=0$, therefore by (24.31) $\alpha(E_\bullet)$ lifts to a unique class $\tilde{\alpha}(E_\bullet) \in K^0(D(X,A,X),X_0)$. Note that if we restrict $\alpha(E_\bullet)$ to X_1 (or equivalently, $\tilde{\alpha}(E_\bullet)$ to the pair $(X_1,\emptyset)$) then we get $[E_0]-[E_1]$.

It follows formally that if $E_\bullet \sim_h E'_\bullet$ in $\mathcal{C}\mathrm{h}_1(X,A)_A$ then the corresponding α classes are equal. We next check that if $E_\bullet \sim_{st} E'_\bullet$ then $\alpha(E_\bullet) = \alpha(E'_\bullet)$. To this end, suppose that J is a vector bundle on X. Then $\mathcal{B}(E_1 \oplus J, E_0 \oplus J, d \oplus \mathrm{id}) = \mathcal{B}(E_1,E_0,d) \oplus \pi^*J$, and likewise $\mathcal{B}(E_0 \oplus J, E_0 \oplus J, \mathrm{id} \oplus \mathrm{id}) = \mathcal{B}(E_0,E_0,\mathrm{id}) \oplus \pi^*J$. Therefore

$$\alpha(E_\bullet \oplus D_0(J)) = [\mathcal{B}(E_0,E_0,\mathrm{id})] + [\pi^*J] - \big([\mathcal{B}(E_1,E_0,d)] + [\pi^*J]\big) = \alpha(E_\bullet).$$

We have therefore proven that α gives a well-defined map $\mathcal{M}_1(X,A)_A \to K^0(D(X,A,X))$, and it is clearly a group homomorphism. The composite $j_0^* \circ \alpha$ is zero, and so α factors through $K^0(D(X,A,X),X_0)$. Define $\chi_1 \colon \mathcal{M}_1(X,A)_A \to K^0(X,A)$ to be the composition of this map with the sequence of isomorphisms from (24.32). Note that some authors also call $\chi_1(E_\bullet)$ the "difference bundle" associated to $E_\bullet$.

Remark 24.33. [Atiyah5, Lemma 2.6.7] gives this construction for pairs (X,A) where both X and A are compact Hausdorff, and uses $X \amalg_A X$ instead of our

$D(X, A, X)$. When $A \hookrightarrow X$ is a cofibration these two spaces are homotopy equivalent. In Atiyah's situation the bundle on $X \amalg_A X$ is constructed using the closed-cover case of Corollary 11.7 corresponding to hypothesis (i) of Proposition 11.5. He then uses an isomorphism $K^0(X \amalg_A X, X) \cong K^0(X, A)$ which seems to implicitly require a condition like $A \hookrightarrow X$ being a cofibration.

Our construction of χ_1 is clearly natural in pairs (X, A).

Proposition 24.34. *The map $\chi_1 \colon \mathcal{M}_1(X, A)_A \to K^0(X, A)$ is an isomorphism whenever X and A are cofibrant and also homotopically compact (e.g. (X, A) is a finite CW-pair).*

Proof. The proof proceeds in three steps.

Step 1: $A = \emptyset$. This step is trivial, since $\mathcal{M}_1(X, \emptyset)_\emptyset$ is readily identified with the Grothendieck group of vector bundles on X (see Exercise 24.3). Since X is both cofibrant and homotopically compact, it is homotopy equivalent to a finite CW-complex. So the natural map $K^0_{Grt}(X) \to K^0(X)$ is an isomorphism (Proposition 17.17).

Step 2: $A = *$. Here we consider the diagram of groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_1(X, *)_* & \xrightarrow{\alpha} & \mathcal{M}_1(X, \emptyset)_\emptyset & \xrightarrow{\beta} & \mathcal{M}_1(*, \emptyset)_\emptyset & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_1} & & \\ 0 & \longrightarrow & K^0(X, *) & \longrightarrow & K^0(X) & \longrightarrow & K^0(*) & \longrightarrow & 0. \end{array}$$

The bottom row is exact, and the middle and right vertical maps are isomorphisms by Step 1. It will suffice to show that the top row is exact, since then the left vertical map is also an isomorphism.

For the remainder of this step let $x \in X$ denote the basepoint.

Surjectivity of β is trivial. For exactness at the middle spot, note that $\mathcal{M}_1(\{x\}, \emptyset)_\emptyset \cong \mathbb{Z}$ via $(V_1, V_0) \to \operatorname{rank} V_0 - \operatorname{rank} V_1$. So if $E_\bullet$ is in the kernel of β then $\operatorname{rank}_x E_1 = \operatorname{rank}_x E_0$, and therefore there certainly exists an isomorphism $(E_1)_x \cong (E_0)_x$. Such an isomorphism determines an element of $\mathcal{M}_1(X, x)_{\{x\}}$ that is a preimage for $E_\bullet$.

Finally, let $E_\bullet \in \mathcal{C}\mathrm{h}_1(X, x)_{\{x\}}$ and assume that $E_\bullet$ is in the kernel of α. Using the isomorphism $\mathcal{M}_1(X, \emptyset)_\emptyset \cong K^0(X)$, this means that $[E_1] = [E_0]$ in $K^0(X)$. Since X is homotopically compact and cofibrant, this implies there exists a vector bundle Q such that $E_1 \oplus Q \cong E_0 \oplus Q$ (Proposition 17.17). Adding $\mathrm{id} \colon Q \to Q$ to $E_\bullet$ and using this isomorphism, we may assume that $E_\bullet$ satisfies $E_1 = E_0$. Our isomorphism of fibers over x is then an element $\sigma \in \mathrm{Aut}((E_0)_x) \cong GL_n(\mathbb{C})$ for $n = \operatorname{rank}(E_0)$. As $GL_n(\mathbb{C})$ is connected, choose a path from the identity element to σ in $\mathrm{Aut}((E_0)_x)$. If $\pi \colon X \times I \to X$ is the projection, this path defines an isomorphism $(\pi^* E_0)|_{\{x\} \times I} \to (\pi^* E_0)|_{\{x\} \times I}$ and therefore an object in $\mathcal{C}\mathrm{h}_1(X \times I, \{x\} \times I)_{\{x\} \times I}$. Restricting this element to $X \times \{0\}$ and $X \times \{1\}$ shows that $E_\bullet$ equals $E_0 \xrightarrow{\mathrm{id}} E_0$ in $\mathcal{M}_1(X, x)_{\{x\}}$. But the latter element is zero, by the stability relation.

[NOTE: When proving the result for $\mathbb{R}$-vector bundles an extra argument is needed here, because $GL_n(\mathbb{R})$ is not path-connected. If σ is in the path component of the identity, the argument is the same as above. If not, add the elementary complex

$D_0(\mathbb{R}^2)$ onto $E_\bullet$ and then use the isomorphism of vertical complexes

$$\begin{array}{ccc} E_0 \oplus \underline{2} & \xrightarrow[\cong]{\mathrm{id}\oplus t} & E_0 \oplus \underline{2} \\ \downarrow{\scriptstyle \sigma\oplus\mathrm{id}} & & \downarrow{\scriptstyle \sigma\oplus t} \\ E_0 \oplus \underline{2} & \xrightarrow[\cong]{\mathrm{id}} & E_0 \oplus \underline{2} \end{array}$$

where $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the map that interchanges the standard basis vectors. The isomorphism $\sigma \oplus t$ is now in the path component of the identity and so we are back in the previous case; therefore, our complex represents zero in $\mathcal{M}_1(X,x)_{\{x\}}$.]

Step 3: General case. Let $j\colon A \hookrightarrow X$ be the inclusion and recall the projections $\pi\colon \mathrm{Cyl}(j) \to X$ and $p\colon \mathrm{Cyl}(j) \to \mathrm{Cone}(j)$. Consider the diagram

$$\begin{array}{ccccc} \mathcal{M}_1(X,A)_A & \xrightarrow{\pi^*} & \mathcal{M}_1(\mathrm{Cyl}(j),A_T)_{A_T} & \xleftarrow{p^*} & \mathcal{M}_1(\mathrm{Cone}(j),*)_{\{*\}} \\ \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \cong}\ {\scriptstyle \chi_1} \\ K^0(X,A) & \xrightarrow[\cong]{\pi^*} & K^0(\mathrm{Cyl}(j),A_T) & \xleftarrow[\cong]{p^*} & K^0(\mathrm{Cone}(j),*). \end{array} \tag{24.35}$$

The right vertical map is an isomorphism by Step 2, since $\mathrm{Cone}(j)$ is cofibrant and homotopically compact because X and A were. The lower p^* and π^* are isomorphisms because K is a cohomology theory (using excision and homotopy invariance, respectively).

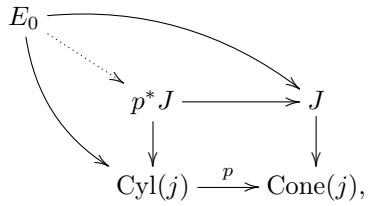
Since $p^* \circ \chi_1$ is an isomorphism, the upper p^* is injective. We next show it is surjective. Let $E_\bullet$ be an object in $\mathcal{C}\mathrm{h}_1(\mathrm{Cyl}(j),A_T)_{A_T}$. The space $\mathrm{Cyl}(j)$ is cofibrant, since both A and X are. It is also homotopy equivalent to X, which is homotopically compact. So by Proposition 17.17 there exists a bundle Q on $\mathrm{Cyl}(j)$ such that $E_1 \oplus Q$ is trivial. We can add the elementary complex $D_0(Q)$ to $E_\bullet$ without changing its class in $\mathcal{M}_1$, and in this way we can assume that E_1 is trivial, i.e. $E_1 = \underline{n}$ for some n. The differential $d\colon (E_1)|_{A_T} \to (E_0)|_{A_T}$ is therefore a trivialization $A_T \times \mathbb{C}^n \xrightarrow{\cong} (E_0)|_{A_T}$. Define J to be the pushout in $\mathcal{T}op$ of

$$\mathbb{C}^n \longleftarrow (E_0)|_{A_T} \hookrightarrow E_0$$

where the left map is the composite $(E_0)|_{A_T} \xrightarrow{d^{-1}} A_T \times \mathbb{C}^n \xrightarrow{\pi_2} \mathbb{C}^n$. In words, J is constructed by gluing all of the fibers of E_0 above A_T together into one, using the identification provided by the trivialization. Clearly we have the map $J \to \mathrm{Cone}(j)$, and this is a vector bundle by Exercise 10.25: here it is important that d can be extended to a trivialization of $(E_0)|_{A_T \times [\frac{1}{2},1]}$, which in turn follows from the usual bijection $\mathrm{Vect}_n(A_T) \cong \mathrm{Vect}_n(A_T \times I)$ of Corollary 14.2(b) since $A = A_T$ is cofibrant and hence paracompact Hausdorff.

Write x for the cone point in $\mathrm{Cone}(j)$. Consider the element of $\mathcal{M}_1(\mathrm{Cone}(j),x)_{\{x\}}$ given by $\underline{n} \xrightarrow{d'} J$, where d' is the isomorphism $\mathbb{C}^n \to J_x$ that is built into the definition of J. We claim that $[E_\bullet] = p^*([\underline{n} \to J])$. The universal property of

pullbacks gives us a dotted arrow in the diagram



where all of the other maps are the evident ones. This arrow is a bundle map and a fiberwise isomorphism, hence an isomorphism of bundles. Now one readily checks that $\underline{n} \xrightarrow{d} E_0$ is isomorphic to p^* applied to $\underline{n} \xrightarrow{d'} J$, hence our complex is in the image of p^* and we have proven surjectivity. So the upper p^* in (24.35) is an isomorphism, and therefore the same is true of the middle χ_1.

Now we analyze the upper π^* map in (24.35). Since $\mathrm{Cyl}(j) \to X$ is part of a homotopy equivalence (and all the spaces are paracompact Hausdorff, being cofibrant), every bundle on $\mathrm{Cyl}(X)$ is isomorphic to one that is pulled back from X. So any element in $\mathcal{M}_1(\mathrm{Cyl}(j), A_T)_{A_T}$ is represented by a complex of the form $\pi^*E_1 \xrightarrow{d} \pi^*E_0$ for some bundles E_1 and E_0 on X. Since $(\pi^*E_i)|_{A_T}$ can be canonically identified with $(E_i)|_{A_T}$, the differential d can be regarded as a map $(E_1)|_A \to (E_0)_A$. Thus we have described an element of $\mathcal{M}_1(X, A)_A$, and it is clear that applying π^* yields our original class. So π^* is surjective.

To prove injectivity of π^* we will construct an explicit left inverse. Suppose given an object $d\colon E_1 \to E_0$ in $\mathcal{C}\mathrm{h}_1(\mathrm{Cyl}(j), A_T)$. Restricting to the cylinder $A \times I \hookrightarrow \mathrm{Cyl}(j)$ gives two bundles on $A \times I$ and an isomorphism between them on $A \times \{0\}$ (we let time 0 denote the top of the cylinder, and time 1 the bottom). By Proposition 14.8 this isomorphism can be extended to an isomorphism D on $A \times I$, and then restricted to time 1 to give $D_1\colon (E_1)|_{A\times 1} \to (E_0)|_{A\times 1}$. The bundle map $(E_1)|_X \xrightarrow{D_1} (E_0)|_X$ represents an element $\alpha(E_\bullet) \in \mathcal{M}_1(X, A)_A$. Proposition 14.8 also says that a different extension D' gives an isomorphism D'_1 that is homotopic to D_1, which implies that there is an element in $\mathcal{C}\mathrm{h}_1(X \times I, A \times I)_{A\times I}$ that restricts to $(E_1)|_X \xrightarrow{D_1} (E_0)|_X$ and $(E_1)|_X \xrightarrow{D'_1} (E_0)|_X$ at the two ends (this takes a little thought but is not hard). Therefore $\alpha(E_\bullet)$ does not depend on the choice of D.

Adding an elementary complex to $E_\bullet$ clearly results in the addition of an elementary complex to $\alpha(E_\bullet)$. Finally, suppose given an element $\mathcal{E} \in \mathcal{C}\mathrm{h}_1(\mathrm{Cyl}(j) \times I, A_T \times I)_{A_T\times I}$ and set $E = \mathcal{E}|_{\mathrm{Cyl}(j)\times 0}$ and $E' = \mathcal{E}_{\mathrm{Cyl}(j)\times 1}$. As above we can extend the isomorphism $E|_{A_T\times I} \to E'|_{A_T\times I}$ to $E|_{\mathrm{Cyl}(A)\times I} \to E'|_{\mathrm{Cyl}(A)\times I}$, and then restricting to time 1 defines an object in $\mathcal{C}\mathrm{h}_1(X \times I, A \times I)_{A\times I}$ The restriction of this object to $X \times 0$ represents $\alpha(E_\bullet)$ and the restriction to $X \times 1$ represents $\alpha(E'_\bullet)$, and so $\alpha(E_\bullet) = \alpha(E'_\bullet)$ in $\mathcal{M}_1(X, A)_A$. This shows that α is a well-defined map $\mathcal{M}_1(\mathrm{Cyl}(j), A_T)_{A_T} \to \mathcal{M}_1(X, A)_A$. It is clear that $\alpha \circ \pi^* = \mathrm{id}$, so this completes the proof that π^* is a bijection. It follows that the left vertical χ_1 is a bijection, which is what we needed. □

24.36. **Wrapping things up.** We can now bring together all of the work in this section to prove the main results. The hard work has already been done, and now it is just a matter of fitting together some small details. Let us first start with a simple lemma:

Lemma 24.37. *Suppose that $\eta\colon K^0(X,A) \to K^0(X,A)$ is a natural transformation defined on all finite CW-pairs (X,A) that is equal to the identity when $A = \emptyset$. Then η is the identity on all pairs.*

Proof. Just use the natural isomorphism $\pi^*\colon K^0(X/A,*) \to K^0(X,A)$ together with the natural short exact sequence $0 \to K^0(X/A,*) \to K^0(X/A) \to K^0(*) \to 0$. □

When (X,A) is a cofibrant pair we have all the tools to construct a map $\mathcal{K}(X,A) \to K^0(X,A)$. Standard homotopical techniques allow us to reduce to this case, leading to the following result:

Proposition 24.38. *There is a natural transformation $\chi\colon \mathcal{K}(X,A) \to K^0(X,A)$, defined for all pairs (X,A), having the property that for $A = \emptyset$ one has $\chi([E_\bullet]) = \sum_i (-1)^i [E_i]$. All such natural transformations agree on pairs (X,A) where X and A are homotopically compact, and are an isomorphism on pairs that are homotopy equivalent to a finite CW-pair.*

Proof. We will use the functorial factorizations on the model category $\mathcal{T}op$, though this is not strictly necessary. Given a pair (X,A) factor $\emptyset \to A$ as $\emptyset \rightarrowtail \tilde{A} \xrightarrow{\sim} A$. Then factor the composite $\tilde{A} \to A \to X$ as $\tilde{A} \rightarrowtail \tilde{X} \xrightarrow{\sim} X$. In this way we functorially produce a map of pairs $(\tilde{X},\tilde{A}) \to (X,A)$ where the former is a cofibration between cofibrant spaces. Define $\chi\colon \mathcal{K}(X,A) \to K^0(X,A)$ to be the following composite:

$$\begin{array}{ccc}
\mathcal{K}(X,A) & \xrightarrow{\quad\chi\quad} & K^0(X,A) \\
\downarrow & & \cong\downarrow \\
\mathcal{K}(\tilde{X},\tilde{A}) \xleftarrow[(24.4)]{\cong} \mathcal{M}_\infty(\tilde{X},\tilde{A}) \xrightarrow[(24.12)]{\cong} \mathcal{M}_\infty(\tilde{X},\tilde{A})_{\tilde{A}} \xleftarrow[(24.23)]{\cong} \mathcal{M}_1(\tilde{X},\tilde{A})_{\tilde{A}} & \xrightarrow{\chi_1} & K^0(\tilde{X},\tilde{A}).
\end{array}$$

This gives a natural transformation of functors. When $A = \emptyset$ then $\tilde{A} = \emptyset$ and it is easy to check that the composite across the bottom row sends a bounded chain complex of vector bundles to the alternating sum of the terms. It follows that the top map must do the same thing.

Isomorphism. If $A' \hookrightarrow X'$ is a cofibration between cofibrant objects with a map of pairs $(X',A') \to (X,A)$ that is a weak equivalence on both spaces, then there is a commutative triangle of weak equivalences

$$\begin{array}{ccc}
(X',A') & \longrightarrow & (\tilde{X},\tilde{A}) \\
& \searrow & \downarrow \\
& & (X,A)
\end{array}$$

(this is an easy exercise). One readily sees that the map χ defined above by passing through $(\tilde{X},\tilde{A})$ coincides with the similarly-defined map that passes through (X',A') (just chase around a big commutative diagram). In particular, the exact choice of $(\tilde{X},\tilde{A})$ does not affect the definition of χ. As a result, when $A \hookrightarrow X$ is a cofibration between cofibrant objects we can take $\tilde{X} = X$ and $\tilde{A} = A$ and so χ is just the composite across the bottom row of the above diagram. But for finite CW-pairs χ_1 was proven to be an isomorphism in Proposition 24.34, and so χ is also an isomorphism here.

Now suppose that (X, A) is homotopy equivalent to a finite CW-pair (X', A'). Let $f\colon (X', A') \to (X, A)$ be part of such a homotopy equivalence and consider the diagram

$$\begin{array}{ccc} \mathcal{K}(X, A) & \xrightarrow{\chi} & K^0(X, A) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{K}(X', A') & \xrightarrow{\chi} & K^0(X', A'). \end{array}$$

Both of the vertical maps are isomorphisms (the left one by Proposition 24.7) and we have already proven that the bottom χ is an isomorphism, therefore the top one is as well.

Uniqueness. Now suppose that χ' is any other natural transformation that equals χ when $A = \emptyset$. Since χ is an isomorphism on finite CW-pairs we can define $\eta = \chi' \circ \chi^{-1}$ on such pairs, thereby obtaining natural maps $\eta\colon K^0(X, A) \to K^0(X, A)$. When $A = \emptyset$ these are identity maps by our assumption on χ', so by Lemma 24.37 we must have $\eta = \mathrm{id}$ on all pairs. Therefore $\chi = \chi'$ on all finite CW-pairs.

Now assume (X, A) is a pair where both X and A are homotopically compact. Then by Lemma C.2 there is a finite CW-pair (X', A') and a map of pairs $(X', A') \to (X, A)$ that is a weak equivalence on each term. The diagram

$$\begin{array}{ccc} \mathcal{K}(X, A) & \xrightarrow{\chi} & K^0(X, A) \\ \downarrow & & \downarrow{\scriptstyle \cong} \\ \mathcal{K}(X', A') & \xrightarrow{\chi} & K^0(X', A') \end{array}$$

must commute both for χ (shown) and for χ'. Since χ and χ' are equal for (X', A'), they must also be equal for (X, A).

□

Remark 24.39. Note that we have not proven that χ is an isomorphism whenever X and A are homotopically compact. It is not clear whether this is true.

24.40. **Compatibility of products.** As we have observed before, if $E_\bullet \in \mathcal{C}\mathrm{h}(X, A)$ and $F_\bullet \in \mathcal{C}\mathrm{h}(Y, B)$ then $E_\bullet \hat{\otimes} F_\bullet \in \mathcal{C}\mathrm{h}(X \times Y, (A \times Y) \cup (X \times B))$. This is readily seen to induce pairings

$$\mu\colon \mathcal{K}(X, A) \otimes \mathcal{K}(Y, B) \to \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)).$$

We would like these pairings to be compatible (via χ) with the similar pairings on K^0. The following result covers many cases of interest, but unfortunately is not the general statement that one would hope for.

Proposition 24.41. *Let (X, A) and (Y, B) be pairs where all of A, B, X, and Y are homotopically compact. If (X, A) and (Y, B) are mateable then the diagram*

$$\begin{array}{ccc} \mathcal{K}(X, A) \otimes \mathcal{K}(Y, B) & \xrightarrow{\mu} & \mathcal{K}(X \times Y, (A \times Y) \cup (X \times B)) \\ \downarrow{\scriptstyle \chi \otimes \chi} & & \downarrow{\scriptstyle \chi} \\ K^0(X, A) \otimes K^0(Y, B) & \longrightarrow & K^0(X \times Y, (A \times Y) \cup (X \times B)) \end{array}$$

is commutative, where the bottom horizontal map is the product on K-theory. [Recall in particular that the mateable hypothesis holds if any of the following conditions are satisfied:

(1) (X, A) and (Y, B) are finite CW-pairs.
(2) A is open in X and B is open in Y.
(3) We are working in a convenient category of topological spaces and A is cofibrant and $B \to Y$ is a cofibration.

See Proposition 19.7.]

The idea of the proof can be stated in two lines: For multiplicative cohomology theories, the product for CW-pairs is uniquely determined by the product on pairs $(X, \emptyset)$, using the isomorphisms $E^*(X, A) \cong E^*(X/A, *)$ and naturality. Since $\mathcal{K}(X, A) \to K^0(X, A)$ is an isomorphism when (X, A) is a finite CW-pair, the multiplication on $\mathcal{K}$ induces a multiplication on K^0—which therefore must equal the multiplication coming from topology, since they agree when $A = \emptyset$. The proof we give below is just an expansion of these two lines.

Proof of Proposition 24.41. We first prove the result when (X, A) and (Y, B) are finite CW-pairs. It is trivial to check that the diagram commutes when $A = B = \emptyset$. The general case follows formally from this one using naturality together with the fact that certain maps are isomorphisms. First check that it works for $A = B = *$ using the diagram

$$\begin{array}{ccccc} K^0(X,*) \otimes K^0(Y,*) & \longrightarrow & K^0(X \times Y, (X \times *) \cup (* \times Y)) & \xleftarrow{\cong} & K^0(X \wedge Y, *) \\ \downarrow & & \downarrow & \swarrow & \\ K^0(X) \otimes K^0(Y) & \longrightarrow & K^0(X \times Y) & & \end{array}$$

and the similar one for $\mathcal{K}(-,-)$ that maps to it. Here the diagonal map is an injection (for any cohomology theory) because stably the product $X \times Y$ splits as $X \vee Y \vee (X \wedge Y)$. Consequently, the central vertical map is an injection. The desired commutativity now follows by a diagram chase.

When (X, A) and (Y, B) are finite CW-pairs consider the diagram

$$\begin{array}{ccc} K^0(X,A) \otimes K^0(Y,B) & \longrightarrow & K^0(X \times Y, (A \times Y) \cup (X \times B)) \\ \uparrow \cong & & \uparrow \cong \\ K^0(X/A,*) \otimes K^0(Y/B,*) & \longrightarrow & K^0(X/A \times Y/B, X/A \vee Y/B) \end{array}$$

and its $\mathcal{K}$-analog that maps to it. Since (X, A) is a finite CW-pair we know by Proposition 24.38 that $\mathcal{K}(X, A) \to K^0(X, A)$ and $\mathcal{K}(X/A, *) \to K^0(X/A, *)$ are both isomorphisms, and similarly for (Y, B). The desired commutativity now follows from an easy diagram chase. *Note: This is the central step that requires all of the spaces to be homotopically compact.*

Now suppose that (X, A) and (Y, B) are more general pairs but with the requirement that all of the spaces are homotopically compact. Let $(X', A') \to (X, A)$ and $(Y', B') \to (Y, B)$ be weak equivalences where (X', A') and (Y', B') are finite CW-pairs (these exist by Proposition C.1). The diagram from the statement of the proposition for (X, A) and (Y, B) maps to the similar one for (X', A') and (Y', B'), and the maps $K^0(X, A) \to K^0(X', A')$ and $K^0(Y, B) \to K^0(Y', B')$ are isomorphisms. The desired result will follow from what has already been proven if we know that

$$K^0(X \times Y, A \times Y \cup X \times B) \to K^0(X' \times Y', A' \times Y' \cup X' \times B')$$

is an isomorphism. The map $X' \times Y' \to X \times Y$ is certainly a weak equivalence, but it is not clear that the same can be said for the relative term. This is the tricky point.

We are thereby led to examine the diagram

$$\begin{array}{ccccc} A \times Y & \longleftarrow & A \times B & \longrightarrow & X \times B \\ \downarrow & & \downarrow & & \downarrow \\ A' \times Y' & \longleftarrow & A' \times B' & \longrightarrow & X' \times B'. \end{array}$$

The vertical maps are all weak homotopy equivalences, and we would like to know that the induced map on pushouts is a weak homotopy equivalence. This will follow if both of those pushouts are in fact homotopy pushouts. That is automatic for the lower row, because both of those inclusions are cofibrations. The issue is the upper row. But the condition that (X, A) and (Y, B) be mateable is precisely the condition that the pushout of the top row is a homotopy pushout. □

Remark 24.42. Note that the tensor product of complexes does not give a pairing $\mathcal{M}_\infty(X, A)_A \otimes \mathcal{M}_\infty(Y, B)_B \to \mathcal{M}_\infty(X \times Y, A \times Y \cup X \times B)_{A \times Y \cup X \times B}$, as the resulting differential would only be defined on $A \times B$. So for products it is important to use $\mathcal{M}(X, A)$ rather than the A-relative version $\mathcal{M}(X, A)_A$. This is the reason Proposition 24.41 does not have a more appealing statement and proof. The definition of χ passes through the A-relative groups, but the product does not.

The hypothesis in Proposition 24.41 that all of the spaces are homotopically compact feels like it should be unnecessary, but I do not know how to remove it. The above argument seems to be the extent of how far one can get without having to do a very unpleasant, hands-on comparison of the two products.

24.43. **The proofs of Theorems 20.14 and 20.17.** To finally close out our main goal for this section, let us just observe that Theorem 20.14 is a combination of Proposition 24.38 and Proposition 24.41. And Theorem 20.17 is a combination of Propositions 24.4 and 24.12. So the proofs have now been completed.

24.44. **The role of the homotopy relation.** [Atiyah5] uses a version of our $\mathcal{M}_n(X, A)_A$ construction but where the homotopy relation is left out. Let $\mathcal{L}_n(X, A)_A$ be this set: it consists of equivalence classes of objects in $\mathcal{C}\mathrm{h}_n(X, A)_A$ up to isomorphism and stability. It is a monoid under direct sum, and we have the natural surjection $\mathcal{L}_n(X, A)_A \twoheadrightarrow \mathcal{M}_n(X, A)_A$.

Proposition 24.45. *Assume that $A \hookrightarrow X$ is a cofibration and X is paracompact Hausdorff. Then the map $\mathcal{L}_1(X, A)_A \to \mathcal{M}_1(X, A)_A$ is a bijection.*

Proof. Suppose given an $\mathcal{E} \in \mathcal{C}\mathrm{h}_1(X \times I, A \times I)_{A \times I}$ and let $E = \mathcal{E}|_{X \times 0}$ and $E' = \mathcal{E}|_{X \times 1}$. We need to show that $[E] = [E']$ in $\mathcal{L}_1(X, A)_A$. Since X is paracompact Hausdorff, the bundles on $X \times I$ are all isomorphic to ones pulled back along the projection $X \times I \to X$ (Corollary 14.2(b)). So we can assume all the bundles in $\mathcal{E}$ are of this form, which implies $E_0 = E'_0$ and $E_1 = E'_1$.

Write d_A and d'_A for the differentials on E and E', and note that these are isomorphisms. The homotopy $d'_A \sim d_A$ induces a homotopy $\mathrm{id} \sim (d'_A)^{-1} \circ d_A$.

Consider the fiber bundle $\underline{\mathrm{Iso}}(E_1, E_1) \to X$ and the diagram

$$\begin{array}{ccc} (X \times 0) \cup (A \times I) & \xrightarrow{\mathrm{id} \cup H} & \underline{\mathrm{Iso}}(E_1, E_1) \\ \simeq \downarrow & & \downarrow \\ X \times I & \xrightarrow{\pi} & X \end{array}$$

where id is the identity section and H is our homotopy on A. This diagram must have a lift, and restricting it to time 1 gives an isomorphism $\phi\colon E_1 \to E_1$ that restricts to $(d'_A)^{-1} d_A$ on A. Thus, we have the following isomorphism in $\mathcal{C}\mathrm{h}_1(X, A)_A$:

$$\begin{array}{ccc} E_1 & \xrightarrow[\cong]{\phi} & E_1 \\ {\scriptstyle d_A}\downarrow & & \downarrow{\scriptstyle d'_A} \\ E_0 & \xrightarrow[\cong]{\mathrm{id}} & E_0. \end{array}$$

This proves that $[E] = [E']$ in $\mathcal{L}_1(X, A)_A$ and we are done. □

Corollary 24.46. *Suppose that (X, A) is a cofibrant pair. Then for all $1 \le n \le \infty$ the map $\mathcal{L}_n(X, A)_A \to \mathcal{M}_n(X, A)_A$ is a bijection.*

Proof. Consider the commutative square

$$\begin{array}{ccc} \mathcal{L}_1(X, A)_A & \longrightarrow & \mathcal{L}_n(X, A)_A \\ \cong \downarrow & & \downarrow \\ \mathcal{M}_1(X, A)_A & \xrightarrow{\cong} & \mathcal{M}_n(X, A)_A. \end{array}$$

The left vertical map is an isomorphism by Proposition 24.45 and the lower horizontal map is an isomorphism by Proposition 24.23. It follows that the upper horizontal map is an injection. But use of the Γ-construction shows easily that the upper horizontal map is surjective: the argument is a much-simplified form of the proof of Proposition 24.23, in that we don't actually need to construct the inverse map we just need to produce preimages for elements. So the upper horizontal map is a bijection, which implies the same for the right vertical map. □

We close by returning briefly to the following diagram listing all of the bijections we established in this section:

$$\begin{array}{ccccccc} & & \mathcal{M}_1(X, A) & \longrightarrow & \mathcal{M}_n(X, A) & \longrightarrow & \mathcal{M}_\infty(X, A) \xrightarrow{(24.4)} \mathcal{K}(X, A) \\ & & {\scriptstyle (24.12)}\downarrow & & {\scriptstyle (24.12)}\downarrow & & {\scriptstyle (24.12)}\downarrow \qquad \downarrow \\ K^0(X, A) & \xleftarrow[(24.34)]{\chi_1} & \mathcal{M}_1(X, A)_A & \xrightarrow[(24.23)]{} & \mathcal{M}_n(X, A)_A & \xrightarrow[(24.23)]{} & \mathcal{M}_\infty(X, A)_A \xrightarrow{(24.4)} \mathcal{K}(X, A)_A \\ & & {\scriptstyle (24.45)}\uparrow & & \uparrow{\scriptstyle (24.46)} & & \uparrow{\scriptstyle (24.46)} \\ & & \mathcal{L}_1(X, A)_A & \longrightarrow & \mathcal{L}_n(X, A)_A & \longrightarrow & \mathcal{L}_\infty(X, A)_A. \end{array}$$

With the exception of χ_1, all of the maps are isomorphisms whenever (X, A) is a cofibrant pair—the labels on the maps give the references, and for the unlabelled maps it follows by the commutative diagram. The map χ_1 is an isomorphism when X and A are cofibrant and homotopically compact.

Note that the product, defined in terms of tensor product of chain complexes, only makes sense for $\mathcal{M}_\infty(X, A)$ and $\mathcal{K}(X, A)$.

∘ Exercises ∘

Exercise 24.47.

(a) Let (V, d) be an exact chain complex of finite-dimensional real vector spaces. Suppose each V_i has an inner product, and let d^* be the adjoints of the d maps. Define $\Delta_i \colon V_i \to V_i$ be $\Delta_i = dd^* + d^*d$. Prove that each Δ_i is an isomorphism, that the Δ's together give a chain map $\Delta \colon V \to V$, and that the d^* are a chain homotopy between Δ and the zero map.

(b) Let (X, A) be a pair and let $(E_\bullet, d)$ be a bounded A-relative exact complex on X. Suppose that all of the E_i are equipped with inner products. Show that if one modifies the complex $\Gamma(E_\bullet)$ by using d^* rather than e, all of the arguments in this section still work to show that $E_\bullet$ and $\Gamma(E_\bullet)$ represent the same element in $\mathcal{KO}(X, A)_A$.

(c) By iterating the procedure from (b), show that

$$0 \longrightarrow \bigoplus_{i \text{ odd}} E_i \xrightarrow{d+d^*} \bigoplus_{i \text{ even}} E_i \longrightarrow 0$$

is an A-relative exact complex that represents the same class as $E_\bullet$ in $\mathcal{KO}(X, A)_A$. This proves Proposition 20.20.

(d) Repeat for complex bundles and Hermitian inner products.

Part 5. K-theory and geometry

At this point we have seen that there exist cohomology theories $K^*(-)$ and $KO^*(-)$. To be fair, we have not *proven* the existence of KO, but we have seen that it comes as a consequence of the Bott periodicity theorem $\Omega^8(\mathbb{Z} \times BO) \simeq \mathbb{Z} \times BO$. If the only cohomology theory you have even seen is singular cohomology, this will seem like an amazing thing: suddenly you know three times as many cohomology theories as you used to. But we know from the theory of spectra that there is a bountiful source of cohomology theories—to be a little poetic about it, they are as plentiful as grains of sand on the beach. What is rare, however, is to have cohomology theories with a close connection to geometry: and both K and KO belong to this (vaguely-defined) class. In the following sections we will begin to explore what this means.

To some extent we have a "geometric" understanding of $K^0(-)$ and $KO^0(-)$ in terms of Grothendieck groups of vector bundles, at least for compact Hausdorff spaces. We also have similar "geometric" descriptions of the relative K^0-groups in terms of Grothendieck groups of chain complexes with exactness conditions, and we know that any $K^n(-)$ (or $KO^n(-)$) group can be shifted to a $K^0(-)$ group using the suspension isomorphism and Bott periodicity. One often hears a slogan like "The connection between K-theory and geometry is via vector bundles". This doesn't really say very much, though; our goal will be to develop a more detailed story along these lines.

One way to encode geometry into a cohomology theory is via Thom classes for vector bundles. Such classes give rise to fundamental classes for submanifolds and a robust connection wth intersection theory. In the next section we begin our story by recalling how all of this works for singular cohomology.

25. Thom classes, Thom isomorphism, and Thom spaces

The theory of Thom classes begins with the cohomological approach to orientations. Recall that

$$H^*(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H^*(D^n, S^{n-1}) \cong \tilde{H}^*(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, an orientation on $\mathbb{R}^n$ determines a generator for $H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \mathbb{Z}$. (For a review of how this correspondence works, see the proof of Lemma 26.2 in the next section).

Now consider a vector bundle $p\colon E \to B$ of rank n. Let $\zeta\colon B \to E$ be the zero section, and write $E-0$ as shorthand for $E-\mathrm{im}(\zeta)$. We know that $H^n(E_x, E_x - 0) \cong \mathbb{Z}$, and an orientation of the fiber gives a generator. We wish to consider the problem of giving compatible orientations for all the fibers at once; this can be addressed through the cohomology of the pair $(E, E - 0)$.

For a neighborhood V of x, let $E_V = E|_V = p^{-1}(V)$. If E_V is trivializable then there is an isomorphism $E_V \cong V \times \mathbb{R}^n$, and $(E_V - 0) \cong V \times (\mathbb{R}^n - 0)$. Hence, $H^*(E_V, E_V - 0) \cong H^*(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0))$. If V is contractible (which we will temporarily assume), this gives that

$$H^*(E_V, E_V - 0) \cong H^*(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \begin{cases} \mathbb{Z} & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Pick a generator $\mathcal{U}_V \in H^n(E_V, E_V - 0) \cong \mathbb{Z}$. For all $x \in V$ the inclusion $j_x\colon (E_x, E_x - 0) \hookrightarrow (E_V, E_V - 0)$ gives a map $j_x^*\colon H^*(E_V, E_V - 0) \to H^*(E_x, E_x - 0)$. Since we are assuming that V is contractible, j_x^* is an isomorphism. So $\mathcal{U}_V$ gives rise to generators in $H^n(E_x, E_x - 0)$ for all $x \in V$. We think of $\mathcal{U}_V$ as orienting all of the fibers simultaneously.

Even when V is not contractible the conclusions of the last paragraph still hold. One has that $H^*(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0)) \cong H^*(V) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$ by the Künneth Theorem, and so

$$H^i(V \times \mathbb{R}^n, V \times (\mathbb{R}^n - 0)) \cong H^{i-n}(V) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \begin{cases} H^{i-n}(V) & \text{if } i \geq n, \\ 0 & \text{if } i < n. \end{cases}$$

Let $\mathcal{U}_V \in H^n(E_V, E_V - 0)$ be an element that corresponds to $1 \in H^0(V)$ under the above isomorphism. Then one checks that $j_x^*(\mathcal{U}_V)$ is a generator for $H^n(E_x, E_x - 0)$ for every $x \in V$.

Next suppose that we have two open sets $V, W \subseteq B$ where E is trivializable over each one, together with classes $\mathcal{U}_V \in H^n(E_V, E_V - 0)$ and $\mathcal{U}_W \in H^n(E_W, E_W - 0)$ that restrict to generators (orientations) on the fibers E_x for every $x \in V$ and every $x \in W$, respectively. We would like to require that these orientations match: so we require that the images of $\mathcal{U}_V$ and $\mathcal{U}_W$ in $H^n(E_{V\cap W}, E_{V\cap W} - 0)$ coincide. Consider the (relative) Mayer-Vietoris sequence:

$$\begin{array}{ccccc} & & & & H^{n-1}(E_{V\cap W}, E_{V\cap W} - 0) \\ & & & & \downarrow \\ H^n(E_{V\cap W}, E_{V\cap W} - 0) & \longleftarrow & \begin{array}{c} H^n(E_V, E_V - 0) \\ \bigoplus \\ H^n(E_W, E_W - 0) \end{array} & \longleftarrow & H^n(E_{V\cup W}, E_{V\cup W} - 0) \end{array}$$

Under our requirement of compatibility between $\mathcal{U}_V$ and $\mathcal{U}_W$, the class $\mathcal{U}_V \oplus \mathcal{U}_W$ maps to zero; so it is the image of a class $\mathcal{U}_{V\cup W}$. Since $H^{n-1}(E_{V\cap W}, E_{V\cap W} - 0) = 0$ (see the computation in the previous paragraph), the class $\mathcal{U}_{V\cup W}$ is unique. Note that the Mayer-Vietoris sequence also shows that $H^*(E_{V\cup W}, E_{V\cup W} - 0) = 0$ for $* < n$, which leaves us poised to inductively continue this argument. In other words, the argument shows that we may patch more and more $\mathcal{U}$-classes together, provided that they agree on the regions of overlap. This is the kind of behavior one would expect for orientation classes.

The above discussion suggests the following definition:

Definition 25.1. *Given a (constant) rank n bundle $E \to B$, a* ***Thom class*** *for E is an element $\mathcal{U}_E \in H^n(E, E - 0)$ such that for all $x \in B$, $j_x^*(\mathcal{U}_E)$ is a generator in $H^n(E_x, E_x - 0)$. (Here $j_x\colon E_x \hookrightarrow E$ is the inclusion of the fiber).*

There is no guarantee that a bundle has a Thom class. Indeed, consider the following example:

Example 25.2. Let $M \to S^1$ be the Möbius bundle. Take two contractible open subsets V and W of S^1, where $V \cup W = S^1$. We can choose a Thom class for $M|_V$, and one for $M|_W$, but the orientations won't line up correctly to give us a Thom class for M. In fact, notice that by homotopy invariance $H^*(M, M - 0)$ is

the cohomology of the Möbius band relative to its boundary. But collapsing the boundary of the band gives an $\mathbb{R}P^2$



and we know $H^1(\mathbb{R}P^2) = 0$. So a Thom class cannot exist in this case.

If a bundle $E \to B$ has a Thom class then the bundle is called **orientable**. Said differently, an **orientation** on a vector bundle $E \to B$ is simply a choice of Thom class in $H^n(E, E-0; \mathbb{Z})$. One can readily prove that this notion of orientability agrees with other notions one may have encountered, and we leave this to the reader.

One can also talk about Thom classes with respect to the cohomology theories $H^*(-; R)$ for any ring R. Typically one only needs $R = \mathbb{Z}$ and $R = \mathbb{Z}/2$, however. In the latter case, note that any n-dimensional real vector space V has a *canonical* orientation in $H^n(V, V-0; \mathbb{Z}/2)$. It follows that local Thom classes always patch together to give global Thom classes, and so every vector bundle has a Thom class in $H^*(-; \mathbb{Z}/2)$.

Finally, note that we can repeat all that we have done for complex vector spaces and complex vector bundles. However, a complex vector space V of dimension n has a *canonical* orientation on its underlying real vector space, and therefore a canonical generator in $H^{2n}(V, V-0)$. Just as in the last paragraph, this implies that local Thom classes always patch together to give global Thom classes; so every complex vector bundle has a Thom class.

The following theorem summarizes what we have just learned:

Theorem 25.3. *Suppose (purely for convenience) that B is connected.*
(a) Every complex bundle $E \to B$ of rank n has a Thom class in $H^{2n}(E, E-0)$.
(b) Every real bundle $E \to B$ of rank n has a Thom class in $H^n(E, E-0; \mathbb{Z}/2)$.

The Mayer-Vietoris argument preceding Definition 25.1 shows that if $p\colon E \to B$ is a rank n orientable real vector bundle then $H^*(E, E-0)$ vanishes for $* < n$ and equals $\mathbb{Z}$ for $* = n$ (when B is non-compact a certain limiting argument is also needed, but we ignore this for now). A careful look at the argument reveals that it also gives a complete determination of the cohomology groups for $* > n$. We describe this next.

For any $z \in H^*(B)$, we may first apply p^* to obtain an element $p^*(z) \in H^*(E)$. We may then multiply by the Thom class $\mathcal{U}_E$ to obtain an element $p^*(z) \cup \mathcal{U}_E \in H^{*+n}(E, E-0)$. This gives a map $H^*(B) \to H^*(E, E-0)$ that increases degrees by n.

Theorem 25.4 (Thom Isomorphism Theorem)**.** *Suppose that $p\colon E \to B$ has a Thom class $\mathcal{U}_E \in H^*(E, E-0)$. Then the map $H^*(B) \to H^*(E, E-0)$ given by*

$$z \mapsto p^*(z) \cup \mathcal{U}_E$$

is an isomorphism of graded abelian groups that increases degrees by n. (The isomorphism can also be regarded as one of left $H^*(B)$-modules, where the module structure on $H^*(E, E-0)$ is via p^*; so $H^*(E, E-0)$ is a free $H^*(B)$-module of rank one, generated by $\mathcal{U}_E$.)

Proof. If the bundle is trivial, then $E = B \times \mathbb{R}^n$, and $E - 0 = B \times (\mathbb{R}^n - 0)$. Here the suspension and Künneth isomorphisms yield

$$H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0)) \cong H^{*-n}(B).$$

One readily checks that the map from the statement of the theorem gives the isomorphism.

For the case of a general bundle one can attempt to use Mayer-Vietoris and the Five Lemma to reduce to the case of trivial bundles. For a compact base space, where this argument can be made with a finite number of steps, it is straightforward. For non-compact bases one has to work a bit more to patch the stories over compact subspaces together; see [MS, Theorems 10.2 and 10.4], or Proposition 27.22 below for a different approach. Another proof, similar to the one in Milnor-Stasheff, can be found in [Hatcher1, Theorem 4D.8 and Corollary 4D.9]. □

Exercise 25.5. Write out the details for the above proof.

25.6. **Thom spaces.** The relative groups $H^*(E, E-0)$ coincide with the reduced cohomology groups of the mapping cone of the inclusion $E-0 \hookrightarrow E$. This mapping cone is sometimes called the **Thom space** of the bundle $E \to B$, although that name is more commonly applied to more geometric models that we will introduce next (the various models are all weakly equivalent). For the most common model we require that the bundle have an inner product (see Section 11.10).

Definition 25.7. *Suppose that $E \to B$ is a bundle with an inner product. Define the **disk bundle** of E as $D(E) = \{v \in E \,|\, \langle v, v\rangle \leq 1\}$, and the **sphere bundle** of E as $S(E) = \{v \in E \,|\, \langle v, v\rangle = 1\}$.*

If E has rank n over each component of B, note that $D(E) \to B$ and $S(E) \to B$ are fiber bundles with fibers D^n and S^{n-1}, respectively. Note also that we have the following diagram:

$$\begin{array}{ccc} E - 0 & \rightarrowtail & E \\ \simeq \uparrow & & \uparrow \simeq \\ S(E) & \rightarrowtail & D(E). \end{array}$$

This diagram shows that $E - 0 \hookrightarrow E$ and $S(E) \hookrightarrow D(E)$ have weakly equivalent mapping cones. Unlike $E-0 \hookrightarrow E$, however, the map $S(E) \hookrightarrow D(E)$ is a cofibration (under the mild condition that B is a CW-complex, say): so the mapping cone is weakly equivalent to the quotient $D(E)/S(E)$. This quotient is what is most commonly meant by the term 'Thom space':

Definition 25.8. *For a bundle $E \to B$ with inner product, the **Thom space** of E is $\operatorname{Th} E = D(E)/S(E)$.*

Remark 25.9. The notation B^E is also commonly used in the literature to denote the Thom space.

If B is compact Hausdorff then $\operatorname{Th} E$ is homeomorphic to the one-point compactification of the space E. Recall that if X is locally compact Hausdorff then up to homeomorphism there is a unique space $\hat{X}$ that is compact Hausdorff, contains X as a subspace, and has $\hat{X} - X$ equal to a single point (see [Munkres, Theorem

29.1]). We can identify E with the open unit disk $D^\circ(E) \subseteq D(E)$, and then observe that $D(E)/S(E)$ has the properties to be the one-point compactification of $D^\circ(E)$.

Another perspective on the Thom space is to first compactify each of the fibers separately, which amounts to forming the pushout of $B \leftarrow S(E) \to D(E)$. The inclusion from B into the pushout P is the 'section at infinity', and the quotient P/B is the Thom space $D(E)/S(E)$. When B is compact Hausdorff this space is also compact Hausdorff, contains E as a subspace, and has exactly one point outside of E—so it is the one-point compactification.

Example 25.10. We will show that $\mathrm{Th}(nL \to \mathbb{R}P^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$, where L is the tautological line bundle. First we define a homeomorphism of spaces over $\mathbb{R}P^k$:

$$\begin{array}{ccc} \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} & \xrightarrow{\cong} & nL^* \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & \mathbb{R}P^k. & \end{array}$$

Consider $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{n+k}$ as embedded via the last n coordinates. Take a point $\ell = [x_0 : \cdots : x_k : y_1 : \cdots : y_n] \in \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1}$, and note that at least one x_i is nonzero. The map $\pi\colon \mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \to \mathbb{R}P^k$ is defined to send ℓ to $[x_0 : \cdots : x_k]$.

Regard ℓ as a line in $\mathbb{R}^{n+k+1}$, and $\pi(\ell)$ as a line in $\mathbb{R}^{k+1}$. The formula $(x_0, \ldots, x_k) \mapsto y_1$ specifies a unique functional $\pi(\ell) \to \mathbb{R}$ (obtained by extending linearly). Likewise, we obtain n functionals on $\pi(\ell)$ via the formulas

$$(x_0, \ldots, x_k) \mapsto y_1, \qquad \ldots \quad (x_0, \ldots, x_k) \mapsto y_n.$$

Note also that these functionals are independent of the choice of the homogeneous coordinates for ℓ: multiplying all the x_i's and y_j's by λ gives rise to the same functionals. We have therefore described a continuous map $\mathbb{R}P^{n+k} - \mathbb{R}P^{n-1} \to nL^*$, and this is readily checked to be a homeomorphism.

Since the Thom space is the one-point compactification, we get that

$$\mathrm{Th}(nL^* \to \mathbb{R}P^k) \cong \widehat{(nL^*)} \cong (\mathbb{R}P^{n+k} - \mathbb{R}P^{n-1})^\wedge \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}.$$

We know by Corollary 11.15 that any real vector bundle over a paracompact space is isomorphic to its dual. So $nL^* \cong nL$, and we have shown that $\mathrm{Th}(nL \to \mathbb{R}P^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$.

Remark 25.11. Note the case $n = 1$ in the above example: $\mathrm{Th}(L \to \mathbb{R}P^k) \cong \mathbb{R}P^{k+1}$.

Remark 25.12. A similar analysis to above shows that $\mathrm{Th}(nL^* \to \mathbb{C}P^k) \cong \mathbb{C}P^{n+k}/\mathbb{C}P^{n-1}$, but note that unlike the real case the dual is important here.

There is another approach to Thom spaces that does not require a metric for the bundle. If $E \to B$ is any vector bundle, let $\mathbb{P}(E) \to B$ be the corresponding fiber bundle of projective spaces where the fiber of $\mathbb{P}(E) \to B$ over a point b is $\mathbb{P}(E_b)$ (see Exercise 11.25). Another definition of Thom space is then

$$\mathrm{Th}\, E = \mathbb{P}(E \oplus \underline{1})/\mathbb{P}(E).$$

Note that this definition does not require a metric.

To see that our definitions are equivalent, note that if V is a vector space then there is a canonical inclusion $V \hookrightarrow \mathbb{P}(V \oplus \mathbb{R})$ given by $v \mapsto \langle v \oplus 1\rangle$. A little thought shows that we get a diagram

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{P}(V \oplus \mathbb{R}) \\ \downarrow & & \downarrow \\ \hat{V} & \xrightarrow{\cong} & \mathbb{P}(V \oplus \mathbb{R})/\mathbb{P}(V) \end{array}$$

where the bottom map is a homeomorphism. Extending this to the bundle setting, it is clear that the pushout of $B \longleftarrow \mathbb{P}(E) \to \mathbb{P}(E \oplus \underline{1})$ is the fiberwise one-point compactification of E. Then $\mathbb{P}(E \oplus \underline{1})/\mathbb{P}(E)$ is obtained by taking this fiberwise one-point compactification and collapsing the section at infinity: this clearly agrees with the other descriptions we have given of the Thom space.

It is sometimes useful to be able to connect the pairs $(\mathbb{P}(E \oplus \underline{1}), \mathbb{P}(E))$ and $(E, E - 0)$ in a way that doesn't make use of any metric. To do so, observe that every vector space V gives rise to a commutative diagram

$$\begin{array}{ccccc} V - 0 & \rightarrowtail & \mathbb{P}(V \oplus \mathbb{R}) - * & \xleftarrow{\sim} & \mathbb{P}(V) \\ \downarrow & & \downarrow & & \downarrow \\ V & \rightarrowtail & \mathbb{P}(V \oplus \mathbb{R}) & = & \mathbb{P}(V \oplus \mathbb{R}). \end{array}$$

Here $* \in \mathbb{P}(V \oplus \mathbb{R})$ is the line formed by the distinguished copy of $\mathbb{R}$, and $V \to \mathbb{P}(V \oplus \mathbb{R})$ is the map $v \mapsto \langle v \oplus 1\rangle$. All the other maps are the evident inclusions. To see that the indicated map is a homotopy equivalence, use the projection $V \oplus \mathbb{R} \to V$ with kernel $\mathbb{R}$. This gives a map $\mathbb{P}(V \oplus \mathbb{R}) - * \to \mathbb{P}(V)$ that is readily checked to be a deformation retraction. The left square is an open covering diagram (with open sets along the antidiagonal and their intersection in the upper left corner), and therefore a homotopy pushout. Taking homotopy cofibers of the three columns therefore yields a zig-zag of weak equivalences between the homotopy cofiber of $V - 0 \hookrightarrow V$ and the homotopy cofiber of $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V \oplus \mathbb{R})$. The latter is weakly equivalent to its cofiber, because $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V \oplus \mathbb{R})$ is a cofibration.

Now consider a fiberwise version of the above diagram. If $E \to B$ is a real bundle then we have maps

$$\begin{array}{ccccc} E - 0 & \rightarrowtail & \mathbb{P}(E \oplus \underline{1}) - B & \xleftarrow{\sim} & \mathbb{P}(E) \\ \downarrow & & \downarrow & & \downarrow \\ E & \rightarrowtail & \mathbb{P}(E \oplus \underline{1}) & = & \mathbb{P}(E \oplus \underline{1}). \end{array}$$

The only difference worth noting is that $B \hookrightarrow \mathbb{P}(E \oplus \underline{1})$ is the evident section that in each fiber selects out the distinguished line determined by the trivial bundle $\underline{1}$. The left square is again a homotopy pushout square, and so taking homotopy cofibers of the columns gives a zig-zag of weak equivalences between the homotopy cofibers of $E - 0 \hookrightarrow E$ and $\mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus \underline{1})$.

25.13. **Thom spaces for virtual bundles.** Thom spaces behave in a very simple way in relation to adding on trivial bundles:

Proposition 25.14. *For any real bundle $E \to X$ one has $\mathrm{Th}(E \oplus \underline{n}) \cong \Sigma^n \mathrm{Th}(E)$. For a complex bundle $E \to X$ one has $\mathrm{Th}(E \oplus \underline{n}) \cong \Sigma^{2n} \mathrm{Th}(E)$.*

Proof. We only prove the statement for real bundles, as the complex case works the same (and is even a consequence of the real case). Also, we will give the proof assuming the bundle has a metric, although even without this one can get the result with a weak equivalence instead of the homeomorphism.

We have the homeomorphism $D(E) \times D^n \cong D(E \oplus \underline{n})$ given by

$$(v, w) \mapsto \begin{cases} \frac{\max\{|v|,|w|\}}{\sqrt{|v|^2+|w|^2}}(v, w) & \text{if } (v, w) \neq (0, 0), \\ (0, 0) & \text{if } v = w = 0. \end{cases}$$

This restricts to a homeomorphism

$$(S(E) \times D^n) \amalg_{S(E)\times S^{n-1}} (D(E) \times S^{n-1}) \cong S(E \oplus \underline{n}).$$

From this one readily sees that

$$D(E \oplus \underline{n})/S(E \oplus \underline{n}) \cong [D(E)/S(E)] \wedge [D^n/S^{n-1}] \cong \mathrm{Th}(E) \wedge S^n.$$

□

Proposition 25.14 allows one to make sense of Thom spaces for virtual bundles, provided that we use *spectra*. This material will only be needed briefly in the rest of the book, but we include it here because these Thom spectra play a large role in modern algebraic topology.

Assume that X is compact Hausdorff and let $E \to X$ be a bundle. Then by Proposition 12.10 E embeds in some trivial bundle $\underline{N}$. Let Q denote the quotient, so that we have $E \oplus Q \cong \underline{N}$. Assuming that $\mathrm{Th}(-E)$ had some meaning then we would expect

$$\mathrm{Th}(Q) = \mathrm{Th}(\underline{N} - E) = \Sigma^N \mathrm{Th}(-E).$$

This suggests the definition

$$\mathrm{Th}(-E) = \Sigma^{-N} \mathrm{Th}(Q),$$

where the negative suspension must of course be interpreted as taking place in a suitable category of spectra.

Our definition seems to depend on the choice of embedding $E \hookrightarrow \underline{N}$. To see that this dependence is an illusion, let $E \hookrightarrow \underline{N}'$ be another embedding and let Q' be the quotient. Then $\underline{N}' \oplus Q \cong Q' \oplus E \oplus Q \cong Q' \oplus \underline{N}$. On Thom spaces this gives $\Sigma^{N'} \mathrm{Th}(Q) \cong \Sigma^N \mathrm{Th}(Q')$, or $\Sigma^{-N} \mathrm{Th}(Q) \simeq \Sigma^{-N'} \mathrm{Th}(Q')$.

The above discussion can be extended to cover any element $\alpha \in KO(X)$. Write $\alpha = E - F$ for vector bundles E and F, and choose an embedding $F \hookrightarrow \underline{N}$. Let Q denote the quotient $\underline{N}/F$. Note that $\alpha + \underline{N} = (E - F) + (F + Q) = E + Q$. If $\mathrm{Th}(\alpha)$ makes sense then we would expect $\Sigma^N(\mathrm{Th}\,\alpha) \simeq \mathrm{Th}(\alpha + \underline{N}) \simeq \mathrm{Th}(E + Q)$, and so this suggests the definition

$$\mathrm{Th}(\alpha) = \Sigma^{-N} \mathrm{Th}(E \oplus Q).$$

Again, one readily checks that up to homotopy this does not depend on the choice of E, F, N, or the embedding $F \hookrightarrow \underline{N}$.

25.15. **An application to stunted projective spaces.** To demonstrate the usefulness of Thom spaces we give an application to periodicities amongst stunted projective spaces. This material is also needed in the solution of the vector fields on spheres problem presented [D7].

Consider the space $\mathbb{R}P^{a+b}/\mathbb{R}P^a$. This has a cell structure with exactly b cells (not including the zero cell), in dimensions $a + 1$ through $a + b$. The space

$\mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}$ has a similar cell structure, although here the cells are in dimensions $a+1+r$ through $a+b+r$. The natural question arises: fixing a and b, what values of r (if any) satisfy

$$\Sigma^r[\mathbb{R}P^{a+b}/\mathbb{R}P^a] \simeq \mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}?$$

One can use singular cohomology and Steenrod operations to produce some necessary conditions here. For example, integral singular homology easily yields that if if $b \geq 2$ then r must be even. Use of Steenrod operations produces more stringent conditions (we leave this for the reader to think about).

We will use Thom spaces to provide some *sufficient* conditions for a stable homotopy equivalence between stunted projective spaces. We begin with a simple lemma:

Lemma 25.16. *The element $\lambda = [L] - 1 \in \widetilde{KO}(\mathbb{R}P^n)$ satisfies $\lambda^2 = -2\lambda$ and $\lambda^{n+1} = 0$. Consequently, $\lambda^k = (-2)^{k-1}\lambda$ for all k, and $2^n\lambda = 0$.*

Proof. The square of any real line bundle is trivializable (Exercise 16.31), so $L^2 \cong \underline{1}$. This immediately yields $\lambda^2 = -2\lambda$. The second statement follows from the fact that $\mathbb{R}P^n$ may be covered by $n+1$ contractible sets $U_0, \ldots, U_n$. (With respect to homogeneous coordinates $[x_0 : \cdots : x_n]$ on $\mathbb{R}P^n$, one may take U_i to be the open set $x_i \neq 0$). The element $\lambda \in KO(\mathbb{R}P^n, *)$ lifts to a class $\lambda_i \in KO(\mathbb{R}P^n, U_i)$, and therefore λ^{n+1} is the image of $\lambda_0\lambda_1\cdots\lambda_n$ under the natural map

$$KO(\mathbb{R}P^n, U_0 \cup \cdots \cup U_n) \to KO(\mathbb{R}P^n).$$

But since $\cup_i U_i = \mathbb{R}P^n$, the domain of the above map is zero; hence $\lambda^{n+1} = 0$.

Finally, since $\lambda^2 = -2\lambda$ it follows that $\lambda^e = (-2)^{e-1}\lambda$ for all e. In particular, $(-2)^n\lambda = \lambda^{n+1} = 0$. □

Proposition 25.17. *Let r be any positive integer such that $r([L]-1) = 0$ in $\widetilde{KO}(\mathbb{R}P^{b-1})$. Then for any $a > 0$ there is a stable homotopy equivalence*

$$\Sigma^r[\mathbb{R}P^{a+b}/\mathbb{R}P^a] \simeq \mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}.$$

Proof. The assumption $r([L]-1) = 0$ implies that $rL \oplus \underline{s} \cong \underline{r} \oplus \underline{s}$ for some $s \geq 0$. We have

$$\begin{aligned}
\mathbb{R}P^{a+b}/\mathbb{R}P^a \cong \operatorname{Th}\begin{pmatrix} (a+1)L \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} &\simeq \Sigma^{-r-s}\operatorname{Th}\begin{pmatrix} (a+1)L\oplus\underline{r+s} \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} \\
&\simeq \Sigma^{-r-s}\operatorname{Th}\begin{pmatrix} (a+1+r)L\oplus\underline{s} \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} \\
&\simeq \Sigma^{-r}\operatorname{Th}\begin{pmatrix} (a+1+r)L \\ \downarrow \\ \mathbb{R}P^{b-1}\end{pmatrix} \\
&\simeq \Sigma^{-r}\big[\mathbb{R}P^{a+b+r}/\mathbb{R}P^{a+r}\big].
\end{aligned}$$

The first and last steps use the identification of stunted projective spaces with a corresponding Thom space—see Example 25.10 for this. □

Combining Lemma 25.16 and Proposition 25.17 we see that stunted projective spaces with b cells have a periodicity of 2^{b-1}:

$$\Sigma^{2^{b-1}}\big[\mathbb{R}P^{a+b}/\mathbb{R}P^a\big] \simeq \mathbb{R}P^{a+b+2^{b-1}}/\mathbb{R}P^{a+2^{b-1}}$$

(here $\simeq$ means stable homotopy equivalence). However, this is not the best result along these lines: we will get a better result by finding the exact order of $[L]-1$ in $\widetilde{KO}(\mathbb{R}P^{b-1})$. This was determined by Adams; see [D7, Section 7].

◦ Exercises ◦

Exercise 25.18. For the vector bundle $\underline{0}$ one has $\mathrm{Th}(\underline{0}) \cong X_+$. Think through this isomorphism from the viewpoint of each of the different perspectives on the Thom space. [Hint: Remember that X/A is the pushout of $* \longleftarrow A \longrightarrow X$.]

Exercise 25.19. Let X be a compact Hausdorff space and let $A \subseteq X$ be closed. Prove that $(X-A)^+ \cong X/A$. (This identification gets used frequently.)

Exercise 25.20. Construct an example of a non-compact space B and a vector bundle $E \to B$ for which $\mathrm{Th}\, E$ is not homotopy equivalent to $\widehat{E}$. Is there a natural comparison map in one direction (say, in the homotopy category)?

Exercise 25.21. Let $E \to X$ be a rank n real vector bundle. Prove that E has a Thom class in $H^n(E, E-0)$ if and only if E is orientable in the sense of Exercise 15.26.

Exercise 25.22. For the tangent bundle TS^2 construct a cell decomposition $TS^2 \cong S^2 \cup_f e^4$, where the S^2 is the compactification of one of the fibers. What information would you need in order to identify the homotopy type of the attaching map $f \in \pi_3(S^2)$?

Exercise 25.23. Let $\gamma_{k-1} \to \mathrm{Gr}_{k-1}(\mathbb{R}^{n-1})$ be the tautological bundle, and let $\gamma_{k-1}^{\perp}$ be its orthogonal complement inside the trivial bundle $\underline{n-1}$. Prove that there is a homeomorphism $\mathrm{Th}(\gamma_{k-1}^{\perp}) \cong \mathrm{Gr}_k(\mathbb{R}^n)/\mathrm{Gr}_k(\mathbb{R}^{n-1})$, and show that the same result is true with $\mathbb{R}$ replaced by $\mathbb{C}$. [Hint: Start with the map $\mathrm{Gr}_k(\mathbb{R}^n) - \mathrm{Gr}_k(\mathbb{R}^{n-1}) \to \mathrm{Gr}_{k-1}(\mathbb{R}^{n-1})$ given by $V \mapsto V \cap \mathbb{R}^{n-1}$, then use Example 25.10 as a partial guide.]

26. Thom classes and intersection theory

In this section we will see how Thom classes give rise to fundamental classes for submanifolds, and we will develop the connection between products of such classes and intersection theory.

Let $E \to B$ be a real vector bundle of rank n. In general, E may not have a Thom class; and if it does have a Thom class, it actually has *two* Thom classes (since $H^n(E, E-0) \cong \mathbb{Z}$ by the Thom Isomorphism Theorem). The situation is familiar, as it matches the usual behavior of orientations. It is, of course, possible—and necessary!—to do geometry in a way that includes keeping track of orientations and computing signs according to whether orientations match up or not. But it is easier if we are in a situation where we don't have to keep track of quite so much, and there are two situations with that property: we can work always with mod 2 coefficients, or we can work in the setting of complex geometry. In either case we have canonical Thom classes all the time. In this section, and for the bulk of the book, we choose to work in the setting of complex bundles and complex geometry. But it is important that almost everything works verbatim for real bundles if we use $\mathbb{Z}/2$ coefficients, and many things can be made to work for oriented real bundles if one is diligent about keeping track of signs.

For a rank n complex bundle $E \to B$ we have a canonical Thom class $\mathcal{U}_E \in H^{2n}(E, E-0)$. The following result gives two useful properties:

Proposition 26.1.

(a) *(Naturality) Suppose $E \to B$ is a rank n complex vector bundle, and $f\colon A \to B$. Consider the pullback*

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B. \end{array}$$

Then $\bar{f}^*\colon H^{2n}(E, E-0) \to H^{2n}(f^*E, f^*E-0)$ sends $\mathcal{U}_E$ to $\mathcal{U}_{f^*E}$.

(b) *(Multiplicativity) Suppose that $E \to B$ is a rank n complex vector bundle, and $E' \to B'$ is a rank k complex vector bundle, with Thom classes $\mathcal{U}_E \in H^{2n}(E, E-0)$ and $\mathcal{U}_{E'} \in H^{2k}(E', E'-0)$. Then $\mathcal{U}_E \times \mathcal{U}_{E'} = \mathcal{U}_{E \hat{\oplus} E'}$ in $H^{2n+2k}(E \times E', (E \times E') - 0)$, where $E \hat{\oplus} E' = E \times E' \to B \times B'$ is the external direct sum.*

Proof. Recall that the Thom class of a rank n complex bundle $E \to B$ is the unique class in $H^{2n}(E, E-0)$ that restricts to the canonical generator in $H^{2n}(E_x, E_x - 0)$ for every fiber E_x. Part (a) follows readily from this characterization. Using the same reasoning, part (b) is reduced to the case where $B = B' = *$; this is checked in the lemma below. □

Lemma 26.2. *Let V and W be two real vector spaces, of dimensions n and k, respectively. Assume given orientations on V and W, and let $V \oplus W$ have the product orientation. Let $U_V \in H^n(V, V-0)$, $U_W \in H^k(W, W-0)$, and $U_{V\oplus W} \in H^{n+k}(V \oplus W, (V \oplus W) - 0)$ be the corresponding orientation classes. Then $U_{V\oplus W} = U_V \times U_W$.*

Proof. Let $v_1, \ldots, v_n$ be an oriented basis for V, and let $\sigma_V\colon \Delta^n \to V$ be the affine simplex whose ordered list of vertices is $0, v_1, \ldots, v_n$. Let σ_V^t denote any translate of σ that contains the origin of V in the interior. Then $[\sigma_V^t]$ is a generator for $H_n(V, V-0)$, and any relative cocycle in $C^*_{sing}(V, V-0)$ that evaluates to 1 on σ_V^t is a generator (in fact, the same generator) for $H^n(V, V-0)$. This is how an orientation of V determines a generator of $H^n(V, V-0)$.

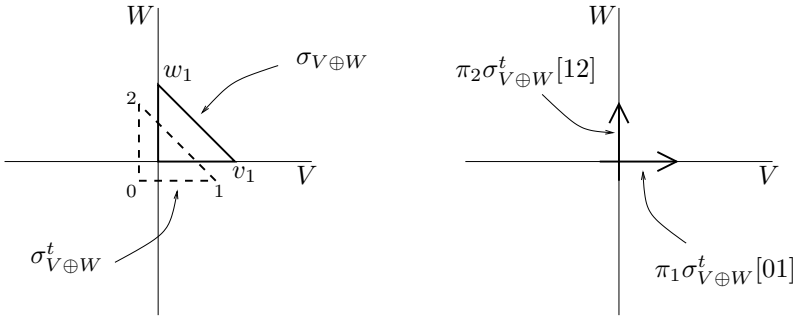
Now let $w_1, \ldots, w_k$ be an oriented basis for W. Let $\sigma_{V\oplus W}\colon \Delta^{n+k} \to V \oplus W$ be the affine simplex whose ordered list of vertices is $0, v_1, \ldots, v_n, w_1, \ldots, w_k$ (note that omitting 0 gives an oriented basis for $V \oplus W$). Again, let $\sigma^t_{V\oplus W}$ denote a translate of $\sigma_{V\oplus W}$ that contains the origin in its interior.

Recall that $U_V \times U_W = (\pi_1)^*(U_V) \cup (\pi_2^*)(U_W)$, where $\pi_1\colon V \times W \to V$ and $\pi_2\colon V \times W \to W$ are the two projections. The definition of the cup product gives

$$\begin{aligned}(U_V \times U_W)(\sigma^t_{V\oplus W}) &= (\pi_1^* U_V)(\sigma^t_{V\oplus W}[01\cdots n]) \cdot (\pi_2^* U_W)(\sigma^t_{V\oplus W}[n \cdots (n+k)]) \\ &= U_V(\pi_1 \circ \sigma^t_{V\oplus W}[01\cdots n]) \cdot U_W(\pi_2 \circ \sigma^t_{V\oplus W}[n \cdots (n+k)]).\end{aligned}$$

It is clear that $\pi_1 \circ \sigma^t_{V\oplus W}[01\cdots n]$ gives a simplex in the same homology class as σ_V^t, and so U_V evaluates to 1 on this simplex. Similarly, $\pi_2 \circ \sigma^t_{V\oplus W}[n\cdots(n+k)]$ gives a simplex in the same homology class as σ_W^t, and so U_W evaluates to 1 here. Since $1 \cdot 1 = 1$, we see that $U_V \times U_W$ satisfies the defining property of $U_{V\oplus W}$.

Given that a picture is worth a thousand words, here is a picture showing what is happening in the smallest nontrivial case:



□

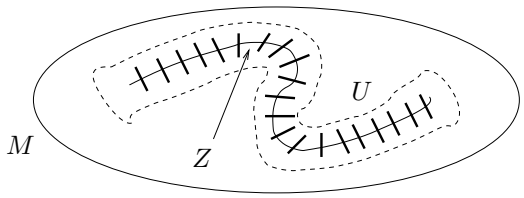
Remark 26.3. If $E \to B$ and $E' \to B$ are two bundles over the same base, then as spaces $E \oplus E' = E \times_B E' \hookrightarrow E \times E'$. So here the appropriate statement about Thom classes is $\mathcal{U}_{E\oplus E'} = i^*(\mathcal{U}_E \times \mathcal{U}_{E'})$ where $i\colon E \times_B E' \hookrightarrow E \times E'$ is the inclusion. This follows from the pullback diagram

$$\begin{array}{ccc} E \oplus E' & \xrightarrow{i} & E \hat{\oplus} E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

together with both parts of Proposition 26.1.

26.4. **Fundamental classes.** Next we use the Thom isomorphism to define fundamental classes for submanifolds. Let M be a differentiable manifold. Recall that a (differntiable) submanifold is a closed set $Z \subseteq M$ with the property that for every $z \in Z$ there exists a coordinate chart for M around z such that Z is equal to the vanishing locus of some subset of the coordinates. The tangent bundle of Z is canonically embedded in the tangent bundle of M. The normal bundle of Z in M is defined to be the quotient $(TM|_Z)/TZ$; we write it as $N_{M/Z}$. If M has a Riemannian metric then the normal bundle can also be defined as the orthogonal complement of TZ inside $TM|_Z$.

The **tubular neighborhood theorem** says that if $Z \subseteq M$ is a submanifold of codimension c then there is an open set U containing Z and a diffeomorphism $\phi\colon U \to N_{M/Z}$ such that the evident triangle involving $Z \hookrightarrow U$ and $\zeta\colon Z \hookrightarrow N_{M/Z}$ is commutative. See [Kosinski, Corollary III.2.3] for a proof. The open set U is of course called a "tubular neighborhood". Keep in mind the following rough picture:



The diffeomorphism ϕ is not unique in any useful sense, and we need to be careful about this. For example, if $M = \mathbb{R}$ and $Z = \{0\}$ one immediately sees two very

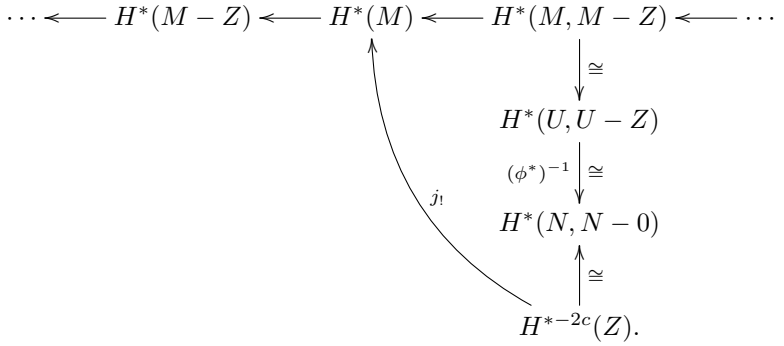
different choices for ϕ, differing by a sign. A convenient way to fix a choice is to put a metric on M and then use the exponential map: for vectors in $(T_zZ)^\perp$ we get unique paths (defined for small time intervals) having those tangent vectors at $t = 0$. A small neighborhood V of the zero-section in $(TZ)^\perp$ exponentiates to give a small neighborhood around Z, and we precompose this with a fiberwise-dilation map that shrinks all of $(TZ)^\perp$ into V. Since the space of metrics on M is contractible, following this procedure gives a ϕ that is unique up to isotopy. We will always implicitly use these ϕ's for our diffeomorphisms.

Now let M be a complex manifold. A complex submanifold is defined to be a differentiable submanifold Z such that all of the tangent spaces T_zZ are complex subspaces of T_zM. The normal bundle $N_{M/Z}$ then carries a complex structure induced from TM, and if M is equipped with a Hermitian metric then we can again identify $N_{M/Z}$ with $(TZ)^\perp$. We still have a tubular neighborhood U and a diffeomorphism ϕ to the normal bundle (none of that depended on the complex structure), and again we always choose the ϕ's that come from exponential maps associated to some choice of Hermitian metric. Be warned, though, that it is almost never possible to choose ϕ to be biholomorphic. That is, we cannot identify $N_{M/Z}$ with U as *complex manifolds*. This will not matter for our applications, but it is something to be careful about. See Remark 26.14 for more details.

Let M be a complex manifold with complex submanifold Z, of codimension c, having normal bundle N and tubular neighborhood U. Then we have that $H^*(U, U - Z) \cong H^*(N, N - 0)$. Since $N \to Z$ is a complex bundle of rank c it has a Thom class $\mathfrak{U}_N \in H^{2c}(N, N - 0)$, and so by the Thom Isomorphism we get $H^{i-2c}(Z) \cong H^i(N, N-0)$. Also, by excision one has $H^*(M, M-Z) \cong H^*(U, U-Z)$. So we have isomorphisms

$$H^{i-2c}(Z) \xrightarrow{Thom} H^i(N, N-0) \cong H^i(U, U-Z) \xleftarrow{\cong} H^i(M, M-Z).$$

Now consider the long exact sequence for the pair $(M, M-Z)$, but use the above isomorphisms to rewrite the relative groups $H^*(M, M-Z)$ as $H^{*-2c}(Z)$:



Note that the map ϕ^* is independent of any choices, since ϕ is well-defined up to isotopy. If $j\colon Z \hookrightarrow M$ is the inclusion, then the indicated composition in the above diagram is denoted $j_!$ and called a **pushforward map** or **Gysin map**. We can rewrite the long exact sequence to get the **Gysin sequence**, also called a **localization sequence** by algebraic geometers:

$$\cdots \longleftarrow H^i(M - Z) \longleftarrow H^i(M) \xleftarrow{j_!} H^{i-2c}(Z) \longleftarrow H^{i-1}(M - Z) \longleftarrow \cdots$$

Definition 26.5. *Let Z be a codimension c submanifold of the complex manifold M. Let $j_!$ be the Gysin map described above. We define the* ***fundamental class*** *of Z to be $[Z]_M = j_!(1) \in H^{2c}(M)$, where $1 \in H^0(Z)$ is the unit. We also define the* ***relative fundamental class*** *$[Z]_{rel} \in H^{2c}(M, M-Z)$ to be the image of 1 under the chain of isomorphisms from $H^0(Z)$ to $H^{2c}(M, M-Z)$. Note that $j^*([Z]_{rel}) = [Z]$, where j^* denotes the induced map in cohomology associated to the inclusion $(M, \emptyset) \hookrightarrow (M, M-Z)$.*

On an intuitive level one should think of $[Z]$ as being the Poincaré dual of the usual fundamental class of Z in $H_*(M)$. The point, however, is that we don't need to think through the hairiness of the Poincaré duality isomorphism; this has been replaced with the machinery of vector bundles and Thom classes.

One must of course prove a collection of basic results showing that the classes $[Z]$ really do behave as one would expect fundamental classes to behave, and have the expected ties with geometry. We will do a little of this, just enough to give the reader the idea that it is not hard. Before tackling this let us do the most trivial example:

Exercise 26.6.

(a) Check that the relative fundamental class of the origin in $\mathbb{C}^d$ is the canonical generator: i.e., $[0]_{rel} \in H^{2d}(\mathbb{C}^d, \mathbb{C}^d - 0)$ is the canonical generator provided by the complex orientation on $\mathbb{C}^d$.

(b) Let M be a d-dimensional complex manifold. If $a, b \in M$ are path-connected, verify that $[a] = [b]$. Hint: Reduce to the case where a and b belong to a common chart U of M, with $U \cong \mathbb{C}^d$. Let I be a line joining a and b inside of U, and consider the diagram

$$\begin{array}{ccccc} H^*(M, M-a) & \xrightarrow{\cong} & H^*(M, M-I) & \xleftarrow{\cong} & H^*(M, M-b) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H^*(U, U-a) & \xrightarrow{\cong} & H^*(U, U-I) & \xleftarrow{\cong} & H^*(U, U-b). \end{array}$$

Using an argument similar to that in the proof of Lemma 26.2, show that $[a]_{rel,U}$ and $[b]_{rel,U}$ map to the same element in $H^*(U, U-I)$.

(c) Suppose M is compact and connected. Verify that if $a \in M$ then $[a] \in H^{2d}(M)$ is a generator.

(d) Give a different proof of (b) as follows. Let $\sigma\colon I \to M$ be a smooth path from a to b, and let $Z \subseteq M \times I$ be the set $Z = \{(\sigma(t), t) \,|\, t \in I\}$. Produce a relative fundamental class $[Z]_{rel} \in H^{2d}(M \times I, (M \times I) - Z)$ and show that the restrictions to $(M \times \{0\}, (M-a) \times \{0\})$ and $(M \times \{1\}, (M-b) \times \{1\})$ give $[a]_{rel}$ and $[b]_{rel}$. Explain why $[a] = [b]$ follows almost immediately. (Between (b) and (d), which proof do you like better?)

The following theorem connects our fundamental classes to intersection theory. It is far from the most general statement along these lines, but it will suffice for our applications later in the text. The diligent reader will find that the proof readily generalizes to tackle more complicated situations, for example where the intersection is not discrete.

Theorem 26.7. *Let M be a connected complex manifold. Suppose that Z and W are submanifolds of M that intersect transversely in d points. Assume also that W is connected (this is mostly for convenience in the statement of (b)). Then*

(a) $[Z]_M \cup [W]_M = d[*]_M$
(b) $j^*([Z]_M) = d[*]_W$*, where* $j\colon W \hookrightarrow M$*.*

Proof. We begin by proving (a). Suppose that $\dim Z = k$ and $\dim W = \ell$, so that $\dim M = k + \ell$. Let $Z \cap W = \{p_1, \ldots, p_d\}$, and for each i let U_i be a Euclidean neighborhood of p_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Consider the following diagram:

$$\begin{array}{ccc}
H^{2l}(M, M-Z) \otimes H^{2k}(M, M-W) & \longrightarrow & H^{2l}(M) \otimes H^{2k}(M) \\
\downarrow{\scriptstyle \cup_{rel}} & & \downarrow{\scriptstyle \cup} \\
H^{2k+2l}(M, M-(Z\cap W)) & \longrightarrow & H^{2k+2l}(M) \\
\| & & \\
H^{2k+2l}(M, M-\{p_1,\ldots,p_d\}) & & \\
\uparrow{\scriptstyle \cong} & & \\
\oplus_r H^{2k+2l}(M, M-\{p_r\}) & \nearrow &
\end{array}$$

Since $[Z]$ and $[W]$ lift to relative classes $[Z]_{rel}$ and $[W]_{rel}$, it will suffice to show that if we take $[Z]_{rel} \cup [W]_{rel}$ and take its projection to the rth factor $H^{2k+2l}(M, M - \{p_r\})$ of the direct sum then we get $[p_r]_{rel}$. From this it will follow from the diagram that $[Z] \cup [W] = [p_1] + \ldots + [p_d]$ in $H^{2k+2l}(M)$. Since we have already seen in Exercise 26.6 that $[p_i] = [p_j]$ for any i and j, this will complete the proof of (a).

Next, fix an index r and consider the second diagram

$$\begin{array}{ccc}
H^{2l}(U_r, U_r - Z) \otimes H^{2k}(U_r, U_r - W) & \longleftarrow & H^{2l}(M, M-Z) \otimes H^{2k}(M, M-W) \\
\downarrow{\scriptstyle \cup_{rel}} & & \downarrow{\scriptstyle \cup_{rel}} \\
H^{2k+2l}(U_r, U_r - \{p_r\}) & \longleftarrow & H^{2k+2l}(M, M-\{p_1,\ldots,p_d\}) \\
& \swarrow{\scriptstyle \cong} & \uparrow \\
& & H^{2k+2l}(M, M-\{p_r\}).
\end{array}$$

Thanks to this diagram, it is enough to replace M by U_r, Z by $Z \cap U_r$, and W by $W \cap U_r$, and to prove that $[Z]_{rel} \cup [W]_{rel} = [p_r]_{rel}$.

But now M is just $\mathbb{C}^{k+l}$. By choosing our neighborhood small enough, we can find local coordinates so that Z is just $\mathbb{C}^k \times \{0\}$ and W is just $\{0\} \times \mathbb{C}^l$, intersecting transversely at the origin (we will write $\mathbb{C}^k \times 0$ and $0 \times \mathbb{C}^l$ for brevity). We need to compute $[\mathbb{C}^k \times 0]_{rel} \cup [0 \times \mathbb{C}^l]_{rel} \in H^{2(k+l)}(\mathbb{C}^{k+l}, \mathbb{C}^{k+l} - 0)$. Decompose $\mathbb{C}^{k+l} = \mathbb{C}^k \times \mathbb{C}^l$ and let $\pi_1\colon \mathbb{C}^{k+l} \to \mathbb{C}^k$ and $\pi_2\colon \mathbb{C}^{k+l} \to \mathbb{C}^l$ be the two projections. Then $[\mathbb{C}^k \times 0]_{rel}$ coincides with $(\pi_2)^*(\mathcal{U}_{\underline{l} \to *})$ and $[0 \times \mathbb{C}^l]_{rel}$ coincides with $(\pi_1)^*(\mathcal{U}_{\underline{k} \to *})$ (just look at the restriction to any normal fiber). So we have

$$\begin{aligned}
[\mathbb{C}^k \times 0]_{rel} \cup [0 \times \mathbb{C}^l]_{rel} &= (\pi_2)^*(\mathcal{U}_{\underline{l} \to *}) \cup (\pi_1)^*(\mathcal{U}_{\underline{k} \to *}) \\
&= (\pi_1)^*(\mathcal{U}_{\underline{k} \to *}) \cup (\pi_2)^*(\mathcal{U}_{\underline{l} \to *}) \\
&= (\mathcal{U}_{\underline{k} \to *}) \times (\mathcal{U}_{\underline{l} \to *}) \\
&= \mathcal{U}_{\underline{k+l} \to *} \quad \text{(by Lemma 26.2)} \\
&= [0]_{rel}.
\end{aligned}$$

This completes the proof of (a).

The proof of (b) is very similar. One considers the diagram

$$\begin{array}{ccccc}
 & & H^{2l}(W) & \xleftarrow{\ j^*\ } & H^{2l}(M) \\
 & & \uparrow & & \uparrow \\
\oplus_i H^{2l}(W, W-p_i) & \xrightarrow{\cong} & H^{2l}(W, W-\{p_1,\dots,p_d\}) & \longleftarrow & H^{2l}(M, M-Z) \\
\downarrow & & \downarrow & & \downarrow \\
H^{2l}(W, W-p_r) & \xrightarrow{\cong} & H^{2l}(W\cap U_r, (W\cap U_r)-p_r) & \longleftarrow & H^{2l}(U_r, U_r-p_r)
\end{array}$$

where r is an arbitrary choice of index. The top square implies that it suffices to show that the projection of $j^*([Z]_{rel})$ to $H^{2l}(W, W-p_r)$ equals $[p_r]_{rel}$, for any choice of r. The bottom square then allows us to replace M by U_r and Z and W by $Z\cap U_r$ and $W\cap U_r$. That is, we are again reduced to the case where $M=\mathbb{C}^{k+l}$, $Z=\mathbb{C}^k\times 0$, and $W=0\times\mathbb{C}^l$. Here we are considering the map

$$H^{2l}(\mathbb{C}^l, \mathbb{C}^l-0) \xleftarrow{\ j^*\ } H^{2l}\big(\mathbb{C}^k\times\mathbb{C}^l, (\mathbb{C}^k\times\mathbb{C}^l)-(\mathbb{C}^k\times 0)\big)$$

and must show that the image of $[\mathbb{C}^k\times 0]_{rel}$ is the canonical generator in the target. But if we identity $\mathbb{C}^k\times\mathbb{C}^l$ with the bundle $\underline{l}\to\mathbb{C}^k$ then $[\mathbb{C}^k\times 0]_{rel}$ is just the Thom class $\mathcal{U}_{\underline{l}}$, and the map j^* is restriction to the fiber over $0\in\mathbb{C}^k$; so it becomes the canonical generator by definition of the Thom class. □

Exercise 26.8. If W were not assumed to be connected in Theorem 26.7, what would need to change in the statement of part (b)?

Exercise 26.9. When computing $[\mathbb{C}^k\times 0]_{rel}\cup[0\times\mathbb{C}^l]_{rel}$ in the above proof, it might cause some worry that we relied on commutativity of the cup product. The classes were even-dimensional and so it didn't matter, but it feels like the proof shouldn't depend on this. Think about what would happen for $[\mathbb{R}^k\times 0]_{rel}\cup[0\times\mathbb{R}^l]_{rel}$, giving all the subspaces their natural orientations. Prove that $[0\times\mathbb{R}^l]_{rel}=(-1)^{kl}\pi_1^*(\mathcal{U}_{\underline{k}\to *})$, so that the use of commutativity exactly cancels this sign and still leads to $[\mathbb{R}^k\times 0]_{rel}\cup[0\times\mathbb{R}^l]_{rel}=[0]_{rel}$. [Given an oriented submanifold of an oriented manifold, the convention is to orient the normal spaces so that the induced orientation of $T\oplus N$ matches the global orientation.]

It is important to notice that the proof of Theorem 26.7, for the most part, used nothing special about singular cohomology—we only used the basic properties of Thom classes, together with generic properties that hold in *any* cohomology theory. In the proof of Lemma 26.2 we apparently used particular details about the definition of the cup product, but in fact what we needed could have been written in a way that doesn't reference the peculiar definition of the cup product at all. Indeed, we have the identifications $H^*(\mathbb{C}^n, \mathbb{C}^n-\mathbb{C}^k)=H^*(\mathbb{C}^k\times\mathbb{C}^{n-k}, \mathbb{C}^k\times(\mathbb{C}^{n-k}-0))=H^*(\mathbb{C}^{n-k}, \mathbb{C}^{n-k}-0)=H^*(D^{2n-2k}, \partial D^{2n-2k})\cong\tilde{H}^*(S^{2(n-k)})$ (for the second identification we use the map induced by projection $\mathbb{C}^k\times\mathbb{C}^{n-k}\to\mathbb{C}^{n-k}$, and for the third identification we use the induced map of any orientation-preserving embedding of the disk into $\mathbb{C}^{n-k}$ that sends the boundary onto the unit sphere). Similarly, we have a canonical identification $H^*(\mathbb{C}^n, \mathbb{C}^n-\mathbb{C}^{n-k})=\tilde{H}^*(S^{2k})$. Considering the

commutative diagram

$$\begin{array}{ccc} H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^k) \otimes H^*(\mathbb{C}^n, \mathbb{C}^n - \mathbb{C}^{n-k}) & \xrightarrow{\quad\mu\quad} & H^*(\mathbb{C}^n, \mathbb{C}^n - 0) \\ \| & & \| \\ \tilde{H}^*(S^{2(n-k)}) \otimes \tilde{H}^*(S^{2k}) & \xrightarrow{\quad\mu\quad} & \tilde{H}^*(S^{2n}) \end{array}$$

where μ denotes our product, the property needed for the proof of Lemma 26.2 boils down to the requirement that

$$\sigma^{2(n-k)}(1) \otimes \sigma^{2k}(1) \xrightarrow{\mu} \sigma^{2n}(1).$$

In other words, the computation comes down to the fact that the product behaves well with respect to the suspension isomorphism.

Example 26.10. To demonstrate Theorem 26.7 we will be content with the usual first example. Let $Z \hookrightarrow \mathbb{C}P^n$ be a codimension c complex submanifold. Then $[Z] \in H^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}$. A generator for this group is $[\mathbb{C}P^{n-c}]$, so $[Z] = d[\mathbb{C}P^{n-c}]$ for a unique integer d. This integer is called the **degree** of the submanifold Z. A generic, c-dimensional, linear subspace of $\mathbb{C}P^n$ will intersect Z transversely in finitely many points, say e of them. Theorem 26.7 gives that $[Z] \cup [\mathbb{C}P^c] = e[*]$, but we also have $d[\mathbb{C}P^{n-c}] \cup [\mathbb{C}P^c] = d[*]$ since $[\mathbb{C}P^{n-c}] \cup [\mathbb{C}P^c] = [*]$ (again by Theorem 26.7). So $d = e$, and this gives the geometric description of the degree: the number of intersection points with a generic linear subpace of complementary dimension.

The following result is the evident generalization of Theorem 26.7.

Theorem 26.11. *Let M be a connected complex manifold. Suppose that Z and W are submanifolds of M that intersect transversely. Then*
(a) $[Z]_M \cup [W]_M = [Z \cap W]$;
(b) $j^*([Z]_M) = [Z \cap W]$, where $j\colon W \hookrightarrow M$.

Outline of proof. We omit the details here, since the proof is largely similar to that of Theorem 26.7. For (a) use the relative fundamental classes $[Z]_{rel}$ and $[W]_{rel}$, and show that $[Z]_{rel} \cup [W]_{rel} = [Z \cap W]_{rel}$ in $H^*(M, M - (Z \cap W))$. For this, restrict to a tubular neighborhood and then show that both classes restrict to the canonical generators on the fibers of the normal bundle. For $[Z \cap W]_{rel}$ this is the definition, and for $[Z]_{rel} \cup [W]_{rel}$ this is a computation with the cup product. The proof of (b) is similar. □

26.12. **Topological intersection multiplicities.** We can now use our machinery to give a topological definition of intersection multiplicity. Suppose that Z and W are complex submanifolds of the complex manifold M, and that Z and W have an isolated point of intersection at p. Let k denote the complex codimension of Z in M, and l denote the complex codimension of W in M. Let d be the complex dimension of M.

Let U be a Euclidean neighborhood of p that contains no other points of $Z \cap W$. Consider the classes $[Z]_{rel} \in H^{2k}(M, M - Z)$ and $[W]_{rel} \in H^{2l}(M, M - W)$. Then the cup product $[Z]_{rel} \cup [W]_{rel}$ lies in the group $H^{2(k+l)}(M, M - (Z \cap W))$. This group is zero if $k + l \neq d$, and in this case we define the intersection multiplicity $i(Z, W; p)$ to be zero. In the case where $k + l = d$ we have that

$$H^{2(k+l)}(M, M - (Z \cap W)) = H^{2d}(M, M - \{p\}) = \mathbb{Z},$$

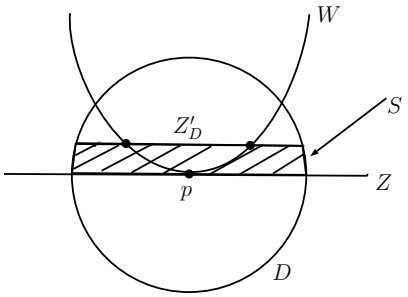
which has the preferred generator $[p]_{rel}$. Here we define $i(Z, W; p)$ to be the unique integer such that

$$[Z]_{rel} \cup [W]_{rel} = i(Z, W; p)[p].$$

The same proof as for Theorem 26.7 shows that when Z and W intersect transversely at p we have $i(Z, W; p) = 1$.

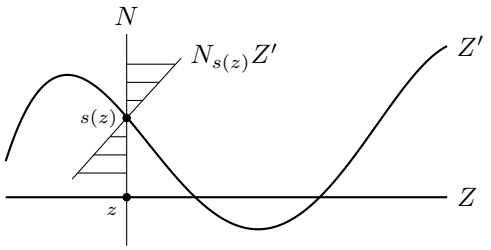
The final subject we turn to is the invariance of the intersection multiplicity under small deformations. There are different ways one might approach this; we just give one version. Assume we are in the case where Z and W have complementary dimensions: $k + l$ is the complex dimension of M. Let E be a Euclidean neighborhood of p in which Z is the vanishing locus of the first $2k$ real coordinates. The normal and tangent bundles for Z inherit the evident trivializations inside this neighborhood, and by choosing a smaller E if necessary we can use E for our chosen tubular neighborhood of Z. Fix a closed disk $D = D^{2d} \subseteq E$ where p is in the interior. Let $Z_D = Z \cap D$ and $W_D = W \cap D$.

Next suppose we have a smooth section s of the normal bundle of Z_D, let $Z' \subseteq E$ be the image of s, and set $Z'_D = Z' \cap D$. Consider the associated homotopy $h\colon Z_D \times I \to D$ given by $h(z, t) = ts(z)$. Let S be the image of h. Assume that $S \cap W \subseteq \mathrm{int}(D)$ and that $Z'_D \cap W$ has only finitely-many points $q_1, \ldots, q_r$. The following picture shows an example of this setup:



Note that the picture shows the *simplest* example of the setup, where the section s is constant. If s has a nonempty vanishing locus then Z' can cross Z or even overlap with it in some regions.

The space Z'_D need not be a complex submanifold of M, but it comes with a trivialization of its normal bundle. The resulting Thom class is all that is needed to define $[Z'_D]_{rel}$ and the intersection multiplicities $i(Z'_D, W; q_j)$. To demonstrate the trivialization, consider the picture below (where this time we have depicted a more interesting Z'):



The vertical lines N are the normal directions to Z, and Z' is the graph of a smooth function $Z \to N$. For such a graph the normal spaces $N_{s(z)}Z'$ will never intersect

$N^{\perp}$, and so orthogonal projection onto N gives an isomorphism $N_{s(z)}Z' \to N$. This trivializes the normal bundle of Z', and our standard orientation on N thereby determines an orientation on this bundle—that is, a Thom class. The following diagram might be helpful in parsing this line of reasoning. Let $N = N_z Z$, $z' = s(z)$, $N' = N_{z'}Z'$ and let NN' by the cone obtained by taking the union of all the straight-line paths from elements of N' to their orthogonal projections onto N. Then we have:

$$\begin{array}{ccccc}
H^{2k}(D, D - Z'_D) & \longrightarrow & H^{2k}(N, N - z') & \xrightarrow{\cong} & H^{2k}(N, N - zz') \\
\cong\downarrow & \searrow & \uparrow\cong & & \cong\uparrow \\
H^{2k}(N', N' - z') & \xleftarrow{\cong} & H^{2k}(NN', NN' - z') & & H^{2k}(N, N - z)
\end{array}$$

where all of the maps are induced by inclusions of spaces (the dotted map might only exist if we pass to a small neighborhood of z', but that is okay). The orientation on N determines a generator for $H^{2k}(N, N - z)$, which by chasing through the isomorphisms determines a generator for $H^{2k}(N', N' - z')$. The class $[Z'_D]_{rel}$ is the one that lifts this local orientation class for N', and the diagram shows this is equivalent to lifting the local orientation class for N at z'.

Proposition 26.13. *In the above setting, we have*

$$i(Z, W; p) = \sum_{j=1}^{r} i(Z'_D, W; q_j).$$

In particular, if Z'_D meets W transversely at each point q_j then $i(Z, W; p) = r$.

Proof. We give a sketch and leave some of the details to the reader. Let $d = k + l$ and consider the following diagram:

$$\begin{array}{ccccc}
H^{2k}(D, D - Z_D) \otimes H^{2l}(D, D - W) & \xrightarrow{\cup} & H^{2d}(D, D - p) & \longrightarrow & H^{2d}(D, \partial D) \\
\downarrow & & \downarrow & & \| \\
H^{2k}(D, D-S) \otimes H^{2l}(D, D-W) & \xrightarrow{\cup} & H^{2d}(D, D - (S \cap W)) & \longrightarrow & H^{2d}(D, \partial D) \\
\uparrow & & \uparrow & & \| \\
H^{2k}(D, D - Z'_D) \otimes H^{2l}(D, D - W) & \xrightarrow{\cup} & H^{2d}(D, D - \{q_1, \ldots, q_r\}) & \to & H^{2d}(D, \partial D)
\end{array}$$

Note that in the middle line we are using that $S \cap W$ does not meet ∂D. In the top left corner we have $[Z_D]_{rel} \otimes [W_D]_{rel}$, and in the bottom left corner we have $[Z'_D]_{rel} \otimes [W_D]_{rel}$. The diagram shows that these map to the same element in $H^{2d}(D, \partial D) = \mathbb{Z}$, provided that $[Z_D]_{rel}$ and $[Z'_D]_{rel}$ map to the same element in $H^{2k}(D, D - S)$. Assuming this for the moment, along the top row the class maps to $i(Z, W; p)$ and along the bottom row the class maps to $\sum_j i(Z', W; q_j)$ (using the same argument as in the proof of Theorem 26.7).

So it only remains to analyze the two maps

$$H^{2k}(D, D - Z_D) \longrightarrow H^{2k}(D, D - S) \longleftarrow H^{2k}(D, D - Z'_D)$$

and see that $[Z]_{rel}$ and $[Z']_{rel}$ map to the same element in the middle. Note that in our setup the inclusions $D - S \hookrightarrow D - Z_D$ and $D - S \hookrightarrow D - Z'_D$ are homotopy

equivalences, so the above two maps are isomorphisms. Pick any point $z \in Z_D$ and let $N_z \subseteq D$ be the fiber of the normal bundle to Z. Consider the diagram

$$\begin{array}{ccccc} H^{2k}(D, D - Z_D) & \xrightarrow{\cong} & H^{2k}(D, D - S) & \xleftarrow{\cong} & H^{2k}(D, D - Z'_D) \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ H^{2k}(N_z, N_z - 0) & \xrightarrow{\cong} & H^{2k}(N_z, N_z - (N_z \cap S)) & \xleftarrow{\cong} & H^{2k}(N_z, N_z - (N_z \cap Z'_D)) \end{array}$$

where the indicated maps are all isomorphisms (and therefore every map is an isomorphism). The elements $[Z_D]_{rel}$ and $[Z'_D]_{rel}$ map to the orientation classes in the left and right groups from the bottom row, and these map to the same class in the middle precisely because we oriented the normal directions to Z' in the way that was determined by the given orientation on the normal spaces of Z. □

There is a moral to these kinds of arguments, which is that there is a certain art to transferring geometric intuition into the language of cohomology theories and commutative diagrams. But once a person has developed the basic skills for doing that, the arguments are mostly routine.

Remark 26.14. Since we used tubular neighborhoods so much in this section, it is worth saying a bit more about the failure of the tubular neighborhood isomorphism to preserve the complex structure. For a complex manifold M with a complex submanifold Z, one might hope that an open neighborhood of Z is biholomorphic to an open neighborhood of the zero section in $N_{M/Z}$. If this were true, then the standard exact sequence of holomorphic vector bundles

$$0 \to TZ \to (TM)|_Z \to N_{M/Z} \to 0$$

would split holomorphically: the vector bundle projection $N_{M/Z} \to Z$ induces a holomorphic retraction of the tubular neighborhood onto Z, which on tangent spaces yields the splitting $(TM)|_Z \to TZ$. But the theory of holomorphic vector bundles gives obstructions to such splittings. For example, a classical theorem of [van de Ven] says that for complex subvarieties $Z \hookrightarrow \mathbb{C}P^n$ the corresponding sequence only splits when Z is a linear subspace.

∘ Exercises ∘

Exercise 26.15. Let M be a complex manifold, let $Z \hookrightarrow M$ be a complex submanifold, and let $h\colon Z \times I \to M$ be an isotopy where each $h_t(Z)$ is a complex submanifold. Let $Z' = h_1(Z)$. Prove that $[Z] = [Z']$ in $H^*(M)$. [Hint: Consider the associated embedding $Z \times I \hookrightarrow M \times I$ and show that this has a relative fundamental class. Then argue as in Exercise 26.6(d).]

Exercise 26.16. Suppose that M is a complex manifold, Z is a complex submanifold of M of codimension c, and X is a complex submanifold of Z of codimension d. Let $k\colon X \hookrightarrow Z$ and $j\colon Z \hookrightarrow M$ be the inclusions. Prove that $(jk)_! = j_! \circ k_!$ as maps $H^*(X) \to H^{*+2c+2d}(M)$. [Use $N_{M/X} = N_{Z/X} \oplus k^* N_{M/Z}$, and write down a large commutative diagram.]

27. Thom classes in K-theory

In the last section we saw how Thom classes for complex vector bundles give rise to cohomological fundamental classes for submanifolds, and we saw that these fundamental classes have the expected connections to geometry. The discussion was carried out in the case of singular cohomology, but very little specific information about this cohomology theory was actually used. In fact, once we showed that Thom classes existed almost everything else followed formally. In this section we adapt these ideas to generalized cohomology theories and explain why they apply to complex K-theory.

27.1. **Complex-oriented cohomology theories.** Let $\mathcal{E}$ be a multiplicative generalized cohomology theory (Definition 19.10).

Definition 27.2. *Let $E \to B$ be a rank n complex vector bundle, and for each $x \in B$ let $i_x \colon E_x \hookrightarrow E$ be the inclusion of the fiber. A **Thom class** for E is an element $\mathcal{U}_E \in \mathcal{E}^{2n}(E, E-0)$ such that for all $x \in B$ one has $i_x^*(\mathcal{U}_E)$ mapping to 1 under the string of isomorphisms*

$$\mathcal{E}^{2n}(E_x, E_x{-}0) \cong \mathcal{E}^{2n}(\mathbb{C}^n, \mathbb{C}^n{-}0) \cong \mathcal{E}^{2n}(D^{2n}, \partial D^{2n}) \cong \tilde{\mathcal{E}}^{2n}(S^{2n}) \cong \tilde{\mathcal{E}}^0(S^0) = \mathcal{E}^0(pt).$$

In the above definition note that the first isomorphism depends on an identification $E_x \cong \mathbb{C}^n$, which is equivalent to a choice of basis in E_x. However, any two such bases give homotopic maps of pairs $(\mathbb{C}^n, \mathbb{C}^n - 0) \to (E_x, E_x - 0)$ (since $GL_n(\mathbb{C})$ is path-connected) and therefore induce the same map on $\mathcal{E}^*$.

Definition 27.3. *A **complex orientation** for $\mathcal{E}$ is a choice, for every rank n complex bundle $E \to B$, of a Thom class $\mathcal{U}_E \in \mathcal{E}^{2n}(E, E-0)$ such that*

1. *(Naturality) $\mathcal{U}_{f^*E} = f^*(\mathcal{U}_E)$ for every map $f \colon A \to B$;*
2. *(Multiplicativity) $\mathcal{U}_{E \hat{\oplus} E'} = \mathcal{U}_E \cdot \mathcal{U}_{E'}$, whenever $E' \to B'$ is a complex bundle and $E \hat{\oplus} E'$ is the external direct sum.*

A given cohomology theory may or may not admit a complex orientation—most likely, it will not. The complex-orientable cohomology theories are a very special class. Once a complex orientation is provided one gets the Thom isomorphism, Gysin sequences, and fundamental classes for complex submanifolds just as before—as well as the same connections to intersection theory. (To be fair, our earlier proof of the Thom isomorphism result referred to [MS] which for noncompact bases uses a specialized technique that is not available outside singular cohomology; see Proposition 27.22 below for a more robust version of that proof.)

Our goal in this section is the following:

Theorem 27.4. *Complex K-theory is a multiplicative cohomology theory that admits a complex orientation.*

We will spend a long time exploring the geometric consequences of Theorem 27.4 (essentially all of Part 6 of this book), but let us go ahead and give one example right away. Let $Z \hookrightarrow \mathbb{C}P^n$ be a complex submanifold of codimension c. The above theorem implies that we have a fundamental class $[Z] \in K^{2c}(\mathbb{C}P^n)$, just as we did in the case of singular cohomology. Whereas $H^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}$ and only leads to a single integer-valued invariant for submanifolds, we will find that $K^{2c}(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$. This is a much larger group, and so there is suddenly the potential for detecting more

information: the K-theoretic fundamental class $[Z]$ is an $(n+1)$-tuple of integers rather than just a single integer. Of course it might end up that all of these new invariants are just zero, or some algebraic function of the invariant we already had—we will have to do some computations to find out. But this demonstrates the general situation: K-theory has an inherent ability to detect more information than singular cohomology did.

We discussed in Section 19 why K-theory is a multiplicative cohomology theory. So to prove Theorem 27.4 we need to give a construction, for every rank n complex vector bundle $E \to B$, of a Thom class in $K^{2n}(E, E-0)$. By Bott periodicity this group is the same as $K^0(E, E-0)$. We saw in Section 20 that one can produce classes in such relative K^0-groups using chain complexes of vector bundles on E that are exact on $E-0$. Most of what we are about to do simply amounts to discovering that there is a natural algebraic source of such objects.

27.5. **Koszul complexes.** Let V be a complex vector space of dimension n. For any $v \in V$ consider the chain complex

$$0 \longrightarrow \Lambda^0 V \xrightarrow{v\wedge -} \Lambda^1 V \xrightarrow{v\wedge -} \cdots \xrightarrow{v\wedge -} \Lambda^{n-1} V \xrightarrow{v\wedge -} \Lambda^n V \longrightarrow 0.$$

Denote this chain complex by $J_{V,v}$. It is easy to see that this is exact when $v \neq 0$: indeed, pick a basis $e_1, \ldots, e_n$ for V where $e_1 = v$, then use the usual induced basis for the exterior products. It is clear that if $e_1 \wedge \omega = 0$ then all the basis elements appearing in ω have an e_1 in them.

Exercise 27.6. Check that $J_{V,v} \otimes J_{W,w} \cong J_{V\oplus W, v\oplus w}$, and that the isomorphism is canonical.

For various reasons we will need to consider the dual of $J_{V,v}$. Recall the existence of a natural isomorphism $\Lambda^k(V^*) \to (\Lambda^k V)^*$: for any $\phi_1, \ldots, \phi_k \in V^*$ it sends $\phi_1 \wedge \cdots \wedge \phi_k$ to the functional on $\Lambda^k V$ given by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{\sigma \in \Sigma_k} (-1)^\sigma \phi_1(v_{\sigma(1)}) \phi_2(v_{\sigma(2)}) \cdots \phi_k(v_{\sigma(k)}).$$

If $e_1, \ldots, e_n$ is a basis for V, let $e_1^*, \ldots, e_n^*$ denote the dual basis for V^*. As a basis for $\Lambda^k V$ use the standard basis of wedge products $e_{i_1 \cdots i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k}$ where $i_1 < \cdots < i_k$, and write $e^*_{i_1 \cdots i_k}$ for the corresponding elements of the dual basis for $(\Lambda^k V)^*$. One readily checks that our map sends $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ to the functional $e^*_{i_1 \cdots i_k}$, which shows that we have an isomorphism.

Using these isomorphisms, the dual complex of $J_{V,v}$ has the form

$$0 \to \Lambda^n V^* \xrightarrow{d_v} \Lambda^{n-1} V^* \xrightarrow{d_v} \cdots \xrightarrow{d_v} \Lambda^1 V^* \xrightarrow{d_v} \Lambda^0 V^* \to 0$$

We denote this by $J^*_{V,v}$, and it is called a **Koszul complex**. Here is a description of the differential:

Proposition 27.7. *Let $e_1, \cdots, e_n$ be a basis for V and write $v = \sum v_i e_i$. Let $e_1^*, \ldots, e_n^*$ be the dual basis for V^*. Then the differential in $J^*_{V,v}$ is given by*

$$d_v(e^*_{i_0} \wedge \cdots \wedge e^*_{i_k}) = \sum_{j=0}^{k} (-1)^j v_{i_j} e^*_{i_0} \wedge \cdots \wedge \widehat{e^*_{i_j}} \wedge \cdots \wedge e^*_{i_k},$$

where the hat indicates that that term is omitted from the wedge.

Proof. Left to the reader. □

Exercise 27.8. Prove that $J_{V,v}$ and $J^*_{V,v}$ are isomorphic as chain complexes. A good exercise is to try to do this without any help, starting with a specific map $\Lambda^0 V \to \Lambda^n V^*$ and figuring out what the other maps would have to be to get a map of complexes. This essentially leads one to discover the so-called "Hodge star-operator". But for good measure we also give an outline of how to do this.

Fix a basis $e_1, \ldots, e_n$ of V, and let $\omega = e_1 \wedge \cdots \wedge e_n \in \Lambda^n V$. For $I = (i_1, \ldots, i_k)$, let $\hat{I} = (j_1, \ldots, j_{n-k})$ denote any tuple of indices for which $e_I \wedge e_{\hat{I}} = \omega$. Note that $e_{\hat{I}} \in \Lambda^{n-k}(V)$ only depends on $e_I \in \Lambda^k(V)$ and not the choice of ordering in either tuple. Define $S\colon \Lambda^k(V) \to \Lambda^{n-k} V^*$ by

$$S(e_I) = (-1)^{\binom{k}{2}} \cdot e^*_{\hat{I}}.$$

A few examples are: $S(1) = \omega$, $S(e_1) = e^*_{2\cdots n}$, $S(e_2) = -e^*_{13\cdots n}$, and $S(e_{12}) = -e^*_{3\cdots n}$. Prove that S gives the desired isomorphism of complexes. (Note that the isomorphism between $J_{V,v}$ and $J^*_{V,v}$ is not canonical, though, as it depends on the chosen basis of V.)

Recall the perspective that vector bundle theory is about doing linear algebra fiberwise over a base space. Anything canonical that we can do for vector spaces can be done for vector bundles as well. So let $E \to B$ be a rank n complex vector bundle, and let $s\colon B \to E$ be a section. We get a chain complex of vector bundles

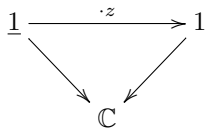
$$0 \longrightarrow \Lambda^0 E \xrightarrow{s\wedge -} \Lambda^1 E \xrightarrow{s\wedge -} \cdots \xrightarrow{s\wedge -} \Lambda^{n-1} E \xrightarrow{s\wedge -} \Lambda^n E \longrightarrow 0$$

which we will denote $J_{E,s}$. For $x \in B$ this chain complex is exact over x provided that $s(x) \neq 0$. Thus it determines an element in $K^0(B, B - s^{-1}(0))$. We can just as well consider the dual complex, which also determines a (likely different) element $[J^*_{E,s}] \in K^0(B, B - s^{-1}(0))$.

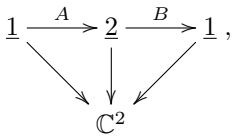
Now let V be a complex vector space of dimension n. Consider the vector bundle $\pi_1\colon V \times V \to V$, with section given by the diagonal map $\Delta\colon V \to V \times V$. Our Koszul complex $J^*_{V\times V,\Delta}$ is exact on $V - 0$, and so defines an element

$$\beta(V) = [J^*_{V\times V,\Delta}] \in K^0(V, V - 0).$$

Example 27.9. One readily checks that $\beta(\mathbb{C})$ is the complex



where the fiber over $z \in \mathbb{C}$ is the chain complex $0 \to \mathbb{C} \xrightarrow{z} \mathbb{C} \to 0$ (multiplication by z). The Koszul complex $\beta(\mathbb{C}^2)$ has the form



where over a point $(z, w) \in \mathbb{C}^2$ we have

$$A = \begin{bmatrix} -w \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} z & w \end{bmatrix}.$$

Finally we look at $\beta(\mathbb{C}^3)$, which has the form

$$\underline{1} \xrightarrow{A} \underline{3} \xrightarrow{B} \underline{3} \xrightarrow{C} \underline{1}$$

where the fiber over $(z, w, u) \in \mathbb{C}^3$ has

$$A = \begin{bmatrix} u \\ -w \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -w & -u & 0 \\ z & 0 & -u \\ 0 & z & w \end{bmatrix}, \quad C = \begin{bmatrix} z & w & u \end{bmatrix}.$$

Let us return to our element $\beta(V) \in K^0(V, V-0)$. If we pick a basis for V then we get isomorphisms

$$K^0(V, V-0) \cong K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^0(D^{2n}, \partial D^{2n}) \cong \widetilde{K}^0(S^{2n}) \\ \cong \widetilde{K}^{-2n}(S^0) = K^{-2n}(pt).$$

Moreover, one checks that any two choices of basis for V give rise to the same isomorphism (essentially because a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ is orientation-preserving). So we may regard $\beta(V)$ as giving us an element of $K^{-2n}(pt)$. Using Exercise 27.6 we have $\beta(V \oplus W) = \beta(V) \cdot \beta(W)$.

The $\beta(\mathbb{C}^n)$ elements are actually *generators* for the groups $K^{-2n}(pt)$:

Theorem 27.10.
$K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^{-2n}(pt) \cong \mathbb{Z}$ and $\beta(\mathbb{C}^n) = (\beta(\mathbb{C}))^n$ is a generator.

Proof. Identify S^2 with $\mathbb{C}P^1$, and regard $\mathbb{C}$ as a subspace of $\mathbb{C}P^1$ in the usual way. Consider the maps

$$\widetilde{K}^0(\mathbb{C}P^1) = K^0(\mathbb{C}P^1, \infty) \xleftarrow{\cong} K^0(\mathbb{C}P^1, \mathbb{C}P^1 - 0) \xrightarrow{\cong} K^0(\mathbb{C}^1, \mathbb{C}^1 - 0).$$

The first is an isomorphism because $\mathbb{C}P^1 - 0$ is contractible, and the second is an isomorphism by excision. Let B be the complex $0 \longrightarrow L \xrightarrow{d} 1 \longrightarrow 0$ on $\mathbb{C}P^1$ that on the fiber over $[z_0 : z_1]$ sends the point (z_0, z_1) to z_0. That is, if we identify L with $\mathbb{C}^2 - 0$ then d is really just the first-coordinate function. This complex is exact on $\mathbb{C}P^1 - 0$ and so defines a class in $K^0(\mathbb{C}P^1, \mathbb{C}P^1 - 0)$.

Mapping B into $K^0(\mathbb{C}P^1)$ gives the class $1 - [L]$, which we know is a generator for $\widetilde{K}^0(\mathbb{C}P^1)$ by Corollary 21.7. Therefore B generates $K^0(\mathbb{C}P^1, \mathbb{C}P^1 - 0)$, and hence its image in $K^0(\mathbb{C}^1, \mathbb{C}^1 - 0)$ is also a generator. But that image is precisely $\beta(\mathbb{C}^1)$.

The comparison isomorphisms

$$K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \xleftarrow{\cong} K^0(D^{2n}, \partial D^{2n}) \xleftarrow{\cong} \widetilde{K}^0(S^{2n})$$

are compatible with products, so Theorem 20.14 and Corollary 21.7 imply that $K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$ is generated by the nth external tensor power of the generator of $K^0(\mathbb{C}^1, \mathbb{C}^1 - 0)$. So $\beta(\mathbb{C}^1)^n$ is a generator, and this class is the same as $\beta(\mathbb{C}^n)$. $\square$

The element $\beta = \beta(\mathbb{C}) \in K^{-2}(pt)$ is often called the **Bott element**, although sometimes this name is applied to $\beta^{-1} \in K^2(pt)$ instead.

Let $p\colon E \to B$ be a rank n complex vector bundle. Consider the pullback p^*E, which is $\pi_1\colon E \times_B E \to E$. This bundle has an evident section given by the diagonal map $\Delta\colon E \to E \times_B E$, and we may consider the Koszul complex with respect to this section. Since Δ is nonzero away from the zero-section of E, this gives us an element in $K^0(E, E-0)$: we define

$$\mathcal{U}_E = [J^*_{p^*E,\Delta}] \in K^0(E, E-0). \tag{27.11}$$

Note that if $x \in B$ and $j_x\colon E_x \hookrightarrow E$ is the inclusion of the fiber, it is completely obvious that $j_x^*(\mathcal{U}_E) = \beta(E_x) \in K^0(E_x, E_x - 0)$.

The element $\mathcal{U}_E$ is not quite our desired Thom class, since the Thom class is supposed to live in $K^{2n}(E, E-0)$ rather than $K^0(E, E-0)$. Of course these groups are the same because of Bott periodicity. To be completely precise, we should define our Thom class to be $\mathcal{U}_E = \beta^{-n} \cdot [J^*_{p^*E,\Delta}]$. However, it is common practice to leave off the factors of β and just do constructions in K^0. We will often follow this practice, but sometimes we will put the factors of β back into the equations in order to emphasize a point. Hopefully this won't be too confusing.

We have now completed our construction of a system of Thom classes in K-theory—that is, we have shown that K-theory admits a complex orientation and thereby proven Theorem 27.4.

Remark 27.12. A choice was made to define the Thom class $\mathcal{U}_E$ in terms of J^* instead of J, and at this point in the story it is not at all clear why that choice was made. In some sense it is just a convention, but we will eventually see in Section 31 that this convention leads to fundamental classes that have a more streamlined connection with algebraic geometry.

Remark 27.13. One can write down Koszul complexes over $\mathbb{R}$ and thereby produce elements in $KO^0(D^n, \partial D^n)$, but for $n \geq 3$ these elements are all equal to zero. This follows from the ring structure and the fact that $KO^0(D^3, \partial D^3) = \widetilde{KO}^0(S^3) = 0$. But the Koszul complexes for $n = 1$ and $n = 2$ do give the generators in those degrees—see the argument at the end of Section 24.

For the quaternionic case one can only write down the rank one Koszul complex $\mathbb{H} \xrightarrow{\cdot q} \mathbb{H}$ over D^4, as the lack of a tensor product is an obstruction to going further. But this is indeed the generator of $KSp^0(D^4, \partial D^4)$.

27.14. **Koszul complexes in algebra.** Now that we have seen Koszul complexes in geometry it seems worthwhile to also look at how they appear in algebra. They turn out to be very important tools in homological algebra.

Let R be a commutative ring, and let $x_1, \ldots, x_n \in R$. Define the Koszul complex $K(x_1, \ldots, x_n; R)$ to be the complex

$$0 \longrightarrow \Lambda^n R^n \xrightarrow{d} \Lambda^{n-1} R^n \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^2 R^n \xrightarrow{d} \Lambda^1 R^n \xrightarrow{d} \Lambda^0 R^n \longrightarrow 0,$$

where the differential d is given by

$$d(e_{i_0} \wedge \cdots \wedge e_{i_k}) = \sum_{j=0}^{k} (-1)^j x_{i_j} (e_{i_0} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}).$$

Note that d is the unique derivation such that $d(e_i) = x_i$. Define the **Koszul homology groups** as $H_*(x_1, \dots, x_n; R) = H_*(K(x_1, \dots, x_n; R))$. We will often abbreviate the sequence $x_1, \dots, x_n$ to just $\underline{x}$, and write $K(\underline{x}; R)$ and so forth. In degree 0 we have $H_0(\underline{x}; R) = R/(x_1, \dots, x_n)$.

In some cases the Koszul complex $K(\underline{x}; R)$ is actually a resolution of $R/(x_1, \dots, x_n)$, and this is perhaps the main reason it is useful. To explain when this occurs we need a new definition. The sequence $x_1, \dots, x_n$ is said to be a **weakly regular sequence** if x_i is a nonzerodivisor in $R/(x_1, \dots, x_{i-1})$ for every $1 \leq i \leq n$ (in particular, x_1 is a nonzerodivisor in R). If in addition $R/(x_1, \dots, x_n) \neq 0$ then one says that $\underline{x}$ is a **regular sequence**. For example, in the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$ the indeterminates $z_1, \dots, z_n$ are a regular sequence.

Remark 27.15. For nice rings R—such as regular rings—a sequence $\underline{x}$ is weakly regular if and only if $\dim R/(x_1, \dots, x_i) = \dim R - i$ for all i. See Exercise 27.30. One can think of this as being a kind of transversality condition: iteratively intersecting with the vanishing locus of each x_i cuts the dimension by exactly one.

Theorem 27.16. *Let $x_1, \dots, x_n \in R$.*

(a) If $x_1, \dots, x_n$ is a weakly regular sequence, then $H_i(\underline{x}; R) = 0$ for all $i \geq 1$.

(b) Suppose R is local Noetherian and $x_1, \dots, x_n \in m$, where m is the maximal ideal. Then $x_1, \dots, x_n$ is a regular sequence if and only if $H_i(\underline{x}; R) = 0$ for all $i \geq 1$.

Proof. The subalgebra of $\Lambda^* R^n$ generated by $e_1, \dots, e_{n-1}$ is a subcomplex of $K(x_1, \dots, x_n; R)$, and is isomorphic to $K(x_1, \dots, x_{n-1}; R)$. The quotient complex has a free basis consisting of wedge products that contain e_n; and in fact wedging with e_n gives an isomorphism between $K(x_1, \dots, x_{n-1}; R)$ and this quotient complex that shifts degrees by one. We can summarize this by saying that there is a short exact sequence of chain complexes

(27.17)
$$0 \to K(x_1, \dots, x_{n-1}; R) \hookrightarrow K(x_1, \dots, x_n; R) \twoheadrightarrow \Sigma K(x_1, \dots, x_{n-1}; R) \to 0.$$

Denote the sequence $x_1, \dots, x_n$ by $\underline{x}$ and $x_1, \dots, x_{n-1}$ by $\underline{x}'$.

Our short exact sequence induces a long exact sequence in homology groups:

$$\cdots \to H_i(\underline{x}'; R) \to H_i(\underline{x}; R) \to H_{i-1}(\underline{x}'; R) \xrightarrow{d} H_{i-1}(\underline{x}'; R) \to H_{i-1}(\underline{x}; R) \to \cdots$$

and one easily checks that the connecting homomorphism is multiplication by $\pm x_n$ (we leave this as Exercise 27.18).

Our proof of part (a) now proceeds by induction on the length of the sequence n. When $n = 1$ the Koszul complex is $0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow 0$, so $H_1(\underline{x}; R) = \operatorname{Ann}_R x_1 = 0$ since x_1 is a nonzerodivisor.

Now assume that we know part (a) for all regular sequences of length $n-1$. By the induction hypothesis and the above long exact sequence, it is easy to see that $H_i(\underline{x}; R) = 0$ for $i \geq 2$. So we only need to worry about $H_1(\underline{x}; R)$, for which we have

$$H_1(\underline{x}'; R) \to H_1(\underline{x}; R) \to H_0(\underline{x}'; R) \xrightarrow{\pm x_n} H_0(\underline{x}'; R) \to H_0(\underline{x}; R) \to 0$$

By induction $H_1(\underline{x}'; R) = 0$, and we know $H_0(\underline{x}'; R) = R/(x_1, \dots, x_{n-1})$. Since x_n is a nonzerodivisor in this ring, the kernel of the map labelled $\pm x_n$ is zero—hence $H_1(\underline{x}; R) = 0$ as well. This completes the proof of (a).

For (b), the point is that the above argument is almost reversible. For $n = 1$ the other direction works without any assumptions on R, because $H_1(x; R) = \mathrm{Ann}(x)$. So assume by induction that the result holds for sequences of length $n-1$. It follows from the long exact sequence we saw in part (a) that there are short exact sequences

$$0 \to H_i(\underline{x}'; R)/x_n H_i(\underline{x}'; R) \to H_i(\underline{x}; R) \to \mathrm{Ann}_{H_{i-1}(\underline{x}'; R)}(x_n) \to 0.$$

The assumption that $H_i(\underline{x}; R) = 0$ implies that $x_n H_i(\underline{x}'; R) = H_i(\underline{x}'; R)$. But $x_n \in m$, so by Nakayama's Lemma this yields $H_i(\underline{x}'; R) = 0$. This holds for all $i \geq 1$, so induction gives that $\underline{x}'$ is a regular sequence. The assumption $H_i(\underline{x}; R) = 0$ also yields (using the other half of the above short exact sequence) that x_n is a nonzerodivisor on $H_{i-1}(\underline{x}'; R)$; so for $i = 1$ this says that x_n is a nonzerodivisor on $R/(x_1, \ldots, x_{n-1})$. Since Nakayama's Lemma gives that $R/(x_1, \ldots, x_n) \neq 0$, we have shown that $\underline{x}$ is a regular sequence. □

Exercise 27.18. In the above proof, work through the definition of the connecting homomorphism for the short exact sequence (27.17) to check that it is multiplication by $\pm x_n$ (the sign can be determined, we just don't care about it).

We can use our knowledge of Koszul complexes to prove the Hilbert Syzygy Theorem, which we encountered back in Section 2:

Theorem 27.19 (Hilbert Syzygy Theorem)**.** *Let L be a field. Then every finite-generated module over $L[x_1, \ldots, x_n]$ has a finite projective resolution.*

Proof. We first prove the result in the graded case. Let $R = L[x_1, \ldots, x_n]$, and grade R by setting $\deg(x_i) = 1$. Assume that M is a finitely-generated, graded R-module. We construct the so-called "minimal resolution" of M: Start by picking a minimal set of homogeneous generators $w_1, \ldots, w_k$ for M. Define $F_0 = R^k$, graded so that the ith generator has degree equal to $\deg(w_i)$. Let $d_0 \colon F_0 \to M$ be the map sending e_i to w_i, and let K_0 be the kernel. Then d_0 preserves degrees, so K_0 is again a graded module. Repeat this process to construct $F_1 \twoheadrightarrow K_0$, let K_1 be the kernel, repeat to get $F_2 \twoheadrightarrow K_1$, and so forth. This constructs a free resolution $F_\bullet \to M$ of the form

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to M \to 0$$

Each differential, when written as a matrix, has entries in the ideal $(x_1, \ldots, x_n)$: this property is independent of the free bases used and follows from the fact that at each stage we chose a *minimal* set of generators.

Next, form the complex $F_\bullet \otimes_R R/(x_1, \ldots, x_n)$ and take homology. Tensoring with $R/(x_1, \ldots, x_n)$ kills all the entries of the matrices and changes every R to an L; so we have

$$L^{b_i} \cong H_i(F_\bullet \otimes_R R/(x_1, \ldots, x_n)) = \mathrm{Tor}_i(M, R/(x_1, \ldots, x_n)).$$

Now we use the fact that we can also compute Tor by resolving $R/(x_1, \ldots, x_n)$ and tensoring with M. By Theorem 27.16(a) $R/(x_1, \ldots, x_n)$ is resolved by the Koszul complex, which has length n: so this immediately yields that $\mathrm{Tor}_i(M, R/(x_1, \ldots, x_n)) = 0$ for $i > n$. It follows that $b_i = 0$ for $i > n$, which says that $F_\bullet$ was actually a finite resolution.

Now we prove the general case, for modules that are not necessarily graded. Choose a presentation of the module

$$R^{b_1} \xrightarrow{A} R^{b_0} \twoheadrightarrow M$$

where A is a matrix with entries in R. Now introduce a new variable x_0 and homogenize A to $\tilde{A}$: that is, multiply factors of x_0 onto the monomials appearing in the entries of A so that all the entries have the same degree. Put $S = L[x_0, \dots, x_n] = R[x_0]$, and let $\widetilde{M}$ be the cokernel of $\tilde{A}$:

$$S^{b_1} \xrightarrow{\tilde{A}} S^{b_0} \longrightarrow \widetilde{M} \longrightarrow 0\ .$$

Note that $\widetilde{M}$ is a graded module over S, and $\widetilde{M} \otimes_S S/(1-x_0) \cong M$.

What we have already proven in the graded case guarantees a finite $S-$free resolution $\widetilde{F}_\bullet \to \widetilde{M} \to 0$. Let $F_\bullet = \widetilde{F}_\bullet \otimes_S (S/(1-x_0))$. This is an R-free chain complex, and $H_0(F_\bullet) \cong M$. Note that $H_i(F) = \operatorname{Tor}_i^S(\widetilde{M}, S/(1-x_0))$, and the Tor-module can also be computed by resolving $S/(1-x_0)$. We use the resolution $0 \to S \xrightarrow{1-x_0} S \to 0$ and immediately find that $H_i(F) = 0$ if $i \geq 2$. We also have that $H_1(F) \cong \operatorname{Ann}_{\widetilde{M}}(1-x_0)$, but such an annihilator is zero for any finitely-generated, graded module. So $F_\bullet \to M$ is a finite free resolution over R. □

Remark 27.20. In the above proof, the deduction of the general case from the graded case was taken from [Eisenbud, Corollary 19.8].

27.21. **The generalized Thom isomorphism.** This final discussion is for readers who like technical details, but can otherwise be skipped. We explain how to prove the Thom isomorphism theorem for complex-oriented cohomology theories when the base space of the bundle is noncompact.

Proposition 27.22. *Let $\mathcal{E}$ be a complex-oriented cohomology theory and let $E \xrightarrow{p} B$ be a rank n complex vector bundle. Then the Thom map $\mathcal{E}^*(B) \to \mathcal{E}^{*+2n}(E, E-0)$ is an isomorphism.*

Proof. When B is compact the result is readily proven by choosing a finite open cover that trivializes the bundle and inducting via Mayer-Vietoris. This same proof works for any finite-dimensional CW-complex, by passing from the skeleton B_{n-1} to B_n via the use of a cover $\{U, V\}$ where V consists of the top halves (plus a little more) of all of the n-cells and U is an open set consisting of B_{n-1} and the bottom-halves (plus a little more) of all the n-cells. So the only question is how to handle the noncompact/infinite-dimensional case. For this, we need to assume that $\mathcal{E}$ comes from a structured ring spectrum in a robust category of spectra that has smash products and function objects $\mathcal{F}(-,-)$. See [BGH] for background material.

Start by letting $f\colon \tilde{B} \to B$ be a CW-approximation to B. It suffices to prove the result for $f^*E \to \tilde{B}$, since $\mathcal{E}^*(-)$ takes weak equivalences to isomorphisms. So at this point we can just replace B with $\tilde{B}$ and thereby assume that B is a CW-complex. By the argument we have already mentioned, we know the result for all of the skeleta B_n. Write E_n for the restriction of E to B_n.

The key step is to realize our Thom map as being induced by a natural map of spectra. For this proof, let $\operatorname{Th} E$ be the homotopy cofiber of $E - 0 \hookrightarrow E$. The Thom class of E is a map $S \to \Sigma^{2n}\mathcal{F}(\operatorname{Th} E, \mathcal{E})$, where S is the sphere spectrum.

For any map $f\colon X \to Y$, if Cf is the mapping cone then there is a canonical map $\tilde{\Delta}\colon Cf \to Y \wedge Cf$ obtained by extending the diagonal on Y. Precisely, start

with composite $Y \to Y \wedge Y \to Y \wedge Cf$. The map obtained by precomposing with f has a canonical null-homotopy (use the cone in the Cf factor) and using that gives the induced map $\tilde{\Delta}\colon Cf \to Y \wedge Cf$. In particular, we have $\tilde{\Delta}\colon \operatorname{Th} E \to E \wedge \operatorname{Th} E$. We can therefore write down the composite

$$\begin{array}{ccccc}
\mathcal{F}(B,\mathcal{E}) & \xrightarrow{\quad p^* \quad} & \mathcal{F}(E,\mathcal{E}) & = & \mathcal{F}(E,\mathcal{E}) \wedge S \\
 & & & & \downarrow{\scriptstyle \mathrm{id} \wedge \operatorname{Th} E} \\
 & & & & \mathcal{F}(E,\mathcal{E}) \wedge \Sigma^{2n}\mathcal{F}(\operatorname{Th} E,\mathcal{E}) \\
 & & & & \downarrow \\
\Sigma^{2n}\mathcal{F}(\operatorname{Th} E,\mathcal{E}) & \xleftarrow{\tilde{\Delta}^*} & \Sigma^{2n}\mathcal{F}(E \wedge \operatorname{Th} E,\mathcal{E}) & \longleftarrow & \Sigma^{2n}\mathcal{F}(E \wedge \operatorname{Th} E, \mathcal{E} \wedge \mathcal{E}).
\end{array}$$

This gives a map of spectra $th_B\colon \mathcal{F}(B,\mathcal{E}) \to \Sigma^{2n}\mathcal{F}(\operatorname{Th} E,\mathcal{E})$ whose induced map on homotopy groups is the Thom homomorphism. This construction of th_B is clearly natural, so that it gives us a map of towers

$$\mathcal{F}(B_\bullet,\mathcal{E}) \xrightarrow{th_{B_\bullet}} \Sigma^{2n}\mathcal{F}(\operatorname{Th} E_\bullet,\mathcal{E})$$

and the homotopy limit is th_B. But the stages th_{B_n} are all known to be weak equivalences, so the homotopy limit is also a weak equivalence. (One can also think of this argument as writing down the Milnor sequences for $\mathcal{E}^*(B)$ and $\mathcal{E}^*(\operatorname{Th} E)$ and arguing that the Thom map gives a commutative diagram of Milnor sequences.) $\square$

∘ Exercises ∘

Regular sequences turn out to be an enormously important concept in commutative algebra. We do not have time to delve into this material in detail, but these exercises contain a very quick tour.

Exercise 27.23. Let R be a ring and let $x_1, \dots, x_n$ be a regular sequence. For any $e_i \geq 1$ prove that $x_1^{e_1}, \dots, x_n^{e_n}$ is also a regular sequence. [Hint: It suffices to handle the case where only one of the e_i's is different than 1.]

Exercise 27.24. Let (R, m) be a local ring. A set of generators for a module M is called **minimal** if its image in M/mM forms a vector space basis over R/m. A free resolution $F_\bullet \to M$ is called a **minimal resolution** if the image of F_i is contained in mF_{i-1}, for all $i \geq 1$.

(a) Prove that a resolution $F_\bullet \to M$ is minimal if and only if any free basis of F_i maps to a minimal set of generators for $\operatorname{im}(F_i \to F_{i-1})$, for all $i \geq 0$ (here $F_{i-1} = M$ by convention).
(b) Prove that minimal resolutions always exist.
(c) Prove that any two minimal resolutions for M are isomorphic.
(d) If $F_\bullet \to M$ is a minimal resolution prove that $\operatorname{rank} F_i = \dim_{R/m} \operatorname{Tor}_i(M, R/m)$, for all i.
(e) Prove that if $J_\bullet \to M$ is any free resolution then it contains the minimal free resolution as a direct summand. [Hint: One hands-on method is to use row- and column-reduction of the matrices for the differentials.]

Exercise 27.25. Let R be a Noetherian ring, let M be a finitely-generated R-module, and let P be a prime ideal. Write $\mathrm{pd}_R M$ for the projective dimension of M over R, i.e. the infinum of lengths of projective resolutions of M.

(a) If M_P is free prove that there is a Zariski open neighborhood U of P such that M_Q is free for all $Q \in U$.
(b) Prove that $\mathrm{pd}_R M \leq n$ if and only if for every exact sequence $Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to M \to 0$ where all Q_i are projective, the kernel of $Q_{n-1} \to Q_{n-2}$ is projective. [Hint: Reduce to the case where R is local. Patch in a minimal free resolution of the kernel and compute $\mathrm{Tor}_*(M, R/m)$.]
(c) Prove that if $\mathrm{pd}_{R_P} M_P \leq n$ then there is a Zariski open neighborhood U of P such that $\mathrm{pd}_{R_Q} M_Q \leq n$ for all $Q \in U$.

Exercise 27.26. Let (R, m) be a Noetherian local ring. Prove that the following are equivalent:

(1) R is regular (every finitely-generated module has finite projective dimension).
(2) R/m has a finite free resolution.
(3) m is generated by a regular sequence.

[Hint: (2)⇒(3) is quite challenging, and perhaps the exercise is really just to go study this in a commutative algebra text. Most approaches follow something like the following outline:

- If m is an associated prime of R and the minimal resolution of R/m is finite, prove that $m = 0$.
- If R has associated primes $P_1, \ldots, P_k$ and m is not one of them, then by prime avoidance $m \not\subseteq m^2 \cup \bigcup_i P_i$. Pick $x \in m$ that is not in the union. Show that R/m has a finite free resolution over R/xR, and then use induction on the dimension of the ring.

But be warned that the second step is not easy.]

Exercise 27.27. Let R be a Noetherian regular local ring. Prove that R_P is also regular, for every prime ideal P.

Exercise 27.28. Let R be a Noetherian ring. Prove that the following are equivalent:

(1) R is regular.
(2) R_P is regular for every prime ideal P.
(3) R_m is regular for every maximal ideal m.

[Hint: For (2)⇒(1) use that $\mathrm{Spec}\, R$ is quasi-compact, i.e. that every open cover has a finite subcover.]

There is a useful class of rings that is weaker than regular but still has many of their nice properties. Let R be a Noetherian ring. Then the following are equivalent:

(1) For every maximal ideal m, $\dim R_m$ equals the length of a maximal regular sequence in R_m.
(2) Every ideal I generated by $\mathrm{ht}\, I$ elements is unmixed (every associated prime of I is a minimal prime of I).

Rings satisfying these properties are called **Cohen-Maculay rings**. For the equivalence, see [BH, Theorem 2.1.6]. It is known that if R is Cohen-Macaulay and P is a prime ideal, then R_P is still Cohen-Macaulay. Also, if R is Cohen-Macaulay and

$x_1, \ldots, x_d$ is a regular sequence then $R/(x_1, \ldots, x_d)$ is still Cohen-Macaulay. For both of these, see [BH, Theorem 2.1.3].

Here are some other nice properties of Cohen-Macaulay rings. If R is Cohen-Macaulay then:

(i) For any ideal $I \subseteq R$, ht I is equal to the maximal length of a regular sequence contained in I.
(ii) R is catenary: if $P \subseteq Q$ are primes, then any two saturated chains of prime ideals connecting P to Q have the same length (this length is therefore equal to $\dim R_Q/PR_Q$).
(iii) If R is local then ht $I = \dim R - \dim R/I$ for every ideal I.
(iv) If R is local then it is equidimensional: $\dim R/P$ is the same for all minimal primes P of R.

See [BH, Corollary 2.1.4, Theorem 2.1.12] and also note that (iv) follows from (iii).

Exercise 27.29.

(a) Check that every regular ring is Cohen-Macaulay.
(b) Find an example of a Cohen-Macaulay ring that is not regular.
(c) If k is a field prove that $k[x,y]/(x^2, xy)$ and $k[x,y,z]/(xy, xz)$ are not Cohen-Macaulay.

Exercise 27.30. Let R be a Noetherian Cohen-Macaulay ring. Prove that $x_1, \ldots, x_n$ is a weakly regular sequence if and only if $\dim R/(x_1, \ldots, x_i) = \dim R - i$ for all $1 \leq i \leq n$. [Hint: Use that the set of zero divisors in R is the union of the associated primes, together with the fact that (0) is unmixed.]

28. Interlude: More algebraic geometry

In preparation for some arguments in Section 31 we need to develop a little more algebraic geometry: we need the concepts of schemes and coherent sheaves, as well as some experience with specific examples. The good news is that we only require the basic ideas here. An extremely brief summary is:

(i) The category of affine schemes is the opposite category of commutative rings. Affine algebraic geometry is just commutative algebra.
(ii) Schemes are affine schemes that have been pasted together along open inclusions, analogously to the way one passes from open subsets of Euclidean space to manifolds.
(iii) Here is an SAT[1]-style analogy:

$$[R : R\text{-modules}] \quad :: \quad [X : \text{quasi-coherent } \mathcal{O}_X\text{-modules}].$$

That is, the category of quasi-coherent $\mathcal{O}_X$-modules is the algebraic geometers' version of the category of R-modules. When $X = \operatorname{Spec} R$ the categories are equivalent. *Coherent* $\mathcal{O}_X$-modules are the analog of *finitely-presented* R-modules, but we will always deal with Noetherian situations and so one might as well read *finitely-generated* here. If you know and love the theory of modules over a ring, the theory of quasi-coherent $\mathcal{O}_X$-modules will soon be your good friend.

This section will give an introduction to these topics.

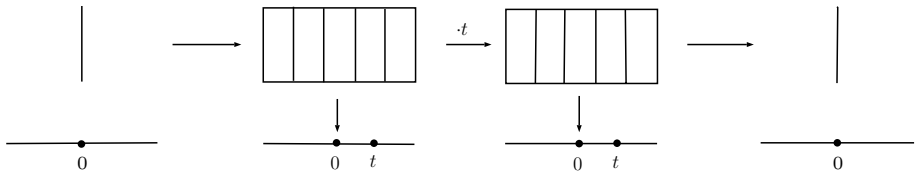
[1]College admissions exam frequently used in the U.S.

28.1. **Background.** Let us begin by recalling what we have learned so far about the correspondence between geometry and algebra:

Geometry	**Algebra**
$\mathbb{C}^n$ or $\mathbb{A}^n_{\mathbb{C}}$	$\mathbb{C}[x_1, \ldots, x_n] = R$
Points $(q_1, \ldots, q_n)$ in $\mathbb{A}^n$	Maximal ideals $(x_1 - q_1, \ldots, x_n - q_n)$
Algebraic sets	Radical ideals
Irreducible algebraic sets	Prime ideals
subvarieties $X = V(P) \subseteq \mathbb{A}^n$	$\mathbb{C}[x_1, \ldots, x_n]/P = R/P$
(Closed) Points in X	Maximal ideals in R/P
Algebraic subsets $V(I) \subseteq X$	Radical ideals in R/P
Irreducible algebraic sets $V(Q) \subseteq X$	Prime ideals in R/P
algebraic vector bundles E on X	f.g. projective R/P-modules $\mathcal{E}$
fiber E_x at the closed point $x \in X$	$\mathcal{E}/m_x\mathcal{E}$ where m_x is the maximal ideal corresponding to x

One of the questions we aim to answer is this: if finitely-generated R/P-projectives on the Algebra side correspond to algebraic vector bundles over $V(P)$ on the Geometry side, what do more general finitely-generated R/P-modules correspond to? The answer will be that they represent "coherent sheaves", but our job will be to figure out what these are.

Here is a related issue that will drive our discussion. Once we have the idea that a chain complex of projectives corresponds to a chain complex of algebraic vector bundles, we immediately notice an issue: exactness in the module setting does not match our usual fiberwise notion of exactness in the geometric setting. Recall that the most basic example is the map of trivial bundles $f\colon \underline{1} \to \underline{1}$ on $\mathbb{A}^1$ that corresponds to the map $\mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x]$. So on the fiber over $t \in \mathbb{C}$ this map is multiplication by t. On fibers these are isomorphisms for every $t \neq 0$, but at $t = 0$ we have both a kernel and a cokernel. We might draw the following picture:



Notice that on the module side $\mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x]$ we have a cokernel, namely $Q = \mathbb{C}[x]/(x)$, but zero kernel. In other words, regarding these maps as chain complexes concentrated in degrees 0 and 1, the complex on the module side is exact in degree 1 but the complex on the geometric side is not. The module Q satisfies $Q/mQ = 0$ for every maximal ideal $m \subseteq \mathbb{C}[x]$ other than $m = (x)$, and $Q/(x)Q = \mathbb{C}$. So the "fibers" of Q are matching the cokernel in our picture, but something is not lining up when we look at kernels.

The fix for this—that is to say, the way to get the geometry to better match the algebra—is that we need to be looking at *stalks* rather than fibers. So let us pause for a moment to say what these are.

If $p\colon E \to X$ is an algebraic vector bundle and $x \in X$, then the stalk $E_{(x)}$ is the vector space of germs of sections of p: that is, it consists of pairs (U, s) where U is a Zariski neighborhood of x and $s\colon U \to X$ is a section of p, subject to the identification $(U, s) = (V, s')$ if $s|_{U\cap V} = s'|_{U\cap V}$. Note that the stalk contains much more information about sections than the fiber does: the fiber just records the values of sections, but the stalk remembers all of the local information (for example, the values of all of the derivatives). For our example of $E = \underline{1}$ on $\mathbb{A}^1$, the stalk $E_{(t)}$ may be identified with the localization $\mathbb{C}[x]_{(x-t)}$.

Returning to our map of bundles $f\colon E \to E$ on $\mathbb{A}^1$, on the stalk over t we have

$$\mathbb{C}[x]_{(x-t)} \xrightarrow{\cdot x} \mathbb{C}[x]_{(x-t)}.$$

When $t \neq 0$ the element x is a unit in $\mathbb{C}[x]_{(x-t)}$, therefore this map is an isomorphism—kernel and cokernel are both zero. When $t = 0$ we have the map $\mathbb{C}[x]_{(x)} \xrightarrow{x} \mathbb{C}[x]_{(x)}$, which has cokernel $\mathbb{C}$ but which is an *injective* map.

The lesson is that if we are to create a geometric version of the category of modules, which in some way expands on our existing picture of algebraic vector bundles, we need to be paying attention to stalks and not just fibers. This provides some motivation for the notions of sheaves and quasi-coherent $\mathcal{O}_X$-modules that we survey below.

28.2. **Crash course on schemes.** We start with the category of affine schemes over $\mathbb{C}$, which is defined to be the opposite of the category of $\mathbb{C}$-algebras. For R a $\mathbb{C}$-algebra we define $\operatorname{Spec} R$ to be the associated affine scheme. Maps from $\operatorname{Spec} R$ to $\operatorname{Spec} T$ are the same as maps of $\mathbb{C}$-algebras from T to R. We define the underlying topological space of $\operatorname{Spec} R$ to be the set of prime ideals in R, equipped with the Zariski topology.

It will be most convenient for us to build up the category of schemes by working backwards. So let us assume given a category of objects called "$\mathbb{C}$-schemes" which will be our preferred setting for doing algebraic geometry. It should contain the affine schemes over $\mathbb{C}$ as a full subcategory, but it should also contain any Zariski open subset of an affine variety. The category should also contain objects, like complex projective space, that are constructed by gluing affine varieties together in certain allowable ways. We will have the property that for any $\mathbb{C}$-scheme X and any point $p \in X$ there is a neighborhood of p that is affine. Denote this imagined category by $\mathcal{S}ch/\mathbb{C}$. Since we will be working entirely over $\mathbb{C}$ throughout this section, we shorten $\mathbb{C}$-scheme to just "scheme" and write $\mathcal{S}ch$ instead of $\mathcal{S}ch/\mathbb{C}$.

Let $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$. We have maps $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ called addition and multiplication, corresponding to the maps of rings $\mathbb{C}[t] \to \mathbb{C}[t_1, t_2]$ given by $t \mapsto t_1 + t_2$ and $t \mapsto t_1 t_2$. For any scheme X we can consider $\mathcal{S}ch(X, \mathbb{A}^1)$, which is a ring via the operations induced by the above two maps. We denote this ring as $\mathcal{O}(X)$ and call it the "ring of algebraic functions" on X. For any open covering $\{U_i\}$ of X we would expect to have an isomorphism

$$\mathcal{O}(X) \xrightarrow{\cong} \lim\Big[\prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U_i \cap U_j)\Big] \tag{28.3}$$

where the top map sends a tuple (f_i) to $(f_i|_{U_i\cap U_j})_{ij}$, and the bottom map sends the same tuple to $(f_j|_{U_i\cap U_j})_{ij}$. The isomorphism captures the notion that functions patch well: giving an algebraic function defined on all of X is the same as giving functions defined on each U_i which agree on the pairwise intersections.

Observe that when $X = \operatorname{Spec} R$ we have

$$\mathcal{O}(X) = \mathcal{S}ch(X, \mathbb{A}^1) = \mathbb{C}-\mathrm{Alg}(\mathbb{C}[t], R) = R.$$

That is, the ring of algebraic functions on $\operatorname{Spec} R$ is just R.

The isomorphism of (28.3) is called a **descent condition**. It is useful to consider other assignments satisfying these conditions, so define a **presheaf** F on X to be an assignment $U \mapsto F(U)$ that maps each open set $U \subseteq X$ to a set (or abelian group, or ring) $F(U)$, together with restriction maps $F(V) \to F(U)$ whenever $U \subseteq V$. Say that a presheaf F is a **sheaf** if it satisfies the descent condition saying that for every open covering $\{U_i\}$ of X the natural map

$$F(X) \xrightarrow{\cong} \lim\Big[\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\Big] \tag{28.4}$$

is an isomorphism. The functor $\mathcal{O}$, when restricted to open subsets of X, is usually written $\mathcal{O}_X$. It is a sheaf of rings on X. More generally, for any scheme Z the functor $U \mapsto \mathcal{S}ch(U, Z)$ will give a sheaf on X.

For any presheaf F and any point $x \in X$, define the **stalk** of F at x to be

$$F_{(x)} = \operatorname*{colim}_{x\in U^{open}\subseteq X} F(U),$$

where the colimit is taken over the inverse system of Zariski neighborhoods of x in X. The stalk represents the truly local information in F near x. The stalk $(\mathcal{O}_X)_{(x)}$ is usually written as $\mathcal{O}_{X,x}$ and called the ring of **germs of algebraic functions** at x. In the case $X = \operatorname{Spec} R$ and $p \in \operatorname{Spec} R$ one has $\mathcal{O}_{X,p} = R_P$

Starting with a presheaf F, there is a universal way to impose the descent condition, resulting in a universal map $F \to \tilde{F}$ where $\tilde{F}$ is a sheaf. This is called the **sheafification** of F. We will not explain the construction, but it is not hard (consult any introductory text on sheaf theory, e.g. [Bredon1]). In addition to the universality, the main property is that $F_{(x)} \to \tilde{F}_{(x)}$ is an isomorphism for every point x. So F and $\tilde{F}$ contain the same local information.

One can mimic the definition of topological vector bundle and define an *algebraic vector bundle* to be a map of schemes $p\colon E \to X$ with certain extra structures, that is locally trivial in the evident way. We omit the details only because it is cumbersome to write them all out. For U an open subset of X write $\Gamma_E(U)$ for the subset of $\mathcal{S}ch(U, E)$ consisting of sections of p. Then Γ_E will be a sheaf on X. Even more, each $\Gamma_E(U)$ will be an abelian group under pointwise addition from the fibers of E, and an $\mathcal{O}_X(U)$-module under pointwise multiplication. Let us generalize this scenario and define a **sheaf of $\mathcal{O}_X$-modules** to be a sheaf of abelian groups F on X together with maps $\mathcal{O}_X(U) \otimes F(U) \to F(U)$ that make $F(U)$ into an $\mathcal{O}_X(U)$-module and are compatible with the restriction maps in $\mathcal{O}_X$ and F. Often one just says "$\mathcal{O}_X$-module" as shorthand for "sheaf of $\mathcal{O}_X$-modules".

Observe that Γ_E isn't just any $\mathcal{O}_X$-module, but it has the additional property of being **locally-free**: for any point x in X there is an open set $x \in U$ such that $\Gamma_E|_U \cong (\mathcal{O}_X|_U)^{\oplus(n)}$ where n is the rank of E. Consequently, note that for any point $x \in X$ one has an isomorphism of stalks $(\Gamma_E)_{(x)} \cong \mathcal{O}^n_{X,x}$.

Now assume $X = \operatorname{Spec} R$ is an affine scheme, and let M be an R-module. For any open set $U \subseteq X$ we have the map of rings $R = \mathcal{O}_X(X) \to \mathcal{O}_X(U)$, and so we can define

$$F_M(U) = \mathcal{O}_X(U) \otimes_R M.$$

This might not be a sheaf, but one can sheafify it to obtain $\widetilde{F_M}$. This will be a sheaf of $\mathcal{O}_X$-modules. We will write $\widetilde{M}$ instead of $\widetilde{F_M}$. For a prime ideal $P \in \operatorname{Spec} R$ one has a natural isomorphism $\widetilde{M}_{(P)} \cong M_P$. That is, the stalks of $\widetilde{M}$ are the usual localizations of M. Of course when $M = R$ one has $\widetilde{R} = \mathcal{O}_X$.

If X is a scheme, define a **quasi-coherent $\mathcal{O}_X$-module** to be a sheaf F of $\mathcal{O}_X$-modules such that for every $x \in X$ there is an affine open neighborhood $x \in U$ for which $F|_U$ is isomorphic to $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M. See Remark 28.5 below for a more intrinsic characterization. Often one says "quasi-coherent sheaf" for "quasi-coherent $\mathcal{O}_X$-module". One can prove that when $X = \operatorname{Spec} R$ the category of quasi-coherent sheaves on X is equivalent to the category of R-modules, via the functor $M \mapsto \widetilde{M}$. A **coherent $\mathcal{O}_X$-module** is a quasi-coherent $\mathcal{O}_X$-module that is locally isomorphic to $\widetilde{M}$ for M a *finitely-generated* $\mathcal{O}_X(U)$-module (recall that by convention all of our rings are Noetherian).

Let qcMod_X and cMod_X denote the categories of quasi-coherent and coherent $\mathcal{O}_X$-modules, respectively. These are both abelian categories, for any scheme X.

Remark 28.5. Here are equivalent versions of the definitions that in some ways are more appealing in that they don't explicitly refer back to the categories of rings and modules on an affine chart. They also don't require any Noetherian conditions. A sheaf of $\mathcal{O}_X$-modules F is **quasi-coherent** if for each $x \in X$ there is an open neighborhood $x \in U$ such that $F|_U$ is a cokernel of free sheaves of $\mathcal{O}_X$-modules:

$$\bigoplus_{\mathcal{I}} (\mathcal{O}_X)|_U \longrightarrow \bigoplus_{\mathcal{J}} (\mathcal{O}_X)|_U \longrightarrow F|_U \longrightarrow 0$$

where $\mathcal{I}$ and $\mathcal{J}$ are indexing sets. Said differently, an $\mathcal{O}_X$-module is quasi-coherent when it is locally presentable. A sheaf of $\mathcal{O}_X$-modules F is **finite-type** if for each $x \in X$ there is an open neighborhood $x \in U$, an $n \geq 0$, and a surjection $(\mathcal{O}_X)^n|_U \twoheadrightarrow F|_U$. Finally, a sheaf of $\mathcal{O}_X$-modules F is **coherent** if it is finite-type and for every open set U, every $n \geq 0$, and every surjection $(\mathcal{O}_X^n)|_U \twoheadrightarrow F$, the kernel is also finite-type.

We will not need these conditions, but include them to give the reader some assurance that the theory we are outlining can all be worked out in rigorous detail and in quite general settings.

An $\mathcal{O}_X$-module M is said to be **locally free** if every $x \in X$ has a neighborhood U such that $M|_U \cong (\mathcal{O}_X|_U)^n$, for some n (which might depend on x). The analog of Swan's theorem says that the assignment $E \mapsto \Gamma_E$ gives an equivalence of categories between algebraic vector bundles over X and the locally-free coherent sheaves on X. In this context it is often called the **Serre-Swan Theorem**. Note that when $X = \operatorname{Spec} R$ this is an equivalence between algebraic vector bundles on X and finitely-generated projective R-modules. Thus, one may regard the category of quasi-coherent sheaves as an expansion of the category of algebraic vector bundles into an abelian category, analogous to the way the category of R-modules is an expansion of the category of finitely-generated projectives. It is often useful to

think of a quasi-coherent sheaf as being like a vector bundle but where there can be certain kinds of "jumps" in the fibers.

The category of quasi-coherent sheaves on a scheme X is the algebraic geometer's version of the category of modules over a ring. It is an abelian category with enough injectives, and so one can do homological algebra. When the scheme is not affine the locally free sheaves are not necessarily projective (see Exercise 28.25), but they are flat and for all the schemes we will consider there are "enough" of them: any coherent sheaf is a quotient of a locally free sheaf, and so one can produce resolutions by locally free sheaves and thereby define left derived functors of tensor product. In short, most of the constructions of commutative algebra carry over from modules to quasi-coherent (or coherent) sheaves, though sometimes requiring a little extra care. See Exercises 30.19 through 30.22 in the next section for some experience with this.

At this point we must face up to the fact that we have not explicitly defined the category $\mathcal{S}ch$. Nevertheless, we have established the main ideas. The usual way to construct it is to take the basic properties we have learned and essentially force them into the definitions. One defines a *locally-ringed space* to be a topological space with a sheaf of rings for which the stalks are rings with unique maximal ideals. Given a ring R one can construct a locally-ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, and then a scheme is defined to be a locally-ringed space that is locally isomorphic to one of these affines. Consult any basic text on algebraic geometry for details, e.g. [Hartshorne] or [Shafarevich2].

Remark 28.6. For a scheme X write $\mathcal{O}(X) = \mathcal{O}_X(X)$. This is the *ring of global functions* on X. There is an adjunction

$$\mathcal{S}ch \underset{\operatorname{Spec}}{\overset{\mathcal{O}(-)}{\rightleftarrows}} \operatorname{Ring}^{op}$$

with $\mathcal{O}$ the left adjoint. That is, we have bijections

$$\mathcal{S}ch(X, \operatorname{Spec} R) = \operatorname{Ring}^{op}(\mathcal{O}(X), R) = \operatorname{Ring}(R, \mathcal{O}(X)). \tag{28.7}$$

The slogan to keep in mind is that $\operatorname{Spec} R$ is "good to map into". The prototypical example is when k is a field, we are working with k-schemes, and we use $\mathbb{A}^1 = \operatorname{Spec} k[t]$: then $\mathcal{S}ch_k(X, \mathbb{A}^1) = k-\operatorname{Alg}(k[t], \mathcal{O}(X)) = \mathcal{O}(X)$.

The above adjunction gives a canonical map $X \to \operatorname{Spec} \mathcal{O}(X)$. This is an isomorphism if and only if X is affine.

Example 28.8. Like the category of topological spaces, the category of schemes admits certain pathological objects. Here is just one example. Take two copies of $\mathbb{A}^n$ and glue them together along the open set $\mathbb{A}^n - 0$, via the identity map. This gives a perfectly fine scheme: every point has an affine neighborhood, namely one of the two $\mathbb{A}^n$'s. But it is a weird object, usually referred to as "affine n-space with the origin doubled". Working over $\mathbb{C}$ and giving the closed points the analytic topology would yield a non-Hausdorff space, typically not something we want to consider. We will sometimes want to rule out such oddities, and the technical word here is that this is a "non-separated" scheme. See Section 28.17 below for this term and others.

28.9. **Local schemes.** If P is a prime in R then we have the localization map $R \to R_P$, which corresponds to a map of schemes $\operatorname{Spec} R_P \to \operatorname{Spec} R$. So if X is a scheme and $p \in X$ then by choosing an affine neighborhood of p we find the canonical map $\operatorname{Spec} \mathcal{O}_{X,p} \to X$.

Recall that $\mathcal{O}_{X,p}$ is the ring of germs of algebraic functions at p. The residue field is denoted $\kappa(p)$; it can loosely be thought of as the ring of *values* of algebraic functions at p. We have the two maps

$$\operatorname{Spec} \kappa(p) \to \operatorname{Spec} \mathcal{O}_{X,p} \to X.$$

In general, a map from Spec of a field into X is called a **geometric point** of X (in contrast to the set-theoretic points of X). Every set-theoretic point p has a canonical geometric point associated to it, given by the above composite. If $\kappa(p) \subseteq F$ is a field extension then the composite $\operatorname{Spec} F \to \operatorname{Spec} \kappa(p) \to X$ is a different geometric point that has the same image, i.e. the same underlying set-theoretic point.

When we say "point" with no modifying adjective, we mean either a set-theoretic point or the canonical geometric point associated to it.

The scheme $\operatorname{Spec} \mathcal{O}_{X,p}$ behaves like an infinitesimal Zariski neighborhood of p. This construction has no real parallel in topology, though the intuitive idea is appealing. For a point p consider the inverse system of all open neighborhoods of p, where the maps are the inclusions. For typical spaces these neighborhoods appear to "converge" in some vague sense—one gets the impression that there is information common to all of them. One might naively hope to isolate this common information by passing to the inverse limit. In topology, though, taking the actual inverse limit just gives the one-point space $\{p\}$ and so has no information in it. But in schemes the inverse limit is nontrivial, for two reasons. First, Zariski neighborhoods are "big" and so there is more than just a single (set-theoretic) point common to all of the neighborhoods. Second, schemes remember not just the points but the rings of functions; the inverse limit translates to the direct limit at the ring level, and such direct limits are usually nontrivial.

Exercise 28.10. Prove that $\operatorname{Spec} \mathcal{O}_{X,p}$ is the inverse limit of the system of all Zariski open neighborhoods of p in X. [Hint: Use (28.7) and the fact that R_P is the direct limit of the localizations R_f for $f \notin P$.]

28.11. **Projective space.** Continuing at a fairly intuitive and non-rigorous level, let us consider what $\mathbb{C}P^n$ will be like as a scheme. The closed points will of course correspond to points $[x_0 : \cdots : x_n]$ with $x_i \in \mathbb{C}$ and not all x_i equal to zero, subject to the usual relation that $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for any nonzero λ. There should be Zariski open sets U_i corresponding to the condition $x_i \neq 0$, and these will cover $\mathbb{C}P^n$.

Note that the fractions $\frac{x_j}{x_i}$ represent algebraic functions defined on U_i. We expect $\mathcal{O}(U_i)$ to be the polynomial ring $\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$, reflecting the fact that U_i is isomorphic to $\mathbb{A}^n$.

It is somewhat cumbersome, but one can define the scheme $\mathbb{P}^n$ to be the result of pasting the affine schemes $U_0, \ldots, U_n$ together along their evident intersections. For example, $\mathbb{P}^1$ consists of the two affines $\operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}]$ and $\operatorname{Spec} \mathbb{C}[\frac{x_1}{x_0}]$ glued together along the identity map of $\operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}, \frac{x_1}{x_0}]$; or said differently, this is two copies of $\operatorname{Spec} \mathbb{C}[t]$ glued together along the map $\operatorname{Spec} \mathbb{C}[t, t^{-1}] \to \operatorname{Spec} \mathbb{C}[t, t^{-1}]$ that sends $t \mapsto \frac{1}{t}$.

To give a quasi-coherent sheaf on $\mathbb{P}^n$ we need to give sheaves on each U_i which patch over the intersections. Each U_i is affine so a quasi-coherent sheaf there is the same as a module over $\mathcal{O}(U_i)$. So what it boils down to is that for each i we need a module M_i over $\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ such that for all i and j, $(\frac{x_j}{x_i})^{-1}M_i = (\frac{x_i}{x_j})^{-1}M_j$ as modules over $\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}] = \mathcal{O}(U_i \cap U_j)$. One evident way to do this is to start with a graded $\mathbb{C}[x_0, \ldots, x_n]$-module M, and to define M_i to be the degree zero elements in $M[x_i^{-1}]$. The compatibilities then become automatic. Call this quasi-coherent sheaf F_M.

Let R denote the graded ring $\mathbb{C}[x_0, \ldots, x_n]$. Consider the graded module $M = R(d)$, which is a copy of R but with the generator 1 shifted into degree $-d$ (so that $M_k = R_{k+d}$). Then the degree 0 piece of $M[x_i^{-1}]$ is the span of the degree d monomials in $x_0, \ldots, x_n, x_i^{-1}$. These monomials are the same as algebraic sections of the line bundle $(L^{\otimes d})^* \to \mathbb{C}P^n$, where $L \to \mathbb{C}P^n$ is the tautological bundle: the monomial m corresponds to the section that sends $[x_0 : \cdots : x_n]$ to the functional given by $\underline{x} \otimes \cdots \otimes \underline{x} \mapsto m(\underline{x})$, where $\underline{x} = (x_0, \ldots, x_n)$. Note that the functional that sends $\lambda\underline{x} \otimes \cdots \otimes \lambda\underline{x} \mapsto m(\lambda\underline{x})$ is the same functional, so that this is well-defined. The $\mathcal{O}_{\mathbb{C}P^n}$-module F_M (from the last paragraph) is locally free and corresponds under the Serre-Swan theorem to the algebraic line bundle $(L^*)^{\otimes d}$.

Lkewise, now consider $M = R(-d)$, with the generator 1 up in degree d. Then the degree 0 piece of $M[x_i^{-1}]$ consists of degree $-d$ monomials in $x_0, \ldots, x_n, x_i^{-1}$, and these correspond to algebraic sections of $L^{\otimes d} \to \mathbb{C}P^n$: a monomial m corresponds to the section sending $[x_0 : \cdots : x_n]$ to the point $m(\underline{x}) \cdot \underline{x} \otimes \cdots \otimes \underline{x}$ (d tensor factors). Again, note that in this description $[\lambda x_0 : \cdots : \lambda x_n]$ is sent to the same point, because a λ^d from the tensors cancels the λ^{-d} from the monomial. The locally free sheaf F_M corresponds to $L^{\otimes d}$ under the Serre-Swan theorem.

Algebraic geometers use the notation $\mathcal{O}(d)$ for the coherent sheaf corresponding to $R(d)$ and $\mathcal{O}(-d)$ for the coherent sheaf corresponding to $R(-d)$. The sheaf $\mathcal{O}(1)$ (identified with the corresponding algebraic vector bundle) is called the **hyperplane bundle** on projective space, whereas $\mathcal{O}(-1)$ is the **tautological line bundle**.

Example 28.12. Maps of quasi-coherent sheaves are the evident globalization of maps between modules. Giving a map $\mathcal{O} \to \mathcal{O}(-1)$ is equivalent to specifying, for each $0 \leq i \leq n$, a map of $\mathcal{O}(U_i)$-modules

$$\mathcal{O}(U_i) \to \mathbb{C}\langle \text{degree -1 monomials in } x_0, \ldots, x_n, x_i^{-1} \rangle$$

such that we have agreement on the overlaps. But the above $\mathcal{O}(U_i)$-module map is determined by the image of 1, so this amounts to giving a collection of degree -1 homogeneous polynomials $\phi_i \in \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]$ with the property that any pair ϕ_i and ϕ_j map to the same element in the ring where x_i and x_j are both inverted. It's easy to see that this only happens if all ϕ_i are zero. This shows $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(-1)) = 0$.

In contrast, let us next consider $\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O})$. To give a map $\mathcal{O}(-1) \to \mathcal{O}$ is equivalent to giving compatible $\mathcal{O}(U_i)$-module maps

$$\mathbb{C}\langle \text{deg. -1 monomials in } x_0, \ldots, x_n, x_i^{-1} \rangle \to \mathbb{C}\langle \text{deg. 0 monomials in } x_0, \ldots, x_n, x_i^{-1} \rangle.$$

Multiplication by any element of $\mathbb{C}\langle x_0, \ldots, x_n \rangle$ gives such a compatible collection, and a moment's thought shows that this is the only way to do so. Therefore $\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \cong \mathbb{C}\langle x_0, \ldots, x_n \rangle$.

Exercise 28.13. Generalize the above example to show that $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j))$ is isomorphic to the degree $j - i$ piece of the graded ring $\mathbb{C}[x_0, \ldots, x_n]$.

28.14. **The secret behind gradings.** Why do *graded* rings and modules appear when considering projective space? This is often a source of puzzlement when first learning this subject. Here is a sketch of an explanation.

Consider the scheme $\mathbb{A}^1 - 0 = \operatorname{Spec} \mathbb{C}[t^{\pm 1}]$. There is a multiplication map $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to \mathbb{A}^1 - 0$ making this into a group scheme: the dual map of rings is $\mathbb{C}[t^{\pm 1}] \mapsto \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ given by $t \mapsto t_1 t_2$. The unit is the point $\operatorname{Spec} \mathbb{C} \to \mathbb{A}^1 - 0$, corresponding to the map of rings $\mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ sending $t \to 1$. When thought of as a group one often writes this scheme as $\mathbb{G}_m$, for the "multiplicative group".

The scheme $\mathbb{C}P^n$ should be obtained from $\mathbb{A}^{n+1} - 0$ by quotienting out by the evident $\mathbb{G}_m$-action. So we can try to understand $\mathbb{C}P^n$ via the projection map $\pi\colon \mathbb{A}^{n+1} - 0 \to \mathbb{C}P^n$. For example, algebraic functions on $U \subseteq \mathbb{C}P^n$ should be algebraic functions on $\pi^{-1}(U)$ that are invariant under the $\mathbb{G}_m$-action.

Let $X = \operatorname{Spec} R$ be an affine scheme with a $\mathbb{G}_m$-action. We claim that the action induces a natural $\mathbb{Z}$-grading on R (and in fact is equivalent to a $\mathbb{Z}$-grading). The action is a map $\mu\colon \mathbb{G}_m \times X \to X$, which on the level of rings is $\mu^*\colon R \to R[t, t^{-1}]$. Write $\mu^*(r) = \sum r_n t^n$. The fact that the action is unital translates to the condition $\sum_n r_n = r$, and the action being associative translates into the condition that $(r_n)_k$ is r_n if $k = n$ and 0 otherwise. Setting $R_n = \{r_n \,|\, r \in R\}$ then gives a grading for R.

Let $f \in R$, and regard this as a function $f\colon X \to \mathbb{A}^1$. This map is equivariant (with $\mathbb{A}^1$ having the trivial action) if the following diagrams commute:

$$\begin{array}{ccc} \mathbb{G}_m \times X & \xrightarrow{\mu} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & \mathbb{A}^1 \end{array} \qquad\qquad \begin{array}{ccc} R[t, t^{-1}] & \xleftarrow{\mu^*} & R \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle f^*} \\ R & \xleftarrow{f^*} & \mathbb{C}[s] \end{array}$$

(the second diagram is the dual of the first). In the second diagram t maps to $\sum_n f_n t^n$ along the up-top route, and to f along the bottom-up route. So f is equivariant precisely when $f_n = 0$ for all $n \neq 0$, or equivalently $f \in R_0$. More generally, $f \in R_n$ corresponds to the condition that can be written as $f(\lambda x) = \lambda^n f(x)$ for all $\lambda \in \mathbb{C} - 0$ and $x \in X$.

Exercise 28.15. Verify the last sentence of the above remark. [Hint: First determine the ring map $\mathbb{C}[s] \to \mathbb{C}[s, t, t^{-1}]$ that corresponds to the map of schemes $\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $(\lambda, u) \mapsto \lambda^n u$.]

Exercise 28.16. Consider $\mathbb{A}^1$ as a monoid scheme, where the operation is multiplication.

(a) Check that giving an action $\mathbb{A}^1 \times \operatorname{Spec} R \to \operatorname{Spec} R$ is equivalent to giving an $\mathbb{N}$-grading on the ring R.
(b) Let $R = \mathbb{C}[x, y]$ where x and y have degree 1, and let $S = \mathbb{C}[x, y]$ where x has degree 0 and y has degree 1. This gives two actions of $\mathbb{A}^1$ on the affine plane. Explain geometrically what the difference is between the two actions.

Now let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring. Then $X = \operatorname{Spec} R$ has a $\mathbb{A}^1$-action (hence also a $\mathbb{G}_m$-action) and we can try to construct an associated projective space $\operatorname{Proj} R$. Note that R_0 is a retract of R and therefore $\operatorname{Spec} R_0$ is a retract of X. This

subspace has trivial $\mathbb{G}_m$-action and plays the role of an "origin" inside X, which we will want to ignore when constructing the projective space. The subvarieties of $\operatorname{Proj} R$ will correspond to subvarieties of X that are invariant under the $\mathbb{G}_m$-action, but excluding the "irrelevant" subvarieties of $\operatorname{Spec} R_0$. This translates to prime ideals of R that are *graded* (equal to the direct sum of their homogeneous pieces) and that do not contain the ideal $R_+ = \bigoplus_{n\geq 1} R_n$. So as a set we define $\operatorname{Proj} R$ to be the set of such graded prime ideals, and we equip this with the usual Zariski topology where the closed sets are of the form $V(I)$ for $I \subseteq R$ a graded ideal. To describe $\operatorname{Proj} R$ as a scheme we need to give the sheaf of regular functions $\mathcal{O}_{\operatorname{Proj} R}$, but it is enough to describe this on a basis of open sets. For a homogeneous element $f \in R$ let $D(f)$ be the set of primes in $\operatorname{Proj} R$ that do not contain f, which is a Zariski open. The ring $\mathcal{O}_{\operatorname{Proj} R}(D(f))$ will then be defined to be the ring of degree 0 elements in the localization R_f (note that these correspond to functions on $\operatorname{Spec} R_f \subseteq X$ that are invariant under the $\mathbb{G}_m$-action).

We have gone quickly through the details of the construction of $\operatorname{Proj} R$ because we will only need that such a thing exists, and very little about the intricacies behind it.

As two basic examples we note that if k is a field and $R = k[x_0, \ldots, x_n]$ with the x_i in degree 1 then $\operatorname{Proj} R$ is projective space $\mathbb{P}^n_k$. If $R = k[x_1, \ldots, x_n, y_0, \ldots, y_r]$ with the x_i in degree 0 and the y_i in degree 1, then $\operatorname{Proj} R$ is $\mathbb{A}^n_k \times \mathbb{P}^r_k$. We will see further examples in Section 29.1 where we talk about blowups.

28.17. **Properties of schemes.** We collect here a few common properties of schemes that will be useful to us.

integral: every affine open is Spec of an integral domain (equivalently, X is connected and every point has an affine open neighborhood that is Spec of a domain).

locally Noetherian: every point has an affine open neighborhood that is Spec of a Noetherian ring (equivalently, every affine open is Spec of a Noetherian ring).

Noetherian: the scheme admits a finite cover by open affines that are each Spec of a Noetherian ring (equivalently, the scheme is locally Noetherian and also quasi-compact as defined below.)

regular: every point has an affine open neighborhood that is Spec of a regular Noetherian ring (equivalently, X is locally Noetherian and all its local rings are regular; equivalently, every affine open is Spec of a Noetherian regular ring).

quasi-compact: the underlying topological space is compact (equivalently, X has a finite cover by affine opens). [The prefix "quasi" is used as a reminder that the underlying topological spaces need not be Hausdorff.]

Similarly to the category of topological spaces, the category of schemes admits plenty of pathological objects. The following terms are largely about eliminating pathologies of various types. It is convenient to phrase these as properties of morphisms, though keep in mind that in many applications the target scheme will be just $\operatorname{Spec} k$.

quasi-compact morphism: $X \to Y$ is quasi-compact if the inverse image of every quasi-compact open is quasi-compact (equivalently, Y has an affine open covering $\{U_i\}$ such that each $f^{-1}(U_i)$ is quasi-compact).

separated morphism: $X \to Y$ is separated if the diagonal map $X \to X \times_Y X$ is a closed immersion (the immersion property is automatic, so this is equivalent to saying that the image is Zariski closed). [When $Y = \operatorname{Spec} k$ (k a field) this should be thought of as an analog of the Hausdorff condition in topology; it removes the pathology of having two points that are "infinitely close" to each other, like in Example 28.8. In topology, a space X is Hausdorff if and only if the diagonal $X \to X \times X$ has closed image.]

finite-type morphism: $X \to Y$ is finite-type if it is quasi-compact and for every $x \in X$ there is an affine neighborhood $\operatorname{Spec} S = U \subseteq X$ and an affine open $\operatorname{Spec} R = V \subseteq Y$ such that $f(U) \subseteq V$ and the associated map $R \to S$ exhibits S as a finitely-generated R-algebra.

quasi-separated morphism: $X \to Y$ is quasi-separated if the diagonal map $X \to X \times_Y X$ is quasi-compact.

Remark 28.18. The quasi-separated property is one that has no familiar analog in topology, and the name is confusing because there is no apparent connection to a separation condition. When $Y = \operatorname{Spec} R$, it turns out that a map $X \to Y$ is quasi-separated if and only if the intersection of any two affine opens in X is quasi-compact. This is a technical condition that is exactly what is needed to make certain kinds of patching arguments work. If $X \to Y$ is separated it has the further property that the intersection of any two affine opens is affine, so separated implies quasi-separated.

The following term is not about ruling out pathologies, but rather is an attempt to capture the topological idea of a map with compact fibers (see Section A.23 for topological background):

proper morphism: a morphism that is separated, finite type, and universally closed (the latter condition means that the map is closed—sends closed subsets to closed subsets—and that every pullback is also closed).

Remark 28.19. The above definition of proper morphism feels esoteric, and it can be difficult to see the meaning here. It is a good idea to look up the topological notions of separated and proper maps in Appendix A for comparison. In topology "separated" is the relative form of the Hausdorff property, and "proper" is the relative form of compactness. Note, however, that in algebraic geometry the term "proper" is the analog of what topologists would call "separated and proper". This can be more than a little confusing.

Standard examples of proper morphisms are inclusions of closed subschemes, projection maps $\mathbb{P}^n_X \to X$, as well as maps built up from these via compositions or pullbacks.

Schemes over a base

Often one has a fixed base scheme B and works in the general context of schemes over B, i.e. schemes X with a structure map $X \to B$. This category is written as $\mathcal{S}ch \downarrow B$. Often B is $\operatorname{Spec} k$ where k is a field, and in this book we usually have $B = \operatorname{Spec} \mathbb{C}$. The phrase "$B$-scheme" is synonymous with "scheme over B". A B-scheme is said to be separated if the structure map $X \to B$ is a separated morphism, and likewise for any other property of morphisms.

Schemes over a field

When k is a field the category of k-schemes (or schemes over $\operatorname{Spec} k$) is $\mathcal{S}ch \downarrow \operatorname{Spec} k$. The following terminology is used in this setting:

projective: a scheme that is isomorphic to a closed subscheme of some projective space $\mathbb{P}^n_k$.

quasi-projective: a scheme that is isomorphic to an open subscheme of a projective scheme.

variety: an integral scheme such that the map $X \to \operatorname{Spec} k$ is separated and of finite type.

complete variety: a variety such that $X \to \operatorname{Spec} k$ is proper. [The phrases "complete variety" and "proper variety" are therefore synonymous. The adjective "complete" has historical precedence, dating from a time before the modern notion of properness was established.]

smooth variety: this is a condition about nondegeneracy of certain Jacobian matrices formed from the equations defining the scheme, and can also be described in terms of sheaves of differentials. Over characteristic zero fields it is equivalent to the variety being regular, and so we will use the two terms interchangeably in this context.

Remark 28.20. At its most fundamental level, algebraic geometry is about understanding solutions sets of algebraic equations. This perhaps gives the impression that all schemes of interest are quasi-projective, and it should be warned that this is not the case. In dimension three and above it is very easy to glue quasi-projective schemes together in mild ways and end up with something that is not quasi-projective. See [Shafarevich2, Chapter 6.2.3] for a nice discussion.

28.21. **Pulling back and pushing forward $\mathcal{O}_X$-modules.** If $R \to S$ is a map of rings then there are two associated functors on the categories of modules. The "extension-of-scalars" functor $(-) \otimes_R S \colon \operatorname{Mod}_R \to \operatorname{Mod}_S$ takes finitely-generated modules to finitely-generated modules and is right exact. The "restriction-of-scalars" functor $\operatorname{Mod}_S \to \operatorname{Mod}_R$, that takes an S-module and regards it as an R-module, is exact but only preserves finite-generation if S is module-finite over R.

For a map of schemes $f \colon Y \to X$ one has the **pullback functor** $f^* \colon \operatorname{qcMod}_X \to \operatorname{qcMod}_Y$, that locally is modeled by the tensor product. That is to say, if $\mathcal{F}$ is a

quasi-coherent $\mathcal{O}_X$-module and $\operatorname{Spec} R = V \subseteq X$ is an affine open with an R-module M and an isomorphism $\widetilde{M} \cong \mathcal{F}|_V$, and $\operatorname{Spec} S = U \subseteq Y$ is an affine open such that $f(U) \subseteq V$, then there is an isomorphism $\widetilde{(S \otimes_R M)} \cong (f^*\mathcal{F})|_V$. The pullback functor is right exact, and when X and Y are locally Noetherian it sends coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_Y$-modules.

If $f\colon Y \to X$ is slightly nice (quasi-compact and quasi-separated, see Section 28.17)) there is also the corresponding **direct-image functor** $f_*\colon \operatorname{qcMod}_Y \to \operatorname{qcMod}_X$. This is the right adjoint to f^*. In general f_* does not preserve coherence, but it does in the case where X is locally Noetherian and f is proper [EGA3, Theorem 3.2.1].

For careful definitions of f^* and f_* one place to start is [Hartshorne, Section II.5]. But the main thing to remember is that locally these are the familiar extension and restriction functors from commutative algebra.

Schemes have open and closed subschemes, which restrict to the usual notions for affine schemes. If $i\colon Z \hookrightarrow X$ is a closed subscheme then in an affine chart $U = \operatorname{Spec} R$ for X, Z corresponds to $\operatorname{Spec} R/I$ for some ideal I. In this case the direct image functor i_* is the analog of taking an R/I-module and thinking of it as an R-module. If $\mathcal{F}$ is a quasi-coherent sheaf on Z then the stalks of $i_*\mathcal{F}$ satisfy

$$(i_*\mathcal{F})_{(x)} = \begin{cases} \mathcal{F}_{(x)} & \text{if } x \in Z, \\ 0 & \text{if } x \notin Z. \end{cases} \tag{28.22}$$

For this reason $i_*\mathcal{F}$ is also called the "extension-by-zero" of $\mathcal{F}$ (note that for inclusions of non-closed subschemes the direct image and the extension-by-zero are different, so be careful).

The structure sheaf $\mathcal{O}_Z$ is a sheaf on Z, and the direct image $i_*\mathcal{O}_Z$ is often also written as $\mathcal{O}_Z$, by abuse. (This is the analog of taking the ring R/I and denoting it by the same symbol when thinking of it as an R-module). When $Z = \{z\}$ is a closed point then $i_*\mathcal{O}_Z$ is a sheaf with a single nonvanishing stalk, referred to as the **skyscraper sheaf** on X supported at z with stalk $\kappa(z)$.

∘ Exercises ∘

Exercise 28.23. Let R be an $\mathbb{N}$-graded ring. For any $d > 0$ define $R\langle d\rangle \subseteq R$ to be the subring $\bigoplus_{k\geq 0} R_{dk}$. Prove that the induced map $\operatorname{Proj} R \to \operatorname{Proj} R\langle d\rangle$ is an isomorphism of schemes.

Exercise 28.24. Let $R = \mathbb{F}_2[X,Y]/(X^2 + XY + Y^2)$, regarded as a graded ring where X and Y have degree 1. The ring R is one-dimensional, so $\operatorname{Proj} R$ is a zero-dimensional scheme. Prove that $\operatorname{Proj} R \cong \operatorname{Spec} \mathbb{F}_4$.

Exercise 28.25. Over affine schemes, locally free $\mathcal{O}_X$-modules are projectives in the category of quasi-coherent $\mathcal{O}_X$-modules. This need not be true for more general schemes; in fact, it need not even be true that $\mathcal{O}_X$ is itself projective. When $X = \mathbb{P}^1$ with coordinates $[x : y]$ prove that the diagram

$$\begin{array}{ccc} & & \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ & & \big\downarrow{\scriptstyle (x,y)} \\ \mathcal{O} & \xrightarrow{\quad\text{id}\quad} & \mathcal{O} \end{array}$$

does not have a lifting, despite the vertical map being a surjection. (See Example 28.12 for the components of the vertical map.)

Exercise 28.26. Let G be a finite group acting on a commutative ring R. It was shown in Exercise 4.22 that as a map of topological spaces $\operatorname{Spec} R \to \operatorname{Spec} R^G$ is the quotient map by the G-action. Prove that this is also true in the category of schemes: that is, prove that if Z is a scheme with a trivial G-action then a map $\operatorname{Spec} R \to Z$ descends in a unique way to give a map $\operatorname{Spec} R^G \to Z$. One approach is to follow these steps:

(a) For any $f \in R^G$ prove that the natural map $f^{-1}(R^G) \to (f^{-1}R)^G$ is an isomorphism. [Hint: Surjectivity is a little tricky. Prove that if r/f is G-invariant then there exists N such that $f^N r$ is G-invariant in R.]
(b) Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} R^G$ and $\pi\colon X \to Y$ be the projection. For $f \in R^G$ write $D_{Y,f} = Y - V(f)$ and $D_{X,f} = X - V(f) = \pi^{-1}(D_{Y,f})$. Prove that for any $f \in R^G$ one has $\mathcal{O}_Y(D_{Y,f}) = \mathcal{O}_X(D_{X,f})^G$.
(c) Prove that for any open set $U \subseteq Y$ one has $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}U)^G$.
(d) Think through the mechanics of proving that our original map $f\colon X \to Z$ factors uniquely through Y. Let W be an affine open in Z, let $V = f^{-1}W$, and let $U = \pi(V)$. Prove that U is open in Y and then use Remark 28.6 coupled with the previous parts of the exercise.

Exercise 28.27. Let R be an $\mathbb{N}$-graded ring and let G be a finite group acting on R. Prove that $\operatorname{Proj}(R^G)$ is the categorical quotient $(\operatorname{Proj} R)/G$ in the category of schemes.

29. Interlude, Part 2: Blowups and normal cones

If X is a smooth scheme and $Z \hookrightarrow X$ is a closed, smooth subscheme, we will need to work with the algebraic normal bundle of Z in X. Even when the schemes are not smooth there are analogs called the *normal family of vector spaces* and the *normal cone*: these coincide at nonsingular points and agree with what one expects for the normal bundle there, but at the singular points they give two different levels of information. As the name suggests, the fibers of the normal cone are just cones (spaces with an $\mathbb{A}^1$-action), not vector spaces.

A related idea is that of the *blowup* of X at Z, where one removes Z but then sews in a copy of the projectivized normal cone instead. Points on Z are "blown up" and replaced by the set of normal directions emanating from them. This is a fundamental construction in algebraic geometry. Somewhat unexpectedly, it is a little more convenient to talk about blowups *first* and then circle back to the discussion of normal cones.

29.1. **Blowups.** In the next section we will need to use blowup varieties, so in preparation we discuss them here. Even if one is mostly interested in affine schemes, blowups are an important tool and the full power of general schemes is needed to discuss them.

Let X be a smooth complex variety and let $A \hookrightarrow X$ be a closed subvariety, which for convenience we assume to be smooth as well. The blow-up $\operatorname{Bl}_A(X)$ of X at A is an algebraic variety that homotopically corresponds to removing A and then sewing in a copy of the projectived normal bundle in its place. That is, let V be a tubular

neighborhood of A, with associated homeomorphism $V \cong N_{X/A}$. Then there is a homotopy equivalence

$$\mathrm{Bl}_A(X) \simeq (X - A) \amalg_{(V-A)} \mathbb{P}(N) \tag{29.2}$$

Here the map $V - A \to \mathbb{P}(N)$ is the map $N - 0 \to \mathbb{P}(N)$ that sends any nonzero element of a fiber N_a to the corresponding line it spans, regarded as an element of $\mathbb{P}(N_a)$. The homotopy equivalence in (29.2) can in fact be upgraded to a homeomorphism, but the usual point-set model for the blowup is a little different and so this can create confusion. We describe the usual model next.

Let us start with the case $X = \mathbb{C}^n$ and $A = \{\underline{0}\}$. The idea of the blowup $\mathrm{Bl}_{\underline{0}}\,\mathbb{C}^n$ is that away from $\underline{0}$ it is exactly $\mathbb{C}^n - \{\underline{0}\}$, but the point $\underline{0}$ is "blown up" into a whole family of points representing the normal directions around $\underline{0}$. We can construct this blowup by doing something clever with projective space. Begin by observing that given a point $\underline{x} \in \mathbb{C}^n$ and a line $[\underline{Y}] = [Y_1 \colon \ldots \colon Y_n] \in \mathbb{C}P^{n-1}$ then $\underline{x} \in [\underline{Y}]$ if and only if $x_i Y_j = x_j Y_i$ for all pairs $1 \leq i, j \leq n$. To see this, note that for $\underline{x} = \underline{0}$ both conditions are vacuous; but if instead some $x_i \neq 0$ then the equations are equivalent to $\underline{Y} = (\frac{Y_i}{x_i})\underline{x}$, which in particular implies that $Y_i \neq 0$ and hence $\underline{x} = (\frac{x_i}{Y_i})\underline{Y}$.

Define $B = \mathrm{Bl}_{\underline{0}}\,\mathbb{C}^n$ to be the subset of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ consisting of pairs $(\underline{x}, [\underline{X}])$ such that $\underline{x} \in [\underline{X}]$, or equivalently $x_i X_j = x_j X_i$ for all i, j. Observe that when $\underline{x} \neq 0$ then $[\underline{X}]$ is determined uniquely and is equal to $[\underline{x}]$, whereas when $\underline{x} = 0$ then $[\underline{X}]$ can be any line in $\mathbb{C}^n$. The point $\underline{x} = 0$ has been "blown up" into an entire $\mathbb{C}P^{n-1}$ in B. Note that B, being defined by algebraic equations, is a subvariety of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$.

The projection map $\pi_1 \colon \mathbb{C}^n \times \mathbb{C}P^{n-1} \to \mathbb{C}^n$ restricts to give $\pi \colon B \to \mathbb{C}^n$, called the standard projection associated to the blowup. Observe that the fiber over $\underline{0}$ is $E = \mathbb{C}P^{n-1}$; this is usually called the **exceptional divisor** (we discuss this terminology in a moment). Also note that we have the pushout diagram

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{\underline{0}\} & \longrightarrow & \mathbb{C}^n. \end{array}$$

That is, squashing E back down to a point recovers the original space $\mathbb{C}^n$.

Exercise 29.3. It is useful to think about the case $n = 2$. Consider the sequence $p_n = (\frac{1}{n}, \frac{3}{n})$, which of course converges to $(0, 0)$ in $\mathbb{C}^2$. This sequence lifts uniquely into B. What is the limit in B?

Exercise 29.4. Let $\tilde{B} = \{\underline{x} \in \mathbb{C}^n \mid 1 \leq |\underline{x}|\} \cup_{S^{2n-1}} \mathbb{C}P^{n-1}$, where each $\underline{x} \in S^{2n-1}$ is identified with the associated point $[\underline{x}] \in \mathbb{C}P^{n-1}$. This is the first description we gave of the blowup, see (29.2), where one takes out a tubular neighborhood and sews in a copy of the projectivized normal bundle. Define a map $f \colon \tilde{B} \to B$ as follows:

$$\begin{cases} f(\underline{x}) = \big((|\underline{x}| - 1)\underline{x}, [\underline{x}]\big) & \text{for } \underline{x} \in \mathbb{C}^n \text{ with } |\underline{x}| \geq 1, \\ f([\underline{x}]) = (\underline{0}, [\underline{x}]) & \text{for } [\underline{x}] \in \mathbb{C}P^{n-1}. \end{cases}$$

Verify that f is a homeomorphism.

Returning to general properties of $B = \mathrm{Bl}_{\underline{0}}\,\mathbb{C}^n$, consider the region of B defined by $X_i \neq 0$. This is an open set U_i; it consists of all points $\underline{x}$ where $x_i \neq 0$ together with the pairs $(\underline{0}, [\underline{X}])$ where $X_i \neq 0$. The equation $x_i = 0$ therefore cuts out $E \cap U_i$

inside of U_i. That is to say, E is defined by a single equation inside of U_i. Since this holds for each i, we see that E is locally cut out by a single equation—algebraic geometers call such an object a *divisor*.

Remark 29.5. There is a clever way of encoding the equations $x_iX_j = x_jX_i$. Introduce a formal variable t and set $X_i = x_it$; then the validity of those equations is automatic. This technique proves useful when talking about more general blowups, as we will see below. For now, let us just introduce the algebra behind it. If R is a ring and I is an ideal, the **Rees ring** is defined to be the graded ring

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots \subseteq R[t].$$

That is, $R[It]$ consists of polynomials in t where the coefficient of each t^r lies in I^r. One can check that when $R = \mathbb{C}[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_n)$ then $R[It]$ is isomorphic to the quotient ring $\mathbb{C}[x_1, \ldots, x_n, X_1, \ldots, X_n]/(x_iX_j - x_jX_i \,|\, 1 \leq i, j \leq n)$ via $X_i = x_it$, where each x_i has degree 0 and each X_i has degree 1. Taking Proj of this graded ring yields the blowup $\mathrm{Bl}_{\underline{0}}\,\mathbb{C}^n$: this is just the fact that Proj of $\mathbb{C}[x_1, \ldots, x_n, X_1, \ldots, X_n]$ is $\mathbb{A}^n \times \mathbb{C}P^{n-1}$, coupled with $\mathrm{Bl}_{\underline{0}}\,\mathbb{C}^n$ being the closed subscheme defined by the equations $x_iX_j = x_jX_i$.

Let us take another look at $\mathrm{Bl}_{\underline{0}}\,\mathbb{C}^3 \subseteq \mathbb{C}^3 \times \mathbb{C}P^2$ (taking $n = 3$ just to keep the numbers small), this time keeping the Rees ring in mind. Points of the blowup are pairs $((x, y, z), [X : Y : Z])$. We can think of x, y, and z as being functions on the original space $\mathbb{C}^3$, with the corresponding X, Y, and Z functions on the normal space to $\underline{0}$. What is confusing is that in some sense x and X are the same function, but with an artificial distinction being thrown in regarding their domains: x is a function on $\mathbb{C}^3$ whereas X is a function on $N_{\underline{0}}\mathbb{C}^3$ (which happens to be another copy of $\mathbb{C}^3$). As soon as one allows for this distinction one can write down functions like xX, x^2yZ^3, and so forth. Note that x^2, xX, and X^2 are all different.

The way this plays out in the Rees ring is that rather than distinguishing the two types of functions via lowercase and uppercase letters, one introduces the artificial symbol t that says "regard this as a function on the normal space". So we have the functions x and xt (formerly called X), and then (for example) the functions x^2, $x(xt) = x^2t$, and $(xt)(xt) = x^2t^2$. The Rees ring is simply the universal ring constructed in this way, consisting of the original elements of a ring R together with elements of an ideal I that are "promoted" to having a separate existence, here regarded as functions on the normal spaces to $V(I)$.

Now let us return to more advanced blowups. Increasing our level of sophistication, let us keep $X = \mathbb{C}^n$ and now let A be the linear subspace defined by the equations $x_1 = x_2 = \cdots = x_c = 0$. Each point on A has a normal space of dimension c, and we will blow up that normal space at the origin—in fact, we will do this simultaneously at all the points of A. Here define $\mathrm{Bl}_A\,\mathbb{C}^n$ to be the subset of $\mathbb{C}^n \times \mathbb{C}P^{c-1}$ consisting of pairs $(\underline{x}, [\underline{X}])$ such that $x_iX_j = x_jX_i$ for all $1 \leq i, j \leq c$. Again observe that for any $\underline{x}$ not in A there is exactly one possible $[\underline{X}]$, namely $[\underline{X}] = [x_1 : \cdots : x_c]$, whereas for $\underline{x}$ in A any $[\underline{X}]$ is possible.

Again, we can keep track of the equations $x_iX_j = x_jX_i$ by introducing a formal variable t and setting $X_i = tx_i$ for all i. The blowup $\mathrm{Bl}_A\,\mathbb{C}^n$ is Proj of the Rees ring $R[It]$ where $R = \mathbb{C}[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_c)$.

Finally, for an arbitrary pair (X, A) consisting of a smooth complex manifold and a smooth submanifold of codimension c, at each point in A we can find local

analytic coordinates in which A is defined by the vanishing of the first c of them. So we can build the blowup $\mathrm{Bl}_A X$ by doing the above construction locally around each point in A and patching the results together. The key facts about $\mathrm{Bl}_A X$ are that it comes with a projection map $\pi\colon \mathrm{Bl}_A X \to X$, the map $\pi^{-1}(X-A) \to X-A$ is a homeomorphism, the fibers $\pi^{-1}(a)$ for $a \in A$ are projective spaces $\mathbb{C}P^{c-1}$, and moreover $\pi^{-1}(A)$ is precisely the projectivized normal bundle of A in X. The closed submanifold $\pi^{-1}(A) \subseteq \mathrm{Bl}_A X$ is locally cut out by a single equation (i.e., it is a divisor) and is called the *exceptional divisor*. Finally, we have a pushout square

$$\begin{array}{ccccc} \mathbb{P}(N_{X/A}) & = & \pi^{-1}(A) & \longrightarrow & \mathrm{Bl}_A X \\ & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & A & \longrightarrow & X. \end{array}$$

We have described the construction of the blowup in the setting of complex manifolds because that is the best for building intuition, but in fact all of this can be done entirely in the realm of algebraic geometry and without the smoothness conditions. If X is any scheme and $A \hookrightarrow X$ is a closed subscheme there is a blowup $\mathrm{Bl}_A X$ with similar properties to what we described above. Without smoothness one does not have a normal *bundle*, but one always has a normal *cone*—similar to a vector bundle, but where some of the fibers can degenerate into a cone of normal directions rather than a subspace of normal directions. For $X = \operatorname{Spec} R$ and $A = V(I)$, I an ideal of R, the blowup $\mathrm{Bl}_A X$ is defined to be $\operatorname{Proj} R[It]$, and for general schemes one patches together these local constructions. See Section 29.12 for a precise definition.

The idea behind the Rees ring to keep in mind is that we have the functions on the original variety (this is the R) and then additional functions on the normal directions: these are the functions ft for $f \in I$, and products of such. Recall that 'ft' is not a true product, and the t just plays the formal role of distinguishing the f that is a function on the normal directions from the f that is a function on X.

29.6. **Closed immersions and normal cones.** Given an R-algebra S we get the map of schemes $\operatorname{Spec} S \to \operatorname{Spec} R$. We can perform this construction locally on a scheme X: if $\mathcal{S}$ is a sheaf of $\mathcal{O}_X$-algebras then for every affine chart $U = \operatorname{Spec} R$ on X we form $\operatorname{Spec} \mathcal{S}(U) \to U$ and paste these together in the natural way. This produces a scheme we will write as $\operatorname{Spec}_X \mathcal{S}$, and it comes with a natural map $\operatorname{Spec}_X \mathcal{S} \to X$. We won't have to do any heavy lifting with this construction; the point is just that it exists, and we understand it locally.

As an application, if $\mathcal{I} \subseteq \mathcal{O}_X$ is a sheaf of ideals then we can form $V(\mathcal{I}) = \operatorname{Spec}_X \mathcal{O}_X/\mathcal{I}$. The associated map $V(\mathcal{I}) \to X$ identifies $V(\mathcal{I})$ with a closed subscheme of X. Writing $Z = V(\mathcal{I})$, the map $Z \to X$ is called a **closed immersion**.

A **regular immersion** is a closed immersion where $\mathcal{I}$ is locally generated by a regular sequence—that is to say, where $\mathcal{I}_{(x)} \subseteq \mathcal{O}_{X,x}$ is generated by a regular sequence, for every $x \in X$ (it is equivalent to require the existence of an affine cover where this is true on each chart). One also says that Z is a **local complete intersection** in X (lci for short).

Here is another important example of the Spec_X construction. Given a quasi-coherent sheaf $\mathcal{F}$ on X we can form the sheaf of $\mathcal{O}_X$-algebras $\operatorname{Sym} \mathcal{F}$, which locally is modeled on the assignment $M \mapsto \operatorname{Sym}_R(M)$. We will call $\operatorname{Spec}_X \operatorname{Sym} \mathcal{F} \to X$ the **family of vector spaces co-associated to $\mathcal{F}$**. The language is slightly awkward

due to the fact that the sections of this family give $\mathcal{F}^*$, not $\mathcal{F}$: locally, this is the fact that algebra-splittings of $R \hookrightarrow \mathrm{Sym}_R(M)$ correspond to R-module maps $M \to R$, i.e. to elements of M^*. Observe that the fiber of our family over a point $x \in X$ is $\operatorname{Spec} \mathrm{Sym}_{\kappa(x)} \mathcal{F}_x$, which is an affine space of dimension $\dim_{\kappa(x)} \mathcal{F}_x$.

When $\mathcal{F}$ is reflexive, in the sense that the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, then $\mathcal{F}$ is identified with the sheaf of sections of $\operatorname{Spec}_X \mathrm{Sym}(\mathcal{F}^*) \to X$. But this doesn't work without the reflexivity.

This gives another perspective on our equivalence between locally free coherent sheaves and algebraic vector bundles. For $\mathcal{F}$ a locally free coherent sheaf (which is necessarily reflexive), $\operatorname{Spec}_X \mathrm{Sym}(\mathcal{F}^*) \to X$ is the corresponding algebraic vector bundle.

Example 29.7. Let us look at an example where $\mathcal{F}$ is *not* locally free. Take $X = \operatorname{Spec} k[t]$ and $\mathcal{F}$ the sheaf associated to $M = k[t]/(t)$—the structure sheaf of the origin. If we set $x = \bar{1} \in M$ then each $M^{\otimes(r)}$ is generated by $x^{\otimes(r)}$ and so $\mathrm{Sym}_{k[t]} M = k[t,x]/(tx)$. Then $\operatorname{Spec}_X \mathrm{Sym}(\mathcal{F}^*) \to X$ has the affine line as its fiber over the origin, and all other fibers are zero.

Remark 29.8. Harkening back to Definition 10.1 in the topological context, the maps $\operatorname{Spec}_X \mathrm{Sym}(\mathcal{F}^*) \to X$ do indeed have the structure of a "vector space object over X". The grading on the symmetric algebra gives the $\mathbb{A}^1$-action, and the addition is induced by the standard coproduct on a symmetric algebra (see Exercise 29.13).

The phrase "family of vector spaces" is not used in algebraic geometry, but there seems to be no common term for this type of object. In [EGA2, Definition 1.7.8] the term "vector bundle" was used—even though the object is not a bundle in any usual sense—but that is too confusing and never caught on. In [BF] it is called an *abelian cone*, since it has both a fiberwise $\mathbb{A}^1$-action (the *cone* part) and a fiberwise additive structure (the *abelian* part).

When $Z \hookrightarrow X$ is a closed immersion with ideal sheaf $\mathcal{I}$, then $\mathcal{I}/\mathcal{I}^2$ is a sheaf on Z called the **conormal sheaf** of the immersion. The dual $(\mathcal{I}/\mathcal{I}^2)^*$ is called the **normal sheaf**. Then $\operatorname{Spec}_Z \mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)$ is the family of vector spaces co-associated to the conormal sheaf, and we will denote it as $\mathcal{N}_{X/Z}$. We call this the **normal family of vector spaces** of Z in X. In many cases $\mathcal{I}/\mathcal{I}^2$ is locally free and then we will call both $(\mathcal{I}/\mathcal{I}^2)^*$ and $\mathcal{N}_{X/Z}$ the **normal bundle** of Z in X, as usual blurring the distinction between an algebraic vector bundle and its associated locally free coherent sheaf.

We can also form the associated graded algebra $\operatorname{gr}_{\mathcal{I}} \mathcal{O}_X = \bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$. We have a canonical surjection $\mathrm{Sym}_{\mathcal{O}_X/\mathcal{I}} \mathcal{I}/\mathcal{I}^2 \twoheadrightarrow \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$ of sheaves of rings on Z, and this gives us a closed immersion

$$\operatorname{Spec}_Z \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X \hookrightarrow \mathcal{N}_{X/Z}.$$

The scheme $\operatorname{Spec}_Z \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$ is called the **normal cone** of Z in X and denoted $C_{X/Z}$. Being Spec of an $\mathbb{N}$-graded ring it comes with an $\mathbb{A}^1$-action, thus the name "cone" is appropriate; but unlike $\mathcal{N}_{X/Z}$ here we do not have any fiberwise addition operation. One should think of the normal family as only depending on *linear* normal information (just $\mathcal{I}/\mathcal{I}^2$), whereas the normal cone sees the higher-order information contained in all of the $\mathcal{I}^n/\mathcal{I}^{n+1}$ objects. So the normal cone is potentially a finer invariant.

Because the terminology can get overwhelming, here is a summary:

- $\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf of $Z \hookrightarrow X$. If $\mathcal{I}/\mathcal{I}^2$ is locally free then one calls it the conormal *bundle*.
- Likewise, $(\mathcal{I}/\mathcal{I}^2)^*$ is the *normal sheaf*, or—when locally free—the normal bundle.
- $\mathcal{N}_{X/Z} = \operatorname{Spec}_Z \operatorname{Sym}(\mathcal{I}/\mathcal{I}^2)$ is the **normal family of vector spaces** of Z in X. When $\mathcal{I}/\mathcal{I}^2$ is locally free then $\mathcal{N}_{X/Z}$ is an algebraic vector bundle, called the **normal bundle** of $Z \hookrightarrow X$.
- Note that "normal bundle" is used both for the locally free coherent sheaf $(\mathcal{I}/\mathcal{I}^2)^*$ and for the associated algebraic vector bundle, but this is fine because of the Serre-Swan equivalence of categories.
- The **normal cone** $C_{X/Z} \hookrightarrow \mathcal{N}_{X/Z}$ is the closed subscheme corresponding to the quotient of (sheaves of) algebras $\operatorname{Sym} \mathcal{I}/\mathcal{I}^2 \twoheadrightarrow \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$.

In a moment we will discuss several examples which show the differences between the normal cone and the normal family of vector spaces.

The following result connects an immersion being regular to certain nice properties of its normal cone and normal family.

Proposition 29.9. *Let X be a Noetherian scheme and $Z \hookrightarrow X$ a closed immersion.*

(a) If $Z \hookrightarrow X$ is a regular immersion then $\mathcal{I}/\mathcal{I}^2$ is locally free.

(b) $Z \hookrightarrow X$ is a regular immersion if and only if $\mathcal{I}/\mathcal{I}^2$ is locally free and the normal cone equals the normal family.

(c) When X is regular, $Z \hookrightarrow X$ is a regular immersion if and only if $\mathcal{I}/\mathcal{I}^2$ is locally free.

This is just the global version of the following result from commutative algebra:

Proposition 29.10. *Let R be a Noetherian ring and let $I = (x_1, \ldots, x_n)$.*

(a) If $x_1, \ldots, x_n$ is a regular sequence then the natural map $R/I[X_1, \ldots, X_n] \to \operatorname{gr}_I R$ sending $X_i \to \overline{x_i}$ is an isomorphism.

(b) If $x_1, \ldots, x_n$ is a regular sequence then I/I^2 is a free R/I-module.

(c) If (R, m) is local and $I \subseteq m$, then the converse of (a) is true.

(d) Suppose that (R, m) is regular local. Then I is generated by a regular sequence if and only if I/I^2 is free as an R/I-module.

Proof. For (a) see [BH, Theorem 1.1.8]. Note that (b) is a consequence of (a). For (c) see [Eisenbud, Exercise 17.16(b)]. For (d) see [BH, Theorem 2.2.8] or [Eisenbud, Exercise 20.23]. □

Example 29.11. We next discuss several examples that show the difference between the normal cone and the normal bundle. In all of these $X = \operatorname{Spec} R$ is the ambient scheme and $Z = \operatorname{Spec} R/I$ is the closed subscheme.

(1) Let $R = k[x, y]/(xy)$ and $I = (x, y)$. So X is the union of the two coordinate axes and Z is the singular point. Then $R/I = k$ and $I/I^2 = k\langle x, y\rangle$. So $\operatorname{Sym}_{R/I}(I/I^2) = k[x, y]$ and thus the normal bundle to our point Z is a plane. Even though X is one-dimensional, we are getting a larger normal fiber because Z was a singular point.

But $\operatorname{gr}_I R = k[\bar{x}, \bar{y}]/(\bar{x}\bar{y})$: for example, in degree two one sees $I^2/I^3 = (x^2, y^2)/I^3$. So here the normal cone is one-dimensional, it is essentially just the union of the two evident normal directions.

(2) Now let $R = k[x,y]/(y^2 - x^3)$ and $I = (x,y)$. Then X is the cuspidal cubic and Z is the singular point. Again, $R/I = k$ and $I/I^2 = k\langle x, y\rangle$. So I/I^2 is free as an R/I-module and the normal bundle to Z is a plane.

But $\operatorname{gr}_I R = k[\bar{x},\bar{y}]/(\bar{y}^2)$: for example, in degree two one sees $I^2/I^3 = (x^2, xy, y^2)/(x^3, x^2y, xy^2, y^3) = k\langle x^2, xy\rangle$, leading to the relation $\bar{y}^2 = 0$. So the normal cone to Z is the double line tangent to the cusp.

(3) Let $R = k[x,y]/(y^2 - x^2 - x^3)$ and $I = (x,y)$. Then X is a nodal cubic and Z is the node. We have $R/I = k$ and $I/I^2 = k\langle x, y\rangle$. Again, the normal bundle to Z is a plane here.

But $\operatorname{gr}_I R = k[\bar{x},\bar{y}]/(\bar{x}^2 - \bar{y}^2) = k[\bar{x},\bar{y}]/((\bar{x}-\bar{y})(\bar{x}+\bar{y}))$, which geometrically is the union of two lines meeting in a point. The key algebraic observation is that we have $x^2 - y^2 = -x^3 \in I^3$ therefore in I^2/I^3 we have $x^2 - y^2 = 0$.

(4) Let $R = k[x,y]$ and $I = (y^2 - x^2 - x^3)$. So here X is the plane and Z is the nodal cubic. Notice that I is generated by a regular sequence, so I/I^2 is a free R/I-module on one generator and $\operatorname{gr}_I R \cong \operatorname{Sym}_{R/I}(I/I^2) \cong R/I[f]$. So we have a one-dimensional trivial normal bundle, which is also the normal cone.

This example can be confusing because we imagine we can "see" the normal fiber at all of the nonsingular points, but not at the node. Keep in mind that the normal spaces that we "see" (which involve a choice of metric) are not *exactly* the normal spaces in the sense of algebraic geometry, though for smooth points they are isomorphic. The algebro-geometric notion of normal is tied to differentials, which have the expected connection to tangent spaces only in the smooth case.

To explain more about the conormal sheaf we need a brief excursion into Kähler differentials. This is just for getting a sense of the ideas, and will not be needed in any of our later work. Let k be a ring and R a k-algebra. Define $\Omega_{R/k}$ to be the quotient of the free R-module on symbols dr for $r \in R$ by the relations

$$d(rs) = r \cdot ds + s \cdot dr, \qquad d(c.1) = 0 \text{ for } c \in k.$$

For $I \subseteq R$ an ideal there is an exact sequence

$$I/I^2 \xrightarrow{d} R/I \otimes_R \Omega_{R/k} \longrightarrow \Omega_{(R/I)/k} \longrightarrow 0.$$

For X a k-scheme one defines the sheaf of differentials Ω_X as being $\Omega_{R/k}$ locally.

For $j\colon Z \hookrightarrow X$ a closed immersion with ideal sheaf $\mathcal{I}$, the above exact sequence becomes

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow j^*\Omega_X \longrightarrow \Omega_Z \longrightarrow 0.$$

This is the so-called **conormal exact sequence**. If X and Z are both smooth then Ω_X and Ω_Z are locally free and dual to the tangent bundles, and the conormal exact sequence is actually short-exact. So applying $\operatorname{Hom}(-, \mathcal{O}_Z)$ to the above gives the familiar exact sequence

$$0 \longrightarrow T_Z \longrightarrow j^*T_X \longrightarrow N_{X/Z} \longrightarrow 0$$

relating tangent and normal bundles. But in more general situations the conormal exact sequence can fail to be left-exact, so we lose control over what happens when we apply $\operatorname{Hom}(-, \mathcal{O}_Z)$. It *is* true (for other reasons) that when X is smooth and $Z \hookrightarrow X$ is a regular immersion we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}(\Omega_Z, \mathcal{O}_Z) \longrightarrow j^*T_X \longrightarrow N_{X/Z} \longrightarrow \mathcal{E}xt^1(\Omega_Z, \mathcal{O}_Z) \longrightarrow 0$$

and this starts to demonstrate how the normal bundle loses its direct geometric connection to tangent spaces.

29.12. **Back to blowups.** Now let us briefly circle back around and complete the discussion of blowups. In the Rees ring $R[It]$ consider the ideal $IR[It] = I \oplus I^2t \oplus I^3t^2 \oplus \cdots$. Then $R[It]/IR[It] = \operatorname{gr}_I R$. Taking Proj then gives the fundamental link between blowups and normal cones: the projectivized normal cone is precisely the exceptional divisor inside the blowup. Equivalently, $\operatorname{Proj} \operatorname{gr}_I R$ is the closed subscheme defined by the homogeneous ideal $IR[It]$ inside of $\operatorname{Proj} R[It]$.

Moving out of the affine case is now just a matter of globalization. Just as we defined a global version of Spec, one can likewise define a global version of Proj for sheaves of graded $\mathcal{O}_X$-algebras. For a closed immersion $Z \hookrightarrow X$ with ideal sheaf $\mathcal{I}$, the blowup $\operatorname{Bl}_Z(X)$ is defined to be $\operatorname{Proj}_X \mathcal{O}_X[\mathcal{I}t]$. The exceptional divisor that appears as the preimage of Z is the closed projective subscheme with ideal sheaf $\mathcal{I}\mathcal{O}_X[\mathcal{I}t]$, so this is $\operatorname{Proj}_Z \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$, i.e. the projectivization $\mathbb{P}(C_{X/Z})$ of the normal cone.

◦ Exercises ◦

Exercise 29.13. Let R be a ring and let M and N be R-modules. Explain how to obtain a canonical isomorphism $\operatorname{Sym}_R(M) \otimes_R \operatorname{Sym}_R(N) \xrightarrow{\cong} \operatorname{Sym}_R(M \oplus N)$. Then use the diagonal map $\Delta\colon M \to M \oplus M$ together with the above isomorphism to produce a coproduct on $\operatorname{Sym}_R(M)$. Check that this makes $\operatorname{Sym}_R(M)$ into a cocommutative Hopf algebra in which every element of M is primitive.

Exercise 29.14. Let $R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[X]/(X^2+3)$ and $I = (2, 1+X)$. We have seen previously that $\operatorname{Spec} R$ is a one-dimensional scheme with a single singular point, namely I (see Exercise 3.26).

(a) Compute I/I^2 as an R/I-module and show that it is free of rank 2.
(b) Compute $\operatorname{gr}_I R$ as a quotient of $\operatorname{Sym}_{R/I}(I/I^2)$ and show that it is $\mathbb{F}_2[A,B]/(A^2+AB+B^2)$.
(c) Compute $\operatorname{Proj} \operatorname{gr}_I R$ and show that this is $\operatorname{Spec} \mathbb{F}_4$.

So I is a $\mathbb{Z}/2$-point in $\operatorname{Spec} R$, which transforms into an $\mathbb{F}_4$-point in the blowup. (Compare to Exercise 3.36).

30. Exactness and fiberwise exactness

Suppose X is a scheme and $\mathcal{L}_\bullet$ is a chain complex of locally free $\mathcal{O}_X$-modules of finite rank. We know this corresponds to a chain complex of algebraic vector bundles on X, and we have discussed the fact that exactness in a certain degree, when regarded as a complex of sheaves (or modules in the case X is affine), is not equivalent to fiberwise exactness in that degree. We now explore the relations between these concepts in more detail.

30.1. **Background from commutative algebra.** If R is a commutative ring, M a finitely-generated R-module, and $P \subseteq R$ a prime ideal, then the following are equivalent:

(1) $M_P = 0$,
(2) $M_f = 0$ for some $f \in R - P$,
(3) $\operatorname{Ann}(M) \not\subseteq P$, i.e. $P \notin V(\operatorname{Ann}(M))$.

Thinking in terms of sheaves on $\operatorname{Spec} R$, the second condition is equivalent to saying that the sheaf is identically zero on some Zariski neighborhood of P; this is because the open sets $D(f) = \operatorname{Spec} R - V(f)$ for $f \in R - P$ are cofinal in all Zariski neighborhoods of P in $\operatorname{Spec} R$.

The **support** of M is the Zariski closed set

$$\operatorname{Supp}(M) = V(\operatorname{Ann}(M)) = \{P \in \operatorname{Spec} R \mid M_P \neq 0\}.$$

One readily extends this to the sheaf setting: if $\mathcal{F}$ is a sheaf on X define

$$\operatorname{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_{(x)} \neq 0\}.$$

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then locally it corresponds to a finitely-generated module and so $\operatorname{Supp}(\mathcal{F})$ is closed in each piece of an affine cover of X—hence it is closed in X.

Returning to the setting of finitely-generated modules, here is another useful set of equivalent statements:

(1) $M = 0$,
(2) $M_P = 0$ for all prime ideals P,
(3) $M_m = 0$ for all maximal ideals m,
(4) $M/mM = 0$ for all maximal ideals m.

Note that the equivalence of (3) and (4) follows from $M/mM = M_m/mM_m$ and Nakayama's lemma. Rephrasing things geometrically and generalizing to schemes, if $\mathcal{F}$ is a coherent sheaf on a scheme X then the following are equivalent:

(1) $\mathcal{F}$ is the zero sheaf,
(2) All stalks of $\mathcal{F}$ are zero,
(3) All stalks of $\mathcal{F}$ at closed points are zero,
(4) All fibers of $\mathcal{F}$ at closed points are zero.

30.2. **Chain complexes of sheaves.** Now let us look at complexes of $\mathcal{O}_X$-modules and their homology sheaves. If $\mathcal{F}_\bullet$ is a complex of $\mathcal{O}_X$-modules and p is a point of X, we will say that $\mathcal{F}_\bullet$ is **exact at p in degree s** if the complex of stalks $(\mathcal{F}_\bullet)_{(p)}$ is exact in degree s. The phrase "locally exact" might be better than just "exact", but the latter is most commonly used.

In contrast, define the **complex of fibers** of $\mathcal{F}_\bullet$ at p to be

$$(\mathcal{F}_\bullet)_p = (\mathcal{F}_\bullet)_{(p)} \otimes_{\mathcal{O}_{X,p}} \kappa(p)$$

where $\kappa(p) = \mathcal{O}_{X,p}/p\mathcal{O}_{X,p}$. Most of the time when we deal with fiber complexes the point p will be closed, but that is not necessary. Say that $\mathcal{F}_\bullet$ is **fiberwise exact at p in degree s** if $H_s((\mathcal{F}_\bullet)_p) = 0$.

Proposition 30.3. *Assume X is a locally Noetherian scheme and $\mathcal{F}_\bullet$ is a complex of coherent $\mathcal{O}_X$-modules. Let $n \in \mathbb{Z}$ and $p \in X$. Then the following statements are equivalent:*

(1) $H_n(\mathcal{F})_{(p)} = 0$;
(2) $\mathcal{F}_\bullet$ is exact at p in degree n;
(3) There exists a Zariski open affine neighborhood U of p such that the complex $\Gamma(U, \mathcal{F}_\bullet)$ is exact in degree n.

Proof. Stalks are computed by directed colimits, and such colimits commute with taking homology. So $H_n(\mathcal{F}_{(p)}) = H_n(\mathcal{F})_{(p)}$ and this yields (1) $\iff$ (2). The implication (3)$\Rightarrow$(2) follows from the fact that localization is exact: if $M' \to M \to M''$ is

a sequence of modules that is exact in the middle then so is $M'_p \to M_p \to M''_p$. For (2)⇒(3) we can take an affine neighborhood $V = \operatorname{Spec} R$ of p, where R is Noetherian, and our complex of sheaves corresponds to a complex $M_\bullet$ of finitely-generated R-modules. The assumption implies that $(M_{n+1})_p \to (M_n)_p \to (M_{n-1})_p$ is exact in the middle. Then $H_n(M)_p = 0$ and since $H_n(M)$ is finitely-generated there exists an $f \in R - P$ such that $H_n(M)_f = 0$. Therefore M_f is exact in degree n. But M_f is exactly the complex $\Gamma(U, \mathcal{F})$ where $U = D(f) = \operatorname{Spec} R - V(f)$, and U is open in $\operatorname{Spec} R$ and therefore also open in X. □

Corollary 30.4. *Assume X is a locally Noetherian scheme and $\mathcal{F}_\bullet$ is a complex of coherent $\mathcal{O}_X$-modules. Let $n \in \mathbb{Z}$. Then the following statements are equivalent:*
(1) $\mathcal{F}_\bullet$ is exact in degree n;
(2) $\mathcal{F}_\bullet$ is exact in degree n at all points of X;
(3) $\mathcal{F}_\bullet$ is exact in degree n at all closed points of X.

Proof. This is just the fact that $H_n(\mathcal{F}) = 0$ if and only if $H_n(\mathcal{F})_{(p)} = 0$ for all points p, if and only if $H_n(\mathcal{F})_{(p)} = 0$ for all closed points p. □

Now that we have a good understanding of exactness and its stalkwise-characterization, let us turn to fiberwise exactness. For complexes of algebraic vector bundles the slogan to remember is that "fiberwise exactness implies stalkwise exactness, but not vice versa".

Proposition 30.5. *Let X be a locally Noetherian scheme and let $\mathcal{L}_\bullet$ be a chain complex of locally free coherent $\mathcal{O}_X$-modules. Let x be a closed point of X. Then if $\mathcal{L}_\bullet$ is fiberwise-exact over x in degree s then $\mathcal{L}_\bullet$ is exact at x in degree s.*

Since the issues are local around x one immediately reduces to a problem in commutative algebra. Here is the version in local algebra:

Proposition 30.6. *Let (R, m) be a local ring, let A, B, and C be free modules of finite rank, and let $A \xrightarrow{f} B \xrightarrow{g} C$ be maps such that $gf = 0$. If $A/mA \to B/mB \to C/mC$ is exact in the middle then the original sequence is exact in the middle.*

Proof. By doing row and column operations we can assume that the matrix for g has the form $\left[\begin{smallmatrix} I & 0 \\ 0 & * \end{smallmatrix}\right]$ where the submatrix in the lower right has entries in m. In other words, we have isomorphisms $B \cong R^j \oplus B'$, $C \cong R^j \oplus C'$, under these isomorphisms g takes the form $\mathrm{id}_{\mathbb{R}^j} \oplus g'$, and the matrix for g' has entries in m. Note that B' and C' are free, being direct sums of free modules (and using that all finitely-generated projectives over local rings are free). Since $gf = 0$ it must be that $\operatorname{im}(f) \subseteq B'$. Again doing row and column operations just as above, we find that there are isomorphisms $A \cong R^s \oplus A'$, $B' \cong R^s \oplus B''$, with A' and B'' free, such that under these isomorphisms f takes the form $\mathrm{id}_{\mathbb{R}^s} \oplus f'$ with the entries for any matrix for f' having entries in m. This is all to say that our original sequence is isomorphic to one of the form

$$\begin{array}{ccccc} R^s & \xrightarrow{\mathrm{id}} & R^s & & \\ \oplus & & \oplus & & \\ A' & \xrightarrow{f'} & B'' & \xrightarrow{g'} & C' \\ & & \oplus & & \oplus \\ & & R^j & \xrightarrow{\mathrm{id}} & R^j \end{array}$$

with the matrices for f' and g' having entries in m. After reducing modulo m the maps f' and g' become zero, and so the only way this sequence could be exact in the middle is if $B'' = 0$. But this implies the original sequence is exact in the middle as well. □

Proof of Proposition 30.5. Working locally around x we reduce the proposition to the following result: When $Q_\bullet$ is a complex of finitely-generated projectives over a Noetherian ring R and m is a maximal ideal, then if Q/mQ is exact in degree s then Q_m is also exact in degree s. This follows immediately from Proposition 30.6 using that $Q/mQ = Q_m/mQ_m$. □

Exercise 30.7. Generalize Proposition 30.6 as follows. Given a bounded below chain complex of finitely-generated free modules over a local ring, prove that $P_\bullet$ is isomorphic to a direct sum of a split-exact sequence (i.e. a sum of elementary complexes) and a complex of finitely-generated free modules where the differentials have matrices in m.

When looking at a specific degree in a chain complex, we have seen that fiberwise exactness implies stalkwise exactness. The converse is not true, as shown by our standard example $\mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t]$: this is stalkwise injective at $t = 0$ but not fiberwise injective there. But stalkwise exactness *does* imply fiberwise exactness when the complexes are bounded below and exactness is considered in all degrees at once:

Proposition 30.8. *Let X be a locally Noetherian scheme and let $\mathcal{L}_\bullet$ be a bounded-below chain complex of locally free coherent $\mathcal{O}_X$-modules. Let x be a closed point of X. Then $\mathcal{L}_\bullet$ is fiberwise exact at x if and only if $\mathcal{L}_\bullet$ is exact at x.*

Proof. This reduces to the claim that if R is a Noetherian ring, m is a maximal ideal, and $Q_\bullet$ is a bounded-below complex of finitely-generated projectives, then Q/mQ is exact if and only if Q_m is exact. Since $Q/mQ = Q_m/mQ_m$ one may as well replace R with R_m and reduce to the case where R is local. The statement that if Q/mQ is exact then Q is exact follows from Proposition 30.6. For the other direction, if Q is exact then it decomposes as a direct sum of elementary complexes (see Lemma 7.3 and the remarks thereafter) and hence $Q \otimes_R (R/m)$ is still exact. □

While we are on this topic, let us record the following:

Proposition 30.9. *Let X be a locally Noetherian scheme and let $\mathcal{L}_\bullet$ be a bounded chain complex of locally free coherent $\mathcal{O}_X$-modules. Then the set of points $p \in X$ such that $\mathcal{L}_\bullet$ is exact at p is Zariski open.*

Proof. The complement of the set in question is the intersection over all $n \in \mathbb{Z}$ of $\operatorname{Supp} H_n(\mathcal{L}_\bullet)$. Each $\operatorname{Supp} H_n(\mathcal{L}_\bullet)$ is closed, and since $\mathcal{L}_\bullet$ is bounded there are only finitely many n where this set is not equal to X. So we have a finite intersection of closed subsets, which is closed. □

It is useful to define

$$\operatorname{Supp} \mathcal{L}_\bullet = \bigcap_n \operatorname{Supp} H_n(\mathcal{L}_\bullet) = \{p \in X \,\big|\, \mathcal{L}_\bullet \text{ is not exact at } p\}.$$

Proposition 30.8 implies that the closed points in $\operatorname{Supp} \mathcal{L}$ coincide with with the closed points of X over which $\mathcal{L}_\bullet$ fails to be fiberwise exact.

There are many interesting phenomena relating the geometry of $\operatorname{Supp}\mathcal{L}_\bullet$ to the algebraic features of $\mathcal{L}_\bullet$. Here is one result along these lines:

Theorem 30.10 (The New Intersection Theorem)**.** *Let X be a locally Noetherian scheme and let $0 \to \mathcal{L}_n \to \cdots \to \mathcal{L}_0 \to 0$ be a bounded complex of locally free coherent sheaves. If $\operatorname{Supp}\mathcal{L}_\bullet$ is zero-dimensional (in particular, nonempty) then $n \geq \dim X$.*

This result has an interesting history. It immediately reduces to the evident analog in local algebra, and in that context it was originally proved by Peskine-Szpiro for characteristic p rings [PS1], and independently by Roberts in the same setting [Roberts1]. Hochster's work on reduction to characteristic p extends the results to all equicharacteristic local rings [Hochster2, Section 6], or alternatively it is explained in [Hochster1, Proposition 3.4 and Section 10A] that the result follows from the existence of big Cohen-Macaulay modules (proven by Hochster in the same setting). In the 1980s Roberts proved the theorem in general using local Chern characters [Roberts3, Roberts4, Roberts5], and in the 2000s a new proof was given by Piepmeyer and Walker using K-theory [PW].

30.11. **Pullbacks and Tor.** We have not defined the tensor product of quasi-coherent sheaves, but the reader can almost guess the main idea: take the tensor product from commutative algebra locally, and globalize. Precisely, $F \otimes_{\mathcal{O}_X} G$ is the sheafification of the assignment $U \mapsto F(U) \otimes_{\mathcal{O}_X(U)} G(U)$. See [Hartshorne, Chapter II.5] or [Vakil, 6.2.F]. If F and G are quasi-coherent sheaves on a scheme X and $U = \operatorname{Spec} R$ is an open affine chart on X such that $F|_U \cong \widetilde{M}$ and $G|_U \cong \widetilde{N}$, then $(F \otimes_{\mathcal{O}_X} G)|_U \cong \widetilde{M \otimes_R N}$. Locally free modules are therefore flat, and one has a canonical isomorphism of stalks $(F \otimes_{\mathcal{O}_X} G)_{(x)} \cong F_{(x)} \otimes_{\mathcal{O}_{X,x}} G_{(x)}$.

Let $f\colon Y \to X$ be a map of schemes, let F be a quasi-coherent sheaf on Y, and let G be a quasi-coherent sheaf on X. The pullback f^*G is given locally by tensoring, and with a little legwork one constructs a map

$$f_*(F \otimes_{\mathcal{O}_Y} f^*G) \to (f_*F) \otimes_{\mathcal{O}_X} G. \tag{30.12}$$

This is not always an isomorphism—for a general map f, the right derived functors of f_* cause an issue (we will discuss these more in Section 43). But the map is an isomorphism in two special cases:

Proposition 30.13 (Projection Formula)**.** *Suppose either that G is locally-free and finite rank, or that f is an affine map. The the canonical map from (30.12) is an isomorphism. In particular, taking $F = \mathcal{O}_Y$ we have the isomorphism*

$$f_*f^*G \cong f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} G.$$

See [Vakil, 14.5.L] or (for the locally-free case) [Hartshorne, Exercise II.5.1(d)]. A map $f\colon Y \to X$ is affine if the preimage of any affine open set is also affine. Common examples are: (1) any map between affine schemes, (2) any closed immersion, (3) the canonical maps $\operatorname{Spec}\mathcal{O}_{X,x} \to X$. Note that open inclusions need not be affine, e.g. $\mathbb{A}^n - 0 \hookrightarrow \mathbb{A}^n$ for $n \geq 2$. We will mostly use the case where f is the closed immersion $j_x\colon \operatorname{Spec}\kappa(x) \to X$ corresponding to a closed point x. Here j_x^*G is the fiber G_x, and $(j_x)_*j_x^*G$ is the skyscraper sheaf at x with G_x as its single fiber. In this setting the projection formula can be written as

$$(j_x)_*(j_x^*)G \cong \mathcal{O}_x \otimes_{\mathcal{O}_X} G. \tag{30.14}$$

Let $\mathcal{L}_\bullet$ be a complex of locally free coherent sheaves on a scheme X. If $f\colon Y \to X$ is a map of schemes then we can consider $f^*\mathcal{L}_\bullet$ and ask how the homology of this pullback complex relates to the homology of $\mathcal{L}_\bullet$. The pullback complex is constructed by tensoring, and so the homology is computed by a spectral sequence involving Tor. For general f this is a bit complicated, but when f is an affine map it reduces to

$$E^2_{p,q} = \operatorname{Tor}_p^{\mathcal{O}_X}(f_*\mathcal{O}_Y, H_q(\mathcal{L}_\bullet)) \Rightarrow f_*H_{p+q}(f^*\mathcal{L}_\bullet). \tag{30.15}$$

Note the relation to the projection formula of Proposition 30.13.

We will not need this spectral sequence, as our applications will only involve the simple case where $\mathcal{L}_\bullet$ is a resolution (i.e., only has homology in degree zero) and then the spectral sequence is just the single isomorphism

$$f_*H_n(f^*\mathcal{L}_\bullet) \cong \operatorname{Tor}_n^{\mathcal{O}_X}(f_*\mathcal{O}_Y, H_0(\mathcal{L}_\bullet)).$$

But this is motivation for us to develop the basics of Tor. This is done in the exercises at the end of the section.

Let $y \in X$ be a closed point and let $j_y\colon \operatorname{Spec}\kappa(y) \to X$ be the associated map of schemes. Then $j_y^*\mathcal{L}_\bullet = (\mathcal{L}_\bullet)_y$ is the complex of fibers over y, and the above isomorphism becomes

$$(j_y)_*H_n((\mathcal{L}_\bullet)_y) \cong \operatorname{Tor}_n^{\mathcal{O}_X}(\mathcal{O}_y, H_0(\mathcal{L}_\bullet)).$$

Again, in some sense the true content here is just a vector space isomorphism. The $(j_y)_*$ just promotes the vector space $H_n((\mathcal{L}_\bullet)_y)$ into a skyscraper sheaf to match the setting of the object on the right. As sheaves, the stalks of the two objects in the isomorphism are zero everywhere except at y.

We can also break the pullback up into two stages:

$$\begin{array}{ccccc} (\mathcal{L}_\bullet)_y & \longrightarrow & (\mathcal{L}_\bullet)_{(y)} & \longrightarrow & \mathcal{L}_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\kappa(y) & \longrightarrow & \operatorname{Spec}\mathcal{O}_{X,y} & \longrightarrow & X \end{array}$$

where the squiggly arrows indicate "being a sheaf over". The right pullback—localization to the stalk at y—is exact, so all of the action is in the left pullback. We find that

$$H_n((\mathcal{L}_\bullet)_y) \cong \operatorname{Tor}_n^{\mathcal{O}_{X,y}}\big(\kappa(y), H_0(\mathcal{L}_\bullet)_{(y)}\big)$$

where here both sides are regarded as $\kappa(y)$-modules. (We could just as well put the $(j_y)_*$ back on the left hand side, and then it would tell us to regard the $\kappa(y)$-module as an $\mathcal{O}_{X,y}$-module via the quotient $\mathcal{O}_{X,y} \to \kappa(y)$.)

Although the above discussion specialized to complexes having homology only in degree 0, that was just for convenience. For more general complexes one uses the spectral sequence from (30.15). The moral is that the homology of the fiber complexes is governed by Tor, over the local rings, with their residue fields.

Remark 30.16. In a certain metaphysical sense, the difference between local exactness and fiberwise exactness is the source of *all* higher Tor modules. That is to say, if we lived in a world where local exactness was equivalent to fiberwise exactness this would force $\operatorname{Tor}_s^R(M,N) = 0$ for all $s > 0$, all Noetherian rings R, and all modules M and N. To see this, first let W be a finitely-generated module. If

$P_\bullet \to W$ is a projective resolution and m is a maximal ideal of R then our assumption would imply that $P_\bullet \otimes R/m$ would still be exact. Therefore the assumption implies that $\operatorname{Tor}_s^R(W, R/m) = 0$ for all $s > 0$, and all finitely-generated modules W. Since every module is a filtered colimit of its finitely-generated submodules, and Tor commutes with such colimits (because $\otimes$ does), it follows that $\operatorname{Tor}_s^R(W, R/m) = 0$ for all modules W.

Assume M and N are finitely-generated. Let $L_\bullet \to R/m$ be a projective resolution and consider the double complex $D_\bullet = P_\bullet \otimes N \otimes L_\bullet$ (all tensors are over R). The two spectral sequences for this double complex take the form

$$\operatorname{Tor}_i(M, \operatorname{Tor}_j(N, R/m)) \Rightarrow H_{i+j}(D)$$

and

$$\operatorname{Tor}_i(\operatorname{Tor}_j(M, N), R/m) \Rightarrow H_{i+j}(D).$$

If all higher Tor's with R/m vanish then the first spectral sequence collapses and gives $H_k(D) \cong \operatorname{Tor}_k(M, N \otimes_R R/m)$ for all k. But since $N \otimes_R R/m$ is a direct sum of copies of R/m these Tors vanish for $k > 0$, thus D is exact in positive degrees. On the other hand, the second spectral sequence collapses to give $H_k(D) \cong \operatorname{Tor}_k(M, N) \otimes R/m$. So for $k > 0$ we have shown that $\operatorname{Tor}_k(M, N) \otimes R/m = 0$ for all maximal ideals m. Since R is Noetherian and M and N are finitely-generated, $\operatorname{Tor}_k(M, N)$ is also finitely-generated. Therefore $\operatorname{Tor}_k(M, N) = 0$, since the tensor products with every R/m give zero (see Section 30.1). Again, this implies the result for all (possibly infinitely-generated) modules M and N by the fact that Tor commutes with filtered colimits.

◦ Exercises ◦

Proving serious theorems in algebraic geometry requires extensive training, but for simple things one can get by just with commutative algebra and the basic technique of working locally. For example, a map $\mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules is surjective if and only if it is surjective on all stalks, if and only if every point $x \in X$ has an affine open U such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. The corresponding statement for modules over a commutative ring is familiar from commutative algebra, and this is the evident globalization.

The following exercises lead up to the definition and properties of Tor for $\mathcal{O}_X$-modules, and they are good training for the technique of "do commutative algebra locally". We will use these results in the next section.

Exercise 30.17. Let X be a locally Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $p \in X$ is such that $\mathcal{F}_{(p)} \cong (\mathcal{O}_{X,x})^n$ as $\mathcal{O}_{X,x}$-modules, where $n \geq 0$. Prove that there exists an open neighborhood U of p such that $\mathcal{F}|_U \cong (\mathcal{O}_X|_U)^{\oplus(n)}$. Conclude that $\mathcal{F}$ is locally free if and only if all of its stalks are free. [Hint: Reduce to the following problem in commutative algebra: if R is Noetherian, $P \subseteq R$ is prime, and M is finitely-generated such that M_P is free over R_P, then there is an $f \in R - P$ such that M_f is free over R_f.]

Exercise 30.18. Let X be a locally Noetherian scheme and let $f\colon \mathcal{M} \to \mathcal{N}$ be a map of coherent $\mathcal{O}_X$-modules. Prove that f is fiberwise surjective if and only if f is surjective (or equivalently, stalkwise surjective). [Hint: Reduce to a commutative algebra statement about maps of modules over a Noetherian ring.]

Exercise 30.19. In module theory, every module is a quotient of a free module and every finitely-generated module is a quotient of a free module of finite rank. It turns out that the analogs in $\mathcal{O}_X$-modules do not always hold.

A scheme X is said to have **enough locally frees** if every quasi-coherent $\mathcal{O}_X$-module is a quotient of a locally free $\mathcal{O}_X$-module. (A scheme without this property will be given in Exercise 30.21 below).

Assume X has enough locally frees.

(a) Prove that every quasi-coherent $\mathcal{O}_X$-module has a resolution by locally free $\mathcal{O}_X$-modules.

(b) Locally free $\mathcal{O}_X$-modules need not be projective, so we do not have the usual lifting properties that we need to do homological algebra. Prove the following local version: given a diagram of quasi-coherent $\mathcal{O}_X$-modules

$$\begin{array}{ccc} & & \mathcal{F} \\ & & \big\downarrow v \\ \mathcal{L} & \xrightarrow{u} & \mathcal{G} \end{array}$$

where v is surjective and $\mathcal{L}$ is locally free, there exists a surjection $\mathcal{L}' \to \mathcal{L}$ with $\mathcal{L}'$ locally free, and a lifting $\mathcal{L}' \to \mathcal{F}$ making the evident triangle commute. [Hint: Form the pullback $u^*\mathcal{F}$ and then use the assumption on X.]

(c) Generalize (a) as follows: prove that if $\mathcal{F}_\bullet$ is a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules then there is a complex $\mathcal{L}_\bullet$ of locally free $\mathcal{O}_X$-modules and a surjective quasi-isomorphism $\mathcal{L}_\bullet \xrightarrow{\sim} \!\!\!\!\rightarrow \mathcal{F}_\bullet$. [Hint: Work inductively, using the technique (but not necessarily the result) from (b).]

(d) Suppose that $\mathcal{M}$ is quasi-coherent and both $\mathcal{L}_\bullet \to \mathcal{M}$ and $\mathcal{J}_\bullet \to \mathcal{M}$ are resolutions by locally free $\mathcal{O}_X$-modules. If these were projective resolutions we would get a comparison map from one resolution to another, but that is too much to expect here. Instead prove that there is a third resolution $\mathcal{E}_\bullet \to \mathcal{M}$ by locally free $\mathcal{O}_X$-modules together with surjective quasi-isomorphisms $\mathcal{E} \twoheadrightarrow \mathcal{L}$ and $\mathcal{E} \twoheadrightarrow \mathcal{J}$. [Hint: Start with the pullback complex $\mathcal{E} \times_{\mathcal{M}} \mathcal{J}$.]

(e) Define $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = H_i(\mathcal{L}_\bullet \otimes \mathcal{N})$ where $\mathcal{L}_\bullet \to \mathcal{M}$ is any resolution by locally free $\mathcal{O}_X$-modules. We need to prove that this is independent of resolution.

  (i) Let $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{L}_\bullet$ be a surjective map of locally free resolutions of $\mathcal{M}$ and let $\mathcal{K}_\bullet$ be the kernel. Explain why $0 \to \mathcal{K} \otimes \mathcal{N} \to \mathcal{E} \otimes \mathcal{N} \to \mathcal{L} \otimes \mathcal{N} \to 0$ is still a short exact sequence of complexes. Prove that each $\mathcal{K}_i$ is locally free and that the complex $\mathcal{K}$ is locally split-exact. Conclude that $\mathcal{K} \otimes \mathcal{N}$ is exact, and therefore $\mathcal{E} \otimes \mathcal{N} \to \mathcal{L} \otimes \mathcal{N}$ is a quasi-isomorphism. [Hint: All of these are an exercise in the art of working locally.]

  (ii) Use (c) to complete the proof that $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is well-defined.

(f) Consider the category $\operatorname{Ar}_{\mathcal{O}_X}$ whose objects are maps $\mathcal{M} \to \mathcal{N}$ of quasi-coherent $\mathcal{O}_X$-modules and whose morphisms are commutative squares. Define a "locally-cofibrant" object to be one of the form $\mathcal{F} \hookrightarrow \mathcal{F} \oplus \mathcal{G}$ where the map is the evident inclusion and both $\mathcal{F}$ and $\mathcal{G}$ are locally free. Prove that every object in $\operatorname{Ar}_{\mathcal{O}_X}$ admits an objectwise surjection from a locally-cofibrant object, and prove the analogs of (b) and (d) in this setting. For a map $f\colon \mathcal{M} \to \mathcal{N}$ use this to define natural maps $f_*\colon \operatorname{Tor}_i(\mathcal{M}, -) \to \operatorname{Tor}_i(\mathcal{N}, -)$. Prove that short exact sequences give rise to long exact Tor-sequences in both variables, and that $\operatorname{Tor}_*^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ can be computed by resolving either $\mathcal{M}$ or $\mathcal{N}$.

Exercise 30.20. This exercise is a lemma for the next one, but the main result is very interesting on its own.

(a) Prove that if $n \geq 2$ then the restriction map $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \to \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n - 0)$ is a bijection. Explain why the result is false for $n = 1$. [Hint: $\mathbb{A}^n - 0$ is not affine, but it is a union of the affine opens U_i given by $x_i \neq 0$.]

(b) Suppose $n \geq 2$ and we have a morphism of coherent sheaves $f\colon (\mathcal{O}^k_{\mathbb{A}^n})|_{\mathbb{A}^n - 0} \to (\mathcal{O}^k_{\mathbb{A}^n})|_{\mathbb{A}^n - 0}$. Prove that this extends uniquely to a morphism $F\colon \mathcal{O}^k_{\mathbb{A}^n} \to \mathcal{O}^k_{\mathbb{A}^n}$, and that if f is an isomorphism then so is F.

(c) When you proved (a), you might have noticed that your proof used suprisingly little. In fact the following generalization is true:
Algebro-geometric Hartogs's theorem: *Let X be a normal, integral, Noetherian scheme and let $Z \subseteq X$ be a closed subscheme of codimension at least 2. Then $\mathcal{O}_X(X) \to \mathcal{O}_X(X - Z)$ is an isomorphism.* [Note: An integral scheme X is normal if and only if all of its local rings $\mathcal{O}_{X,x}$ are normal domains, meaning they are integrally closed in their fraction fields.]

This is an analog of the standard "Hartogs's extension theorem" from multivariable complex analysis. Said in words, if an algebraic function is defined everywhere except on a set of codimension at least two, then it can be extended (uniquely) to be defined everywhere. The "unique" part is essentially obvious, by what we would like to think of as continuity: if an algebraic function is zero on an open set then it is zero everywhere. Prove this in the following form: if X is an integral scheme and $U \subseteq X$ is open and nonempty then $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is injective. Give a counterexample when the scheme is not integral.

(d) The algebro-geometric Hartogs's theorem reduces to the following statement in commutative algebra: if A is a normal domain then $A = \bigcap_{\text{ht } P=1} A_P$. This result is discussed in [Hartshorne, Proposition II.6.3A] and [Vakil, 13.5.19]. Vakil calls this the "Algebraic Hartogs's Lemma". Show that this algebraic statement implies the algebro-geometric statement given in (c).

Exercise 30.21. Not all schemes have enough locally frees. The exercise is to think through, and fill in the details of, the following example. Let $n \geq 2$ and let X be a copy of $\mathbb{A}^n$ with the origin doubled (see Example 28.8). On $\mathbb{A}^n$ all locally free modules are actually free (this is the Quillen-Suslin theorem), and it follows that the same property holds for X (to prove this you will use Exercise 30.20). Any map of free $\mathcal{O}_X$-modules will have to have the same behavior on the fibers over the two origins, by "continuity" (again, see Exercise 30.20). So any $\mathcal{O}_X$-module that is a cokernel of a map of locally free modules must have the same stalks and fibers over the two origins. But it is easy to construct quasi-coherent $\mathcal{O}_X$-modules where this fails: for example, take the structure sheaf of one of the origins.

Exercise 30.22. A scheme X is said to have the **resolution property** if every quasi-coherent $\mathcal{O}_X$-module of finite type (see Remark 28.5) is a quotient of a locally free $\mathcal{O}_X$-module of finite rank. From now on we will abbreviate "locally free of finite rank" to "finite locally free". The name "resolution property" is somewhat inapt, as it is only in certain Noetherian settings that one can iterate the property to construct resolutions. The name was first used in [Totaro2], in the Noetherian context, but it has since been appropriated for the more general situation.

(a) Assume that X is a locally Noetherian scheme (see Section 28.17). Assume as well that X has the resolution property. Prove that every quasi-coherent $\mathcal{O}_X$-module of finite type has a resolution by finite locally free modules.
(b) Let R be a regular local ring of dimension n, let M be a finitely-generated R-module, and let $F_\bullet \to M$ be any resolution of M by finitely-generated free modules. Let K_{n-1} be the kernel of $F_{n-1} \to F_{n-2}$. Prove that $\mathrm{Tor}_i(K_{n-1}, R/m) \cong \mathrm{Tor}_{i+n}(M, R/m)$ for all $i \geq 1$ and use this to deduce that K_{n-1} is free.
(c) Let X be a Noetherian regular scheme with the resolution property, and let $n = \dim X$. If $\mathcal{M}$ is a coherent sheaf and $\mathcal{L}_\bullet \to \mathcal{M}$ is a resolution by finite locally free $\mathcal{O}_X$-modules, let $\mathcal{K}_{n-1}$ be the kernel of $\mathcal{L}_{n-1} \to \mathcal{L}_{n-2}$. Prove that $\mathcal{K}_{n-1}$ is locally free, so that

$$0 \to \mathcal{K}_{n-1} \to \mathcal{L}_{n-1} \to \mathcal{L}_{n-2} \to \cdots \to \mathcal{L}_0 \to \mathcal{M} \to 0$$

is also a resolution of $\mathcal{M}$ by finite locally free $\mathcal{O}_X$-modules. So the punchline is that on a Noetherian regular scheme of dimension n that has the resolution property, every locally free resolution of a coherent sheaf, levelwise of finite rank, can be truncated appropriately to give a locally free resolution of length $\leq n$.

Remark 30.23. The example from Exercise 30.21 is clearly pathological, in the sense that one is not likely to encounter that scheme in the course of ordinary life. In fact, most reasonable schemes have enough locally frees—and in particular this will include all schemes that we naturally encounter in this book. To make a precise statement, the following types of schemes all have enough locally frees:

(1) All affine schemes.
(2) All open subschemes of closed subschemes of projective space $\mathbb{P}^n_X$ where X is affine.
(3) Any scheme admitting an ample line bundle, as defined in [EGA2, Definition 4.5.3] (see also [EGA2, 4.5.5])
(4) Any scheme admitting an ample family of line bundles, as defined in [SGA6, II.2.2.4] (see also [TT, Definition 2.1.1]). Such schemes are called *divisorial* [SGA6, II.2.2.5].
(5) Any separated, integral, Noetherian, locally factorial scheme (the latter condition includes regular schemes, since a regular local ring is a UFD).

Note that type (1) is included in (2), type (2) is included in (3), and type (3) is included in (4). Type (5) is also included in (4) by [SGA6, II.2.2.7]. For discussion of the fact that schemes of type (4) have enough locally frees, and also satisfy the resolution property, see [TT, Lemma 2.1.3]. The fact that type (5) schemes have the resolution property is also in [Hartshorne, Exercise III.6.8].

31. The denouement: connecting algebra, topology, and geometry

Although we are far from the end of this book, we have reached the point where we finally have the tools to bring Serre's definition of intersection multiplicities into context. In this section we will show how Serre's definition is naturally linked, via K-theory, to topological intersection multiplicities. Conceptually this is fairly straightforward, but there is one hiccup: when trying to make links between the algebro-geometric and topological worlds, one inevitably runs into the issue that the former is much more rigid. Certain techniques that are easy in the topological world require clever geometric substitutes in the algebro-geometric setting. In this

section we will run into exactly this issue, and will encounter a technique called *deformation to the normal bundle* that appears as a substitute for topological tubular neighborhoods.

The work in this section breaks up into two basic pieces. The first is the local index theorem, which involves a very specialized analysis of the comparison between algebraic and topological K-theory. The second is a more global version of this comparison, where one connects resolutions in algebraic geometry to fundamental classes in topological K-theory. Once these tools are in place it becomes a simple matter to relate Serre's intersection multiplicities to the ones in topological K-theory.

31.1. **The local index.** Suppose that $E_\bullet$ is a bounded chain complex of vector bundles on $\mathbb{C}^n$ that is exact on $\mathbb{C}^n - 0$. Then as we have seen in Theorem 20.14 we get a class $[E_\bullet] \in K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong K^{-2n}(pt)$. But Theorem 27.10 says that this group is $\mathbb{Z}$, generated by a prescribed element β^n. Thus $[E_\bullet] = d \cdot \beta^n$ for a unique integer d. We call this integer the **local index** of the complex $E_\bullet$, and we will denote it $\mathrm{ind}_0(E_\bullet)$. The natural question is: how do we compute this invariant from the data in $E_\bullet$?

I don't know a simple answer to this question, but the problem becomes more tractable if we assume that the complex $E_\bullet$ is algebraic: that is, if we assume that each E_i is an algebraic vector bundle and the maps $E_i \to E_{i-1}$ are algebraic. The phrase "complex of algebraic vector bundles" will always mean such an object.

The following result is an example of an "index theorem", where a topological invariant detected by K-theory is described in terms of more rigid (in this case algebraic) data.

Theorem 31.2. *If $E_\bullet$ is a bounded complex of algebraic vector bundles on $\mathbb{C}^n$ that is exact on $\mathbb{C}^n - 0$, then the local index is given by*

$$\mathrm{ind}_0(E_\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(P_\bullet)$$

where $P_\bullet$ is a complex of finitely-generated, projective $\mathbb{C}[x_1, \ldots, x_n]$-modules such that $P_\bullet(\mathbb{C}) \cong E_\bullet$. (Here $P \mapsto P(\mathbb{C})$ is the functor that associates to every projective $\mathbb{C}[x_1, \ldots, x_n]$-module the corresponding vector bundle over $\mathbb{C}^n$; see Section 13.10, where this was denoted $P \mapsto P_{\mathbb{C}}$.)

The proof of Theorem 31.2 comes down to a comparison between algebraic and topological K-theory groups. Once the machinery for this comparison is in place, the theorem follows by a simple computation. To set up this machinery we will need to recall some ideas from Part 1.

Let $Z \subseteq \operatorname{Spec} R = X$ be Zariski closed (the assumption that X is affine is just temporary, to keep the discussion grounded). Define $K^0_{\mathrm{alg}}(X, X - Z)$ by taking the free abelian group on bounded chain complexes of algebraic vector bundles on X that are exact (in the sense of Section 30.2) at every point in $X - Z$ and quotienting by the following two relations:

(1) $[P_\bullet] = 0$ if $P_\bullet$ is exact on all of X, and
(2) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ for every short exact sequence of chain complexes $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

Note that $K^0_{\text{alg}}(X, X - Z)$ is exactly the same as the group denoted $K(R)_Z$ in Section 6.21; sometimes we will revert to that notation when we want to concentrate on the underlying algebraic perspective.

Exercise 31.3. Prove that if $P_\bullet \to Q_\bullet$ is a quasi-isomorphism of chain complexes that are exact on $X - Z$ then $[P_\bullet] = [Q_\bullet]$ in $K_{alg}(X, X - Z)$. (We have proven similar claims on more than one occasion, the exercise is just to remember how the proof works.)

The assignment $P \mapsto P(\mathbb{C})$ from algebraic to topological vector bundles induces a map of abelian groups

$$\phi\colon K^0_{\text{alg}}(X, X - Z) \to \mathcal{K}^0(X(\mathbb{C}), X(\mathbb{C}) - Z(\mathbb{C}))$$

where $\mathcal{K}^0$ is the Grothendieck group from Definition 20.2. This first uses Proposition 30.8 to know that if $P_\bullet$ is a chain complex that is exact on $X - Z$ then $P_\bullet(\mathbb{C})$ is fiberwise exact at all points in $X(\mathbb{C}) - Z(\mathbb{C})$. Then we need to check that relations (1) and (2) in the definition of $K^0_{\text{alg}}(X, X - Z)$ are preserved, but the first is trivial and the second follows from Proposition 20.10. Composing with the natural map from Theorem 20.14 we get a comparison map—which we will also call ϕ, by abuse—of the form $K^0_{\text{alg}}(X, X - Z) \to K^0(X(\mathbb{C}), X(\mathbb{C}) - Z(\mathbb{C}))$. To avoid cumbersome notation it will be convenient to write the target group as $K^0_{top}(X, X - Z)$, as it looks much more pleasant to write

$$\phi\colon K^0_{\text{alg}}(X, X - Z) \to K^0_{top}(X, X - Z).$$

Sometimes we will drop the "top" and just write $K^0(X, X - Z)$, but we will never drop the "alg".

If M is a finitely-generated R-module, recall that the **support** of M is

$$\operatorname{Supp} M = \{Q \subseteq R \,|\, Q \text{ is prime and } M_Q \neq 0\}.$$

This coincides with $V(\operatorname{Ann} M)$, the set of all primes containing $\operatorname{Ann} M$. In particular, $\operatorname{Supp} M$ is Zariski-closed. Let $G(X)_Z$ (or $G(R)_Z$) denote the Grothendieck group of finitely-generated R-modules M such that $\operatorname{Supp} M \subseteq Z$. We encountered this group back in Section 6.21. Here we should really assume that R is Noetherian, which is always a background assumption but wasn't explicitly needed in this section before now.

We have the usual Euler characteristic map $\chi\colon K^0_{\text{alg}}(X, X - Z) \to G(X)_Z$ that sends $[P_\bullet]$ to $\sum_i (-1)^i [H_i(P)]$. The following result should come as no surprise:

Theorem 31.4. *If R is Noetherian and regular then $\chi\colon K^0_{\text{alg}}(X, X - Z) \to G(X)_Z$ is an isomorphism, for any closed $Z \subseteq \operatorname{Spec} R$.*

Proof. The inverse sends a class $[M]$ to the class $[P_\bullet]$ for any finite projective resolution $P_\bullet$ for M over R. The proof that this is well-defined, and that the maps are inverses, is exactly the same as for Theorem 6.20 □

Our discussion so far has focused on the case where X is affine. Even if one only cares about this case, certain arguments are going to end up forcing us to consider more general schemes. To this end, let us just observe that the definition of the group $K^0_{alg}(X, X - Z)$ can be repeated almost verbatim for any scheme X and subset $Z \subseteq X$, using bounded complexes of locally free coherent sheaves that are exact (on stalks) at points in $X - Z$ (but see Remark 31.7 for a disclaimer).

One usually assumes Z is closed because exactness is an open condition, but this is not technically necessary for the definitions. Likewise, $G(X)_Z$ is defined to be the Grothendieck group of coherent sheaves on X whose support is contained in Z. For Noetherian schemes the Euler characteristic gives a map $\chi\colon K^0(X, X-Z) \to G(X)_Z$, and this is an isomorphism if X is regular and nice enough that we are guaranteed the existence of locally free resolutions (for discussion of this issue see Exercises 30.19, 30.22, and Remark 30.23).

Theorem 31.5. *Assume X is a separated, regular, Noetherian scheme and $Z \subseteq X$. Then $\chi\colon K^0_{alg}(X, X-Z) \to G(X)_Z$ is an isomorphism.*

Exercise 31.6. Prove the above theorem. The proof of Theorem 6.20 doesn't adapt verbatim because of the fact that locally free coherent sheaves are not projective, so we can't refer to standard homological algebra for some of the key steps. The parts below will help you get around this.

A complex of $\mathcal{O}_X$-modules is said to be **strictly perfect** if it is a bounded complex of locally-free coherent sheaves. Throughout this exercise X is assumed to be regular, Noetherian, and separated.

(a) Suppose $\mathcal{L}_\bullet$ is a complex of locally-free coherent sheaves such that $H_i(\mathcal{L}) = 0$ for all $i \geq N$. Use Exercise 30.22(c) to show that $\mathcal{L}_\bullet$ is quasi-isomorphic to a complex of locally-free coherent sheaves that is zero in degrees $N + \dim X$ and higher.

(b) Suppose $C_\bullet$ is a bounded complex of coherent sheaves. Prove that there is a strictly perfect complex $\mathcal{L}_\bullet$ and a quasi-isomorphism $\mathcal{L}_\bullet \to C_\bullet$. We call such a map a "resolution by a strictly perfect complex". [Hint: Build the resolution inductively using the same ideas as in Exercise 30.19(c), at each stage incorporating (in part) a surjection $\mathcal{L}_i \to H_i(C)$ where $\mathcal{L}_i$ is finite locally free. This could potentially construct an unbounded complex, but use (a) to cut it down to size.]

(c) Let $\mathcal{F}$ be a coherent sheaf and let $\mathcal{L}_\bullet \to \mathcal{F}$ and $\mathcal{E}_\bullet \to \mathcal{F}$ be two resolutions by strictly perfect complexes. Prove that there is a third such resolution $\mathcal{G}_\bullet \to \mathcal{F}$ and a zig-zag of quasi-isomorphisms $\mathcal{L}_\bullet \leftarrow \mathcal{G}_\bullet \to \mathcal{E}_\bullet$. [Hint: Start by forming the pullback $\mathcal{L}_\bullet \times_{\mathcal{F}} \mathcal{E}_\bullet$.]

(d) Suppose $\mathcal{F}' \to \mathcal{F}$ is a map of coherent sheaves on X. Prove that there are strictly perfect resolutions $\mathcal{L}_\bullet \to \mathcal{F}'$ and $\mathcal{E}_\bullet \to \mathcal{F}$ together with a map $\mathcal{L}_\bullet \to \mathcal{E}_\bullet$ making the evident square commute. If $\mathcal{F}' \to \mathcal{F}$ is an injection explain how to get a resolution of $\mathcal{F}/\mathcal{F}'$ from this data. [Hint: Same idea as in Exercise 30.19(f).]

(e) Use the previous parts to construct a map $\beta\colon G(X)_Z \to K^0_{alg}(X, X-Z)$ such that $\chi \circ \beta = \mathrm{id}$.

(f) Suppose $0 \to K \to \mathcal{L}_n \to \mathcal{L}_{n-1} \to \cdots \to \mathcal{L}_0 \to 0$ is an exact complex where all of the $\mathcal{L}_i$ are locally free coherent sheaves. Prove that K is also locally free and coherent. Prove that any strictly perfect complex is quasi-isomorphic to one that has nonzero homology in its lowest nonzero degree.

(g) Suppose that $C_\bullet$ is a strictly perfect complex that vanishes in negative degrees. Let $\mathcal{L}_\bullet \to H_0(C)$ be a strictly perfect resolution. Show that there is a strictly perfect complex $\mathcal{E}_\bullet$, a quasi-isomorphism $\mathcal{E}_\bullet \to C_\bullet$, and a map $\mathcal{E}_\bullet \to \mathcal{L}_\bullet$ such

that the induced triangle

$$\begin{array}{ccc} H_0(\mathcal{E}) & \longrightarrow & H_0(\mathcal{L}) \\ {\scriptstyle\cong}\downarrow & \swarrow{\scriptstyle\cong} & \\ H_0(C) & & \end{array}$$

is commutative. [Hint: Use the pullback technique that we saw in (c).]

(h) Use (f) and (g) to prove that β is surjective, and hence a two-sided inverse to χ.

Remark 31.7. A *perfect complex* is a chain complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to a strictly perfect complex. The groups $K^0_{alg}(X, X-Z)$ are often defined using perfect complexes rather than the strictly-perfect complexes that we used here (see, for example, [TT]). For Noetherian schemes with the resolution property the different definitions lead to isomorphic groups, so we will not concern ourselves with the distinction.

Now let us restrict to the case where $R = \mathbb{C}[x_1, \ldots, x_n]$, so that X is affine n-space $\mathbb{A}^n_{\mathbb{C}}$; we will just write $X = \mathbb{C}^n$ for convenience. Let $Z = \{0\} = V(x_1, \ldots, x_n)$ be the closed set consisting only of the origin. Our aim will be to calculate the group $K^0_{\text{alg}}(X, X - Z) = K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0)$ in this case.

Let $m = (x_1, \ldots, x_n)$. It is easy to see that the following conditions on a finitely-generated R-module M are equivalent:

(1) $\operatorname{Supp} M = \{m\}$;
(2) $\operatorname{Ann} M$ is contained in only one maximal ideal, namely m;
(3) $\operatorname{Rad}(\operatorname{Ann} M) = m$;
(4) M is annihilated by a power of m.

Assuming M satisfies these conditions, consider the finite filtration

$$M \supseteq mM \supseteq m^2M \supseteq \cdots \supseteq m^kM \supseteq m^{k+1}M = 0.$$

Then in $G(R)_Z$ we have $[M] = \sum_{i=0}^{k} [m^iM/m^{i+1}M]$. But each quotient is a finite-dimensional R/m-vector space, so $[M]$ is just a multiple of $[R/m]$. This shows that $G(R)_Z$ is cyclic, generated by $[R/m]$. Moreover, each quotient $m^iM/m^{i+1}M$ is finite-dimensional as a $\mathbb{C}$-module (where the module structure is coming from $\mathbb{C} \subseteq R$). It follows that M is also finite-dimensional as a $\mathbb{C}$-module. Since dimension is additive it gives a function

$$\dim\colon G(R)_Z \to \mathbb{Z},$$

which is clearly surjective and hence an isomorphism.

The isomorphism $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \to G(\mathbb{C}^n)_{\{0\}}$ from Theorem 31.4 sends the Koszul complex $K(x_1, \ldots, x_n; R)$ to $[R/m]$, but we have just seen that the target is isomorphic to $\mathbb{Z}$ and $[R/m]$ is a generator. So $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \cong \mathbb{Z}$ and is generated by the Koszul complex.

Now consider the following diagram:

$$\begin{array}{ccccc} K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) & \xrightarrow{\phi} & K^0_{top}(\mathbb{C}^n, \mathbb{C}^n - 0) & \xrightarrow{\cong} & \mathbb{Z}\langle\beta^n\rangle \\ \cong\big\downarrow \chi & & & & \\ G(R)_{\{m\}} & & \xrightarrow[\cong]{\dim} & & \mathbb{Z}. \end{array} \tag{31.8}$$

We know by Theorem 27.10 that the target of ϕ is isomorphic to $\mathbb{Z}$ and is generated by the Koszul complex. Likewise, we have just seen that the domain of ϕ is isomorphic to $\mathbb{Z}$ and is generated by the algebraic Koszul complex. Since ϕ clearly carries the algebraic Koszul complex to the topological one, ϕ is an isomorphism. This is important enough to record:

Proposition 31.9. *The comparison map $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0) \to K^0(\mathbb{C}^n, \mathbb{C}^n - 0)$ is an isomorphism, and both the domain and codomain are infinite cyclic.*

Our desired local index theorem is an immediate consequence of the above discussion:

Proof of Theorem 31.2. Fill in diagram (31.8) with the map $\mathbb{Z}\langle\beta^n\rangle \to \mathbb{Z}$ that sends β^n to 1. The diagram then commutes, because one only has to check this on the Koszul complex that generates $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - 0)$; and here it is obvious. The commutativity of this diagram is exactly the statement of Theorem 31.2. □

31.10. **Resolutions and fundamental classes.** Now we'll use these ideas to do something a bit more sophisticated. Let $Z \hookrightarrow \mathbb{C}^n$ be a closed algebraic subvariety and let $I \subseteq \mathbb{C}[x_1, \dots, x_n] = R$ be the ideal of functions vanishing on Z. So $Z = V(I)$. Assume that Z is smooth of codimension c. Then we have a relative fundamental class $[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$.

Let $P_\bullet$ be a bounded, projective resolution of R/I over $\mathbb{C}[x_1, \dots, x_n]$. Note that if $Q \in \operatorname{Spec} R$ then

$$Q \in Z \iff Q \supseteq I \iff (R/I)_Q \neq 0.$$

So if $Q \notin Z$ then $(R/I)_Q = 0$ and therefore $(P_\bullet)_Q$ is exact. Therefore $P_\bullet$ gives a class $[P_\bullet] \in K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z)$. Using our natural transformation $K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z) \to K^0(\mathbb{C}^n, \mathbb{C}^n - Z)$ we get a corresponding class $[P_\bullet]$ in relative topological K-theory. We can promote this to a class in relative K^{2c} by multiplying by β^{-c}. It is reasonable to expect this class to be related to $[Z]_{rel}$, and that is indeed the case:

Theorem 31.11. *In the above situation we have $[Z]_{rel} = \beta^{-c} \cdot [P_\bullet]$.*

Note that the β^{-c} factor could be dropped if we regarded $[Z]_{rel}$ as a class in $K^0(\mathbb{C}^n, \mathbb{C}^n - Z)$ instead of $K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$.

Theorem 31.11 gives one of the key connections between K-theory and homological algebra: projective (or, in the scheme setting, locally free) resolutions give fundamental classes in K-theory. To prove this theorem, recall that $[Z]_{rel}$ is defined by choosing a tubular neighborhood U of Z in $\mathbb{C}^n$, which includes an identification between U and the normal bundle $N = N_{\mathbb{C}^n/Z}$. The class $[Z]_{rel}$ is the unique class that restricts to the Thom class $\mathcal{U}_N$. This is all encoded in the following diagram:

$$\begin{array}{ccccc}
[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z) & \xleftarrow[\beta^{-c}]{\cong} & K^0(\mathbb{C}^n, \mathbb{C}^n - Z) & \xleftarrow{\phi} & K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z) \\
\text{excision}\downarrow\cong & & \downarrow\cong & & \\
K^{2c}(U, U - Z) & \xleftarrow[\beta^{-c}]{} & K^0(U, U - Z) & & \\
\| & & \| & & \\
\mathcal{U}_N \in K^{2c}(N, N - 0) & \xleftarrow[\beta^{-c}]{} & K^0(N, N - 0) & \xleftarrow{\phi} & K^0_{\text{alg}}(N, N - 0).
\end{array}$$

Our goal is to take $[P_\bullet] \in K^0_{\text{alg}}(\mathbb{C}^n, \mathbb{C}^n - Z)$, push it across the top row and then down, and show that the image is $\mathcal{U}_N$. But note that $\mathcal{U}_N$ is algebraic, as it is represented by the Koszul complex. So $\mathcal{U}_N$ lifts to a class in $K^0_{\text{alg}}(N, N - 0)$. In some sense the most natural idea for our proof would be to stay entirely in the right-most column, and to compare both $[P_\bullet]$ and $\mathcal{U}_N$ on the algebraic side of things. Of course one immediately sees the trouble, which is that the neighborhood U is *not* algebraic—and so we have a missing group in the third column, obstructing our proof. With some work we will establish a clever way around this, using a technique from algebraic geometry called *deformation to the normal bundle*. But let us first pause to look closer at the class $\mathcal{U}_N$ and its algebraic lift.

31.12. **Resolving the zero section of a bundle.** Let $\pi\colon E \to X$ be an algebraic vector bundle. The zero-section $\zeta\colon X \to E$ is a closed immersion; let ζ also denote the image, and consider the structure sheaf $\mathcal{O}_\zeta$. We claim that the Koszul complex $(J_{\pi^*E,\Delta})^*$ from Section 27.5 is a locally-free resolution of $\mathcal{O}_\zeta$:

$$0 \longrightarrow \Lambda^n(\pi^*E)^* \longrightarrow \Lambda^{n-1}(\pi^*E)^* \longrightarrow \cdots \longrightarrow \Lambda^1(\pi^*E)^* \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_\zeta \longrightarrow 0.$$

Let us think first about the case $X = \operatorname{Spec} k$ where k is a field, so that $E = \operatorname{Spec} k[x_1, \ldots, x_n]$ for some n. Let $R = k[x_1, \ldots, x_n]$. The sheaf $\mathcal{O}_\zeta$ corresponds to the module $R/(x_1, \ldots, x_n)$, and the above complex is precisely the Koszul complex $K(x_1, \ldots, x_n; R)$. [Note: The significance of the duals on the π^*E terms is hard to see in this setting. Algebraically it comes down to the difference between $R \otimes_k k^n$ and $R \otimes_k (k^n)^*$, which is almost invisible—it is essentially being absorbed into the combinatorics of the Koszul differentials. But the importance of the duals will become clearer in a moment.]

For the general case, it suffices to work locally on X: so we may assume $X = \operatorname{Spec} A$ for some ring A. Then ΓE is a finitely-generated projective A-module P, and as a scheme $E = \operatorname{Spec}(\operatorname{Sym}_A P^*)$ (the symmetric algebra over A generated by P^*). One can even assume P is a free A-module, by replacing A with a localization. Let $R = \operatorname{Sym}_A(P^*)$. The zero-section (as a subscheme) corresponds to the quotient ring $R/(P^*) = A$, and if $n = \operatorname{rank}(P)$ we can write down the Koszul complex

$$0 \to R \otimes_A \Lambda^n(P^*) \to R \otimes_A \Lambda^{n-1}(P^*) \to \cdots \to R \otimes_A \Lambda^1(P^*) \to R \to 0$$

with the Koszul differential being

$$d(r \otimes (\phi_1 \wedge \cdots \wedge \phi_k)) = \sum_{j=1}^{k} (-1)^{j-1} r\phi_j \otimes (\phi_1 \wedge \cdots \wedge \widetilde{\phi_j} \wedge \cdots \wedge \phi_k).$$

Locally P is a free module with a basis $p_1, \ldots, p_n$ and dual basis $x_1, \ldots, x_n$, and then this is precisely the Koszul complex $K(x_1, \ldots, x_n; A[x_1, \ldots, x_n])$. In particular, it is a resolution of $R/(P^*) = A$.

It only remains to connect this algebraic Koszul complex with $J^*_{\pi^*E,\Delta}$. The key observation is that sections of $\pi^*E \to E$ correspond precisely to elements of $\mathcal{S}ch_X(E,E)$, and this can be analyzed as

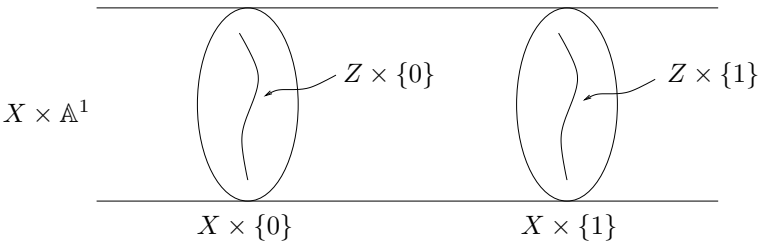
$$\mathcal{S}ch_X(E,E) = \mathrm{Ring}(\mathrm{Sym}_A P^*, \mathrm{Sym}_A P^*) = \mathrm{Hom}_A(P^*, \mathrm{Sym}_A P^*) = P \otimes_A \mathrm{Sym}_A P^*.$$

The last equality is an instance of the canonical map $\mathrm{Hom}_A(Q,A) \otimes_A M \to \mathrm{Hom}_A(Q,M)$, which is an isomorphism when Q is a finitely-generated A-projective (here $Q = P^*$ and we are also using the canonical identification $P^{**} \cong P$). The module of sections of $(\pi^*E)^* \to E$ therefore equals the R-module $\mathrm{Hom}_R(P \otimes_A R, R) = \mathrm{Hom}_A(P,A) \otimes_A R = P^* \otimes_A R$, where the first map is again an example of a canonical map that is an isomorphism because P is finitely-generated projective. Using these isomorphisms one readily checks that $J^*_{\pi^*E,\Delta}$ becomes—upon taking algebraic sections—precisely the Koszul complex from the last paragraph. Putting everything together, we have therefore proven:

Proposition 31.13. *Let $E \to X$ be an algebraic vector bundle. The algebraic Koszul complex $J^*_{\pi^*E,\Delta}$ is a chain complex of locally free coherent sheaves on E that is a resolution of the structure sheaf of the zero section. This is a lifting in $K^0_{\mathrm{alg}}(E, E-0)$ for the Thom class $\mathcal{U}_E \in K^0(E, E-0)$.*

Remark 31.14. When we defined the Thom class $\mathcal{U}_E$ there was a choice to use J^* rather than J, and we noted at the time that it would eventually become clear why we used J^*. Proposition 31.13 is the reason.

31.15. **Deformation to the normal bundle.** Let X be a smooth scheme over $\mathbb{C}$ and let $Z \hookrightarrow X$ be a smooth closed subscheme (feel free to only think about $X = \mathbb{C}^n$). We begin by considering $X \times \mathbb{A}^1$:



We will work with the blow-up of $X \times \mathbb{A}^1$ at the subvariety $Z \times \{0\}$:

$$B = B\ell_{Z\times 0}(X \times \mathbb{A}^1).$$

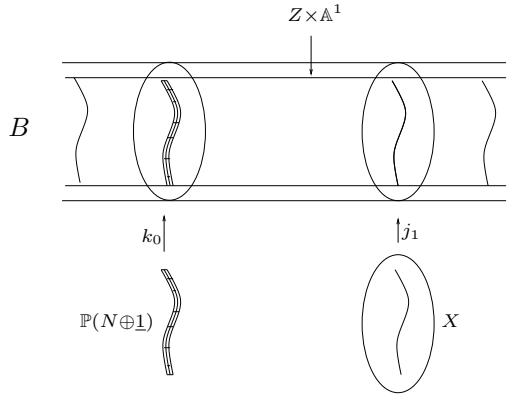
Set $N = N_{X/Z}$ and $N' = N_{X\times\mathbb{A}^1/Z\times\{0\}}$. Note that $N' = N \oplus 1$. Topologically, we have a homeomorphism

$$B \cong \big[(X \times \mathbb{A}^1) - (Z \times \{0\})\big] \amalg_{(V-0)} \mathbb{P}(N')$$

where V is a tubular neighborhood of $Z \times \{0\}$ in $X \times \mathbb{A}^1$.

Let $\pi\colon B \to X \times \mathbb{A}^1$ be the projection associated to the blow-up. Let $j_1\colon X \hookrightarrow B$ be the map $x \mapsto (x,1)$. Let $B_0 = \pi^{-1}(0)$ and let $j_0\colon B_0 \hookrightarrow B$ be the inclusion. Let

$k_0 \colon \mathbb{P}(N \oplus \underline{1}) \hookrightarrow B$ be the inclusion of the exceptional divisor, which actually lands inside B_0. These can be visualized via the following schematic picture:



We claim that $\pi \colon B \to X \times \mathbb{A}^1$ has a section f over $Z \times \mathbb{A}^1$. The definition of this section is completely clear (and unique) on $Z \times (\mathbb{A}^1 - 0)$, the only sublety is the definition on $Z \times \{0\}$; but here we use the canonical section of $\mathbb{P}(N \oplus \underline{1}) \to Z$. A little effort shows that this gives a well-defined map $Z \times \mathbb{A}^1 \to B$, and it is clearly a section (indicated in the above picture).

Example 31.16. Consider the case $X = \mathbb{C}^n$ and $Z = \{\underline{0}\}$. Then B is the blowup of $\mathbb{C}^n \times \mathbb{C}$ at the origin, and we can use the model of B as a subspace of $\mathbb{C}^{n+1} \times \mathbb{C}P^n$. The map $Z \times \mathbb{A}^1 \to B$ on closed points is given by

$$(\underline{0}, s) \mapsto \begin{cases} ((\underline{0}, s), [\underline{0} : s]) & \text{if } s \neq 0, \\ (\underline{0}, 0), [\underline{0} : 1]) & \text{if } s = 0. \end{cases}$$

Observe that this is continuous at $s = 0$.

Returning to our general setting, if X is locally $\operatorname{Spec} R$ and $Z = V(I)$ and $T = R[s]$ then the blowup B is $\operatorname{Proj} T[(I, s)t]$. Let $J \subseteq T[(I, s)t]$ be the ideal (I, It). Then $T[(I, s)t]/J = R/I[s][st]$, with s in degree 0 and st in degree 1. Taking Proj of this graded ring yields $\operatorname{Spec} R/I[s] = Z \times \mathbb{A}^1$, and so this exhibits $Z \times \mathbb{A}^1$ as a closed subscheme of B.

Observe that $B_0 = \operatorname{Bl}_Z(X) \amalg_{\mathbb{P}(N)} \mathbb{P}(N \oplus 1)$ is a decomposition as a union of closed subschemes. The $\mathbb{P}(N \oplus 1)$ is the exceptional divisor, and so of course is closed in B and therefore also in B_0 (since B_0 is itself closed in B). For $\operatorname{Bl}_Z(X)$ we work locally with $X = \operatorname{Spec} R$ and $Z = V(I)$, so that the blowup is Proj of the Rees ring $T = (R[s])[(I, s)t]$ (apologies for the horrible notation). Quotienting out the ideal (s, st) gives the Rees ring $R[It]$, demonstrating that $\operatorname{Bl}_Z(X)$ is closed in B (and therefore also in B_0).

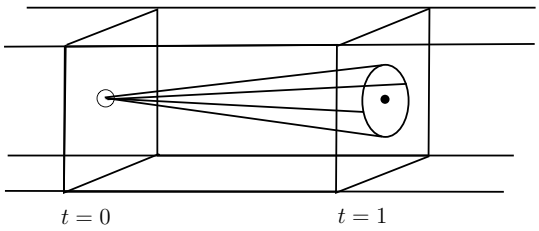
Consider the following (non-commutative) diagram of pairs of spaces:

$$\begin{array}{ccc} (X, X-Z) & \xrightarrow{\quad j_1 \quad} & (B, B-(Z\times\mathbb{A}^1)) \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle j_0} \\ (U, U-Z) & & (B_0, B_0-(Z\times\{0\})) \\ \| & & \uparrow{\scriptstyle k_0} \\ (N, N-0) & \xrightarrow{\quad i \quad} & (\mathbb{P}(N\oplus\underline{1}), \mathbb{P}(N\oplus\underline{1})-\mathbb{P}(\underline{1})) \end{array} \tag{31.17}$$

Here i is the canonical open inclusion that sends a point $v \in N$ to the line spanned by $(v, 1) \in N \oplus \underline{1}$. Note that all of the maps are algebraic except for the inclusion $h\colon N \hookrightarrow X$ (we implicitly identify N with U here).

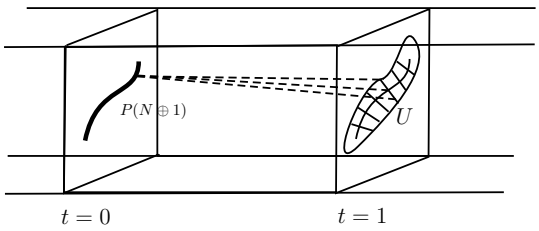
Proposition 31.18. *The diagram in (31.17) commutes up to homotopy. (This homotopy is the so-called "deformation to the normal bundle".)*

Proof. To see this, let us first examine the case $X = \mathbb{C}^n$ and $Z = \{\underline{0}\}$ where we can use the model for the blowup where $B \subseteq \mathbb{C}^{n+1} \times \mathbb{C}P^n$. Let $U \subseteq X$ be the open unit disk around Z and consider the homotopy $U \times I \to B$ given by $(\underline{x}, t) \mapsto (t(\underline{x}, 1), [\underline{x} : 1])$. At $t = 1$ this is the embedding j_1, and at $t = 0$ it is the embedding k_0. The following picture depicts what is happening here:



Just keep in mind that what looks like the cone point is not actually a single point, but rather a $\mathbb{C}P^n$. As a point $\underline{x}$ in the neighborhood U (at $t = 1$) moves towards the origin at $t = 0$, along the indicated line, it converges to the point $[\underline{x} : 1]$ in $\mathbb{C}P^n$.

For the case of a general Z, the homotopy is exactly the one above but taking place in the normal directions to Z. In reference to the crude picture below, each fiber of the normal disk bundle U (at $t = 1$) attempts to collapse—via the straight-line homotopy depicted above—to the corresponding point on Z at $t = 0$, but with points in the fiber converging to the associated point in the blowup that represents the direction of approach.



□

Remark 31.19. The technique of "deformation to the normal bundle", which is essentially embodied in the diagram (31.17) and its homotopy commutativity, was used extensively in papers by Fulton and MacPherson in the 1970s, and has a prominent role in the book [Fulton]. The technique gives a substitute in algebraic geometry for the role played by tubular neighborhoods in topology. A short historical discussion is given at the end of [Fulton, Chapter 5].

With the basics of deformation to the normal bundle under our belt, we are now ready to give the proof of Theorem 31.11. Actually, it is just as easy to prove it in the following generalized form:

Theorem 31.20. *Let X be a smooth variety over $\mathbb{C}$ and let $Z \hookrightarrow X$ be a smooth subvariety of codimension c. Let $\mathcal{L}_\bullet \to \mathcal{O}_Z$ be a bounded resolution by locally free coherent $\mathcal{O}_X$-modules. Then $[Z]_{rel} = \beta^{-c}[\mathcal{L}_\bullet]$ in $K^0_{top}(X, X - Z)$.*

Proof. Apply $K^0(-)$ to the diagram (31.17) to obtain the commutative diagram

$$\begin{array}{ccc} K^0(X, X-Z) & \xleftarrow{\quad j_1^* \quad} & K^0(B, B-(Z\times\mathbb{A}^1)) \\ \Big\downarrow{\scriptstyle h^*} & & \Big\downarrow{\scriptstyle k_0^*} \\ K^0(N, N-0) & \xleftarrow{\quad i^* \quad} & K^0(\mathbb{P}(N\oplus\underline{1}), \mathbb{P}(N\oplus\underline{1}) - \mathbb{P}(\underline{1})). \end{array} \tag{31.21}$$

The left vertical arrow is dotted only as a reminder that it is not algebraic. There is a similar diagram, without the dotted arrow, in which every $K^0(-)$ has been replaced with K^0_{alg}; and this new diagram maps to the one above.

Let $Q_\bullet$ be a resolution of $\mathcal{O}_{Z\times\mathbb{A}^1}$ by locally-free $\mathcal{O}_B$-modules (see Section 28 for terminology and notation here). Then we have the corresponding class $[Q_\bullet] \in K^0_{alg}(B, B - (Z \times \mathbb{A}^1))$. We will show that

(1) $j_1^*(Q_\bullet)$ is a resolution of $\mathcal{O}_Z$ on X, and
(2) $(k_0 \circ i)^*(Q_\bullet)$ is a resolution on N of the structure sheaf of the zero-section.

Statement (1) implies that there is a zig-zag of quasi-isomorphisms $j_1^*(Q_\bullet) \xleftarrow{\simeq} \mathcal{E}_\bullet \xrightarrow{\simeq} P_\bullet$, where $P_\bullet$ is our chosen resolution of $\mathcal{O}_Z$ on X and $\mathcal{E}_\bullet$ is some other bounded locally free resolution of $\mathcal{O}_Z$ (this follows from Exercises 30.19 and 30.22). Hence $j_1^*([Q_\bullet]) = [P_\bullet]$ in $K^0_{\text{alg}}(X, X - Z)$. Statement (2) implies that there is a similar zig-zag of quasi-isomorphisms between $(k_0 \circ i)^*(Q_\bullet)$ and $J^*_{p^*N,\Delta}$, since by Proposition 31.13 both give resolutions of the structure sheaf of the zero section on N. Hence $i^*(k_0^*([Q_\bullet])) = \mathcal{U}_N$ in $K^0_{\text{alg}}(N, N - 0)$, since $\mathcal{U}_N = [J^*_{p^*N,\Delta}]$ by definition. Now push all of this into topological K^0 and use the commutativity of (31.21) to obtain that $h^*([P_\bullet]) = \mathcal{U}_N$. But h^* is an isomorphism, and $[Z]_{rel}$ is defined to be the unique class in $K^0(X, X - Z)$ that maps to $\mathcal{U}_N$ via h^*. So $[P_\bullet] = [Z]_{rel}$.

So the proof reduces to checking the algebraic facts (1) and (2). The geometric intuition here is that the subscheme $Z \times \mathbb{A}^1$ intersects the fibers B_0 and B_1 cleanly enough so that exactness of $Q_\bullet$ is maintained after pulling back to these fibers. Algebraically this boils down to certain Tor calculations that we now explain. Let $\mathcal{F}$ be a coherent sheaf on B and let $\mathcal{L}_\bullet \to \mathcal{F}$ be a resolution by locally free coherent sheaves. Then we have the left derived functor of j_1^* given by $(L_r j_1^*)(\mathcal{F}) = H_r(j_1^*\mathcal{L}_\bullet)$. Apply $(j_1)_*$ and use that this direct image functor is exact, since j_1 is a closed

immersion (stalkwise exactness follows from (28.21)). This yields

$$(31.22)\qquad (j_1)_*(L_r j_1^*(\mathcal{F})) = (j_1)_* H_r(j_1^*\mathcal{L}_\bullet) = H_r((j_1)_* j_1^*\mathcal{L}_\bullet) = H_r(\mathcal{L}_\bullet \otimes_{\mathcal{O}_B} \mathcal{O}_{B_1}) \\ = \operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_{B_1})$$

(the second-to-last equality is the projection formula of Proposition 30.13). The functor $(j_1)_*$ does not change the stalks at the points of B_1, and so $L_r j_1^*(\mathcal{F}) = 0$ if and only if $\operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{F}, \mathcal{O}_{B_1}) = 0$. In particular, (1) is equivalent to the statement that $\operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{O}_{Z\times\mathbb{A}^1}, \mathcal{O}_{B_1}) = 0$ for $r > 0$.

Observe that j_1 factors as $X \xrightarrow{\tilde{j}_1} B \setminus B_0 \xrightarrow{j} B$. Here $B\setminus B_0$ is open in B and so j^* is exact. The scheme $B \setminus B_0$ is just $X \times (\mathbb{A}^1 - 0)$ and $j^*(\mathcal{O}_{Z\times\mathbb{A}^1}) = \mathcal{O}_{Z\times(\mathbb{A}^1-0)}$. We therefore need to prove that $L_r\tilde{j}_1^*(\mathcal{O}_{Z\times(\mathbb{A}^1-0)}) = 0$ for $r > 0$, and by the analog of (31.22) this will be implied by $\operatorname{Tor}_r^{\mathcal{O}_{X\times(\mathbb{A}^1-0)}}(\mathcal{O}_{X\times 1}, \mathcal{O}_{Z\times(\mathbb{A}^1-0)}) = 0$. The vanishing of the Tor_r sheaf can be checked locally (e.g. on stalks) at each point of $X\times(\mathbb{A}^1-0)$, and so in particular we can check it locally on X. That is to say, it suffices to prove the vanishing where X is an open affine $X = \operatorname{Spec} R$, in which case $Z = V(I)$ and we are looking at $\operatorname{Tor}_r^{R[s,s^{-1}]}(R[s,s^{-1}]/(s-1), R/I[s,s^{-1}])$. But $R[s,s^{-1}]/(s-1)$ is resolved by the complex $R[s,s^{-1}] \xrightarrow{s-1} R[s,s^{-1}]$ and $s-1$ is a nonzerodivior on $R/I[s,s^{-1}]$, therefore the indicated Tor_r modules vanish for $r \geq 1$ and we are done.

For statement (2) we need to look at the composite

$$N \xrightarrow{i} \mathbb{P}(N\oplus 1) \xrightarrow{k_0} B_0 \xrightarrow{j} B$$

and compute the left derived functors of pulling back $\mathcal{O}_{Z\times\mathbb{A}^1}$ along this composite. Here i is an open inclusion and both k_0 and j are closed inclusions. We first claim that $k_0 i$ is actually an open inclusion: this is because $B_0 = \operatorname{Bl}_Z X \cup_{\mathbb{P}(N)} \mathbb{P}(N\oplus 1)$ and so $N = \mathbb{P}(N\oplus 1) - \mathbb{P}(N) = B_0 - \operatorname{Bl}_Z X$, together with the fact that $\operatorname{Bl}_Z X$ is closed in B_0. So $(k_0 i)^*$ is exact, and it is enough to prove that $L_r j^*(\mathcal{O}_{Z\times\mathbb{A}^1}) = 0$ for $r > 0$.

Now perform the same analysis from (31.22) but with j_1 replaced by j and deduce that $L_r j^*(\mathcal{O}_{Z\times\mathbb{A}^1}) = 0$ if and only if $\operatorname{Tor}_r^{\mathcal{O}_B}(\mathcal{O}_{Z\times\mathbb{A}^1}, \mathcal{O}_{B_0}) = 0$. Observe that B_0 sits in the pullback diagram

$$(31.23)\qquad \begin{array}{ccc} B_0 & \xrightarrow{j_0} & B \\ \downarrow & & \downarrow{\scriptstyle s} \\ \{0\} & \longrightarrow & \mathbb{A}^1 \end{array}$$

which identifies $\mathcal{O}_{B_0}$ with the cokernel of $\mathcal{O}_B \xrightarrow{\cdot s} \mathcal{O}_B$: locally $B = \operatorname{Spec} T$ and $B_0 = \operatorname{Spec}[T \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/s] = \operatorname{Spec} T/sT$. We claim that

$$(31.24)\qquad 0 \longrightarrow \mathcal{O}_B \xrightarrow{\cdot s} \mathcal{O}_B \longrightarrow \mathcal{O}_{B_0} \longrightarrow 0$$

is exact. Again working locally on X we can assume that $X = \operatorname{Spec} R$, $Z = V(I)$, and then B is the Rees ring $R[s][(I,s)t]$. It is easy to check that s is a nonzerodivisor in this ring. Alternatively, this follows at once from $B \to \mathbb{A}^1$ being a flat map: for flatness over curves, [Hartshorne, Proposition III.9.7] says one can just check that every irreducible component has dense image in the curve, and that is clearly the

case for $B \to \mathbb{A}^1$. So take the resolution $0 \to \mathcal{O}_{\mathbb{A}^1} \xrightarrow{\cdot s} \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_0 \to 0$ and tensor with $\mathcal{O}_B$ over $\mathcal{O}_{\mathbb{A}^1}$ (or said differently, apply π^*) to get the exact sequence above.

Using (31.24) it follows that $\operatorname{Tor}_*^{\mathcal{O}_B}(\mathcal{O}_{B_0}, \mathcal{O}_{Z\times\mathbb{A}^1})$ is the homology of

$$0 \longrightarrow \mathcal{O}_{Z\times\mathbb{A}^1} \xrightarrow{\cdot s} \mathcal{O}_{Z\times\mathbb{A}^1} \longrightarrow 0$$

which is clearly zero in degrees ≥ 1: either realize the map locally as $R[s] \xrightarrow{s} R[s]$ or use the same argument from the last paragraph based on $Z \times \mathbb{A}^1 \to \mathbb{A}^1$ being flat (the two arguments are the same, of course). □

Remark 31.25. In the above proof, the vanishing of the higher Tor's might seem magical. It turns out, though, that the higher Tor's always vanish when dealing with reasonably nice subschemes whose intersection has the expected dimension—see Section 36.8. In both examples from the above proof, we have subschemes of dimension $\dim X$ and $\dim Z+1$ intersecting inside a scheme of dimension $\dim X+1$. The expected dimension for the intersection is $\dim X + (\dim Z+1) - (\dim X+1) = \dim Z$, which matches the actual dimension of the intersection.

The proof of Theorem 31.20 can be adapted to give a more general result. Consider the diagram

$$\begin{array}{ccccc} K^0_{alg}(Z) & \xrightarrow{\cong} & G_0(Z) & \xrightarrow{\cong} & K^0_{alg}(X, X-Z) \\ \downarrow & & & & \downarrow \\ K^0(Z) & & \xrightarrow{\quad j_! \quad} & & K^0(X, X-Z) \end{array} \tag{31.26}$$

where $G_0(Z)$ is the Grothendieck group of coherent sheaves, the top left map sends the class $[E]$ of an algebraic vector bundle to $[E]$ regarded as a coherent sheaf, and the top right map sends the class of a coherent sheaf F on Z to a bounded free resolution of j_*F. The bottom map is the topological pushforward (but neglecting powers of the Bott element). Theorem 31.20 showed that taking $[\mathcal{O}_Z]$ in the upper left corner and pushing it around the diagram in the two possible ways gives the same result. But the stronger statement is the following:

Theorem 31.27. *The rectangle in (31.26) commutes.*

Exercise 31.28. Adapt the proof of Theorem 31.20 to prove Theorem 31.27. [Hint: To get started, let E be a locally free coherent sheaf on Z and write $E \times \mathbb{A}^1$ for the pullback of E to $Z \times \mathbb{A}^1$. If $f\colon Z \times \mathbb{A}^1 \hookrightarrow B$ is the closed inclusion into the blowup then $\mathcal{O}_{Z\times\mathbb{A}^1}$ will be replaced everywhere by $f_*(E \times \mathbb{A}^1)$. The exercise is to see what, if anything, needs to be changed in the argument.]

Theorem 31.27 will be important in Section 43 as part of our treatment of the Grothendieck-Riemann-Roch theorem.

As a consequence of Theorem 31.11 we can now obtain Serre's formula for intersection multiplicities:

Corollary 31.29. *Let Z and W be smooth, closed subvarieties of $\mathbb{C}^n$ such that $Z \cap W = \{0\}$. Then $i(Z, W; 0) = \sum(-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)$, where $R = \mathbb{C}[x_1, \ldots, x_n]$ and I and J are the ideals of functions vanishing on Z and W, respectively.*

Proof. Start with the relative fundamental classes $[Z]_{rel} \in K^{2c}(\mathbb{C}^n, \mathbb{C}^n - Z)$ and $[W]_{rel} \in K^{2d}(\mathbb{C}^n, \mathbb{C}^n - W)$, where c and d are the codimensions of Z and W inside of $\mathbb{C}^n$. Note that since 0 is an isolated point of intersection we must have $c+d \geq n$ (this is geometrically intuitive, but see Lemma 31.31 below for an algebraic proof). There are two cases, depending on whether $c+d = n$ or $c+d > n$. But it will be convenient to handle these two cases essentially simultaneously, only pointing out the small differences that distinguish them.

In the former case, multiplying our fundamental classes together we get

$$[Z]_{rel} \cdot [W]_{rel} \in K^{2n}(\mathbb{C}^n, \mathbb{C}^n - (Z \cap W)) = K^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0).$$

Note that $K^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0) \cong \mathbb{Z}$ and is generated by $[0]_{rel}$. The topological definition of $i(Z, W; 0)$ is that it is the unique integer for which

$$[Z]_{rel} \cdot [W]_{rel} = i(Z, W; 0) \cdot [0]_{rel}. \tag{31.30}$$

This definition works in any complex-oriented cohomology theory.

If $c+d > n$ then it is clear that Z and W may be moved near 0 (in the topological setting) so that they do not intersect at all, and therefore $[Z]_{rel} \cdot [W]_{rel} = 0$. Equation (31.30) is still a valid definition, it just yields that $i(Z, W; 0) = 0$ here.

The key to the proof is simply realizing that all of our constructions can be lifted back into K^0_{alg}. Let $P_\bullet \to R/I$ and $Q_\bullet \to R/J$ be bounded projective resolutions. Then $[Z]_{rel} = \beta^{-c} \cdot [P_\bullet]$ and $[W]_{rel} = \beta^{-d} \cdot [Q_\bullet]$ by Theorem 31.11. So $[Z]_{rel} \cdot [W]_{rel} = \beta^{-c-d} \cdot [P_\bullet \otimes_R Q] \in K^{2(c+d)}(\mathbb{C}^n, \mathbb{C}^n - 0)$. Recall from Theorem 27.10 that $K^0(\mathbb{C}^n, \mathbb{C}^n - 0) \cong \mathbb{Z}$ and is generated by the Koszul complex J^*. Recall as well that $[0]_{rel} = \beta^{-n} \cdot [J^*]$, by definition. If we write $[P \otimes_R Q] = s[J^*]$ for $s \in \mathbb{Z}$, then we have the formula

$$s \cdot \beta^{-c-d} \cdot [J^*] = i(Z, W; 0) \cdot \beta^{-n} \cdot [J^*].$$

If $c+d = n$ then the formula implies $s = i(Z, W; 0)$. If $c+d \neq n$ then the only way the formula can be true is if both sides are zero, in which case $s = 0 = i(Z, W; 0)$. So $s = i(Z, W; 0)$ in either case.

To conclude the proof we just note that the Local Index Theorem (31.2) gives $s = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(P \otimes_R Q)$, and $H_i(P \otimes Q) \cong \operatorname{Tor}_i(R/I, R/J)$. □

Lemma 31.31. *Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let P and Q be prime ideals such that* $\operatorname{Rad}(P+Q)$ *is a maximal ideal. Then* $\dim R/P + \dim R/Q \leq n$, *or equivalently* $\operatorname{codim} R/P + \operatorname{codim} R/Q \geq n$.

Proof. The proof is from [Serre2, Proposition III.17], though we follow the arrangement given in [Hochster5]. Let $\mu\colon R \otimes_k R \to R$ be the multiplication map. This is surjective and the kernel is the ideal $\Delta = (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$, so we can identify R with the quotient $(R \otimes_k R)/\Delta$. Consider the string of isomorphisms

$$R/(P+Q) \cong R/P \otimes_R R/Q \cong (R/P \otimes_k R/Q) \otimes_{R \otimes_k R} R \cong (R/P \otimes_k R/Q)/\Delta.$$

Since Δ is generated by n elements, quotienting out by Δ drops the dimension by at most n. So we obtain

$$\dim R/(P+Q) \geq \dim(R/P \otimes_k R/Q) - n.$$

Since $\operatorname{Rad}(P+Q)$ is a maximal ideal we have that $\dim R/(P+Q) = 0$. So the proof will be completed if we show that $\dim(R/P \otimes_k R/Q) = \dim R/P + \dim R/Q$. For convenience let us prove the analogous statement for any domains D_1 and D_2 that are finitely-generated k-algebras.

By Noether normalization there exist polynomial rings $B_1 \subseteq D_1$ and $B_2 \subseteq D_2$ such that each D_i is module-finite over B_i, and dimension theory tells us that $\dim D_i = \dim B_i$. But then $B_1 \otimes_k B_2 \to D_1 \otimes_k D_2$ is an inclusion (since the tensors are taken over a field), the extension is module-finite, and $B_1 \otimes_k B_2$ is a polynomial ring over k. So $\dim(D_1 \otimes_k D_2) = \dim(B_1 \otimes_k B_2) = \dim B_1 + \dim B_2 = \dim D_1 + \dim D_2$. □

Exercise 31.32. Suppose that Z and W are smooth algebraic subvarieties of $\mathbb{C}^n$ such that $Z \cap W = \{p_1, \ldots, p_d\}$.

(a) Choose a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(p_i) = 0$ for $i > 1$ and $f(p_1) \neq 0$, and let $S = R_f$. Let $U = \operatorname{Spec} R_f \subseteq \mathbb{C}^n$ be the corresponding Zariski open set. Convince yourself that it is reasonable to define the intersection multiplicity $i(Z, W; p_1)$ by the formula

$$[Z]_{rel,U} \cdot [W]_{rel,U} = i(Z, W; p_1) \cdot \beta^{-n} \cdot [p_1]_{rel,U}$$

where $[Z]_{rel,U}$ is the image of $[Z]_{rel}$ under $K^*(\mathbb{C}^n, \mathbb{C}^n - Z) \to K^*(U, U - Z)$, and similarly for $[W]_{rel,U}$ and $[p_1]_{rel,U}$. In particular, convince yourself such an f exists and that $i(Z, W; p_1)$ is independent of the choice of f.

(b) Next, modify the proof of Corollary 31.29 to show that

$$i(Z, W; p_1) = \sum (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_f$$

where I and J are the ideals corresponding to Z and W.

(c) If M is a finitely-generated module over R_f such that $\operatorname{Supp} M = \{p\}$ (p a maximal ideal of R), prove that $M = M_p$. Deduce that

$$i(Z, W; p_1) = \sum (-1)^i \dim_{\mathbb{C}} \operatorname{Tor}_i(R/I, R/J)_{p_1}.$$

Corollary 31.29 (and Exercise 31.32) in some sense brings to a close the main questions we raised at the beginning of this book. We have now seen why Serre's alternating sum of Tor definition of intersection multiplicity is a reasonable one, and how this ties in to the study of K-theory.

Our discussion in this section has been confined to the truly geometric case of affine algebras over $\mathbb{C}$, but recall that Serre's definition works over every regular local ring R. If M and N are finitely-generated R-modules such that $\ell(M \otimes N) < \infty$ then Serre defines $i(M, N) = \sum_{j=0}^{\dim R} (-1)^j \ell(\operatorname{Tor}_j(M, N))$. Serre proved that in this situation one always has $\dim M + \dim N \geq \dim R$ and asked if the the following are true:

(1) $i(M, N) \geq 0$;
(2) $i(M, N) = 0$ if $\dim M + \dim N < \dim R$;
(3) $i(M, N) > 0$ if $\dim M + \dim N = \dim R$.

Using prime filtrations of M and N one reduces easily to the case $M = R/P$ and $N = R/Q$ where P and Q are prime. Serre developed purely algebraic arguments that handle the cases where R is equicharacteristic or mixed characteristic and unramified. In the 1980s Roberts and Gillet-Soulé independently established (2) using ideas related to K-theory, and in the late 1990s Gabber proved (1) in general. See [Roberts2], [Roberts5], [GS] for the work on (2), and for Gabber's proof of (1) see [Hochster6] and [Roberts6]. Note that (3) for mixed characteristic rings remains open at the time of writing.

Part 6. Exploring the apparatus

In the last several sections we developed the basic connections between K-theory and geometry. We have seen that K-theory is a complex-oriented cohomology theory, and we understand "geometric" representatives for the Thom classes and fundamental classes that come with such a theory; in this case "geometric" means that we can write down explicit chain complexes of vector bundles representing the classes. In the following sections our aim is to further explore this general area: now that the basic picture is in place, where does it take us? The topics we cover are somewhat of a hodgepodge, but in some sense they all revolve around the exploration of fundamental classes.

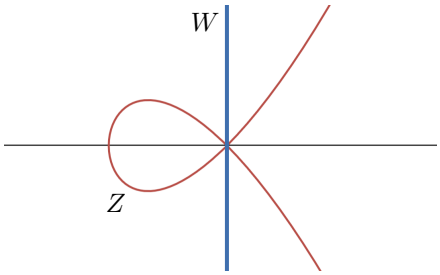
32. More about multiplicities

UNDER CONSTRUCTION

This section is a hodgepodge of different topics that didn't quite fit into the preceding material but need to be discussed before we more forward.

32.1. **Multiplicities in the singular setting.** When we defined topological intersection multiplicities we assumed that Z and W were submanifolds of a smooth manifold X, with P an isolated point of intersection. The smoothness conditions on Z and W were needed to get the fundamental classes $[Z]$ and $[W]$. In algebraic geometry, though, the Serre intersection multiplicity formula works just fine even if Z and W have singularities. So why this difference?

It is informative to work through a simple example. Let $X = \mathbb{A}^2$, let Z be the nodal cubic $y^2 = x^2 + x^3$, and let $W = V(x)$ be the y-axis.



Let us first compute the Serre intersection multiplicity. Since $\mathbb{C}[x,y]/(y^2-x^2-x^3)$ is a domain x is a nonzerodivisor, hence $\operatorname{Tor}_i^{\mathbb{C}[x,y]}(\mathbb{C}[x,y]/(y^2-x^2-x^3), \mathbb{C}[x,y]/(x)) = 0$ for $i \geq 1$. We get that

$$i(Z, W; 0) = \dim \mathbb{C}[x,y]/(x, y^2 - x^2 - x^3) = \dim \mathbb{C}[y]/(y^2) = 2.$$

This agrees with our visual inspection of the above graph, where we see that moving W to $V(x - \epsilon)$ gives two transveral points of intersection.

Moving to the topological framework, since $Z - 0$ is a smooth submanifold of $X - 0$ we have a fundamental class $[Z - 0]_K \in K^0(X - 0)$. Consider the Gysin sequence

$$\cdots \to K^0(pt) \xrightarrow{j_!} K^0(X) \longrightarrow K^0(X - 0) \longrightarrow K^1(pt) = 0.$$

This shows that $[Z-0]$ lifts to a class α in $K^0(X)$, though not canonically. But α is well-defined up to multiples of $[pt]$, and therefore $\alpha \cdot [W]$ is well-defined up to multiples of $[pt]\cdot[W]$. But $[pt]\cdot[W]=0$ for dimension reasons, i.e. we can move the point off of the line. So we find that the topological machinery gives us a way to make sense of $[Z]\cdot[W]$

??? [more discussion of fundamental classes in H^* and K^* for singular varieties]

32.2. **The algebraic degree.** Suppose given $f_1,\dots,f_n \in \mathbb{C}[z_1,\dots,z_n]=R$ such that the vanishing set $V(f_1,\dots,f_n)$ is the single point $(0,\dots,0)$. The map $F=(f_1,\dots,f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ therefore restricts to a map $\mathbb{C}^n-0 \to \mathbb{C}^n-0$, which up to homotopy is a map $S^{2n-1}\to S^{2n-1}$. The unique homotopical invariant of such maps is the degree: it can be computed by looking at the induced map on H_* or H^*, but it can just as well be computed by looking at the induced map on K^0. And we could look at either the induced map on $K^0(\mathbb{C}^n-0)$ or the induced map on $K^0(\mathbb{C}^n,\mathbb{C}^n-0)$, as the two are related by the suspension isomorphism.

To this end, consider $F^*\colon K^0(\mathbb{C}^n,\mathbb{C}^n-0)\to K^0(\mathbb{C}^n,\mathbb{C}^n-0)$. We might as well pass to $K^0_{alg}(\mathbb{A}^n,\mathbb{A}^n-0)$ since the groups are isomorphic. The generator is the standard Koszul complex, and pulling back along F sends this to $K(f_1,\dots,f_n;R)$. We also have the standard isomorphism $K^0_{alg}(\mathbb{A}^n,\mathbb{A}^n-0)\cong\mathbb{Z}$ given by the local index, and so putting everything together we find that

$$\text{topological degree of } F|_{\mathbb{C}^n-0} = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(f_1,\dots,f_n;R).$$

The term on the right is studied in commutative algebra, where it is called the **multiplicity** of the ideal $(f_1,\dots,f_n)$. Because the word "multiplicity" comes up in too many contexts in this book, we are going to call it the *Koszul multiplicity*.

Remark 32.3. Observe that we can repeat the above discussion for $f_1,\dots,f_k$ where $k>n$, again assuming that the vanishing set consists of just the origin (the case $k<n$ is not possible because the vanishing set would have have positive dimension). We then get the function $F\colon(\mathbb{A}^n,\mathbb{A}^n-0)\to(\mathbb{A}^k,\mathbb{A}^k-0)$, and the same arguments show that the induced map on K^0 is again a map $\mathbb{Z}\to\mathbb{Z}$ given by multiplication by $\sum_i(-1)^i\dim_{\mathbb{C}}H_i(\underline{f};R)$. But here we know for topological reasons that this induced map must be zero, because up to homotopy we are looking at a map $S^{2n}\to S^{2k}$ where $n<k$.

Let us move now into the purely algebraic setting. Let (R,m) be a local ring and let $f_1,\dots,f_r\in m$ be such that $\mathrm{Rad}(f_1,\dots,f_r)=m$. Equivalently, this says that some power of m is contained in $I=(f_1,\dots,f_r)$. Define the **Koszul multiplicity** of $f_1,\dots,f_r$ to be

$$e(f_1,\dots,f_r)=\sum_{i=0}^{r}(-1)^i\ell(H_i(\underline{f};R)).$$

Note that the Koszul homology modules are all annihilated by $(\underline{f})$ and therefore by some power of m, and hence are all of finite length.

As described above, this notion of multiplicity has a natural "topological" interpretation in terms of the K^0 functor. The following results can be proven entirely algebraically:

Theorem 32.4. *Let (R,m) be a local ring and $f_1,\dots,f_r\in m$ be such that $\mathrm{Rad}(f_1,\dots,f_r)=m$.*

(a) $e(f_1, \dots, f_r) = 0$ *if* $r > \dim R$.
(b) If $r = \dim R$ *then* $e(f_1, \dots, f_r) > 0$, *depends only on the ideal* $I = (f_1, \dots, f_r)$, *and can be described in the following two (equivalent) ways:*
  (1) $e(f_1, \dots, f_r) = r! \cdot \lim_{t\to\infty} \frac{\ell(R/I^t)}{t^r}$.
  (2) It is $r!$ *times the leading coefficient of the Hilbert-Samuel polynomial that agrees with* $\ell(R/I^t)$ *for* $t \gg 0$.

We will prove this in Section ???, after developing more tools for working with Koszul complexes.

The above notion of multiplicity admits an evident generalization. Let $f_1, \dots, f_r \in m$ and let M be a finitely-generated module such that $M/(f_1, \dots, f_r)$ has finite length (equivalently, $V(\underline{f}) \cap \operatorname{Supp} M = \{m\}$). Define

$$e(f_1, \dots, f_r; M) = \sum_i (-1)^i \ell(H_i(\underline{f}; M)).$$

Note that $\dim M/(f_1, \dots, f_r)M \geq \dim M - r$, so the assumptions imply that $0 \geq \dim M - r$, or $\dim M \leq r$.

The "topological" interpretation is that we are using the pairing

$$K^0_{alg}(X, X - Z) \otimes G(X)_W \to G(X)_{Z\cap W}$$

given by $\langle [E_\bullet], [F] \rangle = \chi(E_\bullet \otimes F)$ and then applying this when $Z = V(\underline{f})$, $W = \operatorname{Supp} M$, $E_\bullet = K_\bullet(\underline{f})$ and $F = M$.

Theorem 32.5. *Let* (R, m) *be a local ring, let* $f_1, \dots, f_r \in m$, *and let* M *be a finitely-generated module such that* $M/(f_1, \dots, f_r)M$ *has finite length. Then*
(a) $e(\underline{f}; M) \geq 0$,
(b) $e(\underline{f}; M) = 0$ *if and only if* $\dim M < r$.
(c) If $r = \dim M$ *then* $e(\underline{f}; M)$ *is* $r!$ *times the leading coefficient of the Hilbert-Samuel polynomial that agrees with* $\ell(M/(\underline{f})^t M)$ *for* $t \gg 0$.

Again, we will prove this (and many other results) in Section 35.

32.6. **Reduction to the diagonal.** The technique of reduction to the diagonal shows that the problem of computing an intersection multiplicity $i(Z, W; P)$ can always be reduced to the case where one of Z and W is very simple. Let Z and W be complex submanifolds of $\mathbb{C}^n$ and let P be an isolated point of intersection. We can then look at $Z \times W \subseteq \mathbb{C}^n \times \mathbb{C}^n$ and the diagonal $\Delta \subseteq \mathbb{C}^n \times \mathbb{C}^n$, which have (P, P) as an isolated point of intersection.

Proposition 32.7 (Reduction to the diagonal)**.** $i(Z, W; P) = i(Z \times W, \Delta; (P, P))$.

Proof. We can use any complex-oriented cohomology theory $\mathcal{E}$ for this. Let c and d be the codimensions of Z and W in $X = \mathbb{C}^n$. We must have $c + d \geq n$, and if $c + d > n$ then both intersection multiplicities are zero since the submanifolds can be perturbed to not intersect at all. So assume $c + d = n$.

There are relative fundamental classes $[Z] \in \mathcal{E}^{2c}(X, X - Z)$ and $[W] \in \mathcal{E}^{2d}(X, X - W)$, giving a product $[Z] \cdot [W] \in \mathcal{E}^{2n}(X, X - (Z \cap W)) = \mathcal{E}^{2n}(X, X - P)$. We know

$$[Z] \cdot [W] = i(Z, W; P)[P].$$

But we can also form the external product $[Z] \otimes [W] \in \mathcal{E}^{2n}(X \times X, ((X - Z) \times X) \cup (X \times (X - W)))$ and we know $[Z] \otimes [W] = [Z \times W]$. For the map of pairs

$\Delta\colon (X, X-(Z\cap W)) \to (X\times X, (X-Z)\times X \cup X\times (X-W))$ we have

$$\Delta^*([Z\times W]) = i(Z\times W, \Delta; (P,P))[P],$$

as in Theorem 26.7. But $\Delta^*([Z]\otimes[W]) = [Z]\cdot[W]$, so the desired identity follows at once. □

In algebra this technique shows up in the following way. Suppose that k is a field and $R = k[x_1,\dots,x_n]$ and M and N are finitely-generated R-modules such that $\operatorname{Supp}(M\otimes_R N) = \{P\}$. The scheme $\mathbb{A}^n\times\mathbb{A}^n$ is $\operatorname{Spec} R\otimes_k R$, and the diagonal is the closed subscheme $V(I_\Delta)$ where $I_\Delta = (x_1\otimes 1 - 1\otimes x_1, \dots, x_n\otimes 1 - 1\otimes x_n)$. Observe that multiplication $R\otimes_k R\to R$ has kernel I_Δ, and so $(R\otimes_k R)/I_\Delta$ is identified with R as $R\otimes_k R$-modules. So reduction to the diagonal suggests the identity

$$\sum_i (-1)^i \dim_k \operatorname{Tor}_i^R(M,N) = \sum_j (-1)^j \dim_k \operatorname{Tor}_j^{R\otimes_k R}(R, M\otimes_k N).$$

This can be proven purely algebraically, and is not even particularly hard. The main observation is that

$$M\otimes_R N = R\otimes_{R\otimes_k R} (M\otimes_k N),$$

which itself is almost tautological. Let $P_\bullet\to M$ and $Q_\bullet\to N$ be projective resolutions over R. Then $P\otimes_k Q$ is an $(R\otimes_k R)$-projective resolution of $M\otimes_k N$, using that k is a field and so all k-modules are flat. Therefore

$$\operatorname{Tor}_i^{R\otimes_k R}(R, M\otimes_k N) = H_i(R\otimes_{R\otimes_k R}(P\otimes_k Q)) = H_i(P\otimes_R Q) = \operatorname{Tor}_i^R(M,N).$$

So the principle of reduction to the diagonal applies not just to the alternating sums but to the actual Tor-modules themselves.

The elements $x_1\otimes 1 - 1\otimes x_1, \dots, x_n\otimes 1 - 1\otimes x_n$ form a regular sequence in $R\otimes_k R$; equivalently, $x_1 - y_1, \dots, x_n - y_n$ is a regular sequence in $k[x_1,\dots,x_n,y_1,\dots,y_n]$. So R, as an $R\otimes_k R$-module, is resolved by the Koszul complex, and we can identify the relevant Tor's with Koszul homology. Thus we find that

$$i(M,N;P) = e(x_1\otimes 1 - 1\otimes x_1, \dots, x_n\otimes 1 - 1\otimes x_n; (M\otimes_k N)_{(P,P)}).$$

That is, we can identify the intersection multiplicity with a certain Koszul multiplicity.

These observations allowed Serre to prove that the $i(M,N)$ multiplicities are well-behaved for any regular local ring R containing a field. The general idea is this:

(1) First complete at the maximal ideal, which doesn't change any of the lengths involved. By the Cohen structure theorem this replaces R with a power series ring $k[[x_1,\dots,x_n]]$.

(2) One can attempt reduction to the diagonal here, but an issue arises because $k[[x_1,\dots,x_n]]\otimes_k k[[x_1,\dots,x_n]]$ is not Noetherian. One wants to end up with $k[[x_1,\dots,x_n,x_1',\dots,x_n']]$, and this can be done with a separate completion process. Briefly, for R-modules M and N define the **complete tensor product**

$$M\hat{\otimes}_k N = \lim_k [M/m^k M\otimes_k N/m^k N].$$

This is naturally a module over $R\hat{\otimes}_k R$. A little work shows that when $R = k[[x_1,\dots,x_n]]$ then $R\hat{\otimes}_k R = k[[x_1\otimes 1,\dots,x_n\otimes 1, 1\otimes x_1,\dots,1\otimes x_n]]$, $M\hat{\otimes}_k N$

is a finitely-generated module over $R\hat{\otimes}_k R$, and one gets the reduction-to-the-diagonal isomorphisms

$$\operatorname{Tor}_i^{R\hat{\otimes}_k R}(R, M\hat{\otimes}_k N) \cong \operatorname{Tor}_i^R(M,N).$$

We refer to [Serre2] for details. Using this, one again reduces the study of $i(M,N)$ to a Koszul multiplicity.

When R does not contain a field but is instead unramified mixed characteristic, then by the Cohen structure theorem completion gives $V[[x_1,\dots,x_n]]$ where V is a discrete valuation ring. Some extra care is needed here in the reduction-to-diagonal argument, since now V has dimension one. If the maximal ideal of V is (π) and π acts as a nonzerodivisor on the modules M and N, then basically everything works as before and one can again reduce $i(M,N)$ to a Koszul multiplicity. Some extra work is required to handle the case where π acts as a zerodivisor, but it is manageable. All of this is in [Serre2].

For the ramified case, though, one does not have enough control over the completion $R^\wedge$ to get very far with this style of proof.

33. $K^*(\mathbb{C}P^n)$ and the K-theoretic analog of the degree

If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold then it has a fundamental class $[Z]$ in $H^*(\mathbb{C}P^n)$. knowing this fundamental class comes down to knowing a single integer, called the degree of Z. As we saw in Example 26.10, the geometric interpretation of the degree is that it equals the number of points of intersection between Z and a generically chosen linear subspace of complementary dimension. In this section we will repeat this line of investigation but replacing H^* with K^*. So we must compute $K^*(\mathbb{C}P^n)$ and investigate what information is encoded in the fundamental class $[Z]_K$. We will find that knowing $[Z]_K$ amounts to knowing *several* integers (not just one); while we can give methods for computing these, their geometric interpretation is somewhat mysterious. We will see that knowing these integers is equivalent to knowing what geometers call the Hilbert polynomial of Z.

33.1. **Calculation of $K^*(\mathbb{C}P^n)$.** We begin with the following easy lemma:

Lemma 33.2. *Let E be any multiplicative cohomology theory. If $x_1,\dots,x_{n+1} \in \tilde{E}^*(\mathbb{C}P^n)$, then $x_1\cdots x_{n+1} = 0$.*

Proof. The key is just that $\mathbb{C}P^n$ can be covered by $n+1$ contractible sets. To be explicit, let $U_i = \{[z_0 : \dots : z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$. Then U_i is open in $\mathbb{C}P^n$ and is homeomorphic to $\mathbb{C}^n$.

Choose our basepoint of $\mathbb{C}P^n$ to be $[1:1:\dots:1]$ (or any other point in the intersection of all the U_i's). The contractibility of U_i implies that $E^*(\mathbb{C}P^n, U_i) \to E^*(\mathbb{C}P^n, *)$ is an isomorphism. So we may lift each x_i to a class $\tilde{x}_i \in E^*(\mathbb{C}P^n, U_i)$.

It now follows that $\tilde{x}_1\tilde{x}_2\cdots\tilde{x}_{n+1}$ lifts $x_1\cdots x_{n+1}$ in the map

$$E^*(\mathbb{C}P^n, U_1\cup\cdots\cup U_{n+1}) \to E^*(\mathbb{C}P^n, *).$$

Since $U_1\cup\cdots\cup U_{n+1} = \mathbb{C}P^n$, the domain is the zero group. So $x_1\cdots x_{n+1} = 0$. □

Recall that $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ where x is a generator in degree 2. It is not hard to see that we may take $x = [\mathbb{C}P^{n-1}]$. This calculation works for any complex-oriented cohomology theory:

Proposition 33.3. *Let E be a complex-oriented cohomology theory. There is an isomorphism of E^*-algebras*

$$E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$$

sending x^i to $[\mathbb{C}P^{n-i}]$.

Proof. Consider the reduced Gysin sequence (as in Section 26.4) for the submanifold $\mathbb{C}P^{n-1} \overset{j}{\hookrightarrow} \mathbb{C}P^n$:

$$\begin{array}{ccccc} \cdots \leftarrow \tilde{E}^k(\mathbb{C}P^n - \mathbb{C}P^{n-1}) & \leftarrow & \tilde{E}^k(\mathbb{C}P^n) & & \\ & & \uparrow{\scriptstyle j_!} & & \\ & & E^{k-2}(\mathbb{C}P^{n-1}) & \leftarrow \tilde{E}^{k-1}(\mathbb{C}P^n - \mathbb{C}P^{n-1}) & \leftarrow \cdots \end{array}$$

Let $x = [\mathbb{C}P^{n-1}] = j_!(1) \in \tilde{E}^2(\mathbb{C}P^n)$. By Lemma 33.2 we know $x^{n+1} = 0$, so we get a map $E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$. We will show that this map is an isomorphism via induction on n. The case $n = 0$ is trivial.

Note that $x^2 = [\mathbb{C}P^{n-2}]$ by intersection theory (specifically, the E-theory analog of Theorem 26.11). The same reasoning gives $x^i = [\mathbb{C}P^{n-i}]$.

The spaces $\mathbb{C}P^n - \mathbb{C}P^{n-1}$ are homeomorphic to $\mathbb{C}^n$ and hence contractible. So the reduced Gysin sequence considered above breaks up into a collection of isomorphisms

$$j_!\colon E^{k-2}(\mathbb{C}P^{n-1}) \xrightarrow{\cong} \tilde{E}^k(\mathbb{C}P^n).$$

Taking all k's together, $j_!$ is a map of $E^*(pt)$-modules and therefore an isomorphism of such modules. By induction $E^*(\mathbb{C}P^{n-1})$ is a free $E^*(pt)$-module generated by the classes $1 = [\mathbb{C}P^{n-1}]$, $[\mathbb{C}P^{n-2}]$, $[\mathbb{C}P^{n-3}], \ldots, [\mathbb{C}P^0]$. Since the pushforward $j_!$ sends $[\mathbb{C}P^{n-i}]$ to $[\mathbb{C}P^{n-i}]$ (this follows from the E-version of Exercise 26.16), we conclude that $\tilde{E}^*(\mathbb{C}P^n)$ is a free $E^*(pt)$-module on $[\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$. If we add 1 to this collection then we get a free basis for $E^*(\mathbb{C}P^n)$ over $E^*(pt)$. This proves that our map $E^*(pt)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$ is an isomorphism. □

In the case of complex K-theory, we can rephrase the above result as saying that $K^{odd}(\mathbb{C}P^n) = 0$ and $K^0(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$, where $X = \beta \cdot [\mathbb{C}P^{n-1}]$. In particular, note that additively we have $K^0(\mathbb{C}P^n) \cong \mathbb{Z}^{n+1}$ with free basis consisting of the powers of X. Suppressing powers of the Bott element as usual, we can write this free basis as $[\mathbb{C}P^n], [\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$.

33.4. **Fundamental classes in $K^*(\mathbb{C}P^n)$.** Now let $Z \hookrightarrow \mathbb{C}P^n$ be a complex, closed submanifold of codimension c. Recall that if we consider fundamental classes in singular cohomology then we have $[Z] = d \cdot [\mathbb{C}P^{n-c}]$ for a unique integer d that is called the **degree** of Z. Geometrically, d is the number of points of intersection of Z with a generically chosen copy of $\mathbb{C}P^c$.

We can now play this same game in the context of K-theory. We have a fundamental class $[Z]_K \in K^0(\mathbb{C}P^n)$, and we can therefore write

$$\begin{aligned} [Z]_K &= d_0 \cdot [\mathbb{C}P^n] + d_1 \cdot [\mathbb{C}P^{n-1}] + \cdots + d_n \cdot [\mathbb{C}P^0] \\ &= d_0 \cdot 1 + d_1 X + d_2 X^2 + \cdots + d_n X^n \end{aligned}$$

for unique integers d_i. These integers are topological invariants of the embedding $Z \hookrightarrow \mathbb{C}P^n$ (not just of Z, as we will see). To explore them further, start by multiplying the above equation by X^n to obtain

$$d_0 X^n = [Z]_K \cdot X^n = [Z]_K \cdot [\mathbb{C}P^0] = [Z \cap \mathbb{C}P^0]$$

where in the last term we mean the intersection of Z with a generically chosen copy of $\mathbb{C}P^0$. But as long as Z is not all of $\mathbb{C}P^n$, this generic intersection will be empty—so if Z is codimension at least one then $d_0 = 0$. We then repeat this argument, but multiplying by X^{n-1} instead of X^n: we get

$$d_1 X^n = [Z]_K \cdot [\mathbb{C}P^1] = [Z \cap \mathbb{C}P^1].$$

Again, if the codimension of Z is at least 2 then the $\mathbb{C}P^1$ can be moved so that it doesn't intersect Z at all, hence $d_1 = 0$. Continuing this argument we find that

$$0 = d_0 = d_1 = \cdots = d_{c-1} \qquad \text{and} \qquad d_c = \deg(Z).$$

The last equality follows because Z intersects a generic $\mathbb{C}P^c$ in $\deg(Z)$ many points. The situation can be summarized as follows: the first non-vanishing d_i coincides with the classical degree of Z, but there is the possibility of the higher d_i's being nonzero. This is what we will investigate next.

Remark 33.5. Notice that what we have done so far works in any complex-oriented cohomology theory E^*. The fundamental class $[Z]$ can be written as a linear combination of the classes $[\mathbb{C}P^{n-i}]$ with coefficients from $E^*(pt)$. The first c of these coefficients vanish, until one gets to the coefficient of $[\mathbb{C}P^{n-c}]$—which must be equal to $\deg(Z)$. After this things become interesting, in the sense that one has invariants that are potentially not detected in singular cohomology.

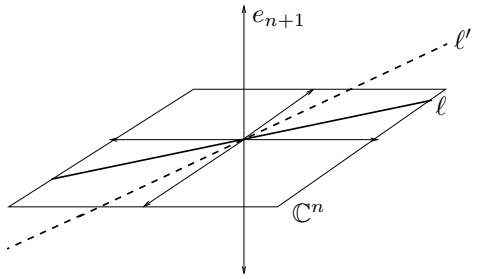
To proceed further with our analysis of the higher d_i's in K-theory, we need to connect our fundamental classes with the vector bundle description of K-theory:

Proposition 33.6. *In $K^0(\mathbb{C}P^n)$ one has $[\mathbb{C}P^{n-1}] = 1 - L$ where $L \to \mathbb{C}P^n$ is the tautological line bundle. Consequently, $[\mathbb{C}P^{n-k}] = (1-L)^k$ for all k and $K^0(\mathbb{C}P^n) = \mathbb{Z}[L]/(1-L)^{n+1}$.*

Proof. We give two explanations. For the first, consider the map of vector bundles $f\colon L \to \underline{1}$ defined as follows: in the fiber over $x = [x_0 : x_1 : \cdots : x_n]$ we send $(x_0, \ldots, x_n)$ to x_n. Note that this is well-defined, since multiplying all the x_i's by a scalar λ yields the same homomorphism $L_x \to \mathbb{C}$.

The map f is exact on all fibers except those where $x_n = 0$. The complex $0 \to L \to \underline{1} \to 0$ is a resolution of the structure sheaf for $\mathbb{C}P^{n-1}$, and hence $1 - L$ represents the associated fundamental class in K-theory by Theorem 31.11. Algebraically, if $R = \mathbb{C}[x_0, \ldots, x_n]$ then we are looking at the resolution $0 \longrightarrow R(-1) \xrightarrow{x_n} R \longrightarrow R/(x_n) \longrightarrow 0$.

Our second explanation takes place entirely in the topological world. The key fact is that the normal bundle to $\mathbb{C}P^{n-1}$ inside $\mathbb{C}P^n$ is L^*. Algebraically, the conormal bundle is $(x_n)/(x_n)^2 \cong R(-1)$, and the normal bundle is therefore $R(1)$. Geometrically, if ℓ is a line in $\mathbb{C}^n$ then moving in the normal direction gives us lines that are graphs of homomorphisms $\ell \to \mathbb{C}\langle e_{n+1}\rangle$, i.e. graphs of functionals on ℓ.



Let U be a tubular neighborhood of $\mathbb{C}P^{n-1}$, and consider the chain of isomorphisms

$$K^0(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^{n-1}) \xrightarrow{\cong} K^0(U, U - \mathbb{C}P^{n-1}) \cong K^0(N, N-0).$$

The relative fundamental class $[\mathbb{C}P^{n-1}]_{rel}$ is the unique class that maps to the Thom class $\mathcal{U}_N$ under the above isomorphisms. But N is a line bundle, and recall from (27.11) that the Thom class is then the Koszul complex $[J^*] = [\pi^*N^* \to 1]$ where $\pi\colon N \to \mathbb{C}P^{n-1}$. If we identify elements of N with pairs (ℓ, ϕ) where $\ell \in \mathbb{C}P^{n-1}$ and ϕ is a functional on ℓ, then on the fiber over (ℓ, ϕ) the map takes an element $v \in \ell$ and sends it $\phi(v)$. Identifying the pair (ℓ, ϕ) with the graph $\ell_\phi = \{(v, \phi(v)) \,|\, v \in \ell\}$, this is the map that takes an element of ℓ_ϕ and sends it to its final coordinate. So the complex $0 \to \pi^*N^* \to 1 \to 0$ is exactly the restriction of $0 \to L \to 1 \to 0$ on $\mathbb{C}P^n$. This latter complex therefore represents $[\mathbb{C}P^{n-1}]_{rel}$, and thus $1 - L$ equals $[\mathbb{C}P^{n-1}]$.

The second set of statements in the proposition follow directly from Proposition 33.3 and Bott periodicity, since those give $K^0(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$ where $x = [\mathbb{C}P^{n-1}]$ and $x^k = [\mathbb{C}P^{n-1}]^k = [\mathbb{C}P^{n-k}]$. □

Example 33.7. In the above proposition we saw the key relation $(1-L)^{n+1} = 0$ in $K^0(\mathbb{C}P^n)$, which of course also follows directly from Lemma 33.2 since $1 - L$ is a reduced class. This relation comes up in many contexts, and it is useful to have a different perspective on it. Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and consider the Koszul complex for the regular sequence $x_0, \ldots, x_n$. It has the form

$$0 \to R(-(n+1)) \to \cdots \to R(-2)^{\binom{n+1}{2}} \to R(-1)^{n+1} \to R \to 0,$$

and we know from Theorem 27.16 that this complex is exact except in degree zero where it has $H_0 = R/(x_0, \ldots, x_n)$. This implies that if one takes the corresponding complex of vector bundles on $\mathbb{C}P^n$ and pulls back along $\mathbb{A}^{n+1} - 0 \to \mathbb{C}P^n$ then the resulting complex of bundles is exact. So the original sequence of bundles on $\mathbb{C}P^n$ is exact, which tells us that

$$0 = 1 - (n+1)L + \tbinom{n+1}{2}L^2 - \cdots + (-1)^{n+1}L^{n+1}$$

in $K^0(\mathbb{C}P^n)$. Of course the expression on the right is precisely $(1-L)^{n+1}$.

As an alternative perspective, the above Koszul complex is the tensor product

$$(L \xrightarrow{x_0} \underline{1}) \otimes (L \xrightarrow{x_1} \underline{1}) \otimes \cdots \otimes (L \xrightarrow{x_n} \underline{1}).$$

A tensor product of complexes of vector spaces is exact as long as any factor is exact, and since any point $[x_0 : \cdots : x_n] \in \mathbb{C}P^n$ has some $x_i \neq 0$ we deduce that our complex of bundles is exact over every point.

We next compute the K-theoretic fundamental classes in a couple of simple examples:

Example 33.8. Let $Z = V(f) \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Let $R = \mathbb{C}[x_0, \ldots, x_n]$. The homogeneous coordinate ring of Z is $R/(f)$, which has the short free resolution given by

$$0 \to R(-d) \xrightarrow{\cdot f} R \to R/(f) \to 0.$$

So $[Z] = 1 - L^d$ in $K^0(\mathbb{C}P^n)$. To write $[Z]$ as a linear combination of the $[\mathbb{C}P^{n-i}]$'s we need to write $1 - L^d$ in terms of powers of $X = 1 - L$. This is easy, of course:

$$\begin{aligned}[Z] = 1 - L^d = 1 - [1 - (1 - L)]^d &= 1 - (1 - X)^d \\ &= 1 - [1 - dX + \tbinom{d}{2}X^2 - \cdots + (-1)^d X^d] \\ &= dX - \tbinom{d}{2}X^2 + \tbinom{d}{3}X^3 - \cdots\end{aligned}$$

So we find that the higher d_i's for a hypersurface are all just (up to sign) binomial coefficients of d. This is somewhat disappointing, as we are not seeing new topological invariants—it is just the degree over and over again, encoded in different ways. This is not actually a surprise, though: it is known that all smooth hypersurfaces of the same degree are actually diffeomorphic. See [Lamotke, 2.3.2], for example.

Things become more interesting in the next example:

Example 33.9. Consider $Z = V(f, g)$ where f, g is a regular sequence of homogeneous elements in $R = \mathbb{C}[x_0, \ldots, x_n]$. Let $d = \deg(f)$ and $e = \deg(g)$. Because f, g is a regular sequence, $R/(f, g)$ is resolved by the Koszul complex:

$$0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/(f, g) \to 0.$$

We can now calculate

$$\begin{aligned}[Z]_K &= \Big[\ 0 \longrightarrow L^{d+e} \longrightarrow L^d \oplus L^e \longrightarrow 1 \longrightarrow 0\ \Big] \\ &= 1 - (L^d + L^e) + L^{d+e} \\ &= 1 - (1-X)^d - (1-X)^e + (1-X)^{d+e} \\ &= \Big[\tbinom{d+e}{2} - \tbinom{d}{2} - \tbinom{e}{2}\Big]X^2 - \Big[\tbinom{d+e}{3} - \tbinom{d}{3} - \tbinom{e}{3}\Big]X^3 + \cdots \\ &= deX^2 + \tfrac{1}{2}de(2-d-e)X^3 + \ldots.\end{aligned}$$

The classical degree of Z is de, but our 'higher invariants' now see more than just this number. In fact, knowing the expansion of $[Z]_K$ as a linear combination of the X^i's implies that we know de and $de(2-d-e)$ (assuming $n \geq 3$), which implies that we know $d + e$. But if we know de and $d + e$ then we know the polynomial $(\xi - d)(\xi - e) = \xi^2 - (d+e)\xi + de$, which means we know its roots. So knowing the expansion of $[Z]_K$ is the same as knowing d and e. This example shows that the K-theoretic fundamental class sees more topological information than the singular cohomology fundamental class does.

Exercise 33.10. Prove that if $Z = V(f_1, \ldots, f_k) \subseteq \mathbb{C}P^n$ is a complete intersection with $\deg f_i = d_i$, then

$$[Z]_K = \left[\prod_{i=1}^{k}(1 - (1-X)^{d_i})\right]_{X^{\leq n}}$$

where $X = 1 - L$ and the notation means that we throw away all terms of degree larger than n. [Hint: Use that the Koszul complex is a tensor product of length one complexes.]

Example 33.9 suggests that when n is large enough knowing $[Z]_K$ is equivalent to knowing the degrees $d_1, \dots, d_k$, but that for small n this need not be the case. We will explore this more in Exercise 33.24 below.

Now that we have seen these simple examples we can return to our main question. Given $Z \hookrightarrow \mathbb{C}P^n$ of codimension c, how does one compute the d_i's in the equation

$$[Z]_K = (\deg Z)[\mathbb{C}P^{n-c}] + d_{c+1}[\mathbb{C}P^{n-c-1}] + d_{c+2}[\mathbb{C}P^{n-c-2}] + \cdots$$

And what do these d_i's mean in terms of geometry? We will see that one answer is given by the Hilbert polynomial.

33.11. **Review of the Hilbert polynomial.** Let $R = \mathbb{C}[x_0, \dots, x_n]$, and regard this as a graded ring where each x_i has degree one. If M is a graded R-module write M_s for the graded piece in degree s. The **Poincaré series** of a finitely-generated graded module M is the formal Laurent series

$$P_M(\xi) = \sum_{s=-\infty}^{\infty} (\dim_{\mathbb{C}} M_s)\xi^s \in \mathbb{Z}[[\xi]]\,[\xi^{-1}].$$

Note that this is evidently an additive invariant of finitely-generated, graded modules: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence then clearly $P_M(\xi) = P_{M'}(\xi) + P_{M''}(\xi)$. We may therefore regard the Poincaré series as a map of abelian groups

$$P\colon G_{grd}(R) \to \mathbb{Z}[[\xi]]\,[\xi^{-1}]$$

where $G_{grd}(R)$ is the Grothendieck group of finitely-generated *graded* modules over R.

We will calculate the Poincaré series of each $R(-k)$, but for this we need the following useful fact:

Lemma 33.12. *The number of monomials of degree d in the variables $z_1, \dots, z_n$ is equal to $\binom{d+n-1}{d}$.*

Proof. Monomials are in bijective correspondence with patterns of "dashes and slashes" that look like

$$- - -/ - -// - - - -/ - .$$

The above pattern corresponds to the monomial $z_1^3 z_2^2 z_4^4 z_5$, and from this the general form of the bijection should be clear. Monomials of degree d will correspond to patterns with d dashes, and if there are n variables then there will be $n-1$ slashes. So we need to count patterns where there will be $d+n-1$ total symbols, of which d are dashes: the number of these is $\binom{d+n-1}{d}$. □

It is an immediate consequence that $P_R(\xi) = \sum_{s\geq 0}\binom{s+n}{n}\xi^s$. For $R(-k)$ we simply shift the coefficients and obtain

$$P_{R(-k)}(\xi) = \sum_{s\geq k}\binom{s+n-k}{n}\xi^s = \xi^k P_R(\xi).$$

The power series $\{\xi^k P_R(\xi) \mid k \in \mathbb{Z}\}$ are obviously linearly independent over $\mathbb{Q}$, and this shows (if it were not clear already) that $G_{grd}(R)$ has infinite rank as an abelian group.

Proposition 33.13. *$G_{grd}(R)$ is isomorphic to $\mathbb{Z}^\infty$, with free basis the set of rank one free modules $\{[R(-d)] \,|\, d \in \mathbb{Z}\}$.*

Proof. The key is the Hilbert Syzygy Theorem. Consider the diagram

$$\begin{array}{ccc} \mathbb{Z}^\infty & \longrightarrow & G_{grd}(R) \\ & \searrow & \downarrow P \\ & & \mathbb{Z}[[\xi]]\,[\xi^{-1}] \end{array}$$

where the horizontal map sends the ith basis element to $[R(-i)]$ and the diagonal map is the composite. We have already seen that this composite is injective, because there is no $\mathbb{Z}$-linear relation amongst the images of the basis elements. So $\mathbb{Z}^\infty \to G_{grd}(R)$ is injective, and it only remains to prove surjectivity. But if M is any finitely-generated, graded R-module then the Syzygy Theorem guarantees a finite, graded, free resolution

$$0 \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0.$$

So $[M] = \sum_i (-1)^i [F_i]$, and each F_i is a sum of elements $[R(-k)]$. This proves surjectivity. □

The Hilbert polynomial is a variant of the Poincaré series that keeps track of less information. At first this might seem to be a bad thing, but we will see that it actually gives us a closer connection to geometry and topology. Here is the main result that gets things started:

Proposition 33.14. *Let M be a finitely-generated graded module over $R = \mathbb{C}[x_0, \ldots, x_n]$. Then there exists a unique polynomial $\mathrm{Hilb}_M(s) \in \mathbb{Q}[s]$ that agrees with the function $s \mapsto \dim_{\mathbb{C}} M_s$ for $s \gg 0$. One has $\deg \mathrm{Hilb}_M(s) \leq n$. The polynomial $\mathrm{Hilb}_M(s)$ is called the **Hilbert polynomial** of M.*

Proof. Consider first the case $M = R$. A basis for M_s consists of all monomials in $x_0, \ldots, x_n$ of degree s, which Lemma 33.12 calculates to be $\binom{n+s}{s} = \binom{n+s}{n}$. This is a polynomial in s of degree n. Next consider $M = R(-k)$. The function $s \mapsto M_s$ is zero for $s < k$, and then for $s \geq k$ it coincides with $\binom{n+s-k}{n}$. This is again a polynomial in s of degree n.

Finally, consider the case of a general M. By the Hilbert Syzygy Theorem M has a finite, graded, free resolution

$$0 \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0.$$

It follows that

$$\dim M_s = \dim(F_0)_s - \dim(F_1)_s + \cdots + (-1)^{n+1} \dim(F_{n+1})_s.$$

But each F_i is a direct sum of finitely-many $R(-k)$'s, and so $s \mapsto \dim(F_i)_s$ has been shown to agree with a polynomial in s of degree at most n, for $s \gg 0$. The desired result follows at once. □

The following calculation was given in the above proof, but we record it below because it comes up so often:

Corollary 33.15. *When $R = \mathbb{C}[x_0, \ldots, x_n]$ and $k \in \mathbb{Z}$ the Hilbert polynomial for $R(-k)$ is $\binom{s+n-k}{n}$.*

Example 33.16. Consider a hypersurface $M = R/(f)$, where $f \in R$ is homogeneous of degree d. We then have the resolution

$$0 \longrightarrow R(-d) \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0$$

from which we find

$$\mathrm{Hilb}_{R/(f)} = \mathrm{Hilb}_R - \mathrm{Hilb}_{R(-d)} = \tbinom{s+n}{n} - \tbinom{s+n-d}{n} = \tfrac{(s+n)\cdots(s+1)}{n!} - \tfrac{(s+n-d)\cdots(s-d+1)}{n!}.$$

The two binomial coefficients have leading terms $s^n/n!$, which therefore cancel. The coefficient of s^{n-1} is

$$\tfrac{1}{n!} \cdot \left[\tfrac{n(n+1)}{2} - \tfrac{(n-2d+1)n}{2} \right] = \tfrac{d}{(n-1)!}.$$

Note that the degree of $\mathrm{Hilb}_{R/(f)}$ is one less than the Krull dimension of $R/(f)$, and the leading coefficient is the degree of f (the geometric degree of the hypersurface) divided by $(n-1)!$. These are general facts that hold for any module: the degree of Hilb_M is one less than the Krull dimension of M, and the leading coefficient times $(\deg \mathrm{Hilb}_M)!$ is always an integer—this integer is called the **multiplicity** of the module M. See Exercise 34.23 below for a proof.

The Hilbert polynomial is an additive invariant of finitely-generated modules: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence then clearly $\mathrm{Hilb}_M(s) = \mathrm{Hilb}_{M'}(s) + \mathrm{Hilb}_{M''}(s)$. We may therefore regard the Hilbert polynomial as a map of abelian groups

$$\mathrm{Hilb}\colon G_{grd}(R) \to \mathbb{Q}[s].$$

This map is clearly not injective, as it kills any module that is finite-dimensional as a $\mathbb{C}$-vector space. The subgroup of $G_{grd}(R)$ generated by such modules is $A = \langle [\mathbb{C}(-d)] \,\big|\, d \in \mathbb{Z} \rangle$, where $\mathbb{C}$ as an R-module is always interpreted as $R/(x_0, \ldots, x_n)$. We may regard Hilb as a map

$$\mathrm{Hilb}\colon G_{grd}(R)/A \to \mathbb{Q}[s].$$

The domain of this map is calculated as follows:

Proposition 33.17. *The group $G_{grd}(R)/A$ is isomorphic to $\mathbb{Z}^{n+1}$, with free basis $R, R(-1), R(-2), \ldots, R(-n)$. The map* $\mathrm{Hilb}\colon G_{grd}(R)/A \to \mathbb{Q}[s]$ *is an injection.*

Proof. Let B be the subgroup of $G_{grd}(R)/A$ generated by $[R], [R(-1)], \ldots, [R(-n)]$. We will show that B is equal to the whole group.

The module $\mathbb{C} = R/(x_0, \ldots, x_n)$ is resolved by the Koszul complex $K(x_0, \ldots, x_n; R)$. This gives the relation in $G_{grd}(R)$ of

$$[\mathbb{C}] = [R] - (n+1)[R(-1)] + \tbinom{n+1}{2}[R(-2)] + \cdots + (-1)^{n+1}[R(-n-1)].$$

In $G_{grd}(R)/A$ we therefore have $[R(-n-1)] \in B$. For $d > 0$ we can tensor the Koszul complex with $R(-d)$ and then apply the same argument to find that $[R(-n-1-d)] \in \langle [R(-d)], \ldots, [R(-n-d)] \rangle \subseteq \langle [R], [R(-1)], \ldots, [R(-n-d)] \rangle$. An inductive argument now shows that $[R(-k)] \in B$ for every $k \geq n+1$.

A similar induction works for $d < 0$ to show that $[R(-d)] \in B$ for all $d \in \mathbb{Z}$. In other words, $B = G_{grd}(R)/A$.

Now consider the map $\mathbb{Z}^{n+1} \to G_{grd}(R)/A$ that sends the ith basis element e_i to $[R(-i)]$ for $0 \leq i \leq n$. We have just shown that this map is surjective. The composite

$$\mathbb{Z}^{n+1} \longrightarrow G_{grd}(R)/A \xrightarrow{\mathrm{Hilb}} \mathbb{Q}[s]$$

takes our basis elements to the polynomials

$$\tbinom{s+n}{n},\ \tbinom{s+n-1}{n},\ \tbinom{s+n-2}{n},\ \dots,\ \tbinom{s}{n}.$$

Evaluating these polynomials at $s = 0$ gives the sequence $1, 0, 0, \dots, 0$. Evaluating at $s = 1$ gives $n + 1, 1, 0, 0, \dots, 0$, and so forth. For $s = i$ the ith polynomial in the list evaluates to 1 and all the subsequent polynomials evaluate to 0. This proves that these polynomials are linearly independent over $\mathbb{Z}$, hence the above composite map is injective. So $\mathbb{Z}^{n+1} \to G_{grd}(R)/A$ is injective, and therefore is an isomorphism. Moreover, the map Hilb is injective. □

The reader might suspect that the $\mathbb{Z}^{n+1}$ in the above result is really an incarnation of the group $K^0(\mathbb{C}P^n)$. This is the foundation for what we do in the next section.

Exercise 33.18. We will not need this, but it is a cute fact: Prove that the image of Hilb: $G(R)/A \to \mathbb{Q}[s]$ equals the set of polynomials $f(s) \in \mathbb{Q}[s]$ having the property that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. That is, the image consists of all rational polynomials that take integer values on integers. (See Section 34 if you get stuck.)

33.19. **K-theory and the Hilbert polynomial.** We now explain how the Hilbert polynomial encodes the same information as the K-theoretic fundamental class.

Given $Z \hookrightarrow \mathbb{C}P^n$ a smooth subvariety of codimension c, recall that we have the fundamental class $[Z] \in K^0(\mathbb{C}P^n)$ and that we may write

$$[Z] = d_c[\mathbb{C}P^{n-c}] + d_{c+1}[\mathbb{C}P^{n-c-1}] + \cdots + d_n[\mathbb{C}P^0].$$

We know that $d_c = \deg(Z)$, and our goal is to understand how to compute the higher d_i's.

Proposition 33.6 says that $[\mathbb{C}P^{n-k}] = (1 - L)^k$ and that these classes for $k = 0, 1, \dots, n$ give a basis for $K^0(\mathbb{C}P^n)$. Clearly one can also use the basis $1, L, L^2, \dots, L^n$. We next introduce an algebraic analogue of $K^0(\mathbb{C}P^n)$. Let $R = \mathbb{C}[x_0, \dots, x_n]$. Take the Grothendieck group $K^0_{grd}(R)$ of finitely-generated, graded, projective R-modules (or equivalently, bounded chain complexes of such modules) and quotient by the subgroup $\tilde{A}$ generated by all complexes $K(x_0, \dots, x_n; R) \otimes R(-d)$ for $d \in \mathbb{Z}$. We obtain a diagram

$$\begin{array}{ccc} K^0_{\mathrm{grd}}(R)/\tilde{A} & \xrightarrow[\cong]{\phi} & K^0(\mathbb{C}P^n) \\ \big\downarrow \cong & & \\ G_{grd}(R)/A & \xrightarrow{\text{Hilb}} & \mathbb{Q}[s] \end{array} \tag{33.20}$$

where ϕ sends $[R(-d)]$ to L^d and where the vertical map is our usual 'Poincaré Duality' isomorphism, in this case sending the class of a projective module to the class of the same module in $G_{grd}(R)/A$. The map ϕ is an isomorphism by inspection: we have computed $G_{grd}(R)/A$ and $K^0(\mathbb{C}P^n)$, both are $\mathbb{Z}^{n+1}$, and ϕ clearly maps a basis to a basis.

Given $Z = V(I)$, we know by Theorem 31.11 that its fundamental class $[Z] \in K^0(\mathbb{C}P^n)$ is represented by a finite, graded, free resolution $F_\bullet \to R/I \to 0$. This resolution (or the alternating sum of its terms) lifts to $K^0_{grd}(R)/\tilde{A}$, and pushing this around the diagram into $\mathbb{Q}[s]$ just gives us $\mathrm{Hilb}_{R/I}(s)$. So the above diagram shows that knowing $\mathrm{Hilb}_{R/I}(s)$ is the same as knowing $[Z]$.

To say something more specific here, consider first the case $Z = \mathbb{C}P^{n-k}$. Then $R/I = R/(x_0, \ldots, x_{k-1})$, which as a graded ring is just $\mathbb{C}[x_k, \ldots, x_n]$. By Corollary 33.15 the Hilbert polynomial is then

$$\mathrm{Hilb}_{\mathbb{C}P^{n-k}}(s) = \tbinom{s+n-k}{n-k}.$$

So pushing our basis $[\mathbb{C}P^n], [\mathbb{C}P^{n-1}], \ldots, [\mathbb{C}P^0]$ around diagram (33.20) into $\mathbb{Q}[s]$ yields the polynomials

$$\tbinom{s+n}{n},\ \tbinom{s+n-1}{n-1},\ \tbinom{s+n-2}{n-2},\ \ldots,\ \tbinom{s}{0}.$$

If $Z = V(I) \hookrightarrow \mathbb{C}P^n$ is now arbitrary, then writing

$$\mathrm{Hilb}_{R/I}(s) = d_0\tbinom{s+n}{n} + d_1\tbinom{s+n-1}{n-1} + d_2\tbinom{s+n-2}{n-2} + \ldots$$

implies that $[Z] = d_0[\mathbb{C}P^n] + d_1[\mathbb{C}P^{n-1}] + d_2[\mathbb{C}P^{n-2}] + \ldots$. In other words, one obtains the expansion of $[Z]$ as a linear combination of the $[\mathbb{C}P^{n-i}]$'s by writing $\mathrm{Hilb}_Z(s)$ as a linear combination of the above-listed binomial functions. We record this fact for future reference:

Proposition 33.21. *Let $Z \hookrightarrow \mathbb{C}P^n$ be a subvariety and let $I \subseteq R = \mathbb{C}[x_0, \ldots, x_n]$ be the corresponding ideal of functions vanishing on Z. Then $\mathrm{Hilb}_{R/I}(s) = \sum_{i=0}^n d_i\binom{s+n-i}{n-i}$ if and only if $[Z]_K = \sum_{i=0}^n d_i[\mathbb{C}P^{n-i}]$.*

We have shown how to calculate the d_i's from the ideal I of equations defining Z: decompose the Hilbert polynomial into a sum of terms $\binom{s+n-i}{n-i}$, and take the resulting coefficients. This is still not exactly a 'geometric' description of the d_i's, although it is at least a description that takes place in the realm of algebraic geometry. We will get another perspective on this material via the Todd genus and the Grothendieck-Riemann-Roch Theorem, to be studied in Section 42. See especially Section 42.29.

Let us make one more observation before completing this section. Recall that $K^0_{alg}(\mathbb{C}P^n)$ denotes the Grothendieck group of locally-free coherent sheaves on $\mathbb{C}P^n$. We know that such sheaves correspond to finitely-generated graded projective modules over $R = \mathbb{C}[x_0, \ldots, x_n]$. $\mathbb{N}$-graded rings behave much like local rings, and the same proof as for Proposition 4.1 shows that all finitely-generated graded projectives are actually free; so these are direct sums of the shifted rank-one free modules $R(n)$ for $n \in \mathbb{Z}$. An exact sequence of graded R-modules gives rise to an exact sequence of sheaves on $\mathbb{C}P^n$, so we have top row of the diagram

$$\begin{array}{ccc} G(\text{f.g. graded free } R\text{-modules}) & \twoheadrightarrow & K^0_{alg}(\mathbb{C}P^n) \\ \downarrow & \nearrow & \downarrow \\ G(\text{f.g. graded free } R\text{-modules})/A & & K^0(\mathbb{C}P^n) \xrightarrow{\cong} \mathbb{Z}^{n+1} \end{array}$$

The Hilbert syzygy theorem implies that the natural map

$$G(\text{f.g. graded free } R\text{-modules}) \to G_{grd}(R)$$

is an isomorphism, by the standard argument. So we have seen earlier in this section that the upper left corner in the above diagram is $\mathbb{Z}^\infty$, using the Poincaré series to map to $\mathbb{Z}[[\xi]][\xi^{-1}]$. The Koszul complex $K(x_0, \ldots, x_n; R)$ is a resolution of $R/(\underline{x})$, and the latter becomes zero as a sheaf on $\mathbb{C}P^n$. So passing to sheaves, the

Koszul complex becomes exact and gives us a relation in $K^0_{alg}(\mathbb{C}P^n)$. Thus, we get the dotted arrow in the diagram. We saw earlier in this section that the domain of that dotted arrow is $\mathbb{Z}^{n+1}$ with generators $\mathcal{O}, \mathcal{O}(-1), \dots, \mathcal{O}(-n)$, and we know these map to a basis in $K^0(\mathbb{C}P^n)$. It follows immediately that the diagonal map and the map $K^0_{alg}(\mathbb{C}P^n) \to K^0(\mathbb{C}P^n)$ are both isomorphisms. We record this for future use:

Proposition 33.22. *The comparison map $K^0_{alg}(\mathbb{C}P^n) \to K^0(\mathbb{C}P^n)$ is an isomorphism, and the sequences $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ and $\mathcal{O}, \mathcal{O}(-1), \dots, \mathcal{O}(-n)$ are both bases for the domain. Using the standard isomorphism $K^0_{alg}(\mathbb{C}P^n) \to G(\mathbb{C}P^n)$ to regard every coherent sheaf as representing a class in $K^0_{alg}(\mathbb{C}P^n)$, the classes $\mathcal{O}_{\mathbb{C}P^0}$, $\mathcal{O}_{\mathbb{C}P^1}, \dots, \mathcal{O}_{\mathbb{C}P^n}$ are also a basis.*

Proof. Perhaps only the final statement has not been explained yet. But $\mathcal{O}_{\mathbb{C}P^i}$ corresponds to the module $R/(x_0, \dots, x_{n-i-1})$, which is resolved by the Koszul complex $K(x_0, \dots, x_{n-i-1}; R)$. So in the K-group we have $[\mathcal{O}_{\mathbb{C}P^i}] = (1-[\mathcal{O}(-1)])^{n-i}$. These are a basis by inspection, or one could map into $K^0(\mathbb{C}P^n)$ and get the elements $(1-L)^{n-i}$ which we already proved are a basis in Proposition 33.6. □

∘ Exercises ∘

Exercise 33.23. The *twisted cubic* is the image of the embedding $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^3$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. It is known that this is the subvariety $C = V(I)$ where $I = (xz - y^2, xw - yz, yw - z^2) \subseteq \mathbb{C}[x, y, z, w] = R$. With a little work one can establish the free resolution

$$0 \longrightarrow R(-3)^2 \xrightarrow{A_2} R(-2)^3 \xrightarrow{A_1} R \longrightarrow R/I \longrightarrow 0$$

where $A_1 = \begin{bmatrix} xz-y^2 & xw-yz & yw-z^2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} z & w \\ -y & -z \\ x & y \end{bmatrix}$. Use this information to calculate $[C]_K$ as a linear combination of the basis elements $[\mathbb{C}P^i]$. Also calculate $\mathrm{Hilb}_C(s)$, and check Proposition 33.21 explicitly in this case. [This example shows that these invariants depend on the embedding of the submanifold, not just on the topological type of the submanifold.]

Exercise 33.24. For $Z = V(f_1, \dots, f_k) \hookrightarrow \mathbb{C}P^n$ a complete intersection, let the corresponding **degree sequence** be $\underline{d} = (\deg f_1, \dots, \deg f_k)$.

(a) Compute $[Z]_K$ for each of the two degree sequences $(1, 6, 6)$ and $(2, 2, 9)$ in $\mathbb{C}P^4$ and check that they are equal. So $[Z]_K$ does not determine the degree sequence.
(b) Compute that $[Z]_K$ is *different* for the degree sequences $(1, 6, 6)$ and $(2, 2, 9)$ in $\mathbb{C}P^5$.
(c) Prove that if $n \geq 3k - 2$ then $[Z]_K$ determines the degree sequence, using the following steps:
    (1) Recall from Exercise 33.10 that $[Z]_K$ is the polynomial $P(X) = \prod_i (1 - (1-X)^{d_i})$, truncated to degrees $\leq n$. Introduce the change of variable $X = 1 - e^{-t}$, or $t = -\log(1-X) = X + \frac{X^2}{2} + \cdots$. Explain why knowing $P(X)$ up through degree n is equivalent to knowing $P(t)$ up through degree n.

(2) Check that $P = \prod_i d_i \cdot t^k \cdot \prod_i [\frac{1-e^{-d_i t}}{d_i t}]$ and so

$$\log\Big(\tfrac{P}{(\prod_i d_i)t^k}\Big) = \sum_i \log[\tfrac{1-e^{-d_i t}}{d_i t}] = -\tfrac{p_1}{2}t + \sum_{r\geq 1}(??)p_{2r}t^{2r}$$

where the p_s are the power sums $p_s = \sum d_i^s$ and where the coefficients are related to Bernoulli numbers (see Exercise E.9).

(3) Prove that knowing $P(X)$ up through degree n is equivalent to knowing $P(t)$ up through degree n, which is in turn equivalent to knowing $\prod_i d_i$, p_1 if $k < n$, and p_{2r} for $2r \leq n-k$.

(4) If y is a formal variable then knowing $d_1, \ldots, d_k$ is the same as knowing $(y-d_1)\cdots(y-d_k)$, which is the same as knowing $\sigma_1, \ldots, \sigma_k$. This in turn is the same as knowing $p_1, \ldots, p_k$, via the Newton identities. If the d_i are all non-negative, prove that knowing σ_k and $p_2, p_4, \ldots, p_{2k-2}$ determines all of the d's.

(5) Deduce that if $n \geq 3k-2$ then knowing $[Z]_K$ determines the degree sequence of Z. (We do not claim that this is the best possible bound on n, in fact that is already not the case when $k = 2$).

(d) When $k = 3$ and $n \geq 5$ prove that $[Z]_K$ determines the degree sequence.

34. Interlude on the calculus of finite differences

If we regard calculus as a set of tools built up from the study of $\frac{f(x+h)-f(x)}{h}$ where h is an infinitesimal, the calculus of finite differences is the analogous set of tools that come into play when h is a finite (non-infinitesimal) object. This is a classical subject whose roots run deep, though it is not exactly a standard part of modern lore. In this section we will discuss the theory as it applies to polynomials of one variable, and for us we will always have $h = 1$. It is somewhat of a surprise that this subject is so relevant to K-theory.

For a classical text on the calculus of finite differences see [MiT]. A quick introduction similar in sprit to what we do here can be found in [Stanley, Section 1.9].

34.1. **The finite difference and sum operators.** If $f \in \mathbb{Q}[t]$ let Δf be the polynomial given by

$$(\Delta f)(t) = f(t+1) - f(t).$$

For example, $\Delta(t^2) = (t+1)^2 - t^2 = 2t+1$, and $\Delta(t^3) = (t+1)^3 - t^3 = 3t^2+3t+1$. Note that $\Delta\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$ is a linear map, and that it lowers degrees by one. We call Δ the **finite difference operator**, and we regard it as an analog of the familiar differentiation operator $D\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$.

The opposite of differentiation is integration, and there is an analogous operator that is the opposite of Δ. If $f \in \mathbb{Q}[t]$ define Sf to be the function $\mathbb{Z}_{\geq 0} \to \mathbb{Q}$ given by

$$(Sf)(t) = f(0) + f(1) + f(2) + \cdots + f(t-1).$$

For example,

$$(St)(t) = 0 + 1 + 2 + \cdots + (t-1) = \tbinom{t}{2} = \tfrac{t(t-1)}{2}$$

and

$$(St^2)(t) = 0^2 + 1^2 + \cdots + (t-1)^2 = \tfrac{(t-1)t(2t-1)}{6}.$$

(Note that in these formulas we are mixing the roles of t as a formal variable and t as a parameter, but this should not cause too much confusion.) One should think of the formula for $(Sf)(t)$ as giving a finite Riemann sum, based on intervals of width 1. It is not immediately clear that Sf is always a polynomial, nor that it raises degrees by one, but we will prove these things shortly. The following two facts are easy, though:

$$\Delta S(f) = f \qquad\qquad D\textstyle\int(f) = f \tag{1}$$

$$S\Delta(f) = f - f(0) \qquad\qquad \textstyle\int D(f) = f - f(0). \tag{2}$$

We have written each identity paired with the corresponding identity for classical differentiation/integration. Note that, like the usual integral, Sf will always have zero as its constant term since $(Sf)(0) = 0$ by definition.

Derivatives and integrals of polynomials are easy to compute because their values on the basic polynomials t^n are very simple. In fact, the point is really that the operators D and $\int$ act very simply on the sequence of polynomials

$$1, \quad t, \quad \tfrac{t^2}{2}, \quad \tfrac{t^3}{3!}, \quad \tfrac{t^4}{4!}, \quad \dots$$

Of course D carries each polynomial in the sequence to the preceding one, and $\int$ carries each polynomial to the subsequent one. In contrast, the operators Δ and S are *not* very well-behaved on this sequence. It is better to use the sequence

$$1, \quad S(1), \quad S^2(1), \quad S^3(1), \quad \dots$$

so let us compute these. It is easy to see that $S(1) = t$, and we previously saw that $S^2(1) = S(t) = \binom{t}{2}$. The following useful lemma implies that $S\binom{t}{k} = \binom{t}{k+1}$, where the binomial coefficient stands for the polynomial $\frac{1}{k!}t(t-1)(t-2)\cdots(t-k+1)$. And so starting with $1 = \binom{t}{0}$ we get $S^k(1) = \binom{t}{k}$,

Lemma 34.2. *For any $d, k \in \mathbb{Z}$ with $k \geq 0$ one has*

$$\Delta\tbinom{t+d}{k} = \tbinom{t+d}{k-1}, \qquad S\tbinom{t+d}{k} = \tbinom{t+d}{k+1} - \tbinom{d}{k+1}.$$

Proof. The statement about Δ follows from Pascal's Identity. Applying S to this equation gives

$$S\tbinom{t+d}{k-1} = S\Delta\tbinom{t+d}{k} = \tbinom{t+d}{k} - \tbinom{d}{k}$$

and changing k to $k+1$ gives the desired identity for S. □

Remark 34.3. The second statement in Lemma 34.2 is equivalent to the identity

$$\tbinom{d}{k+1} + \tbinom{d}{k} + \tbinom{d+1}{k} + \cdots + \tbinom{d+t-1}{k} = \tbinom{t+d}{k+1}.$$

For fun let us give a combinatorial proof of this. Imagine $t+d$ slots labelled 1 through $t+d$ where we are to place $k+1$ asterisks:

$$\underline{\quad} \; \underline{*} \; \underline{\quad} \; \underline{*} \; \underline{*} \; \underline{\quad} \; \underline{*}$$

We can sort these by where the rightmost asterisk is located. There are $\binom{d}{k+1}$ choices where the rightmost is in the first d slots. If the rightmost asterisk is in slight $d+1$, there are $\binom{d}{k}$ choices for placing the others. And more generally, there are $\binom{d+i-1}{k}$ configurations having the rightmost asterisk in spot $d+i$. Summing over $i \in [1, t]$ yields the desired formula.

Exercise 34.4. Pascal's triangle contains the binomial coefficients $\binom{n}{k}$ with $n \geq 0$ and $0 \leq k \leq n$. Extending the picture to include all $0 \leq k$ just adds an infinite trail of zeros to the right side of each row. Write down the first few rows of Pascal's triangle, then use Pascal's identity to work backwards to fill in the numbers $\binom{n}{k}$ for $n \leq 0$ and $k \geq 0$ (should this be called "Pascal's half-plane"?). In some ways this is a silly exercise, but it is worth doing this at least once in your life to watch the infinite sequences create themselves out of nothing.

The sequence of polynomials

$$1 = \tbinom{t}{0}, \quad \tbinom{t}{1}, \quad \tbinom{t}{2}, \quad \tbinom{t}{3}, \quad \dots$$

is clearly a basis for $\mathbb{Q}[t]$, as the kth term has degree equal to k. In relation to the Δ and S operators, this basis plays the role classically taken by the polynomials $\frac{t^d}{d!}$ for D and $\int$. It is now clear that S applied to a polynomial of degree d yields a polynomial of degree $d+1$: a polynomial of degree d is a linear combination of $\binom{t}{d}, \binom{t}{d-1}, \dots, \binom{t}{0}$ with the coefficient on the first term nonzero. Applying S changes each $\binom{t}{k}$ to $\binom{t}{k+1}$, and so this yields a polynomial of degree $d+1$.

For another example of the use of this binomial basis, note the following analog of the Taylor–Maclaurin expansion for writing polynomials in this basis:

Proposition 34.5. *Let $f \in \mathbb{Q}[t]$. Then*

$$f = \sum_{k=0}^{\infty} (\Delta^k f)(0) \cdot \tbinom{t}{k}.$$

Proof. We know $f = \sum_{k=0}^{N} a_k \binom{t}{k}$ for some N and some values $a_k \in \mathbb{Q}$. Plugging in $t = 0$ immediately gives $f(0) = a_0$. Now apply Δ to get $\Delta f = \sum_{k=0}^{N-1} a_{k+1} \binom{t}{k}$ and again plug in $t = 0$: this yields $(\Delta f)(0) = a_1$. Continue. □

Exercise 34.6. Let $\mathbb{Q}[t]_{int} \subseteq \mathbb{Q}[t]$ be the set of all polynomials $f(t)$ such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. Note that $\mathbb{Q}[t]_{int}$ is stable under Δ and S, and use this to prove that $\mathbb{Q}[t]_{int}$ is the $\mathbb{Z}$-linear span of the polynomials $\binom{t}{k}$, $k \geq 0$.

In contrast to the large number of similarities of the pair of operators (Δ, S) to $(D, \int)$, there is an important difference when it comes to the product rule. Of course we have $D(fg) = (Df)g + f(Dg)$, but one readily checks that this does **not** work for Δ. The correct rule is as follows:

Lemma 34.7. *For any $f, g \in \mathbb{Q}[t]$ one has $\Delta(fg) = (\Delta f)g + f(\Delta g) + (\Delta f)(\Delta g)$.*

Proof. A simple calculation, left to the reader. □

In the present section we will not have much use for this product rule, but it is a very important formula whose significance will become larger in subsequent sections. See Proposition 37.17 for a brief hint at this and Section 45 for a more thorough discussion.

34.8. **Translating between (Δ, S) and $(D, \int)$.** If $f \in \mathbb{Q}[t]$ and h is any integer (or even better, a formal variable) we have the Taylor fomula

$$f(t+h) = \sum_{k=0}^{\infty} f^{(k)}(t) \cdot \tfrac{h^k}{k!} = \sum_{k=0}^{\infty} (D^k f)(t) \cdot \tfrac{h^k}{k!}.$$

Note that the sum is actually finite, since large enough derivatives of f are all zero. Taking $h = 1$ and rearranging somewhat we get

$$f(t+1) = \Big(\sum_{k=0}^{\infty} \tfrac{D^k}{k!}\Big) f = (e^D) f.$$

Let us be clear about what this means. The expression $\sum_{k=0}^{\infty} \frac{D^k}{k!}$ makes perfect sense as an *operator*, since evaluating this at any fixed polynomial gives a well-defined answer. It is sensible to denote this operator as e^D.

Taking one more step, we can write $(\Delta f)(t) = f(t+1) - f(t) = (e^D - 1)f$ where 1 denotes the identity operator. Or even more compactly,

$$\Delta = e^D - 1$$

is an identity of operators on $\mathbb{Q}[t]$. This identity gives us Δ as a linear combination of iterates of D.

We are next going to cook up a similar formula for the operator S, but this is a little harder. One does not wish for a formula using higher and higher powers of $\int$, as these operators become more and more complicated. Instead, we say to ourselves that S is very close to being an inverse for Δ (it is a right inverse, but not a left inverse). If it *were* an inverse we might want to write $S = \frac{1}{e^D - 1}$, but it is difficult to make sense of the latter expression as an operator—the trouble is that $e^D - 1$ has no constant term, otherwise we could expand as a power series. While this didn't work, we *can* make sense of the operator

$$\mathcal{B} = \tfrac{D}{e^D - 1}. \tag{34.9}$$

By this we mean write $\frac{x}{e^x - 1} = \sum a_k x^k = 1 - \frac{x}{2} + \frac{x^2}{12} - \cdots$ as a formal power series, and set $\mathcal{B}$ equal to $\sum_{k=0}^{\infty} a_k D^k$. It follows purely formally that $\Delta\mathcal{B} f = Df$ for any polynomial f, and by applying S to both sides we find that

$$\mathcal{B} f - (\mathcal{B} f)(0) = SDf.$$

Replacing f by $\int f$ in the above formula, we get

$$Sf = SD(\textstyle\int f) = \mathcal{B}(\int f) - (\mathcal{B}\int f)(0).$$

Note that $\mathcal{B}(\int f)$ is equal to $\int f - \frac{1}{2} f + \frac{1}{12} Df - \cdots$. So this is as computable as the series $\frac{x}{e^x - 1}$; the coefficients of this series are related to the Bernoulli numbers (see Appendix E). We have

$$\mathcal{B} = \sum_{k=0}^{\infty} \tfrac{B_k}{k!} D^k = 1 - \tfrac{1}{2} D + \tfrac{1}{12} D^2 - \tfrac{1}{720} D^4 + \cdots$$

where the B_k's are the Benoulli numbers, the first few of which are

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0, \quad B_4 = -\tfrac{1}{30}.$$

34.10. **Sums of powers and Bernoulli numbers.** The (Δ, S) pair of operators is useful in a variety of mathematical situations. In a moment we will see an application to determining K-theoretic fundamental classes, but let us first look at a non-topological example. Almost every math student has seen the formulas

$$1 + 2 + \cdots + n = \tfrac{n(n+1)}{2} \quad \text{and} \quad 1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6}.$$

The proof of such formulas by mathematical induction is a common exercise in elementary proof courses. It turns out that there are also formulas for higher powers, of the form

$$1^k + 2^k + \cdots + n^k = p_k(n)$$

where $p_k(n)$ is a polynomial of degree $k+1$. How does one discover the appropriate polynomials? This seems to have first been done by Jacob Bernoulli, the coefficients in these polynomials being closely related to what are now called Bernoulli numbers.

Clearly we may rephrase the problem as that of computing $S(t^k)$, the exact connection being $S(t^k) = p_k(t-1)$. In the last section we developed a formula for S in terms of the usual derivative and integral operators, and we will now use that; but here it is easiest to use it in the form

$$SD(f) = \mathcal{B}f - (\mathcal{B}f)(0) = \left(\sum_{j=0}^{\infty} \tfrac{B_j}{j!} D^j\right) f - (\text{constant term of preceding expression}).$$

We obtain

$$\begin{aligned} S(t^k) = SD\Big(\tfrac{t^{k+1}}{k+1}\Big) &= \tfrac{1}{k+1} \sum_{j=0}^{\infty} \tfrac{B_j}{j!} D^j(t^{k+1}) - (\text{constant term}) \\ &= \tfrac{1}{k+1} \sum_{j=0}^{k} \tfrac{B_j}{j!} (k+1)(k)(k-1)\cdots(k+1-(j-1)) t^{k+1-j} \\ &= \tfrac{1}{k+1} \sum_{j=0}^{k} B_j \cdot \tbinom{k+1}{j} t^{k+1-j}. \end{aligned}$$

Note that the last expression is a polynomial in t, with all of its coefficients clearly visible.

Finally, recalling that $S(t^k) = p_k(t-1)$ we conclude that

$$1^k + 2^k + \cdots + n^k = p_k(n) = S(t^k)\big|_{t=n+1} = \tfrac{1}{k+1} \sum_{j=0}^{k} B_j \cdot \tbinom{k+1}{j} (n+1)^{k+1-j}.$$

Exercise 34.11. Check that the above formula gives the familiar identities in the cases $k = 1$ and $k = 2$, and then see what it gives when $k = 3$. Compare the above formula to (E.3).

34.12. **Back to K-theoretic fundamental classes.** Let $Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface defined by the homogeneous ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n] = R$. We saw in Section 33.19 that the coefficients in

$$[Z]_K = d_0[\mathbb{C}P^n]_K + d_1[\mathbb{C}P^{n-1}]_K + d_2[\mathbb{C}P^{n-2}]_K + \cdots$$

are the same as the coefficients in

$$\text{Hilb}_Z(t) = d_0\tbinom{n+t}{n} + d_1\tbinom{n+t-1}{n-1} + d_2\tbinom{n+t-2}{n-2} + \cdots \tag{34.13}$$

The sequence of polynomials $\binom{n+t}{n}, \binom{n+t-1}{n-1}, \binom{n+t-2}{n-2}, \ldots$ is not quite the standard basis of binomial coefficients we used above, but it is close. Here is a small lemma about expanding polynomials in this basis (compare to Proposition 34.5):

Lemma 34.14. *If $f \in \mathbb{Q}[t]$ then $f = \sum_{k=0}^{\infty} (\Delta^k f)(-k-1) \cdot \binom{t+k}{k}$.*

Proof. The collection of polynomials $\binom{t+k}{k}$ for $k \geq 0$ is clearly a basis for $\mathbb{Q}[t]$ for degree reasons. Write $f = \sum c_k\binom{t+k}{k} = c_0\binom{t}{0} + c_1\binom{t+1}{1} + c_2\binom{t+2}{2} + \cdots$. Plugging in $t = -1$ immediately gives $f(-1) = c_0$. Apply Δ to both sides to get $\Delta f = \sum c_k\binom{t+k}{k-1} = c_1 + c_2\binom{t+2}{1} + \cdots$. Now plugging in $t = -2$ makes all the expressions vanish except the first, so $c_1 = (\Delta f)(-2)$. Continue in this way. □

The following corollary is immediate, by applying the above lemma to (34.13):

Corollary 34.15. *We have* $[Z]_K = \sum_i d_i[\mathbb{C}P^{n-i}]_K$ *where the coefficients are given by* $d_i = (\Delta^{n-i}\operatorname{Hilb}_Z)(-n+i-1)$. *Alternatively, the coefficient of* $[\mathbb{C}P^j]_K$ *in* $[Z]_K$ *is* $(\Delta^j \operatorname{Hilb}_Z)(-j-1)$.

Example 34.16. Let $Z \hookrightarrow \mathbb{C}P^n$ be a hypersurface of degree d, and write $Z = V(f)$. Then $R/(f)$ is resolved by $0 \to R(-d) \to R$ where the map is mutiplication by f. So

$$\operatorname{Hilb}_Z = \operatorname{Hilb}_R - \operatorname{Hilb}_{R(-d)} = \tbinom{n+t}{n} - \tbinom{n+t-d}{n}.$$

But then

$$\Delta^{n-i}\operatorname{Hilb}_Z = \tbinom{n+t}{n-(n-i)} - \tbinom{n+t-d}{n-(n-i)} = \tbinom{n+t}{i} - \tbinom{n+t-d}{i}$$

(using Lemma 34.2 for the first equality), and so

$$d_i = \tbinom{i-1}{i} - \tbinom{i-1-d}{i} = \tbinom{i-1}{i} + (-1)^{i+1}\tbinom{d}{i}$$

where in the last step we have used the identity $\binom{-s}{r} = (-1)^r\binom{r+s-1}{r}$. The expression $\binom{i-1}{i}$ is nonzero only when $i = 0$, so that we get

$$d_i = \begin{cases} 0 & \text{if } i = 0 \\ (-1)^{i+1}\binom{d}{i} & \text{if } i > 0. \end{cases}$$

This of course agrees with what we found in Example 33.8.

Just as in the last example, in practice Hilbert polynomials are often computed by first having a graded free resolution of R/I. To look at what happens in general here, let the (finite) graded free resolution be

$$\cdots \oplus_j R(-e_{2j}) \to \oplus_i R(-e_{1i}) \to \oplus_u R(-e_{0u}) \to R/I \to 0.$$

Then the Hilbert series is given by

$$\operatorname{Hilb}_Z(t) = \sum_{k=0}^{\infty}(-1)^k \sum_j \tbinom{t+n-e_{kj}}{n}$$

(really a finite sum, of course). Then

$$d_i = (\Delta^{n-i}\operatorname{Hilb}_Z)(-n+i-1) = \sum_{k=0}^{\infty}(-1)^k \sum_j \tbinom{i-1-e_{kj}}{i} = (-1)^i \cdot \sum_{k=0}^{\infty}(-1)^k \sum_j \tbinom{e_{kj}}{i}$$

where in the last equality we are using $\binom{i-s}{i} = (-1)^i\binom{s-1}{i}$.

Example 34.17. Let $Z \hookrightarrow \mathbb{C}P^n$ be a complete intersection where the degrees of the defining equations are d and e. Then R/I is resolved by the Koszul complex, which looks like $0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/I \to 0$. We conclude that $d_i = (-1)^i\left[\binom{0}{i} - \binom{d}{i} - \binom{e}{i} + \binom{d+e}{i}\right]$, from which we see that $d_0 = d_1 = 0$ and for $i > 1$

$$d_i = (-1)^{i+1}\left[\tbinom{d}{i} + \tbinom{e}{i} - \tbinom{d+e}{i}\right].$$

Again, this recovers our calculation from Example 33.9.

Recall that if the codimension of $Z \hookrightarrow \mathbb{C}P^n$ is equal to c then we have $d_i = 0$ for $i < c$ and $d_c = \deg(Z)$. So any graded free resolution of R/I must satisfy the identities

$$\sum_k (-1)^k \sum_j \tbinom{e_{kj}}{i} = \begin{cases} 0 & \text{for } i < c \\ (-1)^c \deg(Z) & \text{for } i = c. \end{cases} \tag{34.18}$$

In this way we obtain topological conditions on what graded free resolutions can look like. Conditions such as these seem to have been first observed in [PS1].

Exercise 34.19. Show that the conditions (34.18) on the free resolution are equivalent to

$$\sum_k (-1)^k \sum_j e_{kj}^i = \begin{cases} 0 & \text{for } i < c \\ (-1)^c \deg(Z) \cdot c! & \text{for } i = c. \end{cases}$$

[Hint: This is a purely numerical result. Given two lists $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ of real numbers, compare the conditions $\sum_j \binom{a_j}{i} = \sum_j \binom{b_j}{i}$, $0 \le i < c$, to the conditions $\sum_j a_j^i = \sum_j b_j^i$ in the same range. Apply this to our situation by dividing the e_{kj} into two lists according to the parity of k.]

34.20. **A digression on the Riemann zeta function.** If $Z \hookrightarrow \mathbb{C}P^n$ then the values of the polynomial $\mathrm{Hilb}_Z(t)$ only have an *a priori* significance for $t \gg 0$; recall that these values represent the dimensions of the graded pieces of the homogeneous coordinate ring of Z. Yet, we saw in Corollary 34.15 that the behavior of $\mathrm{Hilb}_Z(t)$ at certain *negative* integers (encoded as the value of a particular finite difference $\Delta^? \mathrm{Hilb}_Z$) gives some naturally-occurring topological invariants of Z. The fact that these negative values have any significance at all is a bit surprising. This situation is somewhat reminiscent of one involving the Riemann zeta function, that coincidentally (or not) is also related to the story of the (Δ, S) operators. We are going to take a moment and talk about this, because of the feeling that it might be related to topology in a way that is not yet fully understood.

Recall that Riemann's $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Standard results from analysis show that this sum converges for $\mathrm{Re}(s) > 1$, and defines an analytic function in that range. It is a non-obvious fact that $\zeta(s)$ can be analytically continued to a meromorphic function on the complex plane with its only pole at $z = 1$. The values on negative integers turn out to be computable and are related to the Bernoulli numbers. We will give an entirely non-rigorous treatment of this computation. Despite its failure to actually make sense, it is nevertheless intriguing.

If $f \in \mathbb{Q}[t]$ then we have seen that $e^D f$ makes sense and is equal to the polynomial $f(t+1)$. It readily follows that $e^{nD} f = f(t+n)$ for any integer $n \ge 0$. Now write

$$f(t) + f(t+1) + f(t+2) + \cdots = [I + e^D + e^{2D} + e^{3D} + \cdots] f = \left[\tfrac{1}{I - e^D}\right] f.$$

Of course none of the three expressions between the equals signs make any sense, but let us pretend for a moment that this is not a problem. Replacing f by Df we can then write

$$Df(t) + Df(t+1) + \cdots = \left[\tfrac{D}{I - e^D}\right] f = -\mathcal{B} f$$

where $\mathcal{B}$ is the Bernoulli operator of (34.9). Evaluating at $t = 0$ we would obtain

$$Df(0) + Df(1) + Df(2) + \cdots = -(\mathcal{B}f)(0).$$

Let us next try to apply this fanciful formula to compute $\zeta(-n) = 1^n + 2^n + 3^n + 4^n + \cdots$. We want $Df(t) = t^n$, and so should take $f(t) = \frac{t^{n+1}}{n+1}$. The above formula then suggests that

$$\zeta(-n) = -\mathcal{B}\big(\tfrac{t^{n+1}}{n+1}\big)(0) = -\left[\sum_{k=0}^{\infty} \tfrac{B_k}{k!} D^k\big(\tfrac{t^{n+1}}{n+1}\big)\right](0) = -\tfrac{B_{n+1}}{(n+1)!} \cdot \tfrac{(n+1)!}{n+1} = -\tfrac{B_{n+1}}{n+1}.$$

Amazingly, this is the correct answer—the same value can be deduced by rigorous arguments from complex analysis. The challenge is to find some explanation for why this wacky argument actually leads to something correct.

∘ Exercises ∘

Exercise 34.21. Return to the twisted cubic from Exercise 33.23 and explicitly check Corollary 34.15 for this subvariety. Also check that the e_{kj} shifts in the free resolution satisfy the equations given in Exercise 34.19.

Exercise 34.22. A certain smooth curve Q in $\mathbb{C}P^3$ has an ideal I generated by one quadric and three cubics. Suppose given that the minimal resolution for R/I has the form

$$0 \longrightarrow F \longrightarrow R(-4)^4 \longrightarrow R(-2) \oplus R(-3)^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where F is a free module that we are not told yet. First, use the given information to determine F. Then compute the degree of Q, as well as $[Q]_K$ and $\mathrm{Hilb}_Q(s)$.

Exercise 34.23. Let R be an $\mathbb{N}$-graded ring that is generated as an algebra over R_0 by elements of degree 1, and is such that R_0 is a local Artinian ring. Note that in this setting there is a unique graded maximal ideal, namely $m_{R_0} + \sum_{i \geq 1} R_i$. The main example to keep in mind is $R = k[x_1, \ldots, x_n]$ where k is a field.

For a finitely-generated graded R-module M define the Hilbert function $F_M \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ to be $t \mapsto \ell_{R_0}(M_t)$. In this exercise we will prove that if $M \neq 0$ then for $t \gg 0$ the Hilbert function agrees with a polynomial in t having degree equal to $\dim M - 1$ (the convention is that the zero polynomial has degree -1).

(a) Most everything from normal commutative algebra works in the graded setting by just putting the adjective "graded" in front of everything. For example, show that every finitely-generated module M has a finite filtration where the quotients all have the from R/P where P is a graded prime ideal. Conclude that $F_M = \sum_i F_{R/P_i}$, and so the result we are after is reduced to the case $M = R/P$.

(b) We will prove the result by induction of $\dim M$. Verify the base case $\dim M = 0$.

(c) Suppose by induction that the result holds for $\dim M < d$, where $1 \leq d$. Let P be a graded prime ideal such that $\dim R/P = d$. Explain why there exists an $x \in R_1 - P$, and then consider $0 \longrightarrow R/P(-1) \xrightarrow{x} R/P \longrightarrow R/(P, x) \longrightarrow 0$. Conclude that $(\Delta F_{R/P})(t-1) = F_{R/(P,x)}$ and complete the proof.

(d) Use the same style of arguments to prove that if P is the Hilbert polynomial of a graded R-module then so is ΔP. Conclude that if P has degree $n \geq 0$ then the leading coefficient has the form $\frac{e}{n!}$ where e is a positive integer.

35. A deeper study of Koszul complexes

Let R be a ring, $x_1, \dots, x_n \in R$, and let M be an R-module. Recall the Koszul complexes $K(\underline{x}; R)$ and $K(\underline{x}; M) = K(\underline{x}; R) \otimes_R M$. We denote their homology modules as $H_*(\underline{x}; R)$ and $H_*(\underline{x}; M)$. Note that permuting the x_i does not change the isomorphism type of $K(\underline{x}; R)$, and so the order of the x's can largely be ignored.

The Koszul complex $K(x_1, \dots, x_n; R)$ is canonically identified with the tensor product $K(x_1; R) \otimes \cdots \otimes K(x_n; R)$. Multiplication by x is chain homotopic to zero on $K(x; R)$, and so multiplication by each x_i is chain homotopic to zero on $K(x_1, \dots, x_n; R)$. The same is of course also true after tensoring with a module M. This yields the following:

Proposition 35.1. *Let $x_1, \dots, x_n \in R$ and let M be an R-module. The Koszul homology modules $H_*(x_1, \dots, x_n; M)$ are all annihilated by $(x_1, \dots, x_n) + \operatorname{Ann} M$.*

Exercise 35.2. It is sometimes useful to have an explicit formula for the null homotopy of multiplication by x_i on $K(\underline{x}; R)$, and it is not hard to give one. We identify $K(\underline{x}; R)$ with the exterior algebra $\Lambda^\bullet(R^n)$ and where d is the unique derivation satisfying $d(e_i) = x_i$ for all i. Define $S_k \colon \Lambda^k R^n \to \Lambda^{k+1} R^n$ by $S_k(\omega) = e_i \wedge \omega$. Check that $dS + Sd$ is multiplication by x_i in each degree.

In addition to $K(\underline{x}; R) \otimes_R M$ one can instead look at $\operatorname{Hom}_R(K(\underline{x}; R), M)$. This is a cochain complex, and the cohomology modules are denoted $H^i(\underline{x}; M)$. However, it turns out that up to re-indexing these are the same as the Koszul homology modules:

Proposition 35.3. *Let R be a ring, $x_1, \dots, x_n \in R$, and let M be an R-module. For all i one has $H^n(\underline{x}; M) \cong H_{n-i}(\underline{x}; M)$.*

Proof. We can write $\operatorname{Hom}_R(K(\underline{x}; R), M) \cong \operatorname{Hom}_R(K(\underline{x}; R), R) \otimes_R M$. The result will follow if we show that $K(\underline{x}; R)$ and $\operatorname{Hom}_R(K(\underline{x}; R); R)$ are isomorphic chain complexes, where in the latter case we convert the cochain complex to a chain complex by reindexing $i \mapsto n - i$. But this is almost obvious, since it reduces to the $n = 1$ case where it is trivial. To give more detail, write $K = K(\underline{x}; R)$, $K_i = K(x_i; R)$, and write $P^* = \operatorname{Hom}_R(P, R)$. Then

$$K^* = (K_1 \otimes_R \cdots \otimes_R K_n)^* \cong (K_1)^* \otimes_R \cdots \otimes_R (K_n)^* \cong K_1 \otimes_R \cdots \otimes_R K_n \cong K.$$

□

35.4. **Koszul homology and regular sequences.** Let M be an R-module. A sequence $x_1, \dots, x_n \in R$ is called **weakly M-regular** if each x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})$. It is called **M-regular** if it is weakly M-regular and in addition $M/(x_1, \dots, x_n)M \neq 0$. We saw these definitions for $M = R$ in Section 27.14.

Given $x_1, \dots, x_d \in R$ write $\underline{x}'$ for the sequence $x_1, \dots, x_{d-1}$. Recall from the proof of Theorem 27.16 that we have a short exact sequence of chain complexes

$$0 \to K(\underline{x}'; R) \to K(\underline{x}; R) \to \Sigma K(\underline{x}'; R) \to 0.$$

Since all of the modules in these complexes are free the sequence is split-exact in each degree, and therefore tensoring with M is still an exact sequence of complexes. The resulting long exact sequence in homology gives

$$\cdots \xrightarrow{x_n} H_i(\underline{x}'; M) \longrightarrow H_i(\underline{x}; M) \longrightarrow H_{i-1}(\underline{x}'; M) \xrightarrow{x_n} H_{i-1}(\underline{x}'; M) \longrightarrow \cdots \tag{35.5}$$

and therefore short exact sequences

$$0 \to H_i(\underline{x}'; M)/x_n H_i(\underline{x}'; M) \to H_i(\underline{x}; M) \to \operatorname{Ann}_{x_n} H_{i-1}(\underline{x}'; M) \to 0. \tag{35.6}$$

Alternatively, one can see this by using

$$K(\underline{x}; R) \otimes_R M = K(\underline{x}'; R) \otimes_R K(x_n; R) \otimes_R M = K(\underline{x}'; M) \otimes_R K(x_n; R)$$

and then using the spectral sequence for the (two-row) double complex in the last term. This inductive connection between $K(\underline{x}; M)$ and $K(\underline{x}'; M)$ leads quickly to the following connection between Koszul homology and regular sequences. This result generalizes Theorem 27.16, which was the case $M = R$.

Proposition 35.7. *Let R be a ring, $x_1, \dots, x_n \in R$, and let M be an R-module.*

(a) If $\underline{x}$ is weakly M-regular then $H_i(\underline{x}; M) = 0$ for all $i \geq 1$.

(b) Suppose (R, m) is local Noetherian, $x_1, \dots, x_n \in m$, and M is finitely-generated. Then $H_1(\underline{x}; M) = 0$ implies $\underline{x}$ is M-regular.

Proof. Both parts are trivial when $n = 1$, and we proceed by induction using (35.6) as the main tool.

If $\underline{x}$ is weakly M-regular then so is $\underline{x}'$, and so by induction $H_i(\underline{x}'; M)$ vanishes except when $i = 0$, where it is $M/(\underline{x}')M$. Part (a) follows directly.

For (b), if $H_1(\underline{x}; M) = 0$ then (35.6) shows that $H_1(\underline{x}'; M)/x_n H_1(\underline{x}'; M) = 0$ and $\operatorname{Ann}_{x_n} M/(\underline{x}')M = 0$. By Nakayama's Lemma the first implies that $H_1(\underline{x}'; M) = 0$ and hence $\underline{x}'$ is M-regular by induction. The vanishing of the annihilator says that x_n is a nonzerodivisor on $M/(\underline{x}')M$, so in fact $\underline{x}$ is M-regular. □

Corollary 35.8. *Suppose (R, m) is local, $x_1, \dots, x_n \in m$, and M is finitely-generated. Then $H_1(\underline{x}; M) = 0$ implies $H_j(\underline{x}; M) = 0$ for all $j \geq 1$.*

Proof. Immediate from Proposition 35.7, since the assumption implies that $\underline{x}$ is M-regular. □

The above property of the Koszul complexes is an example of what is known as *rigidity*: as soon as the homology vanishes in degree 1 then it vanishes in all higher degrees. We will soon see that a similar result holds in higher degree as well.

In a certain sense the Koszul complexes $K(\underline{x}; M)$ are "generic" objects. They depend on M as an abelian group as well as the collection of commuting endomorphisms $M \xrightarrow{\cdot x_i} M$, but on nothing else about the ring R. In fact, let $T = \mathbb{Z}[X_1, \dots, X_n]$ and let $f\colon T \to R$ be the homomorphism that sends each X_i to x_i. Every R-module M becomes a T-module via the map f. We have

$$K(x_1, \dots, x_n; R) = K(X_1, \dots, X_n; T) \otimes_T R$$
$$K(x_1, \dots, x_n; M) = K(X_1, \dots, X_n; T) \otimes_T M = K(X_1, \dots, X_n; M).$$

So the study of Koszul homology can in some sense be reduced to the case of polynomial rings $\mathbb{Z}[X_1, \dots, X_n]$. The only subtlety is that the R-module M, which for us will usually be finitely-generated, need not be finitely-generated as a $\mathbb{Z}[X_1, \dots, X_n]$-module. For this reason one might want to instead use $T' = R[X_1, \dots, X_n]$ and the surjection $f'\colon T' \to R$ that quotients by the ideal $(X_1 - x_1, \dots, X_n - x_n)$. Or sometimes (R, m) is local with all $x_i \in m$, and to get back into the local case one looks at $T'' = R[X_1, \dots, X_n]_{m+(X_1, \dots, X_n)}$ which again maps to R. With these techniques one can always reduce the study of Koszul homology modules $K(\underline{x}; M)$ to the case where $\underline{x}$ is a regular sequence in the underlying ring.

Proposition 35.9. *Let (R, m) be a Noetherian local ring, let $x_1, \dots, x_n \in m$, and let M be a finitely-generated R-module. If $i \geq 1$ and $H_i(\underline{x}; M) = 0$ then $H_j(\underline{x}; M) = 0$ for all $j \geq i$. [This is known as "rigidity of the Koszul complex".]*

Proof. Using the preceding remarks we reduce to the case where $\underline{x}$ is an R-regular sequence. Choose a surjection $F \twoheadrightarrow M$ where F is a finitely-generated free module and let K be the kernel. Since $H_*(\underline{x}; F)$ vanishes in positive degrees, the long exact sequence for $H_*(\underline{x}; -)$ gives that $H_j(\underline{x}; M) \cong H_{j-1}(\underline{x}; K)$ for $j \geq 2$. By iterating this process we can reduce the proposition to the case $i = 1$. But here it follows directly from Proposition 35.7, since $H_1(\underline{x}; M) = 0$ implies that $\underline{x}$ is M-regular and therefore that the higher Koszul homology vanishes as well. □

Corollary 35.10. *Let R be Noetherian, $x_1, \dots, x_n \in R$, and M be a finitely-generated module. Then for all $j \geq i$ one has* $\operatorname{Supp} H_j(\underline{x}; M) \subseteq \operatorname{Supp} H_i(\underline{x}; M)$.

Proof. Use that $H_j(\underline{x}; M)_P = H_j^{R_P}(\underline{x}; M_P)$. If $P \in \operatorname{Supp} H_j(\underline{x}; M)$ then this module is nonzero, and so it follows that all x_i belong to PR_P and by rigidity $H_i^{R_P}(\underline{x}; M_P) \neq 0$ for $1 \leq i \leq j$. So $P \in \operatorname{Supp} H_i(\underline{x}; M)$ for i in this range.

If $P \notin \operatorname{Supp} H_0(\underline{x}; M)$ then $M_P/(\underline{x})M_P = 0$. Since we know all x_i are in PR_P, Nakayama's lemma implies that $M_P = 0$ and this contradicts $P \in \operatorname{Supp} H_j(\underline{x}; M)$. □

When (R, m) is Noetherian local, rigidity says that the Koszul homology modules $H_*(\underline{x}; M)$ are all nonzero until one reaches a certain degree, and then they all vanish. The degree in which that transition happens therefore jumps out as a possibly interesting invariant; we will study this next.

35.11. **Depth.** Let I be an ideal in R. Let $\mathbf{depth}_I M$ denote the supremum of all lengths of M-regular sequences contained in I. As we begin to study this number, the first result characterizes when the depth is zero:

Proposition 35.12. *Let R be a Noetherian ring, $I \subseteq R$ be an ideal, and M be a finitely-generated R-module. The following are equivalent:*

(1) $\operatorname{depth}_I M = 0$.
(2) I is contained in some associated prime of M.
(3) $\operatorname{Hom}(R/I, M) \neq 0$ *(or equivalently, M contains a nonzero element killed by I).*

Proof. The key is that the set of zero-divisors on M equals $\bigcup_{P \in \operatorname{Ass} M} P$. So $\operatorname{depth}_I M = 0$ if and only if $I \subseteq \bigcup_{P \in \operatorname{Ass} M} P$. Prime avoidance then shows that (1) $\iff$ (2). The implication (3)$\Longrightarrow$(1) is also clear—because if (3) holds then there exists an $x \in M - 0$ annihilated by I.

Finally, suppose $I \subseteq P$ where $P \in \operatorname{Ass} M$. Since P is an associated prime, there is an embedding $R/P \hookrightarrow M$. The composite $R/I \twoheadrightarrow R/P \hookrightarrow M$ is then a nonzero homomorphism, showing that (2)$\Rightarrow$(3). □

From now on we will always assume our rings are Noetherian, unless otherwise indicated. Suppose I contains an M-regular element x_1. If I contains an element x_2 such that x_1, x_2 is M-regular, choose such an x_2; otherwise, stop. Continuing in this manner, the process must (for Noetherian rings) eventually stop otherwise we would have an infinite sequence of proper containments $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$. Say that an M-regular sequence in I is **maximal** if it cannot be extended to a

longer M-regular sequence in I. We have just seen that every M-regular sequence contained in I can be extended to a maximal one of finite length.

It is not immediately clear that all maximal sequences obtained in this way have the same length, but the next result will show that this is indeed the case.

The following result is stated in a generality that at first might seem unnecessarily confusing. In most applications $S = R$ and f is the identity map, and the reader should feel free to just think about that case at first. But the generalized situation below allows for M to perhaps *not* be finitely-generated over R, and we will see that this is useful on occasion.

Proposition 35.13. *Let $f\colon R \to S$ be a map of Noetherian rings, I an ideal in R, and M a finitely-generated S-module. Assume $IM \neq M$.*

(a) If $x_1, \dots, x_d \in I$ is a maximal M-regular sequence then $\operatorname{Ext}^i_R(R/I, M)$ is zero for $i \leq d-1$ and nonzero for $i = d$.
(b) $\operatorname{depth}_I R$ equals the smallest integer i such that $\operatorname{Ext}^i_R(R/I, M) \neq 0$.
(c) All maximal M-regular sequences in I have the same length, namely $\operatorname{depth}_I M$.
(d) If $x \in I$ is M-regular then $\operatorname{depth}_I M/xM = \operatorname{depth}_I M - 1$.

Proof. We prove (a) by induction on d. First assume $d = 0$. Our assumption says that no element of I is M-regular, so every $x \in I$ is either a zero divisor on M or else satisfies $xM = M$. But the latter would imply $IM = M$ and so cannot hold. Hence every $x \in I$ is a zero divisor on M.

Let $Q_1, \dots, Q_r \subseteq S$ be the associated primes of M as an S-module. Then $\bigcup_i Q_i$ is the set of zero-divisors on M in S. So $\bigcup_i f^{-1}(Q_i)$ is the set of zero-divisors on M in R. The ideal I is contained in this set, so by prime avoidance we have $I \subseteq f^{-1}(Q_i)$ for some i. Since $Q_i \in \operatorname{Ass}_S M$ we know there is an embedding $S/Q_i \hookrightarrow M$. Now look at the composite $R/I \to S/Q_i \hookrightarrow M$. The image of $\bar{1}$ is clearly nonzero, so $\operatorname{Hom}_R(R/I, M) \neq 0$.

Now suppose $d > 0$. Consider the long exact sequence that comes from applying $\operatorname{Ext}^*(R/I, -)$ to $0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$. Since x_1 kills the $\operatorname{Ext}^*(R/I, -)$ modules, the long exact sequence breaks up into short exact sequences

$$0 \longrightarrow \operatorname{Ext}^i(R/I, M) \longrightarrow \operatorname{Ext}^i(R/I, M/x_1M) \longrightarrow \operatorname{Ext}^{i+1}(R/I, M) \longrightarrow 0.$$

We know that $x_2, \dots, x_d$ is a maximal M/x_1M-regular sequence in I, so induction tells us that the middle group vanishes for $i \leq d-2$ and is nonzero for $i = d-1$. We deduce that $\operatorname{Ext}^i(R/I, M) = 0$ for $i \leq d-1$ and $\operatorname{Ext}^{d-1}(R/I, M/x_1M) \to \operatorname{Ext}^d(R/I, M)$ is an isomorphism, which shows the target group is nonzero. This is what we needed.

Parts (b) and (c) follow immediately from (a). For (d) just observe that if $y_1, \dots, y_d$ is a maximal M/xM-sequence in I then $x, y_1, \dots, y_d$ is a maximal M-sequence in I. □

Exercise 35.14. If $x_1, \dots, x_k$ is an M-regular sequence contained in I prove that $\operatorname{Ext}^i(R/I, M) \cong \operatorname{Ext}^{i-k}(R/I, M/(\underline{x})M)$ for all $i \leq \operatorname{depth}_I M$.

Proposition 35.15. *Let $f\colon R \to S$ be a ring homomorphism, let $I \subseteq R$ be an ideal, and let M be a finitely-generated S-module. Then $\operatorname{depth}_I M \leq \operatorname{depth}_{IS} M$. If R and S are Noetherian then this is an equality.*

Proof. If $x_1, \dots, x_d$ is an M-regular sequence then $f(x_1), \dots, f(x_d)$ is also M-regular (trivially), so this shows $\operatorname{depth}_I M \leq \operatorname{depth}_{IS} M$.

Now assume R and S are Noetherian. Let $x_1, \dots, x_d$ be a maximal M-regular sequence in I. Then $\operatorname{depth}_I M/(x_1, \dots, x_d)M = 0$. By Proposition 35.13 this implies that $\operatorname{Hom}_R(R/I, M) \neq 0$, so there is an element of $M/(x_1, \dots, x_d)M$ that is killed by I. But this element is then also killed by IS, which shows (by the same result) that $\operatorname{depth}_{IS} M/(x_1, \dots, x_d)M = 0$. We already knew that $f(x_1), \dots, f(x_d)$ is an M-regular sequence in IS, so the above statement about depth implies it is a maximal M-regular sequence. So $\operatorname{depth}_{IS} M = d$. □

Remark 35.16. Proposition 35.15 has the following important consequence. Suppose given a Noetherian ring R, ideal $I \subseteq R$, and a finitely-generated R-module M. Pick generators $x_1, \dots, x_r$ for I. Let $T = \mathbb{Z}[X_1, \dots, X_r]$, and let $f\colon T \to R$ be the ring homomorphism sending $X_i \to x_i$. Note that M need not be finitely-generated as a T-module. But Proposition 35.15 still gives $\operatorname{depth}_I M = \operatorname{depth}_{(X_1,\dots,X_r)} M$. The term on the right only depends on M as an abelian group together with the actions of the commuting endomorphisms X_i. This shows that depth captures a certain kind of "universal" aspect of a module, having fairly little to do with the underlying ring. It also suggests looking at the Koszul complex $K(X_1, \dots, X_r; M)$.

Proposition 35.17 (Depth-sensitivity of Koszul homology)**.** *Let R be a Noetherian ring, $I = (x_1, \dots, x_r)$ an ideal, and let M be a finitely-generated R-module. Then the largest degree for which $H_i(\underline{x}; M)$ is nonzero is $i = r - \operatorname{depth}_I M$.*

Proof. Let T and $f\colon T \to R$ be as in Remark 35.16, and set $J = (X_1, \dots, X_r)$. So $\operatorname{depth}_I M = \operatorname{depth}_J M$, and we know by Proposition 35.13 that this coincides with the smallest index for which $\operatorname{Ext}^i_T(T/J, M) \neq 0$.

But $X_1, \dots, X_r$ is a regular sequence in T, so T/J is resolved by the Koszul complex $K(X_1, \dots, X_r; T)$ and

$$\begin{aligned}\operatorname{Ext}^i_T(T/J, M) \cong H^i(\operatorname{Hom}_T(K(\underline{X}; T), M)) = H^i(\underline{X}; M) &\cong H_{r-i}(\underline{X}; M) \\ &\cong H_{r-i}(\underline{x}; M)\end{aligned}$$

where the second-to-last isomorphism is by Proposition 35.3. This completes the proof. □

Remark 35.18. This will not be explained in detail, but $\operatorname{depth}_I R$ should be regarded as a measure of the Zariski homotopical connectivity of the pair $(\operatorname{Spec} R, \operatorname{Spec} R - V(I))$. For example, if $R = \mathbb{C}[x, y, z]$ and $I = (x, y, z)$ then we are looking at $(\mathbb{A}^3, \mathbb{A}^3 - 0)$ and the connectivity/depth is 3. If instead $I = (xy, xz)$ then $V(I)$ is the union of a line and a plane and $(\mathbb{A}^3, \mathbb{A}^3 - V(I))$ has connectivity/depth equal to 1. To help with intution, in the homotopy theory of topological spaces we could access the connectivity by fattening $\operatorname{Spec} R - V(I) \hookrightarrow \operatorname{Spec} R$ into a cofibration and taking the quotient; for the union of a line and a plane in $\mathbb{R}^3$ this gives something that looks (up to homotopy) like $S^1 \vee S^2$. We do not have the time to explore this at length, but we refer the interested reader to the subject of local cohomology. See, for example, [BH, Section 3.5].

Proposition 35.19. *Let R be Noetherian, $I \subseteq R$ an ideal, and M a finitely-generated R-module.*

(a) $\operatorname{depth}_I M = \operatorname{depth}_{\operatorname{Rad}(I)} M$.

(b) For every prime ideal P, $\operatorname{depth}_I M \leq \operatorname{depth}_{IR_P} M_P$.

(c) Suppose (R, m) is local, $I \subseteq m$, and $x \in m$. Then one has $\operatorname{depth}_I M \leq \operatorname{depth}_{(I,x)} M \leq \operatorname{depth}_I M + 1$.

(d) *Suppose (R,m) is local. Then for every prime ideal P one has*

$$\operatorname{depth}_m M \leq \operatorname{depth}_{PR_P} M_P + \dim R/P.$$

(e) *Suppose (R,m) is local. Then for every associated prime P of M we have $\operatorname{depth}_m M \leq \dim R/P$. In particular, if M is nonzero then $\operatorname{depth}_m M \leq \dim M$.*

Proof. Since $I \subseteq \operatorname{Rad}(I)$ one clearly has $\operatorname{depth}_I M \leq \operatorname{depth}_{\operatorname{Rad}(I)} M$. For the inequality in the other direction, if $x_1, \ldots, x_d$ is a maximal M-regular sequence in $\operatorname{Rad}(I)$ then there exists an $r > 0$ such that $x_1^r, \ldots, x_d^r$ are all in I. This is also an M-regular sequence (Exercise 27.23), and this proves (a).

For (b) let $I = (y_1, \ldots, y_k)$. If $IM = M$ then all of the depths are equal to ∞ and so the inequalities are satisfied. So suppose $IM \neq M$. If $IM_P = M_P$ then $\operatorname{depth}_{IR_P} M_P = \infty$ and the inequality from (b) is trivial. If $IM_P \neq M_P$ then we can detect depth using Koszul homology. If $d = \operatorname{depth}_I M$ then $H_i(\underline{y}; M)$ is nonzero for $i = k - d$ and zero for $i > k - d$. But $H_i(\underline{y}; M_P) = H_i(\underline{y}; M)_P$ and so the zeros persist after localization, which implies that $d \leq \operatorname{depth}_{IR_P} M_P$.

The proof of (c) follows from the long exact sequence (35.5) comparing $H_*(\underline{y}, x; M)$ to $H_*(\underline{y}; M)$.

For part (d) let $d = \dim R/P$. Then we may pick $x_1, \ldots, x_d \in m$ such that $R/(P, \underline{x})$ has finite length. Equivalently, $\operatorname{Rad}(P, \underline{x}) = m$. Then

$$\operatorname{depth}_m M = \operatorname{depth}_{(P,\underline{x})} M \leq \operatorname{depth}_P M + d \leq \operatorname{depth}_{PR_P} M_P + d,$$

using (a), (c), and (b) in that order.

For (e) let P be an associated prime of M. Then there is an inclusion $R/P \hookrightarrow M$. After localization this is still an inclusion $R_P/PR_P \hookrightarrow M_P$, so we have that $\operatorname{depth}_{PR_P} M_P = 0$. But $\operatorname{depth}_m M \leq \operatorname{depth}_{PR_P} M_P + \dim R/P$, so we obtain $\operatorname{depth}_m M \leq \dim R/P$. □

Exercise 35.20. Prove part (c) of the above result and explain where the hypothesis that R is local is used. To give a counterexample where R is not local, let $A = k[x_1, \ldots, x_n]$ where k is a field, let $R = k \times A$, let $y = (1, x_2)$, and let $I = (x_2, \ldots, x_n) \subseteq R$. Prove that $\operatorname{depth}_I R = 0$ but $\operatorname{depth}_{(I,y)} R = n$. Give a geometric interpretation along the lines of Remark 35.18, using that $\operatorname{Spec} R = \mathbb{A}^n \amalg pt$.

Corollary 35.21. *Let (R,m) be a Noetherian local ring and let J and M be finitely-generated R-modules. If $\operatorname{Hom}(J, M) \neq 0$ then $\dim J \geq \operatorname{depth}_m M$. In particular, this holds if J is a nonzero submodule of M.*

Proof. Let $f\colon J \to M$ be a nonzero homomorphism, and let $N = f(J)$. Choose a minimal prime P of N such that $\dim N = \dim R/P$. Then $P \in \operatorname{Ass} N \subseteq \operatorname{Ass} M$, hence by Proposition 35.19(e) $\operatorname{depth}_m M \leq \dim R/P = \dim N \leq \dim J$. □

Finally, we give one more corollary that can be useful in surprising ways:

Corollary 35.22. *Let (R,m) be a Noetherian local ring, let M be a finitely-generated module, and let $f\colon M \to M$ be such that $f(M) \subseteq mM$. If $\ker f \neq 0$ then $\dim(\ker f) \geq \operatorname{depth}_m(\operatorname{coker} f)$.*

Proof. Let $K = \ker f$. Since $\bigcap_s m^s M = 0$ and $f^s(M) \subseteq m^s M$ and $K \neq 0$, there is a largest s such that $W \subseteq f^s(M)$. The map f induces

$$M/f(M) \xrightarrow{f} f(M)/f^2(M) \xrightarrow{f} f^2(M)/f^3(M) \xrightarrow{f} \cdots \xrightarrow{f} f^s(M)/f^{s+1}(M)$$

and we claim these are all isomorphisms (but would not be if we went one stage further). In each case surjectivity is evident. If $x \in f^i(M)$ and $f(x) \in f^{i+2}(M)$ then we write $f(x) = f^{i+2}(z)$ for some z. Then $f(x - f^{i+1}z) = 0$, so $x - f^{i+1}z \in K$. As long as $i+1 \leq s$ we have $K \subseteq f^{i+1}(M)$, hence $x \in f^{i+1}(M)$, and so $x = 0$ in $f^i(M)/f^{i+1}(M)$. This shows injectivity.

Finally, since $K \subseteq f^s(M)$ we can consider $K \to f^s(M)/f^{s+1}(M)$. By the choice of s we know this map is nonzero. Therefore by Corollary 35.21 we find that $\dim K \geq \operatorname{depth}_m f^s(M)/f^{s+1}(M) = \operatorname{depth}_m M/f(M)$. □

35.23. **Non-negativity of Koszul multiplicities.** Our next goal will be to show that the Koszul multiplicity $e(\underline{x}; M)$ (defined when $M/(\underline{x})M$ has finite length) is always non-negative. There are different approaches to this, but the most elegant requires us to broaden our perspective a bit. The plan will be

- Prove it for sequences of length 1 (not obvious, but easy enough once one sees the key idea),
- Use a formula

$$e(\underline{x}; M) = e(x_1; e(x_2; e(x_3; \ldots; e(x_n; M)))) \tag{35.24}$$

  to get the general result by induction.

Of course (35.24) does not make sense as written because we have defined the e's as numbers, and also because an expression like $e(x_i; M)$ does not make sense on its own since M/x_1M need not have finite length. This is where the "broadening of perspective" will come in.

Let $I \subseteq R$ be an ideal and let M be a finitely-generated R-module that is killed by I. For $x_1, \ldots, x_n \in R$ define

$$\chi(\underline{x}; M) = \sum_i (-1)^i [H_i(\underline{x}; M)] \in G(R/(I, \underline{x})).$$

This is an additive function, and so we can interpret $\chi(\underline{x}; -)$ as a homomorphism $G(R/I) \to G(R/(I, \underline{x}))$.

Proposition 35.25. *The function $\chi(\underline{x}; -)\colon G(R/I) \to G(R/(I, \underline{x}))$ equals the composite*

$$\begin{array}{ccccccccc} & & G(R/I) & & & & & & \\ & & \downarrow{\scriptstyle \chi(x_n;-)} & & & & & & \\ G(R/(I,x_n)) & \xrightarrow{\chi(x_{n-1};-)} & G(R/(I,x_n,x_{n-1})) & \xrightarrow{\chi(x_{n-2};-)} & \cdots & \xrightarrow{\chi(x_1;-)} & G(R/(I,\underline{x})). \end{array}$$

Proof. The easiest approach is to change from Grothendieck groups of modules to Grothendieck groups of complexes. Let $G^{cplx}(R)_J$ be the Grothendieck group of bounded-below complexes of finitely-generated modules having bounded homology that is killed by J. For Noetherian rings the Euler characteristic $\chi\colon G^{cplx}(R)_J \to G(R/J)$ sending $C_\bullet \to \sum_i (-1)^i [H_i(C)]$ is an isomorphism (Exercise 6.25).

Let $K_i = K(x_i; R)$ and examine the following diagram:

$$\begin{array}{ccccccc} G^{cplx}(R)_I & \xrightarrow{\otimes K_n} & G^{cplx}(R)_{(I,x_n)} & \xrightarrow{\otimes K_{n-1}} & \cdots & \xrightarrow{\otimes K_1} & G^{cplx}(R)_{(I,\underline{x})} \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} & & & & \downarrow{\scriptstyle \chi} \\ G(R/I) & \xrightarrow{\chi(x_n;-)} & G(R/(I,x_n)) & \xrightarrow{\chi(x_{n-1};-)} & \cdots & \xrightarrow{\chi(x_1;-)} & G(R/(I,\underline{x})). \end{array}$$

The maps $(-) \otimes K_i$ are well-defined because K_i is a bounded complex of finitely-generated free modules, so tensoring with K_i preserves quasi-isomorphisms and short exact sequences. The squares all commute: for a complex $C_\bullet$ this can be checked by a spectral sequence (for example), but the proof is easier than this. Since the vertical maps are isomorphisms the groups $G^{cplx}(R)_J$ are generated by chain complexes that are concentrated in degree zero, and for such a complex the commutativity of the square is obvious.

The statement of the proposition follows immediately from the commutativity of the outside rectangle in the diagram. For a finitely-generated R/I-module M start with the chain complex $M[0]$ and track around the outside of the diagram in the two ways. □

Remark 35.26. For a closed set $Z \subseteq \operatorname{Spec} R$ one can also look at $G(R)_Z$, the Grothendieck group of modules supported on Z. When $Z = V(I)$ a finitely-generated module is supported on Z if and only if it is killed by a power of I, so the evident map $G(R/I) \to G(R)_Z$ is an isomorphism by dévissage. We can interpret $\chi(\underline{x}; -)$ as a map $G(R)_Z \to G(R)_{Z \cap V(\underline{x})}$ and all the same results hold in this setting. At some point the support-formulation will be more useful to us than the $G(R/I)$ formulation, but for now we will stick with the latter.

If S is a ring and $u \in G(S)$, write $u \geq 0$ to mean that $u = [M]$ for some finitely-generated S-module M. More generally, for $u, v \in G(S)$ write $u \geq v$ if $u - v \geq 0$ in this sense. Also, if M is a module and $x \in M$ then write $\operatorname{Ann}_M x^\infty$ for $\bigcup_t \operatorname{Ann}_M x^t$. This is the submodule of elements killed by some power of x. If M is Noetherian then the chain $\operatorname{Ann}_M x \subseteq \operatorname{Ann}_M x^2 \subseteq \cdots$ stabilizes, and so $\operatorname{Ann}_M x^\infty = \operatorname{Ann}_M x^r$ for some r.

Proposition 35.27. *Let R be a Noetherian ring, let $x \in R$, and let M be a finitely-generated module. Suppose I is an ideal that kills M.*
(a) If M is killed by a power of x then $\chi(x; M) = 0$ in $G(R/(I, x))$.
(b) $\chi(x; M) = [M/(\operatorname{Ann}_M x^\infty + xM)]$ in $G(R/(I, x))$, and hence $\chi(x; M) \geq 0$.

Proof. Since $\chi(x; M) = [M/xM] - [\operatorname{Ann}_M x]$, if $xM = 0$ then we have $\chi(x; M) = [M] - [M] = 0$. If M is killed by x^r then consider the finite filtration $M \supseteq xM \supseteq x^2M \supseteq \cdots \supseteq x^rM = 0$. Additivity of $\chi(x; -)$ shows that $\chi(x; M) = \sum_i \chi(x; x^iM/x^{i+1}M)$, and all the terms in the sum are zero since x kills all of those quotients. This proves (a).

For (b), first note that if x acts as a nonzerodivisor on M then $\chi(x; M) = [M/xM]$ by definition, and $M/xM = M/(\operatorname{Ann}_M x^\infty + xM)$ in this case. For the general case, observe that x acts as a nonzerodivisor on $M/\operatorname{Ann}_M x^\infty$, and the exact sequence $0 \to \operatorname{Ann}_M x^\infty \to M \to M/\operatorname{Ann}_M x^\infty \to 0$ gives

$$\chi(x; M) = \chi(x; \operatorname{Ann}_M x^\infty) + \chi(x; M/\operatorname{Ann}_M x^\infty).$$

Since the module is Noetherian, $\operatorname{Ann}_M x^\infty = \operatorname{Ann}_M x^r$ for some r. So $\operatorname{Ann}_M x^\infty$ is killed by a power of x, hence $\chi(x; \operatorname{Ann}_M x^\infty) = 0$ by (a). On the other hand, x is a nonzerodivisor on $M/\operatorname{Ann}_M x^\infty$ and so $\chi(x; M/\operatorname{Ann}_M x^\infty) = [M/(\operatorname{Ann}_M x^\infty + xM)]$. □

Let us define $\mathfrak{X}(x; M) = M/(\operatorname{Ann}_M x^\infty + xM)$, and more generally define

$$\mathfrak{X}(\underline{x}; M) = \mathfrak{X}(x_1; \mathfrak{X}(x_2; \cdots; \mathfrak{X}(x_n; M))).$$

It is useful to give this a name, so let us call it the **Koszul multiplicity module**. Note that the order of the x's is important here; changing the order might possibly change the isomorphism type of the module.

In dealing with these modules it is helpful to introduce colon submodules, which are a generalization of annihilators. If $N \subseteq M$ write $N :_M I$ (often just $N : I$ for short) for $\{x \in M \mid Ix \subseteq N\}$. In this notation $\operatorname{Ann}_M x = 0 : x$. Let us also write $N : x^\infty$ for $\bigcup_t N : x^t$. Then by simply re-interpreting the above definition we obtain $\mathcal{X}(\underline{x}; M) = M/N_{\underline{x}}$ where

$$N_{\underline{x}} = \left(\left(((0 : x_n^\infty) + x_n M) : x_{n-1}^\infty \right) + x_{n-1} M \right) : x_{n-2}^\infty + \cdots$$

Proposition 35.28. *Let R be Noetherian, $x_1, \ldots, x_n \in R$, I an ideal of R, and M a finitely-generated module killed by I. Then $\chi(\underline{x}; M) = [\mathcal{X}(\underline{x}; M)]$ in $G(R/(I, \underline{x}))$, and so in particular $\chi(\underline{x}; M) \geq 0$.*

Proof. This is immediate from Proposition 35.25 and Proposition 35.27(b). □

Although we do not know that the module $\mathcal{X}(\underline{x}; M)$ is independent of the ordering of the x's, the above result implies that the class in $G(R/(I, \underline{x}))$ is independent of the order.

Now return to the setting of Koszul multiplicities:

Proposition 35.29. *Let (R, m) be a Noetherian local ring, let $x_1, \ldots, x_n \in m$, and let M be a finitely-generated module such that $M/(\underline{x})M$ has finite length. Then $e(\underline{x}; M) = \ell(\mathcal{X}(\underline{x}; M)) \geq 0$.*

Proof. The finite-length hypothesis implies that $R/(\operatorname{Ann} M, \underline{x})$ is Artinian, and so $\ell \colon G(R/(\operatorname{Ann} M, \underline{x})) \to \mathbb{Z}$ is an isomorphism. We have $e(\underline{x}; M) = \ell(\chi(\underline{x}; M))$, and so Proposition 35.28 implies that $e(\underline{x}; M) = \ell(\mathcal{X}(\underline{x}; M))$. □

Next we turn to the question of when $e(\underline{x}; M)$ vanishes.

Proposition 35.30. *Let R be Noetherian, let $x_1, \ldots, x_n \in R$, and let M be a finitely-generated module. Let $\underline{x}'$ denote the sequence $x_1, \ldots, x_{n-1}$.*

(a) If M is nonzero and $x \in R$ then $\dim \mathcal{X}(x; M) < \dim M$.
(b) If $x_1, \ldots, x_n \in R$ and $\dim M < n$ then $\mathcal{X}(\underline{x}; M) = 0$.
(c) If some power of x_i kills M then $\mathcal{X}(\underline{x}; M) = 0$, and if x_n is a nonzerodivisor on M then $\mathcal{X}(\underline{x}; M) = \mathcal{X}(\underline{x}'; M/x_n M)$.
(d) Suppose (R, m) is local, $x_1, \ldots, x_n \in m$, and $M/(\underline{x})M$ has finite length. Then $e(\underline{x}; M) = 0$ if and only if $\dim M < n$.

Proof. For (a) let $M' = M/\operatorname{Ann}_M x^\infty$, so that $\mathcal{X}(x; M) = M'/xM'$. Since M' is a quotient of M we have $\dim M' \leq \dim M$. If $M' = 0$ then $\mathcal{X}(x; M) = 0$, which has dimension -1; so $\dim \mathcal{X}(x; M) < \dim M$ here. But if $M' \neq 0$ then x is a nonzerodivisor on M', hence $\dim M'/xM' \leq \dim M' - 1 \leq \dim M - 1 < \dim M$.

Part (b) follows from (a): in the iterated sequence $\mathcal{X}(x_1; \mathcal{X}(x_2; \cdots; \mathcal{X}(x_n; M)))$, working from right to left each module is either zero or else the dimension drops by one. If $\dim M < n$, by the time we are done all of $\dim M$ has been used up and we must be left with the zero module.

Part (c) is immediate from the definitions.

Finally, we will prove (d) by induction on n. The case $n = 0$ says that $\ell(M) = 0$ if and only if $\dim M < 0$, and this is true because both conditions

are equivalent to $M = 0$. For $n > 0$ let $A = \operatorname{Ann}_M x_n^\infty$ and $Q = M/A$, so that we have the short exact sequence $0 \to A \hookrightarrow M \to Q \to 0$. The hypothesis that $M/(\underline{x})M$ has finite length says that $\operatorname{Supp} M \cap V(x_1, \dots, x_n) = \{m\}$. Since A is killed by a power of x_n, every prime containing $\operatorname{Ann} A$ also contains x_n. So $(\operatorname{Supp} A) \cap V(x_1, \dots, x_{n-1}) \subseteq \operatorname{Supp} M \cap V(x_1, \dots, x_n) = \{m\}$, and hence $\dim A/(x_1, \dots, x_{n-1})A \leq 0$. This implies $\dim A \leq n-1$. The above short exact sequence then yields that $\dim M < n$ if and only if $\dim Q < n$, since the dimension of the middle term is at most the maximum dimension of the two outer terms.

We also have $e(\underline{x}; M) = e(\underline{x}; A) + e(\underline{x}; Q) = e(\underline{x}; Q) = e(\underline{x}'; Q/x_1Q)$, where we have used (c) twice. So by induction $e(\underline{x}; M) = 0$ if and only if $\dim Q/x_1Q < n-1$. But either $Q = 0$ or x_n is a nonzerodivisor on Q, and the latter yields $\dim Q/x_1Q = \dim Q - 1$ (since R is local). So $e(\underline{x}; M) = 0$ if and only if $\dim Q < n$, which we saw above was equivalent to $\dim M < n$. □

Proposition 35.31. *Let (R, m) be local, $x_1, \dots, x_n \in m$, and let M be a finitely-generated module such that $M/\underline{x}M$ has finite length. Then $e(\underline{x}; M) \leq \ell(M/\underline{x}M)$, and equality holds if and only if $\underline{x}$ is an M-regular sequence.*

Before giving the proof, it is useful to separate out the special case $n = 1$:

Lemma 35.32. *Let (R, m) be local, $x \in m$, and let M be a nonzero finitely-generated module such that M/xM has finite length. Then*

(a) $e(x; M) = 0$ if and only if M is killed by a power of x.
(b) $e(x; M) \leq \ell(M/xM)$, and equality holds if and only if x is a nonzerodivisor on M.

Proof. We know that $e(x; M) = \ell(\mathcal{X}(x; M)) = \ell(M/(\operatorname{Ann} x^\infty + xM))$ by Proposition 35.28. So $e(x; M) = 0$ if and only if $0 = M/(\operatorname{Ann} x^\infty + xM) = (M/\operatorname{Ann} x^\infty)/x(M/\operatorname{Ann} x^\infty)$, and by Nakayama's lemma this is equivalent to $M/\operatorname{Ann} x^\infty = 0$. But this is the same as saying that M is killed by a power of x.

For (b) observe that we have a canonical surjection $M/xM \to \mathcal{X}(x; M)$. This immediately shows $e(x; M) \leq \ell(M/xM)$. Equality holds if and only if our surjection is an isomorphism, which is equivalent to the statement that $\operatorname{Ann} x^\infty \subseteq xM$. We claim this implies (and hence is equivalent to) $\operatorname{Ann} x = 0$.

Let $w \in \operatorname{Ann} x^\infty$. Then $x^r w = 0$ for some $r \geq 1$. But $w \in xM$, so $w = xz$ for some $z \in M$. Then $0 = x^r w = x^{r+1} z$, so $z \in \operatorname{Ann} x^\infty$ and therefore $w \in x \cdot \operatorname{Ann} x^\infty$. This shows that $\operatorname{Ann} x^\infty = x \cdot \operatorname{Ann} x^\infty$, and so by Nakayama's lemma $\operatorname{Ann} x^\infty = 0$. This is equivalent to $\operatorname{Ann} x = 0$. □

Proof of Proposition 35.31. Note that the inequality $e(\underline{x}; M) \leq \ell(M/\underline{x}M)$ is immediate from the surjection $M/\underline{x}M \twoheadrightarrow \mathcal{X}(\underline{x}; M)$. Also, if $\underline{x}$ is M-regular then the higher Koszul homology all vanishes and so $e(\underline{x}; M) = \ell(M/\underline{x}M)$. So we only have to prove that $e(\underline{x}; M) = \ell(M/\underline{x}M)$ implies that $\underline{x}$ is M-regular. We tackle this by induction on n, the case $n = 1$ being Lemma 35.32(b). So assume $n > 1$, and as always write $\underline{x}'$ for the sequence $x_1, \dots, x_{n-1}$.

As elements of $G(R/(\operatorname{Ann} M, \underline{x}))$ we have

$$\chi(\underline{x}; M) = \chi(x_n; \chi(\underline{x}'; M)) = \chi(x_n; \mathcal{X}(\underline{x}'; M))$$

and so

$$\ell(M/\underline{x}M) = e(\underline{x}; M) = e(x_n; \mathcal{X}(\underline{x}'; M)) \leq \ell(\mathcal{X}(\underline{x}'; M)/x_n) \leq \ell(M/\underline{x}M),$$

where the first inequality is by the lemma and the second is from taking the surjection $M/\underline{x}'M \twoheadrightarrow \mathcal{X}(\underline{x}'; M)$ and tensoring with R/x_nR. Since the first and last entries of the chain agree, both inequalities must actually be equalities.

Let J be the kernel of the surjection $M/\underline{x}'M \twoheadrightarrow \mathcal{X}(\underline{x}'; M)$. Then we have

$$e(x_n; M/\underline{x}'M) = e(x_n; J) + e(x_n; \mathcal{X}(\underline{x}'; M)) = e(x_n; J) + \ell(M/\underline{x}M),$$

where the second equality is from the previous paragraph. But as we already know $e(x_n; M/\underline{x}'M) \leq \ell(M/\underline{x}M)$ and $e(x_n; J) \geq 0$, it must be that these are both equalities. By the lemma, x_n is a nonzerodivisor on $M/\underline{x}'M$ and some power of x_n kills J. But $J \subseteq M/\underline{x}'M$, so the only way these statements are compatible is if $J = 0$. This implies that $M/\underline{x}'M = \mathcal{X}(\underline{x}'; M)$, therefore $e(\underline{x}'; M) = \ell(M/\underline{x}'M)$, and so by induction $\underline{x}'$ is M-regular. Putting everything together, we have shown that $\underline{x}$ is M-regular. □

35.33. **Partial Euler characteristics.** Fix $x_1, \ldots, x_n \in R$. For a finitely-generated module M and a fixed $i \geq 0$, let $Z_i = \operatorname{Supp} H_i(\underline{x}; M)$. Define $G(R)_Z$ to be the Grothendieck group of modules that are supported on Z. By rigidity of the Koszul complex (Corollary 35.10) we know $\operatorname{Supp} H_j(\underline{x}; M) \subseteq Z$ for all $j \geq i$, so we can define

$$\chi_i(\underline{x}; M) = \sum_{j \geq i} (-1)^{j-i} [H_j(\underline{x}; M)] \in G(R)_{Z_i}.$$

This is called the ***i*th partial Euler characteristic** (or sometimes "higher" Euler characteristic).

Proposition 35.34. *For all $i \geq 1$ one has $\chi_i(\underline{x}; M) \geq 0$ in $G(R)_{Z_i}$.*

Proof. For length one sequences this is trivial, so we can proceed by induction and assume the length is at least two. As always, let $\underline{x}'$ be the sequence $x_1, \ldots, x_{n-1}$. The short exact sequences from (35.6) show that

$$[H_j(\underline{x}; M)] = [H_j(\underline{x}'; M)/x_n] + [\operatorname{Ann}_{H_{j-1}(\underline{x}'; M)} x_n]$$

for all $j \geq i$ (the short exact sequences also show that all these modules are supported on Z_i). Taking the alternating sum for $j \geq i$, one finds that

$$\begin{aligned} \chi_i(\underline{x}; M) &= [\operatorname{Ann}_{H_{i-1}(\underline{x}'; M)} x_n] + \sum_{j \geq i} (-1)^{j-i} \chi(x_n; H_j(\underline{x}'; M)) \\ &= [\operatorname{Ann}_{H_{i-1}(\underline{x}'; M)} x_n] + \chi(x_n; \chi_i(\underline{x}'; M)). \end{aligned}$$

By induction $\chi_i(\underline{x}'; M) \geq 0$, and then $\chi(x_n; \chi_i(\underline{x}'; M)) \geq 0$ by Proposition 35.27 and Remark 35.26. □

For the rest of this section we work in the setting where (R, m) is local, $x_1, \ldots, x_n \in m$, and M is a finitely-generated module. If $H_i(\underline{x}; M)$ has finite length then we can take $Z = \{m\}$ in the above discussion, in which case $\ell\colon G(R)_Z \to \mathbb{Z}$ is an isomorphism. In this setting we will write

$$e_i(\underline{x}; M) = \sum_{j \geq i} (-1)^{j-i} \ell(H_j(\underline{x}; M)) \in \mathbb{Z},$$

but there is really no difference from the $\chi_i(\underline{x}; M)$ defined earlier for $Z = \{m\}$.

Proposition 35.35. *Let (R, m) be local, let $x_1, \ldots, x_n \in R$, and let M be a finitely-generated R-module such that $H_i(\underline{x}; M)$ has finite length. Then when $i \geq 1$ the following are equivalent:*

(1) $e_i(\underline{x}; M) = 0$,
(2) $H_i(\underline{x}; M) = 0$,
(3) $H_j(\underline{x}; M) = 0$ *for all* $j \geq i$.
Consequently, if $i \geq 1$ *and* $e_i(\underline{x}; M) = 0$ *then* $e_j(\underline{x}; M) = 0$ *for all* $j \geq i$.

Proof. Rigidity of the Koszul complex (Proposition 35.9) implies that (2) and (3) are equivalent, and clearly (3) implies (1). We need to prove that (1)$\Longrightarrow$(2).

By replacing R with a localized polynomial ring we can reduce to the case where $\underline{x}$ is R-regular. Let F be a finitely-generated free module with a surjection $F \twoheadrightarrow M$, and let K be the kernel. Then for $j \geq 2$ we have $H_j(\underline{x}; M) \cong H_{j-1}(\underline{x}; K)$, and this allows us to further reduce to the case $i = 1$. That is, we must show that $e_1(\underline{x}; M) = 0$ implies that $H_1(\underline{x}; M) = 0$. Note that if $M/(\underline{x})M$ has finite length then this follows from Proposition 35.31: just observe that $e_1(\underline{x}; M) = 0$ precisely when $e(\underline{x}; M) = \ell(M/\underline{x}M)$. However, we need to prove the claim under only the hypothesis that $\ell(H_1(\underline{x}; M)) < \infty$.

The case $n = 1$ is trivial, so we assume $n > 1$ and proceed by induction. Start with $K_\bullet(\underline{x}; M) \cong K_\bullet(\underline{x}'; M) \otimes K_\bullet(x_n; R)$ and use one of the two spectral sequences for a double complex. We get an E_2-term that looks like

$$\begin{array}{ccccc}
\cdots & H_2(\underline{x}'; M/x_nM) & H_1(\underline{x}'; M/x_nM) & H_0(\underline{x}'; M/x_nM) \\
 & \searrow^{d_2} & \searrow^{d_2} & \\
\cdots & H_2(\underline{x}'; \operatorname{Ann}_M x_n) & H_1(\underline{x}'; \operatorname{Ann}_M x_n) & H_0(\underline{x}'; \operatorname{Ann}_M x_n).
\end{array}$$

The $E_3 = E_\infty$ page gives an associated graded of $H_*(\underline{x}; M)$; the dotted lines in the above picture indicate the total-degree diagonals. So the assumption that $H_1(\underline{x}; M)$ has finite length implies the same for $H_1(\underline{x}'; M/x_nM)$, which by rigidity implies that $H_j(\underline{x}'; M/x_nM)$ has finite length for all j. This then implies that $H_0(\underline{x}'; \operatorname{Ann}_M x_n)$ must also have finite length, implying the same for all $H_j(\underline{x}'; \operatorname{Ann}_M x_n)$. We conclude from the spectral sequence that

$$e_1(\underline{x}; M) = e_1(\underline{x}'; M/x_nM) + e(\underline{x}'; \operatorname{Ann}_M x_n).$$

We know by Propositions 35.34 and 35.29 that $e_1(\underline{x}'; M/x_nM)$ and $e(\underline{x}'; \operatorname{Ann}_M x_n)$ are nonnegative. So if $e_1(\underline{x}; M) = 0$ then $e_1(\underline{x}'; M/x_nM) = 0$ and $e(\underline{x}'; \operatorname{Ann}_M x_n) = 0$. By induction, the first equation shows that $H_1(\underline{x}'; M/x_nM) = 0$ and hence $\underline{x}'$ is an M/x_nM-regular sequence. By Proposition 35.30 the second equation shows that $\dim \operatorname{Ann}_M x_n < n - 1$.

But if $\operatorname{Ann}_M x_n \neq 0$ then Corollary 35.22 implies that $\dim(\operatorname{Ann}_M x_n) > \operatorname{depth}_m M/x_nM$, which is a contradiction since the first is smaller than $n - 1$ and the second is at least $n - 1$. So $\operatorname{Ann}_M x_n = 0$. Then $\underline{x}$ is M-regular, which implies $H_1(\underline{x}; M) = 0$ (alternatively, look at the E_2-term above and observe that everything except $H_0(\underline{x}'; M/x_nM)$ vanishes). □

The above argument is from [Lichtenbaum].

Say that a sequence of integers $\{a_i\}_{i \geq 0}$ is **finite** if there are only finitely-many nonzero terms. Given such a sequence, the **merged sequence** is $b_i = a_i + a_{i+1}$. Define a **Pascalian sequence** to be any finite sequence $\{a_i\}_{i \geq 0}$ of non-negative integers having the property that $a_i = 0$ for some $i \geq 1$ implies $a_j = 0$ for all $j \geq i$. The merged sequence of a Pascalian sequence is clearly also Pascalian. A familiar example is $a_i = \binom{n}{i}$ for a fixed n, in which case the merged sequence is $b_i = \binom{n+1}{i}$.

Observe that $\underline{a}$ can be recovered from its merged sequence $\underline{b}$ via the formulas $a_i = b_i - b_{i+1} + b_{i+2} - \cdots$. We call this process **separation**. Say that a Pascalian sequence is **well-separable** if it equals the merged sequence of another Pascalian sequence.

Finally, suppose given a sequence $\{d_i\}_{i\geq 0}$ with the property that $d_i = d_{i+1}$ for $i \gg 0$; i.e., the sequence eventually stabilizes. Let d_∞ denote the stabilized value. Say that $\underline{d}$ has the **alternating property** if $d_i \geq d$ for i even, $d_i \leq d$ if i odd, and if some $d_i = d$ then $d_j = d$ for all $j \geq i$.

Proposition 35.36. *Suppose $\{b_i\}_{i\geq 0}$ is a finite sequence of integers. Set $\chi_i = b_i - b_{i+1} + b_{i+2} - \cdots$ and $d_i = b_0 - b_1 + b_2 - \cdots + (-1)^i b_i$. The following are equivalent:*

(1) $\underline{b}$ is well-separable;
(2) $\underline{\chi}$ is Pascalian;
(3) $\underline{d}$ is alternating.

Proof. This is routine. In particular, observe that $\chi_i + \chi_{i+1} = b_i$ and $d_i = \chi_0 + (-1)^i \chi_{i+1}$. □

Exercise 35.37. Let $C_\bullet$ be a bounded, non-negatively-graded chain complex of finite-dimensional vector spaces such that $H_i(C) = 0$ for $i \neq 0$. Prove that the sequence $\{\dim C_i\}$ is well-separable. If R is a domain and $C_\bullet$ instead consists of free R-modules of finite rank (but with the other hypotheses all the same) prove that $\{\operatorname{rank} C_i\}$ is well-separable.

Proposition 35.38. *Let (R, m) be local, $x_1, \ldots, x_n \in m$, and M be a finitely-generated R-module such that $M/(\underline{x})M$ has finite length. Then the sequence $\{\ell(H_i(\underline{x}; M))\}$ is well-separable.*

Proof. By Proposition 35.36 this is implied by the facts that the $\chi_i(\underline{x}; M)$ are all nonnegative (Proposition 35.34), and that if some $\chi_i(\underline{x}; M)$ vanishes for $i \geq 1$ then all of the higher ones do as well (Proposition 35.35). □

36. Multiplicities once again

UNDER CONSTRUCTION

In this section we return one last time to intersection multiplicities. We establish some of the results that were alluded to when we first encountered them in Section 2, and we explore some additional examples.

36.1. **Truncated Euler characteristics and the alternating property.** Let R be a regular local ring and let M and N be finitely-generated modules such that $\ell(M \otimes N) < \infty$. Consider the truncated Euler characteristics

$$d_q(M, N) = \sum_{j=0}^{q} (-1)^j \ell(\operatorname{Tor}_j(M, N)).$$

Of course $d_0(M, N)$ is just another name for the intersection multiplicity $i(M, N)$.

Theorem 36.2. *Suppose that R contains a field. Then $d_q(M, N) \geq i(M, N)$ when q is even, $d_q(M, N) \leq i(M, N)$ when q is odd, and if $d_q(M, N) = i(M, N)$ then $d_j(M, N) = i(M, N)$ for all $j \geq q$.*

Proof. The numbers $\ell(\mathrm{Tor}_j(M,N))$ are not changed by passing to the completions $\hat{R}$, $\hat{M}$, and $\hat{N}$. So we can assume that R is complete, and then by the Cohen structure theorems this implies that R is a power series ring $k[[x_1,\dots,x_n]]$ where k is a field. Now we can use Serre's reduction to the diagonal arguments to reinterpret $\mathrm{Tor}_j(M,N)$ as $H_j(\underline{y};W)$ for $S = k[[x_1,\dots,x_n,x_1',\dots,x_n']]$, $y_i = x_i - x_i'$, and $W = M\hat{\otimes}N$. Then $d_q(M,N) = \chi_q^S(\underline{y};W)$, and the result follows from Propositions 35.38 and 35.36. □

As we saw in Proposition 35.36, the conclusion of the above proposition is equivalent to non-negativity and rigidity of the partial Euler characteristics $\chi_j(M,N) = \sum_{k\geq j}(-1)^k\ell(\mathrm{Tor}_j(M,N))$. For more general regular local rings this is also known in the unramified mixed characteristic case by [Lichtenbaum] and [Hochster4]. At the time of writing the result is open in the ramified case.

36.3. **Cohen-Macaulay modules.** Let X be a regular scheme of dimension n. Our next general theme is this: given subvarieties W and Z of X meeting at an isolated point P, are there interesting bounds on the largest j such that $\mathrm{Tor}_j(\mathcal{O}_W,\mathcal{O}_Z)_P \neq 0$ in terms of local invariants of W and Z at P? We will give two such bounds in the next section, but before tackling those we need a small digression on Cohen-Macaulay modules.

Definition 36.4. *Let M be a finitely-generated module over a local ring (R,m). We say that M is **Cohen-Macaulay** if $\mathrm{depth}_m M = \dim M$.*

One can show that if M is Cohen-Macaulay then M_P is Cohen-Macaulay as an R_P-module, for every prime ideal P. So it makes sense to also make the following definition:

Definition 36.5. *Let M be a finitely-generated module over a Noetherian ring R. Then M is said to be **Cohen-Macaulay** if M_P is Cohen-Macaulay for all prime ideals P.*

Proposition 36.6. *Let R be a Noetherian ring and let M be a finitely-generated R-module. Let $S = R/\operatorname{Ann} M$. Then M is Cohen-Macaulay if and only if for every ideal $I \subseteq R$ one has $\mathrm{depth}_{IS} M = \mathrm{ht}\,(IS)$.*

Proof. ??? □

Proposition 36.7. *Let M be a finitely-generated Cohen-Macaulay module over a local ring (R,m). Then $\mathrm{depth}_I M = \dim M - \dim M/IM$ for all ideals I.*

Proof. We use induction on $d = \mathrm{depth}_I M$. If $d = 0$ then by ??? I is contained in some associated prime P of M. Then R/P is a submodule of M, so $\mathrm{depth}_m M \leq \dim R/P \leq \dim M$, where the first inequality is by ???. Since M is Cohen-Macaulay, these must all be equalities. Then $\dim M = \dim R/P$, so P is a minimal prime of $\operatorname{Ann} M$. Therefore P is also a minimal prime of $\operatorname{Ann} M + I$ and $\dim M/IM = \dim R/(\operatorname{Ann} M + I) = \dim R/P$.

Now suppose that $d > 0$. So there exists an $x \in I$ that is M-regular. Then $\mathrm{depth}_I M/xM = \mathrm{depth}_I M - 1$, $\dim M/xM = \dim M - 1$, and $\mathrm{depth}_m M/xM = \mathrm{depth}_m M - 1$. So M/xM is Cohen-Macaulay, hence by induction we get $\mathrm{depth}_I M/xM = \dim M/xM - \dim M/(I,x)M$. But $(I,x) = I$, so adding 1 to both sides this gives $\mathrm{depth}_I M = \dim M - \dim M/IM$. □

36.8. **Applications to multiplicities.** Suppose Z and W are closed subschemes of a regular scheme X. Then $\dim(Z \cap W) \geq \dim Z + \dim W - \dim X$. The number $\dim Z + \dim W - \dim X$ is referred to as the "expected dimension of intersection", so the inequalty says that the actual intersection can have larger dimension than expected, but not smaller. For example, two curves in 3-space have expected dimension of intersection equal to $1 + 1 - 3 = -1$ (so an empty intersection), but of course their actual intersection could have dimension -1, 0, or 1. The difference

$$\dim(Z \cap W) - (\dim Z + \dim W - \dim X)$$

is called the **excess intersection dimension**.

Proposition 36.9. *Let X be a regular Noetherian scheme and let $Z \hookrightarrow X$ and $W \hookrightarrow X$ be closed subschemes. Suppose Z is a local complete intersection and that W is Cohen-Macaulay. Then $\operatorname{Tor}_i(\mathcal{O}_Z, \mathcal{O}_W) = 0$ when i is larger than the excess intersection dimension, that is when $i > \dim(Z \cap W) - \dim Z - \dim W + \dim X$.*

Proof. The problem is local, so we reduce to the following problem in commutative algebra: R is a regular local ring of dimension n, I is generated by a regular sequence $(x_1, \ldots, x_{n-d})$, and M is a Cohen-Macaulay module of dimension e. We must show that $\operatorname{Tor}_i(R/I, M) = 0$ for $i > n - d - e + \dim(M/IM)$.

Since R/I is resolved by the Koszul complex, $\operatorname{Tor}_i(R/I, M) \cong H_i(\underline{x}; M)$. By the depth-sensitivity of Koszul homology, this vanishes when $i > (n-d) - \operatorname{depth}_I M$. So the desired result will follow if we can prove $\dim M \geq \dim M/IM + \operatorname{depth}_I M$.

Let $S = R/Ann M$. Then we have $\dim S \geq \dim S/IS + \operatorname{ht}(IS)$, as this holds for all ideals in Noetherian rings. But $\dim S = \dim M$, $\dim S/IS = \dim M/IM$, and by Proposition 36.6 the fact that M is Cohen-Macaulay yields that $\operatorname{depth}_I M = \operatorname{ht}(S/IS)$. □

Example 36.10. Consider two lines Z and W in $\mathbb{A}^3$ intersecting in a point. The above result shows that $\operatorname{Tor}_i(\mathcal{O}_Z, \mathcal{O}_W) = 0$ for $i > 1$.

For two curves in $\mathbb{A}^2$ intersecting in an isolated point, both curves are hypersurfaces and therefore complete intersections (and hence Cohen-Macaulay). So the result implies that $\operatorname{Tor}_i(\mathcal{O}_Z, \mathcal{O}_W) = 0$ for $i > 0$.

36.11. **Examples.**

Example 36.12 (The union of two planes in $\mathbb{A}^4$)**.** Consider $\mathbb{C}^4$ with coordinates x, y, z, w. Let $I = (x, y)(z, w) = (xz, xw, yz, yw)$ and $J = (x - z, y - w)$, and set $X = V(I)$ and $Y = V(J)$. Then X is the union of two planes intersecting at the origin, and Y is a third plane that cuts both planes transversely at the origin. Observe that X and Y are both two-dimensional, with a zero-dimensional intersection in $\mathbb{C}^4$. The scheme Y is a complete intersection, so if X were Cohen-Macaulay then Proposition 36.9 would imply that $\operatorname{Tor}_i(\mathcal{O}_X, \mathcal{O}_Y)$ vanishes for $i > 0$. We will calculate and see that this is not the case.

Before doing any calculations, though, let us observe that if P is the origin then we should have $i(X, Y; P) = 2$. This is clear from a topological perspective: if you move Y a little to Y' then the intersection with X becomes two points, namely the points where Y' meets the two planes that make up X.

For Tor_0 we have

$$\operatorname{Tor}_0(\mathcal{O}_X, \mathcal{O}_Y) = \mathbb{C}[x, y, z, w]/(x - z, y - w, xz, xw, yz, yw) = \mathbb{C}[x, y]/(x^2, xy, y^2)$$

and this has dimension 3 over $\mathbb{C}$. So we see already that there must be a nonzero higher Tor, as that is the only way we can get $i(X, Y; P) = 2$. In fact, $\mathcal{O}_Y$ can be resolved by the Koszul complex $K(x - z, y - w; R)$ and so we know automatically that $\mathrm{Tor}_i(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for $i > 2$. So we can predict that there will be a nonvanishing Tor_1.

A good way to do the calculation is to observe the exact sequence

$$0 \longrightarrow R/I \longrightarrow R/(x, y) \oplus R/(z, w) \longrightarrow \oplus R/(x, y, z, w) \longrightarrow 0$$

that exists because $I = (x, y) \cap (z, w)$. Using $K(x - z, y - w; R)$ it is easy to compute the Tor's with the three modules other than R/I, and the long exact sequence becomes

$$\begin{array}{lll} 2) & ? \longrightarrow 0 \oplus 0 \longrightarrow \mathbb{C} & \\ & \hookrightarrow & \\ 1) & ? \longrightarrow 0 \oplus 0 \longrightarrow \mathbb{C}^2 & \\ & \hookrightarrow & \\ 0) & \mathbb{C}^3 \longrightarrow \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C} & \end{array}$$

From this we find that $\mathrm{Tor}_1(\mathcal{O}_X, \mathcal{O}_Y) \cong \mathbb{C}$ and $\mathrm{Tor}_2(\mathcal{O}_X, \mathcal{O}_Y) = 0$, as expected.

It might be useful to exhibit more directly the failure of R/I to be Cohen-Macaulay. Let $S = (R/I)_{(x,y,z,w)}$, which is a two-dimensional local ring. A little work shows that $x - z$ is a nonzerodivisor in S. But $S/(x - z) = \mathbb{C}[x, y, w]_{(x,y,w)}/(x^2, xw, xy, yw)$ and so everything in the maximal ideal is killed by x. So $\mathrm{depth}_m S/(x - z) = 0$, hence $\mathrm{depth}_m S \leq 1$. Therefore S is not Cohen-Macaulay.

Exercise 36.13. In $\mathbb{A}^6$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$ let $I = (x_1, x_2, x_3)(y_1, y_2, y_3)$ and $J = (x_1 - y_1, x_2 - y_2, x_3 - y_3)$. Explain why the intersection multiplicity at the origin should be 2, and then compute that the $\mathrm{Tor}_*(R/I, R/J)$ modules are $\mathbb{C}^4$, $\mathbb{C}^3$, and $\mathbb{C}^1$.

Example 36.14 (A surface in $\mathbb{A}^4$ with an isolated singular point)**.** In $\mathbb{C}^4$ with coordinates u, v, w, c let $Z = V(u^3 - v^2, u^2c - vw, uw - vc, w^2 - uc^2)$ and $W = V(u, c)$. This example was considered back in Example 1.6. It is easy to see that $Z \cap W = \{0\}$.

A good way to think about Z is that it is a family of twisted cubics in $\mathbb{C}^3$ parameterized by c, expanded into $\mathbb{C}^4$ (by adding the c coordinate) so that they don't intersect, and then allowed to degenerate at $c = 0$ into a cusp. To understand this observe that we have a map

$$f\colon \mathbb{A}^2 \to Z, \qquad (s, t) \mapsto (s^2, s^3, st, t).$$

Away from the point $0 = (0, 0, 0, 0) \in Z$ this is a local isomorphism: the inverse sends (u, v, w, c) to $(\frac{v}{u}, c)$. So we find that Z is two-dimensional and smooth away from the origin. Figure 36.14 shows two of the curves making up Z, the first a typical nonsingular example and the second the singular cuspal curve in the $c = 0$ slice.

The map f gives us the identification $\mathcal{O}_Z = \mathbb{C}[s^2, s^3, st, t] \subseteq \mathbb{C}[s, t]$.

Let $R = \mathbb{C}[u, v, w, c]$ and let I and J denote the defining ideals of Z and W. Then

$$\mathrm{Tor}_0(R/I, R/J) = \mathbb{C}[v, w]/(v^2, vw, w^2)$$

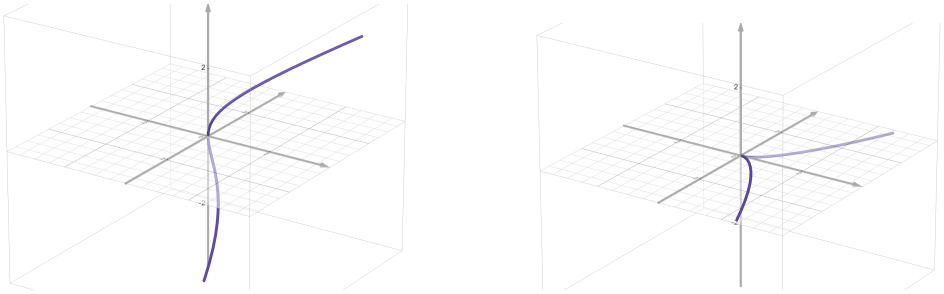


FIGURE 36.14.1. Slices of Z

which has dimension 3 over $\mathbb{C}$. To compute the higher Tor's used the Koszul complex to resolve R/J, then tensor with $S = R/I$ to get

$$0 \longrightarrow S \xrightarrow{\left[\begin{smallmatrix} -c \\ u \end{smallmatrix}\right]} S^2 \xrightarrow{[u\ c]} S \longrightarrow 0.$$

Since S is a domain and $u \neq 0$ in S we have the quasi-isomorphism $K(u, c; S) \simeq K(c; S/u)$. So we get $\operatorname{Tor}_2(R/I, R/J) = 0$ and $\operatorname{Tor}_1(R/I, R/J) = \operatorname{Ann}_{S/u} c$. But $S/u = \mathbb{C}[v, w, c]/(v^2, vw, vc, w^2)$ and therefore $\operatorname{Ann}_{S/u} c = \mathbb{C}\langle v \rangle$. Therefore we compute the intersection multiplicity as

$$i(R/I, R/J) = 3 - 1 = 2.$$

Observe that Z and W are both two-dimensional, with a zero-dimensional intersection in $\mathbb{C}^4$. The scheme W is a complete intersection, so if Z were Cohen-Macaulay then Proposition 36.9 would imply that $\operatorname{Tor}_i(\mathcal{O}_Z, \mathcal{O}_W) = 0$ for $i > 0$. The nonvanishing of our Tor_1 therefore shows that Z is not Cohen-Macaulay. Indeed, there are easier ways to see this. The element c is a nonzerodivisor in $\mathcal{O}_Z = S$, but $S/c = \mathbb{C}[u, v, w]/(u^3 - v^2, vw, uw, w^2)$. When we localize at the origin then S/c becomes a one-dimensional local ring, but since w is killed by all of u, v, w the depth of the maximal ideal is zero. So S/c is not Cohen-Macaulay, which implies that S could not be Cohen-Macaulay either.

Part 7. Characteristic classes and the Riemann-Roch theorem

37. The Euler class

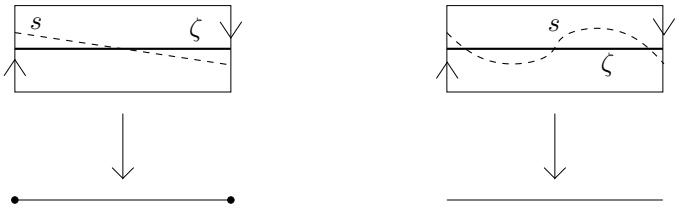
There is a general principle in algebraic topology that all rational cohomology theories detect the same information. The information is not necessarily detected in the same *way*, however, and this makes it hard to formulate the principle precisely. But here is a nice example of it. If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold then we have seen that knowing $[Z] \in K^0(\mathbb{C}P^n)$ is the same as knowing the integers d_i for which $[Z] = \sum d_i[\mathbb{C}P^{n-i}]$. Since $K^0(\mathbb{C}P^n)$ is free abelian, there is no loss of information in regarding this equation as taking place in $K^0(\mathbb{C}P^n) \otimes \mathbb{Q}$. By the above-mentioned principle, the numbers d_i should be able to be detected in rational singular cohomology. The Grothendieck-Riemann-Roch Theorem tells us how to do this, and that will be our next main goal.

To understand Riemann-Roch we need to first understand characteristic classes. We will give a very brief treatment, spread over the next two sections. For a more in-depth exploration some standard references are [MS] and [GH].

The present section deals with the Euler class, which is in some sense the most "primary" of characteristic classes. We discuss two versions: Euler classes in singular cohomology and Euler classes in K-theory.

37.1. **The Euler class for a vector bundle.** We will start with a geometric treatment that is lacking in rigor but shows the basic ideas, and then we will give a more rigorous treatment. Don't worry about verifying all the details in the following, just get a general sense of the geometry.

Start with a real vector bundle $E \to B$ of rank k, where B is a smooth manifold of dimension n. Let ζ denote the zero-section. We will try to construct something like an intersection product $\zeta \cdot \zeta$. To do this, we let s be a section of E that is a "small-perturbation" of ζ, chosen so that s and ζ intersect as little as possible. A good example to keep in mind is the Möbius bundle, shown below with two deformations of the zero section:



The zero locus $s^{-1}(0) \subseteq \mathrm{im}(\zeta)$ may, under good conditions, be given the structure of a cycle—part of this involves assigning multiplicities to the components in a certain way. If we want multiplicities in $\mathbb{Z}$ then we will need to assume the bundle is oriented, whereas if we are willing to use multiplicities only in $\mathbb{Z}/2$ then we can use any bundle. The dimension of this cycle is $\dim \zeta + \dim \zeta - \dim E$, which is $n + n - (n + k) = n - k$. This cycle clearly depends on the choice of s, but a different choice of s gives a homologous cycle. So the associated class in homology is independent of our choices, and is an invariant of E. We call it the **homology Euler class** of E: for orientable bundles we have

$$e_H(E) = \zeta \cdot \zeta = s^{-1}(0) \in H_{n-k}(B),$$

whereas for arbitrary bundles we have $e_H(E) \in H_{n-k}(B; \mathbb{Z}/2)$. The sections s used here are usually referred to as "generic sections" of E.

The following are easy properties of the Euler class construction:

(1) If E has a nonvanishing section, then $e_H(E) = 0$.
(2) $e_H(E \oplus F) = e_H(E) \cdot e_H(F)$ (where $\cdot$ is the intersection product).
(3) If L_1 and L_2 are line bundles on B then $e_H(L_1 \otimes L_2) = e_H(L_1) + e_H(L_2)$.

For (1) we simply note that if σ is a nonvanishing section then the deformation $t \mapsto t\sigma$ (for $t \in [0,1]$) allows us to regard σ as a deformation of the zero-section. Taking $s = \epsilon\sigma$ for small ϵ, the vanishing locus of s is the same as the vanishing locus of σ—which is the empytset. So $e_H(E) = 0$.

For (2), if s is a generic section of E and s' is a generic section of F then $s \oplus s'$ is a generic section of $E \oplus F$. The vanishing locus of $s \oplus s'$ is the intersection of the vanishing loci of s and s'.

For (3), if s_1 and s_2 are generic sections of L_1 and L_2 then $s_1 \otimes s_2$ is a generic section of $L_1 \otimes L_2$. But $s_1 \otimes s_2$ vanishes at points in B where *either* s_1 or s_2 vanish. So the vanishing locus of $s_1 \otimes s_2$ is the union of the vanishing loci of s_1 and s_2.

Example 37.2.

(a) For the Möbius bundle $M \to S^1$ we have $e_H(M) = [*] \in H_0(S^1; \mathbb{Z}/2)$, as is clearly depicted in the pictures above. Note the necessity of $\mathbb{Z}/2$-coefficients here.
(b) If B is an orientable, smooth manifold of dimension d, then $e_H(TB) \in H_0(B)$. So $e_H(TB)$ is a multiple of $[*]$, and this multiple is precisely the Euler characteristic $\chi(B)$: a section of TB is just a vector field on B, and so this is the classical statement that a generic vector field on B vanishes at precisely $\chi(B)$ points. This connection with the Euler characteristic is why e_H is called the Euler class.
(c) Let $L \to \mathbb{R}P^n$ be the tautological line bundle, and recall that $L \cong L^*$ (Corollary 11.15). We examine L^* instead, since it is easier to write down formulas for sections. Generically choose a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and consider the section s_α whose value over $x = [x_0 : \cdots : x_n]$ is the functional sending $(x_0, \ldots, x_n)$ to $\alpha_0 x_0 + \cdots + \alpha_n x_n$. The vanishing locus for this section is a linear subspace of $\mathbb{R}P^n$, and of course we know that all such things are homotopic. So $e_H(L^*) = [\mathbb{R}P^{n-1}]$.
(d) A similar analysis allows one to calculate $e_H(L^*)$ where $L \to \mathbb{C}P^n$ is the tautological line bundle, giving $e_H(L^*) = [\mathbb{C}P^{n-1}]$ (a little thought is required concerning signs, though).
(e) If $Z \hookrightarrow \mathbb{C}P^n$ is a complex submanifold and N is the normal bundle then clearly we have $e_H(N) = [Z] \cdot [Z]$, with the intersection product regarded as a homology class in $H_{n-2}(Z)$.

37.3. **Cohomology version.** As is often the case, working in cohomology instead of homology allows us to make things completely rigorous while avoiding some of the thorny geometric issues that arose in our discussion above.

Let $E \to B$ be an orientable real bundle of rank k; that is, a bundle with a Thom class $\mathfrak{U}_E \in H^k(E, E - 0)$. Note that B need no longer be a manifold. Let $\zeta\colon B \to E$ denote the zero section, as usual. We may interpret ζ as a map of pairs $(B, \emptyset) \to (E, E - 0)$, so that pulling back along ζ gives a cohomology class

$\zeta^*(\mathcal{U}_E) \in H^*(B)$. Define this to be the **(cohomology) Euler class** of E:

$$e^H(E) = \zeta^*(\mathcal{U}_E) \in H^k(B).$$

As $\mathcal{U}_E$ plays the role of a (relative) fundamental class for the 0-section ζ, the pullback $\zeta^*(\mathcal{U}_E)$ captures the spirit of intersecting the 0-section with itself.

The main properties of the Euler class are as follows:

Proposition 37.4. *Let $E \to B$ be an oriented real bundle. Then*

(a) If E has a nonzero section then $e^H(E) = 0$.
(b) $e^H(E \oplus F) = e^H(E) \cup e^H(F)$ for any oriented bundle $F \to B$.
(c) For any map $f\colon Y \to B$ one has $e^H(f^*E) = f^*(e^H(E))$ (naturality under pullbacks).

Analogs of the above properties are all true for non-orientable bundles as long as one uses $\mathbb{Z}/2$-coefficients everywhere.

Proof. Properties (b) and (c) follow from the corresponding properties of Thom classes. To see (a), let s be a nonzero section. Consider the homotopy $H\colon I \times B \to E$ given by $H(t, b) = t \cdot s(b)$. This can be regarded as a homotopy between maps of pairs $(B, \emptyset) \to (E, E - 0)$. It follows that $e^H(E) = \zeta^*(\mathcal{U}_E) = s^*(\mathcal{U}_E)$. But s factors through $E - 0$, and so $s^*(\mathcal{U}_E) = 0$. □

The reader might notice that the above result does not include a formula for $e^H(L_1 \otimes L_2)$, as we had seen for the homology version. Don't worry, it's coming! See Proposition 37.9.

Example 37.5.

(1) For the Möbius bundle $M \to S^1$ we have $e^H(M) = [*] \in H^1(S^1; \mathbb{Z}/2)$. To prove this note that the Thom space of M is $D(M)/S(M) \cong \mathbb{R}P^2$, and the zero section $\zeta\colon S^1 \to \mathrm{Th}(M)$ is just a typical embedding of $\mathbb{R}P^1$ into $\mathbb{R}P^2$. The Thom class $\mathcal{U}_M \in H^1(\mathrm{Th}\, M; \mathbb{Z}/2) = H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ is the unique nonzero class, and we know restricting along $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$ sends this class to the unique nonzero class in $H^1(\mathbb{R}P^1; \mathbb{Z}/2)$.

(2) Let $L \to \mathbb{R}P^n$ be the tautological bundle. If $j\colon \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ is the inclusion, then $j^*L \cong M$. So naturality gives $j^*e^H(L) = e^H(M)$. But $j^*\colon H^1(\mathbb{R}P^n; \mathbb{Z}/2) \to H^1(\mathbb{R}P^1; \mathbb{Z}/2)$ is an isomorphism, and so it follows from (1) that $e^H(L)$ must be the unique nonzero class in $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$.

(3) Let M be a smooth, oriented manifold of dimension n. Then the tangent bundle $TM \to M$ is a rank n oriented bundle. In this case, $e^H(TM) \in H^n(M; \mathbb{Z}) = \mathbb{Z}\langle[*]\rangle$ and so the problem is to determine the integer d for which $e^H(TM) = d[*]$. It is a fact from differential topology that there is a vector field $s\colon M \to TM$ with a finite number of nondegenerate zeros, and that when counted with appropriate signs this number is $\chi(M)$: this is the Poincaré-Hopf theorem. See the discussion in [Milnor5, Section 6]. Let $A = s^{-1}(0) = \{p_1, \ldots, p_r\} \subseteq M$. The deformation $t \mapsto ts$ shows that s is homotopic to the zero section ζ, so that we get the following diagram

$$H^n(M) \xleftarrow{j^*} H^n(M, M - \{p_1, \ldots, p_r\}) \xleftarrow{s^*} H^n(TM, TM - 0)$$

where the composite map is $\zeta^* = s^*$. As is typical in these arguments, we next use that $H^n(M, M - \{p_1, \ldots, p_r\}) \cong \oplus_i H^n(M, M - p_i)$ and that the orientation of M gives canonical generators $[*] \in H^n(M)$ and $[p_i] \in H^n(M, M - p_i)$. The

map j^* sends each $[p_i]$ to $[*]$, and so it is really just a fold map $\mathbb{Z}^r \to \mathbb{Z}$. It remains to see how the Thom class $\mathcal{U}_{TM}$ maps to the canonical generators in $H^n(M, M - p_i)$ under s^*, but this is a local problem—by working through the definitions one sees that $\mathcal{U}_{TM} \mapsto d_i[p_i]$ where d_i is the local index of the vector field at p_i. We finally obtain that $e^H(TM) = (d_1 + \cdots + d_r)[*]$, where $d_1 + \cdots + d_r$ is the sum of the local indices and therefore equal to $\chi(M)$.

The following property of the Euler class is also useful:

Proposition 37.6. *Let M be an oriented manifold and let $j\colon Z \hookrightarrow M$ be an oriented submanifold. Let $N_{M/Z}$ be the normal bundle. Then $e(N_{M/Z}) = j^*([Z])$.*

Proof. Intuitively the result should make sense, since both $e(N_{M/Z})$ and $j^*([Z])$ are modelled by the intersection product of Z with itself inside of M. To give a rigorous proof, let U be a tubular neighborhood of Z in M and $U \cong N$ the associated homeomorphism. The zero section $\zeta\colon Z \hookrightarrow N$ corresponds with the inclusion $j\colon Z \hookrightarrow U$ under this identification.

Let c be the codimension of Z in M. Consider the commutative diagram

$$\begin{array}{ccccccc} H^0(Z) & \xrightarrow{th_N} & H^c(N, N-0) & \xrightarrow{\cong} & H^c(U, U-0) & \xleftarrow{\cong} & H^c(M, M-Z) \longrightarrow H^c(M) \\ & \searrow^{\zeta^*} & \downarrow & & \downarrow & & \downarrow \\ & & H^c(Z) & = & H^c(Z) & = & H^c(Z). \end{array}$$

The image of 1 across the top row is $[Z]$, with $\mathcal{U}_N$ being an intermediate value in the composite. The image of $\mathcal{U}_N$ under ζ^* is $e(N)$, and so the diagram immediately yields $e(N) = j^*[Z]$. □

Example 37.7. Consider the usual embedding $j\colon \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$. We saw in the proof of Proposition 33.6 that the normal bundle is $L^* \to \mathbb{C}P^n$, the dual of the tautological line bundle. By Proposition 37.6 we find that $e(L^*) = j^*([\mathbb{C}P^n])$, and we know the latter is $[\mathbb{C}P^{n-1}]$ by intersection theory. We have shown that $e^H(L^* \to \mathbb{C}P^n) = [\mathbb{C}P^{n-1}]$.

It will be useful to also know $e^H(L \to \mathbb{C}P^n)$. For this observe that there is a commutative pullback square

$$\begin{array}{ccc} L & \longrightarrow & L^* \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \xrightarrow{f} & \mathbb{C}P^n \end{array}$$

where f is induced by complex conjugation. The top map on fibers has the form $\langle v \rangle \to \langle \overline{v} \rangle^*$ and is given by sending w to the functional $z \mapsto \sum z_i w_i$. This is readily checked to be an isomorphism of complex vector spaces.

It follows that $e^H(L) = f^*(e^H(L^*))$. But f^* is multiplication by -1 on $H^2(\mathbb{C}P^n)$ (it suffices to just look at $\mathbb{C}P^1 = S^2$, where f is reflection in the equatorial hyperplane). So $e^H(L) = -[\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n)$. We record this for future use:

Proposition 37.8. $e^H(L \to \mathbb{C}P^n) = -[\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n)$.

We next wish to give a formula for the Euler class of a tensor product of line bundles. At the moment this might seem to be of limited interest, but it turns out to be very significant. We need to be careful about what context we are working

in, however. All orientable real line bundles are trivial (one can write down an evident nonvanishing section), and so using the integral Euler class in this context doesn't lead to anything interesting. So we should work with mod 2 Euler classes and arbitrary real line bundles. Alternatively, if we use *complex* line bundles then they are automatically oriented and then we can indeed use the integral Euler class (of the underlying real plane bundle). So we really get two parallel results, one for the real and one for the complex case:

Proposition 37.9. *If L_1 and L_2 are real line bundles on B then the mod 2 Euler class satisfies*

$$e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2).$$

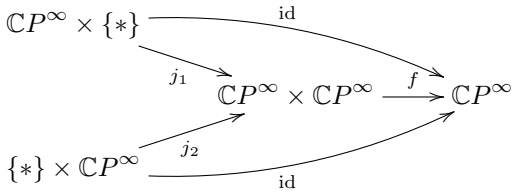
Likewise, if L_1 and L_2 are complex line bundles on B then the integral Euler class satisfies the same formula.

Proof. The proofs of the two parts are basically identical; we will do the complex case. Let $L \to \mathbb{C}P^\infty$ denote the tautological line bundle, and consider the bundle $\pi_1^*(L) \otimes \pi_2^*(L) \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, where $\pi_1, \pi_2 \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ are the two projections. Since L is the universal example of a line bundle, $\pi_1^*(L) \otimes \pi_2^*(L)$ is the universal example of a tensor product of line bundles. Write $E = \pi_1^*(L) \otimes \pi_2^*(L)$, for short. There is a classifying map $f \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ for E, giving a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & L \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{f} & \mathbb{C}P^\infty. \end{array}$$

By naturality $e^H(E) = f^*(e^H(L))$. Set $x = e^H(L) \in H^2(\mathbb{C}P^\infty)$, which we know is a generator for this group by Proposition 37.8.

If $*$ is a chosen basepoint in $\mathbb{C}P^\infty$ then observe that the diagram



commutes up to homotopy. This is because fj_1 classifies $j_1^*(E)$, and this bundle is clearly isomorphic to L; similarly for fj_2.

Note that $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is the free abelian group generated by $x \otimes 1$ and $1 \otimes x$. So $f^*(x) = k(x \otimes 1) + l(1 \otimes x)$ for some integers k and l. The above homotopy commutative diagram forces $k = l = 1$. So we have proven that

$$e^H(E) = e^H(L) \otimes 1 + 1 \otimes e^H(L).$$

Now let L_1 and L_2 be two complex line bundles on a space B. There are maps $g_1, g_2 \colon B \to \mathbb{C}P^\infty$ such that $L_1 = g_1^*(L)$ and $L_2 = g_2^*(L)$. Then $L_1 \otimes L_2 = \gamma^*(E)$, where γ is the composite

$$B \xrightarrow{\Delta} B \times B \xrightarrow{g_1 \times g_2} \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

We obtain

$$e^H(L_1 \otimes L_2) = \gamma^*(e^H(E)) = \gamma^*\big(e^H(L) \otimes 1 + 1 \otimes e^H(L)\big)$$
$$= \Delta^*\big(e^H(L_1) \otimes 1 + 1 \otimes e^H(L_2)\big) = e^H(L_1) + e^H(L_2).$$

□

Remark 37.10. One should note that the above proof is not geometric—this turns out to be important. Rather, the proof in some sense proceeds by showing that there is really not much choice for what $e^H(L_1 \otimes L_2)$ could be, given how small $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is; there is in fact only *one* possibility. We will shortly see that replacing H by other cohomology theories—ones with "more room", so to speak—allows for more to happen here. See Proposition 37.17 below.

Corollary 37.11. *Let $L \to \mathbb{C}P^n$ be the tautological line bundle. Then $e^H(L^*) = [\mathbb{C}P^{n-1}]$ and so*

$$e^H((L^*)^{\otimes k}) = k[\mathbb{C}P^{n-1}] \quad \textit{and} \quad e^H(L^{\otimes k}) = -k[\mathbb{C}P^{n-1}].$$

Proof. The first statement was proven in Example 37.7. All of the other statements follow directly from the first via Proposition 37.9. For $e^H(L)$ use that $L \otimes L^* \cong \underline{1}$ and so $0 = e^H(\underline{1}) = e^H(L) + e^H(L^*)$. □

A nice consequence of all of the above work is that for complex line bundles the Euler class gives a complete invariant:

Corollary 37.12. *Let L_1, L_2 be two complex line bundles over a space X. If $e(L_1) = e(L_2)$ then $L_1 \cong L_2$.*

Proof. Let $f_1, f_2 \colon X \to \mathbb{C}P^\infty$ be classifying maps for the two line bundles: so $f_1^*L \cong L_1$ and $f_2^*L \cong L_2$. Then $e(L_1) = f_1^*e(L)$ and $e(L_2) = f_2^*(e(L))$, by naturality of the Euler class. Our assumption is therefore equivalent to $f_1^*(e(L)) = f_2^*(e(L))$.

But $\mathbb{C}P^\infty$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, and so $[X, \mathbb{C}P^\infty]$ is naturally isomorphic to $H^2(X)$ via the map $f \mapsto f^*(z)$ where z is a chosen generator of $H^2(\mathbb{C}P^\infty)$. Corollary 37.11 gives that $e(L)$ is such a generator, so the fact that $f_1^*(e(L)) = f_2^*(e(L))$ implies that f_1 is homotopic to f_2. But this implies that L_1 is isomorphic to L_2. □

The following easy corollary will be needed often:

Corollary 37.13. *Let $j \colon Z \hookrightarrow \mathbb{C}P^n$ be a degree d hypersurface. Then the normal bundle is isomorphic to $j^*\big((L^*)^{\otimes d}\big)$.*

Proof. By Proposition 37.6 the Euler class of the normal bundle is $e(N) = j^*([Z])$. But we know $[Z] = d[\mathbb{C}P^{n-1}] = e((L^*)^{\otimes d})$, and so $j^*[Z]$ is the Euler class of $j^*((L^*)^{\otimes d})$. Now use Corollary 37.12. □

Remark 37.14. There are of course more direct proofs of the above result, though perhaps requiring more machinery overall. A hypersurface is the vanishing set of a single polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, and then the normal bundle in algebraic geometry is the dual of $(f)/(f^2) \cong R(-d)$. So the normal bundle corresponds to $R(d)$, and is therefore $\mathcal{O}(d) = (L^*)^{\otimes d}$.

37.15. **Euler classes in K-theory.** Our construction of the Euler class $e^H(E)$ can be repeated verbatim in any complex-oriented cohomology theory. In particular, it can be repeated in K-theory. We explore this next.

Let $E \xrightarrow{\pi} X$ be a $\mathbb{C}$-bundle of rank k. We have a Thom class $\mathcal{U}_E \in K^0(E, E-0)$, and so we can mimic the above construction and define

$$e^K(E) = \zeta^*(\mathcal{U}_E) \in K^0(X).$$

This is the **K-theory Euler class** of E. [Note that it is slightly more appropriate to regard $\mathcal{U}_E$ as lying in $K^{2k}(E, E-0)$ and then $e^K(E) \in K^{2k}(X)$, but as usual we take advantage of Bott periodicity to work only in K^0.]

Let us unravel the above definition a bit. First, recall from (27.11) that $\mathcal{U}_E$ is the complex

$$[\ 0 \longrightarrow \textstyle\bigwedge^0(\pi^*E) \xrightarrow{\Delta\wedge -} \bigwedge^1(\pi^*E) \xrightarrow{\Delta\wedge -} \dots \xrightarrow{\Delta\wedge -} \bigwedge^k(\pi^*E) \longrightarrow 0\ ]^*$$

where Δ is the usual diagonal section as shown in the following diagram:

$$\begin{array}{ccccccc} E & = & \zeta^*(\pi^*E) & \longrightarrow & \pi^*E & \longrightarrow & E \\ & & \downarrow & \overset{\Delta}{\nearrow} & \downarrow & & \downarrow \\ & & X & \xrightarrow[\zeta]{} & E & \xrightarrow[\pi]{} & X. \end{array}$$

Note that $\zeta^*(\pi^*(E)) = E$ because $\pi \circ \zeta = \mathrm{id}$. Next let us look at $\zeta^*(\mathcal{U}_E)$. For each j one has $\zeta^*(\bigwedge^j(\pi^*E)) = \bigwedge^j(E)$, and the restriction of Δ to the image of ζ is just the zero section! So the maps in the complex $\zeta^*(\mathcal{U}_E)$ are all zero. In other words, $\zeta^*(\mathcal{U}_E)$ is the Koszul complex for the zero section on E. Therefore

$$\begin{aligned} \zeta^*(\mathcal{U}_E) &= J^*_{E,0} \\ &= [\ 0 \longrightarrow \textstyle\bigwedge^0 E \xrightarrow{0} \bigwedge^1 E \xrightarrow{0} \dots \longrightarrow \bigwedge^k E \longrightarrow 0\ ]^* \\ &= \sum (-1)^i [\textstyle\bigwedge^i (E^*)]. \end{aligned}$$

We see immediately that if E has a nonzero section then $e^K(E) = 0$. Indeed, if s is the nonzero section then we can deform ζ to s, and likewise deform $J^*_{E,\zeta}$ to $J^*_{E,s}$. But this latter complex is exact, and so represents zero in $K^0(X)$.

The analogs of properties (b) and (c) from Proposition 37.4 also hold, as these are simple consequences of corresponding properties of Thom classes. Observe that (b) follows from the algebraic isomorphism $\Lambda^*(E \oplus F) \cong \Lambda^*(E) \otimes \Lambda^*(F)$.

What about the analog of Proposition 37.9? If $L \to X$ is a complex line bundle then we have

$$e^K(L) = 1 - L^*. \tag{37.16}$$

So $e^K(L_1 \otimes L_2) = 1 - L_1^* L_2^*$, which is visibly *not* the same as $e^K(L_1) + e^K(L_2)$. Indeed, one can check the following more complicated formula:

Proposition 37.17. *Let L_1 and L_2 be complex line bundles on a space X. Then*

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2).$$

Proof. We simply observe that $1 - L_1^* L_2^* = (1 - L_1^*) + (1 - L_2^*) - (1 - L_1^*)(1 - L_2^*)$. □

Remark 37.18. The difference between how e^H and e^K behave on tensor products of line bundles turns out to have much more significance than one might expect. In some sense it ends up accounting for all of the differences between H and K, at least in terms of how they encode geometry. See Section 45 for more discussion.

We end this section with some detailed computations of K-theoretic Euler classes:

Example 37.19. Let T be the complex tangent bundle to $\mathbb{C}P^n$. Our goal will be to compute $e^K(T)$ from first principles. Let L denote the tautological line bundle over $\mathbb{C}P^n$. Note that L sits inside of the trivial bundle $\underline{n+1}$ in the evident way (if l is a line in $\mathbb{C}^{n+1}$, then points on l are defacto points in $\mathbb{C}^{n+1}$). Let $L^\perp$ be the orthogonal complement to L relative to the usual Hermitian metric on $\mathbb{C}^{n+1}$. So we have a short exact sequence of bundles

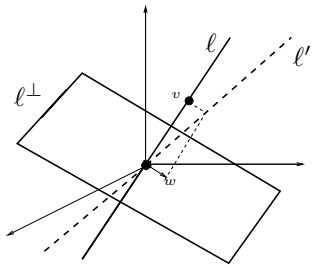
$$0 \to L \hookrightarrow \underline{n+1} \to L^\perp \to 0. \tag{37.20}$$

Since $\mathbb{C}P^n$ is paracompact this sequence is split, and hence $L \oplus L^\perp \cong \underline{n+1}$.

The basis of our computation is the following geometric fact:

$$T \cong \underline{\mathrm{Hom}}(L, L^\perp) \tag{37.21}$$

where $\underline{\mathrm{Hom}}(L, L^\perp)$ is the bundle over $\mathbb{C}P^n$ whose fiber over a point x is the vector space of linear maps $L_x \to L_x^\perp$. To understand this isomorphism, if ℓ is a point in $\mathbb{C}P^n$ then think of the tangent space T_ℓ as giving "local directions" for moving to all nearby points around ℓ. The following picture shows the line ℓ in $\mathbb{C}^{n+1}$ together with its orthogonal complement $\ell^\perp$ and a "nearby line" ℓ':



Note that ℓ' determines a linear map $\ell \to \ell^\perp$ as shown in the picture: a vector $v \in \ell$ is sent to the unique vector $w \in \ell^\perp$ such that $v + w \in \ell'$. This makes sense as long as ℓ' is not orthogonal to ℓ, which will be fine for all nearby lines. We clearly get a bijection between $\mathrm{Hom}(\ell, \ell^\perp)$ and a certain neighborhood of ℓ in $\mathbb{C}P^n$, and it is not hard to extrapolate from this to the isomorphism (37.21). See also Exercise 10.26(b).

Now take the short exact sequence of (37.20) and apply $\underline{\mathrm{Hom}}(L, -)$ to get the short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(L, L) \longrightarrow \underline{\mathrm{Hom}}(L, \underline{n+1}) \longrightarrow \underline{\mathrm{Hom}}(L, L^\perp) \longrightarrow 0.$$

For any line bundle $\mathcal{L}$ one has the identity $\underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{L}) = \underline{1}$: for a one-dimensional vector space V the map $\mathbb{C} \to \mathrm{Hom}(V, V)$ mapping 1 to the identity is a canonical isomorphism, so we can do this fiberwise. Using this, together with the identification $T \cong \underline{\mathrm{Hom}}(L, L^\perp)$, the above short exact sequence can be written as

$$0 \to \underline{1} \to (n+1)L^* \to T \to 0. \tag{37.22}$$

Since there must be a splitting, $\underline{1} \oplus T \cong (n+1)L^*$. Dualizing, we obtain $\underline{1} \oplus T^* \cong (n+1)L$

Recall that $e^K(T) = \sum_i (-1)^i [\bigwedge^i(T^*)]$. We will compute $\bigwedge^i(1 \oplus T^*)$ and then extract formulas for $[\bigwedge^i(T^*)]$.

Of course $\bigwedge^0 \underline{1} = \bigwedge^1 \underline{1} = \underline{1}$ and $\bigwedge^j \underline{1} = 0$ for $j \geq 2$. This allows us to calculate

$$\begin{aligned}\bigwedge^j(\underline{1} \oplus T^*) &= (\bigwedge^0 \underline{1} \otimes \bigwedge^j T^*) \oplus (\bigwedge^1 \underline{1} \otimes \bigwedge^{j-1} T^*) \\ &= \bigwedge^j T^* \oplus \bigwedge^{j-1} T^*.\end{aligned}$$

On the other hand, $\bigwedge^j(\underline{1} \oplus T^*) = \bigwedge^j((n+1)L) = \binom{n+1}{j}(L^{\otimes j})$. So for every j we have

$$[\textstyle\bigwedge^j(T^*)] = \binom{n+1}{j} L^{\otimes j} - [\bigwedge^{j-1}(T^*)]$$

in K-theory.

The evident recursion now gives that

$$\begin{aligned}[\textstyle\bigwedge^0 T^*] &= 1 \\ [\textstyle\bigwedge^1 T^*] &= (n+1)[L] - [\underline{1}] \\ [\textstyle\bigwedge^2 T^*] &= \binom{n+1}{2}[L^{\otimes 2}] - (n+1)[L] + [\underline{1}] \\ [\textstyle\bigwedge^3 T^*] &= \binom{n+1}{3}[L^{\otimes 3}] - \binom{n+1}{2}[L^{\otimes 2}] + (n+1)[L] - [\underline{1}],\end{aligned}$$

and so on. The general formula, obtained by an easy induction, is

$$[\textstyle\bigwedge^j T^*] = \sum_{k=0}^{j} (-1)^{k+j} \binom{n+1}{k} [L^{\otimes k}].$$

Taking the alternating sum of these expressions (from $j = 0$ to $j = n$) now yields

$$\begin{aligned}e^K(T) &= (n+1)[1] - n\tbinom{n+1}{1}[L] + (n-1)\tbinom{n+1}{2}[L]^2 - \cdots + (-1)^n 1 \cdot \tbinom{n+1}{n}[L]^n \\ &= (n+1)\Big([1] - n[L] + \tbinom{n}{2}[L]^2 - \tbinom{n}{3}[L]^3 + \cdots + (-1)^n [L]^n\Big) \\ &= (n+1)(1-[L])^n.\end{aligned}$$

Recall that $(1-[L])^n = [*]$ (Proposition 33.6), and so we have determined that

$$e^K(T) = (n+1)[*].$$

Let us now confess that the result of this computation is not unexpected. Indeed, we saw previously that $e^H(TM) = \chi(M)[*]$ for any smooth manifold M, and in fact the same is true in any complex-oriented cohomology theory (by essentially the same proof). The "$n+1$" in our formula for $e^K(T)$ is just $\chi(\mathbb{C}P^n)$. But note that we computed this without writing down anything remotely resembling a cell structure! In fact, the only geometry in the calculation was in the fact $T \cong \underline{\mathrm{Hom}}(L, L^\perp)$; everything else was some simple linear algebra and then basic algebraic manipulation. It is useful to remember the overall theme of K-theory: do linear algebra fiberwise over a base space X, and see what this tells you about the topology of X. Our computation of $e^K(T_{\mathbb{C}P^n})$ gives an example of this.

The calculation in the above example is a little clunky. One way to streamline it is to introduce the formal power series

$$\lambda_t(E) = \sum_{i=0}^{\infty} t^i [\textstyle\bigwedge^i(E)] = 1 + t[E] + t^2[\bigwedge^2(E)] + \cdots$$

which we regard as living in the ring $K(X)[[t]]$, where $E \to X$ was our vector bundle. Notice that this is actually a polynomial in t, since the exterior powers vanish beyond the rank of E; we consider it as a power series because in that context it has a multiplicative inverse, which we will shortly need.

If L is a line bundle then $\lambda_t(L) = 1 + t[L]$. Also, the formula $\bigwedge^k(E \oplus F) = \bigoplus_{i+j=k} \bigwedge^i(E) \otimes \bigwedge^j(F)$ yields the nice relation

$$\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F).$$

This is what ultimately simplifies our calculations. Finally, notice that the K-theoretic Euler characteristic can be written as $e^K(E) = \lambda_t(E^*)|_{t=-1}$.

Returning to the above calculation, the starting point for the algebra was the bundle isomorphism $\underline{1} \oplus T^* \cong (n+1)L$. Applying λ_t we obtain

$$\lambda_t(\underline{1})\lambda_t(T^*) = \lambda_t(\underline{1} \oplus T^*) = \lambda_t((n+1)L) = (\lambda_t(L))^{n+1}.$$

But $\lambda_t(\underline{1}) = 1 + t$ and $\lambda_t(L) = 1 + t[L]$, so we can write

$$\lambda_t(T^*) = \tfrac{(1+t[L])^{n+1}}{1+t}, \qquad \text{or} \qquad e^K(T) = \tfrac{(1+t[L])^{n+1}}{1+t}\Big|_{t=-1}$$

(and here is where we are using that our power series have multiplicative inverses). Our task is to expand the formula for $e^K(T)$ into powers of t, and then to set $t = -1$. The trick is to do this in a clever way.

In the above formula for $e^K(T)$, both the numerator and denominator vanish at $t = -1$ since $(1-[L])^{n+1} = 0$ in $K^0(\mathbb{C}P^n)$. So let us use L'Hopital's rule to deduce that

$$e^K(T) = (n+1)(1-[L])^n \cdot [L] = (n+1)(1-[L])^n \cdot (1-(1-[L])) = (n+1)(1-[L])^n$$

where in the last equality we have again used that $(1-[L])^{n+1} = 0$. So we have again found that $e^K(T) = (n+1)(1-L)^n = (n+1)[*]$.

Exercise 37.23. Justify the above use of L'Hopital's rule as follows. If R is any commutative ring and $g(t)$ is a polynomial over R such that $g(-1) = 0$, prove that $g(t) = (1+t)u(t)$ for some $u(t) \in R[t]$. Use this to prove that $\frac{g(t)}{1+t}\big|_{t=-1} = g'(-1)$.

Exercise 37.24. In the above analysis, we were lucky to stumble upon L'Hopital's rule as a useful technique. But one is not always so lucky, and it is good to have an idea how to proceed in other cases. As a fallback, we can always write $e^K(T)$ in terms of powers of $(1-L)$, as they give our usual basis for $K^*(\mathbb{C}P^n)$. To this end, regard L as a formal variable and consider $f(L) = (1+tL)^{n+1}/(1+t)$ as a formal power series in two variables (but where we are choosing not to write t in the inputs of f). We can expand in powers of $(L-1)$ via the usual Taylor series:

$$f(L) = f(1) + f'(1)(L-1) + \tfrac{f''(1)}{2}(L-1)^2 + \cdots$$

Compute the derivatives and then see what happens when you plug in $t = -1$.

Example 37.25 (Euler characteristic of a hypersurface)**.** Using the λ_t operators introduced above, we will attempt a harder computation of an Euler class. Let $j\colon Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Our goal is to compute the Euler characteristic $\chi(Z)$. If T_Z and N_Z denote the tangent and normal bundles,

respectively, then $T_Z \oplus N_Z \cong j^*T_{\mathbb{C}P^n}$. We know from Corollary 37.13 that $N_Z \cong j^*\mathcal{O}(d)$, so we have

$$\underline{1} \oplus T_Z \oplus j^*\mathcal{O}(d) \cong \underline{1} \oplus T_Z \oplus N_Z \cong \underline{1} \oplus j^*T_{\mathbb{C}P^n} \cong j^*(\underline{1} \oplus T_{\mathbb{C}P^n}) \cong (n+1)j^*\mathcal{O}(1),$$

the final isomorphism by (37.22). Taking duals and applying λ_t, we obtain

$$\lambda_t(\underline{1}) \cdot \lambda_t(T_Z^*) \cdot \lambda_t(j^*L^d) = \lambda_t(j^*L)^{n+1}.$$

Let $X = j^*L$. Then we can write

$$\lambda_t(T_Z^*) = \tfrac{(1+tX)^{n+1}}{(1+t)(1+tX^d)}.$$

We wish to compute the Euler class $e^K(T_Z) = \lambda_t(T_Z^*)|_{t=-1}$ and then write it in the form $(???) \cdot [*]$, in which case the mystery number in parentheses will be $\chi(Z)$. The technique is to expand in powers of $(X-1)$, since the powers of $1-L$ are our standard generators for $K^0(\mathbb{C}P^n)$.

Consider the power series in X

$$\lambda_t(T^*) = f(X) = \tfrac{(1+tX)^{n+1}}{(1+t)(1+tX^d)} = f(1) + f'(1)(X-1) + \tfrac{f''(1)}{2}(X-1)^2 + \cdots$$

Note that $(1-L)^{n+1}$ is zero in $K^0(\mathbb{C}P^n)$ and therefore $(1-X)^{n+1}$ vanishes in $K^0(Z)$. Even better, $(1-L)^n = [*]$ in $K^0(\mathbb{C}P^n)$ and since $j^*[*] = 0$ by intersection theory it follows that $(1-X)^n = 0$ in $K^0(Z)$. So we don't care about any terms in the above series beyond $(X-1)^{n-1}$. Let us also note at this point that $(1-L)^{n-1} = [\mathbb{C}P^1]$ in $K^0(\mathbb{C}P^n)$, and therefore $(1-X)^{n-1} = j^*\big((1-L)^{n-1}\big) = j^*([\mathbb{C}P^1]) = d[*]$; the last equality holds by intersection theory, since a generic $\mathbb{C}P^1$ will intersect Z in exactly d points.

Before tackling the general calculation let us do the first example, where $n = 2$. Here $f(X) = \frac{(1+tX)^3}{(1+t)(1+tX^d)}$ and we only need the first two terms of the series. Clearly $f(1) = 1+t$ and an easy calculation gives

$$f'(X) = \tfrac{1}{1+t} \cdot \tfrac{(1+tX)^2}{(1+tX^d)^2}\Big[3t(1+tX^d) - (1+tX)tdX^{d-1}\Big]$$

so that

$$f'(1) = (3-d)t.$$

Putting everything together,

$$f(X) = (1+t) + (3-d)t(X-1)$$

and so

$$e^K(T_Z) = f(X)|_{t=-1} = 0 + (d-3)(X-1) = (3-d)(1-X) = (3-d) \cdot d[*].$$

We conclude that $\chi(Z) = d(3-d)$.

For larger n here is how things are going to work. Calculating the derivatives by brute force is doable but unwieldy. If we set $u = X-1$ then we will instead compute the coefficients

$$[u^k]\tfrac{(1+t(u+1))^{n+1}}{(1+t)(1+t(u+1)^d)}$$

via some clever algebra, where $[u^k]p(u)$ denotes the coefficient of u^k in $p(u)$. It will turn out (but is not obvious) that these coefficients vanish for $k < n-1$, so that

$$\begin{aligned} e^K(T_Z) &= [u^{n-1}]\Big[\tfrac{(1+t(u+1))^{n+1}}{(1+t)(1+t(u+1)^d)}\Big]_{t=-1} \cdot (X-1)^{n-1} \\ &= [u^{n-1}]\Big[\tfrac{(1+t(u+1))^{n+1}}{(1+t)(1+t(u+1)^d)}\Big]_{t=-1} \cdot (-1)^{n-1} \cdot d[*]. \end{aligned} \tag{37.26}$$

Then the Euler characteristic $\chi(Z)$ is just the coefficient of $[*]$ in the above expression.

Notice that everything comes down to a purely algebraic problem. Unfortunately we cannot *first* plug in $t = -1$, since our formula has a $1+t$ in the denominator; we have to first do enough algebra to eliminate all such poles. This looks very hairy! We will tackle it via some well-chosen changes of variables.

We are going to sketch the completion of the calculations for the above example, but before diving into that we need to make a confession. It is possible to compute $\chi(Z)$ by doing a similar kind of calculation using singular cohomology instead of K-theory, and in that setting the algebra turns out to be much *easier*! There is a trade-off, which is that the computation cannot be done merely with Euler classes—one needs the complete theory of Chern classes, to be developed in the next section. See Example 38.9 for the computation of $\chi(Z)$ in that context. This situation is fairly typical of the relationship between K-theory and singular cohomology. For calculations that can be done in either theory, usually the singular cohomology version will involve simpler algebra, but more advanced geometric techniques; the K-theory version will involve more advanced algebra, but one needs less geometry. In some sense we saw this phenomenon already in the case of intersection multiplicities.

The completion of our calculation will proceed via the following steps. Set $u = X - 1$ and $a = 1 + t$. We want to find

$$[u^k]f(a,u) = [u^k]\Big[\tfrac{(a+tu)^{n+1}}{a(1+t(1+u)^d)}\Big]$$

Now we make one more change of variable: $u = az$. Then

$$\begin{aligned}[u^k]\Big[\tfrac{(a+tu)^{n+1}}{a(1+t(1+u)^d)}\Big] = a^{-k}\cdot[z^k]\Big[\tfrac{(a+taz)^{n+1}}{a(1+t(1+az)^d)}\Big] &= a^{n-k}\cdot[z^k]\Big[\tfrac{(1+tz)^{n+1}}{a+dtaz+\binom{d}{2}ta^2z^2+\cdots}\Big]\\ &= a^{n-k-1}\cdot[z^k]\Big[\tfrac{(1+tz)^{n+1}}{1+dtz+\binom{d}{2}taz^2+\cdots}\Big].\end{aligned}$$

The coefficient of z^k is the final expression is some polynomial in t and a, so for $k \le n-1$ we can set $t = -1$ (and therefore $a = 0$) to get that $[u^k]f(a,u) = 0$ for $k < n-1$ and

$$\begin{aligned}[u^{n-1}]f(a,u) &= [z^{n-1}]\Big[\tfrac{(1-z)^{n+1}}{1-dz}\Big]\\ &= d^{n-1} - \tbinom{n+1}{1}d^{n-2} + \tbinom{n+1}{2}d^{n-3} - \cdots + (-1)^{n+1}\tbinom{n+1}{n+1}d^0\\ &= \tfrac{(d-1)^{n+1}+(-1)^{n-1}(n+1)d+(-1)^n}{d^2}.\end{aligned}$$

Recalling (37.26) and using what we have just done, we have proven that

$$\begin{aligned}\chi(Z_d \hookrightarrow \mathbb{C}P^n) &= (-1)^{n+1}\tfrac{1}{d}\cdot\Big[(d-1)^{n+1} - (-1)^n((n+1)d-1)\Big]\\ &= \tfrac{(1-d)^{n+1}+(n+1)d-1}{d}.\end{aligned}$$

None of the above formulas are particularly pleasant to look at, and they are difficult to remember. I like to encode the formula in a different way. Let ℓ denote the formal "lowering operator" that sends d^s to d^{s-1}, for each s. Then we may write

$$\chi(Z_d \hookrightarrow \mathbb{C}P^n) = d\cdot(\ell - I)^{n+1}(d^{n-1})$$

where I is the identity operator. For example,

$$\chi(Z_d \hookrightarrow \mathbb{C}P^3) = d \cdot (\ell^4 - 4\ell^3 + 6\ell^2 - 4\ell + I)(d^2) = d \cdot (6 - 4d + d^2).$$

38. Chern classes

Fix a certain collection of vector bundles. A *characteristic class* for this collection assigns to each vector bundle $E \to X$ a cohomology class $b(E)$ belonging to some cohomology theory; the assignment is required to be natural, in the sense that $b(f^*E) = f^*(b(E))$. We have seen essentially two examples so far: for the collection of oriented vector bundles we have the Euler class e^H, and for complex vector bundles we have the K-theoretic Euler class e^K.

The Chern classes are characteristic classes for *complex* vector bundles, that generalize the Euler class in a certain way. Like the Euler class, they have close ties to geometry. Also like the Euler class, there are versions of Chern classes in both singular cohomology and K-theory—indeed, there are versions in any complex-oriented cohomology theory.

In this section we begin with a purely geometric look at the Chern classes, where we again forego all attempts at rigor. Afterwards we will pursue a more rigorous approach, which can even be done axiomatically.

38.1. **Geometric Chern classes in homology.** Let B be a complex manifold of dimension n, and let $E \to B$ be a complex vector bundle of rank k. If s is a generic section of E, then the locus where s vanishes gives a cycle in B that carries the Euler class $e_H(E) \in H_{2(n-k)}(B)$. This homology class will now be renamed as $C_k(E)$ and called the *kth homology Chern class of E*.

Now let s_1 and s_2 be *two* generic sections, chosen so that $s_1(x)$ and $s_2(x)$ are linearly independent on as large a subset of B as possible. We can look at the degeneracy locus

$$D(s_1, s_2) = \{b \in B \,\big|\, s_1(b) \text{ and } s_2(b) \text{ are linearly dependent}\}.$$

Again, for generically chosen s_1 and s_2 it turns out this gives a cycle on B whose associated homology class is independent of any choices. The homology class lies in dimension $2(n-k+1)$, and we call it the *$(k-1)$st homology Chern class* $C_{k-1}(E)$.

At this point it is clear how to continue. For each j in the range $1 \leq j \leq k$, let $s_1, \ldots, s_j$ be sections generically chosen to be as maximally linearly independent as possible. Consider the degeneracy locus

$$D(s_1, \ldots, s_j) = \{b \in B \,\big|\, s_1(b), \ldots, s_j(b) \text{ are linearly dependent}\},$$

which determines a homology class $C_{k-j+1} \in H_{2(n-k+j-1)}(B)$.

These homology classes can be thought of as obstructions to splitting off a trivial bundle. If E contains a trivial bundle of rank r then $0 = C_k(E) = C_{k-1}(E) = \cdots = C_{k-r+1}(E)$. This is clear, as by working inside the trivial subbundle we can choose our "generic sections" so that they are linearly independent everywhere.

The geometric details behind all of this can be found in [GH]. We are skipping them because they are somewhat difficult and are not needed for the cohomological version of the theory that we will describe next. The fact that cohomology gives a shortcut around the dicey geometric issues is part of the magic of homotopy theory, though it can also feel like part of the subject's curse.

38.2. **Chern classes in singular cohomology.** For the cohomological version of the theory we start by adopting an axiomatic approach. For any complex vector bundle $E \to X$ the Chern classes are cohomology classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ for $0 \le i < \infty$ satisfying the following properties:

(1) $c_0(E) = 1$,
(2) $c_i(E) = 0$ if $i >$ rank E,
(3) $c_i(f^*E) = f^*c_i(E)$ (naturality under pullback),
(4) The Whitney Formula: $c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E)c_{k-i}(F)$, for any k,
(5) $c_1(L^* \to \mathbb{C}P^1) = e^H(L^*) = [\mathbb{C}P^0] = [*]$, where $L \to \mathbb{C}P^1$ is the tautological line bundle.

Remark 38.3. The Whitney Formula can be written in a more convenient way using the *total Chern class*, namely

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \cdots \in H^*(X)$$

(notice that this is a finite sum by property (2)). Then the Whitney Formula becomes $c(E \oplus F) = c(E) \cdot c(F)$.

Note that if $E \to X$ is a trivial bundle then $c_i(E) = 0$ for $i > 0$. Indeed, E is the pullback of a bundle on a point: $E \cong \pi^*(\underline{n})$ where $\pi\colon X \to *$ and $n = \mathrm{rank}(E)$. One has $c_i(\mathbb{C}^n \to *) = 0$ for $i > 0$ because a point has no cohomology in positive degrees. The fact that $c_i(E) = 0$ then follows from naturality.

Exercise 38.4. Show that given a theory of Chern classes satisfying the above axioms then for any $n \ge 1$ one has $c_1(L^*) = [\mathbb{C}P^{n-1}]$ for $L \to \mathbb{C}P^n$ the tautological bundle. Then prove that for $L \to \mathbb{C}P^\infty$ one has $c_1(L) = e^H(L)$, and moreoever the same is true for any line bundle on any space. (Axioms (3) and (5) should be enough for this).

Before showing the existence of the Chern classes, let us show that they are uniquely characterized by the above properties:

Proposition 38.5. *There is at most one collection of characteristic classes satisfying properties (1)–(5) above. Moreover, such a collection has the property that if $E \to X$ is a rank n bundle then the top Chern class $c_n(E)$ is equal to the Euler class $e(E)$.*

Proof. Let γ_k be the tautological k-plane bundle on $\mathrm{Gr}_k(\mathbb{C}^\infty)$. Consider the diagram

$$\begin{array}{ccc} \pi_1^*(L) \oplus \pi_2^*(L) \oplus \cdots \oplus \pi_k^*(L) & \longrightarrow & \gamma_k \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty & \xrightarrow{S} & \mathrm{Gr}_k(\mathbb{C}^\infty \times \cdots \times \mathbb{C}^\infty) = \mathrm{Gr}_k(\mathbb{C}^\infty) \end{array}$$

with the obvious maps, e.g. S is the map described in Section 16.18 that sends a collection of k lines in $\mathbb{C}^\infty$ to their direct sum in $(\mathbb{C}^\infty)^k$. Note that there are k copies of $\mathbb{C}P^\infty$ and $\mathbb{C}^\infty$ in the bottom row. Also, $\pi_i : \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the ith projection map. This diagram is a pullback, hence $S^*(\gamma_k) \cong (\pi_1^*L) \oplus \cdots \oplus (\pi_k^*L)$.

Applying cohomology to the map on the bottom row gives

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \xleftarrow{S^*} H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \ .$$

By the Künneth Theorem,

$$H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \cong H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x_1, \ldots, x_k]$$

where $x_i = \pi_i^*(x)$ with $x \in H^2(\mathbb{C}P^\infty)$ being the canonical generator. There is an evident action on $\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$ by the symmetric group Σ_k. This action descends in cohomology to give the statement

$$\begin{aligned} H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty)^{\Sigma_k} &\cong \Big[H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty)\Big]^{\Sigma_k} \\ &= \mathbb{Z}[x_1, \ldots, x_k]^{\Sigma_k} \\ &= \mathbb{Z}[\sigma_1, \ldots, \sigma_k] \end{aligned}$$

where σ_i is the ith elementary symmetric function in the x_i's.

Recall that $[X, \mathrm{Gr}_k(\mathbb{C}^\infty)] \simeq \mathrm{Vect}_k(X)$. Under this bijection, S corresponds to the bundle $E = \oplus_i \pi_i^*(L)$. If $\alpha \in \Sigma_k$ then the map $S \circ \alpha$ corresponds to the direct sum of $\pi_i^*(L)$'s but where the sum is taken in a different order. Since this is isomorphic to the original bundle E, it must be that S and $S \circ \alpha$ are homotopic; in particular, they induce the same map on cohomology. Since this holds for all α, it follows that S^* lands inside the Σ_k invariants. That is, S^* can be regarded as a map

$$H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \xrightarrow{S^*} \Big[H^*(\mathbb{C}P^\infty)^{\otimes k}\Big]^{\Sigma_k} = \mathbb{Z}[\sigma_1, \ldots, \sigma_k]. \tag{38.6}$$

It is a theorem that the above map (38.6) is an isomorphism. We will delay the full proof until Proposition 38.17 below, but the idea is simple enough. The Schubert cell decomposition of $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has all cells in even dimensions, and hence the coboundary maps are all zero; this computes $H^*(\mathrm{Gr}_k(\mathbb{C}^\infty))$ additively, and one readily checks that the groups have the same ranks as in $\mathbb{Z}[\sigma_1, \ldots, \sigma_k]$. The so-called "splitting principle" (which we discuss below) shows that S^* is injective with any coefficients. It follows formally from these two properties that S^* is an isomorphism. See Proposition 38.17.

Using the Whitney Formula (iteratively), we can see that

$$\begin{aligned} c_i(\pi_1^* L \oplus \cdots \oplus \pi_k^* L) &= \sum_\beta c_1(\pi_{\beta(1)}^*(L)) \cdot c_1(\pi_{\beta(2)}^*(L)) \ldots c_1(\pi_{\beta(i)}^*(L)) \\ &= \sum_\beta \pi_{\beta(1)}^*(x) \cdot \pi_{\beta(2)}^*(x) \ldots \pi_{\beta(i)}^*(x) \end{aligned}$$

where the sum ranges over strictly-increasing maps $\beta\colon \{1, \ldots, i\} \to \{1, \ldots, k\}$, and in the second sum $x = c_1(L) \in H^2(\mathbb{C}P^\infty)$. But note that if we write $x_j = \pi_j^*(x)$ then the second sum is simply the elementary symmetric function σ_i in the x_j's.

It follows from the above that $c_i(\gamma_k)$ is the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i under S^*.

Finally, suppose that $E \to X$ is any complex vector bundle, say of rank k. Then there is a map $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ and an isomorphism $f^*\gamma_k \cong E$. It follows that $c_i(E) = f^*(c_i(\gamma_k))$. To complete the proof, assume that c_* and c'_* are two sets of characteristic classes satisfying properties (1)–(5). Then $c_i(\gamma_k)$ and $c'_i(\gamma_k)$ must agree, for they each must be the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i. It then follows from naturality that $c_i(E) = c'_i(E)$ for all bundles E. □

By examining the above proof, one finds that we can *define* the Chern classes in the following way. First, when $\gamma_k \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is the tautological bundle then

for $i \le k$ define $c_i(\gamma_k)$ to be the unique element of $H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty))$ that maps to σ_i under S^*, and for $i > k$ define $c_i(\gamma_k) = 0$. Second, for an arbitrary rank k bundle $E \to X$ let $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ be a classifying map and define $c_i(E) = f^*(c_i(\gamma_k))$. Properties (1)–(3) and (5) are automatic. Property (4) follows from the identity of symmetric functions

$$\sigma_k(x_1, \dots, x_p, x_{p+1}, \dots, x_q) = \sum_{i=0}^{k} \sigma_i(x_1, \dots, x_p)\sigma_{k-i}(x_{p+1}, \dots, x_q).$$

If one regards the monomials in the symmetric functions as being indexed by subsets of the x's, this identity is just the observation that choosing a k-element subset of $\{x_1, \dots, x_q\}$ is the same as choosing an i, an i-element subset of $\{x_1, \dots, x_p\}$, and a $(k-i)$-element subset of $\{x_{p+1}, \dots, x_q\}$.

Remark 38.7. For a bundle $E \to X$ one can use essentially the same method to define K-theoretic Chern classes $c_i^K(E) \in K^0(X)$ (or really in $K^{2i}(X)$, but this is the same by periodicity). It turns out, though, to be slightly easier to follow a different approach that we discuss in Section 38.13. See also the thorough discussion of the K-theoretic Chern classes in Section 47.

Example 38.8. Consider the tangent bundle $T = T\mathbb{C}P^n \to \mathbb{C}P^n$. We saw in Example 37.19 that $1 \oplus T \cong (n+1)L^*$. Then by the Whitney Formula,

$$c(T) = c(1) \cdot c(T) = c(1 \oplus T) = c(L^*)^{n+1} = (1 + [\mathbb{C}P^{n-1}])^{n+1}.$$

Therefore,

$$c_i(T) = \tbinom{n+1}{i}[\mathbb{C}P^{n-i}] = \tbinom{n+1}{i}x^i$$

where $x \in H^2(\mathbb{C}P^n)$ is the canonical generator $[\mathbb{C}P^{n-1}]$. Note that the Euler class is $e(T) = c_n(T) = (n+1)x^n = (n+1)[*]$, and so this again calculates that $\chi(\mathbb{C}P^n) = n+1$.

Example 38.9. Consider a hypersurface $j\colon Z \hookrightarrow \mathbb{C}P^n$ of degree d. Recall from Corollary 37.13 that the normal bundle of this inclusion is $j^*\mathcal{O}(d)$, and we know $T_Z \oplus N_Z \cong j^*T_{\mathbb{C}P^n}$. Then applying total Chern classes we get

$$c(T_Z) \cdot c(N_Z) = j^*c(T_{\mathbb{C}P^n}).$$

But above we calculated that $c(T_{\mathbb{C}P^n}) = (1+x)^{n+1}$ where $x = [\mathbb{C}P^{n-1}]$, and $c(N_Z) = c(j^*(L^*)^d) = j^*(c((L^*)^d)) = 1 + d(j^*x)$. Let $z = j^*x$, so that we have

$$c(T_Z) = \tfrac{(1+z)^{n+1}}{1+dz} = (1 + (n+1)z + \tbinom{n+1}{2}z^2 + \cdots) \cdot (1 - dz + d^2z^2 - \cdots).$$

We can compute $\chi(Z)$ by finding the top Chern class (the Euler class), which in this case is $c_{n-1}(T_Z)$. A direct computation shows that

$$c_{n-1}(T_Z) = z^{n-1} \cdot \big(\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\big).$$

Finally, we need to remember that $x^{n-1} = [\mathbb{C}P^1]$ in $H^*(\mathbb{C}P^n)$ and therefore $z^{n-1} = j^*(x^{n-1}) = d[*]$, since a generic line intersects Z in d distinct points. So we have

$$c_{n-1}(T_Z) = [*] \cdot d \cdot (\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots),$$

thereby yielding

$$\chi(Z) = d \cdot \Big(\tbinom{n+1}{n-1} - \tbinom{n+1}{n-2}d + \tbinom{n+1}{n-2}d^2 - \cdots\Big).$$

For example, a degree d hypersurface in $\mathbb{C}P^2$ has $\chi(Z) = d\cdot(3-d)$ and a degree d hypersurface in $\mathbb{C}P^3$ has $\chi(Z) = d\cdot(6-4d+d^2)$.

38.10. **Stiefel-Whitney classes.** One can repeat almost all of our above work in the setting of real vector bundles, but using $\mathbb{Z}/2$ coefficients everywhere. The analogs of the Chern classes in this setting are called **Stiefel-Whitney classes**. If $E \to X$ is a real vector bundle then the Stiefel-Whitney classes are cohomology classes $w_i(E) \in H^i(X;\mathbb{Z}/2)$, $0 \leq i < \infty$, satisfying the evident analogs of the axioms in Section 38.2. Geometrically, when X is a compact manifold these are Poincaré Duals of certain cycles determined by degeneracy loci, just as in the complex case. In terms of our development, most things go through verbatim but the one exception is the computation $H^*(\mathrm{Gr}_k(\mathbb{R}^\infty);\mathbb{Z}/2) \cong \mathbb{Z}/2[\sigma_1,\ldots,\sigma_k]$. In the complex case this is not too hard, because the standard cell structure on $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has cells only in even dimensions. That is of course not true for $\mathrm{Gr}_k(\mathbb{R}^\infty)$, and so one must work a bit harder here. The isomorphism can be proven using methods developed later in this section; see Exercise 38.32.

Just as for the Chern classes, we will write $w(E)$ for the total Stiefel-Whitney class $1 + w_1(E) + w_2(E) + \cdots$.

Example 38.11. Here is an example where we can use Stiefel-Whitney classes to solve a problem that appeared earlier in this book. Let $L \to \mathbb{R}P^n$ denote the tautological line bundle, and recall that once upon a time we needed to know whether $L \oplus L$ is stably trivial. This came up (for $n = 2$) in Section 17.10 during the course of trying to compute $KO(\mathbb{R}P^2)$.

If $(L \oplus L) \oplus \underline{N} \cong \underline{N+2}$ then applying total Stiefel-Whitney classes gives

$$1 = w(\underline{N+2}) = w(L \oplus L \oplus \underline{N}) = w(L)\cdot w(L)\cdot w(\underline{N}) = w(L)^2.$$

But L is a line bundle, so $w_i(L) = 0$ for $i > 1$ and $w_1(L)$ is the mod 2 Euler class, which we have previously computed to be the generator x on $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$. So $w(L) = 1 + x$ and therefore $w(L)^2 = 1 + x^2$. As long as $n \geq 2$ this is not equal to 1, and hence $L \oplus L$ cannot be stably trivial.

Example 38.12. Let T be the tangent bundle of $\mathbb{R}P^n$. Just as in Example 37.19 there is an isomorphism $\underline{1} \oplus T \cong (n+1)L^*$, where $L \to \mathbb{R}P^n$ is the tautological line bundle. But since we are now in the case of real bundles, $L \cong L^*$ by Corollary 11.15; so we will usually write $\underline{1} \oplus T \cong (n+1)L$. One of course finds that

$$w(T) = w(1 \oplus T) = w((n+1)L) = w(L)^{n+1} = (1+x)^{n+1},$$

similarly to the complex case.

38.13. **The projective bundle approach.** There is another approach to Chern classes that is due to Grothendieck [Grothen2]. This approach adapts to any complex-orientable cohomology theory $\mathcal{E}$ with ease, so we now return to that level of generality.

Given a rank n bundle $p\colon M \to X$ consider the associated projective bundle $\pi\colon \mathbb{P}M \to X$. The pullback bundle π^*M has a tautological line bundle inside of it; we will denote this L_M. To describe this precisely, recall that a point in $\mathbb{P}M$ is a pair (x,ℓ) where $x \in X$ and $\ell \subseteq M_x$ is a line. Then a point in π^*M is a triple (x,ℓ,v) where $\ell \subseteq M_x$ and $v \in M_x$. We define $L_M \subseteq \pi^*M$ to be the subspace of triples (x,ℓ,v) where additionally $v \in \ell$; that is, the subspace of triples (x,ℓ,v) where $v \in \ell \subseteq M_x$.

Since $L_M \subseteq \pi^* M$ we can form the quotient, which we will denote as Q_M. If X is paracompact and Hausdorff then by Proposition 12.2 we have $\pi^* M \cong L_M \oplus Q_M$ (though not canonically, of course).

Starting with $M \to X$ we have produced a map $\pi\colon \mathbb{P}M \to X$ with the property that $\pi^* M$ splits off a line bundle. If we now pass to the space $\mathbb{P}Q_M \to \mathbb{P}M \to X$ then pulling back will result in Q_M splitting off a line bundle, so that M has split off a sum of two lines bundles. By iterating this process, after n steps it will be the case that when we pull back M it will split as a sum of line bundles:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \pi_2^*(L_M) \oplus L_{Q_M} \oplus Q_{Q_M} & \longrightarrow & L_M \oplus Q_M & \longrightarrow & M \\
& \downarrow & & \downarrow & & \downarrow p \\
\cdots \longrightarrow & \mathbb{P}Q_M & \xrightarrow{\pi_2} & \mathbb{P}M & \xrightarrow{\pi} & X.
\end{array}$$

The iterated projective bundles appearing above have a nice description. For example, observe that giving a line $\ell \in \mathbb{C}^n$ and a line $\ell' \in \mathbb{C}^n/\ell$ is equivalent to giving a pair of subspaces $V_1 \subseteq V_2 \subseteq \mathbb{C}^n$ where $\dim V_i = i$: one takes $V_1 = \ell$ and V_2 to be the preimage of ℓ' under $\mathbb{C}^n \to \mathbb{C}^n/\ell$. The sequence $\underline{V}$ is called a **flag** of length 2. Generalizing, define the **flag bundle** $\mathrm{Fl}_k(M)$ by

$$\mathrm{Fl}_k(M) = \{(x, V_1 \subseteq \cdots \subseteq V_k \subseteq M_x) \,\big|\, x \in X, \quad V_i \text{ is a subspace of dimension } i\},$$

topologized appropriately. Then $\mathrm{Fl}_1(M) = \mathbb{P}M$, and $\mathrm{Fl}_k(M)$ is the space appearing at stage k of the above iterative construction. We have tautological line bundles $L_i \to \mathrm{Fl}_k(M)$ whose fiber over $(x, V_\bullet)$ is V_i/V_{i-1}. We have the canonical projection $\pi\colon \mathrm{Fl}_k(M) \to X$, and $\pi^* M$ contains $L_1 \oplus \cdots \oplus L_k$ as a direct summand.

Pulling back our bundle M and having it split off a sum of line bundles is, by itself, nothing to get excited about. After all, if we pull back M along a map $* \to X$ it also splits as a sum of line bundles. What is special about the present situation, though, is that each of the horizontal maps in the bottom row turns out to be an injection on $\mathcal{E}^*$ cohomology whenever $\mathcal{E}$ is complex orientable. So anything interesting about characteristic classes for M—which lie in $\mathcal{E}^*(X)$—will also be seen in the successive spaces as we pull back. This phenomenon is called the "splitting principle":

Proposition 38.14 (The Splitting Principle)**.** *Let $\mathcal{E}$ be a complex-oriented cohomology theory. Let X be paracompact and Hausdorff. Given any rank n complex bundle $M \to X$, the map $\pi\colon Fl_k(M) \to X$ has the properties that π^* is an injection on $\mathcal{E}^*(-)$ and $\pi^* M$ splits off a direct sum of k line bundles. When taking $k = n$ we have that $\pi^* M$ is isomorphic to a sum of n line bundles.*

Oftentimes when applying the splitting principle one doesn't actually care about the exact identity of the space $\mathrm{Fl}_k(M)$, just that there exists some space having the above properties.

By induction the splitting principle will follow as soon as we check it for $\mathbb{P}M$, so we focus on that next. The bundle L_M has an Euler class $e(L_M) \in \mathcal{E}^2(\mathbb{P}M)$ (this is the $\mathcal{E}$-theory Euler class, but we suppress the $\mathcal{E}$ in the notation). In fact let us just write $e_M = e(L_M)$ for short.

Proposition 38.15. *Let $\mathcal{E}$ be a complex-oriented cohomology theory and let $M \to X$ be a rank n complex bundle. Then $\mathcal{E}^*(\mathbb{P}M)$ is free as an $\mathcal{E}^*(X)$-module on generators $1, e_M, e_M^2, \ldots, e_M^{n-1}$. Consequently, the map $\mathcal{E}^*(X) \to \mathcal{E}^*(M)$ is injective.*

Proof. This is just an application of the $\mathcal{E}$-theoretic version of the Leray-Hirsch theorem [Hatcher1, Theorem 4D.1], but we explain how to prove that. It suffices to assume X is a CW-complex. Let X_i be the i-skeleton, let $J = \mathbb{P}M$, and let $J_i = \pi^{-1}(X_i)$. For brevity just write $e = e_M$. The classes e^i in $\mathcal{E}^*(J)$ give corresponding classes in $\mathcal{E}^*(J_i)$, for each i. So we get natural maps $\alpha_i\colon \mathcal{E}^*(X_i)\langle 1, e, \dots, e^{n-1}\rangle \to \mathcal{E}^*(J_i)$, and we will show inductively that they are all isomorphisms.

The base case $i = 0$ is trivial, and for the induction step one uses Mayer-Vietoris in the usual way. It is important that $X_i - X_{i-1}$ is a disjoint union of open disks and hence J is trivializable over that.

The final step is proving that $\alpha\colon \mathcal{E}^*(X)\langle 1, e, \dots, e^{n-1}\rangle \to \mathcal{E}^*(J)$ is an isomorphism once we know the corresponding result for all the α_i. Here we use the Milnor sequences for J and for X. Write $\underline{e}$ as shorthand for $1, e, \dots, e^{n-1}$, and consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim_i{}^1\, \mathcal{E}^{*-1}(X_i)\langle \underline{e}\rangle & \longrightarrow & \mathcal{E}^*(X)\langle \underline{e}\rangle & \longrightarrow & \lim_i \mathcal{E}^*(X_i)\langle \underline{e}\rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim_i{}^1\, \mathcal{E}^{*-1}(J_i) & \longrightarrow & \mathcal{E}^*(J) & \longrightarrow & \lim_i \mathcal{E}^*(J_i) & \longrightarrow & 0.
\end{array}$$

The top sequence is a finite direct sum of Milnor exact sequences, hence exact. The left and right vertical maps are isomorphisms by what has already been proven, so the middle map is also an isomorphism. □

Let us again note that Proposition 38.14 is an immediate consequence of the above result.

Corollary 38.16. *The map $S^*\colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to [\mathcal{E}^*(\mathbb{C}P^\infty)^{\otimes k}]^{\Sigma_k}$ is an injection.*

Proof. We claim that injectivity follows from the splitting principle. For the bundle $\gamma_k \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ there exists a space Z and a map $f\colon Z \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ such that $f^*\gamma_k$ is a sum of line bundles and $f^*\colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to \mathcal{E}^*(Z)$ is injective. Since $f^*\gamma_k$ is a sum of line bundles it is classified by a map $g\colon Z \to (\mathbb{C}P^\infty)^{\times k}$. So the diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\quad f \quad} & \mathrm{Gr}_k(\mathbb{C}^\infty) \\
 & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle S} & \\
 & (\mathbb{C}P^\infty)^{\times k} &
\end{array}$$

commutes up to homotopy. Since f^* is injective on $\mathcal{E}^*(-)$, it follows that the same is true for S^*. □

We can now give a very simple proof of the missing result that was needed in the proof of Proposition 38.5:

Proposition 38.17. *The map $S^*\colon H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to [H^*(\mathbb{C}P^\infty)^{\otimes k}]^{\Sigma_k}$ is an isomorphism.*

Proof. Write $T = [H^*(\mathbb{C}P^\infty)^{\otimes_k}]^{\Sigma_k} = \mathbb{Z}[\sigma_1, \dots, \sigma_k]$ and T^n for the component in degree n. By Proposition 38.16 the maps $S^*_n\colon H^n(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to T^n$ are injections, and by inspection the domain and codomain are free abelian groups of finite rank (using the Schubert cell decomposition for the Grassmannian, which has only even cells and finitely many in each degree). Moreover, the maps remain injections upon

tensoring with $\mathbb{Z}/p$ for any prime p: this comes from applying Proposition 38.16 to the cohomology theory $H^*(-;\mathbb{Z}/p)$. It then follows that the map S_n^* is a split injection of free abelian groups.

Of course everything is zero in odd degrees, so it suffices to look at $n = 2k$. The rank of the domain of S_{2k}^* equals the number of Schubert cells in that degree, which is the number of partitions of the integer k (see Exercise 18.9). This is the same as the rank of T^{2k}, by inspection. So S_{2k}^* is a split injection between free abelian groups having the same finite rank, hence it must be an isomorphism. □

Remark 38.18. We will soon prove the analog of Proposition 38.17 for any complex-oriented cohomology theory, but we proved the version for H separately because one can do it with somewhat less technology.

Corollary 38.16 starts to show how our flag bundles are connected to the map S that we studied in the first part of this section. In fact, the connection can be made extremely directly. A flag $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k$ in $\mathbb{C}^\infty$ can be identified with the data of k mutually-orthogonal lines, by sending $\underline{V}$ to the tuple $(V_1, V_2 - V_1, V_3 - V_2, \ldots, V_k - V_{k-1})$, where here we use $U - W$ to denote the orthogonal complement of W in U. So for $\gamma_k \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ we have that $\mathrm{Fl}_j(\gamma_k)$ is homeomorphic to the subspace of tuples $(N, \ell_1, \ldots, \ell_j)$ where $N \subseteq \mathbb{C}^\infty$ has dimension k and the ℓ_i are mutually-orthogonal lines in M. For $\mathrm{Fl}_k(\gamma_k)$ the N is recoverable from the ℓ's and so $\mathrm{Fl}_k(\gamma_k)$ is simply the space of k orthogonal lines in $\mathbb{C}^\infty$. We therefore have the diagram

$$\begin{array}{ccc} \mathrm{Fl}_k(\gamma_k) & \xrightarrow{\oplus} & \mathrm{Gr}_k(\mathbb{C}^\infty) \\ \downarrow & \nearrow_{S} & \\ (\mathbb{C}P^\infty)^{\times k} & & \end{array}$$

and we claim this commutes up to homotopy. Indeed, pulling back γ_k to $\mathrm{Fl}_k(\gamma_k)$ along either path produces isomorphic bundles. The vertical map is not a homeomorphism, but instead we have the following:

Proposition 38.19. *The inclusion $Fl_k(\gamma_k) \hookrightarrow (\mathbb{C}P^\infty)^{\times k}$ is a homotopy equivalence.*

Proof. Consider the Stiefel manifold $V_k(\mathbb{C}^N)$ of orthonormal frames. This has the evident free action of the Lie group $G = (S^1)^{\times k}$, and $V_k(\mathbb{C}^N)/G = \mathrm{Fl}_k(\gamma_k(\mathbb{C}^N))$. Consider the diagram of fiber bundles

$$\begin{array}{ccccccc} G & \longrightarrow & V_k(\mathbb{C}^N) & \longrightarrow & V_k(\mathbb{C}^N)/G & = & \mathrm{Fl}_k(\gamma_k(\mathbb{C}^N)) \\ \| & & \downarrow & & \downarrow & & \\ G & \longrightarrow & V_1(\mathbb{C}^N)^{\times k} & \longrightarrow & [V_1(\mathbb{C}^N)^{\times k}]/G & = & (\mathbb{C}P^{N-1})^{\times k} \end{array}$$

where the middle vertical map is just the inclusion. Taking the colimit as $N \to \infty$, both $V_k(\mathbb{C}^N)$ and $V_1(\mathbb{C}^N)$ become contractible. So the middle map becomes a weak equivalence, and therefore so does the right map. □

Now we move on to describe Grothendieck's approach to characteristic classes. Given Proposition 38.15, there is a unique relation in $\mathcal{E}^*(\mathbb{P}M)$ of the form

$$e_M^n = b_n + b_{n-1}e_M + b_{n-2}e_M^2 + \cdots + b_1 e_M^{n-1}$$

where $b_i \in \mathcal{E}^{2i}(X)$. We define the ith $\mathcal{E}$-theoretic Chern class of M to be $c_i^{\mathcal{E}}(M) = (-1)^{i-1}b_i$. With this definition the following corollary is immediate:

Corollary 38.20. $\mathcal{E}^*(\mathbb{P}M) = \mathcal{E}^*(X)[e_M]/(e_M^n - c_1^{\mathcal{E}}(M)e^{n-1} + c_2^{\mathcal{E}}(M)e^{n-2} - \cdots)$.

Proof. This follows immediately from Proposition 38.15, since the induced map from the right-hand-side to $\mathcal{E}^*(\mathbb{P}M)$ is a surjection between free $\mathcal{E}^*$-modules of the same rank. □

Remark 38.21. The motivation for the above definition of the $\mathcal{E}$-theoretic Chern classes is not so clear—the definition came out of the blue—but perhaps it will help if we explain for $\mathcal{E} = H$ why this agrees with our previous definition. So let us return to the world where we already have the Chern classes in H^*. When we pull back M to $\mathbb{P}M$ we have the splitting $\pi^*M \cong L_M \oplus Q_M$, and taking total Chern classes gives

$$\pi^*c(M) = c(\pi^*M) = c(L_M \oplus Q_M) = c(L_M)c(Q_M) = (1 + e_M)c(Q_M).$$

Thus, we obtain

$$c(Q_M) = \frac{\pi^*c(M)}{1 + e_M} = \pi^*c(M) \cdot (1 - e_M + e_M^2 - \cdots).$$

The power series might be worrisome, and there are two ways to get past that: (1) assume that X is a finite-dimensonal CW complex, so that $\mathbb{P}M$ also is and therefore high enough powers of e_M will vanish; (2) move from $H^*(X) = \oplus_i H^i(X)$ into $\prod_i H^i(X)$ (the first injects into the second), and note that the power series makes sense in the second (all conclusions will be back in $\oplus_i H^i(X)$, though).

Since Q_M is rank $n-1$ we can now write

$$0 = c_n(Q_M) = c_n(M) - c_{n-1}(M)e_M + c_{n-2}(M)e_M^2 - \cdots + (-1)^n c_0(M)e_M^n$$

and rearranging gives

$$0 = e_M^n - c_1(M)e_M^{n-1} + c_2(M)e_M^{n-2} - \cdots + (-1)^n c_n(M)$$

which is the desired formula (note that we are suppressing the π^* maps here, as $c_i(M)$ should really be $\pi^*c_i(M)$, but this practice is consistent with regarding $H^*(\mathbb{P}M)$ as an $H^*(X)$-module).

We next wish to make a direct connection between the projective bundle approach to characteristic classes and the approach that goes through products of $\mathbb{C}P^\infty$'s and rings of symmetric functions. To describe $\mathcal{E}^*((\mathbb{C}P^\infty)^{\times k})$ we first need a new definition. If R is a graded ring and x is a formal variable of degree d, let $R[[x]]$ denote the graded ring that in degree n consists of homogeneous power series of degree n. We call this the **graded power series ring**. Note that this is different than just the classical ring of power series in x with coefficients in R, which does not have a natural grading. In our version of $R[[x]]$ a power series can only have finitely-many degrees appearing in its expansion. While it can be confusing to have the same notation for both graded power series rings and ungraded power series rings, in practice this won't be a problem: if the ring R is graded, $R[[x]]$ always means graded power series unless otherwise noted.

We have calculated that $\mathcal{E}^*(\mathbb{C}P^n) = \mathcal{E}^*[x]/x^{n+1}$ where $x = e^{\mathcal{E}}(L \to \mathbb{C}P^n)$. In the Milnor sequences

$$0 \to \lim_n{}^1 \mathcal{E}^{k-1}(\mathbb{C}P^n) \to \mathcal{E}^k(\mathbb{C}P^n) \to \lim_n \mathcal{E}^k(\mathbb{C}P^n) \to 0$$

the $\lim^1$ terms vanish by Mittag-Leffler (because all of the maps are surjections), and the term on the right is then identified with degree k homogeneous power series in x with coefficients in $\mathcal{E}^*$. We conclude that $\mathcal{E}^*(\mathbb{C}P^\infty) = \mathcal{E}^*[[x]]$, with $x = e(L \to \mathbb{C}P^\infty)$ having degree 2.

Since $\mathcal{E}^*(\mathbb{C}P^n)$ is free over $\mathcal{E}^*$, the Künneth theorem for generalized cohomology gives that $\mathcal{E}^*(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}) = \mathcal{E}^*[x_1, x_2]/(x_1^{n_1+1}, x_2^{n_2+1})$ where $x_i = e^{\mathcal{E}}(L_i)$ and $L_i = \pi_i^* L$. Then the same arguments as above give that $\mathcal{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \mathcal{E}^*[[x_1, x_2]]$ where $x_i = e^{\mathcal{E}}(L_i)$ and $L_i = \pi_i^* L$. In fact, this works for any number of factors:

Proposition 38.22. *If $\mathcal{E}$ is a complex-oriented cohomology theory then*

$$\mathcal{E}^*((\mathbb{C}P^\infty)^{\times k}) = \mathcal{E}^*[[x_1, \dots, x_k]]$$

where $x_i = e^{\mathcal{E}}(\pi_i^*(L))$ and $\pi_i \colon (\mathbb{C}P^\infty)^{\times k} \to \mathbb{C}P^\infty$ is the ith projection.

Returning to our map $S \colon (\mathbb{C}P^\infty)^{\times k} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$, the same arguments as before show that S^* factors through the ring of symmetric invariants to give $S^* \colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to \mathcal{E}^*[[x_1, \dots, x_k]]^{\Sigma_k}$. The same algebra as for the polynomial case identifies this ring as $\mathcal{E}^*[[\sigma_1, \dots, \sigma_k]]$. We next work towards proving that S^* is an isomorphism.

Before proceeding further we need an algebraic lemma, which we leave to the reader as an amusing exercise.

Exercise 38.23. Let R be a commutative ring. In $R[x_1, \dots, x_n]$ prove that the symmetric polynomials $\sigma_1, \dots, \sigma_n$ are the unique solutions to the n equations

$$x_k^n + \sum_{i=1}^{n} (-1)^i x_k^{n-i} u_i, \quad i = 1, \dots, n$$

in the variables $u_1, \dots, u_n$. Confirm the same result for $R[[x_1, \dots, x_n]]$ where R is graded. [Hint: For the uniqueness, use a Vandermonde determinant.]

Proposition 38.24. *The map $S^* \colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to \mathcal{E}^*((\mathbb{C}P^\infty)^{\times k})$ sends $c_i^{\mathcal{E}}(\gamma_k)$ to the elementary symmetric function σ_i.*

Proof. Let $X = (\mathbb{C}P^\infty)^{\times k}$ and let $M \to X$ be the bundle $L_1 \oplus \cdots \oplus L_k$, where $L_i = \pi_i^* L$. Since $S^* \gamma_k \cong M$, the result boils down to showing that $c_i^{\mathcal{E}}(M) = \sigma_i$.

Observe that there are maps $s_i \colon X \to \mathbb{P}M$ sending $(\ell_1, \dots, \ell_k)$ to the tuple $(\ell_1, \dots, \ell_k, \ell_i)$. That is to say, if $w = (\ell_1, \dots, \ell_k)$ then each ℓ_i gives us a preferred line in M_w. The composites $X \xrightarrow{s_i} \mathbb{P}M \xrightarrow{\pi} X$ are all equal to the identity.

Set $e = e^{\mathcal{E}}(L_M) \in \mathcal{E}^2(\mathbb{P}M)$. Observe that $s_i^* L_M = L_i$, and therefore $s_i^*(e) = x_i$ where as usual $x_i = e^{\mathcal{E}}(L_i)$. Take the relation $e^k - c_1^{\mathcal{E}}(M) e^{k-1} + \cdots = 0$ and apply s_i^*. This gives

$$x_i^k - c_1^{\mathcal{E}}(M) x^{k-1} + c_2^{\mathcal{E}}(M) x^{k-2} - \cdots = 0$$

in $\mathcal{E}^*[[x_1, \dots, x_k]]$. As this equation holds for each i, Exercise 38.23 says that $c_i^{\mathcal{E}}(M) = \sigma_i$ for all i. □

We need one more auxilliary result:

Proposition 38.25. *For any $N \geq 1$, the classes $c_i^{\mathcal{E}}(\gamma_k)$ in $\mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^N))$ are nilpotent for all $i \geq 1$. Consequently, if $M \to X$ is a complex bundle where X is compact Hausdorff then the $c_i^{\mathcal{E}}(M)$ are all nilpotent when $i \geq 1$.*

Proof. Let Q_k be the quotient of the inclusion $\gamma_k \hookrightarrow \underline{N}$. By the Whitney formula we have $c^{\mathcal{E}}(\gamma_k) \cdot c^{\mathcal{E}}(Q_k) = 1$. This identity can be considered as taking place in the $\mathbb{N}$-graded subring $\mathcal{E}^{\geq 0}(\mathrm{Gr}_k(\mathbb{C}^N))$. By the lemma below, all of the classes $c_i^{\mathcal{E}}(\gamma_k)$ must be nilpotent.

The last statement of the proposition follows by using a classifying map for M, since such a map must factor through a finite Grassmannian. □

Lemma 38.26. *Let R be an $\mathbb{N}$-graded ring, either commutative or graded-commutative. Suppose $d_1, d_2, \ldots, d_k$ are homogeneous elements such that each d_i has degree i and $1 + d_1 + \cdots + d_k$ is invertible in R. Then all of the d_i are nilpotent.*

Proof. Let $1 + e_1 + e_2 + \cdots + e_n$ be the inverse of $1 + \sum_i d_i$ and observe that the top product $e_n d_k$ must be zero. Using that idea it is easy to see that if R is a domain then all of the d_i must be zero

Let P be a homogeneous prime ideal. Then R/P is a domain and $1 + \sum_i d_i$ is still invertible, so by the preceding paragraph P must contain all of the d_i. But elements contained in every homogeneous prime ideal are necessarily nilpotent. □

Exercise 38.27. Think through the proof that in a commutative ring the intersection of all prime ideals is the nilradical, and check that this works in the graded setting in both the commutative and graded-commutative cases.

Proposition 38.28. *The map $S^*\colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to \mathcal{E}^*((\mathbb{C}P^\infty)^{\times k})^{\Sigma_k}$ is an isomorphism.*

Proof. We have previously shown that the map is an injection, the target is $\mathcal{E}^*[[\sigma_1, \ldots, \sigma_k]]$, and all of the σ_i are in the image. Proving surjectivity therefore essentially amounts to proving that the domain is complete with respect to the ideal $(c_1^{\mathcal{E}}(\gamma_k), c_2^{\mathcal{E}}(\gamma_k), \ldots)$.

In the rest of the argument we identify $\mathbb{C}^\infty = \mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty$ (k summands) and $\mathbb{C}^{kN} = \mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N$. Consider the Milnor sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim_N{}^1\, \mathcal{E}^{s-1}(\mathrm{Gr}_k(\mathbb{C}^{kN})) & \longrightarrow & \mathcal{E}^s(\mathrm{Gr}_k(\mathbb{C}^\infty)) & \longrightarrow & \lim_N \mathcal{E}^s(\mathrm{Gr}_k(\mathbb{C}^{kN})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle S^*} & & \downarrow{\scriptstyle S^*} & & \downarrow{\scriptstyle S^*} & & \\
0 & \longrightarrow & \lim_N{}^1\, \mathcal{E}^{s-1}((\mathbb{C}P^{N-1})^{\times k}) & \longrightarrow & \mathcal{E}^s((\mathbb{C}P^\infty)^{\times k}) & \longrightarrow & \lim_N \mathcal{E}^s((\mathbb{C}P^{N-1})^{\times k}) & \longrightarrow & 0
\end{array}$$

The lower $\lim^1$ term is zero because all the maps $\mathcal{E}^*(\mathbb{C}P^N) \to \mathcal{E}^*(\mathbb{C}P^{N-1})$ are surjections. We already know the middle S^* is an injection, and so it follows that the upper $\lim^1$ must also be zero. So the two right horizontal maps are isomorphisms.

Return now to our map $S^*\colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to \mathcal{E}^*[[\sigma_1, \ldots, \sigma_k]] = \mathcal{E}^*((\mathbb{C}P^\infty)^{\times k})^{\Sigma_k}$. Let f be a power series in the target, say of homogeneous degree d. Let g be the same formal power series but with each σ_i replaced with $c_i^{\mathcal{E}}(\gamma_k)$. We will show that g defines an element in $\mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty))$ and that S^* sends this element to f.

By Proposition 38.25 the classes $c_i^{\mathcal{E}}(\gamma_k) \in \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^{kN}))$ are nilpotent. So g represents a compatible system of elements in $\mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^{kN}))$ for each N, i.e. an element of $\lim_N \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^{kN}))$. From the above diagram, this determines a unique element in $\mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty))$. The fact that S^* sends it to f is by commutativity of the diagram, together with the bottom right horizontal map being an isomorphism. Thus, we have proven the desired surjectivity statement. □

38.29. **Summary.** The ordering of subjects in this section has perhaps been a bit convoluted, so it is worth summarizing what happened. To construct the Chern classes in a complex-oriented cohomology theory $\mathcal{E}$ a nice path is:

(1) Prove the projective bundle theorem and then define the Chern classes using Grothendieck's approach.
(2) Use the projective bundle theorem to prove the splitting principle, which implies that $S^*\colon \mathcal{E}^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \to \mathcal{E}^*((\mathbb{C}P^\infty)^{\times k})^{\Sigma_k}$ is injective.
(3) Show that S^* sends the Chern classes to the symmetric functions, and use this to prove the Whitney formula and other properties.
(4) Use the already-constructed Chern classes $c_i^{\mathcal{E}}$ to prove that S^* is also surjective.

All of this can be done without any use of a cell structure on $\mathrm{Gr}_k(\mathbb{C}^\infty)$.

In the special case of $\mathcal{E} = H$ one can use the Schubert cell structure to prove surjectivity of S^* without having first constructed the c_i's, and oftentimes this feels like a more straightforward approach. But for this one has to still prove injectivity of S^*, which is essentially the splitting principle, and not so different from the content of the projective bundle theorem. In this section we first took this path, then went back to projective bundles and the general theory of $\mathcal{E}$-Chern classes.

The K-theoretic Chern classes will be explored in much greater detail in Section 47, but see Exercise 38.33 below for an introduction.

∘ Exercises ∘

Exercise 38.30. Let E and F be complex bundles of ranks r and s. Use the splitting principle to prove the following:

(a) $c_1(E \otimes F) = sc_1(E) + rc_1(F)$.
(b) $c_1(\Lambda^k E) = \binom{r-1}{k-1} c_1(E)$.
(c) Try to find formulas for $c_2(\Lambda^2 E)$ and $c_2(E \otimes F)$ (these get a little hairy).
(d) Convince yourself there are universal polynomials that express $c_i(E \otimes F)$ and $c_i(\Lambda^k E)$ in terms of the c_j for $j \le i$, and that finding them amounts to algebra/combinatorics.

Exercise 38.31. Let s_i be the characteristic class that corresponds to the symmetric function that is the ith power sum $x_1^i + x_2^i + \cdots + x_n^i$. Since the power sum can be expressed as a polynomial in the elementary symmetric functions, $s_i(E)$ can be expressed as the same polynomial in the Chern classes: for example, $s_1(E) = c_1(E)$ and $s_2(E) = c_1(E)^2 - 2c_2(E)$. See Appendix F for more information. By convention we let $s_0(E)$ be the rank of E.

Prove that $s_n(E \oplus F) = s_n(E) + s_n(F)$ and $s_n(E \otimes F) = \sum_{i=0}^n \binom{n}{i} s_i(E) s_{n-i}(F)$. [Use the splitting principle.]

Exercise 38.32. Show that the arguments in this section adapt to prove a real version of the projective bundle theorem for $H^*(-; \mathbb{Z}/2)$ and a corresponding splitting principle for real bundles, yielding that $S^*\colon H^*(\mathrm{Gr}_k(\mathbb{R}^\infty); \mathbb{Z}/2) \to [H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)^{\otimes k}]^{\Sigma_k}$ is an injection. Explain how to get the surjectivity too.

Exercise 38.33. Let E be a complex bundle of rank r. Recall from Section 37 that the K-theoretic Euler class is $e^K(E) = \sum_i (-1)^i [\Lambda^i E^*]$. In particular, when J is a line bundle then $e^K(J) = 1 - J^*$.

(a) Show that $c_1^K(E) = r - E^*$ and $c_2^K(E) = \binom{r}{2} - (r-1)E^* + \Lambda^2 E^*$.

(b) Show that $c_n^K(E) = (??) - (??)E^* + (??)\Lambda^2 E^* - \cdots + (-1)^n \Lambda^n E^*$ where the coefficients are for you to determine.
(c) Think about the axiomatic properties of Chern classes and try to give algebraic proofs for why your formulas from (b) satisfy them.

39. Comparing K-theory and singular cohomology

We have seen that singular cohomology and K-theory both encode geometry in similar ways: they have Thom classes, Euler classes, fundamental classes for submanifolds, etc. They can both be used to compute intersection multiplicities. One might hope for a natural transformation from one to the other, that allows one to directly compare what is happening in each theory. Our goal in this section is to construct such a natural transformation, with some caveats that we will discover along the way.

Let us imagine that we have a natural transformation $\phi\colon K^*(-) \to H^*(-)$, and that this is a ring homomorphism. Note first that ϕ cannot preserve the gradings, for $\beta \in K^{-2}(pt)$ is a unit whereas there is no unit in $H^{-2}(pt)$. We can fix this by formally adjoining a unit of the appropriate degree to H^*. When X is a finite CW-complex this amounts to looking at the theory $H^*(X)[t, t^{-1}]$ where t is given degree -2, but that description does not satisfy the wedge axiom. For general spaces X the correct description is $H^*(X)[[t, t^{-1}]]$, using our usual conventions about graded power series. Note that since $H^*(X)$ vanishes in negative degrees a homogeneous power series in t and t^{-1} will only involve finitely-many powers of t^{-1}.

Remark 39.1. If $H\mathbb{Z}$ denote the integral Eilenberg-MacLane spectrum, the natural map $\vee_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Z} \to \prod_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Z}$ is an isomorphism on homotopy groups, and therefore a weak equivalence. The cohomology theory represented by this spectrum is $H^n_{per}(X) = \prod_{i\in\mathbb{Z}} H^{n+2i}(X) = H^*(X)[[t, t^{-1}]]$.

So now we are going to ask for a natural ring homomorphism $\phi\colon K^*(-) \to H^*(-)[[t, t^{-1}]]$. Restricting to $* = 0$ would give a

$$\phi\colon K^0(-) \to H^{ev}(-) = \prod_i H^{2i}(-).$$

We will investigate what this map can look like.

If $J \to X$ is a complex line bundle then we have the element $e^K(J) \in K^0(X)$, in some sense representing the intersection of the zero-section with itself. One's first guess would be that ϕ should send $e^K(J)$ to $e^H(J)$, as the latter represents the same 'geometry' inside of H^*. However, this hypothesis is not compatible with ϕ being a ring homomorphism. Recall from Proposition 37.17 that

$$e^K(J_1 \otimes J_2) = e^K(J_1) + e^K(J_2) - e^K(J_1)e^K(J_2),$$

whereas

$$e^H(J_1 \otimes J_2) = e^H(J_1) + e^H(J_2).$$

These formulas are incompatible.

So it cannot be that ϕ sends $e^K(J)$ to $e^H(J)$. However, it is guaranteed that $e^K(J)$ must be sent to *some* algebraic expression involving $e^H(J)$. Indeed, this is obviously so for the tautological bundle $L \to \mathbb{C}P^\infty$, since $e^H(L)$ is a generator of $H^2(\mathbb{C}P^\infty)$ and everything else in $H^*(\mathbb{C}P^\infty)$ is a polynomial in this generator; the case for general line bundles then follows from naturality.

So we know that ϕ will send $e^K(J) \mapsto f(e^H(J))$ for some $f(x) \in \mathbb{Z}[[x]]$. Note that when X is compact then sufficiently large powers of $e^H(J)$ will be zero (since the classifying map factors through a finite $\mathbb{C}P^n$), so in practice $f(e^H(J))$ is really just a *polynomial* in $e^H(J)$. Even if we were only interested in finite CW-complexes, using a power series allows us to treat all such spaces at once, without assuming some uniform bound on their dimensions.

Let $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$ be the expansion for f. Note that if $J \to X$ is a trivial bundle then both $e^K(J)$ and $e^H(J)$ are zero, and from this it follows that $\alpha_0 = 0$. Next observe that if ϕ is a ring homomorphism then we must have

$$\begin{aligned} f(e^H(J_1) + e^H(J_2)) &= f(e^H(J_1 \otimes J_2)) \\ &= \phi(e^K(J_1 \otimes J_2)) \\ &= \phi(e^K(J_1) + e^K(J_2) - e^K(J_1)e^K(J_2)) \\ &= f(e^H(J_1)) + f(e^H(J_2)) - f(e^H(J_1))f(e^H(J_2)). \end{aligned}$$

This suggests that we're looking for $f(x) \in \mathbb{Z}[[x]]$ such that

$$f(a+b) = f(a) + f(b) - f(a)f(b). \tag{39.2}$$

We can take two approaches to determine the coefficients of such an f.

Approach 1. Substitute $f(x) = \sum_i \alpha_i x^i$ into (39.2) to get

$$\text{LHS} = \alpha_1(a+b) + \alpha_2(a+b)^2 + \dots$$

and

$$\begin{aligned} RHS = [\alpha_1 a + \alpha_2 a^2 + \dots] + [\alpha_1 b + \alpha_2 b^2 + \dots] \\ - [\alpha_1 a + \alpha_2 a^2 + \dots][\alpha_1 b + \alpha_2 b^2 + \dots]. \end{aligned}$$

By expanding and equating coefficients, we can try to determine the coefficients α_i. It turns out there is no equation determining α_1, but looking at the coefficient of ab^{n-1} yields $n\alpha_n = -\alpha_1\alpha_{n-1}$, or $\alpha_n = -\frac{\alpha_1\alpha_{n-1}}{n}$. So by induction $\alpha_n = (-1)^{n-1}\frac{\alpha_1^n}{n!}$. This last equation shows that f cannot have integral coefficients, as we were orginally guessing! So we can only make things work if the target of ϕ is $H^{ev}(-;\mathbb{Q})$.

We have been led to the conclusion $f(x) = 1 - e^{-\alpha_1 x}$, and the reader may readily check that this does indeed yield a power series $f(x)$ satisfying (39.2).

Approach 2. In case you don't like the "equating coefficients" approach, one can also use some basic tools from differential equations to determine f. Recall that we want $f(a+b) = f(a)+f(b)-f(a)f(b)$. Define functions g and h by $g(a,b) = f(a+b)$ and $h(a,b) = f(a) + f(b) - f(a)f(b)$. The partial derivatives are readily computed to be

$$\frac{\partial g}{\partial a}(a,b) = f'(a+b) \quad \text{and} \quad \frac{\partial h}{\partial a}(a,b) = f'(a) - f'(a)f(b).$$

If $g(a,b)$ and $h(a,b)$ are the same function then the above partial derivatives are the same, so that $f'(a+b) = f'(a) - f'(a)f(b)$. Evaluating at $a = 0$ gives the ODE

$$f'(b) = f'(0)[1 - f(b)].$$

Setting $y = f(x)$, this becomes the separable ODE

$$\frac{dy}{dx} = f'(0)(1-y), \qquad \text{or} \qquad \frac{dy}{1-y} = f'(0)\,dx.$$

Integrating both sides yields

$$-\ln(1-y) = f'(0)x + C, \qquad \text{or} \qquad y = 1 - De^{-f'(0)x}$$

where C and D are constants. Since $f'(0) = \alpha_1$, we will write this solution as $f(x) = y = 1 - De^{-\alpha_1 x}$. We did lose some information in the differentiation process, so let's make sure this works by plugging this formula back into (39.2). We get

$$1 - De^{-\alpha_1(a+b)} = [1 - De^{-\alpha_1 a}] + [1 - De^{-\alpha_1 b}] - [1 - De^{-\alpha_1 a}][1 - De^{-\alpha_1 b}],$$

which reduces to

$$De^{-\alpha_1(a+b)} = D^2 e^{-\alpha_1(a+b)}.$$

This implies $D = D^2$, so $D = 0$ or $D = 1$. The case $D = 0$ is uninteresting to us (it corresponds to $f(x) = 1$, and we have already noted that the constant term must be zero for our application). So $D = 1$ and $f(x) = 1 - e^{-\alpha_1 x}$.

We now comment on the fact that α_1 seems to be able to take on any value whatsoever. Note that the presence of the grading on $H^*(X)$ immediately gives rise to a collection of endomorphisms on this theory. Indeed, for any $n \in \mathbb{Z}$ write $\psi_n\colon H^*(X) \to H^*(X)$ for the function that multiplies each $H^i(X)$ by n^i. This is clearly a ring homomorphism, and if we are using rational coefficients then it is even an isomorphism (provided $n \neq 0$). Note that with rational coefficients we actually have maps ψ_q for any $q \in \mathbb{Q}$.

So if we have a natural transformation $\phi\colon K^*(-) \to H^*(-;\mathbb{Q})[t,t^{-1}]$ we can compose it with the natural automorphisms ψ_q to makes lots of other natural transformations. We see that such a ϕ is far from unique. If we had a ϕ whose associated power series f was $f(x) = 1 - e^{-\alpha_1 x}$, then composing with ϕ_q gives one with associated power series $1 - e^{-q\alpha_1 x}$. This is why α_1 could not be explicitly determined.

We can turn these observations around and use them to our advantage. Since we can always compose with a ψ_q, we might as well do so in a way that simplifies things as much as possible. In particular, if we have a ϕ with associated power series $f(x) = 1 - e^{-\alpha_1 x}$ then we can compose with $\psi_{\alpha_1^{-1}}$ to get one with power series $1 - e^{-x}$. We might as well do this, to simplify matters.

Let us summarize what has happened so far. We knew that ϕ, if it exists, must send $e^K(J)$ to some power series in $e^H(J)$, for any line bundle $J \to X$. The different equations for $e(J_1 \otimes J_2)$ in K-theory versus singular cohomology then forced what this power series must be: $\phi(e^K(J)) = 1 - e^{-\alpha_1 x}|_{x=e^H(J)}$, for some $\alpha_1 \in \mathbb{Q}$. We then saw that we might as well assume $\alpha_1 = 1$, since by composing with a certain "trivial" automorphism one can arrange for this.

So now we are looking at an imagined natural transformation ϕ that sends $e^K(J)$ to $1 - e^{-x}|_{x=e^H(J)}$ for any line bundle $J \to X$. Recall that $e^K(J) = 1 - J^*$, and so $\phi(J^*) = e^{-x}|_{x=e^H(J)}$. But $e^H(J^*) = -e^H(J)$ (use that $J \otimes J^* \cong \underline{1}$, and so $e^H(J) + e^H(J^*) = e^H(\underline{1}) = 0$). So we have $\phi(J^*) = e^x|_{x=e^H(J^*)} = e^{c_1(J^*)}$. Since

this must hold for any line bundle J, we might as well just write it as

$$\phi(J) = e^{c_1(J)}. \tag{39.3}$$

We next claim that ϕ is completely determined by formula (39.3). Recall the direct sum map $S\colon (\mathbb{C}P^\infty)^{\times k} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ from Section 16.18, classifying the bundle $\bigoplus_i \pi_i^*(L)$. Consider the diagram

$$\begin{array}{ccc} K^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) & \xleftarrow{S_K^*} & K^0(\mathrm{Gr}_k(\mathbb{C}^\infty)) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ H^{ev}(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty; \mathbb{Q}) & \xleftarrow{S_H^*} & H^{ev}(\mathrm{Gr}_k(\mathbb{C}^\infty); \mathbb{Q}). \end{array}$$

We know that $S^*\gamma_k \cong \pi_1^*(L) \oplus \cdots \oplus \pi_k^*(L)$, and therefore we see that

$$S_H^*(\phi(\gamma_k)) = \phi(S_K^*\gamma_k) = \sum_{i=1}^k \phi(\pi_i^*(L)) = \sum_{i=1}^k \pi_i^*(\phi(L)) = \sum_{i=1}^k e^{\pi_i^*(c_1(L))}.$$

Clearly this expression is invariant under the action of Σ_k, and we know from Proposition 38.17 that S_H^* maps its domain isomorphically onto the subring of Σ_k-invariants. Thus, $\phi(\gamma_k)$ is determined by this formula.

Let $x_i = \pi_i^*(c_1(L))$. The power sum $x_1^r + \cdots + x_k^r$ can be written uniquely as a polynomial $s_r(\sigma_1, \ldots, \sigma_k)$ in the elementary symmetric functions of the x_i's. Here s_r is called the **rth Newton polynomial**; see Appendix F for a review of these. The first few Newton polynomials are

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

If $E \to X$ is a vector bundle then define $s_r(E) = s_r(c_1(E), \ldots, c_k(E)) \in H^{2r}(X)$, where $k = \operatorname{rank} E$. This is a characteristic class for bundles, but it doesn't seem to have a common name. We have seen that

$$\phi(\gamma) = \sum_{k=0}^\infty \frac{1}{k!} s_k(\gamma).$$

But if $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is a classifying map for E then the commutative diagram

$$\begin{array}{ccc} K^0(X) & \xleftarrow{f^*} & K^0(\mathrm{Gr}_k(\mathbb{C}^\infty)) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ H^{ex}(X; \mathbb{Q}) & \xleftarrow{f^*} & H^{ev}(\mathrm{Gr}_k(\mathbb{C}^\infty); \mathbb{Q}) \end{array}$$

gives

$$\phi(E) = \phi(f^*(\gamma)) = f^*(\phi(\gamma)) = f^*\Big(\sum_{k=0}^\infty \tfrac{s_k(\gamma)}{k!}\Big) = \sum_{k=0}^\infty \tfrac{s_k(f^*\gamma)}{k!} = \sum_{k=0}^\infty \tfrac{s_k(E)}{k!}.$$

We have, at this point, reasoned as follows. IF there is a natural transformation of rings $\phi\colon K^0(-) \to H^{ev}(-; \mathbb{Q})[[t, t^{-1}]]$ THEN there is one that is given by the above formula. One can turn this around, by starting with the above formula and proving that it is a natural transformation of rings. This is not hard, and we will

leave it to the reader. This natural transformation is called the **Chern character**, and is usually denoted $\mathrm{ch}\colon K^0(-) \to H^{ev}(-,\mathbb{Q})$. The defining formula is

$$\mathrm{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} s_k(E). \tag{39.4}$$

Exercise 39.5. Prove directly from the properties of Chern classes that the definition from (39.4) gives a ring homomorphism $K^0(X) \to H^{ev}(X;\mathbb{Q})$. [See Exercise 38.31].

Since ch is a natural transformation, it of course maps $\widetilde{K}^0(X)$ into $\tilde{H}^{ev}(X;\mathbb{Q})$. Replacing X with ΣX and shifting indices, we get

$$\mathrm{ch}\colon K^{-1}(X) \to H^{odd}(X;\mathbb{Q}).$$

By periodicity we might as well regard the Chern character as giving maps

$$\mathrm{ch}\colon K^n(X) \to \prod_p H^{n+2p}(X;\mathbb{Q}).$$

Equivalently, ch as a map of graded rings $K^*(X) \to H^*(X;\mathbb{Q})[[t,t^{-1}]]$ where t has degree -2.

Theorem 39.6. *The induced maps $K^n(X)\otimes\mathbb{Q} \to \prod_p H^{n+2p}(X;\mathbb{Q})$ are isomorphisms, for all spaces X.*

Proof. We have a natural transformation between two cohomology theories, so it is sufficient just to check that we have an isomorphism when X is a point. The isomorphism for general spaces then follows formally using long exact sequences, homotopy invariance, direct limits, etc.

When X is a point our map is $\mathbb{Q}[\beta,\beta^{-1}] \to \mathbb{Q}[t,t^{-1}]$, so it will suffice just to check that β is not send to zero. So we need to analyze $\mathrm{ch}\colon K^{-2}(pt)\otimes\mathbb{Q} \to H^{-2}_{per}(pt;\mathbb{Q})$. Via the suspension isomorphism, this is the same as $\mathrm{ch}\colon \widetilde{K}^0(S^2)\otimes\mathbb{Q} \to \widetilde{H}^0_{per}(S^2;\mathbb{Q}) = t\cdot H^2(S^2;\mathbb{Q})$. The generator for the domain is $1-L$, and $\mathrm{ch}(1-L) = -c_1(L)$ which is the generator of $H^2(S^2;\mathbb{Z})$. So $\mathrm{ch}\colon K^{-2}(pt;\mathbb{Q}) \to H^{-2}_{per}(pt;\mathbb{Q})$ is an isomorphism, and we are done. □

Of course a natural transformation of cohomology theories immediately extends to a natural transformation defined on pairs, using the standard techniques. If (X,A) is a pair let Cj denote the mapping cone of the inclusion $j\colon A \hookrightarrow X$. Let $\{U_j, V_j\}$ be the usual open cover of Cj where U_j consists of X and a little more than the bottom half of the cone and V_j is a little more than the top half of the cone (so that $U_j \simeq X$, $V_j \simeq *$, and $U_j \cap V_j \simeq A$). Note that all of this structure is functorial in (X,A). We have the zig-zag of natural isomorphisms

$$K^*(X,A) \xrightarrow{\cong} K^*(U_j, U_j\cap V_j) \xleftarrow{\cong} K^*(C_j, V_j) \xrightarrow{\cong} K^*(C_j, pt) = \widetilde{K}^*(C_j)$$

and likewise for $H^*(-;\mathbb{Q})[[t,t^{-1}]]$. Starting with our existing map $\mathrm{ch}\colon \widetilde{K}^*(C_j) \to H^*(C_j;\mathbb{Q})[[t,t^{-1}]]$ we define $\mathrm{ch}\colon K^*(X,A) \to H^*(X,A;\mathbb{Q})[[t,t^{-1}]]$ to be the unique map that is compatible with the above isomorphisms.

This raises the following interesting question. Suppose $E_\bullet$ is a complex of vector bundles on X that is exact on A. Then $\mathrm{ch}([E_\bullet])$ is a well-defined element of $H^*(X,A;\mathbb{Q})$. How does one describe what this element is? See [Iversen] for

the topological case, and Baum, Fulton, and MacPherson [BFM1, BFM2] for the algebro-geometric setting using Chow groups. A generalization of the latter appears in [Fulton], and a very algebraic description is in [Roberts5].

The following result is often useful:

Proposition 39.7. *For any $n \geq 1$, the image of* $\mathrm{ch}\colon \widetilde{K}^0(S^{2n}) \to \widetilde{H}^{ev}(S^{2n};\mathbb{Q})$ *is precisely $\widetilde{H}^{ev}(S^{2n};\mathbb{Z})$.*

Proof. Recall from Theorem 27.10 that $\widetilde{K}^0(S^{2n})$ is generated by $\beta^{\times n}$ where $\beta = 1-L \in \widetilde{K}^0(S^2)$. We can compute that $\mathrm{ch}(\beta) = 1-\mathrm{ch}(L) = 1-(1+c_1(L)) = -c_1(L)$, but this is a generator of $H^2(S^2;\mathbb{Z})$. Multiplicativity of the Chern character gives

$$\mathrm{ch}(\beta^{\times n}) = \big(\mathrm{ch}(\beta)\big)^{\times n} = c_1(L)^{\times n},$$

but the nth external product of a generator for $H^2(S^2;\mathbb{Z})$ gives a generator of $H^{2n}(S^{2n};\mathbb{Z})$. This completes the proof. □

Proposition 39.8. *If X is any $(2n-1)$-connected space and $E \to X$ is a complex vector bundle then $c_n(E) \in H^{2n}(X;\mathbb{Z})$ is a multiple of $(n-1)!$.*

Proof. We first verify the result when $X = S^{2n}$. Note that if $E \to S^{2n}$ is a complex vector bundle then $\mathrm{ch}(E) = \frac{1}{n!}\cdot s_n(E)$, by definition of the Chern character. The Newton identities from Proposition F.1 show that $s_n(E) = (-1)^{n+1}n\cdot c_n(E)$, since in this case $c_1(E),\dots,c_{n-1}(E)$ must all vanish. By Proposition 39.7 we know that ch takes its image in $H^{2n}(S^{2n};\mathbb{Z})$, and so $\frac{nc_n(E)}{n!}$ is integral; that is, $c_n(E)$ is a multiple of $(n-1)!$.

Now let X be any $(2n-1)$-connected space. Replacing X by a weakly equivalent space, we can assume X has a cell structure with no cells of degree smaller than $2n$; that is, the $2n$-skeleton is a wedge of $2n$-spheres. Consider the cofiber sequence $\vee S^{2n} \hookrightarrow X \to C$ where C is the cofiber. In cohomology we get

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \prod \widetilde{K}^0(S^{2n}) & \longleftarrow & \widetilde{K}^0(X) & \longleftarrow & \widetilde{K}^0(C) \longleftarrow \cdots \\
 & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\
\cdots & \longleftarrow & \prod \widetilde{H}^{ev}(S^{2n};\mathbb{Q}) & \xleftarrow{j^*} & \widetilde{H}^{ev}(X;\mathbb{Q}) & \longleftarrow & \widetilde{H}^{ev}(C;\mathbb{Q}) \longleftarrow \cdots
\end{array}$$

Given a complex vector bundle $E \to X$ we know by the commutativity of the diagram that $j^*(\mathrm{ch}_n(E))$ lies in $\prod \widetilde{H}^{ev}(S^{2n};\mathbb{Z})$. But since $H^{2n}(C;\mathbb{Q}) = 0$ the map j^* is injective in degree $2n$, and it is easy to see that $(j^*)^{-1}(H^{2n}(\vee S^{2n};\mathbb{Z})) = H^{2n}(X;\mathbb{Z})$ (if a cellular $2n$-cochain takes integral values on all of the $2n$-cells, it is integral). So $\mathrm{ch}_n(E)$ is an integral class. The same computation as in the previous paragraph shows that $\mathrm{ch}_n(E) = \pm\frac{c_n(E)}{(n-1)!}$, and this completes the proof. □

The space $\mathbb{C}P^1$ is a complex manifold whose underlying topological manifold is S^2. Can any other spheres be given the structure of a complex manifold? Of course this is only interesting for the even spheres. A simple corollary of the previous result rules out almost all possibilities:

Corollary 39.9. *If $n \geq 4$ then there is no complex structure on S^{2n}. Even more, there is no complex vector bundle on S^{2n} whose underlying real bundle is the tangent bundle $T_{S^{2n}}$ (said differently, the tangent bundle $T_{S^{2n}}$ does not admit a complex structure).*

Proof. The second statement clearly implies the first. Let $T = T_{S^{2n}}$ and suppose that T has a complex structure. By Proposition 39.8 we know that $c_n(T)$ is a multiple of $(n-1)!$ in $H^{2n}(S^{2n};\mathbb{Z})$. But $c_n(T)$ is the Euler class of the underlying real bundle, and therefore it is twice a generator since $\chi(S^{2n}) = 2$. This implies that $\frac{2}{(n-1)!}$ is an integer, which clearly cannot happen if $n \geq 4$. □

Remark 39.10. It is also known that S^4 cannot have a complex structure on its tangent bundle (and is therefore not a complex manifold); we will give a proof in Example 42.13 below, using the Todd genus. So the last remaining sphere we have not yet discussed is S^6.

It turns out that the tangent bundle to S^6 *does* admit a complex structure—one says that S^6 is an *almost complex manifold*. The construction is outlined in Exercise 39.13. Whether or not S^6 admits the structure of a complex manifold is a famous open problem.[1]

We close this section with an example showing how the Chern character can help us carry out the calculation of K-groups.

Example 39.11. Recall that $\mathbb{C}P^2$ is the mapping cone of the Hopf map $\eta\colon S^3 \to S^2$. Since the suspension of η is 2-torsion ($\pi_4(S^3) \cong \mathbb{Z}/2$), a choice of null-homotopy for $\eta \circ 2$ gives a map $f\colon \Sigma^3\mathbb{R}P^2 \to S^3$ which coincides with η when restricted to the bottom cell. Let X be the cofiber of f; this is a cell complex with a 3-cell, a 5-cell, and a 6-cell. The 5-skeleton of X is $\Sigma\mathbb{C}P^2$. Our goal will be to compute the groups $\widetilde{K}^*(X)$. [Note: This choice of X, which seemingly has come out of nowhere, is motivated by the fact that this is in some sense the smallest space for which $\widetilde{K}^*(X)$ and $\widetilde{H}^*(X)$ have different torsion subgroups—see the discussion of the Atiyah-Hirzebruch spectral sequence in [D7] for a deeper perspective.]

There are two cofiber sequences that we can exploit: $S^3 \hookrightarrow X \to \Sigma^4\mathbb{R}P^2$ and $\Sigma\mathbb{C}P^2 \hookrightarrow X \to S^6$. We leave it to the reader to compute that $\widetilde{K}^0(\mathbb{R}P^2) = \mathbb{Z}/2$ and $K^1(\mathbb{R}P^2) = 0$, using that $\mathbb{R}P^2$ is the cofiber of $S^1 \xrightarrow{2} S^1$. Using this, the first cofiber sequence gives

$$0 \longleftarrow\!\!= \widetilde{K}^0(S^3) \longleftarrow \widetilde{K}^0(X) \longleftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z} \longleftarrow K^1(X) \longleftarrow 0$$

(recall that $K^1(X) = K^{-1}(X)$ by periodicity). Note that we immediately deduce $K^1(X) \cong \mathbb{Z}$, and $\widetilde{K}^0(X)$ is either 0 or $\mathbb{Z}/2$. However, it is not clear how to analyze the map $\mathbb{Z} \to \mathbb{Z}/2$. The second cofiber sequence gives

$$0 \longleftarrow \widetilde{K}^0(X) \longleftarrow \mathbb{Z} \xleftarrow{\delta_K} \mathbb{Z}^2 \longleftarrow K^1(X) \longleftarrow 0$$

where the labelled map is the connecting homomorphism $\delta_K\colon \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2) \to \widetilde{K}^0(S^6)$. Again, we are left with the task of determining this map; agreement with the previous (partial) calculation demands that the cokernel either be 0 or $\mathbb{Z}/2$, and we need to determine which one. The good news is that because the domain and target are both torsion-free, there is a chance that the Chern character

[1] An AI-generated solution to this problem—in the affirmative—was just recently announced, August 2026.

will give us the information we need. We will examine the commutative square

$$\begin{array}{ccccc} \mathbb{Z}^2 & \xrightarrow{\cong} & \widetilde{K}^{-1}(\Sigma\mathbb{C}P^2) & \xrightarrow{\delta_K} & \widetilde{K}^0(S^6) \xrightarrow{\cong} \mathbb{Z} \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ & & \widetilde{H}^{odd}(\Sigma\mathbb{C}P^2;\mathbb{Q}) & \xrightarrow{\delta_H} & \widetilde{H}^{ev}(S^6;\mathbb{Q}). \end{array}$$

The two Chern character maps are injective because they are rational isomorphisms and the domains are torsion-free.

Recall that $K^0(\mathbb{C}P^2) = \mathbb{Z}[Y]/(Y^3)$ where $Y = 1 - L$. Let u be the standard generator for $H^2(\mathbb{C}P^2)$, so that $c_1(L) = -u$. We have

$$ch(Y) = 1 - \mathrm{ch}(L) = 1 - (1 - u + \tfrac{u^2}{2}) = u - \tfrac{u^2}{2}, \qquad \mathrm{ch}(Y^2) = \mathrm{ch}(Y)^2 = u^2.$$

Write Y_1 and Y_2 for the suspensions of Y and Y^2, lying in $\widetilde{K}^1(\Sigma\mathbb{C}P^2)$; likewise, write u_1 and u_2 for the suspensions of u and u^2 in $H^*(\Sigma\mathbb{C}P^2;\mathbb{Z})$. Compatibility of the Chern character with suspension shows that

$$\mathrm{ch}(Y_1) = u_1 - \tfrac{u_2}{2}, \qquad \mathrm{ch}(Y_2) = u_2.$$

We must next compute the images of these classes under δ_H. But this is easy from the long exact sequence for $\Sigma\mathbb{C}P^2 \hookrightarrow X \to S^6$: one finds that $\delta(u_1) = 0$ and $\delta(u_2)$ is twice a generator in $H^6(S^6)$. So the subgroup $\langle \delta_H(\mathrm{ch}(Y_1)), \delta_H(\mathrm{ch}(Y_2))\rangle \subseteq H^*(S^6;\mathbb{Q})$ equals the subgroup $H^6(S^6;\mathbb{Z}) \subseteq H^6(S^6;\mathbb{Q})$. Finally, recall from Proposition 39.8 that the image of $\mathrm{ch}\colon \widetilde{K}^0(S^6) \to H^{ev}(S^6;\mathbb{Q})$ is also equal to $H^6(S^6;\mathbb{Z})$. It follows that δ_K is surjective, and so the cokernel of δ_K is zero. This completes our calculation: $\widetilde{K}^0(X) = 0$.

To appreciate the significance of this example, note that $\widetilde{H}^{ev}(X) \cong \mathbb{Z}/2$ (concentrated in degree 6) and $\widetilde{H}^{odd}(X) \cong \mathbb{Z}$ (concentrated in degree 3). The corresponding K-groups are $\widetilde{K}^0(X) \cong 0$ and $\widetilde{K}^1(X) \cong \mathbb{Z}$. It is a general fact that all torsion-free summands in $H^*(X)$ will also appaear in $K^*(X)$, as this follows from Theorem 39.6. But the present example demonstrates that the torsion subgroups of $H^*(X)$ and $K^*(X)$ can be different.

Exercise 39.12. If one is willing to work stably, there is a map $f\colon S^4 \to \mathbb{C}P^2$ whose cofiber is $\Sigma^{-1}X$. The induced map $f^*\colon \widetilde{K}^0(\mathbb{C}P^2) \to \widetilde{K}^0(S^4)$ is the map $\mathbb{Z}^2 \to \mathbb{Z}$ from the above example. Use the Chern character to give an alternative argument that this map is nonzero ("alternative" in the sense of not just quoting what we have already done, though the steps will be very similar).

∘ Exercises ∘

Exercise 39.13. In this exercise you will prove that TS^6 may be given a complex structure—that is, that S^6 is an almost complex manifold. The construction will make use of the $\mathbb{R}$-algebra $\mathbb{O}$ of octonions. This algebra is not associative but it is alternative: $a(ab) = (aa)b$ and $(ab)b = a(bb)$ for all a and b. The algebra comes with an involution $x \mapsto x^*$ that is $\mathbb{R}$-linear and satisfies $(xy)^* = y^*x^*$ and $x + x^* \in \mathbb{R}$. For any $x \in \mathbb{O}$ the element xx^* is a nonnegative real multiple of the identity element 1, and is zero only when $x = 0$. Define $||x|| = \sqrt{xx^*}$

Define the real part of an octonion by $\mathrm{Re}(x) = \frac{x+x^*}{2}$, and the imaginary part by $\mathrm{Im}(x) = x - \mathrm{Re}(x)$. Note that $\mathrm{Re}(x^*) = \mathrm{Re}(x)$ for all x. Two important properties are $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ and $\mathrm{Re}(x(yz)) = \mathrm{Re}((xy)z)$ for all $x, y, z \in \mathbb{O}$. [Note that this does not imply, for example, that $\mathrm{Re}(a(bc)) = \mathrm{Re}(a(cb))$.]

(a) Define a bilinear form on $\mathbb{O}$ by $\langle x, y\rangle = \mathrm{Re}(xy^*)$. Check that this is symmetric and positive definite, and the associated norm is $||-||$. If p is an imaginary unit vector check that $\langle px, py\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{O}$.
(b) Verify that if $x \in \mathbb{O}$ is an imaginary unit vector then $x^2 = -1$.
(c) Regard S^6 as the unit sphere inside the imaginary part of $\mathbb{O}$. Define $J\colon TS^6 \to TS^6$ by $J_p(v) = pv$. Check that this is well-defined: if $p \in S^6$ and $v \in T_pS^6$ then $pv \in T_pS^6$ as well. (Hint: Given that $\langle 1, v\rangle = 0 = \langle p, v\rangle$ prove that $\langle 1, pv\rangle = 0 = \langle p, pv\rangle$.)
(d) Verify that $J^2 = -Id$, so that J gives a complex structure on the tangent bundle to S^6.

40. Interlude: complex-oriented maps

In mathematics, sometimes a phenomenon appears where at first we only see its shadows. Eventually we are convinced that there is something real there, but our attempts to pin it down with mathematical rigor feel clunky and inefficient. Only over time do we eventually find the "right" way of approaching it, where the mathematics feels like poetry.

For the notion of complex-oriented maps that we develop in this section, things are at the "clunky" stage (at least the way we present them here). The idea is simple enough. For a closed embedding of manifolds with a complex structure on the normal bundle, we can make pushforward maps in complex-oriented cohomology theories. Are there related hypotheses that can be put on maps that are not embeddings, that still lead to pushforwards? The answer is "yes"! To any smooth map of manifolds f one can attach a so-called "stable normal bundle", which is really a homotopy-theoretic construct and not a geometric bundle. A complex-orientation on f is in some sense a "complex structure on the stable normal bundle", except that while this sounds very geometric it doesn't exactly make sense except at a homotopy-theoretic level. Our goal in this section is to pull this ethereal concept from the heavens down into the concrete world of geometry, but this is where the clunkiness comes in.

This "clunkiness" often occurs when trying to force homotopy theory and geometry to interact, and maybe it is inevitable. But the definition of complex-oriented map that we give in this section is definitely jury-rigged, and it is natural to wonder if we just don't yet know the right way of accessing this concept. It will sometimes be painfully apparent that we are having to work too hard to establish the basic facts.

A good way to approach these next two sections is to read them for the basic ideas, but don't dwell on the fine details until the day you really need them.

40.1. **Pullback squares of smooth manifolds.** We start with some background from smooth manifold theory that will be needed frequently over the next few sections. One piece of notation: when $i\colon X \hookrightarrow Y$ is a smooth embedding it will often be convenient to write N_i for the normal bundle $N_{Y/X}$. Recall that this is the quotient of the inclusion $Di\colon T_X \to i^*T_Y$.

Consider a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{40.2}$$

where X, Y, and Y' are smooth manifolds and where f and g are smooth. We say this is a **transverse pullback** if for every pair $x \in X$ and $y' \in Y'$ such that $f(x) = g(y') = y$, we have

$$Df_x(T_xX) + Dg_{y'}(T_{y'}Y') = T_yY.$$

Two common examples of transverse pullbacks are (1) any pullback square where either f or g is a submersion (a surjection on all tangent spaces), and (2) f and g are inclusions of submanifolds having the property that at any point of intersection the two tangent spaces span the ambient tangent space. Observe that the above pullback square is transverse if and only if the associated pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'\times f'} & X \times Y' \\ {\scriptstyle fg'=gf'}\downarrow & & \downarrow{\scriptstyle f\times g} \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

is transverse—this is another form of the "reduction to the diagonal" principle. So the theory of transverse pullback squares is in some sense reduced to the case where the g in (40.2) is a closed embedding.

If (40.2) is transverse this guarantees that X' is a smooth manifold, g' and f' are smooth, and the evident sequence

$$0 \to T_{(x,y')}X' \to T_xX \oplus T_{y'}Y' \to T_yY \to 0$$

is exact, for all x, y, y' as above. This follows from the above reduction-to-the-diagonal move together with [GP, first Theorem of Chapter 1.5] (or see [Kosinski, IV.1.4] for the same result). The exactness of the above sequence is equivalent to saying that

$$\begin{array}{ccc} T_{(x,y')}X' & \xrightarrow{Dg'} & T_xX \\ {\scriptstyle Df'}\downarrow & & \downarrow{\scriptstyle Df} \\ T_{y'}Y' & \xrightarrow{Dg} & T_yY \end{array} \tag{40.3}$$

is both a pullback and a pushout, and this is often a more convenient way to package the information. In particular, it yields the following useful fact:

Proposition 40.4. *Assume that (40.2) is a transverse pullback square where f is a closed embedding. Then f' is a closed embedding and the map $Dg\colon N_{f'} \to (g')^*N_f$ is an isomorphism of bundles.*

Proof. The pullback of a closed embedding is always another closed embedding. The normal bundles are the cokernels of $TX' \to (f')^*TY'$ and $TX \to f^*TY$, and so the desired isomorphism follows from (40.3) being a pushout. □

Remark 40.5 (Clean pullbacks)**.** A pullback square (40.2) is said to be **clean** if all of the spaces are smooth manifolds, all the maps are smooth, and all the squares (40.3) are pullbacks. This is weaker than being transverse; a good example to keep in mind is two lines in $\mathbb{R}^3$ intersecting in a single point. A non-example is the parabola $y = x^2$ and the x-axis intersecting at the origin in $\mathbb{R}^2$.

For a clean pullback square we have the exact sequence of bundles

$$0 \to TX' \to (g')^*TX \oplus (f')^*TY' \to (fg')^*TY.$$

Exactness implies that the last map must have locally constant rank, and so the cokernel is a vector bundle on X' (Corollary 12.8). This is called the **excess bundle** (or **excess normal bundle**) for the pullback square. The pullback is transverse if and only if the excess bundle is zero. Loosely speaking, for a point in X' the fiber of the excess bundle is the space of tangent directions in Y that are not involved in the "intersection problem" represented by the pullback. For the example of two lines in $\mathbb{R}^3$ intersecting in a point, the excess bundle over the intersection point is the orthogonal complement of the plane containing the two lines. See Section 41.24 for more discussion of clean intersections.

Now consider a transverse pullback square (40.2) where f and f' are closed embeddings. Let N and N' be the corresponding normal bundles, and let c denote their (equal) rank. Let U be a tubular neighborhood of X in Y. There is a tubular neighborhood U' of X', and by shrinking it appropriately we can assume $U' \subseteq g^{-1}(U)$. Assume that $\mathcal{E}$ is a cohomology theory where we have Thom classes for N and N', and that the isomorphism $Dg\colon N' \to (g')^*N$ is oriented in the sense that $(Dg)^*$ sends $(g')^*\mathcal{U}_N$ to $\mathcal{U}_{N'}$.

We can now write down the following diagram, where the composites of the two rows define $f_!$ and $f'_!$:

(40.6)

$$\begin{array}{ccccccc}
\mathcal{E}^*(X) & \xrightarrow{th_N} & \mathcal{E}^{*+2c}(N, N-0) & = & \mathcal{E}^{*+2c}(U, U-X) & \xleftarrow{\cong} & \mathcal{E}^{*+2c}(Y, Y-X) \\
\downarrow{\scriptstyle (g')^*} & & \downarrow{\scriptstyle (Dg)^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\
\mathcal{E}^*(X') & \xrightarrow{th_{N'}} & \mathcal{E}^{*+2c}(N', N'-0) & = & \mathcal{E}^{*+2c}(U', U'-X) & \xleftarrow{\cong} & \mathcal{E}^{*+2c}(Y', Y'-X').
\end{array}$$

Note that g is a map of pairs $(Y', Y'-X') \to (Y, Y-X)$ because we started with a pullback square, and Dg is a map of pairs $(N', N'-0) \to ((g')^*N, (g')^*N-0)$ using that we have a *transverse* pullback square (so that Proposition 40.4 applies). In the above diagram, the left square commutes because of the assumption that $(Dg)^*$ preserves the Thom classes, and the right square commutes because of naturality. The middle square takes some work, though. Under the indentification of normal bundles with their tubular neighborhoods, g is a (non-linear) map $N' \to N$ that carries the zero section $\zeta_{N'}$ to ζ_N, and Dg is a linear map that agrees with g on the zero section. The key result is the following:

Proposition 40.7. *In the above setting, the two maps of pairs $g, Dg\colon (N', N'-0) \to (N, N-0)$ are homotopic.*

Proof. Let $p\colon N \to X$ and $p'\colon N' \to X'$ be the bundle maps. For every $t \in \mathbb{R}$ let $m_t\colon N \to N$ and $m'_t\colon N' \to N'$ denote the multiplication-by-t maps. Define

$H\colon N' \times I \to N$ by

$$H(v,t) = \begin{cases} m_{t^{-1}}(g(m'_t(v))) & \text{if } t > 0, \\ (Dg)_{p'v}(v) & \text{if } t = 0. \end{cases}$$

In the first formula, note that if v starts out in the fiber over x' then $g(m'_t(v))$ might end up in a different fiber. We have written the formulas in terms of the m functions to make continuity clear when $t > 0$.

To check continuity of H, the only question is at $t = 0$. Since continuity is a local property, we can focus on Euclidean neighborhoods $W' \subseteq X'$ and $W \subseteq X$ where N' and N are trivial and $g(W') \subseteq W$. Choose local coordinates in each case of the form (u, v) where u is in our coordinate chart of the base and v is in $\mathbb{R}^c$ (recall $c = \operatorname{rank} N = \operatorname{rank} N'$). In these coordinates g is a map $W' \times \mathbb{R}^c \to W \times \mathbb{R}^c$ given by $g(x', v) = (A(x', v), B(x', v))$ for some smooth functions A and B. Observe that $B(x', 0) = 0$ for all x'. The map Dg is then given by $(Dg)(x', v) = (A(x', 0), (Dg)_{x'}(v)) = (A(x', 0), (D_2B)_{(x',0)}(v))$.

We have

$$H_t(x', v) = \begin{cases} (A(x', tv), \frac{1}{t}B(x', tv)) & \text{if } t > 0, \\ (A(x', 0), (D_2B)_{(x',0)}(v)) & \text{if } t = 0. \end{cases}$$

Continuity in the first coordinate is clear, so we focus on the second. The spaces are all Euclidean, so it suffices to prove sequential continuity: if $(x'_i, v_i) \mapsto (x', v)$ and $t_i \mapsto 0$ then $\frac{1}{t_i}B(x'_i, t_iv_i) \mapsto (D_2B)_{(x',0)}(v)$. The standard technique is to use that if $F\colon \mathbb{R}^a \to \mathbb{R}^b$ is smooth then

$$F(tv) - F(0) = \int_0^t (DF)_{sv}(v)\, ds = t \cdot \int_0^1 (DF)_{rtv}(v)\, dr.$$

Therefore

$$\tfrac{1}{t_i}B(x'_i, t_iv_i) = \int_0^1 (D_2B)_{(x'_i, st_iv_i)}(v_i)\, ds \mapsto \int_0^1 (D_2B)_{(x',0)}(v)\, ds = (D_2B)_{(x',0)}(v),$$

where the convergence uses that the function $W' \times \mathbb{R}^c \times I \to \mathbb{R}^c$ given by $(x', v, t) \mapsto (D_2B)_{(x',tv)}(v)$ is continuous.

Finally, we need to check that H is a homotopy of maps of pairs: that is, we need that $H_t(N' - 0) \subseteq N - 0$ for all t. For $t = 0$ this is because $Dg\colon N' \to (g')^*N$ is an isomorphism of bundles. For $t > 0$ the question is whether a point $v \in N - 0$ could have the property that $g(tv)$ lies on the zero section—i.e., equals $f(x)$ for some $x \in X$. But then (tv, x) would represent a point in the pullback, i.e. a point in X'. This would imply that tv was itself on the zero section of N, contradicting our assumption. □

To summarize our discussion, we have now proven the following result. This is the prototype for more general versions of "push-pull" formulas we will see in the next section.

Corollary 40.8. *Suppose given a transverse pullback diagram (40.2) where the maps f and f' are closed embeddings, and a cohomology theory $\mathcal{E}$ for which the two normal bundles are oriented and where the isomorphism $Dg\colon N_{f'} \to (g')^*N_f$ preserves the orientation. Then the diagram (40.6) containing $f_!$ and $(f')_!$ is commutative. Briefly, we have $g^*f_! = (f')_!(g')^*$.*

There is one more useful "move" in studying transverse pullbacks as in (40.2). Suppose that Z is a smooth manifold and we have a smooth embedding $i\colon X \hookrightarrow Z$. Let $j = (f, i)\colon X \to Y \times Z$. This is also a smooth embedding, and it is a *closed* embedding if either f is proper or i is closed (Exercise A.29); we will assume that one of these conditions is satisfied. Consider the two squares

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle j'=(f',ig')}\downarrow & & \downarrow{\scriptstyle j=(f,i)} \\ Y' \times Z & \xrightarrow{g\times \mathrm{id}} & Y \times Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

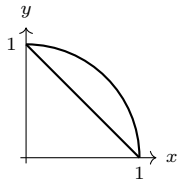
Both are pullback squares. The bottom square is automatically transverse because π_1 is a submersion, and it is easy to see that transversality of (40.2) translates into transversality of the top square. In this way we have decomposed the original transverse pullback square into two others, the first with the property that the vertical maps are closed embeddings and the second with the property that they are submersions (in fact, a particularly simple kind of submersion). This technique will be used repeatedly throughout the next several sections to reduce theorems for general maps to the cases of closed embeddings and projections.

Remark 40.9. In our applications Z will always be something simple: $\mathbb{R}^N$ or $\mathbb{C}^N$ or $\mathbb{C}P^N$ (each is contained in the next). Recall that any smooth manifold X admits a closed embedding $i\colon X \hookrightarrow \mathbb{R}^N$ for some N by the Whitney embedding theorem [Lee, Theorem 10.11]. If we want to use $Z = \mathbb{C}P^N$ then we cannot embed noncompact manifolds X as closed subspaces, and so here is where we would need to insist that $f\colon X \to Y$ be proper in order for $j = (f, i)$ to be closed.

40.10. **Smooth homotopies.** We need one more background concept before starting on the main work of this section. Let X and Y be smooth manifolds. In dealing with homotopies of maps we run into the slight annoyance that $X \times I$ is a manifold-with-boundary, not simply a manifold. To deal with this let us define a **smooth homotopy** to be a map $H\colon X \times I \to Y$ such that $X \times (0, 1) \to Y$ is smooth and that also has the property that for some $\epsilon > 0$, $H(x, t) = H(x, 0)$ for all $t < \epsilon$ and $H(x, t) = H(x, 1)$ for all $t > 1 - \epsilon$, for all $x \in X$. This "constant at the ends" property is just a technical condition so that we don't have to worry about smoothness issues at the boundary of $X \times I$, and so that smooth homotopies can be concatenated.

From now on, when we talk about restricting a smooth homotopy to times $t = 0$ or $t = 1$ what we really mean is restricting it to time ϵ or $1 - \epsilon$ when ϵ is sufficiently small. In some sense this is just a linguistic issue, but in another sense it serves to make sure we are always in a situation where smoothness and tangent spaces have a clear meaning.

Example 40.11. Suppose X is a smooth manifold and $i, i'\colon X \to \mathbb{R}^N$ are two smooth maps. The usual straight-line homotopy $H\colon X \times I \to \mathbb{R}^N$ given by $H(x, t) = (1 - t)i(x) + ti(x')$ is not a smooth homotopy in our sense, because of the behavior at the boundary. A fix is to replace the coefficients $(1-t, t)$ by the parameterization of a curve such as in this picture:



Any smooth curve that is flat at the ends will do, but of course $(\cos\frac{\pi t}{2}, \sin\frac{\pi t}{2})$ comes to mind as an easy choice. Concatenating the map $(x,t) \mapsto (\cos\frac{\pi t}{2})i(x) + (\sin\frac{\pi t}{2})i'(x)$ with brief stationary homotopies on the two ends gives a smooth homotopy from i to i'. We will loosely refer to such a construction as the "smooth version of the straight-line homotopy".

Proposition 40.12. *Let X be a smooth manifold and let $i, i' \colon X \hookrightarrow \mathbb{R}^n$ be closed embeddings. Then for $N \geq 2n$ there is a smooth isotopy H between the maps $i \oplus \underline{0}, i' \oplus \underline{0} \colon X \hookrightarrow \mathbb{R}^N$, and moreover the map $X \times I \to \mathbb{R}^N$ may be chosen to be proper. If $N \geq 2n+1$ then H can also be chosen to be a closed embedding.*

Proof. It suffices to prove this for $N = 2n$. Start by using the smooth version of a straight-line homotopy from $i \oplus \underline{0}$ to $\underline{0} \oplus i$. Then concatenate this with a smooth version of a straight-line homotopy from $\underline{0} \oplus i$ to $i' \oplus \underline{0}$. This gives the desired smooth isotopy H.

For any $k > 0$, let $D_k \subseteq \mathbb{R}^{2n}$ be the closed disk of radius k centered at 0. To prove that H is proper we will show that $H^{-1}(D_k)$ is compact for every k. Notice that $|H(x,t)|$ is always at least as large as $\min\{|i(x)|, |i'(x)|\}$. So $H^{-1}(D_k) \subseteq [i^{-1}(D_k) \cup (i')^{-1}(D_k)] \times I$. Since i and i' are closed embeddings they are proper, and hence both $i^{-1}(D_k)$ and $(i')^{-1}(D_k)$ are compact. So $H^{-1}(D_k)$ is a closed subset of a compact Hausdorff space, hence compact.

For the final statement just note that the associated map $\tilde{H} \colon X \times I \to \mathbb{R}^{2n} \times \mathbb{R}$ given by $(x,t) \mapsto (H(x,t), t)$ will be a proper injection and therefore a closed embedding (Exercise A.29). □

It will be convenient to write $I^\circ = (0,1)$. Given a smooth homotopy H as above, let $\tilde{H} \colon X \times I \to Y \times I$ be given by $\tilde{H}(x,t) = (H(x,t), t)$. This construction will appear often, so it needs a name: we call $\tilde{H}$ the **time dispersal** of H. Instead of having the homotopy take place within Y it gets spread out over $Y \times I$. If H is an isotopy (meaning each H_t is an embedding) then $\tilde{H}$ is an embedding; but be warned that if all H_t are closed embeddings it does not follow that $\tilde{H}$ is a closed embedding.

The pullback squares

$$\begin{array}{ccc} X \times \epsilon & \longrightarrow & X \times I^\circ \\ {\scriptstyle H_\epsilon}\downarrow & & \downarrow{\scriptstyle \tilde{H}} \\ Y \times \epsilon & \longrightarrow & Y \times I^\circ \end{array} \tag{40.13}$$

are transverse for all $\epsilon \in I^\circ$. In particular, when $\tilde{H}$ is a closed embedding then Proposition 40.4 gives us a natural isomorphism between the restriction to $t = 0$ (or $t = 1$) of the normal bundle of $\tilde{H}$ and the normal bundle to H_0 (or H_1)—and again, this is to be interpreted as $t = \epsilon$ or $t = 1 - \epsilon$ for ϵ sufficiently small. (This is also easy to see by just thinking about $D\tilde{H}$ directly.)

40.14. **Complex-oriented maps.** Recall that an almost complex manifold is a smooth manifold X together with a complex structure on its tangent bundle TX. A *stably almost complex manifold* is, roughly, a smooth manifold X together with a complex structure on some $TX \oplus \mathbb{R}^k$, up to a certain notion of equivalence. The complex-oriented maps we will define in this section can be thought of as a relative version of stably almost complex manfiolds.

We will write D^{2n} for the disk in $\mathbb{R}^{2n} = \mathbb{C}^n$ and $\mathring{D}^{2n}$ for its interior. In the following discussion it is best to think of $\mathring{D}^{2n}$ as a copy of $\mathbb{C}^n$, but one that has been shrunk down so that we have access to the boundary sphere when we sometimes need it.

Let $f\colon X \to Y$ be a smooth map between smooth manifolds. Consider the data (n, i, J) where $n \geq 0$, i is a closed embedding $X \hookrightarrow Y \times \mathring{D}^{2n}$ such that f is the composite $X \hookrightarrow Y \times \mathring{D}^{2n} \to Y$, and J is a complex structure on the normal bundle N_i. Let $\sim$ be the equivalence relation on this data generated by the following:

(i) $(n, i, J) \sim (n+1, i \oplus 0, J \oplus J_{\mathbb{C}})$ where $i \oplus 0$ is the embedding $X \hookrightarrow Y \times \mathring{D}^{2n} \hookrightarrow Y \times \mathring{D}^{2(n+1)}$, and the complex structure on the normal bundle is the evident one coming from the isomorphism $N_{i\oplus 0} = N_i \oplus \underline{1}$ and using J for the first summand and the standard complex structure for the second.

(ii) $(n, i, J) \sim (n, i', J')$ if there is a smooth homotopy $H\colon X \times I \to Y \times \mathring{D}^{2n}$ between i and i' such that each H_t is a closed embedding, a complex structure $\mathcal{J}$ on $N_{\tilde{H}}$ where $\tilde{H}\colon X \times I \to Y \times I$ is the associated time dispersal of H, such that restricting $(N_{\tilde{H}}, \mathcal{J})$ to $t = 0$ and $t = 1$ gives (N_i, J) and $(N_{i'}, J')$,

An equivalence class of triples (n, i, J) is called a **complex orientation** of the map f.

Observe that a complex orientation can exist only if $\dim X - \dim Y$ is even. In this case it will be convenient to define the **complex dimension** of f to be $\dim f = \frac{1}{2}(\dim X - \dim Y)$. When Y is a point the map $X \to *$ has dimension equal to the complex dimension of X.

Remark 40.15. One can give the same definition as above but with $\mathbb{C}^n$ replacing $\mathring{D}^{2n}$ everywhere, and this doesn't change anything. But for our work in the next section it is more convenient to use $\mathring{D}^{2n}$.

Also, in [Quillen3] a slightly different definition of complex-orientations is given, where $Y \times \mathring{D}^{2n} = Y \times \mathbb{C}^n \to Y$ is replaced by a complex vector bundle $E \to Y$. Because manifolds have finite covering dimension, any such E will embed in a trivial vector bundle by Proposition 12.13. So there is no loss of generality in assuming that E is trivial as we have done.

Let us give a few examples:

(a) If $f\colon X \hookrightarrow Y$ is a closed embedding and we are given a complex structure J on the normal bundle, then $(0, f, J)$ defines a complex-orientation on f.

(b) For the projection map $p\colon Y \times \mathbb{C}^n \to Y$, the triple $(n, \mathrm{id}, -)$ defines a complex-orientation on p (the complex structure is unique because the normal bundle is the zero bundle).

(c) More generally, let $p\colon E \to Y$ be a complex vector bundle. Since Y is bundcompact there is an inclusion of vector bundles $i\colon E \hookrightarrow \underline{n}$ for some n. The normal bundle to i is $p^*(\underline{n}/E)$, which comes with the complex structure J it inherits from $\underline{n}$. Then (n, i, J) defines a complex-orientation of p. We leave it as an exercise to check that this does not depend on the choice of n and i.

(d) Let $i\colon X \hookrightarrow \mathbb{C}^n$ be a smooth complex submanifold. We have the short exact sequence $0 \to TX \to \underline{n}_{\mathbb{C}} \to N \to 0$ and N inherits a complex structure J as the quotient $\underline{n}_{\mathbb{C}}/TX$. Then (n, i, J) represents a complex-orientation of the map $X \to pt$. (We will soon generalize this to the case where X is any almost complex manifold).

Remark 40.16. There is an instance of relation (ii) above that is particularly useful. Let $E \to X$ be a real vector bundle with two complex structures J and J'. Say that these are homotopic if there is a complex structure on π^*E, where $\pi\colon X \times I \to X$ is projection, that restricts to J and J' on the two ends. This is equivalent to saying that in the space of complex structures on E (suitably interpreted) the points J and J' are path-connected.

If (n, i, J) and (n, i, J') are triples defining complex orientations on f and J is homotopic to J', then relation (ii) implies $(n, i, J) \sim (n, i, J')$. To see this, let $H\colon X \times I \to Y \times \mathring{D}^{2n}$ be the constant homotopy $H_t = i$ for all t. Then for the time dispersal $\tilde{H}$ we have $N_{\tilde{H}} = \pi^* N_i$. So our assumption is that there is a complex structure on $N_{\tilde{H}}$ that restricts to J and J' at times $t = 0$ and $t = 1$. This precisely says that $(n, i, J) \sim (n, i, J')$ via relation (ii).

Definition 40.17. *A **complex-oriented map** is a smooth map $f\colon X \to Y$ between smooth manifolds that is equipped with a chosen complex-orientation.*

Remark 40.18 (Manifolds with boundary)**.** It will occasionally be useful to generalize all of the above ideas to allow X to be a smooth manifold with boundary. For this change, the closed embedding i should be a map $i\colon X \hookrightarrow Y \times D^{2n}$ that is *neat*: this means that $i(X) \cap (Y \times \partial D^{2n}) = i(\partial X)$ and also that $i(X)$ meets $Y \times \partial D^{2n}$ transversely. The generalization of the Whitney embedding theorem says that any smooth manifold with boundary admits a neat embedding into D^{2n} for some n. The generalization of the tubular neighborhood theorem says that a neatly embedded manifold-with-boundary has a tubular neighborhood.

In proving theorems about complex orientations it will be useful to separate out relations (i) and (ii) and deal with them independently. This is offputting at first, particularly because of the required notation, but it allows us to maximally take advantage of the various pieces of structure.

Define $\mathrm{CO}_n(f)$ to be the set of pairs (i, J) where $i\colon X \hookrightarrow Y \times \mathring{D}^{2n}$ is a smooth closed embedding and J is a complex structure on N_i. There are stabilization maps $\mathrm{CO}_n(f) \to \mathrm{CO}_{n+1}(f)$ that send (i, J) to $(i \oplus 0, J \oplus J_{\mathbb{C}})$, and we let $\mathrm{CO}_\infty(f)$ be the colimit as $n \to \infty$.

If (40.2) is a transverse pullback square we claim there is a natural map $\mathrm{CO}_n(f) \to \mathrm{CO}_n(f')$. Given (i, J), form the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y \times \mathring{D}^{2n} & \longrightarrow & Y \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g\times \mathrm{id}} & & \uparrow{\scriptstyle g} \\ X' & \xrightarrow{i'} & Y' \times \mathring{D}^{2n} & \longrightarrow & Y' \end{array}$$

where both squares are pullbacks. The fact that the original square is transverse implies the same for the left square, and so by Proposition 40.4 we have a canonical isomorphism $N_{i'} \to (g')^* N_i$. So the complex structure J on N_i induces a complex

structure J' on $N_{i'}$, and we define (i', J') to be the image of (i, J). It is clear that this is compatible with stabilization and so we also get $\mathrm{CO}_\infty(f) \to \mathrm{CO}_\infty(f')$.

Let $f \times I$ be the map $f \times \mathrm{id}_I \colon X \times I \to Y \times I$. We define $\mathrm{CO}_n(f \times I)$ analogously to $\mathrm{CO}_n(f)$ except that the embedding $X \times I \hookrightarrow Y \times I \times \mathring{D}^{2n}$ is required to be a smooth homotopy (equivalently, the projection to $\mathring{D}^{2n}$ is a smooth homotopy). We also get $\mathrm{CO}_\infty(f \times I)$. Transverse pullback to $t = 0$ and $t = 1$ gives restriction maps

$$\mathrm{CO}_\infty(f \times I) \rightrightarrows \mathrm{CO}_\infty(f)$$

and we define $\mathrm{CO}(f)$ to be the coequalizer. This is precisely the set of complex-orientations on f as defined above.

For a transverse pullback square (40.2) we now get the diagram

$$\begin{array}{ccccc}
\mathrm{CO}_\infty(f \times I) & \rightrightarrows & \mathrm{CO}_\infty(f) & \longrightarrow & \mathrm{CO}(f) \\
\downarrow & & \downarrow & & \vdots \\
\mathrm{CO}_\infty(f' \times I) & \rightrightarrows & \mathrm{CO}_\infty(f') & \longrightarrow & \mathrm{CO}(f')
\end{array}$$

where all of the solid-arrow maps in the left square are induced by pullbacks and therefore obviously commute. So we get an induced map on the coequalizers.

There is an important principle here: the homotopy relation is induced by pullbacks, so if we can show that a construction behaves well with respect to pullbacks then it necessarily also behaves well with respect to homotopy.

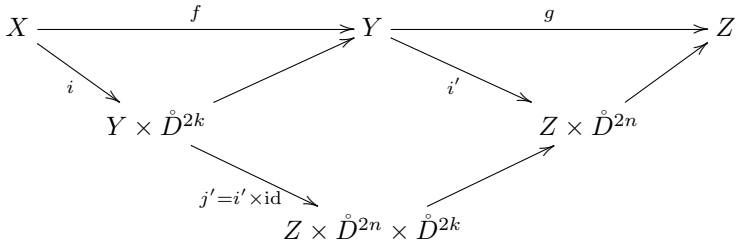
It will also be useful to define $\mathrm{CO}^h_n(f)$ to be the coequalizer of the two maps $\mathrm{CO}_n(f \times I) \rightrightarrows \mathrm{CO}_n(f)$.

Here are two important properties of complex-oriented maps:

(1) The composite of two complex-oriented maps has a natural complex orientations, and if f, g, h are composable complex-oriented maps then the complex orientations on $h(gf)$ and $(hg)f$ are equal.
(2) A smooth map between almost complex manifolds has a canonical complex orientation, and these are compatible with the natural complex orientations inherited by transverse pullbacks and compositions.

One way to interpret these facts is that there exists a category whose objects are smooth manifolds and where the morphisms are the complex-oriented maps. The category of almost complex manifolds and smooth maps occurs naturally as a subcategory.

Justifying (1) and (2) takes a little legwork. For (1), let $f \colon X \to Y$ and $g \colon Y \to Z$ be smooth maps and choose triples (k, i, J) and (n, i', J') that represent orientations on f and g. Use the diagram

(40.19)


where all of the projections are the evident ones. We use the standard homeomorphism $\mathring{D}^{2n} \times \mathring{D}^{2k} \cong \mathring{D}^{2n+2k}$ given by $(x, y) \mapsto \frac{1}{\sqrt{|x|^2+|y|^2}}(x, y)$. The rectangle is a transverse pullback, so the complex structure on $N_{i'}$ determines a corresponding complex structure on $N_{i'\times \mathrm{id}}$. There is a natural short exact sequence

$$0 \to N_i \to N_{j'i} \to i^* N_{i'\times \mathrm{id}} \to 0.$$

This sequence is split—but not naturally—and all choices of splitting are homotopic. The given complex structure on N_i and $N_{i'}$ therefore determine a deformation class of complex structure on $N_{j'i}$, and hence an element in $\mathrm{CO}^h_{n+k}(gf)$. These considerations define a map $\mathrm{CO}_n(g) \times \mathrm{CO}_k(f) \to \mathrm{CO}^h_{n+k}(gf)$. A little thought shows it is compatible with stabilization and so gives $\mathrm{CO}_\infty(g)\times\mathrm{CO}_\infty(f) \to \mathrm{CO}(gf)$. But it is also clearly compatible with transverse pullback, in the sense that if we have a ladder diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \\ \uparrow & & \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

with both squares transverse pullbacks then

$$\begin{array}{ccc} \mathrm{CO}_\infty(g') \times \mathrm{CO}_\infty(f') & \longrightarrow & \mathrm{CO}(g'f') \\ \downarrow & & \downarrow \\ \mathrm{CO}_\infty(g) \times \mathrm{CO}_\infty(f) & \longrightarrow & \mathrm{CO}(gf) \end{array}$$

is commutative. So taking $g' = g \times I$ and $f' = f \times I$ we get $\mathrm{CO}(g) \times \mathrm{CO}(f) \to \mathrm{CO}(gf)$ as desired.

Associativity of composition comes from writing down a larger version of (40.19) with (f, g, h) across the top row. It takes a little thought, but is not hard.

Remark 40.20. If $f\colon X \to Y$ is a smooth embedding with a complex structure J on the normal bundle then $(0, f, J)$ gives a canonical complex orientation on f. Likewise, if Y is a smooth manifold then the projection $p\colon Y \times \mathring{D}^{2n} \to Y$ also has a canonical complex orientation given by the triple $(n, \mathrm{id}, -)$ (the complex structure is uniquely determined because here the normal bundle is the zero bundle). It is routine to check that if (n, i, J) is a triple for f then in the category of complex-oriented maps f is the composite $X \hookrightarrow Y \times \mathring{D}^{2n} \to Y$ where the two factors are given their canonical complex orientations.

Turning to (2) now, suppose that X and Y are almost complex manifolds and $f\colon X \to Y$ is a smooth map. Choose a smooth, closed embedding $i\colon X \hookrightarrow \mathring{D}^{2n}$ for some n, as in Remark 40.9. Then we have the corresponding factorization of f as $X \to Y \times \mathring{D}^{2n} \to Y$ where the first map is the closed embedding $j = (f, i)$. The map j yields the canonical exact sequence

$$0 \to T_X \to f^* T_Y \oplus \underline{n}_{\mathbb{C}} \to N_j \to 0$$

of real vector bundles on X. By Proposition 12.2 this exact sequence has a splitting, and the space of all splittings is contractible. A choice of splitting gives an isomorphism of real bundles $T_X \oplus N_j \cong f^* T_Y \oplus \underline{n}_{\mathbb{C}}$. Since X and Y are almost

complex manifolds, T_X and T_Y come with complex structures. Since X is a manifold and therefore has finite covering dimension, there is an embedding of bundles $u\colon T_X \hookrightarrow \underline{k}_{\mathbb{C}}$ for some k (Proposition 12.13). We equip $\underline{k}_{\mathbb{C}}$ with the standard Hermitian metric and set $Q = (T_X)^{\perp}$. Then $Q \oplus T_X = \underline{k}_{\mathbb{C}}$, and therefore we get isomorphisms of real bundles

$$\underline{k}_{\mathbb{C}} \oplus N_j \cong Q \oplus T_X \oplus N_j \cong Q \oplus f^*T_Y \oplus \underline{n}_{\mathbb{C}}.$$

The complex structure on $Q \oplus f^*T_Y \oplus \underline{n}_{\mathbb{C}}$ can now be transferred across the above isomorphisms to a complex structure J on $\underline{k}_{\mathbb{C}} \oplus N_j$. So the triple $(n+k, j \oplus \underline{0}, J)$ represents a complex orientation on f. We claim that this complex orientation is independent of all the choices made:

Proposition 40.21. *The above complex orientation on f only depends on X, Y, and f.*

Proof. The choices made were for the pair (n, i), a choice of splitting for the tangent-normal exact sequence, and for the pair (k, u).

Since any two choices of splitting are homotopic, we get an associated homotopy between the corresponding complex structures on $\underline{k}_{\mathbb{C}} \oplus N_j$. So this choice is now accounted for.

Replacing (k, u) with $(k+1, u \oplus 0)$ clearly will not affect the resulting complex-orientation, using relation (i) from the definition. So given two choices (k, u) and (k', u'), we can stabilize them and thereby assume that $k = k'$. We can further stabilize them to $(2k, u \oplus \underline{0})$ and $(2k, u' \oplus \underline{0})$. Let Q and Q' be the corresponding bundles obtained from the two embeddings. We have the two straight-line homotopies from $u \oplus \underline{0}$ to $\underline{0} \oplus u$ and from $\underline{0} \oplus u$ to $u' \oplus \underline{0}$. Let H be the concatenation. If $\pi\colon X \times I \to X$ is the projection then H defines an embedding $H\colon \pi^*T_X \hookrightarrow \underline{k} \oplus \underline{k}$ of complex bundles over $X \times I$. Let $\tilde{Q}$ be the quotient bundle. Then $\tilde{Q}$ is a complex bundle that restricts to Q and Q' at times 0 and 1. This gives us a deformation between the resulting complex structures on $\underline{2k}_{\mathbb{C}} \oplus N_j$, and so these represent the same complex orientation on f.

At this point we can talk about *the* complex orientation on f determined by a choice (n, i). Let us write it as $o(f, n, i) \in \mathrm{CO}(f)$. Our last task is to show that it doesn't actually depend on (n, i).

With slight modifications the same work we have already done, but applied to the map $f \times I\colon X \times I \to Y \times I$, shows that for any smooth, closed embedding $v\colon X \times I \hookrightarrow \mathring{D}^{2m}$ we have an element $o(f \times I, m, v) \in \mathrm{CO}(f \times I)$. Let us explain the modifications. Let $j\colon X \times I \hookrightarrow Y \times I \times \mathring{D}^{2m}$ be $(x, t) \mapsto (f(x), t, v(x, t))$. Let $\pi\colon X \times I \to X$ be projection and consider

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{X\times I} & \longrightarrow & (f\times I)^*T_{Y\times I} \oplus \underline{n}_{\mathbb{C}} & \longrightarrow & N_j & \longrightarrow & 0 \\
 & & \uparrow & & \downarrow & & & & \\
0 & \longrightarrow & \pi^*T_X & \dashrightarrow & \pi^*(f^*T_Y \oplus \underline{n}_{\mathbb{C}}). & & & &
\end{array}$$

Both $T_{X\times I}$ and $T_{Y\times I}$ canonically split off a trivial line bundle for the I direction, which give the vertical inclusion and projection in the diagram; the dotted map is the evident composite. A little thought shows that the cokernel of that dotted map is canonically N_j. A splitting for the resulting exact sequence then gives $\pi^*T_X \oplus N_j \cong \pi^*(f^*T_Y \oplus \underline{n}_{\mathbb{C}})$, an isomorphism of bundles over $X \times I$. Next we choose

an embedding of complex bundles $u\colon \pi^*T_X \hookrightarrow \underline{k}_{\mathbb{C}}$ for some k, let Q be the quotient, and then proceed exactly as before. This defines our element $o(f \times I, m, v) \in \mathrm{CO}(f \times I)$.

Now we wrap up the proof by showing that $o(f, n, i)$ does not depend on (n, i). Given an (n, i), it is clear that passing to $(n+1, i \oplus 0)$ will not affect $o(f, n, i)$. So given choices (n, i) and (n', i') we can increase n and/or n' and thereby assume $n = n'$. We can further pass to $(2n+1, i \oplus \underline{0})$ and $(2n+1, i' \oplus \underline{0})$. Let $H\colon X \times I \to \mathring{D}^{2(n+1)}$ be a smooth homotopy between these that is also a closed embedding, as guaranteed by Proposition 40.12.

Consider the element $o(f \times I, 2n+1, H) \in \mathrm{CO}(f \times I)$. Pulling back to times 0 and 1 gives $o(f, 2n+1, i \oplus 0)$ and $o(f, 2n+1, i' \oplus 0)$, by construction. But the two maps $\mathrm{CO}(f \times I) \to \mathrm{CO}(f)$ obtained by pulling back to times 0 and 1 are necessarily equal, by the definition of $\mathrm{CO}(f)$. It follows that $o(n, i) = o(n', i')$, as desired. □

Remark 40.22. Given $f\colon X \to Y$, the normal bundles for different factorizations give a well-defined class in $KO^{st}(X)$—or equivalently, a homotopy class $X \to BO$. This is called the **stable normal bundle of** f and denoted ν_f. As a class in $\widetilde{KO}^0(X)$ it is equal to $[f^*T_Y] - [T_X] + 2\dim f$. A complex-orientation of f is sometimes described as a "complex structure on the stable normal bundle", but one must be careful in making sense of this phrase. We have the map $r\colon BU \to BO$, and one possible interpretation is a lifting of ν_f to a class in $[X, BU] = \widetilde{K}^0(X)$; but that interpretation is too weak. The correct interpretation is that there is a bijection

$$\mathrm{CO}(f) \longleftrightarrow \pi_0(\mathrm{hofib}(BU^X \to BO^X))$$

where the homotopy fiber is taken over the point represented by ν_f. An element of the set on the right gives an element in $\pi_0(BU^X) = \widetilde{K}^0(X)$ that lifts ν_f, but it contains more information than just that.

We will not need to develop these ideas further, but we wanted to warn the reader about the above common phrasing and the associated issues of interpretation.

As one final point, we will also need a notion of homotopy for complex-oriented maps. It is easy to write one down:

Definition 40.23. *Let $f, g\colon X \to Y$ be complex-oriented. An **oriented homotopy** from f to g is a smooth homotopy $H\colon X \times I \to Y$ together with a complex-orientation on the time-dispersal map $\tilde{H}\colon X \times I \to Y \times I$ that restricts, via pullback at times 0 and 1, to the orientations on f and g.*

40.24. **The Pontryagin-Thom collapse.** Before constructing pushforwards in the next section we need one more tool. Let $j\colon X \hookrightarrow Y$ be a smooth, closed embedding of manifolds. Let $p\colon N \to X$ be the normal bundle and let U be a tubular neighborhood. Choose a metric on N (and hence on U). Call a continuous function $s\colon X \to (0, \infty)$ a **radius function**. Define $D_s(N) = \{v \in N \mid s(p(v)) \geq |v|\}$ and $S_s(N) = \{v \in N \mid s(p(v)) = |v|\}$. Then $D_s(N)$ and $S_s(N)$ are like the disk and sphere bundles of N, but where the fiberwise radius of the disks is controlled by the function s. Multiplication by s gives a homeomorphism of pairs $(D(N), S(N)) \xrightarrow{\cong} (D_s(N), S_s(N))$. Also, both $D_s(N)$ and $S_s(N)$ are closed in N.

Here is an easy exercise: if W is an open subset of N containing the zero section then there is a continuous (and even smooth) radius function s such that $D_s(N) \subseteq$

W. This can be proven by constructing s locally and then patching the results together with a partition of unity.

Since Y is normal and $X \subseteq Y$ is closed, there is an open set $X \subseteq W$ such that $\overline{W} \subseteq U$. Here $\overline{W}$ is the closure in Y. Then there is a radius function s such that $D_s(N) \subseteq W$. Since $D_s(N)$ is closed in U, it is also closed in $\overline{W}$. Therefore $D_s(N)$ is also closed in Y.

Let $D_s(N)^\circ = D_s(N) - S_s(N)$. The map $D_s(N) \hookrightarrow Y$ induces $D_s(N)/S_s(N) \to Y/(Y - D_s(N)^\circ)$, which is a continuous bijection. Because $D_s(N)$ is closed in Y, this is a closed map and hence a homeomorphism. The **Pontryagin-Thom collapse map** is the map ρ obtained as the composite

$$Y \longrightarrow Y/[Y - D_s(N)^\circ] \xleftarrow{\cong} D_s(N)/S_s(N) \xleftarrow{\cong} D(N)/S(N) = \operatorname{Th} N\ .$$

It does depend on some choices—the metric on N and the radius function s—but different choices lead to homotopic maps ρ.

Remark 40.25. The case where X is compact is a little easier, as here $D(N)$ is compact and therefore automatically closed in Y. So the extra step of passing to $D_s(N)$ can be omitted here. See Exercise 40.27 for an example showing why this cannot be avoided in the non-compact case.

Note that it was important that $X \hookrightarrow Y$ is a *closed* embedding. Without this we cannot be assured that some $D_s(N)$ is closed in Y, and without closedness we do not have one of the homeomorphisms used in the construction of the Pontryagin-Thom map.

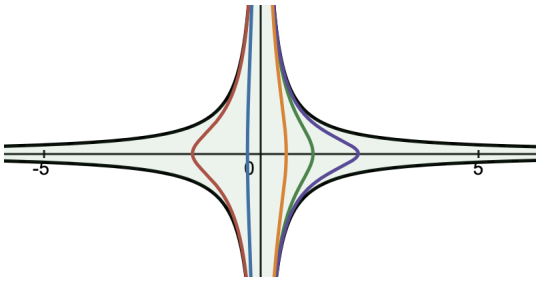
∘ Exercises ∘

Exercise 40.26. Let $f\colon X \to Y$ and $g\colon X' \to Y'$ be complex-oriented maps. Prove that $f \times g$ inherits a natural complex orientation.

Exercise 40.27. We saw in our treatment of the Pontryagin-Thom map that the unit disk bundle $D(N)$ in a tubular neighborhood is not necessarily closed in the ambient space. This exercise gives an example. Let $X = \mathbb{R}$, let $Y = \mathbb{R}^2$, and let $j\colon X \hookrightarrow Y$ be the closed embedding $j(x) = (x, 0)$. The normal bundle is $N = \mathbb{R}^2$. Let $U = \{(x, y) : |y| < \frac{1}{|x|}\}$, where if $|x| = 0$ then $\frac{1}{|x|}$ is interpreted as ∞. Consider the map $\phi\colon N \to U$ given by

$$\phi(x, t) = \left(\frac{x}{\sqrt{1 + t^2x^2}}\,,\ t\right).$$

The following picture shows U as the shaded region (not including the boundary), and the vertical curves are the images of some selected fibers of N:



Prove that ϕ is a diffeomorphism and verify that $\phi(D(N))$ is not closed in Y. Find an appropriate radius function s having the property that $\phi(D_s(N))$ is closed in Y.

Exercise 40.28. Prove the following fact that was needed in this section. Let $E \to X$ be a bundle over a paracompact Hausdorff space, with a chosen metric. If $U \subseteq E$ is an open set containing the zero-section, prove that there is a radius function $s\colon X \to (0,\infty)$ such that $D_s(E) \subseteq U$.

41. Pushforwards in complex-oriented cohomology theories

So far in this book we have seen that complex-oriented cohomology theories admit pushforward maps $j_!$ for closed embeddings of manifolds $j\colon X \hookrightarrow Y$ where there is a given complex structure on the normal bundle (Section 26.4). Recall that the construction goes through a tubular neighborhood U, multiplication by the Thom class of the normal bundle N, and excision:

$$\mathcal{E}^*(X) \xrightarrow{th_N} \mathcal{E}^{*+2c}(N, N-0) = \mathcal{E}^{*+2c}(U, U-X) \xleftarrow{i^*} \mathcal{E}^{*+2c}(Y, Y-X)$$

(c is the codimension of X in Y). It turns out that one can generalize this to get pushforwards for all complex-oriented maps, via a reduction to the closed embedding case—the notion of complex-oriented map was essentially built around this idea.

The work in this section is a prelude to our discussion of the Grothendieck-Riemann-Roch theorem, where these generalized pushforwards will have a main role. In this section we will construct the pushforwards, establish their basic properties, and discuss some examples.

41.1. **Properties of pushforwards.** Let $\mathcal{E}$ be a complex-oriented cohomology theory. We claim that for any complex-oriented map $f\colon X \to Y$ that is proper one can construct an induced map of $\mathcal{E}^*$-modules $f_!\colon \mathcal{E}^*(X) \to \mathcal{E}^{*+2\dim f}(Y)$ such that the following properties are satisfied:

(1) If $g\colon Y \to Z$ is also complex-oriented and proper then $(gf)_! = g_! \circ f_!$.
(2) When $f\colon X \to Y$ is a smooth, closed embedding with a complex structure on its normal bundle the map $f_!$ equals the one constructed in Section 26.4.
(3) [Projection formulas] If $a \in \mathcal{E}^*(Y)$ and $b \in \mathcal{E}^*(X)$ then
$$f_!(f^*a \cdot b) = a \cdot f_!b \qquad \text{and} \qquad f_!(a \cdot f^*b) = f_!(a) \cdot b.$$
(Each follows from the other using graded-commutativity). Said differently, $f_!$ is a map of $\mathcal{E}^*(Y)$-bimodules.
(4) [Push-pull formula] Suppose
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
is a transverse pullback diagram of smooth maps where f' has the complex-orientation induced from f. Then for every $a \in \mathcal{E}^*(X)$ one has $g^*f_!a = (f')_!(g')^*a$.

The push-pull formula is also called a **base-change formula**, especially in algebraic geometry. Observe that the push-pull formula starts with a class a in the

corner that corresponds to the domain of the proper map—it would not be equivalent to start with a class $a \in \mathcal{E}^*(Y')$. It is important to pay attention to this, as sometimes the pullback squares get drawn with the proper maps horizontal rather than vertical.

The following property follows formally from the ones above:

Proposition 41.2. *Let $f\colon X \to Y$ and $g\colon X' \to Y'$ be proper, complex-oriented maps, and give $f \times g$ the product orientation (Exercise 40.26). Then for $a \in \mathcal{E}^*(X)$ and $b \in \mathcal{E}^*(X')$ we have*

$$(f \times g)_!(a \otimes b) = (f_!a) \otimes (g_!b).$$

Proof. By writing $f \times g = (f \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \times g)$ one reduces to showing that $(f \times \mathrm{id}_{Y'})(a \otimes b) = (f_!a) \otimes b$, as the similar formula for $(\mathrm{id}_X \times g)$ follows by symmetry and the general formula is just a combination of the two.

Now consider the transverse pullback square

$$\begin{array}{ccc} X \times Y' & \xrightarrow{f\times \mathrm{id}} & Y \times Y' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

The push-pull formula gives that $(f \times \mathrm{id})_!(\pi_1^* a) = \pi_1^*(f_! a)$, which can also be written as $(f \times \mathrm{id})_!(a \otimes 1) = (f_!a) \otimes 1$. The projection formula finishes the proof:

$$\begin{aligned} (f \times \mathrm{id})_!(a \otimes b) = (f \times \mathrm{id})_!(\pi_1^* a \cdot \pi_2^* b) &= (f \times \mathrm{id})_!(\pi_1^* a \cdot (f \times \mathrm{id})^* \pi_2^* b) \\ &= (f \times \mathrm{id})_!(\pi_1^* a) \cdot \pi_2^* b \\ &= \pi_1^*(f_! a) \cdot \pi_2^* b \\ &= (f_! a) \otimes b. \end{aligned}$$

In the second equality the π_2 maps are slightly different (they have different domains), and we are using that $\pi_2 = \pi_2 \circ (f \times \mathrm{id})$. □

Properties (1)–(4) also imply invariance of the pushforwards under proper homotopy:

Proposition 41.3. *Let $f, g\colon X \to Y$ be proper complex-oriented maps, and let $H\colon X \times I \to Y$ be a smooth homotopy between them (see Definition 40.23). Assume further that H is proper. Then $f_! = g_!$.*

Proof. Let $\tilde{H}\colon X \times I \to Y \times I$ be the time dispersal map $\tilde{H}(x,t) = (H(x,t),t)$. This is still proper by Exercise A.29. For any $\epsilon \in (0,1)$ the square

$$\begin{array}{ccc} X \times \epsilon & \xrightarrow{i_\epsilon} & X \times I^\circ \\ {\scriptstyle H_\epsilon}\downarrow & & \downarrow{\scriptstyle \tilde{H}} \\ Y \times \epsilon & \xrightarrow{i_\epsilon} & Y \times I^\circ \end{array}$$

is a transverse pullback. So push-pull gives $i_\epsilon^* \tilde{H}_! = (H_\epsilon)_! i_\epsilon^*$. If we take ϵ small enough then $H_\epsilon = f$ and $H_{1-\epsilon} = g$. But also note that i_ϵ^* is independent of ϵ, since it equals $(\pi^*)^{-1}$ where $\pi\colon X \times I^\circ \to X$ is projection. So now we can write (for small enough ϵ)

$$f_! i_\epsilon^* = i_\epsilon^* \tilde{H}_! = i_{1-\epsilon}^* \tilde{H}_! = g_! i_{1-\epsilon}^* = g_! i_\epsilon^*.$$

Since i_ϵ^* is an isomorphism we get $f_! = g_!$. □

41.4. **Construction of pushforwards.** Now we give the construction of the pushforward map $f_!$ for a proper complex-oriented map $f\colon X \to Y$. Let $c = \dim f$. The complex orientation on f is represented by a triple (n, i', J) where $i'\colon X \hookrightarrow Y \times \mathring{D}^{2n}$ is a closed embedding and J is a complex structure on the normal bundle. Write i for the composite $X \hookrightarrow Y \times \mathring{D}^{2n} \hookrightarrow Y \times D^{2n}$.

The first factor of i is f, which is proper; so i is proper by Exercise A.29. But i is also an injection, and any proper injection is a closed embedding [loc. cit.]. Let $N = N_i$ be the normal bundle to this embedding, with the chosen complex structure. Choose a tubular neighborhood U of X in $Y \times \mathring{D}^{2n}$.

Define $f_!$ to be the composite of the following chain of maps:

(41.5)

$$\begin{array}{ccccc} \mathcal{E}^*(X) & \xrightarrow{th_N} & \mathcal{E}^{*+2c+2n}(N, N-0) & \xleftarrow[\cong]{j^*} & \mathcal{E}^{*+2c+2n}(Y \times D^{2n}, (Y \times D^{2n}) - X) \\ & & & & \downarrow{\scriptstyle j^*} \\ & & \mathcal{E}^{*+2c}(Y) & = & \mathcal{E}^{*+2c+2n}(Y \times D^{2n}, Y \times \partial D^{2n}). \end{array}$$

The two j^* maps are different, but each is induced by the evident inclusion (in the first case identifying N with U as usual). The final equality is the suspension isomorphism. Note the importance of X being closed in $Y \times D^{2n}$, as this gives the excision isomorphism.

There is an equivalent construction of $f_!$ that uses the Pontryagin-Thom collapse map. Start with the diagram of spaces

$$X \xrightarrow{\zeta} \operatorname{Th} N \xleftarrow{\rho} (Y \times D^{2n})/(Y \times \partial D^{2n}) = Y_+ \wedge S^{2n}.$$

where ρ is induced by Pontryagin-Thom collapse. We have the corresponding chain of algebraic maps:

$$\mathcal{E}^*(X) \xrightarrow{th_N} \widetilde{\mathcal{E}}^{*+2c+2n}(\operatorname{Th} N) \xrightarrow{\rho^*} \widetilde{\mathcal{E}}^{*+2c+2n}(Y_+ \wedge S^{2n}) = \mathcal{E}^{*+2c}(Y).$$

The composite is equal to the $f_!$ constructed above.

Exercise 41.6. Construct a large(ish) commutative diagram showing that the two constructions of $f_!$ are the same.

We need to prove that $f_!$ does not depend on any of the choices we made in the construction, and then also prove that these pushforwards satisfy properties (1)–(4) above. Writing this out in complete detail is very tedious and wordy, but the ideas themselves are fairly simple. We will sketch the basic reasons and then leave the reader to ponder the fine details.

(A) If we increase n to $n+1$ and compose i' with the standard embedding $\mathring{D}^{2n} \hookrightarrow \mathring{D}^{2(n+1)}$ then this results in adding a trivial complex line bundle to N and also tacking on an extra desuspension at the end. The multiplicative property of Thom classes shows that these two things cancel out.

(B) Now suppose we have two triples (n, i, J) and (n, i', J') together with a homotopy H as in (ii) of the defining relation for complex orientations. Write $N = N_i$, $N' = N_{i'}$, $\mathcal{N} = N_{\tilde{H}}$, $D = D^{2n}$. To make the following diagram semi-readable we suppress all product symbols, so e.g. $XI = X \times I$, $YI\partial D = Y \times I \times \partial D$, and we suppress cohomological indices (writing $\mathcal{E}(X)$

instead of $\mathcal{E}^*(X)$):

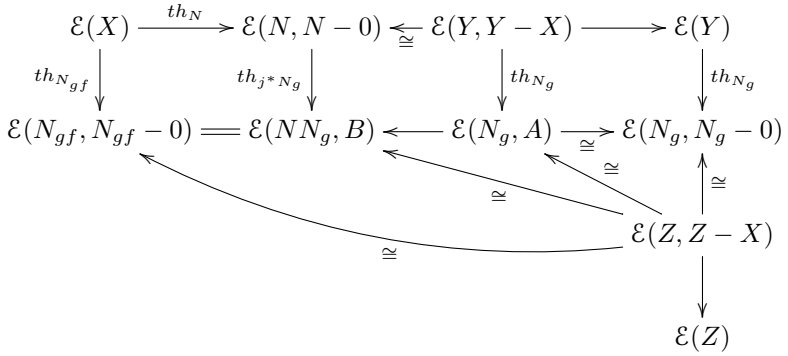
$$\begin{array}{ccccccccc}
\mathcal{E}(X) & \xrightarrow{th_N} & \mathcal{E}(N, N-0) & \longleftarrow & \mathcal{E}(YD, YD-X) & \longrightarrow & \mathcal{E}(YD, Y\partial D) & \longrightarrow & \mathcal{E}(Y) \\
\uparrow{\scriptstyle i_0^*} & & \uparrow{\scriptstyle i_0^*} & & \uparrow{\scriptstyle i_0^*} & & \uparrow{\scriptstyle i_0^*} & & \uparrow{\scriptstyle i_0^*} \\
\mathcal{E}(XI) & \xrightarrow{th_{\mathcal{N}}} & \mathcal{E}(\mathcal{N}, \mathcal{N}-0) & \longleftarrow & \mathcal{E}(YDI, YDI-(XI)) & \longrightarrow & \mathcal{E}(YID, YI\partial D) & \longrightarrow & \mathcal{E}(YI) \\
\downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_1^*} \\
\mathcal{E}(X) & \xrightarrow{th_{N'}} & \mathcal{E}(N', N'-0) & \longleftarrow & \mathcal{E}(YD, YD-X') & \longrightarrow & \mathcal{E}(YD, Y\partial D) & \longrightarrow & \mathcal{E}(Y).
\end{array}$$

Everything commutes, the composite across the top row is $f_!$ constructed using (n, i, J), and the composite across the bottom row is $f_!$ constructed using (n, i', J'). The vertical maps i_0^* and i_1^* at the two ends of the diagram are isomorphisms, and are equal to each other (that is, the two on the left are equal and the two on the right are equal). It now follows formally that the composite across the top and bottom rows are equal.

(C) [Functoriality] This is a little tricky. Suppose we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ where both maps are complex-oriented and proper. First consider the case where f and g are closed embeddings. Choose tubular neighborhoods for f and g and identify these with the normal bundles as usual. Write $N = N_f$, write q for all the bundle projections and j for all inclusions, and let NN_g be the pullback

$$\begin{array}{ccccc}
NN_g & \longrightarrow & N_g & \rightarrowtail & Z \\
\downarrow & & \downarrow{\scriptstyle q} & & \\
N & \xrightarrow{j} & Y & &
\end{array}$$

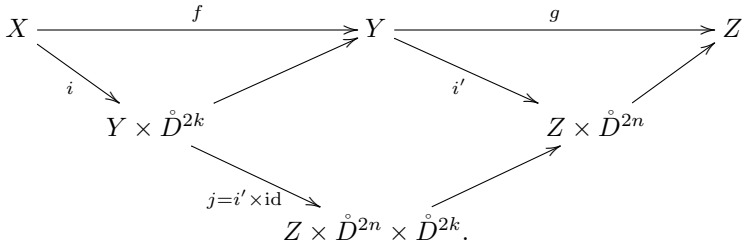
where j is the inclusion. Here j is an open inclusion, so NN_g is an open subset of N_g and can be viewed as a tubular neigborhood for X in Z, i.e. as the normal bundle N_{gf}. Consider the diagram



where $A = (N_g - 0) \cup (N_g|_{Y-X}) = N_g - \zeta(X)$ and $B = [(N-0) \times_Y N_g] \cup [N \times_Y (N_g - 0)]$, and all the unlabelled maps are induced by inclusions of spaces. The composite across the top and then down is $g_! f_!$, whereas the composite along the left side of the diagram is $(gf)_!$. Every region of the diagram is readily checked to commute except for the upper left square, which requires an argument. The key points are that the maps $j, fq \colon N \to Y$ are homotopic and so one can identify the Thom classes $\mathcal{U}_{j^*N_g}$ and $\mathcal{U}_{q^*f^*N_g}$. Then use the

exact sequence $0 \to N \to N_{gf} \to f^*N_g \to 0$ and multiplicativity of Thom classes to get the commutativity.

Now move to the general case. Let (k, i, J) and (n, i', J') be triples that represent the complex orientations on f and g, respectively. Consider the following diagram:



All of the maps that look like projections are the evident projections. The bottom square is a transverse pullback. With a little thought one reduces to the functoriality property for the two embeddings i and j, which was verified above.

(D) [Agreement with the closed embedding case] This part is obvious from the construction, since we may take $n = 0$ and $i' = f$.

(E) [Projection formula] We use the version of $f_!$ given in (41.5). The second j^* and the suspension isomorphism are both maps of $\mathcal{E}^*(Y)$-modules, so it suffices to just deal with the $(j^*)^{-1} \circ th_N$ piece. Let $q\colon N \to X$ be the projection for the normal bundle. The projection formula comes from the following calculation:

$$\begin{array}{ccccc}
a \cdot f^*b & \xrightarrow{th_N} & q^*(a \cdot f^*b) \cdot \mathcal{U}_N & = & q^*a \cdot q^*(f^*b) \cdot \mathcal{U}_N \\
 & & & & \| \\
(j^*)^{-1}[q^*a \cdot \mathcal{U}_N] \cdot (j^*)^{-1}q^*f^*b & & \xrightarrow{j^*} & & q^*a \cdot \mathcal{U}_N \cdot q^*(f^*b) \\
\| & & & & \\
(j^*)^{-1}th_N(a) \cdot (j^*)^{-1}q^*f^*b. & & & &
\end{array}$$

No sign appears in the use of commutativity because the Thom class is in even degree. The commutative square

$$\begin{array}{ccc}
N & \xrightarrow{j} & Y \times D^{2n} \\
{\scriptstyle\zeta}\uparrow & & \downarrow{\scriptstyle\pi} \\
X & \xrightarrow{f} & Y
\end{array}$$

shows that $\zeta^* j^* \pi^* b = f^* b$. But ζ^* is an isomorphism with inverse q^*, so $j^*\pi^*b = q^*f^*b$. Making this subsitution in the final term above yields that $(j^*)^{-1} \circ th_N$ is a map of right $\mathcal{E}^*(Y)$-modules, and this is what we needed.

(F) [Push-pull] For the push-pull formula, start with a transverse pullback square as in (40.2). The complex orientation on f is represented by a triple (n, i, J).

Define i' to be the pullback:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ Y' \times \mathring{D}^{2n} & \xrightarrow{g\times \mathrm{id}} & Y \times \mathring{D}^{2n}. \end{array}$$

Write $N = N_i$, $N' = N_{i'}$, and for convenience in the following diagram write YD^{2n} for $Y \times D^{2n}$, $Y\partial D^{2n}$ for $Y \times \partial D^{2n}$, and so on. We know $D(g\times id)\colon N' \to (g')^*N$ is an isomorphism by Proposition 40.4, and we have the diagram

$$\begin{array}{ccccccc} \mathcal{E}^*(X) & \longrightarrow & \mathcal{E}^*(N, N-0) & \xleftarrow{\cong} & \mathcal{E}^*(YD^{2n}, YD^{2n} - X) & \longrightarrow & \mathcal{E}^*(YD^{2n}, Y\partial D^{2n}) \\ \downarrow{\scriptstyle (g')^*} & & \downarrow{\scriptstyle (Dg)^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ \mathcal{E}^*(X') & \longrightarrow & \mathcal{E}^*(N', N'-0) & \xleftarrow{\cong} & \mathcal{E}^*(Y'D^{2n}, Y'D^{2n} - X) & \longrightarrow & \mathcal{E}^*(Y'D^{2n}, Y'\partial D^{2n}) \end{array}$$

(we are not writing the shift in cohomological indices, because that is not important here). The two left squares were dealt with in Proposition 40.7 and its corollary, where we showed that both squares commute (the left is easy, the middle required some work). The right square commutes just by naturality, and the rightmost g^* is compatible with the suspension isomorphisms to $\mathcal{E}^*(Y) \xrightarrow{g^*} \mathcal{E}^*(Y')$. So commutativity of the outer diagram is precisely the formula $(f')_!(g')^* = g^* f_!$.

Exercise 41.7. Flush out the missing details in the above explanations.

Remark 41.8. It can be useful to try to expand the definition of pushforwards to maps where the domain and codomain are manifolds-with-boundary. If $f\colon X \to Y$ is such that $f^{-1}(\partial Y) \subseteq \partial X$ then one hopes for $f_!\colon \mathcal{E}^*(X, \partial X) \to \mathcal{E}^{*+2\dim f}(Y, \partial Y)$. One obvious advantage of having this is that in our previous factorization of $X \to Y$ (with $\partial X = \partial Y = \emptyset$) as $X \xrightarrow{i} Y \times D^{2n} \xrightarrow{p} Y$ one can talk about $i_!$ and $p_!$ and observe that $f_! = p_! i_!$. Such a decomposition is clearly what is happening in the background anyway.

While we will not take the time to extend carefully to the manifolds-with-boundary case, the reader might find it useful to consider this. On occasion we will write an argument using this approach because it leads to a more compact and appealing presentation. Usually it is evident how one can rewrite the argument to avoid manifolds with boundary if one really wanted to.

Exercise 41.9. Think about how the above remark might simplify the proof of functoriality for pushforwards.

41.10. **Examples of pushforwards.** We now have our pushforward maps, but they might seem more than a little mysterious. We will spend the rest of the section working out various examples, concentrating on smooth maps $f\colon X \to Y$ where X and Y are almost complex manifolds.

Example 41.11. It is interesting to take $Y = pt$. Then for $f\colon X \to pt$ to be proper we need that X is compact. The pushforward $f_!$ gives $\mathcal{E}^*(X) \to \mathcal{E}^{*-2\dim X}(pt)$, and so $f_!(1) \in \mathcal{E}^{-2\dim X}(pt)$. So when $\mathcal{E}$ is complex-oriented we obtain (potentially) interesting invariants of compact almost complex manifolds taking values in the coefficient ring $\mathcal{E}^*$. We call $f_!(1)$ the **$\mathcal{E}$-genus** of X and denote it $\mathrm{genus}_{\mathcal{E}}(X)$.

In the case of K-theory we have $K^{-2\dim X}(pt) \cong \mathbb{Z}$, and so $\text{genus}_K(X)$ is an integer-valued invariant. It is more commonly called the **Todd genus** of X (for reasons that we will shortly come to), denoted here as Td-genus(X).

Example 41.12. Let $p\colon X \times \mathbb{C}P^n \to X$ be projection. Then $p_!\colon \mathcal{E}^*(X \times \mathbb{C}P^n) \to \mathcal{E}^{*-2n}(X)$, and the domain is $\mathcal{E}^*(X) \otimes \mathcal{E}^*(\mathbb{C}P^n)$ (either by the Künneth theorem for $\mathcal{E}$ or by the projective bundle theorem). Let $x = [\mathbb{C}P^{n-1}] \in \mathcal{E}^2(\mathbb{C}P^n)$, so that $\mathcal{E}^*(X) \otimes \mathcal{E}^*(\mathbb{C}P^n)$ is spanned by elements $a \otimes x^i$ for $a \in \mathcal{E}^*(X)$ and $0 \le i \le n$. We find that

$$p_!(a \otimes x^i) = p_!(p^*a \cdot \pi_2^*(x^i)) = a \cdot p_!(\pi_2^*(x^i)) = a \cdot \pi_!(x^i)$$

where $\pi\colon \mathbb{C}P^n \to pt$ and the last equality is by push-pull. But $x^i = [\mathbb{C}P^{n-i}] = j_!(1)$ where $j\colon \mathbb{C}P^{n-i} \hookrightarrow \mathbb{C}P^n$ and so $\pi_!(x^i) = \pi_!(j_!(1)) = \pi_!(1) = \text{genus}_{\mathcal{E}}(\mathbb{C}P^{n-1})$. So $p_!$ is determined by the formulas

$$p_!(a \otimes x^i) = a \cdot \text{genus}_{\mathcal{E}}(\mathbb{C}P^{n-i}). \tag{41.13}$$

Since every proper map $f\colon X \to Y$ can be factored as $X \hookrightarrow Y \times \mathbb{C}P^n \to Y$, we see that general pushforwards $f_!$ are determined by the pushforwards of closed embeddings together with (41.13).

Example 41.14. Since $H^*(pt)$ vanishes in negative degrees, the H-genus for any positive-dimensional manifold is necessarily zero. If X is zero-dimensional and $f\colon X \to pt$ then we need to calculate $f_!\colon H^0(X) \to H^0(pt) = \mathbb{Z}$. Let $x_1, \dots, x_k$ be the points of X, and also write $x_i\colon pt \hookrightarrow X$ for the corresponding inclusion. Then $1 = \sum [x_i]$ as elements of $H^0(X)$, and also $[x_i] = (x_i)_!(1)$. So

$$f_!(1) = \sum_i f_!([x_i]) = \sum_i f_!((x_i)_!(1)) = \sum_i (f \circ x_i)_!(1) = \sum_i 1 = k.$$

So

$$H\text{-genus}(X) = \begin{cases} 0 & \text{if } \dim X > 0, \\ \#X & \text{if } \dim X = 0. \end{cases}$$

This is a silly invariant, but it is what the theory gives us.

Exercise 41.15. Let X and Y be almost complex manifolds. Prove that $\text{genus}_{\mathcal{E}}(X \amalg Y) = \text{genus}_{\mathcal{E}}(X) + \text{genus}_{\mathcal{E}}(Y)$ and $\text{genus}_{\mathcal{E}}(X \times Y) = \text{genus}_{\mathcal{E}}(X) \cdot \text{genus}_{\mathcal{E}}(Y)$.

While we are on this subject, let us explore the pushforwards in H a little more. If $f\colon X \to pt$ then $f_!$ is a map

$$f_!\colon H^i(X) \to H^{i-2\dim X}(pt).$$

The target is zero except when $i = 2\dim X$. But when X is connected $H^{2\dim X}(X) = \mathbb{Z}$, generated by $[*] = j_!(1)$ where $j\colon pt \hookrightarrow X$ is any choice of point. So $f_!([*]) = f_!(j_!(1)) = 1$. We have therefore shown that $f_!\colon H^*(X) \to H^*(pt)$ sends an element $\alpha \in H^*(X)$ to the coefficient of $[*]$ appearing in its $2\dim(X)$-dimensional homogeneous piece. Usually it will be convenient to just say "$f_!(\alpha)$ is the top-dimensional piece of α", though we really mean the coefficient of $[*]$ in that top-dimensional piece. That coefficient is often denoted $\int \alpha$, because this is how the integral works in de Rham cohomology. So one would write

$$f_!(\alpha) = \int \alpha,$$

but keep in mind that the integral symbol is just notation here—it doesn't depend on an analytical theory of integration in the background.

If $F \to E \xrightarrow{p} B$ is a fiber bundle where all of the spaces are oriented manifolds and F is compact, there is the "integration over the fiber" map $\int_F \colon H^i_{dR}(E) \to H^{i-\dim F}_{dR}(B)$ on de Rham cohomology. It can be proven that when the manifolds are almost complex then $\int_F$ agrees with the pushforward $p_!$ after tensoring with $\mathbb{R}$. We will never need this fact, but one should be aware that the pushforwards $f_!$ in $\mathcal{E}$-theory are often described colloquially as "integration over the fiber" for this reason.

Example 41.16. We will next calculate the Todd genus of $\mathbb{C}P^n$ by elementary means. This is not an idle example—the calculation will be used in our proof of the GRR theorem.

The trick for doing this, due to Quillen, is to instead do a more general calculation. Let $B = (\mathbb{C}P^\infty)^{\times r}$, let $L_i = \pi_i^*(L)$, and let $E = \oplus_{i=1}^r L_i$. Let $\pi \colon \mathbb{P}E \to B$ be the projective bundle. We will prove that $\pi_!(1) = 1$ in $K^0(B)$.

Recall from Proposition 38.22 that $K^0(B) = \mathbb{Z}[[x_1, \ldots, x_r]]$ where $x_i = e(L_i) = 1 - L_i^*$. Since $L_i L_i^* = 1$ in $K^0(B)$, L_i is invertible. We will write L_i^* and L_i^{-1} interchangeably.

Let $s_i \colon B \to \mathbb{P}E$ be the canonical section corresponding to the bundle L_i. This is an embedding, and the normal bundle is $N_i = \mathrm{Hom}(L_i, E/L_i) = \mathrm{Hom}(L_i, \oplus_{j\neq i} L_j) = \oplus_{j\neq i} L_i^* \otimes L_j$. (If you are worried about B and $\mathbb{P}E$ being infinite-dimensional, work with $\mathbb{C}P^M$ instead of $\mathbb{C}P^\infty$ and let $M \to \infty$). For the normal bundle calculation, this is just the fact that if $\ell_1, \ldots, \ell_r$ are complex lines and $V = \oplus_i \ell_i$ then the tangent space to ℓ_i in $\mathbb{P}(V)$ is naturally identified with $\mathrm{Hom}(\ell_i, V/\ell_i)$. We saw the argument for this in Example 37.19.

Let $e_i = e(N_i) \in K^0(B)$. Since N_i is a sum of line bundles the Whitney formula for the top Chern class gives $e_i = \prod_{j\neq i} e(L_i^* L_j) = \prod_{j\neq i}(1 - L_i L_j^*) = \prod_{j\neq i}(1 - L_i/L_j)$. Since $K^0(B)$ is a domain and $L_i \neq L_j$ (since $x_i = 1 - L_i^{-1}$ and $x_i \neq x_j$), we see that e_i is nonzero.

The pullbacks along the sections $s_1, \ldots, s_r$ together assemble to give a map $\underline{s}^* \colon K^0(\mathbb{P}E) \to K^0(B)^{\oplus(r)}$. Use the projective bundle theorem to identify $K^0(\mathbb{P}E)$ with $K^0(B)\langle 1, e, \ldots, e^{r-1}\rangle$ where $e = e(\mathcal{L}_E)$ and $\mathcal{L}_E$ is the tautological line bundle. Then $\underline{s}^*$ sends a polynomial $p(e)$ to $(p(x_1), \ldots, p(x_r))$, since $s_i^* \mathcal{L}_E = L_i$ and so $s_i^*(e(\mathcal{L}_E)) = e(L_i) = x_i$. So if $\underline{s}^*(p) = 0$ then the vector of coefficients of p is annihilated by the Vandermonde matrix $V(x_1, \ldots, x_r)$. But the determinant of this matrix is $\prod_{i<j}(x_j - x_i)$, which is a nonzerodivisor in $K^0(B)$. It follows that the coefficients of p must all vanish, and hence $\underline{s}^*$ is injective.

After the above build-up, we can now get to the heart of the argument. We start with the formulas

$$s_i^*((s_i)_!(1)) = e_i, \qquad s_j^*((s_i)_!(1)) = 0 \text{ if } i \neq j. \tag{41.17}$$

The first is from Proposition 37.6 (recall this works in any complex-oriented cohomology theory). The second follows from the fact that s_i and s_j have disjoint images, which implies that s_j factors through $\mathbb{P}E - s_i(B)$. So we can write down

the diagram

$$\begin{array}{ccccccc} K^0(B) & \xrightarrow{(s_i)^{rel}_!} & K^0(\mathbb{P}E, \mathbb{P}E - s_i(B)) & \xrightarrow{i^*} & K^0(\mathbb{P}E) & \longrightarrow & K^0(\mathbb{P}E - s_i(B)) \\ & & & & \downarrow{\scriptstyle s_j^*} & \swarrow{\scriptstyle s_j^*} & \\ & & & & K^0(B). & & \end{array}$$

Use that $(s_i)_! = i^*(s_i)^{rel}_!$ and that the composite of the right two horizontal maps is zero.

Let $e_{all} = \prod_j e_j$. Then (41.17) implies that

$$\underline{s}^*\Big(\sum_i \tfrac{e_{all}}{e_i}(s_i)_!(1)\Big) = (e_{all}, \dots, e_{all}) = \underline{s}^*(e_{all})$$

so by injectivity of $\underline{s}^*$ we have $\sum_i \frac{e_{all}}{e_i}(s_i)_!(1) = e_{all}$ in $K^0(\mathbb{P}E)$. Observe that the fractions $\frac{e_{all}}{e_i}$ can be taken as shorthand for $\prod_{j\neq i} e_j$, though one could also work in the localized ring obtained by inverting all of the e_i.

Applying $\pi_!$ to the last formula and using the projection formula multiple times gives

$$e_{all} \cdot \pi_!(1) = \pi_!(e_{all}) = \sum_i \tfrac{e_{all}}{e_i}\pi_!(s_i)_!(1) = \sum_i \tfrac{e_{all}}{e_i}.$$

The last equality uses that $\pi \circ s_i = \mathrm{id}$ for all i.

We next claim that there is a purely algebraic identity in $K^0(B)$ of the form

$$\sum_i \tfrac{e_{all}}{e_i} = e_{all}.$$

In fact, this can just be proven as a universal identity in the ring of Laurent polynomials $\mathbb{Z}[L_1^{\pm 1}, \dots, L_r^{\pm 1}]$, obtained by manipulating Vandermonde determinants. See Exercise 41.18 below.

We now have that $e_{all} \cdot \pi_!(1) = e_{all}$. But $K^0(B)$ is a domain and e_{all} is nonzero, so we can cancel to get $\pi_!(1) = 1$.

Finally, let us return to the original setting of $\mathbb{C}P^n \to *$. If we set $r = n+1$ then there is a transverse pullback diagram

$$\begin{array}{ccc} \mathbb{C}P^n & \xrightarrow{F} & \mathbb{P}E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ * & \xrightarrow{f} & B \end{array}$$

and so $\pi_!(F^*(1)) = f^*(\pi_!(1))$. So now we have

$$\text{Td-genus}(\mathbb{C}P^n) = \pi_!(1) = \pi_!(F^*(1)) = f^*(\pi_!(1)) = f^*(1) = 1.$$

This completes our calculation.

We will return to these ideas in Section 46, where we generalize them to all complex-oriented cohomology theories.

Exercise 41.18. Let $R = \mathbb{Z}[L_1^{\pm 1}, \dots, L_r^{\pm 1}]$. Define $\Delta(a_1, \dots, a_s) = \prod_{i<j}(a_j - a_i)$, and set $\Delta = \Delta(L_1, \dots, L_r)$ and $\Delta(i) = \frac{\Delta}{\Delta(L_1, \dots, \hat{L}_i, \dots, L_r)}$. Let $e_i = \prod_{j\neq i}(1 - L_i/L_j)$.

(a) Check that $\Delta(i) = (-1)^{i-1}\prod_{j\neq i}(L_j - L_i)$.

(b) Set $L_{all} = \prod_i L_i$. Prove that $e_i = \frac{L_i}{L_{all}}(-1)^{(i-1)}\Delta(i)$.

(c) Consider the Vandermonde matrix $V(L_1, \dots, L_r)$, whose determinant is Δ. Think about what happens when you scale the ith column by L_i^{-1}, for all i simultaneously. Use a cofactor expansion of the resulting determinant to prove the identity

$$\sum_i \tfrac{\Delta}{e_i} = \Delta$$

in R.

(d) Check that $e_{all} = \prod_i e_i$ is equal to $(-1)^{\binom{r}{2}} L_{all}^{1-r} \cdot \Delta$ and conclude that $\sum_i \frac{e_{all}}{e_i} = e_{all}$. This is the identity used in the above proof.

[Note: There is of course a temptation to just invert Δ (or equivalently, e_{all}) and write these formulas as $\sum_i \frac{1}{e_i} = 1$ in the localized ring. There is nothing wrong with that, but the arguments are basically the same either way.]

Remark 41.19. The above proof calculated $\pi_!(1)$ where $\pi\colon \mathbb{P}E \to B$. The same proof does not work if we specialize to $B = *$ and $E = \mathbb{C}^{n+1}$, as then the Euler classes $e_i = e(N_i)$ are all zero and most of the formulas that appeared just say $0 = 0$. The magic of the above proof is to work in a generic setting where all of the e_i are nonzerodivisors, and then only do the specialization at the very end.

41.20. **Pushforwards and Euler classes.** Let X be an almost complex manifold and let $M \to X$ be a complex bundle. Suppose that $s\colon X \to J$ is a smooth section such that the pullback

$$\begin{array}{ccc} s^{-1}(0) & \xrightarrow{i} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{\zeta} & M \end{array}$$

is transverse. Then

$$[s^{-1}(0)] = j_!(1) = j_!(i^*1) = \zeta^* s_!(1) = \zeta^* \zeta_!(1) = \zeta^* \mathcal{U}_M = e^{\mathcal{E}}(M).$$

The fourth equality is because $\zeta \simeq s$, using Proposition 41.3, and the fifth equality is because M is the normal bundle of the embedding ζ. We have therefore proven the following:

Proposition 41.21. *Let $M \to X$ be a complex bundle over an almost complex manifold. If $s\colon X \to J$ is a smooth section that intersects the zero-section transversely, then $e^{\mathcal{E}}(M) = [s^{-1}(0)]$.*

Exercise 41.22. Check that the straight-line homotopy $H\colon X \times I \to M$ given by $H(x,t) = ts(x)$ is a proper map, so that Proposition 41.3 does indeed apply.

Exercise 41.23. Let $n = \dim X$ and suppose $\operatorname{rank} M > 2n$. Use a dimension count to show that the only way the above pullback square can be transverse is if $s^{-1}(0) = \emptyset$. This forces the Euler class to vanish. In case this seems confusing, show via homotopical methods that any complex bundle of rank larger than $2n$ must split off a trivial line bundle and therefore have $e^{\mathcal{E}}(M) = 0$. [Hint: Use the connectivity of the map $BU(r) \to BU(r+1)$.]

41.24. **The excess intersection formula.** Recall the notion of clean pullbacks from Remark 40.5. As one final topic in this section, we observe that the push-pull formulas for transverse pullbacks generalize to clean pullbacks in a certain way. To make the statement we first need to address an orientation issue. Suppose

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{41.25}$$

is a clean pullback diagram, where f is complex-oriented. Let (n, i, J) be a triple representing the complex structure. As usual, we pull back $i\colon X \hookrightarrow Y \times \mathring{D}^{2n}$ to $i'\colon X' \hookrightarrow Y' \times \mathring{D}^{2n}$. The fact that the above square is a clean pullback implies that

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Y' \times \mathring{D}^{2n} & \xrightarrow{g\times\mathrm{id}} & Y \times \mathring{D}^{2n} \end{array}$$

is also a clean pullback. This then gives that $D(g \times \mathrm{id})\colon N_{i'} \to N_i$ is a fiberwise injection, or equivalently that we have an injection of real bundles $D(g\times\mathrm{id})\colon N_{i'} \to (g \times \mathrm{id})^* N_i$. The operator J gives a complex structure on the target.

Definition 41.26. *For a clean pullback square as above, say that f and f' are* ***compatibly complex-oriented*** *if there exists an (n, i, J) representing the complex orientation of f such that $D(g\times\mathrm{id})(N_{i'})$ is a complex subbundle of $(g\times\mathrm{id})^* N_i$ and, if J' is the induced complex structure on $N_{i'}$, the triple (n, i', J') represents the complex orientation on f'.*

Remark 41.27. The most common application is where (41.25) is a diagram of complex algebraic varieties and f and f' are closed immersions, with the canonical complex structure on their normal bundles. In this case the compatibility is self-evident.

Proposition 41.28. *Let $\mathcal{E}$ be a complex-oriented cohomology theory. Suppose given a clean pullback diagram (41.25) where f is proper and the pair (f, f') is compatibly complex-oriented. Let $\eta \to X'$ denote the excess normal bundle. Then for $a \in \mathcal{E}^*(X)$ we have*

$$g^* f_!(a) = (f')_!((g')^* \cdot e^{\mathcal{E}}(\eta)).$$

When the clean pullback square is actually transverse then η is the zero bundle, $e^{\mathcal{E}}(\eta) = 1$, and therefore the excess intersection formula reduces to the usual push-pull formula.

It is useful to think about the case where f and f' are closed embeddings (the proof quickly reduces to this case anyway). The clean condition implies that $Dg\colon N_{f'} \to N_f$ is injective on fibers, and then it readily follows that

$$0 \longrightarrow N_{f'} \xrightarrow{Dg} (g')^* N_f \longrightarrow \eta \to 0$$

is an exact sequence. This justifies the name "excess normal bundle". The key observation is the following:

Lemma 41.29. *Let A and B be complex vector bundles on X with structure maps p_A and p_B, and let $i\colon A \hookrightarrow A \oplus B$ be the inclusion. Then in any complex-oriented cohomology theory $\mathcal{E}$ we have $i^*\mathcal{U}_{A\oplus B} = \mathcal{U}_A \cdot e(B)$, where the product is via the algebra map $p_A^*\colon \mathcal{E}^*(X) \to \mathcal{E}^*(A)$.*

Proof. We know that $\mathcal{U}_{A\oplus B}$ is the external product of $\mathcal{U}_A$ and $\mathcal{U}_B$. This means it equals the product of $\pi_1^*(\mathcal{U}_A) \in \mathcal{E}^*(A\oplus B, (A\oplus B) - (0\oplus B))$ and $\pi_2^*(\mathcal{U}_B) \in \mathcal{E}^*(A\oplus B, (A\oplus B) - (A \oplus 0))$. Now apply i^* and use that $\pi_1 \circ i = \mathrm{id}$, whereas $\pi_2 \circ i = \zeta_B \circ p_A$. We get that

$$i^*(\mathcal{U}_{A\oplus B}) = \mathcal{U}_A \cdot p_A^*(\zeta_B^*(\mathcal{U}_B)) = \mathcal{U}_A \cdot p_A^*(e(B)) = \mathcal{U}_A \cdot e(B).$$

□

Proof of Proposition 41.28. Choose a triple (n, i, J) representing the complex orientation on f and where (n, i', J') represents the complex orientation on f' (definitions in the above discussion). Write $N = N_i$ and $N' = N_{i'}$ for the normal bundles. Suppressing superscripts in our diagrams, we have

$$\begin{array}{ccccc}
\mathcal{E}^*(X) & \xrightarrow{i_!} & \mathcal{E}^*(Y \times D^{2n}, Y \times \partial D^{2n}) & \xrightarrow{p_!} & \mathcal{E}^*(Y) \\
\downarrow{\scriptstyle (g')^*} & & \downarrow{\scriptstyle (g\times \mathrm{id})^*} & & \downarrow{\scriptstyle g^*} \\
\mathcal{E}^*(X') & \xrightarrow{i'_!} & \mathcal{E}^*(Y' \times D^{2n}, Y \times \partial D^{2n}) & \xrightarrow{p'_!} & \mathcal{E}^*(Y')
\end{array}$$

where the horizontal composites are $f_!$ and $f'_!$ (see Remark 41.8). The right square commutes by the usual push-pull for the square of projections (which is transverse), so we can focus on the left square. Choosing tubular neighborhoods U and U' such that $U' \subseteq g^{-1}(U)$, the left square expands into

$$\begin{array}{ccccccc}
\mathcal{E}^*(X) & \xrightarrow{th_N} & \mathcal{E}^*(N, N-0) & = & \mathcal{E}^*(U, U-0) & \xleftarrow{\cong} & \mathcal{E}^*(YD^{2n}, YD^{2n} - X) \\
\downarrow{\scriptstyle (g')^*} & & \downarrow{\scriptstyle (Dg)^*} & & \downarrow{\scriptstyle (g\times \mathrm{id})^*} & & \downarrow{\scriptstyle (g\times \mathrm{id})^*} \\
\mathcal{E}^*(X') & \xrightarrow{th_{N'}} & \mathcal{E}^*(N', N'-0) & = & \mathcal{E}^*(U', U'-0) & \xleftarrow{\cong} & \mathcal{E}^*(Y'D^{2n}, Y'D^{2n} - X')
\end{array}$$

(as we have done before, we are suppressing product symbols so that $YD = Y \times D$, etc.) The same proof as for Proposition 40.7 shows that the middle square commutes. The left square does not commute, but writing q for the bundle projections we instead have

$$\begin{aligned}
(Dg)^*(th_N(a)) = (Dg)^*(q^*a \cdot \mathcal{U}_N) &= (Dg)^*q^*a \cdot (Dg)^*\mathcal{U}_N \\
&= (q')^*(g')^*a \cdot \mathcal{U}_{N'} \cdot (q')^*e(\eta) \\
&= (q')^*((g')^*a \cdot e(\eta)) \cdot \mathcal{U}_{N'} \\
&= th_{N'}((g')^*a \cdot e(\eta)).
\end{aligned}$$

where the third equality is by Lemma 41.29 using that $(g')^*N \cong N' \oplus \eta$. So we get

$$(g \times \mathrm{id})^* i_!(a) = i'_!((g')^*a \cdot e(\eta))$$

and this quickly yields the desired identity. □

Example 41.30. Let $\mathrm{Gr} = \mathrm{Gr}_2(\mathbb{C}^4)$. Fix a $V \subseteq \mathbb{C}^4$ with $\dim V = 3$ and a line $\ell \subseteq V$. Let $A = \{M \in \mathrm{Gr} \mid \ell \subseteq M\}$ and $B = \{M \in \mathrm{Gr} \mid M \subseteq V\}$. Then $A \cong$

$\mathbb{P}(\mathbb{C}^4/\ell) \cong \mathbb{C}P^2$ and $B \cong \mathrm{Gr}_2(\mathbb{C}^3) \cong \mathbb{C}P^2$. Generically $A \cap B$ should have dimension zero, but instead we have $A \cap B = \{M \in \mathrm{Gr} \mid \ell \subseteq M \subseteq V\} \cong \mathbb{P}(V/\ell) \cong \mathbb{C}P^1$.

We have the two fundamental classes $[A], [B] \in H^2(\mathrm{Gr})$. We claim that $[A]\cdot[B] = 0$. One way to see this is to prove that the homology classes are independent of the choice of ℓ and V; if we move ℓ outside of V then A moves to a new A' where we have $A' \cap B = \emptyset$ instead. But let us instead do the calculation via the excess intersection formula, which avoids having to move anything.

Consider the square of inclusions

$$\begin{array}{ccc} A \cap B & \xrightarrow{i'} & B \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{i} & \mathrm{Gr}\,. \end{array}$$

For any $M \in A \cap B$ one can interpret the tangent spaces algebraically as

$$T_M \,\mathrm{Gr} = \mathrm{Hom}(M, \mathbb{C}^4/M), \quad T_M A = \mathrm{Hom}(M/\ell, \mathbb{C}^4/M),$$
$$T_M B = \mathrm{Hom}(M, V/M), \quad T_M(A \cap B) = \mathrm{Hom}(M/\ell, V/M).$$

Here we regard $\mathrm{Hom}(M/\ell, \mathbb{C}^4/M)$ as a subset of $\mathrm{Hom}(M, \mathbb{C}^4/M)$ in the evident way, and similar for the other inclusions. Clearly the square of tangent spaces is a pullback, and so we have a clean pullback diagram. If η is the excess normal bundle then the excess intersection formula gives that $i^*(j_!(1)) = (j')_!((i')^*(1) \cdot e(\eta)) = (j')_!(e(\eta))$. Then

$$[A] \cdot [B] = i_!(1) \cdot j_!(1) = i_!(1 \cdot i^*(j_!(1))) = i_!((j')_!(e(\eta))).$$

It remains to calculate the Euler class $e(\eta)$. Observe that the canonical map $\mathrm{Hom}(M, \mathbb{C}^4/M) \to \mathrm{Hom}(\ell, \mathbb{C}^4/V)$ (obtained by restriction and projection) annhilates both $T_M A$ and $T_M B$, hence descends to a map

$$\eta_M \to \mathrm{Hom}(\ell, \mathbb{C}^4/V).$$

Both the domain and codomain are one-dimensional, and the map is readily seen to be surjective. This shows that $\eta \cong \underline{1}$, hence $e(\eta) = 0$, and therefore $[A] \cdot [B] = 0$ as we expected.

Exercise 41.31. Explain the algebraic models of the four tangent spaces in the above example.

42. The topological Grothendieck-Riemann-Roch Theorem

As we present it here, the Grothendieck-Riemann-Roch (GRR) Theorem really has two components: one that is purely topological, and one that is algebro-geometric. The topological part is a comparison between the complex-oriented structures on K-theory and singular cohomology, and gives precise formulas for how they line up under the Chern character. From the perspective we have adopted in this book, this topological GRR theorem is fairly easy. The algebro-geometric component, on the other hand, is of a somewhat different nature; in our presentation it is a comparison between algebraic and topological K-theory, showing that certain topologically-defined maps are compatible with purely algebraic ones that at first glance appear quite different. This second part of the GRR theorem lets us see that certain algebraic constructions actually give topological invariants, whereas

the first part leads to precise (although often complicated) topological formulas for these invariants.

In the present section we discuss the topological GRR theorem and some of its consequences. The next section will deal with the algebro-geometric version.

42.1. **The Todd class.** We have seen that for a line bundle $J \to X$ one has

$$\mathrm{ch}(e^K(J)) = (1 - e^{-x})|_{x=e^H(J)}.$$

The power series $1 - e^{-x}$ is a multiple of x, which means that the right-hand-side can be written as $e^H(J)$ multiplied by a 'correction factor':

$$\mathrm{ch}(e^K(J)) = e^H(J) \cdot \Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=e^H(J)}.$$

It is useful to have a name for this correction factor; for historical reasons, the name is actually attached to its inverse. We define the **Todd class** of J to be

$$\begin{aligned}\mathrm{Td}(J) = \Big(\frac{x}{1-e^{-x}}\Big)\Big|_{x=c_1(L)} &= \frac{1}{1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^3}{24}+\cdots}\Big|_{x=c_1(J)} \\ &= \Big(1+\tfrac{x}{2}+\tfrac{x^2}{12}-\tfrac{x^4}{720}+\cdots\Big)\Big|_{x=c_1(J).}\end{aligned}$$

The coefficients in this power series are related to Bernoulli numbers, and we refer the reader to Appendix E for a review of the basics about these. The Bernoulli numbers are defined by $\frac{x}{e^x-1} = \sum_i \frac{B_i}{i!}x^i$, and so we have

$$\mathrm{Td}(J) = \sum_i (-1)^i \tfrac{B_i}{i!} c_1(J)^i.$$

Next observe that if $E \to X$ is a sum of line bundles $J_1 \oplus \cdots \oplus J_k$ then

$$\begin{aligned}\mathrm{ch}(e^K(E)) &= \mathrm{ch}(e^K(J_1)\cdots e^K(J_k)) \\ &= \mathrm{ch}(e^K(J_1))\cdots\mathrm{ch}(e^K(J_k)) \\ &= [e^H(J_1)\cdots e^H(J_k)]\cdot\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(J_1)}\cdots\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(J_k)} \\ &= e^H(E)\cdot\prod_i\Big[\frac{1-e^{-x}}{x}\Big]\Big|_{x=c_1(J_i)}.\end{aligned}$$

It therefore makes sense to define $\mathrm{Td}(E)$ to be the inverse of the product in the final formula. More generally, if E is a bundle of rank k then the Todd class of E is

$$\mathrm{Td}(E) = \prod_{i=1}^k \Big(\frac{x_i}{1-e^{-x_i}}\Big)$$

where $c_i(E) = \sigma_i(x_1,\ldots,x_k)$. In other words, take the expression on the right and write each homogeneous piece as a polynomial in the elementary symmetric functions. Then replace those symmetric functions with the Chern classes of E, and one gets the Todd class. For example, if $\mathrm{rank}\, E = 2$ then we would expand

$$\big(1+\tfrac{x}{2}+\tfrac{x^2}{12}-\tfrac{x^4}{720}+\cdots\big)\cdot\big(1+\tfrac{y}{2}+\tfrac{y^2}{12}-\tfrac{y^4}{720}+\cdots\big)$$

to get

$$1+\tfrac{1}{2}(x+y)+\tfrac{1}{12}(x^2+y^2)+\tfrac{1}{4}xy+\tfrac{x^2y+xy^2}{24}-\tfrac{x^4+y^4}{720}+\tfrac{x^2y^2}{144}+\cdots$$

and then write this as

$$1 + \tfrac{1}{2}\sigma_1 + \tfrac{1}{12}(\sigma_1^2 - 2\sigma_2) + \tfrac{1}{4}\sigma_2 + \tfrac{1}{24}(\sigma_1\sigma_2) + \tfrac{1}{720}(-\sigma_1^4 + 4\sigma_1^2\sigma_2 - 2\sigma_2^2) + \tfrac{\sigma_2^2}{144}.$$

Then replace each σ_i with $c_i(E)$ to get the formula for $\mathrm{Td}(E)$.

The first few terms of the Todd class of an arbitrary bundle are

$$\mathrm{Td}(E) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 - c_4 + c_1 c_3 + 3c_2^2}{720} + \cdots \tag{42.2}$$

(Check that when $c_3 = c_4 = 0$ this agrees with what we calculated above.) The homogeneous components of this series are called the **Todd polynomials**. The fourth Todd polynomial, seen as the last term in the formula above, gives an indication of the growing complexity—particularly in the size of the denominators.

We will have more to say about computing the Todd class in Section 42.19 below. But let us now turn to the study of how it measures the comparison between K-theory and singular cohomology. As we have remarked above, one should think of the Todd class as a 'correction factor'. The most basic formula where it enters is

$$\mathrm{ch}(e^K(E)) = e^H(E) \cdot \mathrm{Td}(E)^{-1}.$$

In our above analysis we showed this when E is a sum of line bundles, but the general case readily follows from this one using the splitting principle. A very similar formula, which actually implies the above one, is the following:

Proposition 42.3. *Let $E \to X$ be a complex vector bundle. Then*

$$\mathrm{ch}(\mathcal{U}_E^K) = \mathcal{U}_E^H \cdot \mathrm{Td}(E)^{-1}.$$

That is to say, applying the Chern character to a K-theoretic Thom class does not quite give the H-theoretic Thom class—one needs the Todd class correction factor. Note, by the way, that it does not matter whether we write $\mathcal{U}_E^H \cdot \mathrm{Td}(E)^{-1}$ or $\mathrm{Td}(E)^{-1} \cdot \mathcal{U}_E^H$ in the above result, since both the Thom class and the Todd class are concentrated in even degrees.

Proof. The proof has four steps:

Step 1: If the result is true for sums of line bundles, it is true for all bundles.
Step 2: If the result is true for line bundles, it is true for all sums of line bundles.
Step 3: If the result is true for the tautological line bundle over $\mathbb{C}P^\infty$, it is true for all line bundles.
Step 4: The result is true for the tautological line bundle $L \to \mathbb{C}P^\infty$.

Step 1 is a direct consequence of the splitting principle (Proposition 38.14). Indeed, if $E \to X$ is a bundle then choose a map $p\colon \widetilde{X} \to X$ such that p^*E is a sum of line bundles and such that p^* induces a monomorphism in singular cohomology. If $\widetilde{E} = p^*E$, the claim follows at once from the commutative diagram

$$\begin{array}{ccccc}
K^0(E, E-0) & \xrightarrow{\mathrm{ch}} & H^*(E, E-0) & \xleftarrow[\cong]{th_E} & H^*(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p^*} \\
K^0(\widetilde{E}, \widetilde{E}-0) & \xrightarrow{\mathrm{ch}} & H^*(\widetilde{E}, \widetilde{E}-0) & \xleftarrow[th_{\widetilde{E}}]{\cong} & H^*(\widetilde{X}).
\end{array}$$

The middle vertical map is injective because because the right p^* is.

Step 2 follows from the fact that $\mathcal{U}_{L_1\oplus L_2\oplus\cdots\oplus L_r} = \mathcal{U}_{L_1}\times\mathcal{U}_{L_2}\times\cdots\times\mathcal{U}_{L_r}$, $\mathrm{Td}(\oplus_i L_i) = \prod_i \mathrm{Td}(L_i)$, and the fact that ch is multiplicative. Step 3 follows at once from naturality and the fact that every line bundle is pulled back from the tautological line bundle.

So we are reduced to Step 4, which is a calculation. Consider the zero section $\zeta\colon \mathbb{C}P^\infty \hookrightarrow L$ and the composite of natural maps

$$H^*(L, L-0) \xrightarrow{j^*} H^*(L) \xrightarrow{\zeta^*} H^*(\mathbb{C}P^\infty).$$

Recall that this composite sends $\mathcal{U}_L$ to the Euler class $e^H(L)$. The map ζ^* is an isomorphism by homotopy invariance, and the map j^* is also an isomorphism: the latter follows from the long exact sequence for the pair $(L, L-0)$ together with the fact that as spaces $L - 0 \cong \mathbb{C}^\infty - 0$ and is therefore contractible.

Consider the two elements $\mathrm{ch}(\mathcal{U}_L^K)$ and $\mathcal{U}_L^H\cdot\mathrm{Td}(L)^{-1}$ in $H^*(L, L-0)$. Applying the composite $\zeta^*\circ j^*$ sends the first to $\mathrm{ch}(e^K(L))$, by naturality. Likewise, the second is sent to $e^H(L)\cdot\mathrm{Td}(L)^{-1}$. We have already computed that these two images are the same (indeed, this is how we started off this section); since $\zeta^*\circ j^*$ is an isomorphism this means $\mathrm{ch}(\mathcal{U}_L^K) = \mathcal{U}_L^H\cdot\mathrm{Td}(L)^{-1}$. □

42.4. **The Grothendieck-Riemann-Roch Theorem for embeddings.** Let $j\colon X\hookrightarrow Y$ be a smooth embedding of smooth manifolds, equipped with a complex structure on the normal bundle. Let c be the complex dimension of the normal bundle. We saw in Sections 26 and 27 that any complex-oriented cohomology theory admits pushforward maps $j_!$, and in particular one gets $j_!\colon K^*(X)\to K^{*+2c}(Y)$. We will take advantage of Bott periodicity to write $j_!$ as a map $K^0(X)\to K^0(Y)$.

Consider the square

$$\begin{array}{ccc} K^0(X) & \xrightarrow{j_!} & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X;\mathbb{Q}) & \xrightarrow{j_!} & H^{ev}(Y;\mathbb{Q}). \end{array}$$

This square does not commute; while this might seem strange, the point is just that the $j_!$ maps are defined using Thom classes and ch doesn't preserve these. But since ch almost preserves Thom classes, up to a correction factor, it follows that the above square almost commutes—up to the same factor. The precise result is as follows:

Proposition 42.5. *Let $j\colon X\hookrightarrow Y$ be a smooth embedding of manifolds, equipped with a complex structure on the normal bundle. Then for any $\alpha\in K^0(X)$ one has*

$$\mathrm{ch}(j_!\alpha) = j_!(\mathrm{Td}(N_{Y/X})^{-1}\cdot\mathrm{ch}(\alpha)).$$

Proof. Simply consider the diagram

$$\begin{array}{ccccccc} K^0(X) & \xrightarrow{th_N^K} & K^0(N,N-0) & \xleftarrow{\cong} & K^0(Y,Y-X) & \longrightarrow & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X) & \xrightarrow{th_N^H} & H^{ev}(N,N-0) & \xleftarrow{\cong} & H^{ev}(Y,Y-X) & \longrightarrow & H^{ev}(Y) \end{array}$$

where all singular cohomology groups have rational coefficients. The left horizontal arrows are the Thom isomorphism maps, and so the leftmost square does not

commute; but the other two squares do. The compositions across the two rows are the pushforward maps $j_!$ in K-theory and singular cohomology, respectively. The desired result is now an easy application of Proposition 42.3. □

The next result records how the Chern character behaves on fundamental classes:

Corollary 42.6. *Let $j\colon X \hookrightarrow Y$ be a smooth embedding with a complex structure on its normal bundle. Then* $\mathrm{ch}([X]_K) = j_!(\mathrm{Td}(N_{Y/X})^{-1}) = [X]_H +$ (*higher degree terms*).

Proof. Recall that $[X]_K = j_!(1)$, and so the first equality is just Proposition 42.5 applied to $\alpha = 1$. The second equality then follows directly from the fact that the Todd class of a bundle has the form $1 +$ higher degree terms together with $j_!(1) = [X]_H$. □

Example 42.7. Let $j\colon Z \hookrightarrow \mathbb{C}P^n$ be a hypersurface of degree d, and consider the GRR square

$$\begin{array}{ccc} K^0(Z) & \xrightarrow{j_!} & K^0(\mathbb{C}P^n) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(Z;\mathbb{Q}) & \xrightarrow{j_!} & H^{ev}(\mathbb{C}P^n;\mathbb{Q}). \end{array}$$

Recall that $K^0(\mathbb{C}P^n) = \mathbb{Z}[y]/(y^{n+1})$ where $y = 1 - L = [\mathbb{C}P^{n-1}]_K$, and that $H^*(\mathbb{C}P^n;\mathbb{Q}) = \mathbb{Q}[x]/(x^{n+1})$ where $x = [\mathbb{C}P^{n-1}]_H = c_1(L^*)$. One has

$$\mathrm{ch}(y) = \mathrm{ch}(1-L) = \mathrm{ch}(1) - \mathrm{ch}(L) = 1 - e^{c_1(L)} = 1 - e^{-x}.$$

We will determine a formula for $[Z]_K$ by using the GRR statement $\mathrm{ch}([Z]_K) = j_!(\mathrm{Td}(N)^{-1})$. The normal bundle is $N = j^*\big((L^*)^{\otimes d}\big)$ (Corollary 37.13). So

$$\mathrm{Td}(N)^{-1} = j^*\left(\frac{1-e^{-dx}}{dx}\right).$$

Recall that for any α one has the projection formula $j_!(j^*(\alpha)) = j_!(j^*(\alpha)\cdot 1) = \alpha \cdot j_!(1)$, and of course we know that $j_!(1) = dx$. We therefore conclude that

$$\mathrm{ch}([Z]_K) = \left(\frac{1-e^{-dx}}{dx}\right)\cdot j_!(1) = \left(\frac{1-e^{-dx}}{dx}\right)\cdot dx = 1 - e^{-dx}.$$

We can now work backwards to determine $[Z]_K$. If we write

$$[Z]_K = a_1 y + a_2 y^2 + \cdots + a_n y^n$$

then

$$1 - e^{-dx} = \mathrm{ch}([Z]_K) = a_1(1-e^{-x}) + a_2(1-e^{-x})^2 + \cdots + a_n(1-e^{-x})^n.$$

Let $\alpha = e^{-x}$; then to determine the a_i's we need to expand $1-\alpha^d$ in terms of powers of $1-\alpha$. To do this, simply write

$$1 - \alpha^d = 1 - (1-(1-\alpha))^d = \sum_{k=1}^{d} (-1)^{k-1}\tbinom{d}{k}(1-\alpha)^k.$$

We conclude $[Z]_K = dy - \binom{d}{2}y^2 + \binom{d}{3}y^3 - \cdots$.

Of course, we have seen this calculation before in a slightly different form—see Example 33.8. But notice that GRR allowed us to carry it through without knowing anything about $[Z]_K$, whereas before we relied on the connection between

K-theoretic fundamental classes and resolutions (and our ability to write down an appropriate resolution in this case).

42.8. **The full topological GRR theorem.** Constructing pushforwards $f_!$ for proper maps f was a nontrivial undertaking in Section 41, but now that we have them the full version of GRR is not much harder to state than the version for closed embeddings:

Theorem 42.9 (Grothendieck-Riemann-Roch, full version)**.** *Let X and Y be almost complex manifolds and let $f\colon X \to Y$ be a smooth proper map equipped with its canonical complex orientation. Then in the square*

$$\begin{array}{ccc} K^0(X) & \xrightarrow{f_!} & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{ev}(X;\mathbb{Q}) & \xrightarrow{f_!} & H^{ev}(Y;\mathbb{Q}). \end{array}$$

one has

$$\mathrm{ch}(f_!\alpha)\cdot \mathrm{Td}(T_Y) = f_!(\mathrm{ch}(\alpha)\cdot \mathrm{Td}(T_X))$$

for all $\alpha \in K^0(X)$, where T_X and T_Y are the tangent bundles of X and Y.

Exercise 42.10. Check that when $X \hookrightarrow Y$ is an embedding then the above version of GRR is equivalent to the version given in Proposition 42.5.

In fact, we can state a version of Theorem 42.9 for any proper complex-oriented map f: we don't need X and Y to be almost complex manifolds. If f is complex-oriented then we have a complex normal bundle N that is well-defined up to the addition of trivial bundles, and so $\mathrm{Td}(N)$ is a well-defined element of $H^*(X;\mathbb{Q})$. Thus the following statement makes sense:

Theorem 42.11 (Full GRR, Version 2)**.** *Let $f\colon X \to Y$ be a complex-oriented map that is proper. Then for any $\alpha \in K^0(X)$ one has*

$$\mathrm{ch}(f_!\alpha) = f_!(\mathrm{ch}\,\alpha \cdot \mathrm{Td}(N)^{-1}).$$

Exercise 42.12. When X and Y are almost complex manifolds and f has the canonical complex orientation, prove that f^*T_Y and $T_X \oplus N$ are stably isomorphic. Deduce that $\mathrm{Td}(N) = \frac{f^*\,\mathrm{Td}(T_Y)}{\mathrm{Td}(T_X)}$ and explain how GRR Version 2 implies the version in Theorem 42.9.

We will prove the version in Theorem 42.9 and leave the reader the exercise of adapting the proof to get Theorem 42.11. But before tackling this let us explore some consequences. First consider $f\colon X \to pt$ where X is compact. Then GRR gives

$$\mathrm{ch}(f_!^K(1)) = f_!^H(\mathrm{Td}\,T_X).$$

On the left-hand-side, $f_!^K(1) = \text{Td-genus}(X)\cdot 1 \in K^0(*)$ and so $\mathrm{ch}(f_!^K(1)) = \text{Td-genus}(X)\cdot 1 \in H^0(pt;\mathbb{Q})$. To analyze the right-hand-side we recall from Example 41.14 above that $f_!^H\colon H^{ev}(X) \to H^{ev}(pt)$ sends a class α to the coefficient of $[*]$ in its top-dimensional piece (the component in dimension $2\dim X$). So GRR says that

$$\text{Td-genus}(X) = \text{top-dimensional coefficient of } \mathrm{Td}(T_X) = \int \mathrm{Td}(T_X).$$

That is, the Todd genus of X is the coefficient of the top-dimensional piece of the Todd class—this explains why the K-genus is called the "Todd genus".

One of the surprises here is that the right-hand-side is not *a priori* an integer: recall that the definition of the Todd class contains complicated denominators. The resulting integrality conditions can lead to some nonexistence results in topology, as demonstrated in the following example.

Example 42.13. We claim that there is no almost complex manifold whose underlying topological manifold is S^4. If S^4 were an almost complex manifold then it would have a Todd genus, which we know will be an integer. But GRR tells us that the Todd genus is also the top-dimensional component of $\mathrm{Td}(T)$, where T denotes the complex tangent bundle of our fictitious complex manifold. But $c_1(T) = 0$ because $H^2(S^4) = 0$, and $c_2(T) = 2[*]$ because $c_2(T)$ is the Euler class and $\chi(S^4) = 2$. Plugging into (42.2) we find that $\mathrm{Td}(T) = 1 + \frac{c_2}{12}$ and so the top-dimensional piece is $\frac{1}{6}$. As this is not an integer, we have arrived at a contradiction.

A similar argument shows that S^{4n} is not a complex manifold for any n, although to follow through with this we will need to get better at computing terms in the Todd class. We return to this problem in Proposition 42.24 below.

As one more sample computation let us use GRR to compute the Todd genus of $\mathbb{C}P^n$.

Example 42.14 (Todd genus of $\mathbb{C}P^n$)**.** Recall that $\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \cdot \mathrm{Td}(F)$, and that $1 \oplus T_{\mathbb{C}P^n} \cong (n+1)L^*$. So

$$\mathrm{Td}(T_{\mathbb{C}P^n}) = \mathrm{Td}(T_{\mathbb{C}P^n}) \cdot \mathrm{Td}(1) = \mathrm{Td}\big((n+1)L^*\big) = \big[\mathrm{Td}(L^*)\big]^{n+1}.$$

Recall that $c_1(L^*) = [\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n)$. Call this generator x, for short. Then

$$\mathrm{Td}(T_{\mathbb{C}P^n}) = \Big(\frac{x}{1-e^{-x}}\Big)^{n+1} = \Big(1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots\Big)^{n+1}.$$

Let's look at some examples. When $n = 1$ we have $x^2 = 0$ and so $\mathrm{Td}(T_{\mathbb{C}P^1}) = 1 + x$. When $n = 2$ we have $x^3 = 0$ and

$$\mathrm{Td}(T_{\mathbb{C}P^2}) = \Big(1 + \tfrac{x}{2} + \tfrac{x^2}{12}\Big)^3 = 1 + \tfrac{3}{2}x + x^2.$$

Finally, for $n = 3$ we have $x^4 = 0$ and

$$\mathrm{Td}(T_{\mathbb{C}P^3}) = \Big(1 + \tfrac{x}{2} + \tfrac{x^2}{12}\Big)^4 = 1 + 2x + \tfrac{11}{6}x^2 + x^3.$$

One discernible pattern in these polynomials is that the leading coefficient is always 1. This is an amusing exercise that we leave to the reader. It shows that Td-genus$(\mathbb{C}P^n) = 1$ for all n.

Exercise 42.15. Complete the above example by proving that the coefficient of x^n in $\big(\frac{x}{1-e^{-x}}\big)^{n+1}$ is equal to 1, for all n. One method is to interpret the coefficient as a residue:

$$\mathrm{Res}_{x=0}\big(\tfrac{1}{(1-e^{-x})^{n+1}}\big) = \tfrac{1}{2\pi i}\int_C \tfrac{1}{(1-e^{-x})^{n+1}}\,dx$$

where C is a small counterclockwise circle around the origin. Use the substitution $z = 1 - e^{-x}$ to convert this to a different residue that is easily computed. You will need to convince yourself that the mapping $x \mapsto 1 - e^{-x}$ takes C to another small counterclockwise loop around the origin.

Exercise 42.16. Let $Z \hookrightarrow \mathbb{C}P^n$ be a smooth hypersurface of degree d. Show that

$$\text{Td-genus}(Z) = d - \tbinom{d}{2} + \tbinom{d}{3} - \cdots + (-1)^{n+1}\tbinom{d}{n} = 1 - (-1)^n\tbinom{d-1}{n}.$$

Conclude that if $d \leq n$ then Td-genus$(Z) = 1$. [Hint: Use the expression of $[Z]_K$ as a linear combination of the $[\mathbb{C}P^i]_K$ classes. The second equality comes from a standard formula for alternating sums of binomial coefficients, but a direct way to prove it is to use that both sides are degree n polynomials in d and so it suffices to check the identity on n carefully chosen test values.]

42.17. **Proof of the full GRR theorem.**

Proof of Theorem 42.9. The proof is via the steps listed below. We will outline arguments in each case, but leave some of the details to the reader.

Step 1: The result is true when f is an embedding.
Step 2: The result is true when f is $\mathbb{C}P^n \to *$.
Step 3: The result is true when f is the projection $Y \times \mathbb{C}P^n \to Y$, for any almost complex manifold Y.
Step 4: The result is true in general.

Step 1 was handled in Proposition 42.5. Step 2 is just a computation, where one computes both sides of the GRR formula and sees that they are the same. Use that $K^0(\mathbb{C}P^n)$ is generated by the classes $[\mathbb{C}P^i]$. For the left side of GRR use that $[\mathbb{C}P^i]$ is mapped to 1 via $f_!$, as Td-genus$(\mathbb{C}P^i) = 1$ (Example 41.16). For the right side use that $\text{ch}([\mathbb{C}P^i]) = (1-e^{-x})^{n-i}$ and compute that the coefficient of x^n in the series $(1-e^{-x})^{n-i} \cdot \left(\frac{x}{1-e^{-x}}\right)^{n+1}$ is equal to 1. For this final piece use a method similar to what we did in Exercise 42.15 above.

For Step 3 consider the product map $K^0(Y) \times K^0(\mathbb{C}P^N) \to K^0(Y \times \mathbb{C}P^N)$. This is an isomorphism by the projective bundle theorem (Corollary 38.20), applied to the trivial bundle $\underline{N+1} \to Y$ and using the fact that the classes c_i^K of a trivial bundle all vanish. It therefore suffices to verify the GRR formula on classes of the form $\alpha = (p_1)^*(\beta) \cdot (p_2)^*(\gamma)$ where $\beta \in K^0(Y)$, $\gamma \in K^0(\mathbb{C}P^N)$, and p_1 and p_2 are the projections of $Y \times \mathbb{C}P^N$ onto Y and $\mathbb{C}P^N$, respectively. Use the diagram

$$\begin{array}{ccc} Y \times \mathbb{C}P^N & \xrightarrow{p_2} & \mathbb{C}P^N \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_1} \\ Y & \xrightarrow{\pi_2} & * \end{array}$$

and the formulas

$$(p_1)_!\big[p_1^*\beta \cdot p_2^*\gamma\big] = \beta \cdot (p_1)_!(p_2^*\gamma) = \beta \cdot \pi_2^*\big((\pi_1)_!\gamma\big)$$

(for both K-theory and singular cohomology)—note that the second equality is from property (4) of pushforwards. The desired result follows from Step 2 and the fact that $T_{Y\times\mathbb{C}P^N} \cong p_1^*T_Y \oplus p_2^*T_{\mathbb{C}P^N}$.

Finally, for Step 4 choose a smooth embedding $k\colon X \hookrightarrow \mathbb{C}^N$ for some N, by the Whitney embedding theorem [Lee, Theorem 10.11]. Let $j'\colon X \to Y \times \mathbb{C}^N$ be the map (f,k), which is injective. Let j be the composite $X \xrightarrow{j'} Y \times \mathbb{C}^N \hookrightarrow Y \times \mathbb{C}P^N$. The map j is proper by Exercise A.29, therefore closed, and since it is injective this makes it a closed embedding. In this way we have factored $f\colon X \to Y$ as

$X \xrightarrow{j} Y \times \mathbb{C}P^N \xrightarrow{\pi} Y$ where j is a smooth, closed embedding and π is projection. Use the diagram

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{j_!} & K^0(Y \times \mathbb{C}P^N) & \xrightarrow{\pi_!} & K^0(Y) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(X;\mathbb{Q}) & \xrightarrow{j_!} & H^*(Y \times \mathbb{C}P^N;\mathbb{Q}) & \xrightarrow{\pi_!} & H^*(Y;\mathbb{Q}), \end{array}$$

where the horizontal composites are $f_!$. We established GRR for the two squares, by Steps 1 and 3. Deduce the general GRR by putting these two squares together. □

The above proof is from [AH3], which itself was an adaptation of Grothendieck's proof in the algebro-geometric setting.

Exercise 42.18. Think through what—if anything—in the above proof must be changed in order to prove Theorem 42.11.

42.19. **Computing the Todd class.** In order to get better at computing the Todd class, let us step back for a moment and study the algebra in more generality. Let $a_1, a_2, \ldots$ be indeterminates and write $Q(x) = 1 + a_1 x + a_2 x^2 + \cdots$. Let

$$Q(\underline{x}) = Q(x_1, \ldots, x_n) = Q(x_1)Q(x_2)\cdots Q(x_n)$$

where the x_i's are formal variables of degree 1. This gives us a power series that is invariant under permutations of the x_i's, and so it may be written as a power series in the elementary symmetric functions $\sigma_i = \sigma_i(x_1, \ldots, x_n)$, $1 \le i \le n$. Our hope would be to give a formula, in terms of the a_i's, for the coefficient of any given monomial $m = \sigma_1^{m_1} \cdots \sigma_n^{m_n}$.

Notice that the degree of m is $m_1 + 2m_2 + 3m_3 + \cdots + nm_n$; call this number N. The coefficient of m in $Q(\underline{x})$ will not involve any terms a_i for $i > N$, so we might as well just assume that $a_i = 0$ for $i > N$. In this case write

$$Q(x) = 1 + a_1 x + \cdots + a_N x^N = (1 + t_1 x)(1 + t_2 x)\cdots(1 + t_N x)$$

as a formal factorization of $Q(x)$ (or if you like, we are working in the algebraic closure of $\mathbb{Q}(a_1, \ldots, a_N)$). Then

$$Q(\underline{x}) = \prod_{i=1}^{n} Q(x_i) = \prod_{i=1}^{n}\prod_{j=1}^{N}(1 + t_j x_i).$$

The evident next step is to reverse the order of the products, so let

$$Q_j = \prod_{i=1}^{n}(1 + t_j x_i) = 1 + t_j\sigma_1 + t_j^2\sigma_2 + \cdots + t_j^n\sigma_n$$

and observe that $Q(\underline{x}) = \prod_{j=1}^{N} Q_j$. Consider the process of multiplying out all factors in

$$(1 + \sigma_1 t_1 + \cdots + \sigma_n t_1^n)\cdot(1 + \sigma_1 t_2 + \cdots + \sigma_n t_2^n)\cdots(1 + \sigma_1 t_N + \cdots + \sigma_n t_N^n).$$

The first few terms are

$$1 + \sigma_1[t_1] + \sigma_2[t_1^2] + \sigma_1^2[t_1 t_2] + \sigma_3[t_1^3] + \sigma_1\sigma_2[t_1 t_2^2] + \sigma_1^3[t_1 t_2 t_3] + \cdots$$

where the bracket notation means to sum the terms in the Σ_N-orbit of the monomial inside the brackets (see Appendix F). Each of these brackets is a polynomial in the t_i's that is invariant under the symmetric group, and therefore can be written (in

a unique way) as a polynomial in the variables $\sigma_i(t) = a_i$. These are the desired coefficients of $Q(\underline{x})$. The general result, whose proof has basically just been given, is the following:

Proposition 42.20. *The coefficient of $\sigma_1^{m_1}\cdots\sigma_n^{m_n}$ in $Q(\underline{x})$ is*

$$[t_1 t_2\cdots t_{m_1} t_{m_1+1}^2\cdots t_{m_1+m_2}^2 t_{m_1+m_2+1}^3\cdots t_{m_1+m_2+m_3}^3\cdots t_{m_1+\cdots+m_n}^n].$$

The bracketed expression in the above result looks horrible, but it is simpler than it appears. The subscripts can basically be ignored. The idea is to write down a product of powers of the t_i's where no index i appears more than once and where the number of t_i's raised to the kth power is m_k. For example, here are a few σ-monomials and their associated coefficients:

$$\sigma_4 : [t_1^4], \qquad \sigma_1\sigma_3 : [t_1 t_2^3], \qquad \sigma_2\sigma_3^2 : [t_1^2 t_2^3 t_3^3], \qquad \sigma_1\sigma_4^2\sigma_6 : [t_1 t_2^4 t_3^4 t_4^6].$$

Said briefly, the sequence of *degrees* of the t's equals the sequence of *subscripts* of the σ's.

For Proposition 42.20 to be useful one has to write the bracketed expression as a polynomial in the elementary symmetric functions $\sigma_i(t) = a_i$. This is, of course, an unpleasant process. One case where it is not *so* bad is for the power sum $[t_1^k]$, since here we have the Newton polynomials s_k described in Appendix F.

Corollary 42.21. *For any $k \geq 1$, the coefficient of σ_k in $Q(\underline{x})$ is $s_k(a_1,\ldots,a_k)$, where s_k is the kth Newton polynomial. By Proposition F.4 this is also equal to the two expressions*

$$(-1)^k\cdot\Big[\textit{coeff. of } x^k \textit{ in } 1 - x\frac{d}{dx}\big(\log Q(x)\big)\ \Big] = (-1)^{k-1}\cdot\Big[\textit{coeff. of } x^{k-1} \textit{ in } \frac{Q'(x)}{Q(x)}\Big].$$

Now let us specialize to $Q(x) = \frac{x}{1-e^{-x}} = \sum_i (-1)^i \frac{B_i}{i!}x^i$. Then writing $Q(x_1,\ldots,x_n)$ as a power series in the elementary symmetric functions exactly yields an expression for the Todd class of a rank n vector bundle in terms of its Chern classes. To apply Corollary 42.21 to this situation, we must compute $Q'(x)/Q(x)$. This is easy enough:

$$Q'(x) = \tfrac{1}{1-e^{-x}} - \tfrac{xe^{-x}}{(1-e^{-x})^2} = \tfrac{1}{x}Q(x) - \tfrac{e^{-x}}{1-e^{-x}}Q(x) = \big(\tfrac{1}{x} + 1 - \tfrac{1}{1-e^{-x}}\big)\cdot Q(x)$$

and so

$$\frac{Q'(x)}{Q(x)} = \frac{1}{x} + 1 - \frac{1}{1-e^{-x}} = \frac{1}{x}\cdot\left[1 + x - \frac{x}{1-e^{-x}}\right] = \frac{1}{x}\left[1 + x - \sum_i (-1)^i \tfrac{B_i}{i!}x^i\right]$$
$$= \frac{1}{x}\left[\frac{1}{2}x + \sum_{i\geq 2}(-1)^{i-1}\tfrac{B_i}{i!}x^i\right].$$

Corollary 42.21 now gives:

Corollary 42.22. *Let $E \to X$ be a rank n vector bundle. Then for any $2 \leq k \leq n$, the coefficient of c_k in the formula for $\mathrm{Td}(E)$ is equal to $\frac{B_k}{k!}$, whereas the coefficient of c_1 is $-\frac{B_1}{1!} = \frac{1}{2}$.*

42.23. **An application of GRR.** In Example 42.13 we used Todd classes to prove that S^4 does not admit an almost complex structure. The following proposition generalizes this to cover all spheres S^{4n}. While this is (mostly) weaker than what we proved back in Corollary 39.9, we offer it as another sample application of the Todd genus.

Proposition 42.24. *No sphere S^{4n} admits the structure of an almost complex manifold.*

Proof. Assume S^{4n} is an almost complex manifold. Then it has a Todd genus, which is necessarily an integer by definition. This will be the coefficient of $[*]$ in the top component of the Todd class $\mathrm{Td}(T)$, where T denotes the complex tangent bundle to S^{4n}. To compute this Todd class directly from its definition, first note that $c_i(T) = 0$ for $i < 2n$, because $H^{2i}(S^{4n}) = 0$ in this range. We must also have $c_{2n}(T) = e(T) = 2[*]$, because the Euler characteristic of an even sphere equals 2. These facts let us easily write down $\mathrm{Td}(T_{S^{4n}})$. Let us consider some examples of this.

When $n = 1$ we would have $\mathrm{Td}(T_{S^4}) = 1 + \frac{c_2}{12}$. The Todd genus of S^4 would then be $\frac{2}{12}$, which is not an integer. When $n = 2$ we would have $\mathrm{Td}(T_{S^8}) = 1 - \frac{c_4}{720}$, and so the Todd genus of S^8 would be $-\frac{2}{720}$; again, not an integer. These examples give the general idea, and the denominators only get worse as n gets larger.

To be specific, one will have $\mathrm{Td}_{S^{4n}} = 1 + Mc_{2n}$ where M is a mystery number that must be computed from the definition of the Todd class. It is a consequence of Corollary 42.22 that $M = \frac{B_{2n}}{(2n)!}$. The Todd genus of S^{4n} will then be $2 \cdot B_{2n}/(2n)!$. By Theorem E.6 the number 3 divides the lowest-terms-denominator of B_{2n}, and so this expression cannot be an integer. This is our contradiction. □

Remark 42.25. Notice why our proof of Proposition 42.24 does not extend to cover spheres S^{4n+2}: the Chern class c_{2n+1} does not appear by itself in the formula for the Todd class, because the odd Bernoulli numbers are zero. If S^{4n+2} has a complex structure one can conclude that its Todd genus is zero, but this by itself does not produce a contradiction.

42.26. **The arithmetic genus.** We now discuss the problem of computing the Todd genus for smooth algebraic subvarieties $Z \hookrightarrow \mathbb{C}P^n$. We will see that it can be described entirely in terms of algebro-geometric data. This material foreshadows much of what we do in Section 43.

Let $p\colon \mathbb{C}P^n \to *$ and $q\colon Z \to *$ be the squash maps, and consider the composition

$$K^0(Z) \xrightarrow{j_!} K^0(\mathbb{C}P^n) \xrightarrow{p_!} K^0(pt).$$

The composite is $q_!$ and therefore sends 1 to $\text{Td-genus}(Z) \cdot [*]$. On the other hand, if we write

$$j_!(1) = [Z] = a_{n-1}[\mathbb{C}P^{n-1}] + a_{n-2}[\mathbb{C}P^{n-2}] + \cdots + a_0[\mathbb{C}P^0]$$

then since $p_!([\mathbb{C}P^{n-i}]) = \text{Td-genus}(\mathbb{C}P^{n-i}) \cdot [*] = [*]$ we have

$$p_!(j_!(1)) = (a_{n-1} + a_{n-2} + \cdots + a_0)[*].$$

So $\text{Td-genus}(Z) = \sum_i a_i$.

Recall from Proposition 33.21 that knowing $[Z]$ is the same as knowing the Hilbert polynomial of Z. We wish to ask the question: how can the Todd genus be extracted from the Hilbert polynomial? To answer this, start by recalling the diagram

$$\begin{array}{ccc} K^0_{\mathrm{alg}}(\mathbb{C}P^n) & \xrightarrow[\cong]{\phi} & K^0(\mathbb{C}P^n) \\ \cong\big\downarrow & & \\ G_{grd}(\mathbb{C}[x_0,\ldots,x_n])/\langle[\mathbb{C}]\rangle & \xrightarrow{\text{Hilb}} & \mathbb{Q}[s] \end{array}$$

from Section 33.19. The image of the function Hilb is the $\mathbb{Z}$-submodule of $\mathbb{Q}[s]$ generated by $\binom{s+n}{n}, \binom{s+n-1}{n-1}, \dots, \binom{s}{0}$. If one takes $[\mathbb{C}P^{n-i}] \in K^0(\mathbb{C}P^n)$ and pushes it around the diagram, we have seen in Section 33.19 that the corresponding Hilbert polynomial is $\binom{s+n-i}{n-i}$.

The Todd genus can be thought of as the unique homomorphism $K^0(\mathbb{C}P^n) \to \mathbb{Z}$ sending all the classes $[\mathbb{C}P^{n-i}]$ to 1. We look for a similar function $\mathrm{im}(\mathrm{Hilb}) \to \mathbb{Z}$ that sends $\binom{s+n-i}{n-i}$ to 1, for all i. A moment's thought shows that the map "evaluate at $s = 0$" has this property. We have therefore proven the following:

Proposition 42.27. *Let $Z \hookrightarrow \mathbb{C}P^n$ be an algebraic subvariety. Then the Todd genus of Z is* $\mathrm{Hilb}_Z(0)$.

In particular, the above implies that $\mathrm{Hilb}_Z(0)$ only depends on Z and not on the embedding $Z \hookrightarrow \mathbb{C}P^n$, even though the Hilbert polynomial itself definitely depends on the embedding.

Example 42.28. Let $Z \hookrightarrow \mathbb{C}P^2$ be a degree d hypersurface. The Riemann-Hurwitz theorem gives that the genus of Z as a real 2-manifold is $g = \binom{d-1}{2}$. The Hilbert polynomial is $\mathrm{Hilb}_Z(s) = \binom{s+2}{2} - \binom{s+2-d}{2}$ (Example 33.16). So $\mathrm{Hilb}_Z(0) = 1 - \binom{2-d}{2} = 1 - \binom{d-1}{2} = 1 - g$. Since $\mathrm{Hilb}_Z(0)$ does not depend on the embedding, this calculates that the Todd genus of a Riemann surface of genus g is equal to $1 - g$.

In algebraic geometry, the invariant $\mathrm{Hilb}_Z(0)$ is repackaged into the so-called **arithmetic genus**, which is defined to be $(-1)^{\dim Z}(\mathrm{Hilb}_Z(0) - 1)$. The above example shows that for projective curves the arithmetic genus coincides with the classical genus. The invariant $\mathrm{Hilb}_Z(0)$, which equals the Todd genus of Z, is also sometimes called the *Hirzebruch genus*, or the *holomorphic Euler characteristic* (see Section 43 below for more information about this).

42.29. **Fundamental classes and the Todd genus.** Again let $Z \hookrightarrow \mathbb{C}P^n$ be a complex submanifold of codimension c and consider the fundamental class $[Z] \in K^0(\mathbb{C}P^n)$. Write

$$[Z] = a_{n-c}[\mathbb{C}P^{n-c}] + a_{n-c-1}[\mathbb{C}P^{n-c-1}] + \cdots + a_0[\mathbb{C}P^0].$$

We have seen that a_{n-c} is the degree of Z, which has a simple geometric interpretation as the number of intersection points of Z with a generic linear subspace of dimension c. But the question remains as to how to give a geometric interpretation for the other a_i's. We will now explain how the Todd genus gives an answer to this (although perhaps not an entirely satisfactory one).

As we saw in the last section, $\text{Td-genus}(Z) = \sum_{0 \le i} a_i$. But note that multiplying the equation for $[Z]$ by $[\mathbb{C}P^{n-1}]$ gives $[Z] \cdot [\mathbb{C}P^{n-1}] = a_{n-c}[\mathbb{C}P^{n-c-1}] + \cdots + a_1[\mathbb{C}P^0]$ and therefore

$$\text{Td-genus}(Z \cap \mathbb{C}P^{n-1}) = \sum_{1 \le i} a_i.$$

Here $Z \cap \mathbb{C}P^{n-1}$ indicates the intersection of Z with a generic hyperplane in $\mathbb{C}P^n$.

Likewise we have $[Z] \cdot [\mathbb{C}P^{n-j}] = a_{n-c}[\mathbb{C}P^{n-c-j}] + \cdots + a_j[\mathbb{C}P^0]$, and hence

$$\text{Td-genus}(Z \cap \mathbb{C}P^{n-j}) = \sum_{j \le i} a_i.$$

So the partial sums $\sum_{j\le i} a_i$ for $j = 0, 1, \dots, n-c$ are the same as the Todd genera of $Z, Z \cap \mathbb{C}P^{n-1}, \dots, Z \cap \mathbb{C}P^c$. (This gives another viewpoint on why a_{n-c} is the degree of Z). We immediately obtain the formulas

$$a_i = \text{Td-genus}(Z \cap \mathbb{C}P^{n-i}) - \text{Td-genus}(Z \cap \mathbb{C}P^{n-i-1}), \tag{42.30}$$

where again the intersections are interpreted to be generic. This is our desired geometric description of the a_i's. Note, however, that whether or not this is indeed "geometric" depends on whether one feels that this adjective applies to the Todd genus.

43. The algebro-geometric GRR theorem

Let X be a compact almost complex manifold and $E \to X$ a complex vector bundle. If π denotes the projection $X \to *$ then we get an element $\pi_!([E]) \in K^0(pt) = \mathbb{Z}$. This gives an integer-valued invariant of the bundle E, which we will call the **Todd number** of E. We will write it as

$$\text{Td-num}_X(E) = \pi_!([E]).$$

The topological GRR theorem identifies this number as $\int_X \text{Td}(T_X) \cdot \text{ch}(E)$, and so we can calculate it in terms of the Chern classes of E and T_X. Note that the Todd number of the trivial bundle $\underline{1}$ is the Todd genus of X.

If X is an algebraic variety and $E \to X$ is an algebraic vector bundle then there is another way to compute the Todd number of E, in terms of algebro-geometric invariants. This identification of invariants is an example of the algebro-geometric GRR theorem. Although the theorem covers far more than just the Todd number, we will concentrate on this special case before stating the more general result.

For this section we need certain portions of the vast machinery that makes up sheaf cohomology in algebraic geometry. Developing that machinery carefully would take much longer than we wish to spend on it. More so than in other parts of this book, our approach in this section will be to sketch the general ideas but just assert many of the basic facts without proof. Readers can always consult [Hartshorne], [Vakil], or [Stacks] for careful details.

43.1. **Sheaf cohomology.** Given a sheaf $\mathcal{F}$ on a scheme X, modern algebraic geometry shows how to obtain sheaf cohomology groups $H^i(X; \mathcal{F})$. These will be of particular interest when $\mathcal{F}$ is the locally-free coherent sheaf associated to an algebraic vector bundle. The general theory is technical (although not incredibly hard), and would take too long to recount here; briefly, these are the right derived functors of the global sections functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$, and so can be studied with the standard tools of homological algebra. But as is often the case with homotopical constructions, there are methods for *computing* the sheaf cohomology groups that do not require the high-tech definitions.

Let $\mathcal{F}$ be quasi-coherent. Suppose we have a Zariski open cover $\{U_\alpha\}$ of X with the property that each U_α is affine, and moreover assume that each iterated intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is affine (for each $k \ge 1$). Write $\Gamma(U_\alpha; \mathcal{F}) = \mathcal{F}(U_\alpha)$. Then we may form the Čech complex

$$0 \to \bigoplus_{\alpha} \Gamma(U_\alpha; \mathcal{F}) \to \bigoplus_{\alpha_1, \alpha_2} \Gamma(U_{\alpha_1} \cap U_{\alpha_2}; \mathcal{F}) \to \cdots$$

and the sheaf cohomology group $H^i(X;\mathcal{F})$ is isomorphic to the ith cohomology group of this complex.

Example 43.2 (Cohomology of $\mathcal{O}(k)$ on $\mathbb{C}P^n$)**.** We will now compute $H^i(\mathbb{C}P^n;\mathcal{O}(k))$ for all i and k. While somewhat of a long read, it is worth going through this one example very carefully. Let $x_0,\dots,x_{n+1}$ be homogeneous coordinates on $\mathbb{C}P^n$, and for each $0\leq j\leq n$ let $U_j\subseteq\mathbb{C}P^n$ be the open subscheme defined by $x_j\neq 0$. Write $U_{j_1\cdots j_r}=U_{j_1}\cap\cdots\cap U_{j_r}$, and note that all of these are affine. Indeed, U_j is the spectrum of $\mathbb{C}[\frac{x_0}{x_j},\frac{x_1}{x_j},\dots,\frac{x_n}{x_j}]$, $U_{j,k}$ is the spectrum of the localization of this ring at x_k/x_j, and so forth.

Let $S=\mathbb{C}[x_0,\dots,x_n]$, regarded as a graded ring where all x_i's have degree 1. Let $R_j=\Gamma(U_j,\mathcal{O})=\mathbb{C}[\frac{x_0}{x_j},\frac{x_1}{x_j},\dots,\frac{x_n}{x_j}]$. This is the degree zero homogeneous piece of the localization S_{x_j}. Further, observe that each $\mathcal{O}(k)$ is trivializable over U_j, and so $\Gamma(U_j,\mathcal{O}(k))$ will be a free R_j-module of rank 1. We can identify $\Gamma(U_j,\mathcal{O}(k))$ with the degree k homogeneous component of the localization S_{x_j}; that is, it is the $\mathbb{C}$-linear span of all degree k monomials $x_0^{a_0}\cdots x_n^{a_n}$ where $a_j\in\mathbb{Z}$ and all other $a_i\geq 0$. This is readily checked to coincide with the cyclic R_j-module $x_j^kR_j$.

The analogs of the above facts work for any open set $U_{\underline{j}}=U_{j_1}\cap\cdots\cap U_{j_r}$. The sections $\Gamma(U_{\underline{j}},\mathcal{O}(k))$ form the degree k homogeneous component of the ring $S_{x_{j_1}\cdots x_{j_r}}$. We want to examine the Čech complex $\check{C}(U_\bullet,\mathcal{O}(k))$ for each value of k, but it is more convenient to take the direct sum over all values for k and consider them all at once.

For a collection of indices $\sigma\subseteq\{0,\dots,n\}$ let S_σ be the localization of S at the element $\prod_{i\in\sigma}x_i$. Consider the augmented Čech complex

$$C^\bullet:\quad 0\to S\to\oplus_i S_i\to\oplus_{i<j}S_{ij}\to\cdots\to S_{01\cdots n}\to 0$$

where the S is in degree -1 and where the differentials are all induced by the inclusions $S_\sigma\hookrightarrow S_{\sigma'}$ for $\sigma\subseteq\sigma'$. That is to say, if we have a tuple $\alpha=(\alpha_\sigma\in S_\sigma)_{\#\sigma=r}$ then $d\alpha$ is the tuple whose value at $\sigma'=\{i_0,\cdots,i_r\}$ (with the entries ordered from least to greatest) is

$$(d\alpha)_{\sigma'}=\sum_{k=0}^{r}(-1)^k\alpha_{i_0\cdots\widehat{i_k}\cdots i_r}.$$

The Čech complex for computing cohomology is obtained from $C^\bullet$ by omitting the S in degree -1, but we will quickly see why it is convenient to have that S around.

It is easy to compute the cohomology group $H^n(C^\bullet)$. The ring $S_{01\cdots\widehat{i}\cdots n}$ is generated as a vector space by monomials $x_0^{a_0}\cdots x_n^{a_n}$ where $a_i\geq 0$. So the image of $C^{n-1}\to C^n$ is the span of all monomials where *some* a_i is nonnegative. The monomials that are not in the image have the form $x_0^{-1}\cdots x_n^{-1}\cdot(x_0^{-b_0}\cdots x_n^{-b_n})$ where all $b_i\geq 0$. So $H^n(C^\bullet)$ only has terms in degree $k\leq-(n+1)$, and in such a degree the cohomology is isomorphic as a $\mathbb{C}$-vector space to $\mathrm{Sym}^{-k-(n+1)}(\mathbb{C}^{n+1})$. This computation will be subsumed by the more general one in the next paragraph, but it is useful to see this particular case by itself.

To compute the cohomology of $C^\bullet$ in all dimensions it is useful to regard S, and each of its localizations, as multigraded by the group $\mathbb{Z}^{n+1}$ (by the multidegrees of monomials). The maps in the complex preserve this multigrading, so we might as well look at one multidegree $\underline{a}=(a_0,\dots,a_n)\in\mathbb{Z}^{n+1}$ at a time. Let $\tau=\{i_0,\dots,i_u\}$ be the complete list of indices for which $a_i<0$. Note that S_σ is zero

in multidegree $\underline{a}$ unless $\sigma \supseteq \tau$: that is, we will only have monomials of multidegree $\underline{a}$ if we have inverted the x_i's for $i \in \tau$. It is not hard to check that $C^\bullet_{\underline{a}}$ (the portion of $C^\bullet$ in multidegree $\underline{a}$) coincides—after shifting degrees by n—with the augmented simplicial chain complex for $\Delta^{n-\#\tau}$ with coefficients in $\mathbb{C}$. The point is that the rings S_σ that are nonzero in degree $\underline{a}$ correspond to precisely those σ that contain τ, and these correspond in turn to subsets of $\{0,1,\dots,n+1\}-\tau$. Subsets of $\{0,1,\dots,n+1\}-\tau$ also index the simplices of $\Delta^{n-\#\tau}$, and we leave it to the reader to verify that the complexes do indeed coincide.

The augmented simplicial cochain complex for $\Delta^{n-\#\tau}$ has zero cohomology except in one extreme case—when $\#\tau = n+1$ we have the augmented cochain complex of the emptyset, and this has a single $\mathbb{Z}$ in its cohomology. This corresponds to those multidegrees $\underline{a}$ in which all $a_i < 0$; for these the total degree satisfies $\sum a_i \leq -(n+1)$.

We have seen that $C^\bullet$ is exact except in cohomological degree n, and there we get a single copy of $\mathbb{C}$ in every multidegree $\underline{a}$ for which $\sum a_i \leq -(n+1)$. So for a fixed integral degree $k \leq -(n+1)$ the kth homogeneous component of $H^n(C)$ is the $\mathbb{C}$-linear span of monomials $\underline{x}^{\underline{a}}$ where all $a_i < 0$ and $\sum a_i = k$. This is what we saw earlier in the argument as well, and it gives us that $H^n(C)_k \cong \mathrm{Sym}^{-k-(n+1)}(\mathbb{C}^{n+1})$.

Finally, let us turn to our original Čech complex by removing the S from degree -1 of $C^\bullet$. In doing so we introduce homology in degree 0, and the graded homology groups exactly coincide with the homogeneous components of S. That is, the kth graded piece of H^0 is isomorphic to $\mathrm{Sym}^k(\mathbb{C}^{n+1})$.

We have now proven that

$$H^i(\mathbb{C}P^n; \mathcal{O}(k)) \cong \begin{cases} \mathrm{Sym}^k(\mathbb{C}^{n+1}) & \text{if } i = 0, \\ \mathrm{Sym}^{-k-(n+1)}(\mathbb{C}^{n+1}) & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Table 43.1 shows these sheaf cohomology groups. Note that these vanish except for $0 \leq i \leq n$. In the table we write $\mathcal{O}_k$ instead of $\mathcal{O}(k)$, for typographical reasons, and we write $V = \mathbb{C}^{n+1}$. The dth symmetric power $\mathrm{Sym}^d V$ will be abbreviated to $S^d V$. Note that this is isomorphic to the space of degree d homogeneous polynomials in $x_0, \dots, x_n$.

TABLE 43.2. Cohomology groups $H^i(\mathbb{C}P^n; \mathcal{O}(k))$

i	$\mathcal{O}_{-(n+3)}$	$\mathcal{O}_{-(n+2)}$	$\mathcal{O}_{-(n+1)}$	$\mathcal{O}_{-n}$	$\cdots$	$\mathcal{O}_{-1}$	$\mathcal{O}$	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$
0	0	0	0	0	$\cdots$	0	$\mathbb{C}$	S^1V	S^2V	S^3V
1	0	0	0	0	$\cdots$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	0	0	0	0	$\cdots$	0	0	0	0	0
n	S^2V	S^1V	$\mathbb{C}$	0	$\cdots$	0	0	0	0	0

The evident "duality" that one finds in the above table is an example of Serre duality, which is an incarnation of Poincaré duality in algebraic geometry. We will learn more about this next.

43.3. **General facts about sheaf cohomology.** Sheaf cohomology is constructed as the right derived functor of the global sections functor $\Gamma\colon \mathrm{Shv}(X) \to \mathcal{A}b$. So

$H^0(X;\mathcal{F}) = \Gamma(X,\mathcal{F})$ and if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves on X then one gets a long exact sequence in sheaf cohomology groups of the form

$$0 \to H^0(X;\mathcal{F}') \to H^0(X;\mathcal{F}) \to H^0(X;\mathcal{F}'') \to H^1(X;\mathcal{F}') \to \cdots$$

Here are some other basic properties:

(1) If X is affine and $\mathcal{F}$ is quasi-coherent then $H^i(X;\mathcal{F}) = 0$ for all $i > 0$. [Stacks, Tag 01XB].
(2) If X is Noetherian then $H^i(X;\mathcal{F}) = 0$ for all $i > \dim X$. [Stacks, Tag 02UZ].
(3) If X is a k-scheme and $\mathcal{F}$ is quasi-coherent then the cohomology groups $H^i(X;\mathcal{F})$ are naturally k-vector spaces. If $X \to \operatorname{Spec} k$ is also proper and $\mathcal{F}$ is coherent then they are finite-dimensional over k. [Stacks, Tag 02O6].
(4) If $f\colon X \to Y$ is affine and $\mathcal{F}$ is quasi-coherent over X then $H^i(X;\mathcal{F}) \cong H^i(Y; f_*\mathcal{F})$ for all $i \geq 0$. [Observe that this generalizes (1) for affine k-schemes.] [Stacks, Tag 089W].
(5) [Serre duality] Suppose X is smooth and projective over a ground field k. Let T_X be the tangent bundle of X and let $\omega_X = \Lambda^{\dim X} T_X^*$, which is called the **canonical line bundle** over X. Then for $\mathcal{L}$ a locally-free coherent sheaf there are isomorphisms
$$H^i(X;\mathcal{L})^* \cong H^{\dim X - i}(X; \mathcal{L}^* \otimes \omega_X)$$
for all i. [Hartshorne, Theorem III.7.6].
(6) [Serre vanishing] Suppose X is a projective variety over a ground field k, $j\colon X \hookrightarrow \mathbb{P}^n$ is a closed immersion, and $\mathcal{F}$ is a coherent sheaf on X. Then there exists an $m_0 \geq 0$ such that $H^i(X; \mathcal{F} \otimes j^*\mathcal{O}(m)) = 0$ for all $i > 0$ and all $m \geq m_0$. [Stacks, Tag 0B5U].

Careful discussions of these properties can be found in [Hartshorne] or [Vakil].

Remark 43.4. Serre duality is the analog of Poincaré duality. Modern topologists mostly learn Poincaré duality only for orientable manifolds, but there is also a version in the non-orientable case. If M is an n-manifold one can define the **orientation bundle** o_M to be the principal $\mathbb{Z}/2$-bundle whose fiber over $x \in M$ consists of the two possible generators of $H_n(M, M-x)$ (with o_M topologized appropriately). There is a corresponding local coefficient system $\mathbb{Z}_M = o_M \times_{\mathbb{Z}/2} \mathbb{Z}$. When M is compact, Poincaré duality takes the form $H^i(M;A) \cong H_{n-i}(M; \mathbb{Z}_M \otimes A)$ for any abelian group A (or really, for any local coefficient system $\mathcal{A}$). Of course if M is orientable then o_M is trivial, $\mathbb{Z}_M = \mathbb{Z}$, and this reduces to the standard version of Poincaré duality. When M is smooth, o_M can be identified with the principal $\mathbb{Z}/2$-bundle associated to the line bundle $\Lambda^n T_M$. Note that $o_M \cong o_M^*$, for the same reason that real line bundles are isomorphic to their duals.

If coefficients are taken in a field k then there is a version of Poincaré duality that involves only cohomology. Here we set $k_M = \mathbb{Z}_M \otimes_{\mathbb{Z}} k$ and the statement is that we have perfect pairings

$$H^i(M;\mathcal{A}) \otimes H^{n-i}(M; \mathcal{A}^* \otimes k_M) \to H^n(M; k_M) \cong k.$$

So this gives natural isomorphisms $H^i(M;\mathcal{A})^* \cong H^{n-i}(M; \mathcal{A}^* \otimes k_M)$, and we get the same kind of statement as in Serre duality (again, noting that $k_M \cong k_M^*$ in this topological setting).

Example 43.5. For $\mathbb{C}P^n$ we know that there is a short exact sequence $0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbb{C}P^n} \to 0$ (see (37.20)–(37.21)). This gives that $\Lambda^n T_{\mathbb{C}P^n} \cong \Lambda^{n+1}(\mathcal{O}(1)^{n+1}) \cong \mathcal{O}(n+1)$, and hence $\omega_{\mathbb{C}P^n} \cong \mathcal{O}(-n-1)$. Serre duality yields

$$H^i(\mathbb{C}P^n; \mathcal{O}(k))^* \cong H^{n-i}(\mathbb{C}P^n; \mathcal{O}(-k) \otimes \mathcal{O}(-n-1)) \cong H^{n-i}(\mathbb{C}P^n; \mathcal{O}(-k-n-1)).$$

This recovers the duality we saw in Table 43.1.

43.6. **Sheaf cohomology and the Todd number.** When X is a complex projective variety and E is an algebraic vector bundle then (identifying E with a locally-free coherent sheaf) we learned in the last section that

- Each $H^i(X; E)$ is finite-dimensional over $\mathbb{C}$;
- $H^i(X; E)$ vanishes when $i > \dim X$;
- A short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ gives rise to a long exact sequence of sheaf cohomology groups.

The first two properties allow us to define the sheaf-theoretic Euler characteristic

$$\chi(X; E) = \sum_i (-1)^i \dim H^i(X; E),$$

and the third property yields that $\chi(X; E) = \chi(X; E') + \chi(X; E'')$. So $\chi(X; -)$ gives a homomorphism $K^0_{\text{alg}}(X) \to \mathbb{Z}$. We already had the topologically-defined pushforward map $\pi_!\colon K^0(X) \to \mathbb{Z}$, and perhaps it will come as no surprise that these two maps take the same values on algebraic vector bundles:

Proposition 43.7. *If X is a complex projective algebraic variety and $E \to X$ is an algebraic vector bundle then* $\text{Td-num}_X(E) = \chi(X; E)$.

This gives us our algebro-geometric interpretation of the Todd number—it equals to so-called "holomorphic Euler characteristic". We postpone the proof for the moment, preferring to obtain this as a corollary of our general GRR theorem below. But let us at least check the proposition in the important example of $\mathbb{C}P^n$. Recall from Proposition 33.22 that $K^0_{\text{alg}}(\mathbb{C}P^n)$ is the free abelian group generated by $[\mathcal{O}], [\mathcal{O}(1)], [\mathcal{O}(2)], \ldots, [\mathcal{O}(n)]$; so we can verify the result for all algebraic vector bundles E on $\mathbb{C}P^n$ by checking it for these particular $n+1$ cases. Luckily we have already computed the vector spaces $H^i(\mathbb{C}P^n; \mathcal{O}(k))$. From Table 43.1 we find that

$$\chi(\mathbb{C}P^n; \mathcal{O}(k)) = \begin{cases} \dim \operatorname{Sym}^k \mathbb{C}^{n+1} & \text{if } k \geq 0, \\ 0 & \text{if } -n \leq k < 0, \\ (-1)^n \dim \operatorname{Sym}^{-k-(n+1)} \mathbb{C}^{n+1} & \text{if } k < -n. \end{cases}$$

We leave the reader to check that all three cases in the above formula can be unified into the simple statement $\chi(\mathbb{C}P^n; \mathcal{O}(k)) = \binom{n+k}{n}$.

It remains to compute the Todd number of $\mathcal{O}(k)$. We use the topological GRR formula of Theorem 42.9 together with the by-now-familiar technique from Exercise 42.15. Recall that $[x^k]p(x)$ denotes the coefficient of x^k in $p(x)$.

$$\begin{aligned}\text{Td-num}_X(E) = \int \Big[\mathrm{Td}(T_X)\cdot \mathrm{ch}(\mathcal{O}(k))\Big] &= [x^n]\Big[\big(\tfrac{x}{1-e^{-x}}\big)^{n+1}\cdot e^{kx}\Big]\\ &= \mathrm{Res}_{x=0}\big(\tfrac{1}{1-e^{-x}}\big)^{n+1}\cdot e^{kx}\,dx\\ &= \mathrm{Res}_{z=0}\big(\tfrac{1}{z^{n+1}}\cdot\tfrac{1}{(1-z)^{k+1}}\,dz\big)\\ &= [z^n]\big((1-z)^{-(k+1)}\big)\\ &= (-1)^n\tbinom{-(k+1)}{n}\\ &= \tbinom{n+k}{n}.\end{aligned}$$

Note that the substitution $z = 1-e^{-x}$ was used for the fourth equality.

43.8. **Pushforwards in algebraic geometry.** Let $f\colon X\to Y$ be a map of schemes. The induced functor $f_*\colon \mathrm{Shv}(X)\to\mathrm{Shv}(Y)$ is left exact, and so we can consider the right derived functors R^if_*. When $f\colon X\to \operatorname{Spec} k$ and k is a field then $R^if_*\mathcal{F} = H^i(X;\mathcal{F})$. In the more general case the sheaf $R^if_*\mathcal{F}$ is is the sheaf associated to $U\mapsto H^i(f^{-1}U,\mathcal{F})$, and so the stalks are given by $(R^if_*\mathcal{F})_y = \mathrm{colim}_{y\in U} H^i(f^{-1}U,\mathcal{F})$. One should think of $R^if_*\mathcal{F}$ as being a relative form of cohomology, or a certain kind of bundling of the cohomology of the fibers.

Suppose that X and Y are both Noetherian. Here are some useful facts about the derived pushforwards:

(1) Each R^if_* takes quasi-coherent sheaves to quasi-coherent sheaves. [Stacks, Tag 01XJ].
(2) If f is affine then f_* is actually exact on quasi-coherent sheaves, and so if $\mathcal{F}$ is quasi-coherent then $R^if_*\mathcal{F} = 0$ for all $i>0$.
(3) Suppose f is proper, $\mathcal{F}$ is a quasi-coherent sheaf on X, and $d\geq \dim f^{-1}(y)$ for every $y\in Y$. Then $R^if_*\mathcal{F}=0$ for $i>d$. (In particular, this holds when $d=\dim X$.) [Stacks, Tag 0E7D].
(4) If f is proper then R^if_* takes coherent sheaves to coherent sheaves. [Stacks, Tag 02O5].
(5) If $\mathcal{F}$ is quasi-coherent on X and $\mathcal{G}$ is a locally-free coherent sheaf on Y then there are natural isomorphisms
$$\mathcal{G}\otimes_{\mathcal{O}_Y} R^if_*\mathcal{F}\xrightarrow{\cong} R^if_*(f^*\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{F})$$
for all $i\geq 0$. [Stacks, Tag 01E8].
(6) [Flat base change] Suppose given a pullback diagram of schemes

(43.9)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X\\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f}\\ Y' & \xrightarrow{g} & Y\end{array}$$

and a quasi-coherent sheaf $\mathcal{F}$ on X. There is a natural map $g^*R^if_*\mathcal{F}\to R^if'_*((g')^*\mathcal{F})$, and this is an isomorphism if g is flat. [Stacks, Tag 02KH].

The [Stacks] references often include hypotheses like f being quasi-compact and quasi-separated, but note that those are automatic when X and Y are Noetherian.

When X is a Noetherian scheme recall that $G(X)$ denotes the Grothendieck group of coherent $\mathcal{O}_X$-modules. Suppose $f\colon X \to Y$ is a proper map between Noetherian schemes. The assignment $\mathcal{F} \mapsto \sum_i (-1)^i [R^i f_* \mathcal{F}] \in G(Y)$ is well-defined because the $R^i f_* \mathcal{F}$ vanish for $i > \dim X$, and because all of the $R^i f_* \mathcal{F}$ are coherent if $\mathcal{F}$ is. The assignment is additive in $\mathcal{F}$ by the usual argument using long exact sequences. So we get an induced map $Rf_*\colon G(X) \to G(Y)$. When X and Y are regular we can use the isomorphism with K^0-groups to write this as

$$f_!\colon K^0_{alg}(X) \to K^0_{alg}(Y).$$

In this context the map $f_!$ is called the **geometric pushforward** or **geometric Gysin map**. We will also denote it as $f_!^{alg}$.

Example 43.10. Consider the case where $Y = \operatorname{Spec} \mathbb{C}$, so that $f_!\colon K^0_{alg}(X) \to K^0_{alg}(\operatorname{Spec} \mathbb{C}) = \mathbb{Z}$. If E is an algebraic vector bundle on X then $R^i f_* E = H^i(X; E)$, and so $f_!([E]) = \chi(X; E)$.

Exercise 43.11. It is not obvious from the above definition that $f_!$ is functorial in f, but it is. Prove this using the Grothendieck spectral sequence $R^p g_*(R^q f_* \mathcal{F}) \Rightarrow R^{p+q}(gf)_* \mathcal{F}$ (look this up if need be, though the details are mostly not necessary for the argument).

Proposition 43.12. *Let $f\colon X \to Y$ be a proper map between regular Noetherian schemes.*

(a) [Projection formula] *$f_!(f^* a \cdot b) = a \cdot f^* b$ for $a \in K^0_{alg}(Y)$ and $b \in K^0_{alg}(X)$.*

(b) [Push-pull formula] *Suppose given a pullback square (43.9) where g is flat. Then for any $a \in K^0_{alg}(X)$ one has $g^* f_!(a) = f'_!((g')^* a)$.*

Proof. Immediate from properties (5) and (6) above. □

Remark 43.13. Observe that the above push-pull formula comes to us under slightly different hypotheses from the push-pull formula we developed in the topological case: here we need the hypothesis of flatness on g, whereas in the topological case we assumed that the pullback square was transverse. These hypotheses are doing similar things for us. There might be a generalization of both that brings them closer to even footing, but the present formulations will suffice for our purposes.

At this point we have geometrically-defined maps $f_!^{alg}\colon K^0_{alg}(X) \to K^0_{alg}(Y)$, and topologically-defined maps $f_!^{top}\colon K^0(X) \to K^0(Y)$. It is certainly not clear from the definitions that these have any relation to each other, but they do:

Theorem 43.14 (GRR, algebro-geometric component)**.** *Let $f\colon X \to Y$ be a proper map between smooth complex projective varieties. Then the square*

$$\begin{array}{ccc} K^0_{alg}(X) & \xrightarrow{f_!^{alg}} & K^0_{alg}(Y) \\ \downarrow & & \downarrow \\ K^0(X) & \xrightarrow{f_!^{top}} & K^0(Y) \end{array}$$

is commutative.

Observe that when $Y = pt$ both groups on the right are $\mathbb{Z}$ and the vertical map is an isomorphism. So the commutativity of the square says precisely that $\chi(X;E) = \text{Td-num}_X(E)$ for any algebraic vector bundle $E \to X$, which was earlier stated as Proposition 43.7.

Proof of Theorem 43.14. First consider the case where f is a closed immersion. Then f is an affine map and so for $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module one has $R^i f_* \mathcal{F} = 0$ for $i > 0$. The map $f_!\colon K^0_{alg}(X) \to K^0_{alg}(Y)$ therefore sends a locally-free coherent $\mathcal{O}_X$-module E to a bounded locally-free resolution of f_*E. Commutativity of the square was proven back in Theorem 31.27 using deformation to the normal bundle.

Next consider the case where f is the projection $\mathbb{C}P^n \to *$. This case was actually checked by hand right after the statement of Proposition 43.7, but let us do it again by another method that is a little more efficient. The class $1 = [\mathcal{O}_{\mathbb{C}P^n}]$ is sent by $f_!^{alg}$ to $\sum_i (-1)^i \dim H^i(\mathbb{C}P^n; \mathcal{O}) = \dim H^0(\mathbb{C}P^n, \mathcal{O}) = 1$ (see Table 43.1). Going around the diagram in the other way, $f_!^{top}$ sends 1 to Td-genus$(\mathbb{C}P^n)$, which we have computed equals 1 (Example 41.16).

But we also know from Proposition 33.22 that $K^0_{alg}(\mathbb{C}P^n)$ is generated by the classes $[\mathcal{O}_{\mathbb{C}P^i}]$. Since $[\mathcal{O}_{\mathbb{C}P^i}] = [j_*\mathcal{O}_{\mathbb{C}P^i}] = j_!^{alg}(1)$ where $j\colon \mathbb{C}P^i \hookrightarrow \mathbb{C}P^n$ is the usual inclusion, the commutativity of the diagram for this class follows from what has already been proven for $\mathbb{C}P^i \to *$. So now we know the theorem for $\mathbb{C}P^n \to *$, for all n.

Next consider the case where f is the projection $Y \times \mathbb{C}P^n \to Y$. For this case let us refer to the maps in the diagram

$$\begin{array}{ccc} Y \times \mathbb{C}P^n & \xrightarrow{q\times \mathrm{id}} & \mathbb{C}P^n \\ {\scriptstyle f = \mathrm{id}\times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y \times pt & \xrightarrow{q} & pt. \end{array}$$

The group $K^0_{alg}(Y \times \mathbb{C}P^n)$ is spanned by classes $f^*a \otimes (q \times \mathrm{id})^*b$ (see below). The projection formula is true in both the algebraic and topological contexts, so we have

$$f_!^{alg}(f^*a \otimes (q\times \mathrm{id})^*b) = a \cdot f_!^{alg}((q\times \mathrm{id})^*b) = a \cdot q^*\pi_!^{alg} b$$

(the second equality is the push-pull formula and uses that q is flat) and

$$f_!^{top}(f^*a \otimes (q\times \mathrm{id})^*b) = a \cdot f_!^{top}((q\times \mathrm{id})^*b) = a \cdot q^*\pi_!^{top} b.$$

Thus, the result follows from the fact that $\pi_!^{alg} b = \pi_!^{top} b$, which was the case $\mathbb{C}P^n \to *$ proven above.

Finally, for the general case choose a closed immersion $i\colon X \hookrightarrow \mathbb{C}P^N$. Factor f as $X \xrightarrow{f\times i} Y \times \mathbb{C}P^N \xrightarrow{p} Y$. The first map is a closed immersion, so the theorem is known for both $f \times i$ and for p. Taking compositions gives the theorem for f. $\square$

The above proof is essentially the one in [AH3, Theorem 4.2], though our treatement of the case where f is a closed immersion is different. We needed the following special case of the algebraic projective bundle theorem (in fact all we really needed was the surjectivity part):

Proposition 43.15. *Let Y be a smooth complex variety and consider the maps $u_i\colon K^0_{alg}(Y) \to K^0_{alg}(Y \times \mathbb{C}P^n)$ given by $\mathcal{F} \mapsto \mathcal{F}\hat{\otimes}\mathcal{O}_{\mathbb{C}P^i} = \pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{O}_{\mathbb{C}P^i} = \pi_1^*\mathcal{F} \otimes \mathcal{O}_{Y\times\mathbb{C}P^i}$ (here we are identifying K^0_{alg} with G in the usual manner). Then the direct*

sum $u\colon K^0_{alg}(Y)^{n+1} \to K^0_{alg}(Y \times \mathbb{C}P^n)$ *given by* $(a_0, \dots, a_{n+1}) \mapsto \sum_i u_i(a_i)$ *is an isomorphism.*

Proof. Surjectivity follows from induction on n using the localization sequences

$$G(Y \times \mathbb{C}P^{n-1}) \xrightarrow{j_*} G(Y \times \mathbb{C}P^n) \xrightarrow{i^*} G(Y \times \mathbb{A}^n) \to 0$$

together with homotopy invariance $G(Y \times \mathbb{A}^n) \cong G(Y)$. The classes $\mathcal{F} \otimes \mathcal{O}_{Y \times \mathbb{C}P^n}$ are sent by i^* to $\mathcal{F} \otimes \mathcal{O}_{\mathbb{A}^n}$, and these generate $G(Y \times \mathbb{A}^n)$ by homotopy invariance. Also $j_*(\mathcal{F} \otimes \mathcal{O}_{Y \times \mathbb{C}P^i}) = \mathcal{F} \otimes \mathcal{O}_{Y \times \mathbb{C}P^i}$ by the projection formula. The rest is immediate from induction.

For injectivity we argue using the pushforward $(\pi_1)_!\colon K^0_{alg}(Y \times \mathbb{C}P^n) \to K^0_{alg}(Y)$. Let $z = \mathcal{O}_{Y \times \mathbb{C}P^{n-1}} = \pi_2^*(\mathcal{O}_{\mathbb{C}P^{n-1}})$ so that $z^i = \mathcal{O}_{Y \times \mathbb{C}P^{n-i}}$. Suppose $0 = \sum_{i=0}^n u_i(a_i) = \sum_{i=0}^n \pi_1^* a_i \cdot z^{n-i}$ in $K^0_{alg}(\mathbb{C}P^n)$. Apply $(\pi_1)_!$ and use the projection formula to get

$$0 = \sum_{i \geq 0} a_i \cdot (\pi_1)_!(z^{n-i}).$$

But $(\pi_1)_! z^{n-i} = (\pi_1)_! \pi_2^* \mathcal{O}_{\mathbb{C}P^i} = q^* \pi_! \mathcal{O}_{\mathbb{C}P^i}$, the last by push-pull where $q\colon Y \to pt$ and $\pi\colon \mathbb{C}P^i \to pt$. Since $\pi_!(\mathcal{O}_{\mathbb{C}P^i}) = \text{Td-genus}(\mathbb{C}P^i) = 1$, we have $(\pi_1)_!(z^{n-i}) = 1$ for all i. So $\sum_{i \geq 0} a_i = 0$.

Now we make one more clever move. We know that $z^{n+1} = 0$ because this relation holds down in $K^0_{alg}(\mathbb{C}P^n)$. Multiplying the equation $0 = \sum_{i=0}^n \pi_1^* a_i \cdot z^{n-i}$ by z gives $0 = \sum_{i=1}^n \pi_1^* a_i \cdot z^{n-i+1}$, which by the above arguments then yields $\sum_{i \geq 1} a_i = 0$. Continuing in this way, we get $\sum_{i \geq j} a_i = 0$ for every j. This implies that all of the a_i vanish. □

We should explain that historically GRR developed in a very different way than we have explained it in the last two sections. Let us use the following diagram to frame our discussion:

$$\begin{array}{ccc} K^0_{alg}(-) & \longrightarrow & K^0(-) \\ \vdots & & \downarrow \text{ch} \\ \text{CH}^*(-;\mathbb{Q}) & \longrightarrow & H^{ev}(-;\mathbb{Q}) \end{array}$$

Hirzebruch's original work on Riemann-Roch involved connecting the holomorphic Euler characteristics to characteristic classes in singular cohomology. In reference to the above diagram, one should think of this as connecting stuff in the world of the upper left corner to stuff in the lower right corner—though with the disadvantage that $K^0_{alg}(X)$ did not actually exist at that time. Grothendieck invented $K^0_{alg}(X)$ and built the connection with the Chow group $\text{CH}^*(X;\mathbb{Q})$ (which we have largely avoided in our discussion). He developed the entire GRR machinery for that half of the diagram; see [BS] and [SGA6]. Later Atiyah-Hirzebruch introduced $K^0(X)$ and developed the topological analogs there [AH1, AH2, AH3], and so gradually the tools were built for making connections throughout the entire square. The only proviso is that there are some small gaps in the Atiyah-Hirzebruch papers where results are asserted but proofs deferred to future papers which never appeared, or else ad hoc constructions are used that do not exactly line up with modern definitions.

43.16. **GRR and the Hilbert polynomial.** Let $j\colon X \hookrightarrow \mathbb{C}P^N$ be a smooth subvariety. Write $\mathcal{O}_X(1) = j^*\mathcal{O}(1)$ and $\mathcal{O}_X(s) = \mathcal{O}_X(1)^{\otimes(s)} = j^*\mathcal{O}(s)$. GRR tells us that

$$\begin{aligned}\chi(X, \mathcal{O}_X(s)) = \int \mathrm{Td}(T_X) \cdot \mathrm{ch}(\mathcal{O}_X(s)) &= \int \mathrm{Td}(T_X) \cdot e^{c_1(\mathcal{O}_X(s))} \\ &= \int \mathrm{Td}(T_X) \cdot e^{s\cdot c_1(\mathcal{O}_X(1))}.\end{aligned}$$

Write $z = c_1(\mathcal{O}_X(1)) \in H^2(X)$ and $n = \dim X$. Since $\mathrm{Td}(T_X) = 1 +$ (higher order terms) and $e^{sz} = 1 + sz + \cdots + \frac{s^n z^n}{n!}$ (higher terms vanish in $H^*(X)$) we find that the final expression in the above chain is visibly a polynomial in s:

$$\chi(X, \mathcal{O}_X(s)) = \tfrac{\int z^n}{n!} s^n + (\text{lower order terms}).$$

We claim that $\int z^n$ is precisely the degree d of the embedding $X \hookrightarrow \mathbb{C}P^N$. If $r = \int z^n$ then $z^n = r[*]$ in $H^*(X)$. So $j_!(z^n) = r[*]$ in $H^*(\mathbb{C}P^N)$. Now let $x = c_1(\mathcal{O}(1)) = [\mathbb{C}P^{N-1}] \in H^2(\mathbb{C}P^N)$. We know that $j_!(1) = d[\mathbb{C}P^n] = dx^{N-n}$. Using the projection formula we get

$$\begin{aligned}j_!(z^n) = j_!((j^*x)^n) = j_!((j^*x)\cdot(j^*x)^{n-1}) = x \cdot j_!((j^*x)^{n-1}) = \cdots &= x^n \cdot j_!(1) \\ &= x^n \cdot dx^{N-n} \\ &= dx^N \\ &= d[*].\end{aligned}$$

So $r = d$, as claimed.

We have now seen that $\chi(X, \mathcal{O}_X(s))$ is a polynomial in s of degree n whose leading coefficient is $\frac{d}{n!}$. This is, in fact, the Hilbert polynomial of X: Serre vanishing (Section 43.3) implies that for $s \gg 0$ one has $H^i(X, \mathcal{O}_X(s)) = 0$ for all $i > 0$, and so in this range $\chi(X, \mathcal{O}_X(s)) = \dim H^0(X, \mathcal{O}_X(s)) = \mathrm{Hilb}_X(s)$.

Note the punchline here: $\dim H^0(X, \mathcal{O}_X(s))$ agrees with a polynomial in s for $s \gg 0$, but that polynomial is precisely $P(s) = \chi(X, \mathcal{O}_X(s))$. So one can think of the Hilbert polynomial in either way. This discussion also makes it clear that the coefficients of the Hilbert polynomial can all be described as integrals of algebraic combinations of characteristic classes of T_X and the element $z = c_1(\mathcal{O}_X(1))$.

We will hold off on GRR exercises until after first seeing in the next section how the theorem plays out for Riemann surfaces. But see Exercise 44.33 for a nice, extended application of GRR.

44. Interlude: The classical Riemann-Roch theorem

The classical Riemann-Roch theorem relates to the problem of counting dimensions of spaces of meromorphic functions, on a fixed Riemann surface, with prescribed bounds on their zeros and poles. It is certainly not immediately clear how this is connected to the fancy Grothendieck-Riemann-Rochs theorems we encountered in the previous two sections. In this section we will explain the classical problems and how their solutions fit into the modern perspective.

Let X be a Riemann surface, i.e. a compact, connected complex manifold of dimension one. A map $X \to \mathbb{C}$ is said to be holomorphic if it restricts to a holomorphic map on every open set in a coordinate chart for X. There aren't very many such maps! In fact we have:

Proposition 44.1. *On a Riemann surface all holomorphic maps are constant.*

Proof. Let $f\colon X \to \mathbb{C}$ be holomorphic. By continuity, the image of f is compact. By the Open Mapping Theorem from complex analysis, if f is not constant then its image is open in $\mathbb{C}$ (see below). But the image of f cannot be both open and compact, so we have a contradiction. Hence f must be a constant map.

To use the Open Mapping Theorem, choose an open cover $U_1, \ldots, U_r$ of X where each U_i is a connected open subset of $\mathbb{C}$. Since X is connected we can order the U's so that each U_i intersects $U_1 \cup \cdots \cup U_{i-1}$. Write $U_{\leq j} = U_1 \cup \cdots \cup U_j$ and consider the images $f(U_{\leq 1})$, $f(U_{\leq 2}) \ldots$. At each stage we have $f(U_{\leq i}) = f(U_{\leq (i-1)}) \cup f(U_i)$. If f is constant on U_i then $f(U_i) \subseteq f(U_{\leq (i-1)})$, since U_i intersects $U_{\leq (i-1)}$ by our choice of ordering. So in this case $f(U_{\leq i}) = f(U_{\leq (i-1)})$. It follows that as long as f is not constant on all of X we have $f(X) = \cup_{i'} f(U_{i'})$ where i' ranges over the indices for which f is not constant on $U_{i'}$. The Open Mapping Theorem says that each $f(U_{i'})$ is open, so $f(X)$ is open. □

Instead of holomorphic functions let us instead consider meromorphic functions. There are uncountably many of these: for example, on $\mathbb{C}P^1$ we have the functions $\frac{1}{(z-a)^n}$ for every $a \in \mathbb{C}$ and $n \in \mathbb{Z}$. To cut things down to size, let us fix the locations of the possible poles and also the maximum degree of each pole. To this end, let us introduce some terminology. A **divisor** on X is a formal linear combination $D = \sum m_i[P_i]$ where the P_i are finitely-many points in X and each $m_i \in \mathbb{Z}$. We can also write $D = \sum_{P \in X} m_P[P]$ where it is understood that $m_P \neq 0$ only for finitely many P. Say that $D_1 \geq D_2$ if $m_P(D_1) \geq m_P(D_2)$ for all points P.

If $f\colon X \to \mathbb{C}$ is meromorphic and not identically zero then in local coordinates near a point P the function looks like $f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots$ with $a_k \neq 0$, for some $k \in \mathbb{Z}$. Set $\mathrm{ord}_P(f) = k$ and call this the "order of vanishing of f at P". Note that P is a zero of f if and only if $\mathrm{ord}_P(f) \geq 1$, and P is a pole of f if and only if $\mathrm{ord}_P(f) < 0$. When $n \geq 0$ it will be convenient to adopt the following language: $\mathrm{ord}_P(f) = n$ will be phrased as "f has n zeros at P", and $\mathrm{ord}_P(f) = -n$ will be phrased as "f has n poles at P".

Define

$$\mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f)[P].$$

Note that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.

Definition 44.2. *For a divisor D, let*

$$L(D) = \{f\colon X \to \mathbb{C} \mid f \text{ is meromorphic and } \mathrm{div}(f) \geq -D\}.$$

Set $\ell(D) = \dim L(D)$ (which a priori might be infinite).

For example, $L(5[P] - 2[Q])$ is the set of meromorphic functions having at most 5 poles at P, at least 2 zeros at Q, and no other poles. In general, functions in $L(D)$ have upper bounds on how many poles they have at each point and lower bounds on how many zeros they have at each point.

Exercise 44.3. Check that in this language f being holomorphic at a point P is equivalent to saying that "f has at least 0 zeros at P", and also equivalent to saying that "f has at most 0 poles at P". (The former version is a little awkward but technically okay.)

Proposition 44.4. *For any divisor D and point P one has $L(D) \subseteq L(D+[P])$, and moreover $\dim L(D+[P])/L(D) \leq 1$. Consequently, $\ell(D) \leq \ell(D+P) \leq \ell(D)+1$.*

Proof. Write $D = \sum_{Q \in X} m_Q[Q]$. If $f, g \in L(D+P)\backslash L(D)$ then f and g have order of vanishing exactly $-(m_P+1)$ at P. So some linear combination $af+bg$ has order of vanishing at least $-m_P$ at P. At all other points Q the order of vanishing of f and g are both at least $-m_Q$, and so this is also true for $af+bg$. Therefore $af+bg \in L(D)$. □

Corollary 44.5. *For every divisor D, $L(D)$ is finite-dimensionsal. Even more, $\ell(D) \leq 1 + \sum_{m_P > 0} m_P$.*

Proof. If $D \leq 0$ then $L(D)$ consists entirely of holomorphic (hence constant) functions, and so $L(D)$ is either $\{0\}$ or $\mathbb{C}$. Adding summands $[P]$ to D can increase the dimension of $L(D)$ by at most one. □

Our main goal will be to compute $\ell(D)$ for every divisor D. These numbers give interesting invariants of the Riemann surface. This is a tall order, though!

Define the **degree** of a divisor D by $\deg(D) = \sum_{P \in X} m_P(D)$.

Proposition 44.6. *For any meromorphic function f on X, $\deg(\operatorname{div} f) = 0$. (In other words, f has the same number of zeros and poles).*

Proof. This is really a theorem of topology. The map $f\colon X \to \mathbb{C}P^1$ has a topological degree $d \geq 0$, meaning that the induced map $f_*\colon H_2(X) \to H_2(\mathbb{C}P^1)$ sends the fundamental class $[X]$ to $d[\mathbb{C}P^1]$. Note that $d \geq 0$ because f is locally orientation-preserving. The degree can be computed locally by counting (with multiplicity) the preimages of any point. If m_P is the local degree of f at a point P then we have

$$\sum_{P \in f^{-1}(\infty)} m_P = d = \sum_{Q \in f^{-1}(0)} m_Q.$$

Now just observe that the local topological degree at a zero or pole agrees with the absolute value of its algebraic order, by a standard computation. So we have $\operatorname{div}(f) = \sum_{Q \in f^{-1}(0)} m_Q[Q] - \sum_{P \in f^{-1}(\infty)} m_P[P]$, and thus the degree of this advisor is zero. □

Corollary 44.7. *If $\deg(D) < 0$ then $\ell(D) = 0$.*

Proof. This is easiest to explain via an example. The space $L(P+Q-3R)$ consists of functions with at least 3 zeros at R, but possible poles only at P and Q and each of order at most 1. But if there are at least three zeros at R the function must have at least three poles as well, by Proposition 44.6. So there are no functions in $L(P+Q-3R)$ other than the zero function.

In general, if $\deg(D) < 0$ then $L(D)$ consists of functions with more zeros than poles, and by Proposition 44.6 no such (nonzero) functions exist. □

We also obtain the following improvement to the inequality in Corollary 44.5:

Corollary 44.8. *If $\deg(D) \geq 0$ then $\ell(D) \leq \deg(D)+1$.*

Proof. Let P be a point in X and consider $E = D - (\deg(D) + 1)[P]$. Then $\deg(E) < 0$, so $\ell(E) = 0$ by Corollary 44.7. But then

$$\ell(D) = \ell(E + (\deg(D) + 1)[P]) \leq \ell(E) + \deg(D) + 1$$

where the inequality is by repeated application of Proposition 44.4. □

Let us compute $\ell(P)$ for any point P in our Riemann surface. Elements of $L(P)$ are meromorphic functions having at most a single pole, at P. So $\ell(P) \geq 1$, since $L(P)$ constains the constant functions. When $X = \mathbb{C}P^1$ then at a point $P = a$ we have $L(P) = \langle 1, \frac{1}{z-a} \rangle$ and thus $\ell(P) = 2$. The following result covers all other cases:

Proposition 44.9. *If the genus of X is positive then $\ell(P) = 1$ for all $P \in X$. Moreover, $\ell(P - Q) = 0$ for all $P \neq Q$.*

Proof. Suppose $f \in L(P)$ is non-constant. Then $f \colon X \to \mathbb{C}P^1$ has degree equal to 1, and is therefore injective (since the degree can be computed as the number of elements in the preimage of any point, counted with multiplicity). The image of f must be open by the Open Mapping Theorem (see the argument given in the proof of Proposition 44.1). But the image must also be compact (since X is), hence closed, and so the image of f is all of $\mathbb{C}P^1$. Then f is a bijection between compact Hausdorff spaces, hence a homeomorphism; so the genus of X is zero.

If X has positive genus we have now shown that $L(P)$ is the space of constant functions. But $L(P-Q) \subseteq L(P)$ is the subspace of functions that vanish at Q, and such a function is necessarily zero. □

Proposition 44.10. *Let f be a meromorphic function on X. Then for any divisor D, the assignment $g \mapsto fg$ is an isomorphism $L(D) \xrightarrow{\cong} L(D - \operatorname{div}(f))$.*

Proof. Immediate, as $h \mapsto \frac{h}{f}$ is the inverse. □

Definition 44.11. *Divisors D_1 and D_2 are **linearly equivalent**, written $D_1 \sim D_2$, if $D_1 - D_2 = \operatorname{div}(f)$ for some meromorphic function f. The **divisor class group** (often just called the class group) is $\operatorname{Cl}(X) = \mathbb{Z}\langle X \rangle / \langle \operatorname{div}(f) \mid f \in \operatorname{Mer}(X) \rangle$.*

Proposition 44.12. *If $D_1 \sim D_2$ then $\ell(D_1) = \ell(D_2)$.*

Proof. Immediate from Proposition 44.10. □

The elements of $\operatorname{Cl}(X)$ can be thought of as parameterizing the different "compute $\ell(D)$" problems.

Example 44.13. For $X = \mathbb{C}P^1$ the function $\frac{1}{z-a}$ shows that $[a] \sim [\infty]$. It follows that $[a] \sim [b]$ for every $a, b \in \mathbb{C}P^1$, and so $\operatorname{Cl}(X) \cong \mathbb{Z}$ via the degree map. Hence, $\ell(D)$ depends only on $\deg(D)$. We can now compute:

$$\ell(D) = \begin{cases} 0 & \text{if } \deg(D) < 0, \\ 1 + \deg(D) & \text{if } \deg(D) \geq 0. \end{cases}$$

The case of $\deg(D) < 0$ is by Corollary 44.7. For the other case it suffices to compute $\ell(n[0])$, which counts meromorphic functions whose only pole is at 0 and has order at most n. One can write down a basis for such functions: $\frac{1}{z^k}$ for $0 \leq k \leq n$. So $\ell(n[0]) = n + 1$.

For X of positive genus it turns out that $\operatorname{Cl}(X)$ is always uncountable, so the $\mathbb{C}P^1$ case is deceptively simple.

44.14. **Meromorphic 1-forms.** 1-forms on X are constructs ω that look locally in a coordinate patch like $f(z)dz$. We will say that ω is holomorphic or meromorphic when the corresponding $f(z)$ functions are so. For a meromorphic 1-form ω define

$$\operatorname{div}(\omega) = \sum_P \operatorname{ord}_P(f)[P].$$

If ω and η are two meromorphic 1-forms then $\omega = h \cdot \eta$ for a unique meromorphic function h. Indeed, if in local coordinates $\omega = f(z)dz$ and $\eta = g(z)dz$ then we take $h = \frac{f(z)}{g(z)}$ (check that this does not depend on the choice of coordinate). Consequently, we obtain

$$\operatorname{div}(\omega) = \operatorname{div}(h) + \operatorname{div}(\eta)$$

and therefore $[\operatorname{div}(\omega)] = [\operatorname{div}(\eta)]$ in $\operatorname{Cl}(X)$. This common class (the divisor of any meromorphic 1-form) is called the **canonical class** of X, and usually denoted K_X or just K.

Example 44.15. On $\mathbb{C}P^1$ we use the standard coordinates given by z near 0 and $w = \frac{1}{z}$ near ∞. We have $dw = -\frac{1}{z^2}dz$. Let η be the form that is $dz = -\frac{1}{w^2}dw$. Then $K = \operatorname{div}(\eta) = -2[\infty]$.

If f is any meromorphic function on X then one has the associated 1-form df. If f is not a constant map then df is nonzero, and so $\operatorname{div}(df)$ represents the canonical divisor class. For example, if $f(z) = z^2$ on $\mathbb{C}P^1$ then df is the 1-form that is $2zdz$ on the z-coordinate patch and $-\frac{2}{w^3}dw$ in the $w = \frac{1}{z}$ coordinate patch. So $\operatorname{div}(d(z^2)) = [0] - 3[\infty]$. Since $[0] = [\infty]$ in $\operatorname{Cl}(\mathbb{C}P^1)$ this divisor class is the same as $-2[\infty]$.

The degree of the canonical class is a numerical invariant of the Riemann surface X. It will come as no surprise that it is related to the genus:

Proposition 44.16. *If X has genus g then* $\deg(K) = 2g - 2 = -\chi(X)$.

Proof. This is another case where the result is topological. Let f be a non-constant meromorphic function, regarded as a map $f\colon X \to \mathbb{C}P^1$. Let d be the topological degree of f, and recall from the proof of Proposition 44.6 that we have

$$d = \sum_{P \text{ a pole}} (-\operatorname{ord}_P(f)).$$

The map f is a branched cover, so we can apply the Riemann-Hurwitz formula. Let BP be the set of points in X where f branches, and let $BL = f(BP) \subseteq \mathbb{C}P^1$ be the branch locus. For each $y \in BP$ let $\operatorname{ram}(y)$ denote the ramification index; intuitively this is the number of branches that are coming together at y. Then Riemann-Hurwitz says that

$$\chi(X) = 2d - \sum_{y \in BP} [\operatorname{ram}(y) - 1].$$

Let us recall where this formula comes from, since it is simple. If we remove BP and BL then $X - BP \to \mathbb{C}P^1 - BL$ is a d-fold cover, and so $\chi(X - BP) = d \cdot \chi(\mathbb{C}P^1 - BL)$. Since we are dealing with surfaces, removing a point decreases the Euler characteristic by 1 (it is like removing a triangle from a triangulation). So we get

$$\chi(X) - \#BP = \chi(X - BP) = d \cdot \chi(\mathbb{C}P^1 - BL) = d \cdot (2 - \#BL)$$

and this becomes

$$\chi(X) = 2d + \#BP - d\cdot\#BL = 2d + \#BP - \sum_{y\in BP} \operatorname{ram}(y) = 2d - \sum_{y\in BP} (\operatorname{ram}(y) - 1)$$

where in the second-to-last equality we have used that adding up the ramification indices in each fiber over a point in BL must yield the number d.

But if f is holomorphic at y then y is a branch point if and only if $f'(y) = 0$, and $\operatorname{ram}(y) = \operatorname{ord}_y(f') + 1$ (see Exercise 44.30 below). Likewise, if y is a pole of f then y is a branch point if and only if $\operatorname{ord}_y(f) \le -2$, and $\operatorname{ram}(y) = -\operatorname{ord}_y(f)$. So our Riemann-Hurwitz formula now becomes

$$\begin{aligned}
2 - 2g = \chi(X) &= 2d - \left[\sum_{\substack{P \text{ pole} \\ \operatorname{ord}_P(f)\le -2}} (-\operatorname{ord}_P(f) - 1) + \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \right] \\
&= 2\cdot \sum_{P \text{ pole}} \Big(-\operatorname{ord}_P(f)\Big) + \sum_{\substack{P \text{ pole} \\ \operatorname{ord}_P(f)\le -2}} (\operatorname{ord}_P(f) + 1) - \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \\
&= \sum_{\substack{P \text{ simple} \\ \text{pole}}} 2 + \sum_{\substack{P \text{ pole} \\ \operatorname{ord}_P(f)\le -2}} (-\operatorname{ord}_P(f) + 1) - \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \\
&= \sum_{P \text{ pole}} (-\operatorname{ord}_P(f) + 1) - \sum_{r \text{ root of } f'} \operatorname{ord}_r(f') \\
&= -\deg(\operatorname{div}(df)).
\end{aligned}$$

Since $\operatorname{div}(df)$ is a canonical divisor, this completes the proof. □

We can now state the classical Riemann-Roch Theorem:

Theorem 44.17 (Riemann-Roch)**.** *For any divisor D on a Riemann surface X,*

$$\ell(D) - \ell(K - D) = 1 + \deg(D) - g = \deg(D) + \frac{\chi(X)}{2}.$$

The first equality, with $1 + \deg(D) - g$, is the classical statement. The second form of the equality, with $1 - g$ replaced by $\frac{\chi(X)}{2}$, connects better with modern versions of the Riemann-Roch theorem, with the two terms divided more clearly into contributions from D and contributions from X.

At first the Riemann-Roch theorem might not appear very useful for our problem, as it does not appear to calculate $\ell(D)$. It only gives us a relation between $\ell(D)$ and $\ell(K - D)$, and if we cannot calculate either one of them then we are out of luck. But sometimes we get lucky. For example, the following is immediate:

Corollary 44.18. *If $\deg(D) > \deg(K)$ then $\ell(D) = 1 + \deg(D) - g$.*

Proof. The hypothesis implies $\deg(K - D) < 0$, and so $\ell(K - D) = 0$ by Corollary 44.7. □

To give some additional perspective on the $\ell(K - D)$ term, consider the vector spaces

$$\Omega^1(D) = \{\text{meromorphic 1-forms } \omega \text{ such that } \operatorname{div}(\omega) \ge D\}.$$

For example, $\Omega^1(5[P] - 2[Q])$ consists of 1-forms that have at least an order 5 zero at P and at most an order 2 pole at Q (and no other poles). Note that is backwards of how things worked for functions in place of 1-forms.

If ω is a meromorphic 1-form, then the assignment $f \mapsto f\omega$ gives a map $L(E) \to \Omega^1(K-E)$. Indeed, if $\operatorname{div}(f) \geq -E$ then $\operatorname{div}(f\omega) = \operatorname{div}(f) + K \geq K - E$. This map is readily seen to be an isomorphism. Setting $E = K - D$ we can rephrase this result as follows:

Proposition 44.19. *For any divisor D, $L(K-D) \cong \Omega^1(D)$ via multiplication by any meromorphic 1-form ω.*

So the Riemann-Roch theorem for Riemann surfaces can also be stated as

$$\dim L(D) - \dim \Omega^1(D) = 1 + \deg(D) - g.$$

44.20. **Applications of Riemann-Roch.** We still need to prove the Riemann-Roch theorem, but let us come back to that after seeing a few applications.

Let X be a Riemann surface of genus 1 and choose $P \in X$. Note that $\deg(K) = -\chi(X) = 0$ here. Consequently we obtain $\ell(nP) = n$ for any $n > 0$, by Corollary 44.18. The following is therefore a sequence of proper inclusions:

$$L(P) \subsetneq L(2P) \subsetneq L(3P)$$

(the vector spaces having dimensions 1, 2, and 3, respectively). Choose $x \in L(2P)$ that is not a constant function, and choose $y \in L(3P) \backslash L(2P)$. Define $f\colon X \to \mathbb{C}P^2$ by

$$u \mapsto \begin{cases} [x(u) : y(u) : 1] & \text{if } u \neq P, \\ [0:1:0] & \text{if } u = P. \end{cases}$$

Note that $[x : y : 1] = [\frac{x}{y} : 1 : \frac{1}{y}]$ and the two fractions are holomorphic near P, in fact with zeros at P. So f is well-defined and holomorphic.

The vector space $L(6P)$ has dimension 6 but contains the functions y^2, x^3, x^2, xy, x, y, and 1. So there is a nontrivial linear relation

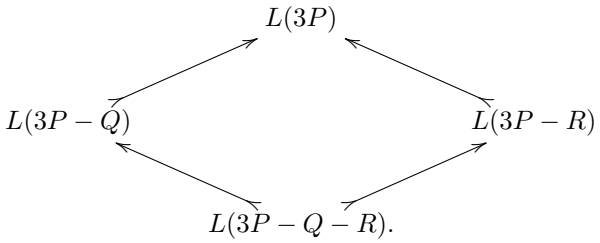
$$ay^2 + bx^3 + cx^2 + dxy + ex + gy + h = 0$$

for some $a, b, c, d, e, g, h \in \mathbb{C}$. This shows that the image of f lies in the subspace of $\mathbb{C}P^2$ defined by the homogenization of the above equation, namely

$$az_1^2 z_2 + bz_0^3 + cz_0^2 z_2 + dz_0 z_1 z_2 + ez_0 z_2^2 + gz_1 z_2^2 + hz_2^3 = 0 \tag{44.21}$$

where $[z_0 : z_1 : z_2]$ are homogeneous coordinates on $\mathbb{C}P^2$.

We claim that f is also injective. It is clear from the definition that $f(P)$ cannot coincide with any $f(Q)$ for $Q \neq P$, since the former has vanishing third coordinate whereas the latter does not. So instead consider two distinct points Q and R in $X - \{P\}$. For any point T the homogeneous coordinates of $f(T)$ are given by $x(T)$, $y(T)$, and 1. If we can find a homogeneous combination of these functions that vanishes for $T = Q$ but not for $T = R$, we will have shown that $f(Q) \neq f(R)$. To this end, consider the subspaces



Corollary 44.18 calculates the dimensions as 3, 2, 2, and 1 (proceeding from top to bottom). Note that $1, x, y \in L(3P)$ and are independent by our choice of x and y, so $L(3P) = \langle 1, x, y\rangle$. The dimension calculations imply that there exists a $g \in L(3P - Q)\backslash L(3P - Q - R)$. Then this function vanishes at Q but not R. Since $g \in L(3P)$ we can write $g = mx + ny + s$ for $m, n, s \in \mathbb{C}$. The fact that $g(Q) = 0$ then implies that $f(Q)$ belongs to the subspace $mz_0 + nz_1 + sz_2 = 0$ of $\mathbb{C}P^2$, whereas $g(R) \neq 0$ implies that $f(R)$ does not belong to this subspace. So $f(Q) \neq f(R)$.

We have proven that f maps X injectively into the subvariety C of $\mathbb{C}P^2$ defined by the cubic equation (44.21). Since the image of f is compact, and hence closed, it must be that f is a homeomorphism onto its image. It takes a little more work to show that the image is precisely C (and not something smaller)—if C is smooth one can use the Open Mapping Theorem as we have done before, but we do not know a priori that C is smooth. Another approach is to use the Remmert proper mapping theorem, which says that $f(X)$ is an analytic subvariety of C; a dimension argument then gives $f(X) = C$. We will not pursue this point further, though.

As our next application we use Riemann-Roch to prove that the points of X form an abelian group. Fix a point $\mathcal{O}$ in X.

For $P, Q \in X$ we will define a new point $P*Q$. We take $P*\mathcal{O} = P$ and $\mathcal{O}*Q = Q$, so assume $P \neq \mathcal{O}$ and $Q \neq \mathcal{O}$. Consider $L(P + Q - \mathcal{O})$, which has dimension 1 by Corollary 44.18. So there exists a meromorphic function g on X having a zero at $\mathcal{O}$ and possibly simple poles at P and/or Q (but no other poles). Recall from Proposition 44.9 that $L(P)$ has dimension 1 and therefore consists only of constant functions, and so $L(P - \mathcal{O}) = 0$ (the only constant function having a zero at $\mathcal{O}$ is the zero function). So $g \notin L(P - \mathcal{O})$, and therefore g has a simple pole at Q. Similarly, g has a simple pole at P. Since the number of zeros and poles for g must be equal (Proposition 44.6), this means g has one extra zero—call it S (and note that S might be $\mathcal{O}$). Define $P * Q = S$. Since g was unique up to a scalar multiple, S is uniquely determined by P and Q.

Since $\operatorname{div}(g) = [\mathcal{O}] + [S] - [P] - [Q]$ we have that $[P] + [Q] = [\mathcal{O}] + [S]$ in $\operatorname{Cl}(X)$, and in fact we could take this as the definition of S. For if $[P] + [Q] = [\mathcal{O}] + [T]$ in $\operatorname{Cl}(X)$ then there is a meromorphic function h having simple zeros at $\mathcal{O}$ and T and simple poles at P and Q. Then $h \in L(P + Q - \mathcal{O})$, but we have already remarked that this space has dimension 1. So h is a multiple of g, and therefore $S = T$.

To summarize: $P * Q$ is defined to be the unique point S such that $[P] + [Q] = [\mathcal{O}] + [S]$ in $\operatorname{Cl}(X)$. This definition also works when either P or Q is $\mathcal{O}$. Associativity of the $*$ operation is now immediate, as is the fact that $\mathcal{O}$ is the identity. We turn to the question of inverses.

Given a point P, consider $L(2\mathcal{O} - P)$. This has dimension 1 by Corollary 44.18, and $L(\mathcal{O} - P) = 0$ by Proposition 44.9. So if $g \in L(2\mathcal{O} - P)$ is nonzero then g has a double pole at $\mathcal{O}$. This implies g has exactly two zeros, one at P and one at a point R (which possibly equals P). Then $[P] + [R] = [\mathcal{O}] + [\mathcal{O}]$ in $\operatorname{Cl}(X)$, and so $P * R = \mathcal{O}$.

Exercise 44.22. Write out the proof of associativity that was omitted above.

44.23. **Classical proof of the Riemann-Roch theorem.** Our aim here is just to give an outline of the main ideas, as some of the details require a good deal

of work. The modern approach doesn't exactly short-cut this work, but instead sweeps it into the sheaf theory technology where some of it gets hidden.

We first concentrate on the case of a divisor $D = \sum m_P[P]$ where all $m_P \geq 0$. Let $\{P_1, \ldots, P_r\}$ be the set of points for which $m_P > 0$, and write $m_i = m_{P_i}$. Pick a local coordinate near each P_i, and consider the vector space of "principal parts" of meromorphic functions having poles of at most order m_i at P_i. This is the vector space with basis $\frac{1}{z}, \frac{1}{z^2}, \cdots, \frac{1}{z^{m_i}}$ ("principal part" means the part of the Laurent expansion in negative degrees), so taking the direct sum over all i we get a vector space PP of dimension $\deg(D)$. Consider the sequence of maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & L(D) & \longrightarrow & PP \\ & & & & & & \uparrow \cong \\ & & & & & & \mathbb{C}^{\deg(D)} \end{array}$$

Here the leftmost map is the inclusion of the constant functions, and the next map sends a meromorphic function in $L(D)$ to the collection of its prinicpal parts around each P_i. Clearly we have exactness at $L(D)$. Riemann was interested in characterizing the image of $L(D)$ inside of PP. To do this, we need to recall a bit about residues.

If $\omega \in \Omega^1_{hol}(X)$ and $f \in L(D)$ then we can look at the residue

$$\mathrm{Res}_{z=P_i} f(z)\omega = \frac{1}{2\pi i}\int_{C_i} f \cdot \omega$$

where C_i is a small circle around P_i. Note that

$$\sum_i \mathrm{Res}_{z=P_i} f(z)\omega = 0$$

by the Cauchy Integral Formula, since the C_i collectively bound a region on which $f(z)\omega$ is holomorphic. In this way be obtain from ω a linear equation satisfied by the image of $L(D)$ inside of PP. Specificially, we obtain a linear map $R_\omega \colon PP \to \mathbb{C}$ sending a collection of principal parts $(u_1, \ldots, u_r)$ to $\sum_i \mathrm{Res}_{P_i}(u_i\omega)$, and R_ω vanishes on the image of $L(D)$.

Riemann proved that $\Omega^1_{hol}(X) \cong \mathbb{C}^g$, which means the above procedure yields g linear conditions on the image of $L(D)$ in PP. We can add this to our sequence:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & L(D) & \longrightarrow & P.P. & \xrightarrow{\sum Res} & \mathbb{C}^g & \longrightarrow & Cok & \longrightarrow & 0 \\ & & & & & & \uparrow \cong & & & & & & \\ & & & & & & \mathbb{C}^{\deg(D)} & & & & & & \end{array}$$

Riemann further proved exactness at the PP term: that is, a set of principal parts comes from a meromorphic function in $L(D)$ precisely when the g linear conditions coming from the elements of Ω^1_{hol} are satisfied. Note that the Cok term in the above sequence denotes the cokernel. From this sequence one gets that

$$0 = 1 - \ell(D) + \deg(D) - g + \dim(Cok)$$

or

$$\ell(D) = 1 + \deg(D) - g + \dim(Cok) \geq 1 + \deg(D) - g.$$

This is Riemann's part of the Riemann-Roch theorem. Roch, a student of Riemann, later identified $\dim(Cok)$ with $\ell(K-D)$.

For a general divisor D, write $D = A - B$ where $A, B \geq 0$ and A and B have no points in common. The B component contributes extra conditions to $L(D)$, and we use ideas similar to the ones above to track this. We continue to only give a rough sketch. Let $J(B)$ denote the vector space of "jets" at the points in B: If P appears in B with multiplicity m_P and z is a local coordinate at P then $J(P)$ is the space of polynomials $a_0 + a_1 z + \cdots + a_{m_P-1} z^{m_P-1}$. So a jet is just the initial part of a Taylor expansion at a point, and $J(B)$ is the direct sum of all the jet spaces for all the points in B. We have the map $J\colon L(A) \to J(B)$ that takes a function $f \in L(A)$ and sends it to the collection of all its jets; the kernel is precisely $L(A-B)$.

Given a collection of jets $j = \{j_P\} \in J(B)$ and a 1-form $\omega \in \Omega(D)$, we can form $R_\omega(j) = \sum_P \operatorname{Res}_{z=P}(j_P \omega)$. If $\omega \in \Omega(A)$ then ω is holomorphic at all points in B and so $R_\omega(j) = 0$ (in fact all of the residues in the sum are zero). Therefore R can be considered as a map $J(B) \to (\Omega(D)/\Omega(A))^*$.

If $f \in L(A)$ then it possibly has poles at points of A, but because ω has zeros at those points (of at least the same order as the poles) then $f\omega$ is holomorphic except possibly at the points of B. So $R_\omega(Jf) = 0$ by the residue theorem.

At this point we have produced a chain complex

$$0 \longrightarrow L(A-B) \longrightarrow L(A) \xrightarrow{J} J(B) \xrightarrow{R} (\Omega(D)/\Omega(A))^* \longrightarrow 0 \tag{44.24}$$

that is exact in the first two spots. Exactness at the last spot is not hard: by taking duals, it is equivalent to show that for any $\omega \in \Omega(D) \setminus \Omega(A)$ there is a jet $j \in J(B)$ such that $R_\omega(j) \neq 0$, and this is elementary. The classical Mittag-Leffler theorem gives exactness at $J(B)$: a collection of jets that looks like it comes from a meromorphic function (from the point of view of taking associated residues) actually does—this theorem takes some real work, though.

Exercise 44.25. Assuming you know the Riemann-Roch theorem for A, deduce the Riemann-Roch theorem for $D = A - B$ from the above exact sequence.

Exercise 44.26. Figure out how the exact sequence of (44.24) connects to a long exact sequence from sheaf cohomology, via Serre duality.

44.27. **The modern approach to Riemann-Roch.** The vector space $L(D)$ is the space of holomorphic sections of a holomorphic line bundle $\mathcal{O}_X(D)$ (or equivalently, the space of sections of the corresponding rank one coherent sheaf). We construct this line bundle as follows. Choose a finite collection of coordinate patches $\{U_\alpha\}$ that cover X. On each U_α let g_α be the meromorphic function defined in the local coordinate z by

$$g_\alpha(z) = \prod_{P_i \in U_\alpha} (z - P_i)^{m_i}$$

where $D = \sum_i m_i [P_i]$. Note that this construction ensures that $\frac{g_\beta}{g_\alpha}$ is holomorphic and nonzero on $U_\alpha \cap U_\beta$. Let $\mathcal{O}_X(D)$ be the line bundle obtained by taking the trivial bundle on each U_α and gluing $(z, w) \in U_\alpha \times \mathbb{C}$ to $(z, \frac{g_\beta(z)}{g_\alpha(z)} \cdot w) \in U_\beta \times \mathbb{C}$ for $z \in U_\alpha \cap U_\beta$. This clearly gives a line bundle. If s is a holomorphic section of $\mathcal{O}_X(D)$ then let $s_\alpha\colon U_\alpha \to \mathbb{C}$ be the section written using the local trivialization on

U_α. Then

$$\frac{g_\beta(z)}{g_\alpha(z)} s_\alpha(z) = s_\beta(z), \tag{44.28}$$

or equivalently

$$\frac{s_\alpha(z)}{g_\alpha(z)} = \frac{s_\beta(z)}{g_\beta(z)}.$$

for every $z \in U_\alpha \cap U_\beta$. This implies that the $\frac{s_\alpha}{g_\alpha}$ functions patch together to form a single meromorphic function $\frac{s}{g}$. Since the s_α were holomorphic, we have that $\operatorname{div}(\frac{s}{g}) \geq -D$. In this way we have obtained a map $\Gamma(X, \mathcal{O}_X(D)) \to L(D)$, and it is clearly injective. Surjectivity is just as easy: if $h \in L(D)$ then define $s_\alpha(z) = h|_{U_\alpha} \cdot g_\alpha$, which is necessarily holomorphic by the definition of g_α. Equation (44.28) is readily verified, so that the s_α patch together to give a holomorphic section of $\mathcal{O}_X(D)$.

Let us take a moment to consider the topological type of the line bundle $\mathcal{O}_X(D)$. If we imagine the points P_i in D moving around, this will not change the isomorphism type of the associated (topological) bundle. So we might as well let all the points come together, so that D approaches a divisor $(\deg D)[*]$ (it doesn't matter which point we choose here). This shows that the bundles $\mathcal{O}_X(D)$ and $\mathcal{O}(\deg D[*])$ are isomorphic as topological vector bundles. In particular, note that $c_1(\mathcal{O}_X(D)) = c_1(\mathcal{O}(\deg D[*])) = (\deg D)[*]$.

As we have seen in Proposition 43.7, the holomorphic Euler characteristic $\chi(X, \mathcal{O}_X(D))$ can be described as a K-theoretic pushforward:

$$\chi(X, \mathcal{O}_X(D)) = \pi_!([\mathcal{O}_X(D)]) \in K^0(pt) = \mathbb{Z}$$

where $\pi\colon X \to *$ and $\pi_!\colon K^0(X) \to K^0(*)$. The Grothendieck-Riemann-Roch theorem says that

$$\chi(X, \mathcal{O}_X(D)) = \int \Big(\operatorname{ch}(\mathcal{O}_X(D)) \cdot \operatorname{Td}(X) \Big)$$

where the right-hand side denotes the coefficient of $[*]$ in the top-dimensional piece of $\operatorname{ch}(\mathcal{O}_X(D)) \cdot \operatorname{Td}(X)$. Since X is a Riemann surface the top-dimensional cohomology is $H^2(X)$ and we have

$$\operatorname{ch}(\mathcal{O}_X(D)) = 1 + c_1(\mathcal{O}_X(D)), \qquad \operatorname{Td}(X) = \frac{c_1(T_X)}{1 - e^{-c_1(T_X)}} = 1 + \frac{c_1(T_X)}{2}.$$

It follows that

$$\chi(X, \mathcal{O}_X(D)) = \int \left(c_1(\mathcal{O}_X(D)) + \frac{c_1(T_X)}{2} \right) = \deg(D) + \frac{\chi(X)}{2}.$$

In the last equality we have used that $c_1(\mathcal{O}_X(D)) = \deg(D)[*]$ and $c_1(T_X) = \chi(X)[*]$.

This does not quite complete our story. The final piece is to identify $\dim H^1(X, \mathcal{O}_X(D))$ with Roch's $\ell(K - D)$ term, and for this we use Serre duality (Section 43.3). The latter says that for any algebraic vector bundle E there is an isomorphism of vector spaces

$$H^i(X, E) \cong H^{1-i}(X, \mathcal{O}_X(K) \otimes E^*)^*$$

where $\mathcal{O}_X(K) = T_X^*$ is the canonical line bundle on X (the dual of the tangent bundle). This implies that

$$\begin{aligned}\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(K) \otimes \mathcal{O}_X(-D)) &= \dim H^0(X, \mathcal{O}_X(K - D)) \\ &= \ell(K - D).\end{aligned}$$

Putting everything together, we arrive at

$$\ell(D) - \ell(K - D) = \chi(X, \mathcal{O}_X(D)) = \deg(D) + \frac{\chi(X)}{2}$$

as desired.

Example 44.29. The following tables show the dimensions of $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$ as functions of $\deg(D)$ and the genus. Each table shows the results for a different genus. For convenience let $h^i(D)$ denote these dimensions, for $i \in \{0, 1\}$. Recall $h^0(D) = \ell(D)$ and $h^1(D) = \ell(K - D)$. The only information we need to make these tables is that $h^0 - h^1 = 1 + \deg(D) - g$, $h^0 = 0$ for $\deg(D) < 0$, and that $h^1 = 0$ for $\deg(D) > 2g - 2$ (since $h^1 = \ell(K - D)$).

Genus zero:

$\deg(D)$	-4	-3	-2	-1	0	1	2	3	4
$h^0(D)$	0	0	0	0	1	2	3	4	5
$h^1(D)$	3	2	1	0	0	0	0	0	0

Genus one:

$\deg(D)$	-4	-3	-2	-1	0	1	2	3	4
$h^0(D)$	0	0	0	0	A	1	2	3	4
$h^1(D)$	4	3	2	1	A	0	0	0	0

Here $0 \leq A \leq 1$ by Corollary 44.8, but the value of A will depend on the divisor. For $D = P - Q$ one has $A = 0$ by Proposition 44.9. If we fix P, Q, and R then for $D = P + Q - R - S$ one usually has $A = 0$, but for a unique S one gets $A = 1$.

Genus two:

$\deg(D)$	-4	-3	-2	-1	0	1	2	3	4
$h^0(D)$	0	0	0	0	A	B	C	2	3
$h^1(D)$	5	4	3	2	$A+1$	B	$C-1$	0	0

By Corollary 44.8 we know $0 \leq A \leq 1$, $0 \leq B \leq 2$, and $0 \leq C \leq 3$. But in fact we can argue here that $C \leq 2$: if D were a divisor with $h^0(D) = 3$ then $h^0(D+[P]) \geq 3$ and yet the table shows $h^0(D + [P]) = 2$.

∘ Exercises ∘

Exercise 44.30. This exercise is a supplement to the proof of Proposition 44.16. Recall the holomorphic version of the (local) implicit function theorem, which says that if f is the germ of a holomorphic function $\mathbb{C} \to \mathbb{C}$ defined at 0 and $f'(0) \neq 0$

then f has a holomorphic inverse defined in some neighborhood. Rather than say "defined in some neighborhood" over and over again in this problem we will just use the adjectives "local" or "locally" to signify that.

(a) Suppose f is a local holomorphic function at 0 and $f(0) \neq 0$. Prove that for any $n \geq 2$ there is a local holomorphic function $g(z)$ at 0 such that $g(z)^n = f(z)$.
(b) Suppose f is a local holomorphic function at 0 and that 0 is a zero of order $n-1$ of $f'(z)$. Prove that up to a holomorphic change of coordinate $f(z) = f(0) + z^n$ locally. Conclude that the ramification index of f at 0 is equal to n.
(c) Now suppose that f is a local meromorphic function at 0 and that 0 is a pole of f of order n. By studying $\frac{1}{f}$ instead of f, argue that the ramification index of f at 0 equals n.

Exercise 44.31. Let X be a Riemann surface of genus g.

(a) Fix $P \in X$ and $m \geq g+1$. Prove that there is a meromorphic function on X having a pole at P of order at most m and no other poles.
(b) If $m \geq 2g$ strengthen (a) by showing that there is such a function having P a pole of order exactly m.
(c) Prove that there is a nonconstant map $X \to \mathbb{C}P^1$ having topological degree at most $g+1$.

Exercise 44.32. Let X be a Riemann surface of genus g and let $\mathcal{L} \to X$ be an algebraic (equivalently, holomorphic) line bundle.

(a) Use GRR and Serre duality to deduce the formula

$$\dim \Gamma(X, \mathcal{L}) - \dim \Gamma(X, \mathcal{L}^* \otimes \omega_X) = 1 - g + \int c_1(\mathcal{L}).$$

(b) Prove that $\mathcal{L}$ is isomorphic to $\mathcal{O}(D)$ for some divisor D such that $\deg D = \int c_1(\mathcal{L})$. The integer $\int c_1(\mathcal{L})$ is often called the **degree** of the line bundle. Explain why bundles of negative degree do not have any holomorphic sections (other than the zero section).
(c) Compute $\deg \omega_X$.
(d) We saw in this section that there is a bijection

$$\Gamma(X, \mathcal{O}(D)) \longleftrightarrow \{\text{meromorphic functions } f \text{ on } X \text{ such that } \operatorname{div}(f) \geq -D\}.$$

For a section s of $\mathcal{O}(D)$ and P a point of X, explain why s vanishing at P is different than the corresponding meromorphic function f vanishing at P.
(e) If $P \in X$ prove that $\Gamma(X, \mathcal{L} \otimes \mathcal{O}(-P))$ is isomorphic to the subspace of $\Gamma(X, \mathcal{L})$ consisting of sections that vanish at P. [Hint: One way is to create a short exact sequence $0 \to \mathcal{O}(-P) \to \mathcal{O} \to \mathcal{O}_P \to 0$.]
(f) If $\deg \mathcal{L} \geq 2g$ prove that for every $P \in X$ there is a holomorphic section s of $\mathcal{L}$ such that $s(P) \neq 0$. (Geometers would say "$\mathcal{L}$ is generated by global sections".)

The following exercise starts to explore GRR for complex algebraic surfaces (algebraic varieties of complex dimension two).

Exercise 44.33. Let X be a compact almost complex manifold. We have the canonical line bundle $\omega_X = \Lambda^{\dim X} T_X^*$, and the canonical class K_X is defined to be $K_X = c_1(\omega_X)$.

(a) Prove that $K_X = -c_1(T_X)$. [Hint: Use the splitting principle.]

(b) For the rest of this problem assume that $\dim X = 2$. Then the characteristic classes $c_i(T_X)$ give rise to two integer invariants of X: $\int c_1(T_X)^2 = \int K_X^2$ and $\int c_2(T_X) = \chi(X)$. We will follow geometers and typically leave off the integral signs, so that K_X^2 denotes the integer k for which $K_X \cdot K_X = k[*]$. We will also abbreviate $c_i(T_X)$ to just c_i.

When $X = \mathbb{C}P^2$ compute that $K_X = -3[\mathbb{C}P^1]$, $K_X^2 = 9$, and $c_2 = 3$. When $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ compute that $K_X = -2[\mathbb{C}P^1 \times *] - 2[* \times \mathbb{C}P^1]$, $K_X^2 = 8$, and $c_2 = 4$.

(c) Use the GRR theorem to prove that $\text{Td-genus}(X) = \frac{1}{12}(c_1^2 + c_2) = \frac{1}{12}(K_X^2 + c_2)$ (this is known as *Noether's formula*). In particular, notice that $K_X^2 + c_2$ must always be a multiple of 12.

(d) Fix $d \geq 1$, set $Y = \mathbb{C}P^2$, and let $L = \mathcal{O}(d)$. Fix a section $s \in H^0(Y, L^{\otimes 2})$, or equivalently $s\colon \mathcal{O}_Y \to L^{\otimes 2}$. Using this data we will construct an interesting smooth algebraic surface X that comes as a 2-fold branched cover $X \to Y$. Then we will compute the topological invariants of X.

Let $\mathcal{A} = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$, and make this into a sheaf of $\mathbb{Z}/2$-graded algebras by letting the nontrivial part of the multiplication be $s^*\colon \mathcal{L}^{-1} \otimes \mathcal{L}^{-1} \to \mathcal{O}_Y$. Let $\sigma\colon \mathcal{A} \to \mathcal{A}$ be the involution of algebras that is the identity on $\mathcal{O}_Y$ and multiplication by -1 on $\mathcal{L}^{-1}$. Note that $\mathcal{A}^{\mathbb{Z}/2} = \mathcal{O}_Y$. Let $X = \operatorname{Spec}_Y \mathcal{A}$ and let $f\colon X \to Y$ be the associated projection. Then f is an affine map, it is finite of degree two, $\sigma\colon X \to X$ is such that $f\sigma = \sigma$, and $X/(\mathbb{Z}/2) \cong Y$. Locally on Y the map f looks like $f_R\colon \operatorname{Spec} R[t]/(t^2 - r) \to R$, and from this one readily checks that X is smooth and that the branch locus of f is the curve $C = V(s)$, a degree $2d$ hypersurface in $\mathbb{C}P^2$.

This part of the exercise is just to work out the details of the above paragraph.

(e) Use a Riemann-Hurwitz style argument (as in the proof of Proposition 44.16) to prove that $\chi(X) = 2\chi(\mathbb{C}P^2) - \chi(C) = 6 - 2d(3 - 2d)$.

(f) Explain why $f^*([*]) = 2[*]$ in $H^4(X)$, and use this to show $f_!(1) = 2$ in $H^0(Y)$. Deduce that $f^*\colon H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is injective.

(g) The map $f^*\colon H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ factors through $H^*(X;\mathbb{Q})^{\mathbb{Z}/2}$. Let us take as granted that X can be given a CW-structure where $\mathbb{Z}/2$ acts cellularly. This implies that the map $f^*\colon H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})^{\mathbb{Z}/2}$ is an isomorphism. Explain why $\sigma^* T_X \cong T_X$ and conclude that the Chern classes $c_i(T_X)$ are in the image of f^*.

(h) We know $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{C}P^2} + \mathcal{O}(-d)$ by the construction of X, and since f is affine we also know $R^i f_* \mathcal{O}_X = 0$ for $i > 0$. So $f_![\mathcal{O}_X] = [f_*\mathcal{O}_X]$ in $K^0(\mathbb{C}P^2)$. Use GRR for f to prove that

$$K_X = f^*(K_{\mathbb{C}P^2} + d[\mathbb{C}P^1]) = f^*((d-3)[\mathbb{C}P^1])$$

and so $K_X^2 = 2(d-3)^2$ (part (g) will be helpful at some point). Also show via GRR (without using (e)) that

$$\text{Td-genus}(X) = 2\,\text{Td-genus}(\mathbb{C}P^2) + \tfrac{1}{2}d(d-3) = \tfrac{d^2-3d+4}{2}.$$

Explain how to recover the result of (e) via these GRR calculations.

[Note in particular that when $d = 3$ one gets an algebraic surface X such that $\chi(X) = 24$, $K_X = 0$, and $\text{Td-genus}(X) = 2$—this is an example of a so-called "K3 surface".]

45. Formal group laws and complex-oriented cohomology theories

We have spent the past several sections exploring the differences in the way that geometric information is encoded in singular cohomology versus complex K-theory. Both of these theories have Thom classes for complex bundles and therefore have Gysin sequences, fundamental classes, Euler classes, and so forth. A key difference we observed was in the formula for the Euler class of a tensor product of line bundles: Proposition 37.9 versus Proposition 37.17. The construction of the Chern character was built around the differences between these formulas, and this difference also led directly to the Todd class and the topological Riemann-Roch theorem.

A natural question arises: Are there *other* complex-oriented cohomology theories lying around, which would then have these same kind of underlying connections to geometry? How does one understand the gamut of such theories? A full discussion of this story is outside the scope of this book, but we want to at least give an introduction to the key element: the purely algebraic notion of *formal group law*. In this section we outline some of the main theory behind what these are and how they connect to complex-oriented cohomology theories.

45.1. **Formal group laws.** We start with a purely algebraic development. Let R be a fixed commutative ring.

Definition 45.2. *A (commutative, one-dimensional)* ***formal group law*** *over R is a power series $F(x,y) \in R[[x,y]]$ having the properties that*

(1) $F(x,y) = F(y,x)$
(2) $F(x,0) = x = F(0,x)$
(3) $F(x,F(y,z)) = F(F(x,y),z)$.

Each part should be interpreted as an identity of formal power series.

It is sometimes useful to adopt the notation $x +_F y = F(x,y)$ and call this the "formal sum" of x and y. The above three properties are then written

(1) $x +_F y = y +_F x$
(2) $x +_F 0 = x = 0 +_F x$
(3) $x +_F (y +_F z) = (x +_F y) +_F z$.

One can talk about non-commutative formal group laws by omitting property (1), but we will not encounter them. Likewise, one can define an n-dimensional formal group law to be a power series in $R[[x_1,\dots,x_n,y_1,\dots,y_n]]$ with properties similar to the above. But as we will not use these either, for us the phrase "formal group law" will always mean one that is commutative and one-dimensional.

The reader will note that while the above axioms encode commutativity, associativity, and the identity condition, there is no mention of inverses. It turns out that in this "formal" setting the existence of inverses is automatic; see Exercise 45.6 below.

Let us write $F(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \cdots$. The identity $F(x,y) = F(y,x)$ shows that $a_1 = a_2$, $a_3 = a_5$, and so on. The identity $F(x,0) = x$ shows that $0 = a_0 = a_3$, $1 = a_1$, etc. So this type of reasoning already shows that we must have

$$F(x,y) = x + y + a_4xy + a_5x^2y + a_5xy^2 + a_6x^3y + a_7x^2y^2 + a_6xy^3 + \cdots$$

Associativity will impose other conditions on the a_i, but we will return to that in a moment.

Example 45.3.

(a) $x +_F y = x + y$ is a formal group law, and certainly the simplest example. It is called the **additive formal group law**.

(b) Consider $x +_F y = x + y - xy = 1 - (1 - x)(1 - y)$. This is called the **multiplicative formal group law**. To verify associativity, observe that

$$1 - (x +_F y) = (1 - x)(1 - y)$$

and so

$$\begin{aligned} 1 - (x +_F (y +_F z)) = (1 - x)(1 - (y +_F z)) &= (1 - x)(1 - y)(1 - z) \\ &= 1 - ((x +_F y) +_F z). \end{aligned}$$

The name "multiplicative formal group law" might not seem intuitive, but see Example 45.12 below for further explanation.

Exercise 45.4.

(a) Verify that $x +_F y = x + y + axy$ is a formal group law, for any $a \in R$.

(b) For $a, b \in R$ verify that $x +_F y = x + y + axy + bx^2y + bxy^2$ is a formal group law only if $0 = 2ab = 2b^2 = b^3 = a^2b + b^2$. So if R is reduced (no nonzero nilpotents), we have a formal group law if and only if $b = 0$.

(c) Assume that R is reduced. For $a, b, c, d \in R$ verify that $x +_F y = x + y + axy + bx^2y + bxy^2 + cx^3y + dx^2y^2 + cxy^3$ is a formal group law only if $b = c = d = 0$. [Warning: Both (b) and (c) can be tedious. Use of a computer algebra system is recommended.]

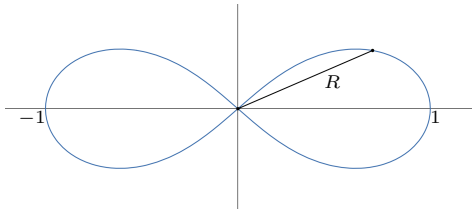
Exercise 45.5. Let $R = \mathbb{F}_3[\epsilon]/(\epsilon^2)$. Prove that $F(x, y) = x + y + \epsilon xy^3$ satisfies conditions (2) and (3) of Definition 45.2 but not condition (1). The fact that R has both torsion and nilpotents is important here: it is a theorem that if R has no torsion nilpotents then all power series satisfying (2) and (3) also satisfy (1). See [Hazewinkel, Chapter 1.6.1], or Exercise 45.23 below for the special case when R is torsion-free.

Exercise 45.6. Let $F(x, y) = x + y + \sum a_{ij}x^iy^j$ be a formal group law over a commutative ring R. Prove that there exists a power series $I(x) \in R[[x]]$ such that $0 = F(x, I(x))$ and find the coefficients of $I(x)$ (in terms of the a_{ij}) up through degree four. [Hint: Set $I(x) = b_1x + b_2x^2 + \cdots$, then expand $F(x, I(x))$ and start solving the equations needed to make each coefficient vanish.] See also Exercise 45.33.

Exercise 45.7. Verify that the inverse series $I(x)$ for the multiplicative formal group law is $I(x) = -\frac{x}{1-x} = -\sum_{k\geq 1} x^k$.

In light of Exercise 45.4 one might get the impression that formal group laws with $F(x, y)$ a *polynomial* are quite rare, and that is indeed the case. So where do interesting examples come from? Historically, the first nontrivial examples probably came from the exploration of elliptic integrals. This is a long story and we will only give a very brief and streamlined introduction.

Consider the lemniscate curve given in polar coordinates by $r^2 = \cos(2\theta)$, drawn here:



This curve seems to have been first investigated in 1694 by Jacob Bernoulli, who realized that it had some properties similar to an ellipse. The arclength from the origin to the point with $r = R$, $0 \leq R \leq 1$, is given by the integral $\int_0^R \frac{1}{\sqrt{1-r^4}}\,dr$ (see Exercise 45.10). Work on these integrals led to the discovery of a surprising addition property, namely

$$\int_0^u \frac{1}{\sqrt{1-r^4}}\,dr + \int_0^v \frac{1}{\sqrt{1-r^4}}\,dr = \int_0^{f(u,v)} \frac{1}{\sqrt{1-r^4}}\,dr \tag{45.8}$$

where

$$f(u,v) = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}. \tag{45.9}$$

Such formulas were first established by Euler, though there were precursors in the work of Fagnano. In fact Euler proved such an addition property for any integral of the form $\int \frac{1}{\sqrt{P(r)}}\,dr$ where $P(r) = 1 + ar^2 - r^4$ for some constant a. These are examples of a certain class now known as "elliptic integrals", the name stemming from the fact that the integral for the arclength of an ellipse was one of the earliest examples.

One can expand $f(u,v)$ as a formal power series in u and v with rational coefficients, and equation (45.8) shows that this gives a formal group law over $\mathbb{Q}$. The symmetry and identity properties are obvious, and associativity follows from the fact that $f(u, f(v,w))$ and $f(f(u,v),w)$ both give the radius of the point on the curve where the arclength equals $\int_0^u \frac{1}{\sqrt{1-r^4}}\,dr + \int_0^v \frac{1}{\sqrt{1-r^4}}\,dr + \int_0^w \frac{1}{\sqrt{1-r^4}}\,dr$.

This entire discussion generalizes properties of the circular integrals $\int_0^u \frac{dr}{\sqrt{1-r^2}}$ that were known classically. See Exercise 45.34.

Exercise 45.10. Start with the arclength integral $\int \sqrt{dx^2 + dy^2}$, translate into polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$, and finally use the lemniscate equation $r^2 = \cos(2\theta)$ to eliminate $d\theta$ and write the arclength entirely as a dr-integral. In this way verify that the arclength is given by $\int \frac{1}{\sqrt{1-r^4}}\,dr$.

Exercise 45.11. Use a computer algebra system to determine the power series $f(u,v)$ from (45.9) up through degree 12 (check: there end up being terms only in degrees 1, 5, and 9). Using only the terms up through degree 12, use the software to check that the identity $f(u, f(v,w)) = f(f(u,v),w)$ holds up through the same degree.

There are many more formal group laws than the few we have just seen, although it is not so easy just to guess what they look like. One source of these gadgets comes from looking at a 1-dimensional algebraic group in an analytic neighborhood of the identity. To understand this, let X be an algebraic curve over $\mathbb{C}$ (for simplicity) and let $O \in X$ be a point. Assume that there is a map of varieties $\mu\colon X \times X \to X$

making X into a group with O as identity. A choice of analytic coordinate near O gives an isomorphism of $\mathbb{C}[[z]]$ with the germs of analytic functions at O on X, and likewise an isomorphism of $\mathbb{C}[[z_1, z_2]]$ with the germs of analytic functions at (O, O) on $X \times X$. The map μ induces

$$\mu^* \colon \mathbb{C}[[z]] \to \mathbb{C}[[z_1, z_2]].$$

Let $F(z_1, z_2) = \mu^*(z)$. The group axioms for μ imply that $F(z_1, z_2)$ is a formal group law.

Example 45.12.

(1) Let $X = \mathbb{A}^1$, with group law $\mu \colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $\mu(x, y) = x + y$. On germs of functions this becomes $\mathbb{C}[[z]] \to \mathbb{C}[[z_1, z_2]]$ given by $z \mapsto z_1 + z_2$, which is the additive formal group law.

(2) Let $X = \mathbb{A}^1 - 0$ with group law $\mu(x, y) = xy$. On rings of functions this is $\mathbb{C}[z, z^{-1}] \to \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ given by $z \to z_1 z_2$. The identity element of X is 1, and a local coordinate near 1 is $t = 1 - z$. One has

$$\mu^*(t) = 1 - \mu^*(z) = 1 - z_1 z_2 = 1 - (1 - t_1)(1 - t_2) = t_1 + t_2 - t_1 t_2.$$

This is the multiplicative formal group law.

(3) We saw in Section 44.20 that any Riemann surface of genus one (i.e., an elliptic curve) can be given a group multiplication. So any elliptic curve gives rise to a formal group law. (In particular, a certain elliptic curve gives rise to the formal group law from (45.9), though we will not explore this here.)

In the theory of algebraic groups, $\mathbb{A}^1$ with the sum operation is called the **additive group** and denoted $\mathbb{G}_a$. The algebraic group $\mathbb{A}^1 - 0$ with the multiplication operation is called the **multiplicative group** and denoted $\mathbb{G}_m$. The names of the corresponding formal group laws make more sense in light of this.

45.13. **The universal formal group law.** Given a ring homomorphism $\alpha \colon R \to S$ and a formal group law $F(x, y)$ over R, we get a formal group law $\alpha_*(F)$ by applying α to all the coefficients of F. If we write $\mathrm{FGL}(R)$ for the set of formal group laws over R, then we have just described a map $\alpha_* \colon \mathrm{FGL}(R) \to \mathrm{FGL}(S)$.

Proposition 45.14. *There is a universal formal group law. That is, there exists a ring L and a formal group law F_{univ} on L having the property that for all rings R, the map*

$$\mathrm{Ring}(L, R) \longrightarrow \mathrm{FGL}(R), \qquad \alpha \mapsto \alpha_*(F_{univ})$$

is a bijection. That is to say, the functor $\mathrm{FGL}(-)$ *on the category of rings is corepresentable.*

Proof. This is easy. We already know that every formal group law looks like

$$F(x, y) = x + y + a_1 xy + a_2 x^2 y + a_2 x y^2 + a_3 x^3 y + a_4 x^2 y^2 + a_3 x y^3 + \cdots$$

Let $S = \mathbb{Z}[a_1, a_2, \ldots]$. Equating coefficients in the associativity identity $F(x, F(y, z)) = F(F(x, y), z)$ yields a collection of polynomial identities in the a_i. Let L be the quotient of S by the ideal generated by these equations. □

Remark 45.15. Note that the ring L can be given a natural grading. Give the formal variables x and y degree -2, and regard $F(x, y)$ as being homogeneous of degree -2. Then writing $F(x, y) = x + y + a_1 xy + a_2 x^2 y + a_2 x y^2 + a_3 x^3 y + a_4 x^2 y^2 + a_3 x y^3 + \cdots$ requires that a_1 have degree 2, a_2 have degree 4, a_3 and a_4 to have

degree 6, and so forth. This defines a grading on S where $F(x,y)$ is homogeneous. The associativity identity is homogeneous, so it leads to homogeneous relations amongst the a_i. So L inherits the grading on S. (Note: We could have assigned x and y to have degree -1 instead of -2. The "unnatural" choice of -2 comes from the way all of this manifests in topology.)

There is also a more satisfying explanation of why one would expect a grading here. Recall from Section 28.14 that a grading on a commutative ring corresponds to a $\mathbb{G}_m$-action on the scheme it represents. If $F(x,y)$ is a formal group law over R and $\lambda \in R$ is a unit, it is easy to check that $\lambda^{-1}F(\lambda x, \lambda y)$ is another formal group law. This is the $\mathbb{G}_m$-action on $\mathrm{FGL}(R)$. This affects the coefficients of F by multiplying the coefficient of $x^i y^j$ by λ^{i+j-1}. So those elements of L would have degree $i+j-1$ (though again, topologists would multiply that by two).

Proposition 45.14 is rather formal, and gives us no indication of what the ring L looks like. A very nontrivial theorem of Lazard completes the picture:

Theorem 45.16 (Lazard)**.** *There is an isomorphism of graded rings $L \cong \mathbb{Z}[u_1, u_2, \ldots]$ where the degree of u_i is $2i$.*

Note that the u_i from Lazard's theorem are not equal to the a_i that appear in the proof of Proposition 45.14. We will not include a proof of Lazard's theorem, but one can be found in [Hazewinkel, Chapter 1.5.2].

45.17. **Isomorphism of formal group laws.**

Definition 45.18. *Suppose F and G are formal group laws over R. A map $F \to G$ is a power series $f(x) \in R[[x]]$ such that $f(0) = 0$ and $f(x +_F y) = f(x) +_G f(y)$.*

Note that if $f_1 \colon F \to G$ and $f_2 \colon G \to H$ then the composition $f_2 \circ f_1$ is the power series composition $f_2(f_1(x))$. This is defined because of the condition $f_1(x) = 0$. By an easy exercise in algebra, the power series f_1 has an inverse under composition if and only if $f_1'(0)$ is a unit. So a map of formal group laws $f \colon F \to G$ is an isomorphism if and only if $f'(0)$ in a unit in R.

Example 45.19. Let $F(x,y) = x+y-xy = 1-(1-x)(1-y)$ and $G(x,y) = x+y$. To produce a map from F to G we need a power series $f(x)$ such that

$$f(1-(1-x)(1-y)) = f(x) + f(y).$$

This suggests the logarithm function, and a moment's thought shows that $f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ works. However, this requires that we have $\mathbb{Q} \subseteq R$ for this power series to exist in $R[[x]]$. As the coefficient of x in $\log(1-x)$ is -1, we actually have an isomorphism of formal group laws. So if $\mathbb{Q} \subseteq R$ then the additive and multiplicative formal group laws are isomorphic.

For future reference, we note that $f(x) = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ is also an isomorphism in this example.

Exercise 45.20. Let $F(x,y) = x+y-xy$ and $G(x,y) = x+y+axy$, where $a \in R$. If a is invertible, check that $f(x) = -\frac{1}{a} \cdot x$ is an isomorphism between F and G. More generally, show that $F(x,y) = x+y+bxy$ and $G(x,y) = x+y+axy$ are isomorphic if $b = ua$ for some unit $u \in R$.

Remark 45.21. We only defined maps between formal group laws defined over the same ring, but one readily extends this as follows. Let F be a FGL defined over

R, and G be a FGL defined over S. A map from F to G is defined to be a pair consisting of a homomorphism $\alpha\colon R \to S$ and a map $f\colon \alpha_* F \to G$.

Over the rational numbers, there is really only one formal group law:

Proposition 45.22. *If $\mathbb{Q} \subseteq R$ then all formal group laws are isomorphic to the additive formal group law.*

Proof. Let F be a formal group law. We look for a power series $f(x) \in R[[x]]$ such that $f(F(x,y)) = f(x) + f(y)$ and $f'(0) = 1$. Apply $\frac{d}{dy}\big|_{y=0}$ to the first equation, to get:

$$\begin{aligned} f'(F(x,y))|_{y=0} \cdot F_2(x,0) &= f'(0) = 1 \\ f'(F(x,0)) \cdot F_2(x,0) &= 1 \\ f'(x) &= \frac{1}{F_2(x,0)} \\ f(x) &= \int \frac{1}{F_2(x,0)}\, dx \end{aligned}$$

where of course F_2 is the partial derivative of F in the second variable. The power series $F_2(x,0)$ has constant term 1, and so is invertible in $R[[x]]$. As long as $R \supseteq \mathbb{Q}$ we can then integrate term-by-term to get $f(x)$. The steps above are not entirely reversible, though (recall that we set $y = 0$ at one point), so we need to work a little to show that this choice of $f(x)$ actually does what we want.

Start with the associativity formula $F(F(x,y),z) = F(x,F(y,z))$. Differentiate both sides with respect to z to get

$$F_2(F(x,y),z) = F_2(x,F(y,z)) \cdot F_2(y,z).$$

Putting $z = 0$ gives $F_2(x +_F y, 0) = F_2(x,y)F_2(y,0)$. The series $F_2(a,b)$ has constant term 1, so it is invertible and we can write

$$f'(x +_F y)F_2(x,y) = \frac{1}{F_2(x +_F y, 0)} \cdot F_2(x,y) = \frac{1}{F_2(y,0)} = f'(y).$$

Integrating both sides with respect to y gives

$$f(x +_F y) = f(y) + g(x)$$

for some power series $g(x)$. But now substitute $y = 0$ and get $f(x) = f(0) + g(x) = g(x)$. So in fact we have $f(x +_F y) = f(y) + f(x)$, as desired. □

Exercise 45.23. The above proof only used the identity and associativity properties of the formal group law $F(x,y)$. Use that fact to prove that any possibly non-commutative one-dimensional formal group law over a torsion-free ring R is necessarily commutative.

Let F_a denote the additive formal group law. If $u \in R$ is a unit, then the power series ux is an automorphism of F_a. Composing with such isomorphisms allow us to adjust any isomorphism $f\colon F \to F_a$ so that it satisfies $f'(0) = 1$. Such isomorphisms are called **strict**. Note that the proof of Proposition 45.22 produced a strict isomorphism.

Example 45.24. Returning to the multiplicative formal group law $F(x,y) = x + y - xy$, we have $F_y(x,0) = 1 - x$, $\frac{1}{F_y(x,0)} = 1 + x + x^2 + \cdots$ and $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x)$.

Definition 45.25. *When F is a formal group law defined over a ring $R \supseteq \mathbb{Q}$, the power series $f(x)$ having the property that $f(x +_F y) = f(x) + f(y)$ and $f(0) = 0$, $f'(0) = 1$ is called the* ***logarithm*** *of F and denoted* $\mathbf{log}_{\boldsymbol{F}}$. *The compositional inverse to $\log_F$ is denoted* $\mathbf{exp}_{\boldsymbol{F}}$ *and called the* ***exponential*** *of F.*

Observe that the exponential for the multiplicative formal group law is the power series $1 - e^{-x}$.

45.26. **Formal group laws in topology.** Let $\mathcal{E}$ be a complex-oriented cohomology theory. We have seen that every complex line bundle $J \to X$ has an Euler class—also called the first Chern class—$c_1^{\mathcal{E}}(J) \in \mathcal{E}^2(X)$. We will show that there is also a unique formal group law $F_{\mathcal{E}}(x, y) \in \mathcal{E}^*[[x, y]]$ having the property that $c_1^{\mathcal{E}}(J_1 \otimes J_2) = F_{\mathcal{E}}(c_1^{\mathcal{E}}(J_1), c_1^{\mathcal{E}}(J_2))$ for all line bundles J_1 and J_2.

Recall from Section 38.13 that if R is a $\mathbb{Z}$-graded ring and x is a formal variable of degree d, then $R[[x]]$ is the ring of graded power series: the R-linear span of homogeneous power series with coefficients in R. That is, $R[[x]] = \oplus_n P_n$ where $P_n = \{a_n + a_{n-d}x + a_{n-2d}x^2 + \cdots \mid a_i \in R_i\}$. One defines $R[[x, y]]$ similarly.

Exercise 45.27. Check that in the case of singular cohomology we have $H^*(pt)[[x]] = \mathbb{Z}[x]$ as a graded ring. So we can write $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[x]]$ or $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$, and they mean the same thing.

Likewise, check that if x is of degree 2 then $K^*(pt)[[x]] = \mathbb{Z}[[\beta x]]\,[\beta, \beta^{-1}]$ where here $\mathbb{Z}[[\beta x]]$ can be interpreted either as graded power series or ungraded power series (they are the same because both $\mathbb{Z}$ and βx are in degree 0).

Exercise 45.28. Check that if R is a $\mathbb{Z}$-graded ring then with our definition of $R[[x]]$ one still has $R[[x]]/(x^n) \cong R[x]/(x^n)$ as graded rings. Show that our $R[[x]]$ coincides with $\lim_n R[x]/(x^n)$ where the limit is taken in the category of *graded* rings.

We begin by recalling the definition of the $\mathcal{E}$-theory Euler class (or first Chern class). Let $L \to \mathbb{C}P^\infty$ be the tautological line bundle and let $\zeta\colon \mathbb{C}P^\infty \hookrightarrow \operatorname{Th} L$ be the zero section. Consider the composite

$$\tilde{\mathcal{E}}^2(\operatorname{Th} L) \xrightarrow{\zeta^*} \tilde{\mathcal{E}}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{\mathcal{E}}^2(\mathbb{C}P^1) = \tilde{\mathcal{E}}^2(S^2) = \mathcal{E}^0(*).$$

Let $x = \zeta^*(\mathcal{U}_L)$, which is an element in $\tilde{E}^2(\mathbb{C}P^\infty)$ that restricts to 1 under the above composition. Then one defines $c_1^{\mathcal{E}}(L) = x$.

By naturality the above determines the first Chern class of all line bundles. Given any complex line bundle $J \to X$, there is a classifying map $f\colon X \to \mathbb{C}P^\infty$ and a pullback diagram

$$\begin{array}{ccc} J & \longrightarrow & L \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{C}P^\infty. \end{array}$$

One defines $c_1^{\mathcal{E}}(J) = f^*(x)$. We will drop the superscript and just write $c_1(J)$ when there is no possibility for confusion.

Recall from Proposition 38.22 that for any $k \geq 1$ we have the canonical isomorphism $\mathcal{E}^*((\mathbb{C}P^\infty)^{\times k}) \cong \mathcal{E}^*[[x_1, \ldots, x_k]]$ where $x_i = \pi_i^*(x)$.

Consider now the external tensor product $L \hat{\otimes} L \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. This is a line bundle, and so is classified by a map $g\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$:

$$\begin{array}{ccc} L\hat{\otimes}L & \longrightarrow & L \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{g} & \mathbb{C}P^\infty \end{array}$$

The map $g^*\colon \mathcal{E}^*(\mathbb{C}P^\infty) \to \mathcal{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is $\mathcal{E}^*[[x]] \to \mathcal{E}^*[[x_1, x_2]]$ and so sends x to a formal power series $F(x_1, x_2)$ of degree 2. By definition we have

$$F(x_1, x_2) = c_1(L\hat{\otimes}L).$$

So if $J_1 \to X$ and $J_2 \to X$ are any two line bundles, with classifying maps $f_1, f_2\colon X \to \mathbb{C}P^\infty$, we have the diagram

$$\begin{array}{ccccccc} J_1 \otimes J_2 & \longrightarrow & J_1\hat{\otimes}J_2 & \longrightarrow & L\hat{\otimes}L & \longrightarrow & L \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X & \xrightarrow{f_1 \times f_2} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{g} & \mathbb{C}P^\infty. \end{array}$$

Therefore

$$\begin{aligned} c_1(J_1 \otimes J_2) = \Delta^*(f_1 \times f_2)^* g^*(x) &= \Delta^*(f_1 \times f_2)^*(F(x_1, x_2)) \\ &= F(f_1^*(x_1), f_2^*(x_2)) \\ &= F(c_1(J_1), c_1(J_2)). \end{aligned}$$

It remains for us to understand why the power series $F(x_1, x_2)$ is a formal group law. The group axioms follow immediately from the following commutative diagrams in the homotopy category:

$$\begin{array}{ccccc} \mathbb{C}P^\infty & \xrightarrow{\mathrm{id} \times *} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{g} & \mathbb{C}P^\infty \\ & \searrow^{\mathrm{id}} & & & \nearrow \end{array} \qquad \begin{array}{ccc} \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{g} & \mathbb{C}P^\infty \\ \downarrow{\scriptstyle t} & \nearrow_{g} & \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & & \end{array}$$

$$\begin{array}{ccccc} & & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty & & \\ & \swarrow^{\mathrm{id} \times g} & & \searrow^{g \times \mathrm{id}} & \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & & & & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \\ & \searrow_{g} & & \swarrow_{g} & \\ & & \mathbb{C}P^\infty. & & \end{array}$$

To see that these diagram commute (up to homotopy), use that $\mathbb{C}P^\infty$ is the classifying space for line bundles. The first diagram just reflects the isomorphism $J \otimes 1 \cong J$, the second diagram reflects that $J_1 \otimes J_2 \cong J_2 \otimes J_1$, and the third reflects that $J_1 \otimes (J_2 \otimes J_3) \cong (J_1 \otimes J_2) \otimes J_3$.

Summarizing, we have now proven the following:

Proposition 45.29. *Let $\mathcal{E}$ be a complex-oriented cohomology theory. Then there exists a formal group law F over $\mathcal{E}^*$ (of homogeneous degree 2 as a power series) having the property that $F(c_1(J_1), c_1(J_2)) = c_1(J_1 \otimes J_2)$ for all complex line bundles J_1 and J_2.*

We have seen in Proposition 37.9 that the formal group law for singular cohomology is the additive one, and in Proposition 37.17 that the formal group law for complex K-theory is the multiplicative one. To be more precise in the latter case, it is the formal group law over $\mathbb{Z}[\beta, \beta^{-1}]$ given by $x + y - \beta xy$.

We have not discussed it so far in this book, but there is a cohomology theory MU called complex cobordism. See Exercises 46.22 and 46.23 for the basic construction. In a certain sense this is the universal complex-oriented cohomology theory. The coefficients are $MU^* = \mathbb{Z}[u_1, u_2, \ldots]$ where $|u_i| = -2i$, by a classical calculation due to Milnor [Milnor4]. In [CF] Conner-Floyd developed the theory of Chern classes in MU—at that stage the notion of a complex-oriented cohomology theory was not yet developed, but one can find all of the basic ideas throughout their work.

The first appearance of the formal group law associated to a complex-oriented cohomology theory is probably in the appendix of [Novikov] due to Mischenko, where it is proven that for the case of complex cobordism the logarithm of the formal group law is $\log_{MU}(X) = \sum_{n\geq 0}[\mathbb{C}P^n]\frac{x^{n+1}}{n+1}$. This result was later generalized by Quillen, who used his version to prove that F_{MU} is the universal formal group law that we encountered in Proposition 45.14. We will tell this story in Section 46.

45.30. **Formal group laws and Leibniz rules.** There is another place that formal group laws appear in an unexpected way, but which relates closely to some of the K-theory calculations we have seen in earlier sections. The differentiation operator D from calculus obeys the Leibniz rule $D(fg) = (Df)g + f(Dg)$. One can write this as

$$D(fg) = \mu[Df \otimes g + f \otimes Dg] = \mu[D \otimes \mathrm{id} + \mathrm{id} \otimes D](f \otimes g).$$

Here $\mu(a \otimes b) = ab$, of course.

We saw in Lemma 34.7 that the finite-difference operator Δ obeys a different form of the Leibniz rule. Written in the above form, it is

$$\Delta(fg) = \mu[\Delta \otimes \mathrm{id} + \mathrm{id} \otimes \Delta + \Delta \otimes \Delta](f \otimes g).$$

One can feel the specter of formal group laws here, and a precise connection can be made as follows. For an operator J, define a "generalized Leibniz rule" to be a formula

$$J(fg) = \mu\Big[\sum_{i,j} a_{ij} J^i \otimes J^j\Big](f \otimes g)$$

where the a_{ij} are coefficients in some underlying ground ring. By setting $x = J \otimes \mathrm{id}$ and $y = \mathrm{id} \otimes J$, the expression inside the brackets can be written as $F(x, y)$ where $F = \sum_{i,j} a_{ij} x^i y^j$. The formal group law conditions on $F(x, y)$ are parallels of the following properties:

- $J(f \cdot 1) = J(f) = J(1 \cdot f)$
- $J(fg) = J(gf)$
- $J((fg)h) = J(f(gh))$

More precisely, the first FGL condition says that applying the Leibniz rule to $J(f \cdot 1)$ gives a formula that just reduces to $J(f)$. The second condition says that applying the Leibniz formula to $J(fg)$ and to $J(gf)$ lead to the same expression, and the third condition says something similar for applying the Leibniz rule twice to $J(fgh)$ (in the two different ways).

The connections between algebraic topology and these generalized differentiation operators has been explored in several papers by Nigel Ray, starting with [Ray1]

and [Ray2]. Here we just refer the reader back to the finite difference operator Δ discussed in Section 34 and especially Lemma 34.7, which we can now see as an incarnation of the multiplicative formal group law.

45.31. **Formal group laws and Riemann-Roch.** Let $\mathcal{E}$ and $\mathcal{J}$ be complex-oriented cohomology theories, and suppose that $\alpha\colon \mathcal{E} \to \mathcal{J}$ is a multiplicative map. On coefficients this gives a ring homomorphism $\mathcal{E}^* \to \mathcal{J}^*$ that we will also denote by α. The map $\alpha\colon \mathcal{E}^*(\mathbb{C}P^\infty) \to \mathcal{J}^*(\mathbb{C}P^\infty)$ is $\mathcal{E}^*[[x]] \to \mathcal{J}^*[[x]]$ and so sends x to a homogeneous power series $f(x) \in \mathcal{J}^*[[x]]$ of degree two. The commutative diagram

$$\begin{array}{ccc} \mathcal{E}^*(\mathbb{C}P^\infty) & \xrightarrow{\alpha} & \mathcal{J}^*(\mathbb{C}P^\infty) \\ \downarrow & & \downarrow \\ \mathcal{E}^*(pt) & \xrightarrow{\alpha} & \mathcal{J}^*(pt) \end{array}$$

shows that $f(0) = 0$, since the vertical arrows are evaluation at $x = 0$. By naturality, for any line bundle $J \to X$ over a paracompact Hausdorff space the map α sends $c_1^E(J)$ to $f(c_1^J(J))$.

Next consider the commutative square

$$\begin{array}{ccc} \mathcal{E}^*(\mathbb{C}P^\infty) & \xrightarrow{\alpha} & \mathcal{J}^*(\mathbb{C}P^\infty) \\ {\scriptstyle\mu^*}\downarrow & & \downarrow{\scriptstyle\mu^*} \\ \mathcal{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) & \xrightarrow{\alpha} & \mathcal{J}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty). \end{array}$$

Going across the top and then down, we have $x \mapsto \alpha(x) = f(x) \mapsto f(x +_{\mathcal{J}} y)$. Going down and across the bottom we have $x \mapsto x +_{\mathcal{E}} y \mapsto f(x) +_{\mathcal{E}} f(y)$ (for the last arrow, use that α is a ring map that sends $x \mapsto f(x)$, $y \mapsto f(y)$). So we must have that

$$f(x +_{\mathcal{J}} y) = f(x) +_{\mathcal{E}} f(y).$$

That is, f is a map of formal group laws from $F_{\mathcal{J}}$ to $F_{\mathcal{E}}$.

So natural transformations between complex-oriented cohomology theories are intimately related to maps between the corresponding formal group laws. We don't claim that any map of formal group laws induces a corresponding natural transformation of cohomology theories, but in cases of interest the connection is closer than one might at first expect. Based on this principle, the isomorphism from F_a to $F_{\mathcal{E}}$ after rationalization should correspond to a map of cohomology theories $\mathcal{E}_{\mathbb{Q}} \to H(\mathbb{Q} \otimes \mathcal{E}^*)$, where the latter is $\bigvee_n H(\mathbb{Q} \otimes \mathcal{E}^n)$. This is an "$\mathcal{E}$-Chern character". We can also compose with $\mathcal{E} \to \mathcal{E}_{\mathbb{Q}}$ and regard this as a map $\mathcal{E} \to H(\mathbb{Q} \otimes \mathcal{E}^*)$. It would send any $c_1^{\mathcal{E}}(L)$ to $\exp_{F_{\mathcal{E}}}(c_1^H(L))$.

Example 45.32. For $\mathcal{E} = K$ the exponential map is $1 - e^{-x}$. So the above says that $c_1^K(L)$ maps to $1 - e^{-c_1(L)}$. Recalling from (37.16) that $c_1^K(L) = 1 - L^*$, we have $L^* \mapsto e^{-c_1(L)} = c^{c_1(L^*)}$. Since this holds for every line bundle L, we can write it more simply as $L \mapsto e^{c_1(L)}$. Of course this is precisely the the classical Chern character.

Just as in our earlier treatment of the topological Riemann-Roch theorem for K-theory, one can develop an analogous Riemann-Roch theorem for $\mathcal{E}$-theory where the $\mathcal{E}$-Todd class is defined using the power series $\frac{x}{\exp_{\mathcal{E}}(x)}$. We leave the details for the interested reader.

◦ Exercises ◦

Exercise 45.33. In Exercise 45.6, very few of the properties of the series $F(x,y)$ were needed to prove the existence of $I(x)$. However, those properties *are* needed to prove that $I(I(x)) = x$. Explain how to do that.

Exercise 45.34. The classical study of the lemniscate revealed the surprise that it behaves in many ways like a circle. To see the analogy, consider the circle $(x-1)^2 + y^2 = 1$ which has polar equation $r = 2\cos\theta$. Prove that the arclength from the point $(0,0)$ to the point with radius u is given by $\int_0^u \frac{dr}{\sqrt{1-r^2}} = \arcsin(u)$. The classical addition law for arcsin is

$$\arcsin(u) + \arcsin(v) = \arcsin(u\sqrt{1-v^2} + v\sqrt{1-u^2})$$

(valid for u and v positive and small). Derive this formula from the addition formula for the sine function.

Exercise 45.35. Let $\mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the map that classifies the assignment $J \to J^*$ for complex line bundles. A concrete model is the complex conjugation map. Let $\mathcal{E}$ be a complex-oriented cohomology theory. Prove that the induced map $\mathcal{E}^*(\mathbb{C}P^\infty) \to \mathcal{E}^*(\mathbb{C}P^\infty)$ is the inverse map I for the formal group law $F_{\mathcal{E}}$ (see Exercise 45.6).

46. Quillen and the invariant differential

On a Lie group one can talk about left- (or right-) invariant differentials, and there is a similar "infinitesimal" version of this concept that exists for formal group laws. Up to multiplication by units in the ground ring there is a unique invariant differential, and one can write an explicit formula for it. Quillen discovered that for a complex-oriented cohomology theory $\mathcal{E}$ the coefficients of the invariant differential are precisely the $\mathcal{E}$-genera of complex projective spaces. This comes out of "Quillen's residue formula", a more sophisticated formula for computing pushforwards along the structure map of projective bundles. After explaining this material we follow Quillen and use it to prove that the formal group law associated to complex cobordism is the universal formal group law.

46.1. **The invariant differential.** For a Lie group G let $L_g\colon G \to G$ be left multiplication by $g \in G$. A differential ω on G is *left-invariant* if $L_g^*\omega = \omega$ for every $g \in G$. A left-invariant differential is completely determined by its value at the identity element $1 \in G$.

In the setting of a formal group law F over a ring R, write $L_y(x) = F(x,y)$. An **invariant differential** is a formal differential $\omega \in R[[x]].dx$ with the property that $L_y^*\omega = \omega$, where here y is an indeterminate (so that a priori $L_y^*\omega \in R[[x,y]]dx$). The set of all such differentials is an R-module, and it is easy to see that this module is free of rank one. Indeed, suppose that $\omega = h(x)dx$ is an invariant differential. The condition $L_y^*\omega = \omega$ becomes

$$h(L_y(x))F_1(x,y) = h(x). \tag{46.2}$$

Putting $x = 0$ gives $h(y)F_1(0,y) = h(0)$ and therefore

$$h(y) = h(0)\cdot\frac{1}{F_1(0,y)} = h(0)\cdot\frac{1}{F_2(y,0)}.$$

For the last equality we have used $F_1(x,y) = F_2(y,x)$, which follows from differentiating $F(x,y) = F(y,x)$ with respect to y.

It remains to verify that $\frac{dx}{F_2(x,0)}$ is actually an invariant differential. Putting $h(x) = F_2(x,0)^{-1}$ into (46.2) and rearranging yields the equivalent formula $F_2(F(x,y),0) = F_1(x,y)F_2(x,0)$, but we saw this identity (or at least, one readily equivalent to it) in the proof of Proposition 45.22. In fact, the reader will observe that $\frac{1}{F_2(x,0)}$ was the formula for $\log_F(x)$ that we discovered in Proposition 45.22.

Since the module of invariant differentials has a naturally occuring preferred generator, it is often called *the* invariant differential (by abuse):

Definition 46.3. *For a formal group law $F(x,y)$, the 1-form $\omega_F = \frac{dx}{F_2(x,0)}$ is called **the invariant differential** for F.*

Note that $\log_F(x)$ is precisely $\int \omega_F$. For the additive formal group law the invariant differential is simply dx. For the multiplicative formal group law it is $\frac{dx}{1-x} = (1 + x + x^2 + \cdots)dx$.

The expression $\frac{1}{F_2(x,0)}$ also comes up in the context of formal differences $x -_F y = F(x, Iy)$, where $I(x)$ is the inverse function for the formal group law F. For ease of notation set $G(x,y) = F(x,Iy)$. As a power series this has the form $G(x,y) = (x-y)$ + higher order monomials. Since $G(x,x) = F(x,Ix) = 0$, we can write $G(x,y) = (x-y)\cdot u(x,y)$ where $u(x,y)$ has constant term 1 and is therefore a unit.

Take the identity

$$x = F(G(x,y),y) = F((x-y)u(x,y),y)$$

and differential both sides with respect to x. This gives

$$1 = F_1((x-y)u(x,y),y)\cdot[u(x,y) + (x-y)u_1(x,y)].$$

Substituting $x = y$ now gives $1 = F_1(0,y)u(y,y) = F_2(y,0)u(y,y)$. Equivalently,

$$u(x,x) = \tfrac{1}{F_2(x,0)}.$$

So morally speaking, $\frac{1}{F_2(x,0)}$ gives the limiting behavior of the expression $\frac{x-_F y}{x-y}$ as (x,y) approaches the diagonal. This will be important for the next topic.

46.4. **Pushforwards via the invariant differential.** The following remarkable result was stated in [Quillen1]:

Proposition 46.5. *Let $\mathcal{E}$ be a complex-oriented cohomology theory and let $F_{\mathcal{E}}$ be the associated formal group law over the ring $\mathcal{E}^*$. Then the $\mathcal{E}$-genus of $\mathbb{C}P^n$ is the coefficient of $x^n dx$ in the invariant differential $\omega_{\mathcal{E}} = \omega_{(F_{\mathcal{E}})}$.*

For example, we saw above that for the multiplicative formal group law all of the coefficients of the invariant differential are equal to 1. This is matches with our previous calculation that the K-genus (or Todd genus) of $\mathbb{C}P^n$ is equal to 1, for all $n \geq 0$.

Quillen obtained Proposition 46.5 from a more general statement about the pushforwards along the structure maps of projective bundles; we saw this idea already when we computed the Todd genus of $\mathbb{C}P^n$ by topological means back in Example 41.16. If $M \to X$ is a complex vector bundle and $\pi\colon \mathbb{P}M \to X$ the associated projective bundle, Quillen found a formula for computing $\pi_!(e^i)$ where e is the Euler class of the canonical quotient line bundle on $\mathbb{P}M$. It is not hard to

see that such a formula must exist: by the splitting principle one reduces to the case of a sum of line bundles, and then by naturality one reduces to the case of $M = \oplus_i \pi_i^*(L)$ over $(\mathbb{C}P^\infty)^{\times k}$. For this case $\pi_!(e^i)$ will be a given by a power series in the elementary symmetric functions, and so one gets a universal formula in terms of the Chern classes of the bundle M.

Proposition 46.6 (Quillen's residue formula)**.** *Let $\mathcal{E}$ be a complex-oriented cohomology theory, and let $F(x,y)$ be the formal group law. Let $I(x)$ be the power series for the inverse. Let $M \to X$ be a complex bundle of rank r and let $\pi\colon \mathbb{P}M \to X$ be the projective bundle of M. Let $\mathcal{L}$ be the tautological line bundle over $\mathbb{P}M$ and let $b = e^{\mathcal{E}}(\mathcal{L}^*)$ be the Euler class of the dual of $\mathcal{L}$. Then for $0 \leq v \leq r-1$*

$$\pi_!(b^v) = \operatorname{res}_{z=0}^{\text{formal}}\left[\frac{z^v \omega_{\mathcal{E}}(z)}{\prod_{j=1}^r F(z, I(x_j))}\right] \tag{46.7}$$

where $x_1, \ldots, x_r$ are the formal Chern roots of M^*.

The denominator in the above formula is perhaps better written as $\prod_j (z -_F x_j)$. In that form one readily sees the "formal poles" at the x_j. The "formal residue" is in some sense just the coefficient of z^{-1}, but interpreting this correctly requires some care that we will describe in a moment.

Remark 46.8. Note that in the above result b is the Euler class of $\mathcal{L}^*$, not of $\mathcal{L}$, and that the characteristic classes that appear in the formula will be $c_i(M^*)$, not $c_i(M)$. This is important. The Chern roots of M are $y_i = I(x_i)$, and so to get a formula involving $c_i(M)$ one could just replace the above denominator with $\prod_j F(z, y_i)$. To get formulas for $\pi_!$ applied to powers of $e(\mathcal{L})$, see Exercise ????.

Now we explain in more detail what is meant by the "formal residue". For convenience let us write $\mathcal{E}^*\{z\}$ for the ring $\mathcal{E}^*[[z]][z^{-1}]$, which can be thought of as the ring of formal meromorphic functions with coefficients in $\mathcal{E}^*$. The numerator in (46.7) belongs to $\mathcal{E}^*[[z]]$. The denominator can be written as a power series $A(z, c_1, \ldots, c_r) \in \mathcal{E}^*[c_1, c_2, \ldots, c_r][[z]]$, but the constant term need not be 1—so the series cannot be inverted. However, let us write $G(z, x_j) = \frac{F(z,x_j)}{z} = 1 + \frac{x_j}{z} + \cdots \in \mathcal{E}^*\{z\}[[x_j]]$. We can invert $G(z, x_j)$ in this ring and thereby write

$$\frac{z^v \omega_{\mathcal{E}}(z)}{\prod_{j=1}^r F(z, Ix_j)} = z^{v-r} \cdot \omega_{\mathcal{E}}(z) \cdot \prod_{j=1}^r G(z, Ix_j)^{-1}$$

where the expression on the right belongs to $\mathcal{E}^*\{z\}[[x_1, \ldots, x_r]]^{\Sigma_r}$. So we can express this as a formal power series in the elementary symmetric functions, with coefficients in $\mathcal{E}^*\{z\}$. We interpret $\operatorname{res}_{z=0}^{formal}$ as taking the usual residue (i.e., z^{-1} term) of each coefficient separately, thereby yielding a power series in $\mathcal{E}^*[[\sigma_1, \ldots, \sigma_r]]$. Replacing each σ_j with $c_j(M^*)$, as usual, gives the desired answer in $\mathcal{E}^*(X)$ with one caveat. If X is compact then all of the $c_j(M^*)$ are nilpotent (Proposition 38.25) and so the power series makes sense as an element in $\mathcal{E}^*(X)$. More generally, it also makes sense when $\mathcal{E}^*(X)$ is complete with respect to $\widetilde{\mathcal{E}^*}(X)$, e.g. when $X = \operatorname{Gr}_r(\mathbb{C}^\infty)$ or $X = (\mathbb{C}P^\infty)^{\times r}$.

As we will eventually see, the above "formal residue" is actually more like a residue at ∞, although it's not quite that either.

The formula for the $\mathcal{E}$-genus of $\mathbb{C}P^n$ follows immediately from Quillen's residue formula:

Proof of Proposition 46.5. Take $M \to X$ to be the bundle $\mathbb{C}^{n+1} \to pt$, so that the associated projective bundle is $\pi\colon \mathbb{C}P^n \to pt$. The Chern roots of M^* can be taken to be all zero, and so Quillen's formula gives

$$\pi_!(1) = \operatorname{res}_{z=0}\left[\tfrac{\omega_{\mathcal{E}}(z)}{z^{n+1}}\right] = \text{coefficient of } z^n\,dz \text{ in } \omega_{\mathcal{E}}(z).$$

□

Let us do one more sample computation:

Example 46.9. Let $M \to X$ be a complex vector bundle of rank $n+1$, and let $\pi\colon \mathbb{P}M \to X$ and b be as in Proposition 46.6. Then for $\pi_!\colon K^0(\mathbb{P}M) \to K^0(X)$ we claim that $\pi_!(b^v) = 1$ for $0 \le v \le n$. Quillen's formula gives

$$\begin{aligned}
\pi_!(b^v) &= \operatorname{res}_{z=0}\left[\frac{\frac{z^v\,dz}{1-z}}{z^{n+1}\prod_{j=1}^{n+1}(1-x_j(1-\frac{1}{z}))}\right] \\
&= \operatorname{res}_{z=0}\left[\frac{dz}{z^{n+1-v}(1-z)}\cdot\prod_j(1+x_j(\tfrac{z-1}{z})+x_j^2(\tfrac{z-1}{z})^2+\cdots)\right] \\
&= \operatorname{res}_{z=0}\left[\tfrac{-dz}{z^{n+1-v}(z-1)}\cdot(1+\text{multiples of } (\tfrac{z-1}{z})^k \text{ for } k>0)\right].
\end{aligned}$$

One could work out the coefficients in the power series in terms of the Chern classes of M^*, but the exact expressions do not matter because for $k \ge 1$ we have

$$\operatorname{res}_{z=0}\frac{dz}{z^{n+1-v}(z-1)}\cdot\Big(\frac{z-1}{z}\Big)^k = [z^{n+k-v}](z-1)^{k-1} = 0,$$

the last equality because $n+k-v > k-1$. So the only contribution to the overall residue is from the $k=0$ term, and that residue is exactly 1 (using $\frac{1}{1-z} = 1+z+z^2+\cdots$).

46.10. **Fun with residues.** Before tackling the proof of Quillen's residue formula it it helpful to take a step back and establish some basic facts about residues. As a quick review, suppose that $a_1,\dots,a_r$ are distinct nonzero complex numbers and $p(z)$ is a holomorphic function. Then the differential

$$\alpha = \frac{p(z)\,dz}{(z-a_1)\cdots(z-a_r)}$$

on $\mathbb{C}P^1$ has poles at $a_1,\dots,a_r$ and possibly at ∞. The residues are

$$\operatorname{res}_{z=a_j}\alpha = \frac{p(a_j)}{\prod_{s\ne j}(a_j-a_s)}.$$

By the residue theorem we get

$$0 = \operatorname{res}_{z=\infty}\alpha + \sum_{j=1}^{r}\operatorname{res}_{z=a_j}\alpha \tag{46.11}$$

so the above formulas determine the residue at ∞. But we can determine this another way, by the change of variables $z = \frac{1}{w}$. Then $dz = -\frac{1}{w^2}dw$ and our form becomes

$$\alpha = \frac{p(\frac{1}{w})\cdot -w^{-2}\cdot w^r\,dw}{(1-wa_1)\cdots(1-wa_r)} = -w^{r-2}p(\tfrac{1}{w})\cdot\prod_{j=1}^{r}(1+wa_j+w^2a_j^2+\cdots).$$

The **complete homogeneous symmetric polynomials** $h_k(x_1, \dots, x_r)$ are defined via the generating function

$$\sum_{k \geq 0} h_k(\underline{x}) t^k = \prod_{i=1}^{r} \tfrac{1}{1-x_i t}.$$

Then $h_k(\underline{x})$ is simply the sum of all monomials of degree k. By convention we set $h_k = 0$ for $k < 0$.

So $\prod_{j=1}^{r}(1 - wa_j)^{-1} = \sum_s h_s(\underline{a}) w^s$ and if we write $p(z) = \sum_m p_m z^m$ then we find that

$$\begin{aligned} \operatorname{res}_{z=\infty} \alpha &= \operatorname{res}_{w=0} -w^{r-2} \sum_{m \geq 0} p_m w^{-m} \cdot \sum_{s \geq 0} h_s(\underline{a}) w^s \\ &= -[w^{1-r}] \sum_{m,s} p_m h_s(\underline{a}) w^{s-m} \\ &= -\sum_{m \geq 0} p_m h_{m+1-r}(\underline{a}). \end{aligned}$$

But also note that

$$\begin{aligned} \operatorname{res}_{z=0}^{\text{formal}} \left[\frac{p(z)\, dz}{\prod_j (z - a_j)} \right] &= [z^{-1}] z^{-r} p(z) \cdot \prod_j (1 - \tfrac{a_j}{z})^{-1} \\ &= [z^{r-1}] \sum_m p_m z^m \cdot \sum_s h_s(\underline{a}) z^{-s} \\ &= [z^{r-1}] \sum_{m,s} p_m h_s(\underline{a}) z^{m-s} \\ &= \sum_m p_m h_{m-r+1}(\underline{a}). \end{aligned}$$

So we see that our "formal residue" agrees—except for a negative sign—with the residue at ∞, and therefore the residue theorem equation (46.11) becomes the identity

$$\operatorname{res}_{z=0}^{\text{formal}} \left[\frac{p(z)\, dz}{\prod_s (z - a_s)} \right] = \sum_{j=1}^{r} \frac{p(a_j)}{\prod_{s \neq j}(a_j - a_s)}. \tag{46.12}$$

One interesting feature of this equation is that the left-hand side is *a priori* a polynomial expression in the a's and the coefficients of p, whereas the summands on the right-hand-side involve denominators; the magic is that these denominators disappear upon taking the sum.

Now here is the punchline. Equation (46.12) can be proven purely algebraically, i.e. without using any complex analysis. To get the generality we will need it will be convenient to clear denominators, so to this end define $\Delta(u_1, \dots, u_m) = \prod_{i<j}(u_j - u_i)$. For a fixed j observe that $\Delta(u_1, \dots, u_m) = \prod_{s \neq j}(u_j - u_s) \cdot (-1)^{r-j} \cdot \Delta(u_1, \dots, \widehat{u}_j, \dots, u_m)$.

Proposition 46.13 (Formal residue theorem)**.** *Let R be a ring containing elements $a_1, \dots, a_r$. If $\prod_{i<j}(a_j - a_i)$ is invertible then (46.12) holds for any polynomial $p(z)$.*

More generally, if $p(z) \in R[[z]]$ then

$$\Delta(a_1,\dots,a_r)\cdot \mathrm{res}_{z=0}^{\mathrm{formal}}\left[\frac{p(z)\,dz}{\prod_j(z-a_j)}\right] = \sum_{j=1}^{r} p(a_j)\cdot(-1)^{r-j}\Delta(a_1,\dots,\widehat{a}_j,\dots,a_r)$$

holds provided that R is complete with respect to the ideal $(a_1,\dots,a_r)$ (and of course not assuming invertibility of the $a_j - a_k$).

Proof. If we perform the partial fraction decomposition

$$\frac{1}{\prod_i(1-x_it)} = \frac{A_1}{1-x_1t} + \cdot + \frac{A_n}{1-x_rt} \tag{46.14}$$

then clearing denominators and substituting $t = x_j^{-1}$ gives that

$$A_j = \prod_{s\neq j}\frac{x_j}{x_j-x_s} = \frac{x_j^{r-1}}{\prod_{s\neq j}(x_j-x_s)}.$$

This gives an identity in the fraction field of $\mathbb{Z}[x_1,\dots,x_r,t]$, and therefore also in the localization of $\mathbb{Z}[\underline{x},t]$ at the elements $1-x_it$ and x_i-x_j (for all $i\neq j$). So in fact it may also be interpreted as an identity in $\mathbb{Z}[\underline{x},\Delta^{-1}][[t]]$ where $\Delta = \Delta(x_1,\dots,x_r)$.

Equating coefficients on both sides of (46.14) leads to the identities

$$h_k(\underline{x}) = \sum_{j=1}^{r}\frac{x_j^{r-1+k}}{\prod_{s\neq j}(x_j-x_s)} \tag{46.15}$$

in $\mathbb{Z}[\underline{x},\Delta^{-1}]$. These are standard formulas for the complete symmetric functions. (For a connection to Vandermonde determinants, see Exercise 46.21). Clearing denominators gives

$$\Delta(x_1,\dots,x_r)\cdot h_k(\underline{x}) = \sum_{j=1}^{r} x_j^{r-1+k}\cdot(-1)^{r-s}\Delta(x_1,\dots,\widehat{x}_j,\dots,x_r).$$

This is an identity in $\mathbb{Z}[\underline{x}]$.

Write $p(z) = \sum_{m\geq 0} p_m z^m$ with $p_m \in R$. We saw above that the formal residue on the left side of (46.12) is equal to $\sum_{m\geq 0} p_m h_{m+r-1}(\underline{a})$. If p is a polynomial then this is a finite sum; if p is a power series then since $h_{m-r+1}(\underline{a}) \in (a_1,\dots,a_r)^{m+r-1}$ the expression represents a unique element of R by the completeness hypothesis.

It remains to analyze the right-hand-side of (46.12). For convenience set $D_j = \prod_{k\neq j}(a_j-a_k)$. Assuming these are all invertible then we have

$$\begin{aligned}\sum_{j=1}^{r}\tfrac{p(a_j)}{D_j} = \sum_{j=1}^{r}\tfrac{1}{D_j}\sum_{m\geq 0} p_m a_j^m &= \sum_{m\geq 0} p_m \sum_{j=1}^{r}\frac{a_j^m}{D_j}\\ &= \sum_{m\geq 0} p_m h_{m-r+1}(\underline{a})\end{aligned}$$

where the last equality is by (46.15).

If we are instead in the power series case then we set $\Delta = \Delta(a_1, \dots, a_r)$ and write

$$\begin{aligned}
\Delta \cdot \sum_{m\geq 0} p_m h_{m-r+1}(\underline{a}) &= \sum_{m\geq 0} p_m \cdot \Delta \cdot h_{m-r+1}(\underline{a}) \\
&= \sum_{m\geq 0} p_m \sum_{j=1}^{r} a_j^m (-1)^{r-j} \Delta(a_1, \dots, \widehat{a}_j, \dots, a_r) \\
&= \sum_{j=1}^{r} \left[\sum_{m\geq 0} p_m a_j^m \right] (-1)^{r-j} \Delta(a_1, \dots, \widehat{a}_j, \dots, a_r) \\
&= \sum_{j=1}^{r} p(a_j)(-1)^{r-j} \Delta(a_1, \dots, \widehat{a}_j, \dots, a_r).
\end{aligned}$$

Every line makes sense because of the assumption that R is complete with respect to $(a_1, \dots, a_r)$. □

46.16. **The proof of Quillen's residue formula.**

Proof of Proposition 46.6. As discussed prior to the statement of the proposition, one immediately reduces to the case where $X = (\mathbb{C}P^\infty)^r$ and $M = \oplus_j \pi_j^*(L)$. We set $L_j = \pi_j^* L$ and $x_j = e^{\mathcal{E}}(L_j^*)$ in $\mathcal{E}^*(X)$.

The proof from here uses the general method that we saw already back in Example 41.16. We have canonical embeddings $s_1, \dots, s_r \colon X \to \mathbb{P}M$ and $s_j^*(b) = x_j$. The normal bundle to s_j is $N_j = \operatorname{Hom}(L_j, M/L_j) = \oplus_{k\neq j} \operatorname{Hom}(L_j, L_k) = \oplus_{k\neq j} L_j^* \otimes L_k$. Set $e_j = e^{\mathcal{E}}(N_j) = \prod_{k\neq j} e^{\mathcal{E}}(L_j^* \otimes L_k) = \prod_{k\neq j} F(x_j, Ix_k)$.

Recall that $x -_F y = (x-y)u(x,y)$ where $u(x,y)$ is a unit in $\mathcal{E}^*[[x,y]]$ such that $u(x,x) = \frac{1}{F_2(x,0)}$. Then $F(x_j, Ix_k) = x_j -_F x_k = (x_j - x_k)u(x_j, x_k)$. Let $\Delta = \prod_{j<k}(x_k - x_j)$. It will be convenient to invert Δ, which the above formula shows is equivalent to inverting all of the e_j. Since $\mathcal{E}^*(X) = \mathcal{E}^*[[x_1, \dots, x_r]]$, there is no loss of information in doing this localization.

We will first prove the desired formula for $\pi_!(b^0)$, and then indicate what needs to change for higher powers.

Just as in Example 41.16 one can prove that $\pi_!(1) = \sum_{j=1}^r \frac{1}{e_j}$. Recall that the point is that $\underline{s}^* \colon \mathcal{E}^*(\mathbb{P}M) \to \bigoplus_{j=1}^r \mathcal{E}^*(X)$ is injective, so it is enough to verify that the identity holds after applying s_j^* for every j. But $(s_i)^*(s_j)_! = 0$ for $i \neq j$ by push-pull, and $(s_j)^*(s_j)_!(1) = e(N_j) = e_j$; the verification is then straightforward.

Let us rewrite the above formula as

$$\pi_!(1) = \sum_{j=1}^{r} \frac{1}{e_j} = \sum_{j=1}^{r} \frac{1}{\prod_{s\neq j}(x_j -_F x_s)}.$$

This is where the formula starts to look like a residue and would get one started thinking in that direction. The right-hand-side looks like the right-hand-side of (46.12) for $p(z) = 1$ except that our formula has the formal differences $x_j -_F x_s$ in place of the actual differences $x_j - x_s$. This is where the invariant differential enters the story, as it arises in the connection between these two expressions.

Observe that

$$\frac{\omega_F(z)}{\prod_{s=1}^r (z -_F x_s)} = \frac{F_2(z,0)^{-1}\, dz}{\prod_s (z - x_s)u(z, x_s)} = \frac{[F_2(z,0)^{-1} \cdot \prod_j u(z, x_s)^{-1}]\, dz}{\prod_j (z - x_s)}.$$

Let $p(z)$ be the power series appearing in the numerator of the expression on the right. Applying Proposition 46.13 therefore gives

$$\begin{aligned}\operatorname{res}_{z=0}^{\text{formal}}\left[\frac{\omega_F(z)}{\prod_{s=1}^r(z-_F x_s)}\right] &= \sum_{j=1}^r \frac{p(x_j)}{\prod_{s\neq j}(x_j-x_s)}\\ &= \sum_{j=1}^r \frac{F_2(x_j,0)^{-1}\cdot\prod_s u(x_s,x_j)^{-1}}{\prod_{s\neq j}(x_j-x_s)}\\ &= \sum_{j=1}^r \frac{\prod_{s\neq j} u(x_s,x_j)^{-1}}{\prod_{s\neq j}(x_j-x_s)}\\ &= \sum_{j=1}^r \frac{1}{\prod_{s\neq j}(x_j-_F x_s)} = \sum_{j=1}^r \frac{1}{e_j} = \pi_!(1).\end{aligned}$$

This is Quillen's residue formula. The first line requires some extra explanation: we are applying Proposition 46.13 in the form without denominators for the ring $R = \mathcal{E}^*(B) = \mathcal{E}^*[[x_1,\dots,x_r]]$, and then inverting $\Delta = \prod_{i<j}(x_j - x_i)$ to get an identity in the localized ring $R[\Delta^{-1}]$. The chain of equalites above are happening in this localized ring. But since both sides of the final chain ($\pi_!(1)$ and the formal residue) both live in R, which is a domain, having the identity in $R[\Delta^{-1}]$ implies the identity in R.

The last thing to explain is how to adapt the above analysis for the calculation of $\pi_!(b^v)$. The starting point is the identity

$$b^v = \sum_j \frac{1}{e_j}(s_j)_!(x_j^v). \tag{46.17}$$

Once again, to prove this it is enough to verify that the identity holds after applying s_i^* for every i. But $(s_i)^*(s_j)_! = 0$ for $i \neq j$, by push-pull. So applying s_i^* gives x_i^v on the left hand side, and on the right it gives

$$\begin{aligned}\tfrac{1}{e_i}(s_i)^*(s_i)_!(x_i^v) &= \tfrac{1}{e_i}(s_i)^*(s_i)_!((s_i)^*(b^v)\cdot 1)\\ &= \tfrac{1}{e_i}(s_i)^*(b^v\cdot(s_i)_!(1))\\ &= \tfrac{1}{e_i}\cdot x_i^v\cdot(s_i)^*(s_i)_!(1)\\ &= \tfrac{1}{e_i}\cdot x_i^v\cdot e_i\\ &= x_i^v,\end{aligned}$$

which is what we needed.

Now apply $\pi_!$ to (46.17) to get

$$\pi_!(b^v) = \sum_j \tfrac{1}{e_j}\pi_!(s_j)_!(x_j^v) = \sum_j \tfrac{x_j^v}{e_j}$$

using $\pi\circ s_j = \mathrm{id}$. From here on, the algebraic analysis is easily adapted from what we did for the $v = 0$ case. □

46.18. **Complex cobordism and the universal formal group law.** The complex cobordism spectrum MU is made up of Thom spaces for universal bundles over complex Grassmannians. See the exercises at the end of the section for an introduction. The coefficient ring MU_* is the bordism ring of compact almost complex

manifolds. We will mostly use it in the form MU^* instead, so a compact almost complex manifold X of dimension d yields a bordism class $[X] \in MU^{-2d}(pt)$.

Quillen provided a geometric description of MU^* that showed that it is complex-oriented, and moreover that in fact it is the *universal* complex-oriented cohomology theory—or equivalently, the universal cohomology with pushforwards satisfying the by-now familiar properties. See [Quillen3]. The class $[X] \in MU^{-2d}(pt)$ is precisely $(\pi_X)_!(1)$ where $\pi_X \colon X \to pt$. Said differently, the MU-genus of X is simply $[X]$.

Early in the history of MU Milnor completely computed the coefficient ring MU^* using homotopical methods [Milnor4]. The facts that we will need are:

(1) $MU^* \cong \mathbb{Z}[m_1, m_2, \ldots]$ where m_i has degree $-2i$.
(2) As an algebra MU^* is generated by the classes of smooth hypersurfaces in products of projective spaces $\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_d}$ (one can be even more specific here, but this is all we will need).

Theorem 46.19 (Quillen)**.** *The formal group law on MU^* is isomorphic to the universal formal group law on the Lazard ring.*

Proof. The proof is from [Quillen1]. Let F_{MU} be the formal group law associated to MU, let L be the Lazard ring, and let F_{univ} be the universal formal group law on L. Then F_{MU} is classified by a map $f \colon L \to MU^*$. Observe that L and MU are graded polynomial algebras with generators in the same degrees, so in each degree they are free abelian groups of the same rank. Hence, it suffices to prove that f is surjective.

The map f sends ω_{univ} to ω_{MU}. By Proposition 46.5 the coefficient of x^n in the latter is equal to $[\mathbb{C}P^n]$, so we know that $[\mathbb{C}P^n]$ is in the image of f for all $n \geq 0$.

Now let M be a smooth hypersurface in $\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}$ of degree $(d_1, \ldots, d_k)$. Then M is the vanishing locus for a generic section s of the line bundle $\mathcal{L}^*$, where $\mathcal{L} = L_1^{\otimes d_1} \otimes \cdots \otimes L_k^{d_k}$ and $L_j = \pi_j^*(L)$ (as always, L is the tautological line bundle on any $\mathbb{C}P^{n_s}$). By Proposition 41.21 this implies that $c_1^{MU}(\mathcal{L}^*) = i_!(1)$ where $i \colon M \hookrightarrow \prod_s \mathbb{C}P^{n_s}$, and hence $[M] = \pi_!(c_1^{MU}(\mathcal{L}^*))$ where $\pi \colon \prod_s \mathbb{C}P^{n_s} \to pt$.

The formal group law F_{MU} gives an algebraic expression for $c_1^{MU}(\mathcal{L}^*)$ as a polynomial expression in the classes $z_i = c_1(L_i^*)$ with coefficients in $\mathrm{im}(f)$. So $[M] = \pi_!(c_1^{MU}(\mathcal{L}^*))$ is an $\mathrm{im}(f)$-linear combination of expressions $\pi_!(z_1^{e_1} \cdots z_k^{e_k})$.

For $j \colon \mathbb{C}P^{n-e} \hookrightarrow \mathbb{C}P^n$ we have $j_!(1) = [\mathbb{C}P^{n-e}] = [\mathbb{C}P^{n-1}]^e = c_1(L^*)^e = z^e$. Using that pushforwards are compatible with products (Proposition 41.2) we find that $z_1^{e_1} \cdots z_k^{e_k} = J_!(1)$ where $J \colon \mathbb{C}P^{n_1-e_1} \times \cdots \times \mathbb{C}P^{n_k-e_k} \hookrightarrow \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}$. So $\pi_!(z_1^{e_1} \cdots z_k^{e_k})$ is precisely $[\mathbb{C}P^{n_1-e_1} \times \cdots \times \mathbb{C}P^{n_k-e_k}]$, using that $\pi_! \circ J_! = \pi_!$ (where the two π's have different domains). In particular, $\pi_!(z_1^{e_1} \cdots z_k^{e_k})$ is in $\mathrm{im}(f)$ because all of the $\mathbb{C}P^i$'s are.

We have therefore shown that $[M]$ is in the image of f, for every hypersurface M in a product of projective spaces. Since those generate MU^*, we get that f is surjective and hence an isomorphism. □

Remark 46.20. In [Quillen3] a separate argument was given that used formal group laws—and other algebro-topological machinery—to obtain Milnor's calculation of MU_* in a new way. This proof shows that F_{MU} is the universal formal group law without having to take as input facts (1) and (2) from above. Another version of this argument is given in [BH]. Finally, Adams gives a somewhat different proof (more homotopical and less geometric) in [Adams5, Chapter II.8].

∘ Exercises ∘

Exercise 46.21. This exercise derives a connection between the complete symmetric polynomials $h_k(x_1, \ldots, x_r)$ and generalized Vandermonde determinants. Set

$$V_m(\underline{x}) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_r \\ \vdots & & & \vdots \\ x_1^{r-2} & x_2^{r-2} & \cdots & x_r^{r-2} \\ x_1^m & x_2^m & \cdots & x_r^m \end{bmatrix}$$

and $\Delta = V_{r-1}(\underline{x}) = \prod_{i<j}(x_j - x_i)$. Also, let $\Delta(i) = V_{r-2}(x_1, \ldots, \widehat{x}_i, \ldots, x_r)$. Finally, set $D_i = \prod_{j \neq i}(x_i - x_j)$.

Prove that $D_i = (-1)^{r-i}\Delta(i)$ and then use a cofactor expansion of a determinant along with (46.15) to prove that $h_k(\underline{x})\Delta = V_{r+k-1}(\underline{x})$. This is more typically written in the form

$$h_k(\underline{x}) = \frac{V_{r+k-1}(\underline{x})}{\Delta}.$$

The following exercises introduce the reader to the cohomology theories MO and MU.

Exercise 46.22. Let $MO(n) = \mathrm{Th}(\gamma_n \to BO(n))$, and note that $MO(0) = S^0$.

(a) Confirm that there is a pullback diagram

$$\begin{array}{ccc} \gamma_n \oplus \underline{1} & \longrightarrow & \gamma_{n+1} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_n(\mathbb{R}^\infty) & \longrightarrow & \mathrm{Gr}_{n+1}(\mathbb{R} \oplus \mathbb{R}^\infty) = \mathrm{Gr}_{n+1}(\mathbb{R}^\infty) \end{array}$$

leading to a canonical (up to homotopy) map $\Sigma MO(n) \to MO(n+1)$. Define MO to be the spectrum made up of the sequence of spaces

$$MO(0),\ MO(1),\ MO(2),\ \cdots$$

and having the above as structure maps. This is called the **real bordism spectrum**.

(b) Choose once and for all a basepoint in $BO(0)$ and take the corresponding basepoint in $BO(n)$ for all n. The fibers of the γ bundles over these basepoints give canonical inclusions $S^n \hookrightarrow MO(n)$. Check that these assemble to give a map of spectra $S \to MO$.

(c) We next investigate homotopy classes of maps $MO \to H\mathbb{Z}/2$. Recall the Milnor sequence

$$0 \longrightarrow \lim{}^1 H^{n-1}(MO(n); \mathbb{Z}/2) \longrightarrow H^0(MO; \mathbb{Z}/2) \longrightarrow \lim_n H^n(MO(n); \mathbb{Z}/2) \longrightarrow 0$$

(Proposition 9.20). Use the connectivity of $MO(n)$ to show that the group on the left vanishes. Then use the canonical Thom classes $\mathfrak{U}_n \in H^n(MO(n); \mathbb{Z}/2)$ to prove that $H^0(MO; \mathbb{Z}/2) \cong \mathbb{Z}/2$. In other words, up to homotopy there is a unique nonzero map of spectra $MO \to H\mathbb{Z}/2$.

Exercise 46.23. Let $MU(n) = \mathrm{Th}(\gamma_n \to BU(n))$.

(a) Confirm that there is a pullback diagram

$$\begin{array}{ccc} \gamma_n \oplus \underline{1} & \longrightarrow & \gamma_{n+1} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_n(\mathbb{C}^\infty) & \longrightarrow & \mathrm{Gr}_{n+1}(\mathbb{C}\oplus\mathbb{C}^\infty) = \mathrm{Gr}_{n+1}(\mathbb{C}^\infty) \end{array}$$

leading to a canonical (up to homotopy) map $\Sigma^2 MU(n) \to MU(n+1)$.

(b) Verify that there is a spectrum made up of the sequence of spaces

$$MU(0),\ \Omega MU(1),\ MU(1),\ \Omega MU(2),\ MU(2),\ \Omega MU(3),\ MU(3),\ \dots$$

where the structure maps are induced by the maps in (a) and the standard counit maps $\Sigma\Omega X \to X$.

[Note that it feels more natural to just use the sequence $MU(0), MU(1), MU(2), \dots$ but that this would give a spectrum-like object where the structure maps involve a two-fold suspension. One can develop such Σ^2-spectra and prove that they give an equivalent theory to regular spectra. While this is somehow the more pleasing approach, it is more common just to use the regular definition of spectra and throw in the "odd" spaces $\Omega MU(n)$ that are formally forced into the picture. But it is good to recognize that one can basically ignore those terms.]

(c) We have $MU = \mathrm{hocolim}_n\, \Sigma^{-2n} MU(n)$, and so we get the Milnor sequence

$$0 \longrightarrow \lim{}^1 H^{2n-1+k}(MU(n)) \longrightarrow H^k(MU) \longrightarrow \lim_n H^{2n+k}(MU(n)) \longrightarrow 0.$$

Using this, verify that $H^0(MU) \cong \mathbb{Z}$ and that the Thom classes $\mathcal{U}_n \in H^{2n}(MU(n))$ assemble to give a map of spectra $MU \to H\mathbb{Z}$ (see Exercise 46.22(c) for the simpler case).

(d) There are canonical classes $\mathcal{U}_n^{MU} \in MU^{2n}(MU(n))$, corresponding to the identity maps $MU(n) \to MU_{2n} = MU(n)$. Verify that these make MU into a complex-oriented spectrum and that the map of cohomology theories $MU \to H\mathbb{Z}$ sends the MU-Thom class for a bundle to its H-Thom class. For the latter, if $E \to X$ is a complex rank n bundle refer to the diagram

$$\begin{array}{ccccccc} MU^{2n}(E, E-0) & \longrightarrow & MU^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0) & = & \widetilde{MU}^{2n}(S^{2n}) & \xleftarrow{\cong} & \widetilde{MU}^{0}(S^0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{2n}(E, E-0) & \longrightarrow & H^{2n}(\mathbb{C}^n, \mathbb{C}^n - 0) & = & \widetilde{H}^{2n}(S^{2n}) & \xleftarrow{\cong} & \widetilde{H}^{0}(S^0). \end{array}$$

(e) Recall that the Thom isomorphism gives $\tilde{H}^*(MU(n)) \cong H^{*-2n}(BU(n))$. Using this together with the Milnor sequence, prove that $H^k(MU) = 0$ whenever k is odd. [Hint: Verify the Mittag-Leffler condition for the vanishing of $\lim^1$.]

(f) Extend part (e) to prove that as graded abelian groups $H^*(MU) \cong H^*(BU) \cong \mathbb{Z}[c_1, c_2, \dots]$. In particular, $H^*(MU)$ is concentrated in even degrees and is free abelian there.

Exercise 46.24. Prove Mischenko's theorem that for the formal group law F_{MU} the associated logarithm is given by

$$\log_{MU}(x) = \sum_{n\geq 0} [\mathbb{C}P^n] \cdot \frac{x^{n+1}}{n+1}.$$

This result first appeared in the appendix of [Novikov].

Exercise 46.25. Let L be the Lazard ring and F_{univ} the universal formal group law. Formal group laws over $\mathbb{Q}$-algebras are uniquely determined by their logarithm series. Without using Lazard's theorem that gives the structure of L, prove that $L \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}[u_1, u_2, \ldots]$ where $\log_{F_{univ}} = x + \sum_{i \geq 2} u_i x^i$. Deduce that $MU^* \otimes \mathbb{Q}$ is the polynomial algebra over $\mathbb{Q}$ on the classes $[\mathbb{C}P^n]$ for $n \geq 1$.

47. K-theoretic Chern classes: the γ-operations

We saw in Section 38 that any complex-oriented cohomology theory admits Chern classes for complex vector bundles, satisfying the usual list of properties. So for K-theory this means that any bundle $E \to X$ has Chern classes $c_i^K(E) \in K^{2i}(X)$. Using Bott periodicity we can shift everything to K^0, as usual. Because $K^0(X)$ is itself made from vector bundles (assuming X is homotopically compact, say), this gives the K-theoretic Chern classes the feeling of *operations*: they take as input a bundle and give as their output a bundle-like object. And since the Chern classes will be determined by their behavior on the universal bundles, it quickly becomes apparent that they must be given by some universal *algebraic* formulas.

We already have a formula for the K-theoretic Euler class: if $E \to X$ has rank n then $c_n^K(E) = e^K(E) = \sum_i (-1)^i [\Lambda^i E^*]$. It is not hard to take this, together with the splitting principle, and determine formulas for $c_r^K(E)$. If E is a sum of line bundles $E = \bigoplus_i J_i$ then

$$c_r^K(E) = \sigma_r(e^K(J_1), \ldots, e^K(J_r)) = \sigma_r(1 - J_1^*, \ldots, 1 - J_n^*)$$

and by expanding out the symmetric function one can write this as a formula in the $\Lambda^i E^*$ classes. The reader should try this as an exercise.

Instead of proceeding in this way, most sources in the literature follow Grothendieck and define the so-called γ-*operations*. The defining formulas, though, are not always well-motivated—they can seem to almost come out of thin air, and the connections to the K-theoretic Chern classes are often left underground. Once one has the formulas for the γ-operations, though, it can be a fun algebraic exercise to derive their basic properties and "rediscover" the K-theoretic Chern classes hidden underneath.

In this section we are going to take a hodgepodge approach. We will start by developing the algebra of γ-operations, but we will do this in a slightly non-traditional way that takes advantage of the ultimate connections with Chern classes to provide more motivation for the eventual definitions.

47.1. **The γ-operations, from scratch.** Let $E \to X$ be a complex vector bundle of rank n. We will define certain natural classes $\tilde{\gamma}^k(E) \in \tilde{K}^0(X)$ with the following properties:

(1) $\tilde{\gamma}^k(E) = 0$ for $k > n$ and $\tilde{\gamma}^0(E) = 1$,
(2) $\tilde{\gamma}^k(E \oplus F) = \sum_{i+j=k} \tilde{\gamma}^i(E)\tilde{\gamma}^j(F)$,
(3) $\tilde{\gamma}^1(\underline{1}) = 0$,
(4) $\tilde{\gamma}^n(E^*) = (-1)^n e^K(E)$, where $e^K(E)$ is the K-theoretic Euler class defined in Section 37.15,
(5) $\tilde{\gamma}^1(E) = [E] - n$.

Note that these properties are (intentionally) overdetermined. Property (3) is a special case of (5), and (5) follows from (1)–(4) via the splitting principle.

Properties (1)–(3), and almost (4), are the familiar properties of Chern classes; in (4) we would not expect the dual or the sign to appear. But note that properties (1)–(3) are unaffected if we add in the duals and signs, so this suggests that the connection to the K-theoretic Chern classes is given by

$$c_i^K(E) = (-1)^i \tilde{\gamma}^i(E^*).$$

Below we will see some further justification of this.

There is a bit of a historical oddity here. One could easily change the (coming) definition of the $\tilde{\gamma}$-operations so that the dual and signs are built in, and then the $\tilde{\gamma}$-operations would be precisely the K-theoretic Chern classes. On some levels this would be more satisfying, but we have chosen to follow the historical conventions.

The following is another familiar property of Chern classes, and is a consequence of the ones above:

Proposition 47.2. *Suppose that X is a paracompact Hausdorff space and the rank n bundle $E \to X$ admits r independent sections. Then $\tilde{\gamma}^{n+1-i}(E) = 0 = c^K_{n+1-i}(E)$ for $1 \leq i \leq r$.*

Proof. The r independent sections give an embedding $\underline{r} \hookrightarrow E$. If Q denotes the quotient, then the exact sequence $0 \to \underline{r} \hookrightarrow E \to Q \to 0$ is split by Proposition 12.2, and so $E \cong \underline{r} \oplus Q$. Then

$$\tilde{\gamma}^{n+1-i}(E) = \sum_{a+b=n+1-i} \tilde{\gamma}^a(\underline{r})\tilde{\gamma}^b(Q) = \tilde{\gamma}^{n+1-i}(Q).$$

But since Q has rank $n-r$, the latter expression vanishes if $n+1-i > n-r$, which is when $i < r+1$.

Since E splits off a copy of $\underline{r}$ if and only if E^* does, we can apply the above reasoning to E^* and deduce the vanishing of the c^K-classes. □

We still need to construct the $\tilde{\gamma}$ classes. We are going to write down a direct but rather unsatisfying definition; then we will spend the rest of the section explaining where the definition comes from.

Definition 47.3. *For $E \to X$ a rank n vector bundle define*

$$\begin{aligned}\tilde{\gamma}^k(E) &= (-1)^k\tbinom{n}{k}[\Lambda^0 E] + (-1)^{k-1}\tbinom{n-1}{k-1}[\Lambda^1 E] + (-1)^{k-2}\tbinom{n-2}{k-2}[\Lambda^2 E] + \cdots \\ &= \sum_{i=0}^{k} (-1)^{k-i}\tbinom{n-i}{k-i}[\Lambda^i E].\end{aligned}$$

Properties (1), (3), (4), and (5) are immediate from this definition. Property (2) can be deduced from the isomorphism $\Lambda^n(E \oplus F) \cong \bigoplus_{a+b=n} \Lambda^a E \otimes \Lambda^b F$ by a brute force calculation, but it requires some combinatorial cleverness. We will instead take a different route which helps us better understand the source of the above formula.

Proposition 47.4. *If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles then*

$$\tilde{\gamma}^k(E) = \bigoplus_{i_1 < \ldots < i_k} (L_{i_1} - 1) \cdots (L_{i_k} - 1).$$

Proof. Let $x_1, \dots, x_n$ be formal variables, and consider the polynomial

$$P_k = \bigoplus_{i_1 < \cdots < i_k} (x_{i_1} - 1) \cdots (x_{i_k} - 1).$$

This is a symmetric function and therefore can be written as a polynomial in the elementary symmetric functions $\sigma_1, \dots, \sigma_n$. The constant term is clearly $\binom{n}{k}(-1)^k$ (put all $x_i = 0$). For $s \leq k$ a given monomial $x_{r_1} \cdots x_{r_s}$ appears in exactly $\binom{n-s}{k-s}$ summands (all k-subsets containing $r_1, \dots, r_s$) with coefficient $(-1)^{k-s}$, so this gives a summand of P_k of the form $(-1)^{k-s}\binom{n-s}{k-s}\sigma_s$. Thus,

$$P_k = \sum_{i=0}^{k} (-1)^{k-i}\tbinom{n-i}{k-i}\sigma_i.$$

When we change x_i to L_i, the σ_i function is precisely $\Lambda^i E$. So this completes the proof. □

Corollary 47.5. *If E and F are sums of line bundles then $\tilde{\gamma}^k(E \oplus F) = \sum_{i+j=k} \tilde{\gamma}^i(E)\tilde{\gamma}^j(F)$.*

Proof. Immediate from Proposition 47.4. □

Remark 47.6. Properties (1)–(4) force the definition of the $\tilde{\gamma}$ classes on sums of line bundles. By applying the argument in the above proof one is led to Definition 47.3 in that special case, and it is natural to then extrapolate to the general case.

At this point one can prove property (2) by using the splitting principle. Such proofs often feel a little unsatisfying, though. We will keep pushing for a better approach. To this end, let us wipe the slate at this point and assume we do not yet have a definition of the $\tilde{\gamma}$ classes. Recall from Example 37.19 the generating function

$$\lambda_t(E) = 1 + t[E] + t[\Lambda^2 E] + \cdots$$

and the fact that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$. We can introduce the analogous generating function

$$\tilde{\gamma}_t(E) = 1 + t\tilde{\gamma}^1(E) + t^2\tilde{\gamma}^2(E) + \cdots$$

and since we want property (2) to hold it is tempting to try to define $\tilde{\gamma}_t(E)$ in terms of $\lambda_t(E)$. For L a line bundle we know $\lambda_t(L) = 1 + t[L]$, whereas we want $\tilde{\gamma}_t(L) = 1 + t([L] - 1) = (1 - t) + t[L]$. This quickly leads us to write

$$\tilde{\gamma}_t(L) = (1 - t)\lambda_{\frac{t}{1-t}}(L)$$

and suggests that for a rank n bundle E we put

$$\tilde{\gamma}_t(E) = (1 - t)^n \lambda_{\frac{t}{1-t}}(E). \tag{47.7}$$

That is, we take this as the *definition* of the $\tilde{\gamma}^i(E)$ classes. With this definition properties (1)—(3) are immediate; (4) is immediate for line bundles and then is deduced in general by applying property (2) and using that $e^K(E \oplus F) = e^K(E)e^K(F)$.

Unravelling equation (47.7) leads to

$$\tilde{\gamma}_t(E) = (1 - t)^n + (1 - t)^{n-1}t[E] + (1 - t)^{n-2}t^2[\Lambda^2 E] + \cdots + t^n[\Lambda^n E].$$

The coefficient of t^k is therefore

$$\tilde{\gamma}^k(E) = (-1)^k\tbinom{n}{k} + (-1)^{k-1}\tbinom{n-1}{k-1}[E] + (-1)^{k-2}\tbinom{n-2}{k-2}[\Lambda^2 E] + \cdots$$

which recovers Definition 47.3.

Now we have finally confirmed that our definition of the $\tilde{\gamma}$ classes satisfies properties (1)–(5).

Remark 47.8. The more standard approach to this material is to first define classes $\gamma^i(E)$, and to get the $\tilde{\gamma}$ classes from those. We briefly recount this. For $x \in K^0(X)$ define

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x), \tag{47.9}$$

regarded as a power series in t with coefficients in $K^0(X)$. The coefficient of t^i is denoted $\gamma^i(x)$. Note that $\gamma_t(x \oplus y) = \gamma_t(x)\gamma_t(y)$ because this property is inherited from the λ_t-construction. So γ_t is a group homomorphism $\gamma_t\colon K^0(X) \to 1 + tK^0(X)[[t]]$, where the target is the multiplicative group of power series with coefficients in $K^0(X)$ and having leading coefficient 1. Note that $\gamma_t(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t} = 1 + t + t^2 + \cdots$. One then defines

$$\tilde{\gamma}_t(x) = \gamma_t(x - \underline{\operatorname{rank} x}) = \frac{\gamma_t(x)}{\gamma_t(\underline{\operatorname{rank} x})} = (1-t)^{\operatorname{rank} x} \cdot \gamma_t(x) = (1-t)^{\operatorname{rank} x} \cdot \lambda_{\frac{t}{1-t}}(x).$$

With this approach the formula (47.9) seems to come out of nowhere, whereas our development provided more motivation.

In the literature one almost never finds the $\tilde{\gamma}^i$ notation, so let us record the formula

$$\tilde{\gamma}^i(x) = \gamma^i(x - \operatorname{rank}(x)) \tag{47.10}$$

that translates from our notation into the classical one. In particular, if x has rank zero then $\gamma^i(x) = \tilde{\gamma}^i(x)$ for all i.

Exercise 47.11. If L is a line bundle calculate that

(i) $\gamma^i(L) = L$ if $i \geq 1$, and $\gamma^0(L) = 1$;

(ii) $\tilde{\gamma}^i(L) = \begin{cases} 1 & \text{if } i = 0, \\ L - 1 & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$

(iii) $\gamma^i(1 - L) = \tilde{\gamma}^i(1 - L) = (1 - L)^i$, for all $i \geq 0$.

(iv) $\gamma^i(L - 1) = \tilde{\gamma}^i(L - 1) = \tilde{\gamma}^i(L)$, for all $i \geq 0$.

Exercise 47.12. Note that $\gamma_t(x)\gamma_t(-x) = \gamma_t(0) = 1$. Use this to calculate that $\gamma^1(-x) = -\gamma^1(x)$ and $\gamma^2(-x) = \gamma^1(x)^2 - \gamma^2(x)$. In general prove that $\gamma^i(-x)$ can be expressed as a homogeneous polynomial of degree i in the classes $1, \gamma^1(x), \gamma^2(x), \ldots, \gamma^i(x)$, where each $\gamma^j(x)$ is regarded as having degree j. In fact prove that $\gamma^i(y - x)$ can be expressed as a homogeneous polynomial of degree i in the $\gamma^a(y)$ and $\gamma^b(x)$ classes. Observe that the analogous result holds for the $\tilde{\gamma}^i$ classes as well.

We have essentially already explained the connection to the K-theoretic Chern classes, but let us record it explicitly:

Proposition 47.13. *For any complex vector bundle $E \to X$ one has $\tilde{\gamma}^i(E) = (-1)^i c_i^K(E^*)$, for all $i \geq 0$.*

Proof. This follows formally from properties (1)–(4), using the splitting principle. □

47.14. **Geometry and the K-theoretic Chern classes.** At this point we have understood the definition of c_r^K from algebraic and topological perspectives. But one would hardly look at any of our definitions—e.g. Definition 47.3—and immediately see some interesting geometry there. Our next aim is to see if we can find that geometry lurking somewhere in the shadows.

If C is a chain complex of complex vector spaces, define the dth symmetric product by the usual formula

$$\mathrm{Sym}^d(C) = (C^{\otimes(d)})/\Sigma_d$$

but note that the Σ_d-action on the tensor product is via the symmetric monoidal structure on chain complexes and so involves the Koszul sign rule. If V is a vector space and $V[i]$ denotes the chain complex with V in dimension i and zeros elsewhere, then

$$\mathrm{Sym}^d(V[i]) \cong \begin{cases} (\mathrm{Sym}^d V)[di] & \text{if } i \text{ is even,} \\ (\Lambda^d V)[di] & \text{if } i \text{ is odd.} \end{cases}$$

If C and D are chain complexes then we have the usual canonical isomorphism

$$\mathrm{Sym}^d(C \oplus D) \cong \bigoplus_{i+j=d} \mathrm{Sym}^i(C) \otimes \mathrm{Sym}^j(D). \tag{47.15}$$

Exercise 47.16. Let C be the chain complex $0 \longrightarrow \mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C} \longrightarrow 0$, concentrated in degrees 0 and -1. It will be convenient to choose a basis f in degree 0, and then df is a basis in degree -1. Show that $\mathrm{Sym}^k C$ is the chain complex $0 \to \mathbb{C} \to \mathbb{C} \to 0$ where the two vector spaces have bases $f^{\otimes(k)}$ and $f^{\otimes(k-1)}\, df$, and where the differential sends the first to k times the second.

If $f\colon V \to W$ is a map of vector spaces write $C'f$ for the chain complex that has $(C'f)_0 = V$, $(C'f)_{-1} = W$, all other chain groups zero, and the differential $(C'f)_0 \to (C'f)_{-1}$ is equal to f (this is essentially the desuspension of the mapping cone of f). Our next goal will be to compute the vector spaces $H_*(\mathrm{Sym}^d(C'f))$. To this end, observe that we can choose direct sum decompositions $V = V_1 \oplus V_2$ and $W = W_2 \oplus W_3$ in such a way that $f(V_1) = 0$, $f(V_2) = W_2$, and $f|_{V_2}\colon V_2 \to W_2$ is an isomorphism. For convenience let $g = f|_{V_2}$, so that $C'f \cong V_1[0] \oplus C'g \oplus W_3[-1]$. We then get that

$$\mathrm{Sym}^d(C'f) \cong \bigoplus_{i+j+k=d} \mathrm{Sym}^i(V_1)[0] \otimes \mathrm{Sym}^j(C'g) \otimes (\Lambda^k W_3)[-k]. \tag{47.17}$$

The next key observation is that $\mathrm{Sym}^j(C'g)$ is acyclic for all $j > 0$. Since $C'g$ is a direct sum of complexes $\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C}$, (47.15) lets us reduce to this particular g. But here the symmetric product was analyzed in Exercise 47.16 above. (note that it is important that we are in characteristic zero, though, otherwise the complexes computed in that exercise can have nonvanishing homology).

Since the tensor product of an acyclic complex and anything is again acyclic, this means the only terms in (47.17) that contribute to homology are the ones with

$j = 0$. So we conclude that

$$\operatorname{Sym}^d(C'f) \simeq \bigoplus_{i=0}^{d} \Big(\operatorname{Sym}^i(V_1) \otimes \Lambda^{d-i}(W_3)\Big)[-(d-i)]. \tag{47.18}$$

This decomposition simplifies dramatically if either V_1 or W_3 is zero:

Proposition 47.19. *Let $f\colon V \to W$ be a nonzero map of vector spaces.*

(a) If $\ker f = 0$ then the homology of $\operatorname{Sym}^d(C'f)$ is concentrated in dimension $-d$ and equals $\Lambda^d(\operatorname{coker} f)$. In particular, $\operatorname{Sym}^d(C'f)$ is exact for $d > \dim \operatorname{coker}(f)$.

(b) If $\operatorname{coker} f = 0$ then the homology of $\operatorname{Sym}^d(C'f)$ is concentrated in dimension 0 and equals $\operatorname{Sym}^d(\ker f)$.

Proof. Immediate from (47.18). □

Remark 47.20. If G is a finite group then over a field k of characteristic zero every exact sequence in $k[G]$-Mod is split (choose any splitting as vector spaces and then average over the group action). It follows that the functor $k[G]$-Mod $\to k$-Mod given by $V \mapsto V/G$ is exact, and therefore $(-)/G$ preserves quasi-isomorphisms of chain complexes. So the functor $C_\bullet \mapsto \operatorname{Sym}^d(C_\bullet)$ also preserves quasi-isomorphisms, and this yields an easy proof of Proposition 47.19. Our development was intentionally more low-tech than this, but it is good to see both approaches.

For us the most important part of Proposition 47.19 is part (a). It shows that if a map is injective then some associated chain complexes will be exact. So that starts to feel like a context where K-theory might provide obstructions.

The above considerations all pass to vector bundles without any trouble. If $f\colon E_1 \to E_2$ is a map of bundles over a space X then one gets the complex of bundles $\operatorname{Sym}^d(Cf)$. The following result is immediate from the preceding one:

Corollary 47.21. *Let $f\colon E_1 \to E_2$ be an injective map of bundles. Then for any $d > \operatorname{rank}(E_2) - \operatorname{rank}(E_1)$ the complex $\operatorname{Sym}^d(C'f)$ is exact.*

This corollary allows us to write down a collection of K-theoretic obstructions to the existence of an embedding $E_1 \hookrightarrow E_2$. If such an embedding f exists then for appropriate d we get that $\operatorname{Sym}^d(C'f)$ is exact, which implies that the alternating sum of the terms is zero in $K^0(X)$. This alternating sum is the same as for $\operatorname{Sym}^d(C'z)$ where $z\colon E_1 \to E_2$ is the zero map, in which case we can use the isomorphism

$$\operatorname{Sym}^d(C'z) \cong \bigoplus_{i=0}^{d} \Big(\operatorname{Sym}^i(E_1) \otimes \Lambda^{d-i}(E_2)\Big)[-(d-i)].$$

So define the class

$$\sigma_d(E_1, E_2) = \operatorname{Sym}^d(E_1 \xrightarrow{0} E_2) = \sum_{i=0}^{d} (-1)^{d-i} \operatorname{Sym}^i(E_1) \otimes \Lambda^{d-i}(E_2) \in K^0(X)$$

where the chain complex $E_1 \xrightarrow{0} E_2$ is in degrees 0 and -1. We have proven the following:

Proposition 47.22. *If there is an embedding of bundles $E_1 \hookrightarrow E_2$ then $\sigma_d(E_1, E_2) = 0$ in $K^0(X)$ for every $d > \operatorname{rank}(E_2) - \operatorname{rank}(E_1)$.*

If E is a vector bundle of rank n then the dth Chern class is supposed to be an obstruction to having $n-d+1$ independent sections, so this suggests that we look at $\sigma_d(n-d+1,E)$. Indeed, this unravels to

$$\begin{aligned}\sigma_d(n-d+1,E) &= \sum_{i=0}^{d}(-1)^{d-i}\operatorname{Sym}^i(n-d+1)\otimes\Lambda^{d-i}(E)\\ &= \sum_{i=0}^{d}(-1)^{d-i}\tbinom{n+i-d}{i}\Lambda^{d-i}(E)\end{aligned}$$

where we have used Lemmma 33.12 to calculate the dimension of the symmetric product. Comparison to Definition 47.3 gives that

$$\sigma_d(n-d+1,E) = (-1)^d\tilde{\gamma}^d(E) = c_d^K(E^*).$$

Suppose that we had started with $\tilde{\gamma}^d(E) = (-1)^d\sigma_d(n-d+1,E)$ as the definition of the $\tilde{\gamma}$-classes. Then of the properties we listed at the beginning of the section all but (2) are immediate. So is Proposition 47.2, even without (2). But what about property (2)? It seems natural to expect this to fall out of the behavior of symmetric products on sums (47.15), but it takes just a little more work than that. We start with the following:

Lemma 47.23. *Write* $\operatorname{Sym}^d(r)$ *for the dimension of* $\operatorname{Sym}^d(V)$ *where* V *is a vector space of dimension* r*. Then*

$$\operatorname{Sym}^d(k+n-d-1) = \sum_{i+j=d}\operatorname{Sym}^i(k-i)\operatorname{Sym}^j(n-j)$$

for all values of d*,* k*, and* n *where the formula makes sense.*

Proof. Recall that $\operatorname{Sym}^d(r) = \binom{r+d-1}{d}$, by Lemma 33.12. Then the identity becomes

$$\tbinom{k+n-2}{d} = \sum_{i+j=d}\tbinom{k-1}{i}\tbinom{n-1}{j}$$

which is the standard Chu-Vandermonde identity for binomial coefficients, coming from the natural ways of breaking up a choice of d objects from a set of $k-1$ red objects and $n-1$ blue objects. □

Proposition 47.24. *Let* E *and* F *be bundles of ranks* e *and* f*. Then*

$$\sigma_d(e+f-d+1, E\oplus F) = \sum_{i+j=d}\sigma_i(e-i+1,E)\sigma_j(f-j+1,F).$$

Consequently, $\tilde{\gamma}^d(E\oplus F) = \sum_{i+j=d}\tilde{\gamma}^i(E)\tilde{\gamma}^j(F)$.

Proof. By reversing the order of the summation the formula for $\sigma_i(e-i+1,E)$ may be rewritten as

$$\sigma_i(e-i+1,E) = \sum_{a=0}^{i}(-1)^a\operatorname{Sym}^{i-a}(e-i+1)\Lambda^a E.$$

We will use this version throughout the calculation. Observe that

$$\begin{aligned}
&\sum_{i+j=d} \sigma_i(e-i+1,E)\sigma_f(f-j+1,F) \\
&= \sum_{i+j=d} \sum_{a=0}^{i} \sum_{b=0}^{j} (-1)^{a+b} \operatorname{Sym}^{i-a}(e-i+1) \operatorname{Sym}^{j-b}(f-j+1)(\Lambda^a E)(\Lambda^b F) \\
&= \sum_{N=0}^{d} \sum_{a+b=N} \sum_{s+t=d-N} (-1)^N \operatorname{Sym}^s(e-a+1-s) \operatorname{Sym}^t(f-b+1-t)(\Lambda^a E)(\Lambda^b F) \\
&= \sum_{N=0}^{d} \sum_{a+b=N} (-1)^N \operatorname{Sym}^{d-N}(e-a+1+f-b+1-(d-N)-1)(\Lambda^a E)(\Lambda^b F) \\
&= \sum_{N=0}^{d} \sum_{a+b=N} (-1)^N \operatorname{Sym}^{d-N}(e+f-d+1)(\Lambda^a E)(\Lambda^b F) \\
&= \sum_{N=0}^{d} (-1)^N \operatorname{Sym}^{d-N}(e+f-d+1)\Lambda^N(E\oplus F) \\
&= \sigma_d(e+f-d+1, E\oplus F).
\end{aligned}$$

In the second equality we are changing the order of the summation and also substituting $s = i - a$ and $t = j - b$. In the third equality we used Lemma 47.23. □

47.25. **The γ-filtration.** To any cohomology group $\mathcal{E}^*(-)$ we may attach the Atiyah-Hirzebruch filtration, defined by

$$F_n\mathcal{E}^p(X) = \{a \in \mathcal{E}^p(X) \,\big|\, f^*(a) = 0 \text{ for every map } f\colon A \to X \text{ where } A \text{ is a CW-complex with } \dim A < n\}.$$

We have

$$\mathcal{E}^p(X) = F_0\mathcal{E}^p(X) \supseteq F_1\mathcal{E}^p(X) \supseteq F_2\mathcal{E}^p(X) \supseteq \cdots$$

with $F_{n+1}\mathcal{E}^p(X) = 0$ if X is a CW-complex of dimension n. Note that when X is 0-connected one has $F_1\mathcal{E}^p(X) = \tilde{\mathcal{E}}^p(X)$.

Given a map $f\colon X \to Y$ the induced map $f^*\colon \mathcal{E}^p(Y) \to \mathcal{E}^p(X)$ clearly sends $F_n\mathcal{E}^p(Y)$ into $F_n\mathcal{E}^p(X)$. So the filtration is functorial. When f is a weak homotopy equivalence the induced map $F_n\mathcal{E}^p(Y) \to F_n\mathcal{E}^p(X)$ is readily checked to be an isomorphism.

When X is a CW-complex it follows from cellular approximation that

$$F_n\mathcal{E}^p(X) = \ker(\mathcal{E}^p(X) \to \mathcal{E}^p(X_{n-1})).$$

But note that our original definition of the filtration shows that it is independent of the choice of CW-structure on X, and indeed does not require a CW-structure on X at all.

Exercise 47.26. If $\mathcal{E}$ is a multiplicative cohomomology theory verify that the Atiyah-Hirzebruch filtration is multiplicative, in the sense that the pairings

$$\mathcal{E}^p(X) \otimes \mathcal{E}^q(Y) \to \mathcal{E}^{p+q}(X \times Y)$$

map the image of $F_n\mathcal{E}^p(X) \otimes F_k\mathcal{E}^q(Y)$ to $F_{n+k}\mathcal{E}^{p+q}(X \times Y)$, for all n and k. Deduce that the filtration on $\mathcal{E}^*(X)$ is also multiplicative under the internal product.

Write $\mathcal{E}_p$ for the representing space for $\mathcal{E}^p(-)$; that is, $\mathcal{E}^p(X) = [X, \mathcal{E}_p]$. Then we can also define the Atiyah-Hirzebruch filtration using the Whitehead tower for the space $\mathcal{E}_p$. Write $\mathcal{E}_p\langle n, \infty\rangle$ for the $(n-1)$-connected cover of $\mathcal{E}_p$, which is the homotopy fiber of the natural map $\mathcal{E}_p \to P_{n-1}\mathcal{E}_p$ whose codomain is the $(n-1)$st Postnikov section of $\mathcal{E}_p$. We get a tower of homotopy fiber sequences

$$\begin{array}{ccccccc} & & K(\pi_2\mathcal{E}_p, 2) & & K(\pi_1\mathcal{E}_p, 1) & & K(\pi_0\mathcal{E}_p, 0) \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & \mathcal{E}_p\langle 2, \infty\rangle & \longrightarrow & \mathcal{E}_p\langle 1, \infty\rangle & \longrightarrow & \mathcal{E}_p \end{array}$$

called the Whitehead tower, where the vertical maps are the canonical maps from a space to its lowest nontrivial Postnikov section: $\mathcal{E}_p\langle n, \infty\rangle \to P_n[\mathcal{E}_p\langle n, \infty\rangle] = K(\pi_n\mathcal{E}_p, n)$. We can now define

$$\mathcal{F}_n\mathcal{E}^p(X) = \text{image of } [X, \mathcal{E}_p\langle n, \infty\rangle] \to [X, \mathcal{E}_p].$$

At first glance this looks totally different from the Atiyah-Hirzebruch construction, but in fact they are equal:

Proposition 47.27. *For every space X one has $\mathcal{F}_n\mathcal{E}^p(X) = F_n\mathcal{E}^p(X)$.*

Proof. It is immediate that $\mathcal{F}_n\mathcal{E}^p(X) \subseteq F_n\mathcal{E}^p(X)$, since if A is a CW-complex with $\dim A < n$ then $[A, \mathcal{E}_p\langle n, \infty\rangle] = 0$. To see the other direction, first replace X by a weakly equivalent CW-complex. Given an element $f \in F_n\mathcal{E}^p(X)$, we can represent it by a map $X \to \mathcal{E}_p$ which is null on X_{n-1}. But then it extends to $\tilde{f}\colon X/X_{n-1} \to \mathcal{E}_p$. Using that $H^i(X/X_{n-1}; \mathcal{A}) = 0$ for $i < n$ and all coefficients $\mathcal{A}$, obstruction theory lets us inductively lift the map $\tilde{f}$ up the tower until it becomes a map into $\mathcal{E}_p\langle n, \infty\rangle$. From there we see that $f \in \mathcal{F}_n\mathcal{E}^p(X)$. □

Now let us apply these observations to K^0, whose representing space is $\mathbb{Z} \times BU$. Since $\pi_*(\mathbb{Z} \times BU)$ is nonzero only in even degrees, notice that the Whitehead tower approach gives that $F_{2n-1}K^0(X) = F_{2n}K^0(X)$ for all X and all n. Because of this, it is tempting to only consider the groups $F_{2n}K^0(X)$ and maybe even re-index them as F_n. We won't do that, but this observation is useful to keep in mind for what we are about to discuss.

Let us next analyze the Whitehead tower for $\mathbb{Z} \times BU$ a bit more closely. Write $BU\langle 2n\rangle = BU\langle 2n, \infty\rangle$, for convenience. Because the odd homotopy groups of BU all vanish we will ignore the odd layers of the tower, so that it takes the form

$$\begin{array}{ccccccccc} & & K(\mathbb{Z}, 6) & & K(\mathbb{Z}, 4) & & K(\mathbb{Z}, 2) & & \mathbb{Z} \\ & & {\scriptstyle \alpha_3}\uparrow & & {\scriptstyle \alpha_2}\uparrow & & {\scriptstyle \alpha_1}\uparrow & & {\scriptstyle \alpha_0}\uparrow \\ \cdots & \longrightarrow & BU\langle 6\rangle & \longrightarrow & BU\langle 4\rangle & \longrightarrow & BU & \longrightarrow & \mathbb{Z} \times BU. \end{array}$$

Here $\alpha_n \in H^{2n}(BU\langle 2n\rangle)$, and there are a couple of ways we can understand this class. Since $BU\langle 2n\rangle$ is $(2n-1)$-connected the Hurewicz map $\pi_{2n}BU\langle 2n\rangle \to H_{2n}(BU\langle 2n\rangle)$ is an isomorphism, and of course $\pi_{2n}BU\langle 2n\rangle \to \pi_{2n}BU$ is an isomorphism by the construction of the Whitehead tower; we write β^n for the generator

of both homotopy groups. So we have the diagram

$$\begin{array}{ccccc} H^{2n}(BU) & \xrightarrow{\cong} & \mathrm{Hom}(H_{2n}(BU),\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(\pi_{2n}(BU),\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow\cong \\ H^{2n}(BU\langle 2n\rangle) & \xrightarrow{\cong} & \mathrm{Hom}(H_{2n}(BU\langle 2n\rangle),\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_{2n}(BU\langle 2n\rangle),\mathbb{Z}). \end{array}$$

The left horizontal maps are universal coefficient isomorphisms. We can define $\alpha_n \in H^{2n}(BU\langle 2n\rangle)$ to be the unique class that is sent to the functional $[\beta^n \to 1]$. If we take $c_n \in H^{2n}(BU)$ and follow it across the top and down, it becomes the map $\beta^n \to c_n(\beta^n) = \pm(n-1)!$: this follows from Proposition 39.7 (the proof is just to calculate $\mathrm{ch}(\beta^n)$ and use the fact that ch is a ring homomorphism). So the left vertical map sends c_n to $\pm(n-1)!\alpha_n$. For convenience let us redefine α_n so that the sign is always positive.

It is common to write $\alpha_n = \frac{c_n}{(n-1)!}$ as a useful piece of notation. One should just remember that the c_n function $BU \to K(\mathbb{Z},2n)$ is not truly divisible by $(n-1)!$, it is only the restriction to $BU\langle 2n\rangle$ that has this divisibility property. To say things very carefully, α_n is a characteristic class for maps into $BU\langle 2n\rangle$, and $(n-1)!$ times this class agrees with taking c_n of the composite into BU.

Note that α_0 represents the rank function $K^0(X) \to \mathbb{Z}$.

If we pass from $K^0(X)$ to $K^0(X)\otimes\mathbb{Q}$ then the Whitehead tower for BU now shows that we can reinterpret the filtration groups $F_{2n}K^0(X)$ in terms of the vanishing of Chern classes:

Corollary 47.28. *For $x \in F_{2n}K^0(X)$ one has $c_i(x) = 0$ for $0 \le i < n$, where in this context $c_0 = \mathrm{rank}$. Rationally we have*

$$F_{2n}K^0(X)\otimes\mathbb{Q} = \{u \in K^0(X)\otimes\mathbb{Q} \,|\, c_i(u) = 0 \text{ for } 0 \le i < n\}.$$

For $n \le 2$ the same result holds before tensoring with $\mathbb{Q}$.

Proof. Immediate from the Whitehead tower and the identification of the α_n classes. □

Following Grothendieck, we will use the $\tilde{\gamma}$-operations to give an approximation to the Atiyah-Hirzebruch filtration on $K^0(X)$. This is the so-called γ-filtration. Grothendieck worked in the context of algebraic geometry, and there one has neither cell decompositions nor Whitehead towers. There are some geometric substitutes one can attempt to use in place of the former, but sometimes it is hard to prove things about those. The motivation for the γ-filtration is that it is defined purely algebraically, and it can be shown to have many of the properties of the Atiyah-Hirzebruch filtration. Rationally they turn out to be equal, as we will see below.

As a prelude to the γ-filtration we start with the following result:

Proposition 47.29. *For any vector bundle E on X one has $\tilde{\gamma}^k(E) \in F_{2k}K^0(X)$ for all k.*

Proof. First note that $\tilde{\gamma}^k$ is a stable operation, in the sense that $\tilde{\gamma}^k(E\oplus 1) = \tilde{\gamma}^k(E)$ for all E. So we may regard it as a map of sets $K^{st}(X) \to \widetilde{K}^0(X)$, or equivalently $\widetilde{K}^0(X) \to \widetilde{K}^0(X)$. We can construct a representing map (or really, homotopy

class of maps) $\tilde{\gamma}^k\colon BU \to BU$ as follows. Since $BU = \operatorname{hocolim}_n BU(n)$ the Milnor sequence gives

$$0 \longrightarrow \lim{}^1 \widetilde{K}^{-1}(BU(n)) \longrightarrow [BU, BU] \longrightarrow \lim_n \widetilde{K}^0(BU(n)) \longrightarrow 0.$$

But we have computed $K^*(\mathrm{Gr}_n(\mathbb{C}^\infty))$ (Proposition 38.28) and know that this is concentrated in even degrees, so the groups $\widetilde{K}^{-1}(BU(n))$ all vanish. For the tautological bundle $\eta_n \to BU(n)$ we have the element $\tilde{\gamma}^k(\eta_n)$, and these elements fit together along the inverse limit to specify a unique element $\tilde{\gamma}^k \in [BU, BU]$.

Use a model for BU where $BU(k-1) \hookrightarrow BU$ is a cofibration (or else replace quotients with homotopy quotients in the following). If E is a bundle of rank less than k then $\tilde{\gamma}^k(E) = 0$, and from this it follows that the composite $BU(k-1) \hookrightarrow BU \xrightarrow{\tilde{\gamma}^k} BU$ is null. So $\tilde{\gamma}^k$ induces a map $g_k\colon BU/BU(k-1) \to BU$. But $\tilde{H}^*(BU/BU(k-1))$ vanishes for $* < 2k$, and so by obstruction theory g_k lifts to a map $BU/BU(k-1) \to (\mathbb{Z} \times BU)\langle 2k\rangle$. Therefore $\tilde{\gamma}^k$ lifts as well, and this is what we needed to prove. □

Define the γ-filtration on $K^0(X)$ to be the smallest multiplicative filtration having the property that each class $\tilde{\gamma}^n(E)$ is in F_n. More specifically, for $n \geq 1$ we set

$$F^\gamma_n K^0(X) = \langle \tilde{\gamma}^{i_1}(E_1)\tilde{\gamma}^{i_2}(E_2)\cdots\tilde{\gamma}^{i_r}(E_r) \,\big|\, r \geq 0,\ i_1 + \cdots + i_r \geq n\rangle$$

where the generators range over all vector bundles $E_1, \ldots, E_r$ on X. (Let us remind the reader once again that $\tilde{\gamma}^i(E) = \gamma^i(E - \mathrm{rank}(E))$ as in (47.10), and that the formulas in the literature are usually written in the latter notation). It follows at once that $F^\gamma_\bullet K^0(X)$ is a descending filtration that is functorial in X. Proposition 47.29 gives that $F^\gamma_n K^0(X) \subseteq F_{2n}K^0(X)$, for all n. Since $\tilde{\gamma}^1(E) = E - \mathrm{rank}(E)$ we also get that $F^\gamma_1 K^0(X) = \widetilde{K}^0(X)$, and so $F^\gamma_1 K^0(X) = \widetilde{K}^0(X) = F_1K^0(X) = F_2K^0(X)$.

Keep in mind when comparing the γ- and Atiyah-Hirzebruch filtrations that the former is indexed at double speed compared to the latter, i.e. we have $F^\gamma_n K^0(X) \subseteq F_{2n}K^0(X)$.

Computing the γ-filtration can feel daunting because one seems to need to know all of the vector bundles on the space X, which in fact almost never happens. The following result offers some simplifications:

Proposition 47.30. *It is always true that*

$$F^\gamma_n K^0(X) = \langle \tilde{\gamma}^{i_1}(x_1)\tilde{\gamma}^{i_2}(x_2)\cdots\tilde{\gamma}^{i_r}(x_r) \,\big|\, r \geq 0,\ i_1 + \cdots + i_r \geq n, x_i \in \widetilde{K}^0(X)\rangle.$$

If $w_1, \ldots, w_N$ are additive generators for $\widetilde{K}^0(X)$ then it is also true that

$$F^\gamma_n K^0(X) = \langle \tilde{\gamma}^{i_1}(w_{j_1})\tilde{\gamma}^{i_2}(w_{j_2})\cdots\tilde{\gamma}^{i_r}(w_{j_r}) \,\big|\, r \geq 0,\ i_1 + \cdots + i_r \geq n, \quad 1 \leq j_1, \ldots, j_r \leq N\rangle.$$

Proof. Note that the second statement is not immediate from the first, since the operations $\tilde{\gamma}^i$ are not additive.

For the first statement, write I_n for the group on the right. Since for a vector bundle E one has $\tilde{\gamma}^i(E) = \tilde{\gamma}^i(E - \mathrm{rank}(E))$ one certainly has $F^\gamma_n \subseteq I_n$. For the other direction, observe that a class $x \in \widetilde{K}^0(X)$ can be written as $x = [E] - [F]$ for some vector bundles E and F. By Exercise 47.12 the class $\gamma^i(x)$ can be written as a homogeneous polynomial of degree i in the classes $\gamma^a(E)$ and $\gamma^b(F)$ (where each

γ^j class is assigned degree j), and from this it follows that $\gamma^i(x) \in F_i^\gamma$. It follows at once that $I_n \subseteq F_n^\gamma$, and so the two are equal.

For the second statement, let J_n be the group on the right. Note that $J_\bullet$ is the descending multiplicative filtration determined by the conditions that $\tilde{\gamma}^i(w_j) \in J_i$ for all i, j. We clearly have $J_n \subseteq I_n$. For the subset in the other direction, first note that by Exercise 47.12 the element $\tilde{\gamma}^i(-w_j)$ is a polynomial of degree i in the classes $\tilde{\gamma}^s(w_j)$. It follows at once that $\tilde{\gamma}^i(-w_j) \in J_i$, for all i and j.

Next we observe that if $x \in \widetilde{K}^0(X)$ and $\tilde{\gamma}^i(x) \in J_i$ for all i, then $\tilde{\gamma}^k(x \pm w_j) \in J_k$ for all k. This follows directly from the formulas

$$\tilde{\gamma}^k(x \pm w_j) = \sum_{t=0}^{k} \tilde{\gamma}^t(x)\tilde{\gamma}^{k-t}(\pm w_j)$$

together with what has already been proven.

Finally, a straighforward induction now proves that $\tilde{\gamma}^i(\sum s_j w_j) \in J_i$ for all i and all $s_1, \ldots, s_N \in \mathbb{Z}$. From this it follows directly that $I_n \subseteq J_n$, and so the two sets are equal. □

Example 47.31. Let us compute the γ-filtration on $K^0(\mathbb{C}P^n)$. Recall from Proposition 33.6 that $K^0(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$ where $x = 1 - L$. As an abelian group this is $\mathbb{Z}^{n+1}$ generated by $1, L, L^2, \ldots, L^n$, and so $\widetilde{K}^0(\mathbb{C}P^n)$ is generated by the classes $1 - L^i$ for $i = 1, \ldots, n$.

By Exercise 47.11 we have $\tilde{\gamma}^s(1 - L^i) = (1 - L^i)^s$, which is a multiple of $(1 - L)^s = x^s$. It follows that $F_s^\gamma \subseteq (x^s)$, for all s (note that we are implicitly using Proposition 47.30 here). But we also have $x^s = (1 - L)^s = \tilde{\gamma}^s(1 - L) \in F_s^\gamma$, and so in fact $F_s^\gamma = (x^s)$ for all s.

Example 47.32. The easiest space where one can see a difference between the γ- and Atiyah-Hirzebruch filtrations is $X = S^6$ (the Whitehead tower for BU gives a sense of why). Here $K^0(S^6) = \mathbb{Z}^2$ with generators 1 and $\beta^{\hat{\otimes}(3)}$. Recall that $\beta = 1 - L \in \widetilde{K}^0(S^2)$, and $\beta^{\hat{\otimes}(3)}$ refers to the external product. For convenience set $v = \beta^{\hat{\otimes}(3)}$.

The class β lies in $\widetilde{K}^0(S^2) = F_2K^0(S^2)$, so by multiplicativity we get that $v \in F_6K^0(S^6)$. So $\widetilde{K}^0(S^6) = \langle v \rangle = F_2 = F_4 = F_6$. But since $\pi_6 BU\langle 8 \rangle = 0$ we know $F_8\widetilde{K}^0(S^6) = 0$.

For the γ-filtration we have $F_1^\gamma = \widetilde{K}^0(S^6) = F_2$ since that holds for all path-connected spaces. But we claim $F_3^\gamma = \langle 2v \rangle$. To see this, use the projection $\pi\colon \mathbb{C}P^3 \to \mathbb{C}P^3/\mathbb{C}P^2 = S^6$ and the induced injection $\pi^*\colon \widetilde{K}^0(S^6) \to \widetilde{K}^0(\mathbb{C}P^3)$ which has $\pi^*(v) = (1 - L)^3$. Then

$$\pi^*(\tilde{\gamma}^s(v)) = \tilde{\gamma}^s(\pi^*(v)) = \tilde{\gamma}^s((1 - L)^3).$$

Now

$$\tilde{\gamma}_t((1-L)^3) = \tilde{\gamma}_t(1-3L+3L^2-L^3) = \frac{\tilde{\gamma}_t(1)\tilde{\gamma}_t(L^2)^3}{\tilde{\gamma}_t(L)^3\tilde{\gamma}_t(L^3)} = \frac{(1 + t(L^2 - 1))^3}{(1 + t(L - 1))^3(1 + t(L^3 - 1))}$$

Substitute $L = 1 - u$ and expand the above fraction into a power series in t and u, remembering that $u^4 = 0$. The algebra is a little hairy, but it yields $\tilde{\gamma}_t(u^3) = 1 + u^3t - 3u^3t^2 + 2u^3t^3$, or equivalently

$$\tilde{\gamma}^1(u^3) = u^3, \quad \tilde{\gamma}^2(u^3) = -3u^3, \quad \tilde{\gamma}^3(u^3) = 2u^3.$$

See Exercise 47.33 below. By naturality we get the analogous formulas for $\tilde{\gamma}^s(v)$ in $\widetilde{K}^0(S^6)$, with u^3 replaced by v everywhere. Then by Proposition 47.30 we find that $F_1^\gamma = \langle v \rangle$, $F_2^\gamma = \langle 3v, 2v \rangle = \langle v \rangle$, and $F_3^\gamma = \langle 2v \rangle$ (note that all products of classes in $\widetilde{K}^0(S^6)$ are zero, e.g. since S^6 is a suspension). So $F_3^\gamma \subsetneq F_6$ here.

Exercise 47.33. Do the missing algebra of computing the power series expansion for $\tilde{\gamma}_t((1-L)^3)$ in the above example. [Hint: One clever way to handle the algebra is to apply log everywhere, use the power series for $\log(1+x)$ and remember that $u^4 = 0$. You should find $tu^3 - t^2(3u^3) + t^3(2u^3)$. Then exponentiate.]

The above example might get one thinking that perhaps the γ- and Atiyah-Hirzebruch filtrations are the same up to integer factors (modulo the issue of the filtrations being indexed at different speeds). Indeed, this is the case:

Proposition 47.34. *Let X be homotopically compact. Then for all $n \geq 0$, $F_n^\gamma K^0(X) \otimes \mathbb{Q} = F_{2n} K^0(X) \otimes \mathbb{Q}$.*

Before giving the proof let us explain the key calculation. Recall that if E is a bundle then $\tilde{\gamma}^n(E)$ is given by an explicit linear combination of the classes $\Lambda^i E$. One can therefore compute $c_t(\tilde{\gamma}^n(E))$ entirely in terms of the Chern classes of E, using the splitting principle and doing the usual algebraic manipulations. The algebra is messy, but one ends up with a very simple answer up through degree n. We will explain the result in a conceptual way that minimizes some of the algebra.

We showed in Proposition 47.29 that $\tilde{\gamma}^n(x) \in F_{2n} K^0(X)$ for any $x \in \widetilde{K}^0(X)$. Then by Corollary 47.28 we have $c_i(\tilde{\gamma}^n(x)) = 0$ for all $0 \leq i < n$. The higher Chern classes might be complicated, but we claim there is a simple formula for $c_n(\tilde{\gamma}^n(x))$. By the splitting principle we can assume that $x = [\oplus_{i=1}^r J_i] - r$ where the J_i are line bundles. Then $\tilde{\gamma}^n(x) = \sum_{i_1 < \cdots < i_n} (J_{i_1} - 1) \cdots (J_{i_n} - 1)$.

Lemma 47.35. *If $J_1, \ldots, J_k$ are line bundles and $x_i = c_1(J_i)$ then*

$$c_t((J_1 - 1) \cdots (J_k - 1)) = 1 + (-1)^{k-1}(k-1)! x_1 \ldots x_k t^k + \textit{higher order terms.}$$

Proof. It suffices to prove this in the universal case where $X = (\mathbb{C}P^\infty)^k$, and so $H^* X$ is torsion-free. This gives us the freedom to look at the Chern character instead. If $u = \prod_i (J_i - 1)$ then

$$\mathrm{ch}(u) = \prod_i (e^{x_i} - 1) = \prod_i (x_i + \tfrac{x_i^2}{2} + \cdots) = x_1 \cdots x_k + \text{h.o.t.}$$

Recall that the jth term of the Chern character is $\frac{1}{j!}$ times the jth Newton polynomial in the Chern classes. The jth Newton polynomial has c_j appearing with coefficient $(-1)^{j-1} j$ by Proposition F.1. So we find inductively that $c_j(u) = 0$ for $j < k$, and that $c_k(u) = (-1)^{k-1}(k-1)! x_1 \ldots x_k$. □

Returning to our calculation, we find that

$$\begin{aligned} c_t(\tilde{\gamma}^n(x)) &= \prod_{i_1 < \cdots < i_n} (1 + (-1)^{n-1}(n-1)! c_1(L_{i_1}) \cdots c_1(L_{i_n}) t^n + \text{h.o.t.}) \\ &= 1 + (-1)^{n-1}(n-1)! c_n(x) t^n + \text{h.o.t.} \end{aligned}$$

So we have now proven:

Proposition 47.36. *For $x \in \widetilde{K}^0(X)$ one has $c_i(\tilde{\gamma}^n(x)) = 0$ for all $i < n$ and $c_n(\tilde{\gamma}^n(x)) = (-1)^{n-1}(n-1)! c_n(x)$.*

Now we can finish our analysis of the rational γ-filtration:

Proof of Proposition 47.34. We have already proven that $F_n^\gamma \subseteq F_{2n}$ integrally for all n (Proposition 47.29). Since X is homotopically compact, $F_n = 0$ for $n \gg 0$. So $F_n^\gamma = F_{2n}$ when n is large, and we can proceed by a descending induction.

From now on everything is tensored with $\mathbb{Q}$, though we will not write the $\mathbb{Q}$'s. Assume n is such that $F_k^\gamma = F_{2k}$ for all $k > n$. Let $f\colon X \to BU$ be an element of F_{2n}. Consider the element $u = (-1)^{n-1}(n-1)!f - \tilde{\gamma}^n(f)$.

Both f and $\tilde{\gamma}^n(f)$ lie in F_{2n} and hence satisfy $c_i = 0$ for $i < n$. Using the Whitney sum formula this implies that $c_i(u) = 0$ for $i < n$ and $c_n(u) = (-1)^{n-1}(n-1)!c_n(f) - c_n(\tilde{\gamma}^n(f))$, which vanishes by Proposition 47.36. It follows that $u\colon X \to BU$ lies in F_{2n+2}, which by induction equals F_{n+1}^γ. But if $(-1)^{n-1}(n-1)!f - \tilde{\gamma}^n(f) \in F_{n+1}^\gamma$ then $(n-1)!f \in F_n^\gamma$, and since we are rational we conclude $f \in F_n^\gamma$. □

When X is homotopically compact recall that $K^0(X) \otimes \mathbb{Q} \cong \bigoplus_i H^{2i}(X;\mathbb{Q})$. What is happening behind the scenes of the above results is that there is really only one natural, multiplicative filtration on this cohomology theory that satisfies some basic nontriviality conditions. So it is not surprising that the naturally-occuring filtrations we find on $K^0(X)$ all end up being the same rationally.

Grothendieck observed that by taking the associated graded groups of the rational γ-filtation one gets a collection of groups that are a convenient substitute for the even-degree singular cohomology groups. This is not so interesting in topology, where the singular groups are readily at hand, but in algebraic geometry this leads to a nice "algebraic" construction of the rational Chow groups that doesn't require any consideration of algebraic cycles.

This completes our introduction to the γ-operations on $K^0(X)$. As the K-theoretic analogs of the Chern classes, they can be used in applications for some of the same purposes that the classical Chern classes are used. Some examples are discussed in [D7, Sections 8 and 9].

48. Chern classes and dependency loci

When we first defined Chern classes in singular cohomology we motivated the ideas by looking at dependency loci of generic sections: for a rank n bundle $E \to X$ the class $c_i(E)$ was the Poincaré dual of the dependency locus of $n-i+1$ generic sections—i.e., the locus of points in X over which the given sections are linearly dependent. This interpretation does *not* work for the K-theoretic Chern classes—or rather, it works for the Euler class (as we saw in Proposition 41.21) but not for the lower Chern classes. Instead one has a formula expressing the fundamental class of the dependency locus in terms of Chern classes of the bundle, but where only the lowest order term is $c_i^K(E)$. Our goal in this section is to explain why K-theory is different than singular cohomology in this way, and to explain how one discovers the required formulas.

As with everything related to K-theory, part of what makes this story interesting is using our different tools for accessing fundamental classes—through the machinery of complex-oriented cohomology theories or through the geometry of locally-free resolutions—and seeing the different perspectives they bring to the problem.

At various points throughout this section we will say a little about how to approach these issues via a general complex-oriented cohomology theory, but here

things are a little dicier. In general the dependency loci are singular, and so do not *a priori* have associated fundamental classes (they do for K-theory and singular cohomology for reasons discussed in Section 32). It is possible that the theory of derived manifolds could provide fundamental classes, but we do not pursue this here.

48.1. **Dependency loci.** Let $M \to X$ be a complex vector bundle of rank r, and suppose given sections $s_1, \dots, s_k$. Define the **dependency locus**

$$D(s_1, \dots, s_k) = \{x \in X \,\big|\, s_1(x), \dots, s_k(x) \text{ are linearly dependent}\}.$$

This can also be described as the vanishing set of the single section $s_1 \wedge \cdots \wedge s_k \colon X \to \Lambda^k M$. In particular, in a neighborhood where M has been trivialized then the sections s_i take values in $\mathbb{C}^r$ and $D(\underline{s})$ can be described as the vanishing locus of the $k \times k$ minors of the matrix whose rows are the sections s_i.

When X is a scheme and M is an algebraic vector bundle, we give $D(\underline{s})$ the closed subscheme structure associated to the ideal generated (locally) by these $k \times k$ minors. Said differently, the sections $s_1, \dots, s_k$ give a map $S \colon \mathcal{O}^k \to M$. Taking exterior powers gives $\mathcal{O} = \Lambda^k \mathcal{O}^k \to \Lambda^k M$, and dualizing gives $\Lambda^k M^* \to \mathcal{O}$. The image of this map is an ideal of $\mathcal{O}$, and locally it is precisely the ideal of $k \times k$ minors of our given sections. The subscheme $D(\underline{s}) \subseteq X$ is the closed subscheme defined by this ideal.

It will be important for us to have some understanding of the dimensions of these dependency loci. For this we can work locally and so assume the bundle M is trivial. Then $s_1, \dots, s_k \colon X \to \mathbb{C}^r$, so this is a map $S \colon X \to M_{k \times r}$. Inside $M_{k \times r}$ we have the space $\mathcal{R}_{k-1}$ of matrices having rank at most $k-1$. For the subset of $\mathcal{R}_{k-1}$ where the final row is a linear combination of the others, the dimension is $(k-1)r + (k-1)$—one can specify the first $k-1$ rows arbitrarily, and then an additional $k-1$ numbers determine the coefficients of the linear combination giving the kth row. So the total dimension is $(k-1)(r+1)$, and therefore the codimension is $rk - (k-1)(r+1) = r - k + 1$.

The space $\mathcal{R}_{k-1}$ is singular and its singular locus is precisely $\mathcal{R}_{k-2}$, the space of matrices having rank at most $k-2$. We postpone the explanation to Section 48.12, which also gives a more detailed version of the codimension calculation. By a similar analysis to above, the dimension of $\mathcal{R}_{k-2}$ equals $(k-2)r + 2(k-2)$. So its codimension in $M_{k \times r}$ is $2(r-k+2)$.

Returning to our algebraic maps $S \colon X \to M_{k \times r}$, generically the images will intersect $\mathcal{R}_{k-1}$ in a space of codimension $r-k+1$, and will intersect $\mathcal{R}_{k-2}$ in a space of codimension $2(r-k+2)$. So if $n < 2(r-k+2)$ then we can avoid the singularities, but not in general. There is also the edge case where $k = 1$, in which case $\mathcal{R}_{k-2} = \emptyset$; here we can always arrange for a smooth dependency locus, no matter what n is.

Going forward, it will be convenient to change our variables and let $q = r - k + 1$: this is the expected codimension of the dependency locus. So the number of sections is $r - q + 1$. For generic sections the dependency locus will be smooth if $n < 2(q+1)$ or if $q = r$.

Remark 48.2. Since $D(\underline{s})$ is the vanishing locus of $s_1 \wedge \cdots \wedge s_{r-q+1} \colon X \to \Lambda^{r-q+1} M$, one might expect the Euler class $e(\Lambda^{r-q+1})$ to be closely related. But here it is very important that $s_1 \wedge \cdots \wedge s_{r-q+1}$ is not at all a "generic" section

of $\Lambda^{r-q+1}M$. In fact, just thinking about an arbitrary exterior product $\Lambda^i V$ one readily sees that the vectors of the form $v_1 \wedge \cdots \wedge v_i$ are very special elements.

The bundle $\Lambda^{r-q+1}M$ has rank $\binom{r}{r-q+1} = \binom{r}{q-1}$, so a generic section would have a vanishing locus of that codimension. In our case the vanishing locus of $s_1 \wedge \cdots \wedge s_{r-q+1}$ has codimension q, which is typically much smaller. So we are getting a vanishing locus that is larger than the expected dimension, and so $s_1 \wedge \cdots \wedge s_{r-q+1}$ does not meet the zero section of Λ^{r-q+1} transversely. Therefore Proposition 41.21 does not apply, and so there is no particular connection between its vanishing locus and the Euler class.

Our next task is to study the class $[\mathcal{O}_{D(\underline{s})}]$ in $K^0(X)$. This prompts us to try to construct a locally-free resolution of $\mathcal{O}_{D(\underline{s})}$. Locally on X we are looking at R/I where I is the ideal of maximal minors of a matrix, so now we have a commutative algebra problem on our hands. The complexes that will solve our problem (at least, in generic situations) are described next.

48.3. **Eagon-Northcott complexes.** For elements $x_1, \ldots, x_n$ in a ring R, the Koszul complex $K(\underline{x}; R)$ is built from the "generic" syzygies one can write down for any such sequence. If the x's themselves are generic enough (specifically, a regular sequence) then these are all the syzygies and the Koszul complex is a resolution. As in Proposition 35.17 we can phrase this as saying that the Koszul complex is a resolution if and only if $\operatorname{depth}_{(\underline{x})} R = n$.

Now instead of looking at a sequence of elements, look at a $g \times n$ matrix A and consider the ideal $I_g(A)$ generated by the $g \times g$ minors (the Koszul case will end up being $g = 1$). It turns out that one can again produce a canonical complex that in some sense contains the generic syzygies for all of these $g \times g$ minors. These are the so-called "Eagon-Northcott complexes", to be described below. Again, when the elements of A are generic enough this is actually a resolution of $R/I_g(A)$.

Before seeing the general version it is helpful to look at the first nontrivial example. Suppose given the matrix

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

with entries in a ring R. The 2×2 minors are

$$\Delta_1 = x_2y_3 - x_3y_2, \quad \Delta_2 = x_1y_3 - x_3y_1, \quad \Delta_3 = x_1y_2 - x_2y_1$$

(the index on the Δ corresponds to the column of A that is deleted). There are two "obvious" relations between these three minors: if one forms the matrices

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

then the determinants are necessarily zero, and a cofactor expansion across the top row gives the two relations

$$x_1\Delta_1 - x_2\Delta_2 + x_3\Delta_3 = 0, \qquad y_1\Delta_1 - y_2\Delta_2 + y_3\Delta_3 = 0.$$

So we may form the complex

$$0 \longrightarrow R^2 \xrightarrow{\left[\begin{smallmatrix} x_1 & y_1 \\ -x_2 & -y_2 \\ x_3 & y_3 \end{smallmatrix}\right]} R^3 \xrightarrow{[\Delta_1\ \Delta_2\ \Delta_3]} R \longrightarrow 0.$$

This is an example of a **Hilbert-Burch complex**. A classical theorem says that it is a resolution of $R/(\Delta_1, \Delta_2, \Delta_3)$ if and only if $\operatorname{depth}_{(\Delta_1,\Delta_2,\Delta_3)} R \geq 2$. This is part of the Hilbert-Burch theorem [Eisenbud, Theorem 20.15], though in modern times one would think of it as a simple example of the Buchsbaum-Eisenbud acyclicity criterion [Eisenbud, Theorem 20.9].

Eagon-Northcott complexes are a massive generalization of the above Hilbert-Burch example, but the basic idea is the same. For a $g \times f$ matrix A with $g \leq f$, one starts with the list of $g \times g$ minors and then finds all of the relations and higher relations that come out of generic multilinear algebra.

Here is how it works. For convenience let us always be in characteristic zero. Let F and G be free modules of ranks f and g with $f \geq g$, and let $A\colon F \to G$ be a map. To help typography we will write $S^i = \operatorname{Sym}^i$, and instead of $M \otimes_R N$ we write $M\big|N$. Then the Eagon-Northcott complex looks like

$$0 \to \Lambda^f F \big| S^{f-g} G^* \big| \Lambda^g G^* \to \Lambda^{f-1} F \big| S^{f-g-1} G^* \big| \Lambda^g G^* \cdots \to \Lambda^g F \big| S^0 G^* \big| \Lambda^g G^* \to R.$$

It is nonzero in degrees 0 through $f - g + 1$. To describe the differentials, recall that for any $v \in G$ we have a directional derivative operator D_v on $\operatorname{Sym} G^*$ that lowers degrees by 1; this can be described as the unique R-linear derivation on $\operatorname{Sym}_R G^*$ having the property that $D_v(\phi) = \phi(v)$ for $\phi \in G^*$. Then except for the first differential (with codomain R) the formulas are

$$d((c_0 \wedge \cdots \wedge c_k) \otimes a \otimes b) = \sum_{i=0}^{k} (-1)^i (c_0 \wedge \cdots \wedge \hat{c}_i \wedge \cdots \wedge c_k) \otimes (D_{Ac_i} a) \otimes b.$$

This is very much like the Koszul differentials. The formula for the first differential is a little different, and this is the only differential that involves the third tensor factor:

$$(c_1 \wedge \cdots \wedge c_g) \otimes 1 \otimes (\phi_1 \wedge \cdots \wedge \phi_g) \mapsto \det [\phi_i(A(c_j))]_{i,j}.$$

Of course the image is precisely the ideal $I_g(A)$.

Exercise 48.4. Prove that the directional derivative operators defined above satisfy $D_v D_w = D_w D_v$, for any $v, w \in G$. Use this to show that the above definitions yield $d^2 = 0$. For the composition involving the differential from degree 1 to degree 0, explain how the $d^2 = 0$ relation connects to the vanishing of determinants for matrices with dependent rows, like we saw in the Hilbert-Burch example.

Exercise 48.5. Check that in the case $g = 1$ the above construction reduces to the usual Koszul complex. Then check that the $f = 3$, $g = 2$ case reproduces the Hilbert-Burch complex given above.

The following result is [EN, Theorem 1]. See also [Eisenbud, Corollary A.2.12] for a modern textbook treatment.

Theorem 48.6 (Eagon-Northcott)**.** *Let R be a Noetherian ring and let A be a $g \times f$ matrix with entries in A where $g \leq f$. Assume $I_g(A) \neq R$. Then $\operatorname{depth}_{I_g(A)} R \leq f - g + 1$, and $f - g + 1 - \operatorname{depth}_{I_g(A)} R$ equals the largest value of i such that the Eagon-Northcott complex has nonzero homology in degree i. Consequently, if $\operatorname{depth}_{I_g(A)} R = f - g + 1$ then the Eagon-Northcott complex is a resolution of $R/I_g(A)$.*

We will leave the reader to chase down the proof from the above sources. Observe how this result generalizes the depth sensitivity of the Koszul complex that we saw in Proposition 35.17.

The Eagon-Northcott construction readily generalizes to the case where $F \to G$ is a map of locally-free coherent $\mathcal{O}_X$-modules over a scheme. One just globalizes what we did above.

Now suppose that M is an algebraic vector bundle of rank r, fix $q \leq r$, and let $s_1, \dots, s_{r-q+1}$ be sections of M. Together these assemble into a map $\mathcal{O}^{r-q+1} \to M$. Dualize to make this $A\colon M^* \to \mathcal{O}^{r-q+1}$, and now take the associated Eagon-Northcott complex. For generic sections this is a resolution of $\mathcal{O}_D$, where $D = D(\underline{s})$ is the vanishing locus of the ideal of maximal minors of A. Here if we set $N = r-q+1$ the Eagon-Northcott complex looks like

$$0 \to \Lambda^r M^* \big| S^{q-1}(N) \to \Lambda^{r-1} M^* \big| S^{q-2}(N) \cdots \to \Lambda^{r-q+1} M^* \big| S^0(N) \to \mathcal{O} \to \mathcal{O}_Z.$$

So we get the formula

$$\begin{aligned} [\mathcal{O}_D] &= 1 + (-1)^{r-q} \sum_{i=r-q+1}^{r} (-1)^i S^{i-N}(N) \Lambda^i M^* \\ &= 1 + (-1)^{r-q} \sum_{i=r-q+1}^{r} (-1)^i \tbinom{i-1}{r-q} \Lambda^i M^*. \end{aligned} \tag{48.7}$$

Here we have used $\mathrm{Sym}^d(n) = \binom{n+d-1}{d} = \binom{n+d-1}{n-1}$ to get the last line. As a reality check, observe that when $q = r$ (the case of 1 generic section) our formula (48.7) reduces to $[\mathcal{O}_D] = \sum_{i=0}^{r} (-1)^i \Lambda^i M^* = e^K(M)$.

To proceed further use the splitting principle to reduce to the case where $M = \oplus_{i=1}^k J_i$ is a sum of line bundles, and set $x_i = e^K(J_i) = 1 - J_i^*$. Then $\Lambda^i M^* = \sigma_i(1-x_1, \dots, 1-x_r) = \sum_{s=0}^{i} (-1)^s \binom{r-s}{i-s} \sigma_s(\underline{x}) = \sum_{s=0}^{i} (-1)^s \binom{r-s}{i-s} c_s^K(M)$. So

$$\begin{aligned} [\mathcal{O}_D] &= 1 + (-1)^{r-q} \sum_{i=r-q+1}^{r} (-1)^i \tbinom{i-1}{r-q} \sum_{s=0}^{i} (-1)^s \tbinom{r-s}{i-s} c_s^K(M) \\ &= 1 + (-1)^{r-q} \sum_{s=0}^{r} c_s^K(M) \cdot \sum_{\substack{i=s \\ i \geq r-q+1}}^{r} (-1)^{i+s} \tbinom{i-1}{r-q} \tbinom{r-s}{i-s} \quad \text{(reverse the sum)} \\ &= 1 + (-1)^{r-q} \sum_{s=0}^{r} c_s^K(M) \cdot \sum_{i} (-1)^{i+s} \tbinom{i-1}{r-q} \tbinom{r-s}{i-s} \qquad \text{(see below)} \\ &= 1 + (-1)^{r-q} \sum_{s=0}^{r} c_s^K(M) \cdot \sum_{j} (-1)^{j} \tbinom{j+s-1}{r-q} \tbinom{r-s}{j} \quad \text{(variable change } j = i-s) \\ &= 1 + (-1)^{r-q} \sum_{s=0}^{r} c_s^K(M) \cdot \begin{cases} (-1)^{r-q+1} & \text{if } s = 0, \\ (-1)^{r-s} \binom{s-1}{q-1} & \text{if } s \geq 1 \end{cases} \qquad \text{(Lemma 48.8 below)} \\ &= \sum_{s=0}^{r} c_s^K(M) \cdot (-1)^{s-q} \tbinom{s-1}{q-1} \\ &= c_q^K(M) - q c_{q+1}^K(M) + \tbinom{q+1}{2} c_{q+2}^K(M) - \cdots. \end{aligned}$$

Note that throughout this calculation we use the convention that $\binom{n}{k}$ is zero when $n < 0$, or when $k < 0$, or when $k > n$. This is used in the third equality to extend the range of the sum to all i. For the fifth equality, Lemma 48.8 applies in the case $s \geq 1$, but for $s = 0$ one must do an independent analysis (the same techniques as in the lemma work).

Lemma 48.8. *Given fixed integers a, b, and c where $a, b \geq 0$, one has $\sum_j (-1)^j \binom{a}{j}\binom{j+b}{c} = (-1)^a \binom{b}{c-a}$.*

Proof. We just calculate as follows:

$$\begin{aligned}\sum_j (-1)^j \tbinom{a}{j}\tbinom{j+b}{c} = \sum_{j=0}^{a} (-1)^j \tbinom{a}{j}[x^c](1+x)^{j+b} &= [x^c](1+x)^b \sum_{j=0}^{a} (-1)^j \tbinom{a}{j}(1+x)^j \\ &= [x^c](1+x)^b(1-(1+x))^a \\ &= [x^c](1+x)^b(-1)^a x^a \\ &= (-1)^a[x^{c-a}](1+x)^b \\ &= (-1)^a \tbinom{b}{c-a}.\end{aligned}$$

□

The kind of analysis we did above first appeared in papers of Fogarty-Rim [FR1, FR2]. They used a different resolution of $\mathcal{O}_D$ then we did: they used a resolution due to Buchsbaum-Rim that is in general much larger than the Eagon-Northcott complex. The differences in the complexes lead to the algebra working out a bit differently, but of course the two approaches describe the same class $[\mathcal{O}_D]$ and so ultimately lead to the same formulas.

48.9. **The complex-oriented perspective.** Now that we have seen how things play out using explicit locally-free resolutions, let us step back and consider another approach from the viewpoint of a general complex-oriented cohomology theory $\mathcal{E}$.

Let $M \to X$ be a rank r bundle, and let $s_1, \ldots, s_{r-q+1}$ be generic sections (it will gradually become clear what we mean by 'generic'). Now pass to the space $X \times \mathbb{C}P^{r-q}$, and let $\mathcal{L}$ be the tautological line bundle: the fiber over (x, ℓ) is the set of points in ℓ. The sections $s_1, \ldots, s_{r-q+1}$ together assemble to give a single section S of $\underline{\mathrm{Hom}}(\mathcal{L}, \pi^*M) = \mathcal{L}^* \otimes \pi^*M$: over $(x, \ell = [a_1 : \cdots : a_{r-q+1}])$ we take the transformation $\ell \to M_x$ that sends $(a_1, \ldots, a_{r-q+1}) \mapsto \sum a_i s_i(x)$. The vanishing locus of S is

$$V(S) = \Big\{ \big(x, [a_1 : \cdots : a_{r-q+1}]\big) \;\Big|\; \sum a_i s_i(x) = 0 \Big\}.$$

In fact, let us take note that

$$\begin{aligned}H^0(X \times \mathbb{C}P^{r-q}, \mathcal{L}^* \otimes \pi^*M) &= H^0(\mathbb{C}P^{r-q}, \mathcal{O}(1)) \otimes H^0(X, M) \\ &= (\mathbb{C}^{r-q+1})^* \otimes H^0(X, M) \\ &\cong H^0(X, M)^{\oplus(r-q+1)}\end{aligned}$$

and the isomorphism from the last to the first is precisely passing from $(s_1, \ldots, s_{r-q+1})$ to S as we described above. In particular, a generic choice of $s_1, \ldots, s_{r-q+1}$ corresponds to a generic section of $\mathcal{L}^* \otimes \pi^*M$. This tells us that choosing $\underline{s}$ generically will result in a smooth $V(S)$, and $[V(S)] = e^{\mathcal{E}}(\mathcal{L}^* \otimes \pi^*M)$ (Proposition 41.21).

Projection onto the first coordinate gives a surjection $p\colon V(S) \twoheadrightarrow D(\underline{s})$. The fiber over x is a singleton if and only if there is at most one (up to scaling) dependency relation on the vectors $s_1(x), \dots, s_{r-q+1}(x)$. Equivalently, in a local trivialization of the bundle M our sections determine an $(r-q+1) \times r$ matrix and the condition is that the rank of this matrix is at least $r-q$. We know from our discussion in Section 48.1 that we cannot be assured of sections where this is simultaneously satisfied for every x, unless $\dim X$ is small enough compared to q.

For arbitrary $x \in D(\underline{s})$, the fiber of p is a projective space whose dimension equals $r - q - \dim\langle s_1(x), \dots, s_{r-q+1}(x)\rangle$. On the smooth locus of $D(\underline{s})$ we know that the sections have rank $r-q$, and so we have singleton fibers; i.e., $p\colon V(S) \to D(\underline{s})$ is a homeomorphism over the smooth part of $D(\underline{s})$. In contrast, points in the singular part of $D(\underline{s})$ are blown up into projective spaces of various dimensions. So $V(S) \to D(\underline{s})$ is a certain kind of resolution of singularities.

From this point on, our discussion splits into two components. The machinery of complex-oriented cohomology theories works very well for analyzing $[V(s)]$ and $\pi_![V(s)]$, and this is one part of the story. In the case of K-theory we can make the extra step of connecting this work to the singular space $D(\underline{s})$, though this requires some nontrivial explanation. Perhaps all of this suggests that $D(\underline{s})$ was somehow the "wrong" object to be studying in the first place, and our focus should have instead been on $V(S)$ all along.

Let us first deal with the connection between $\pi_!([\mathcal{O}_{V(S)}])$ and $[\mathcal{O}_{D(\underline{s})}]$ in K-theory, where $\pi\colon X \times \mathbb{C}P^{r-q} \to X$ is the projection. We claim these two classes are equal, but it takes some work to see this. We will only give a rough outline. For convenience write $V = V(S)$ and $D = D(\underline{s})$. Consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ j\ } & X \times \mathbb{C}P^{r-q} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{\ j\ } & X. \end{array}$$

Both p and π are proper maps. We claim that $p_*\mathcal{O}_V = \mathcal{O}_D$ and $R^ip_*\mathcal{O}_V = 0$ for $i > 0$. The first follows from the fact that $V \to D$ is birational—in fact, we have already seen that it is an isomorphism over the smooth locus of D. The vanishing of the higher direct images is the statement that D has "rational singularities". This is the key result we need, as we can then argue as follows. Since j is a closed immersion and hence j_* is exact, it follows that $(jp)_*\mathcal{O}_V = \mathcal{O}_D$ and $R^i(jp)_*\mathcal{O}_V = 0$ for $i > 0$. So we have

$$[\mathcal{O}_D] = (jp)_!([\mathcal{O}_V]) = (\pi j)_!([\mathcal{O}_V]) = \pi_!([\mathcal{O}_V]).$$

Note that we are avoiding writing $p_!$ or $j_!$ because D is not smooth.

It remains to justify the statement about rational singularities: $R^ip_*\mathcal{O}_V = 0$ for $i > 0$. Intuitively this is coming from the fact that the fibers of p are projective spaces and $H^i(\mathbb{C}P^n; \mathcal{O}) = 0$ for all n and all $i > 0$. But turning this into a rigorous proof is tricky, because R^ip_* depends not just on the fibers of p but on the local structure around those fibers.

There are two routes one can go here, and we will explain what they are but only give a few details. One route is to first prove that the schemes $\mathcal{R}_j$—the universal examples of dependency loci—have rational singularities. This is done in

[Weyman, Proposition 6.1.1]. Then an argument must be made that pulling back along a generic map preserves this property.

Another route is to start with $\mathcal{O}_{V(S)}$, which is resolved over $X \times \mathbb{C}P^{r-q}$ by the Koszul complex associated to the bundle $\mathcal{L} \otimes \pi^* M^*$. Since $\Lambda^i(\mathcal{L} \otimes \pi^* M^*) \cong \mathcal{L}^i \otimes \pi^* \Lambda^i M^*$ this Koszul complex looks like

$$0 \to \mathcal{L}^r \otimes \pi^* \Lambda^r M^* \to \mathcal{L}^{r-1} \otimes \pi^* \Lambda^{r-1} M^* \to \cdots \to \mathcal{L} \otimes \pi^* M^* \to \mathcal{O}_{X \times \mathbb{C}P^{r-q}} \to 0.$$

Now compute the derived pushforward (i.e., apply $R\pi_*$) and use that $R^i\pi_*(\mathcal{L}^j \otimes \pi^* F) \cong (R^i\pi_* \mathcal{L}^j) \otimes F$. Since $\mathcal{L}$ is pulled back from $\mathcal{O}(-1)$ on $\mathbb{C}P^{r-q}$, the computation of $R^i\pi_*\mathcal{L}^j$ boils down to the cohomology of $\mathbb{C}P^{r-q}$ with coefficients in $\mathcal{O}(-j)$. This was computed in Table 43.1, which gives $H^i(\mathbb{C}P^{r-q}; \mathcal{O}(-j)) = 0$ for $1 \le j \le r-q$ and for all i. Also, for $j > r-q$ the only nonvanishing cohomology group is for $i = r-q$, and $H^{r-q}(\mathbb{P}(V); \mathcal{O}(-j)) = \operatorname{Sym}^{j-(r-q+1)} V$. So when applying $R\pi_*$ to the above complex we have that

- The terms $\mathcal{L}^i \otimes \pi^* \Lambda^i M^*$ disappear for $1 \le i \le r-q$.
- For $i > r-q$ the term $\mathcal{L}^i \otimes \pi^* \Lambda^i M^*$ is converted via $R^{r-q}\pi_*$ into $\operatorname{Sym}^{i-(r-q+1)} G \otimes \Lambda^i M^*$ where $G = \mathcal{O}^{r-q+1}$.
- The term $\mathcal{O}_{X \times \mathbb{C}P^{r-q}}$ is converted via $R^0\pi_*$ to $\mathcal{O}_X$.

What results is precisely the Eagon-Northcott complex corresponding to the map $\underline{s}^* \colon M^* \to \mathcal{O}^{r-q+1}$. Of course one has to do a little legwork to check that the differentials match, but this is not too hard.

Since generically the Eagon-Northcott complex is a resolution of $\mathcal{O}_{D(\underline{s})}$, this proves (in the generic setting) that $\pi_!([\mathcal{O}_{V(S)}]) = [\mathcal{O}_{D(\underline{s})}]$ as elements of $K^0(X)$.

Now we turn to the second part of the analysis, where we deal with the Euler class $e^{\mathcal{E}}(\mathcal{L}^* \otimes \pi^* M)$. We will soon specialize to $\mathcal{E} = K$, but let us stay in the general setting for the moment. For generic sections $\underline{s}$ this is equal to $[V(S)] = j_!(1)$ where $j \colon V(S) \hookrightarrow X \times \mathbb{C}P^{r-q}$ is the inclusion (Proposition 41.21). Then

$$\pi_!([V(S)]) = \pi_!(j_!(1)) = \pi_!(e^{\mathcal{E}}(\mathcal{L}^* \otimes \pi^* M)) \in \mathcal{E}^{2q}(X).$$

We will see that in some sense this is the correct intersection-theoretic substitute for the fundamental class of $D(\underline{s})$. For convenience let us write this class as $\tilde{D} = \pi_!([V(S)])$.

Using the Künneth isomorphism $\mathcal{E}^*(X \times \mathbb{C}P^{r-q}) \cong \mathcal{E}^*(X) \otimes_{\mathcal{E}^*} \mathcal{E}^*(\mathbb{C}P^{r-q})$ (or equivalently, the projective bundle theorem) we can write

$$[V(S)] = e^{\mathcal{E}}(\mathcal{L}^* \otimes \pi^* M) = \sum_{i=0}^{r-q} a_{q+i} \otimes [\mathbb{C}P^i]$$

where $a_j \in \mathcal{E}^{2j}(X)$. Of course these a_j classes are characteristic classes for M, and so will be polynomials in the $c_i^{\mathcal{E}}(M)$ classes. Let us write them as $a_j(M)$ to highlight this. Applying $\pi_!$ to the above formula sends each $[\mathbb{C}P^i]$ to $\operatorname{genus}_{\mathcal{E}}(\mathbb{C}P^i)$, and therefore we get

$$\tilde{D} = \pi_![V(S))] = \sum_{i=0}^{r-q} a_{q+i}(M) \cdot \operatorname{genus}_{\mathcal{E}}(\mathbb{C}P^i).$$

It only remains to calculate the $a_{q+i}(M)$ classes. For this, use the splitting principle. Assume that $M = \oplus_{s=1}^r J_s$ where the J_s are line bundles. Then

$$e^{\mathcal{E}}(\mathcal{L}^* \otimes \pi^* M) = \prod_{s=1}^r e^{\mathcal{E}}(\mathcal{L}^* \otimes \pi^* J_s) = \prod_{s=1}^r F_{\mathcal{E}}(e(\mathcal{L}^*), \pi^* e(J_s))$$

where $F_{\mathcal{E}}$ is the formal group law for $\mathcal{E}$.

Now we specialize to K-theory, where $F_K(x,y) = x + y - xy$. Set $u = e^K(\mathcal{L}^*)$ and $x_s = e^K(J_s)$. Then we get

$$\begin{aligned} e^K(\mathcal{L}^* \otimes \pi^* M) = \prod_{s=1}^r (u + x_s - u x_s) &= \prod_{s=1}^r (u + (1-u)x_s) \\ &= u^r + u^{r-1}(1-u)\sigma_1 + \cdots \\ &= \sum_{s=0}^r u^{r-s}(1-u)^s \sigma_s \end{aligned}$$

where $\sigma_s = \sigma_s(\underline{x}) = c_s^K(M)$. Before diving into the algebra let us think a bit. Eventually we will be applying $\pi_!$, which will map each $u^i = [\mathbb{C}P^{r-q-i}]$ to 1 (since the Todd genus of any complex projective space equals 1). However, we cannot simply put $u = 1$ into the above formula—that appears to just give 1. The problem is that u is nilpotent, in fact $u^{r-q+1} = 0$. So we need to expand the above expression in powers of u, throw away all powers larger than u^{r-q}, and *then* set $u = 1$.

As a warm-up, let us look at $r = 4$. The following table shows the coefficients of the powers of u appearing in the summands for different values of s:

s	u^4	u^3	u^2	u^1	u^0
0	1				
1	$-\sigma_1$	σ_1			
2	σ_2	$-2\sigma_2$	σ_2		
3	$-\sigma_3$	$3\sigma_3$	$-3\sigma_3$	σ_3	
4	σ_4	$-4\sigma_4$	$6\sigma_4$	$-4\sigma_4$	σ_4

For $q = 4$ we throw away u^1 and higher, so we just get $\tilde{D}_{1 \text{ section}} = c_4(M)$. For the other q we sum all of the terms in column u^q and lower (to the right), leading to the formulas:

$$\begin{aligned} q = 4: &\qquad \tilde{D}_{1 \text{ section}} = c_4^K(M) \\ q = 3: &\qquad \tilde{D}_{2 \text{ sections}} = c_3^K(M) - 3c_4^K(M) \\ q = 2: &\qquad \tilde{D}_{3 \text{ sections}} = c_2^K(M) - 2c_3^K(M) + 3c_4^K(M) \\ q = 1: &\qquad \tilde{D}_{4 \text{ sections}} = c_1^K(M) - c_2^K(M) + c_3^K(M) - c_4^K(M). \end{aligned}$$

At this point the reader can see the general pattern: for arbitrary r and q we are going to get an expression that starts with $c_q(M)$ and then has higher order Chern classes added on with certain coefficients, and those coefficients are obtained as alternating sums of binomial coefficients. From here it is just a matter of working out the details.

For general r and q, the coefficient of $c_s^K(M)$ in the final answer is

$$\begin{aligned}(-1)^{r-i}\sum_{k=0}^{r-q}(-1)^k\tbinom{s}{k-(r-s)} &= (-1)^{r-s}\sum_{k=r-s}^{r-q}(-1)^k\tbinom{s}{s-(r-k)}\\ &= \sum_{j=0}^{s-q}(-1)^j\tbinom{s}{j} \qquad \text{(change of variables } j=k-r+s)\\ &= (-1)^{s-q}\tbinom{s-1}{s-q} \quad \text{(standard binomial identity)}\\ &= (-1)^{s-q}\tbinom{s-1}{q-1}.\end{aligned}$$

Therefore

$$\tilde{D}_{r-q+1 \text{ sections}} = \sum_{s=q}^{r}(-1)^{s-q}\tbinom{s-1}{q-1}c_s^K(M).$$

This reproduces the same formula we found using the Eagon-Northcott complex.

Exercise 48.10. Return to the setting of a general complex-oriented cohomology theory $\mathcal{E}$. Recall the formulas

$$e^{\mathcal{E}}(\mathcal{L}^*\otimes M) = \prod_{s=1}^{r} F_{\mathcal{E}}(u,x_s), \qquad [D(\underline{s})] = \pi_!\big(e^{\mathcal{E}}(\mathcal{L}^*\otimes M)\big).$$

Since the Euler class lies in $\mathcal{E}^*(X\times\mathbb{C}P^{r-q})$, the coefficients of u^i for $i > r-q$ are irrelevant to the formulas. When writing $e^{\mathcal{E}}(\mathcal{L}^*\otimes M)$ as a power series in u and the $c_i^{\mathcal{E}}(M)$, explain why no $c_i^{\mathcal{E}}(M)$ appears for $i < q$ and $c_q^{\mathcal{E}}(M)$ appears with coefficient equal to $u^{r-q} = [\mathbb{C}P^0]$. Conclude that $\tilde{D}$ is equal to $c_q^{\mathcal{E}}(M)$ + [a polynomial expression in the $c_i^{\mathcal{E}}(M)$ for $i > q$]. [Note: One can give a combinatorial proof of this using the above formulas, but one can also give a less computational proof using the universal complex-oriented theory MU and the fact that MU^* is zero in positive degrees. Try to do it both ways.]

As one final comment on this material, in a general complex-oriented cohomology theory $\mathcal{E}$ we have the submanifold $V(S)\subseteq X\times\mathbb{C}P^{r-q}$ and the fundamental class $[V(S)] = \pi_! e^{\mathcal{E}}(\mathcal{L}^*\otimes\pi^*M)\in\mathcal{E}^{2q}(X)$. It is not clear, though, how to construct a class $[D(\underline{s})]$, since the spaces $D(\underline{s})$ are usually singular. There are theories of "derived manifolds" that admit objects constructed as certain kinds of derived pullbacks, and provide fundamental classes for these objects in complex-oriented theories; see [Spivak] or [Joyce1, Joyce2], for example. But we leave it as a question for the reader whether such theories can be applied to develop a reasonable theory of these dependency loci.

Remark 48.11. Suppose $f\colon E_1\to E_2$ is a map of algebraic vector bundles over X. For each $k\le\min\{\operatorname{rank}E_1,\operatorname{rank}E_2\}$ define $D_k(f) = \{x\in X \mid \operatorname{rank}f(x)\le k\}$. These 'degeneracy loci' (which are typically singular) generalize the dependency loci that we considered throughout this section, where E_1 was always trivial. See [Fulton, Chapter 14] for constructions of associated classes in Chow groups, relations to characteristic classes of E_1 and E_2, and so forth.

48.12. **Determinantal varieties and their singularities.** As an appendix to the preceding work in this section, we give a brief tour of the backround we used about determinantal varieties. Let F be a field, let $k\le r$, and let X be a $k\times r$

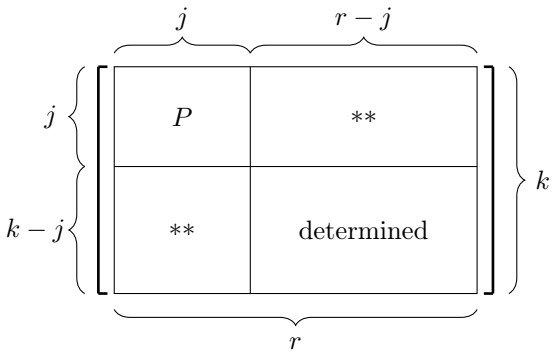
matrix of indeterminants over F. Let $R = F[x_{ij}]$ be the polynomial ring generated by these indeterminants. We will identify the closed points of $\operatorname{Spec} R = \mathbb{A}^{kr}$ with $k \times r$ matrices over F.

For each $0 \leq j \leq k-1$ let $\mathcal{R}_j \subseteq \mathbb{A}^{kr}$ be the closed subscheme defined by the vanishing of all of the $(j+1)\times(j+1)$ minors of X; this is the subscheme of matrices having rank at most j. Observe that we have $\mathcal{R}_{j-1} \subseteq \mathcal{R}_j$ for each j.

We will show that $\mathcal{R}_j$ is irreducible, has codimension $(k-j)(r-j)$, and that the singular set of $\mathcal{R}_j$ is precisely $\mathcal{R}_{j-1}$.

Matrix multiplication is a map $\mathbb{A}^{kj} \times \mathbb{A}^{jr} \to \mathbb{A}^{kr}$, and the image is precisely $\mathcal{R}_j$. Said differently, a linear transformation $F^r \to F^k$ has rank $\leq j$ if and only if it can be factored through F^j. Since $\mathbb{A}^{kj} \times \mathbb{A}^{jr}$ is irreducible, its scheme-theoretic image is also irreducible. But this image is precisely $\mathcal{R}_j$.

The idea of the codimension calculation is the following. For each $j \times j$ minor m let $U_m \subseteq \mathcal{R}_j$ be the open subscheme of matrices for which the minor m is nonzero. Then $\bigcup_m U_m = \mathcal{R}_j - \mathcal{R}_{j-1}$, and since $\mathcal{R}_j$ is irreducible this is a dense open subset of $\mathcal{R}_j$. So it suffices to show that each U_m has codimension $(k-j)(r-j)$ in $\mathbb{A}^{kr}$. By changing coordinates we can assume that m is in the upper left corner of the matrix X. The following schematic diagram shows that $U_m \cong GL_j \times \mathbb{A}^{j(r-j)} \times \mathbb{A}^{(k-j)j}$:



One is free to choose $P \in GL_j$ as well as all of the entries in the two blocks marked $**$, and then the entries of the final block are determined by the requirement that the rank is at most j. We leave it to the reader to write down formulas if they so desire. Since GL_j is an open subset of $\mathbb{A}^{j^2}$, we see that $\dim U_m = j^2 + j(r-j) + (k-j)j = kr - (k-j)(r-j)$, as we claimed.

Perhaps the best way to remember the codimension of $\mathcal{R}_j$ is to observe that in the above picture there are precisely $(k-j)(r-j)$ equations that determine the coefficients of the last block in terms of the others.

To analyze the singular set of $\mathcal{R}_j$, let $F_1, \ldots, F_N$ be the list of $(j+1)\times(j+1)$ minors of X, in some order. Then $R_j = V(F_1, \ldots, F_N)$ and $N = \binom{k}{j+1}\binom{r}{j+1}$. The key observation is that the coefficients of each dF_i are equal to $j \times j$ minors of X; this follows by looking at the cofactor expansion of the minor. If $A \in R_{j-1}$ then all such minors vanish, and so the Jacobian matrix $J(F)$ at A has all entries equal to zero. Therefore A is in the singular locus of R_j.

For the other direction, suppose that $A \notin R_{j-1}$. Then some $j \times j$ minor is nonzero. Up to a change of coordinates we can assume it is the minor in the upper left corner of the matrix. Let α be the value of this minor of A.

Let $d = (k-j)(r-j)$ and order the x's so that $x_1, \dots, x_d$ are in the lower right $(k-j) \times (r-j)$ submatrix of X. Let F_i be the $(j+1) \times (j+1)$ minor obtained by selecting the rows/columns 1 through j together with the row/column of x_i. Then the corresponding $d \times d$ submatrix of $J(F)$ at A consists of the scalar matrix αI_d, and this implies that $J(F)|_A$ has rank at least $d = \dim \mathcal{R}_j$. Hence, R_j is smooth at A.

For more information on determinantal varieties a good place to start is [Weyman].

Part 8. Appendices

Appendix A. Some point-set topology

Here we collect information on the point-set topology (beyond the elementary) that is needed in the body of the text. This material is standard enough that we mostly don't give references, but nonstandard enough that it is typically not included in introductory courses. We present the material as a series of exercises, since the proofs are either routine or else can be done with just a hint or two.

A.1. **Miscellany.**

Exercise A.2. Prove that a colimit of quotient maps is again a quotient map. That is, let I be a small category, let $X, Y \colon I \to \mathcal{T}op$ be diagrams, and let $q \colon X \to Y$ be a natural transformation such that each $q_i \colon X_i \to Y_i$ is a quotient map. Prove that $\operatorname{colim}_I q \colon \operatorname{colim}_I X \to \operatorname{colim}_I Y$ is a quotient map.

Exercise A.3. Let X be a CW-complex. Both the open cells in X and their closures are called "cells" by different authors; to clarify the difference we will refer to the latter as "cell-closures". Prove that the following two conditions are equivalent:

(1) Every point has a neighborhood that meets only finitely many cell-closures.
(2) Each cell-closure meets finitely many other cell-closures.

A CW-complex satisfying these properties is called **locally finite**. Prove that if such a complex is path-connected then it has a countable number of cells.

In this book "topological space" refers to the standard definition. Sometimes algebraic topologists work instead with a smaller subcategory that is better behaved in certain technical ways, but we do not do this here. The main thing to keep in mind when working with the category $\mathcal{T}op$ is that Cartesian products and function spaces are not always adjoint in the way one would like. If Z is a locally compact Hausdorff space then $(-) \times Z$ is left adjoint to $(-)^Z$ (see [Munkres, Theorem 46.11]), but without the assumptions on Z this is not necessarily the case.

Exercise A.4. Let $X \colon I \to \mathcal{T}op$ be a diagram of topological spaces and let Z be locally compact and Hausdorff. Prove that the natural map $\operatorname{colim}_i (X_i \times Z) \to (\operatorname{colim}_i X_i) \times Z$ is a homeomorphism.

A topological space X is a ***k*-space** if it has the following property: a subset $A \subseteq X$ is open if and only if for every compact Hausdorff space K and every continuous map $f \colon K \to X$, $f^{-1}(A)$ is open in K. The standard way to fix the failure of $\mathcal{T}op$ to have the necessary adjunctions is to instead use the category of k-spaces, or perhaps the category of weak Hausdorff k-spaces (also called "compactly-generated spaces"). While we will not follow that practice here, the notion of k-space will occasionally be useful.

Exercise A.5. Prove that a space X is a k-space if and only if it is a colimit of a diagram of compact Hausdorff spaces.

Suppose we have a sequence of spaces $X_1 \to X_2 \to \cdots$ and are studying the colimit X. A common situation in homotopy theory is that we have a map from a space K to the colimit and we want to know that it lifts back into some X_n. The following result covers many cases of interest:

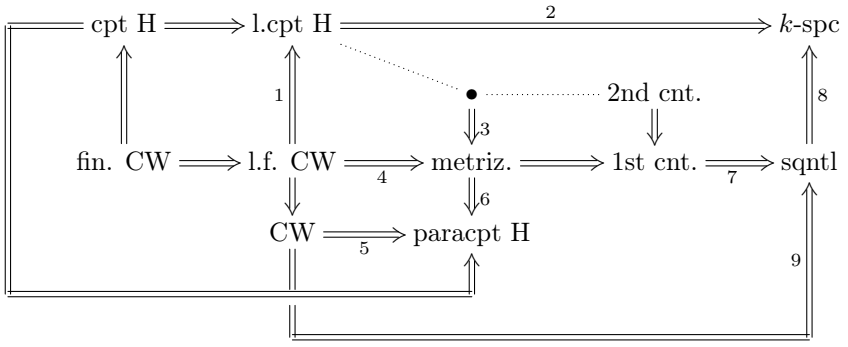
Exercise A.6. Let K be a compact space. For any sequence of closed inclusions $X_1 \hookrightarrow X_2 \hookrightarrow \cdots$ where all of the X_i are T_1-spaces, the natural map

$$\operatorname{colim} \mathcal{T}op(K, X_i) \to \mathcal{T}op(K, \operatorname*{colim}_i X_i)$$

is a bijection. (As an exercise this is slightly challenging as one has to hit upon just the right technique. If you get stuck, consult [Hovey, Proposition 2.4.2]).

The above result is often described by saying that compact spaces are small for closed inclusions of T_1-spaces.

A.7. **Niceness conditions.** As we just saw above, it is common in topology for results to involve various technical conditions on the underlying spaces. In bundle theory paracompactness is often needed, for example, and other conditions arise from time to time. The chart below shows relations between some of the properties that most frequently appear:



Legend: The abbreviations are for the following terms: finite CW-complex, locally-finite CW-complex, CW-complex, metrizable, paracompact Hausdorff, compact Hausdorff, locally compact Hausdorff, first and second countable, sequential, and k-space (for the notion of sequential space, see Section A.14 below). The bullet and dotted lines represent "and": locally compact Hausdorff and second countable together imply metrizable.

Exercise A.8. Justify (or chase down references for) all of the implications in the above diagrams. Some hints:

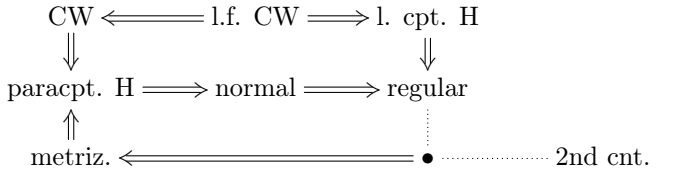
1. Exercise.
2. Exercise.
3. [Munkres, Theorems 29.2 and 40.3]
4. Path-connected locally finite CW-complexes are second countable (use Exercise A.3). Now use (3).

5. [LW, Section II.4] or [FP, Theorem 1.3.5] (but see also [FP, Proposition A.5.1] for a better perspective).
6. [Munkres, Theorem 41.4]
7. Exercise A.19(f) below.
8. Exercise A.19(k) below.
9. Use Exercise A.19(e,f) below.

For further references on some of these see [Geoghegan, Section 10.1] and [FP, Section 1.5].

Exercise A.9. Check that a countable wedge of circles is a CW-complex that is not locally compact and not first-countable.

Exercise A.10. Although we could only fit so much in the above diagram, a useful blow-up of the piece around metrizability is here:



Chase down references or proofs for the arrows that have not yet been fully explained.

A.11. **Semi-cofibrancy and ANRs.** In Section 9 we introduced the term *semi-cofibrant*, and explained that a space was semi-cofibrant if and only if it is homotopy equivalent to a CW-complex. The paper [Milnor3] connected this to the notion of ANR. A space X is said to be an **absolute neighborhood retract** (ANR) if it is metrizable and whenever Z is a metric space having a closed subspace X' that is homeomorphic to X, there exists an open set $X' \subseteq U$ and a retraction $U \to X'$. This concept seems almost unusable, but it turns out to be surprisingly robust:

- Every open subset of an ANR is also an ANR.
- The product of two ANRs is also an ANR.
- (Hanner's theorem) If X is paracompact Hausdorff and it has an open covering by ANRs, then X is an ANR.
- If $X = X_1 \cup X_2$ where X_1 and X_2 are closed, and X_1, X_2, and $X_1 \cap X_2$ are all ANRs, then so is X.

For the above properties see [Sakai, Proposition 6.2.9]. Another useful reference is [Hu], particularly Section III.7. ANRs were originally introduced by Borsuk (with a slightly different definition), and there is much literature in general topology devoted to them.

The following result is often attributed to [Milnor3]:

Proposition A.12. *Every ANR is semi-cofibrant.*

Technically, Milnor used a weaker definition of ANR than is used today. At the time, the common definition of ANRs included the conditions that both the space X and the space Z were second-countable (and this is why countable CW-complexes repeatedly appear in Milnor's paper). It was only later that the second-countable condition was gradually seen as unnecessary; see for example [Dugundji] for movement in that direction. Proposition A.12 using the modern notion is [MarSe, Appendix 1.2.2, Theorem 5].

We restate and prove Proposition 9.6:

Proposition A.13. *Let X be a topological space. If X satisfies any of the hypotheses below then X is an ANR and hence semi-cofibrant:*

(1) X is metrizable and semi-locally contractible and has finite covering dimension (note that any subspace of $\mathbb{R}^n$ satisfies the first and last condition).
(2) X is an open subset of a locally-finite CW-complex.
(3) X is paracompact Hausdorff and every point $x \in X$ has an open neighborhood that is homeomorphic to a space of type (1) or (2).

Proof. For (1) this was proven by Borsuk when X is additionally assumed to be compact (originally in the 1930s, but with a modernized version in [Borsuk, V.10.4]). The generalized version given here is [Dugundji, Theorem 15.1].

Every finite CW-complex is an ANR by (1). Therefore every open subset of a finite CW-complex is an ANR. Every point in a locally-finite complex has a neighborhood that is an open subset of a finite CW-complex, so by Hanner's theorem every locally-finite CW-complex is an ANR. Hence, every open subset of a locally-finite CW-complex is also an ANR.

A space satisfying (3) has an open cover where each set satisfies either (1) or (2). Now use Hanner's theorem. □

A.14. **Nets and sequences.** A **directed set** is a set I with a relation $\leq$ that is reflexive, transitive, and has the property that for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$. Note that some authors include antisymmetry of $\leq$ (e.g. [Munkres]), which better encapsulates our internal notion of "directed". But it turns out not to matter, as including it or not including it leads to equivalent theories in terms of what we are about to do.

Let X be a space. A **net** in X is a directed set I together with a function $x\colon \mathrm{ob}(I) \to X$. If $S \subseteq X$ say that x **eventually settles in** S if there exists an $i \in I$ such that $x_j \in S$ for all $j \geq i$. The net is said to have **limit** $L \in X$ if it eventually settles in every open neighborhood of L—that is, for every open set U containing L there is an $i \in I$ such that $x_j \in U$ for all $j \geq i$. We will say "$(x; L)$ is a convergent net in X" as a synonym for x being a net in X with limit L. Note that in non-Hausdorff spaces one can have convergent nets $(x; L)$ and $(x; L')$ with $L \neq L'$.

Exercise A.15. Recall that $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$ is the set of infinite sequences $(r_1, r_2, \ldots)$ of real numbers. Consider the following net in $\mathbb{R}^\omega$ with the box topology:

$$0,\ e_1,\ \tfrac{1}{2}e_2,\ \tfrac{1}{4}e_3,\ \ldots,\ 0,\ \tfrac{1}{2}e_1,\ \tfrac{1}{4}e_2,\ \tfrac{1}{8}e_3,\ \ldots,\ 0,\ \tfrac{1}{4}e_1,\ \tfrac{1}{8}e_2,\ \tfrac{1}{16}e_3,\ \ldots$$

This net is indexed on the ordinal $\omega \cdot \omega$. Prove that this net does not converge to 0.

Exercise A.16. Let C_n be $\{1, 2, \ldots, n\}$ with the ordering $1 \leq 2 \leq \cdots \leq n \leq 1$. This is a directed set. Prove that a non-constant net $x\colon C_n \to X$ converges only if X is non-Hausdorff.

Exercise A.17 (Crash course on nets)**.**

(a) Check that if $f\colon X \to Y$ is continuous and x is a net with limit L then $f \circ x$ is a net with limit $f(L)$. (Briefly, we say that continuous functions preserve convergent nets).
(b) Let $x \in X$ and consider the category of open neighborhoods of X where the maps are opposite arrows to inclusions. Check that this is directed.

(c) Let $A \subseteq X$. Prove that $u \in \bar{A}$ if and only if there is a net in A whose limit in X is u. That is, $\bar{A}$ is the set of limit points (in X) of nets in A.
(d) Prove that a subset $S \subseteq X$ is open if and only if all convergent nets $(x; L)$ with $L \in S$ eventually settle in S.
(e) Prove that a map $f\colon X \to Y$ is continuous if and only if it preserves convergent nets (as in (a)).
(f) If I is a directed set, define a new directed set $\hat{I} = I \cup \{\infty\}$ by making ∞ larger than every element of I. Make $\hat{I}$ a topological space by taking as basis all the singletons $\{i\}$ as well as the sets $[i, \infty] = \{j \in \hat{I} \mid i \leq j\}$, for $i \in I$. Note that any subset of $\hat{I}$ not containing ∞ is open, and a subset containing ∞ is open if and only if it contains some interval $[i, \infty]$. Prove that a net $x\colon I \to X$ has limit L if and only if the function $\hat{x}\colon \hat{I} \to X$ that extends x and has $\hat{x}(\infty) = L$ is continuous.
(g) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. Suppose that the two topologies have the same class of convergent nets: that is, for any net $x\colon I \to X$ and any point $L \in X$, the net converges to L in $\mathcal{T}_1$ if and only if it converges to L in $\mathcal{T}_2$. Then $\mathcal{T}_1 = \mathcal{T}_2$.
(h) Let X be a set and let $\mathcal{S}$ be a set of pairs (x, L) where x is a net in X and $L \in X$. Define a subset $U \subseteq X$ to be $\mathcal{S}$-open if it has the property that for every $(x, L) \in \mathcal{S}$ with $L \in U$, the net x settles in U. Prove that this defines a topology on X with the property that every element of $\mathcal{S}$ is a convergent net, and prove that it is the finest such topology. Likewise, if a topology on X has the property that every convergent net belongs to $\mathcal{S}$ then it is necessarily finer than this one.

A map between directed sets $f\colon J \to I$ is a function with the property that whenever $j_1 \leq j_2$ one has $f(j_1) \leq f(j_2)$. A map f is **final** if for every $i \in I$ there exists $j \in J$ such that $i \leq f(j)$. In this same setting one says that $f(J)$ is **cofinal** in I (the terminology is a bit confusing). Given a net $x\colon I \to X$, a **refinement** of x is a net of the form $x \circ f$ where J is a directed set and $f\colon J \to I$ is a final map. Refinements are also called **subnets** in the literature, though it is important to note that the map f need not be injective.

Given a net $x\colon I \to X$ and a subset $S \subseteq X$, say that the net **forever returns to** S if $\{i \in I \mid x_i \in S\}$ is cofinal in I—or equivalently, for every $i \in I$ there is a $j \geq i$ such that $x_j \in S$. Given a net $x\colon I \to X$, a point $w \in X$ is an **accumulation point** of x if the net forever returns to every neighborhood of w.

Exercise A.18. Given a net $x\colon I \to X$ and $w \in X$, prove that the following are equivalent:

(1) w is an accumulation point of x;
(2) There is a refinement of x that converges to w;
(3) $(w, \infty) \in \overline{(\mathrm{im}(x \times \mathrm{id}))}$ where $x \times \mathrm{id}\colon I \to X \times I$ sends $i \mapsto (x_i, i)$ and the closure takes place in $X \times \hat{I}$.

A **sequence** is a net indexed on the directed set $\mathbb{Z}_{\geq 0}$. Say that a subset $A \subseteq X$ is **sequentially open** if every convergent sequence $(x; L)$ with $L \in A$ eventually settles in A. Say that $A \subseteq X$ is **sequentially closed** if every convergent sequence $(x; L)$ with values in A has $L \in A$. Note that every open subset is sequentially open, and every closed subset is sequentially closed.

Say that a function $f\colon X \to Y$ is **sequentially continuous** is whenever $(x; L)$ is a convergent sequence in X then $(f(x); f(L))$ is a convergent sequence in Y. Every continuous function is sequentially continuous.

Exercise A.19. Let X be a space.

(a) Check that a subset of X is sequentially open if and only if its complement is sequentially closed. Prove that the sequentially open sets define a topology on X; we will denote this X_{seq}. The identity is a continuous map $X_{seq} \to X$.

(b) Prove that a function $f\colon X \to Y$ is sequentially continuous if and only if $f\colon X_{seq} \to Y$ is continuous.

(c) Prove that the following conditions on X are equivalent:
   (i) Every sequentially open subset is open.
   (ii) Every sequentially closed subset is closed.
   (iii) For every space Y, every sequentially continuous map $f\colon X \to Y$ is continuous.

   A topological space X satisfying the above conditions is called **sequentially determined**, or often just **sequential** for brevity.

(d) Prove that an open subset of a sequential space is sequential, and the same for closed subsets.

(e) Prove that any colimit of a diagram of sequential spaces is again sequential. In particular, any quotient space of a sequential space is sequential.

(f) Prove that any first-countable space is sequential (in particular, any metric space is sequential). Conclude that every locally-finite CW-complex is sequential.

(g) Suppose X has an open cover $\{U_\alpha\}$ where each U_α is sequential. Then X is sequential.

(h) Prove that a retract of a sequential space is sequential. In particular, if X and Y are nonempty and $X \times Y$ is sequential then so are X and Y.

(i) Let $J = \widehat{(\mathbb{Z}_{\geq 0})}$, as defined in Exercise A.17. Recall that a function $x\colon J \to X$ is continuous if and only if $x(\infty)$ is the limit of the sequence $n \mapsto x(n)$. Check that J is homeomorphic to $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\} \subseteq \mathbb{R}$ equipped with the subspace topology.

(j) For any space X consider the category $J \downarrow X$ whose objects are maps $J \to X$ and whose arrows are commutative triangles that involve an endomorphism of J. There is a canonical functor $(J \downarrow X) \to \mathcal{T}op$ sending each $f\colon J \to X$ to J; denote the colimit by $\operatorname{colim}_{J \to X} J$. There is a canonical map $\theta_X\colon \operatorname{colim}_{J \to X} J \to X$ that is readily checked to be a bijection. Prove that X is sequential if and only if θ_X is a homeomorphism.

(k) Prove that every sequential space is a k-space.

(l) Suppose that Z is a space that is locally compact Hausdorff and is such that $J \times Z$ is sequential (for example, the latter holds whenever Z is first-countable). Use the previous parts to show that if X is sequential then so is $X \times Z$, for any space X. (Hint: Use Exercise A.4). *Warning: In general, a product of two sequential spaces need not be sequential.*

While we are on the topic of convergent sequences, the following is very useful:

Exercise A.20. Let $X_1 \hookrightarrow X_2 \hookrightarrow \cdots$ be closed inclusions between T_1-spaces, and let X be the colimit. If $a_1, a_2, \ldots$ is a sequence in X that converges to a, then there

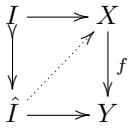
exists an N such that a and all of the a_i belong to X_N and the sequence converges to a in X_N. [Hint: Use the space J from the previous exercise, which is compact.]

A.21. **Separated maps.**

Exercise A.22.

(a) Let X be a topological space. Show that the following conditions are equivalent:
   (1) X is Hausdorff,
   (2) If a net in X converges to two points x_1 and x_2, then $x_1 = x_2$.
   (3) The diagonal $\Delta \subseteq X \times X$ is closed.

(b) Let $f\colon X \to Y$ be a continuous map. Show that the following conditions are equivalent:
   (1) If a net in X converges to two points x_1 and x_2 and $f(x_1) = f(x_2)$, then $x_1 = x_2$.
   (2) For any directed set I, any solid-arrow diagram



   has at most one lifting as shown.
   (3) The diagonal $\Delta \subseteq X \times_Y X$ is closed.

   Maps f satisfying these equivalent conditions are called **separated**.

(c) Observe that a space X is Hausdorff if and only if $X \to *$ is separated. Check that if X is Hausdorff then any map $X \to Y$ is separated.

A.23. **Proper maps.** Properness is a relative form of compactness, so let us begin by stating various equivalent conditions for a space to be compact.

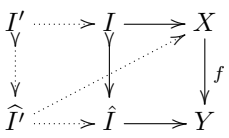
Exercise A.24. Let X be a topological space. Prove that the following conditions are equivalent:

(1) X is compact;
(2) Every net $x\colon I \to X$ has a convergent refinement (or equivalently, an accumulation point);
(3) For every space Z the projection $X \times Z \to Z$ is a closed map.

[Hint: (1)⇒(2)⇒(3) are fairly straightforward. For (2)⇒(1), given a cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ with no finite subcover construct a net indexed on the poset of finite subsets of $\mathcal{A}$. For (3)⇒(2), given a net $x\colon I \to X$ take $Z = \hat{I}$ from Exercise A.17 and use part (3) of Exercise A.18.]

Exercise A.25. Let $f\colon X \to Y$ be a map of topological spaces. Prove that the following are equivalent:

(1) Every net $x\colon I \to X$ whose image in Y converges to a point y has a refinement that converges to a point $x \in f^{-1}(y)$; said differently, every solid-arrow diagram



   admits an extension to the dotted arrow diagram, where $I' \to I$ is final.

(2) For every space Z and map $Z \to Y$ the pullback $X \times_Y Z \to Z$ is a closed map;

(3) For every space W the map $f \times \mathrm{id}\colon X \times W \to Y \times W$ is a closed map;
(4) f is a closed map and for every compact subset $K \subseteq Y$, the preimage $f^{-1}(K)$ is compact;
(5) f is a closed map and every fiber is compact.

[Outline: Prove that maps satisfying (1) are closed, and also that property (1) is stable under pullbacks. Then prove (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(1). These are mostly routine, except perhaps for two of them. For (3)⇒(4), if $x\colon I \to f^{-1}(K)$ is a net without an accumulation point in X then consider the closure of $\mathrm{im}(x \times \mathrm{id})$ in $X \times \hat{I}$, as in Exercise A.18. For (5)⇒(1), if x does not have an accumulation point in $f^{-1}(y)$ then every $w \in f^{-1}(y)$ has a neighborhood that is not forever returned to by x. Choose finitely many of these neighborhoods $U_1, \ldots, U_r$ that cover $f^{-1}(y)$ and then consider $f(X - (U_1 \cup \cdots \cup U_r))$.]

Definition A.26. *Let $f\colon X \to Y$ be a continuous map.*

*(a) f is **proper** if it satisfies any of the equivalent conditions listed in Exercise A.25.*
*(b) f is **strongly proper** if it is proper and separated, i.e. has the additional property that if a net in X converges to two points x_1 and x_2 and $f(x_1) = f(x_2)$ then $x_1 = x_2$.*
*(c) f is **weakly proper** if the preimage of every compact subset of Y is compact.*

Observe that X is compact if and only if $X \to *$ is proper (which is also equivalent to $X \to *$ being weakly proper), and X is compact Hausdorff if and only if $X \to *$ is strongly proper. Also note that in general one has

$$\text{strongly proper} \Rightarrow \text{proper} \Rightarrow \text{weakly proper}$$

and none of these arrows is reversible. If X is a compact non-Hausdorff space (e.g. the Sierpenski space) then $X \to *$ is proper but not strongly proper. For the other arrow, let X be $\{0, 1\}$ with the discrete topology and let Y be the same set with the indiscrete topology. The identity map $X \to Y$ is weakly proper but is not closed, and hence not proper. Hausdorff examples can be obtained by still considering two topologies on the same set X, for example $\mathbb{R}^\infty$ with the colimit and product topologies (see Section A.34).

Note that the terms "strongly proper" and "weakly proper" are not standard, and in fact all three notions are called "proper" in various places in the literature. Exercise A.27 below explains why for nice spaces it is okay to blur the distinction between them. Note that in algebraic geometry the term "proper" is used for the analog of "strongly proper". For one reference on this material see [Geoghegan, Section 10.1] (our "weakly proper" is there called proper, and our "proper" is called perfect).

Exercise A.27. Let $f\colon X \to Y$ be a map that is weakly proper. Prove that if Y is a Hausdorff k-space then f is proper, and if X is also Hausdorff then f is in fact strongly proper. [Hint: It suffices to prove that if $C \subseteq X$ is closed then $f(C)$ is closed. Since Y is a k-space this is equivalent to $g^{-1}(f(C))$ closed in K, for every continuous $g\colon K \to Y$ where K is compact Hausdorff. Play around with $f(C \cap f^{-1}g(K))$.]

Example A.28. Even weaker than "weakly proper" is the condition that a map have compact fibers. Take $X = \{(x, \frac{1}{x}) \mid x > 0\} \subseteq \mathbb{R}^2$ and consider the projection map $\pi_1\colon X \to \mathbb{R}$. This has compact (in fact singleton) fibers but is not weakly

proper. If $X' = X \cup \{(0,0)\}$ and $Y = [0,\infty)$ then the projection map $\pi_1 \colon X' \to Y$ is even a surjective map with compact fibers that is not weakly proper.

The following facts about proper maps are often useful:

Exercise A.29.

(a) A pullback of a proper map is proper.
(b) An injection is proper if and only if it is a closed embedding. [This is almost tauotologous].
(c) Let $f \colon X \to Y$ and $g \colon X \to Z$ be maps, and assume Z is Hausdorff. If f is weakly proper (resp., proper, strongly proper) prove that $(f,g) \colon X \to Y \times Z$ is also weakly proper (resp., proper, strongly proper).

A.30. **Comparing topologies.** For some reason the language mathematicians use for comparing two topologies on a common set is not universally agreed-upon, and also hard to remember. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set S, the following are equivalent statements:

- $\mathcal{T}_1$ has fewer open sets than $\mathcal{T}_2$, i.e. $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$
- The identity map $S_{\mathcal{T}_2} \to S_{\mathcal{T}_1}$ is continuous.
- $\mathcal{T}_2$ is finer than $\mathcal{T}_1$
- $\mathcal{T}_2$ is a refinement of $\mathcal{T}_1$.

Analysts also use "$\mathcal{T}_2$ is stronger than $\mathcal{T}_1$" or "$\mathcal{T}_1$ is weaker than $\mathcal{T}_2$" as equivalent to the above conditions, but unfortunately topologists often reverse the role of weak/strong here. For example, the "W" in "CW-complex" stands for what topologists call the weak topology on an ascending union, which is finer than other topologies one might consider. So the situation is confusing, and as a result we will try to avoid the weak/strong language. (See [Munkres, Section 12] for a similar remark).

If one thinks of a topology as being like a way of distinguishing objects via characteristics—where the open sets are the characteristics—then the fine/coarse terms match up with common usage. A finer topology allows more distinctions, and so has more open sets. It is useful to remember the slogans

$$\text{finer} = \text{more open sets} = \text{easier to map out of.}$$

Finer topologies map to coarser ones.

Example A.31. Recall that if $\{X_\alpha\}$ is a collection of topological spaces then on $\prod_\alpha X_\alpha$ we have the box and product topologies. A basis for the box topology consists of rectangles $\prod U_\alpha$ where each U_α is open in X_α, whereas for the product topology we take rectangles where only finitely many of the U_α differ from X_α. So the box topology has more open sets than the product topology, hence the box topology is the finer one.

Example A.32. More generally, the topology on a limit is the "coarsest topology such that...", whereas the topology on a colimit will be the "finest topology such that...". This can be deduced from the direction of the arrows, remembering always that finer topologies map to coarser ones. In the point-set topology literature the colimit topology is often called the "final topology", and the limit topology the "initial topology", which doesn't help matters; we will not use those terms.

Another way to understand the comparison between two topologies is in terms of the convergent nets. Continuous maps preserve convergent nets, so if $\mathrm{id}\colon S_{\mathcal{T}_2} \to S_{\mathcal{T}_1}$ is continuous and a net $\{x_i\}$ converges to x_∞ in $\mathcal{T}_2$ then it also converges in $\mathcal{T}_1$. So coarser topologies have more convergent nets than finer topologies. In the indiscrete topology everything converges to everything.

Example A.33. Let $\{X_\alpha\}$ be a collection of topological spaces, and consider the box and product topologies on the product $\prod_\alpha X_\alpha$. Write π_α for the projection onto X_α. Then a net $t\colon I \to \prod_\alpha X_\alpha$ is convergent to T in the product topology if and only if $\pi_\alpha(t)$ converges to $\pi_\alpha(T)$ in X_α, for all α.

It is more complex to analyze the situation in the box topology. The net t is convergent to T in the box topology if for every collection of opens U_α containing $\pi_\alpha(T)$, there exists $i \in I$ such that for all $j \geq i$ we have $\pi_\alpha(t_j) \in U_\alpha$ for all α. This is a very strong condition. As an example, let $\{y_i\}$ be a sequence in $\mathbb{R}$ that converges to a number y. Write $\Delta(y)$ for the diagonal element $(y, y, \ldots)$ in $\prod_{i=1}^\infty \mathbb{R}$. Then $\Delta(y_i)$ converges to $\Delta(y)$ in the box topology means that for every countable collection $U_1, U_2, \ldots$ of neighborhoods of y, there is an N such that for all $k \geq N$ one has $y_k \in \bigcap_i U_i$. Since we can arrange for the infinite intersection to be precisely $\{y\}$, this means that $\Delta(y_i)$ converges to $\Delta(y)$ only when the sequence $\{y_i\}$ is eventually constant. More generally, if all the spaces X_α are Hausdorff and if $\{t_i\}$ is a sequence in $\prod X_\alpha$ that converges to T in the box topology, then one can show that there exists a finite set of indices S and a k such that for all $i \geq k$, t_i agrees with T at all coordinates except possibly the ones in S. In other words, the sequence t is eventually constant in all but finitely-many coordinates. This is a very strong condition on the sequence t.

A.34. $\mathbb{R}^\infty$ **with the colimit topology.** Recall that $\mathbb{R}^\infty$ consists of tuples $(x_1, x_2, \ldots)$ with only finitely many nonzero coordinates. As a set we can write

$$\mathbb{R}^\infty = \operatorname*{colim}_n \mathbb{R}^n$$

where $\mathbb{R}^n$ includes into $\mathbb{R}^{n+1}$ as the subset of tuples whose last coordinate is zero. There are at least four topologies that naturally suggest themselves:

- Give each $\mathbb{R}^n$ its standard topology, and then give $\mathbb{R}^\infty$ the induced topology on the colimit. Denote this as $\mathbb{R}^\infty_{\mathrm{colim}}$.
- Regard $\mathbb{R}^\infty \subseteq \prod_{i=1}^\infty \mathbb{R}$ and give $\mathbb{R}^\infty$ the subspace topology induced from the product topology. Denote this as $\mathbb{R}^\infty_{\mathrm{prod}}$.
- Same as above, but this time give $\mathbb{R}^\infty$ the subspace topology induced from the box topology. Denote this by $\mathbb{R}^\infty_{\mathrm{box}}$.
- The metric topology $\mathbb{R}^\infty_{\mathrm{metric}}$, induced by the usual definition of distance between vectors.

The universal property of the colimit shows that $\mathbb{R}^\infty_{\mathrm{colim}}$ will be finer than the other three topologies. In fact the identity map gives us

$$\mathbb{R}^\infty_{\mathrm{colim}} \longrightarrow \mathbb{R}^\infty_{\mathrm{box}} \longrightarrow \mathbb{R}^\infty_{\mathrm{metric}} \longrightarrow \mathbb{R}^\infty_{\mathrm{prod}}.$$

All of these topologies have the property that if U is open than so is any translate $U + x$ for $x \in \mathbb{R}^\infty$, so we can check continuity by looking at open sets around the origin. For the last map it suffices to observe that an open rectangle $B(0, r_1) \times \cdots \times B(0, r_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$ contains $B(\underline{0}, r)$ where r is the minimum of the r's (or

alternatively, the projection maps are continuous for the metric topology). For the second map we observe that $B(\underline{0}, \epsilon)$ contains the product

$$B(0, \tfrac{\epsilon}{2}) \times B(0, \tfrac{\epsilon}{4}) \times B(0, \tfrac{\epsilon}{8}) \cdots$$

by a routine computation.

To see that the last three topologies are all different we can observe that:

- $B(\underline{0}, 1)$ is open in $\mathbb{R}^\infty_{\text{metric}}$ but not in $\mathbb{R}^\infty_{\text{prod}}$;
- $B(0,1) \times B(0, \frac{1}{2}) \times B(0, \frac{1}{4}) \times \cdots$ is open in $\mathbb{R}^\infty_{\text{box}}$ but not in $\mathbb{R}^\infty_{\text{metric}}$.

Or alternatively,

- The sequence e_1, $\frac{1}{\sqrt{2}}(e_1 + e_2)$, $\frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)$, ... converges to $\underline{0}$ in $\mathbb{R}^\infty_{\text{prod}}$ but not in $\mathbb{R}^\infty_{\text{metric}}$;
- The sequence e_1, $\frac{1}{2}e_2$, $\frac{1}{3}e_3$, ... converges to $\underline{0}$ in $\mathbb{R}^\infty_{\text{metric}}$ but not in $\mathbb{R}^\infty_{\text{box}}$.

The reader will have noticed that we have not distinguished the colimit topology from the box topology. That is because they are the same!

Proposition A.35. *The colimit and box topologies on $\mathbb{R}^\infty$ are identical.*

Proof. We need to prove that for every open set U in the colimit topology and every $u \in U$, there is an open set V in the box topology such that $u \in V \subseteq U$. Note that for any $w \in \mathbb{R}^\infty$ the map $(-) + w \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous in either topology, and therefore a homeomorphism in either topology (the inverse being addition by $-w$). So it suffices to assume $u = 0$.

Note the following easily-verified statement:

(*) If $R \subseteq \mathbb{R}^{n-1}$ is a closed rectangle and $V \subseteq \mathbb{R}^n$ is an open set containing $R \times \{0\}$, then there is an open interval J containing 0 such that $R \times \overline{J} \subseteq V$.

This is essentially the standard 'tube lemma', proven using the compactness of R.

$U \cap \mathbb{R}^1$ is an open neighborhood of 0, so there is an open interval M_1 such that $0 \in M_1$ and $\overline{M_1} \subseteq U \cap \mathbb{R}^1$. If we have constructed open intervals $M_1, \ldots, M_k \subseteq \mathbb{R}$, all containing 0, such that $\prod_{i=1}^k \overline{M_i} \subseteq U \cap \mathbb{R}^k$, then using (*) we can produce M_{k+1} with the analogous properties. So by induction we have the existence of such M_i for all $i \geq 1$.

Set $V = \mathbb{R}^\infty \cap (\prod_{i=1}^\infty M_i)$, which is open in the box topology and contains 0. We claim that $V \subseteq U$. If $x \in V$ then $x \in \mathbb{R}^N$ for some N, and so $x \in \prod_{i=1}^N M_i \subseteq U \cap \mathbb{R}^N$. Hence $x \in U$. □

While we are talking about $\mathbb{R}^\infty_{colim}$ let us record the following important fact:

Proposition A.36. *If a sequence $x \colon \mathbb{N} \to \mathbb{R}^\infty_{\text{colim}}$ is convergent with limit L then there exists a $k \geq 1$ such that $L \in \mathbb{R}^k$, $x_i \in \mathbb{R}^k$ for all $i \geq 0$, and the limit of x in $\mathbb{R}^k$ equals L.*

Proof. This follows by Exercise A.20. □

Exercise A.37. A standard property of $\mathbb{R}^\infty_{colim}$ is that if K is a compact space then any map $K \to \mathbb{R}^\infty_{colim}$ factors through some $\mathbb{R}^N$. Prove that this property does not hold for $\mathbb{R}^\infty_{metric}$. [Hint: Use $K = \widehat{(\mathbb{Z}_{\geq 0})}$, as defined in Exercise A.17].

Exercise A.38. Prove that $\mathbb{R}^\infty_{colim}$ is not first-countable and is not locally compact. Prove that $\mathbb{R}^\infty_{metric}$ is first-countable but not locally compact, and likewise for $\mathbb{R}^\infty_{prod}$. Check that all three spaces are sequential.

Exercise A.39. Let $S^\infty \subseteq \mathbb{R}^\infty$ be the set of norm 1 vectors, under the usual Euclidean norm.

(a) Verify that the subspace topologies $S^\infty_{colim} = S^\infty_{box}$, S^∞_{metric}, S^∞_{prod} are all different.
(b) Show that S^∞ is closed in the colimit and metric topologies, but not in the product topology.

Appendix B. Topological vector spaces and tame families

We start by reviewing some standard material about topological vector spaces over $\mathbb{R}$. A classical theorem of Tychonoff says that there are only two topological vector spaces whose underlying vector space is $\mathbb{R}^n$: $\mathbb{R}^n$ with the standard topology and $\mathbb{R}^n$ with the indiscrete topology. Because of this, if V is any finite-dimensional real vector space it makes sense to talk about the **standard topology** on V: it is the topology induced by any linear isomorphism $\mathbb{R}^n \to V$ (all such isomorphisms give the same topology).

For vector spaces of countably infinite dimension things become more complicated:

Exercise B.1. Verify that the vector sum and scalar multiplication maps $\mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{+} \mathbb{R}^\infty$ and $\mathbb{R}\times\mathbb{R}^\infty \to \mathbb{R}^\infty$ are continuous when $\mathbb{R}^\infty$ is given any of the colimit, metric, or product topologies. [Hint: To check that the vector sum is continuous, check that if $x, y\colon I \to \mathbb{R}^\infty$ are nets that converge to a and b then the net $i \mapsto x_i + y_i$ converges to $a + b$. Similar for scalar multiplication.]

One of the deficiencies of both $\mathbb{R}^\infty_{metric}$ and $\mathbb{R}^\infty_{prod}$ is that linear functionals need not be continuous. To see this, consider the linear functional $\phi\colon \mathbb{R}^\infty \to \mathbb{R}$ given by $\underline{x} \mapsto x_1 + x_2 + x_3 + \cdots$. The sequence $s_i = \frac{1}{n}\sum_{i=1}^n e_i$ converges to 0 in both $\mathbb{R}^\infty_{metric}$ and $\mathbb{R}^\infty_{prod}$ but its image under ϕ does not converge to 0.

Definition B.2. *Let V be a topological vector space. Say that V is **reasonable** if every linear functional $\phi\colon V \to \mathbb{R}$ is continuous.*

Exercise B.3. Prove that a reasonable topological vector space is Hausdorff, and conclude that every finite-dimensional subspace is isomorphic to $\mathbb{R}^n$ with the standard topology.

Proposition B.4. *Let V be a reasonable topological vector space. Let $F \subseteq V$ be a finite-dimensional subspace with a chosen complement W. Then the associated projection maps $\pi_F\colon V \to F$ and $\pi_W\colon V \to W$ are continuous.*

Proof. By condition (1) of being reasonable, F is isomorphic to $\mathbb{R}^n$ with the product topology. Composing with this isomorphism, π_F becomes a linear map $V \to \mathbb{R}^n$. Every component of this map is a linear functional, hence continuous by condition (2). So π_F is continuous.

If $j\colon F \hookrightarrow V$ is the inclusion then the map π_W is $\mathrm{id} - j\pi_F$, hence continuous. □

It turns out that there are still many different topologies on $\mathbb{R}^\infty$ that make it into a reasonable topological vector space. We will not go into detail about any of

these, instead referring the interested reader to textbooks on the subject (but see Exercise B.10 below to whet your appetite). Let us work on adding more conditions that try to capture what is unique about $\mathbb{R}^\infty_{colim}$.

Let V be a topological vector space over $\mathbb{R}$. The finite-dimensional subspaces of V, with inclusions, make up a directed category and we have a canonical map

$$\operatorname*{colim}_{J^{fin.dim}\subseteq V} J \longrightarrow V$$

where the colimit is taken in the category of sets. This is a bijection. If we equip each J with the standard topology and the map is a homeomorphism, we refer to V as a **standard** topological vector space.

Example B.5. $\mathbb{R}^n$ with the usual topology is a standard topological vector space, as is $\mathbb{R}^\infty$ with the colimit topology. For the latter, consider the diagram

$$\begin{array}{ccc} \operatorname*{colim}_{J^{fin.dim}\subseteq\mathbb{R}^\infty} J & \longrightarrow & \mathbb{R}^\infty_{\text{colim}}. \\ \uparrow & \nearrow_{\cong} & \\ \operatorname*{colim}_n \mathbb{R}^n & & \end{array}$$

The subspaces $\mathbb{R}^n$ are cofinal in the category of all finite-dimensional subspaces of $\mathbb{R}^\infty$, and so the vertical map is a homeomorphism. Hence the horizontal map is a homeomorphism as well.

Exercise B.6. Check that any standard topological vector space is reasonable.

Proposition B.7. *Let V and W be standard topological vector spaces.*

(a) Every linear map $V \to W$ is continuous.

(b) If V and W have the same dimension (i.e. bases of the same cardinality) then V is isomorphic to W as topological vector spaces.

Proof. If J is finite-dimensional and standard then any linear map $f\colon \mathbb{R}^n \to W$ factors through a finite-dimensional subspace, which is isomorphic as tvs to some $\mathbb{R}^k$. The map $\mathbb{R}^n \to \mathbb{R}^k$ is continuous, as is the inclusion $\mathbb{R}^k \hookrightarrow W$, so f is continuous.

Since a linear map $L\colon V \to W$ is continuous when restricted to every finite-dimensional subspace, it is continuous. This proves (a). Part (b) is an immediate corollary, since a linear isomorphism and its inverse will both be continuous. □

From now on whenever we write $\mathbb{R}^\infty$ we mean for this to be equipped with its standard (i.e. colimit) topology, unless otherwise noted.

Proposition B.8. *For every $n \geq 1$ the topological vector space $(\mathbb{R}^\infty)^n$ is isomorphic (as topological vector spaces) to $\mathbb{R}^\infty$. In particular, the underlying topological spaces are homeomorphic.*

Proof. It suffices to prove this when $n = 2$. Let $\phi\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the map $(\underline{a}, \underline{b}) \mapsto (a_1, b_1, a_2, b_2, \ldots)$. It is immediate that this is an isomorphism of vector spaces, but we need to think about the topology aspect. Consider the maps $f\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $e_i \mapsto e_{2i-1}$ and $g\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $e_i \mapsto e_{2i}$. These are continuous by Proposition B.7. The map ϕ is the composite

$$\mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{f\times g} \mathbb{R}^\infty \times \mathbb{R}^\infty \xrightarrow{+} \mathbb{R}^\infty.$$

so ϕ is continuous.

The inverse ϕ^{-1} is a linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty \times \mathbb{R}^\infty$. The projections to the two factors are continuous by Proposition B.7, therefore ϕ^{-1} is continous. □

Corollary B.9. *The evident map* $\operatorname{colim}_k(\mathbb{R}^k)^n \to (\mathbb{R}^\infty)^n$ *is a homeomorphism.*

Proof. The map is evidently a continuous bijection, but it is not immediate that the inverse is continuous. This does not seem to follow purely from point-set topology, due to the usual incompatibilities between products and colimits.

However, Proposition B.8 implies that $(\mathbb{R}^\infty)^n$ is the colimit of its finite-dimensional subspaces. The family of subspaces $(\mathbb{R}^k)^n$ is cofinal in these, and so the result follows. □

Exercise B.10. Let V be a vector space. Define a topology $\sigma(V, V^*)$ on V to be the one associated to the class of nets $(x; L)$ having the property that $(f(x); f(L))$ is convergent in $\mathbb{R}$ for every linear functional $f\colon V \to \mathbb{R}$ (see Exercise A.17).

(a) Prove that V is a topological vector space and that V is reasonable.
(b) Prove that the standard topology on V is finer than the $\sigma(V, V^*)$ topology.
(c) For each functional $f \in V^*$ and each $\epsilon > 0$, let $W(f, \epsilon) = \{x \in V : |f(x)| < \epsilon\}$. Prove that each $W(f, \epsilon)$ is open in $\sigma(V, V^*)$ and that these sets and their translates form a subbasis for the topology. [Hint: Let T be the topology generated by the $W(f, \epsilon)$ sets and their translates. Prove that the convergent nets for T are exactly the pairs $(x; L)$ for which $(f(x); f(L))$ is convergent in $\mathbb{R}$ for every $f \in V^*$.]
(d) When $V = \mathbb{R}^\infty$ prove that every neighborhood of 0 in the $\sigma(V, V^*)$ topology contains elements of arbitrarily large Euclidean norm.
(e) When $V = \mathbb{R}^\infty$ prove that the standard topology is *strictly* finer than the $\sigma(V, V^*)$ topology. [Hint: Let $V = \mathbb{R}^\infty$ as sets. Let $U = B(\underline{0}, 1)$, the open ball defined using the Euclidean topology. Prove this is open in $\mathbb{R}^\infty_{colim}$ but not in $\sigma(V, V^*)$.]

Exercise B.11. Let V be a topological vector space and let $n \geq 1$.

(a) Prove that every linear map $\mathbb{R}^n \to V$ is continuous.
(b) Give $\operatorname{Hom}(\mathbb{R}^n, V)$ the compact-open topology—that is, topologize it as a subspace of the function space $V^{(\mathbb{R}^n)}$ with the compact-open topology. Then the evaluation map $\operatorname{Hom}(\mathbb{R}^n, V) \to V^n$ given by $L \mapsto (L(e_1), \ldots, L(e_n))$ is continuous. Prove that it is a homeomorphism. [Hint: Use an adjunction to get the inverse.]

B.12. **Tame families of vector spaces.** In this section we will prove Proposition 10.19. If $E \to X$ is a family of vector spaces of finite rank, this says that E is tame if either (a) E is a subfamily of a trivial family, or (b) X is locally compact and E is Hausdorff. The proofs in the two cases are quite different.

Lemma B.13. *Every trivial family of vector spaces of finite rank is tame.*

Proof. Let X be a space, and let $s_1, \ldots, s_n$ be a local weak basis of $E = X \times \mathbb{R}^n$ defined on an open neighborhood U of a point $x \in X$. Write S_i for the composite

$$U \xrightarrow{s_i} E|_U \xrightarrow{\pi_2} \mathbb{R}^n.$$

We can regard $S_1, \ldots, S_n$ as giving a map $F\colon U \to M_{n\times n}(\mathbb{R})$, where the columns are the S_i's. Since $S_1(y), \ldots, S_n(y)$ is a basis for $\mathbb{R}^n$, for every $y \in U$, the image

of F actually lies in $GL_n(\mathbb{R})$. The inverse map $J\colon GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ sending $A \to A^{-1}$ is continuous (a formula is given using determinants), so we can consider the composite

$$U \times \mathbb{R}^n \xrightarrow{S} E|_U \xrightarrow{\mathrm{id}\times JF} U \times \mathbb{R}^n$$

where the second map is $(y, v) \mapsto (y, JF(y)(v))$. This composite is the identity. It follows that the map S is a homeomorphism, which is what we wanted to prove. □

Proof of Proposition 10.19 in case (a). Assume that E is a subfamily of $X \times V$ where V is either $\mathbb{R}^n$ or $\mathbb{R}^\infty_{\mathrm{colim}}$. Let $x \in X$ and let $s_1, \ldots, s_n$ be a local weak basis defined on an open neighborhood U of x. Let S_i be the composite $U \xrightarrow{s_i} E \hookrightarrow X \times V \xrightarrow{\pi_2} V$. The vectors $S_1(x), \ldots, S_n(x)$ are linearly independent, so we can choose a basis $\{f_\alpha\}_{\alpha\in A}$ for V containing these vectors. Let $\beta\colon V \to \mathbb{R}^n$ be the linear map sending each $S_i(x)$ to e_i and all the other f_α to 0. If V is finite-dimensional then β is obviously continuous, whereas if $V = \mathbb{R}^\infty_{\mathrm{colim}}$ the continuity follows from Corollary **??**. We will also write β for the associated map $X \times V \to X \times \mathbb{R}^n$.

Consider now the composites $U \xrightarrow{S_i} U \times V \xrightarrow{\beta} U \times \mathbb{R}^n \xrightarrow{\pi_2} \mathbb{R}^n$. Taken together, they are the columns of a map $F\colon U \to M_{n\times n}(\mathbb{R})$, and we can further consider the composite

$$U \to M_{n\times n}(\mathbb{R}) \xrightarrow{\det} \mathbb{R}.$$

Let $W \subseteq U$ be the preimage of $\mathbb{R} - 0$, which is open and contains x. As in Lemma B.13 consider the composite

$$W \times \mathbb{R}^n \xrightarrow{S} E|_W \hookrightarrow W \times V \xrightarrow{\beta} W \times \mathbb{R}^n \xrightarrow{\mathrm{id}\times JF} W \times \mathbb{R}^n$$

where $J\colon GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is the inverse map. The composition is the identity, therefore S is a homeomorphism, and so $s_1, \ldots, s_n$ is a local strong basis on W. □

The proof for case (b) requires some technical preliminary work. A classical result from the theory of topological vector spaces, originally due to Tychonoff, says that if an n-dimensional topological vector space V is T_1 (points are closed) then $V \cong \mathbb{R}^n$ as topological vector spaces. We take the techniques from that proof and generalize them to families of vector spaces.

Let $E \to X$ be a family of vector spaces and let $W \subseteq E$ be any subset. If $S \subseteq \mathbb{R}$ then write SW for the image of W under the composite $S \times E \hookrightarrow \mathbb{R} \times E \longrightarrow E$. When $S = \{\lambda\}$ we just write λW for $\{\lambda\}W$.

Note that if $\lambda \neq 0$ then the multiplication-by-λ map $E \to E$ is continuous, as is multiplication by λ^{-1}. So these are in fact homeomorphisms, and therefore λW is open whenever W is. In a similar vein, if $s\colon X \to E$ is a section then we can consider the map $+s\colon E \to E$. This is continuous, being the composite

$$E = E \times_X X \xrightarrow{\mathrm{id}\times s} E \times_X E \xrightarrow{+} E.$$

But $-s\colon E \to E$ is also continuous, and is the inverse to $+s$; so again, these are both homeomorphisms. Thus, if W is open in E then so is the set

$$s + W = \{s(x) + w_x \,|\, x \in X, w_x \in E_x \cap W\}.$$

We let $z\colon X \to E$ denote the zero-section. Note that this is a continous map, by the definition of family of vector spaces.

Definition B.14. *Let $E \to X$ be a family of vector spaces. A subset $W \subseteq E$ is called* ***balanced*** *if $[-1, 1]W \subseteq W$.*

Proposition B.15. *Let $E \to X$ be a family of vector spaces. Let U be an open set containing the point $z(x)$, for some $x \in X$. Then there is a balanced open set V such that $z(x) \in V \subseteq U$.*

Proof. Scalar multiplication $\mathbb{R} \times E \to E$ is continuous and $(0, z(x)) \mapsto z(x)$. So there is an $\epsilon > 0$ and an open set W containing $z(x)$ such that $(-\epsilon, \epsilon)W \subseteq U$. Let J be the set of all points v in U having the property that $[-1,1]v \subseteq U$. Then $\frac{\epsilon}{2}W \subseteq J \subseteq U$. Since $\frac{\epsilon}{2}W$ is open and contains $z(x)$, this shows that J contains a neighborhood of $z(x)$. Note also that $[-1,1]J \subseteq J$ by virtue of the definition of J.

Let M be the interior of J. Then M is an open set containing $z(x)$, and $M \subseteq J \subseteq U$. If $0 < |\lambda| \leq 1$ then $\lambda M \subseteq \lambda J \subseteq J$. But λM is open, so in fact $\lambda M \subseteq M$. Note that we only have this for $\lambda \neq 0$, though, so we cannot yet conclude that M is balanced.

For the final step, let $T = z^{-1}(M)$. Since z is continuous, this is an open subset of X containing x. Consider $N = M \cap p^{-1}(T)$. This is open, contains $z(x)$, and is contained in U. For λ nonzero in $[-1,1]$ we have that $\lambda N \subseteq N$ because the same is true for both M and $p^{-1}(T)$. For $v \in N$ we have that $p(v) \in T$, and so $0 \cdot v = z(p(v)) \in M$. Then $0 \cdot v \in N$, thus $0N \subseteq N$ and so N is balanced. □

Proposition B.16. *Let $E \to X$ be a family of vector spaces of finite rank n. If E is Hausdorff and X is locally compact, then any weak basis is a strong basis.*

Proof. The weak basis gives us a continuous bijection $S\colon X \times \mathbb{R}^n \to E$, and we must show that S^{-1} is continuous. So for every open subset $\Omega \subseteq X \times \mathbb{R}^n$ we must prove that $S(\Omega)$ is open in E. Let $y \in S(\Omega)$, so that $y = S(x,u)$ for some $x \in X$ and $u \in \mathbb{R}^n$ with $(x,u) \in \Omega$. There is an open set $x \in U$ and an $\epsilon > 0$ such that $U \times B(u,\epsilon) \subseteq \Omega$, where $B(u,\epsilon)$ denotes the evident open ball.

We first treat the case where $u = 0$. Since X is locally compact, there is a compact set K that contains an open neighborhood W of x. By replacing W with $W \cap U$ we can assume $W \subseteq U$. Let S_ϵ be the sphere in $\mathbb{R}^n$ of radius ϵ, and consider $C = S(K \times S_\epsilon)$. Since $K \times S_\epsilon$ is compact, C is also compact. Since E is Hausdorff, C is therefore closed in E. Observe that the set $S(W \times B(0,\epsilon))$ does not intersect C, since S is a bijection. In particular, $z(x) = S(x,0) \notin C$.

Now apply Proposition B.15 to $z(x) \in E - C$, noting that $E - C$ is open. So there is a balanced open set V of E that contains $z(x)$ and is such that $V \subseteq E - C$. Let $V' = p^{-1}(W) \cap V$, which is still open and balanced. Since V' is balanced and does not intersect C we must have $V' \subseteq S(W \times B(0,\epsilon))$. Indeed, if $v \in V'$ and $v \notin S(W \times B(0,\epsilon))$ then $v = S(w,t)$ for some $w \in W$ and $t \in \mathbb{R}^n$ with $|t| \geq \epsilon$. Since $V \subseteq E - C$, we in fact have $|t| > \epsilon$. But since V' is balanced we must then have $\frac{\epsilon}{|t|}v \in V'$, which contradicts $V \subseteq E - C$ since this vector is $S(w, \frac{\epsilon}{|t|}t)$.

So we have $z(x) \in V' \subseteq S(W \times B(0,\epsilon)) \subseteq S(U \times B(0,\epsilon)) \subseteq S(\Omega)$. Thus, $S(\Omega)$ contains an open neighborhood of $y = z(x)$.

Now consider the general case where $u \neq 0$. Let $k\colon X \to E$ be the section $k(a) = S(a,u)$ for $a \in X$. Then $-k + S(U \times B(u,\epsilon)) = S(U \times B(0,\epsilon))$ and this set contains $z(x) = s(x,0)$. By the case treated already, there is an open set V of E such that $z(x) \in V \subseteq S(U \times B(0,\epsilon))$. Then $k + V$ is an open set such that $y \in k + V \subseteq S(U \times B(u,\epsilon)) \subseteq S(\Omega)$, and again we have proven that $S(\Omega)$ contains an open neighborhood of y. Since this holds for all y, we have that $S(\Omega)$ is open. □

Proof of Proposition 10.19 in case (b). This follows immediately from Proposition B.16, using that an open subset of a locally compact space is again locally compact. □

Appendix C. Homotopically compact pairs

By a **pair of topological spaces** we mean an ordered pair (X, A) where A is a subspace of A. A map of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$, and such a map is said to be a weak equivalence if both $f\colon X \to Y$ and $f|A\colon A \to B$ are weak equivalences. Two maps $f, g\colon (X, A) \to (Y, B)$ are said to be homotopic if there is a map $H\colon (X \times I, A \times I) \to (Y, B)$ such that $H|_{X\times 0} = f$ and $H|_{X\times 1} = g$.

Define a topological space X to be **homotopically compact** if it is weakly equivalent to a finite CW-complex. Likewise, define a pair of topological spaces (X, A) to be homotopically compact if there exists a finite CW-pair (X', A') and a weak equivalence $(X', A') \to (X, A)$. In this case we call (X', A') a **finite model** for (X, A).

Proposition C.1. *A pair (X, A) is homotopically compact if and only if both X and A are homotopically compact.*

Proof. The "only if" direction is trivial, and the other direction is an immediate consequence of the slightly more general lemma below. □

Lemma C.2. *Let $f\colon A \to X$ be a map, where both A and X are homotopically compact. Let $\gamma_a\colon \widetilde{A} \to A$ be any finite model for A. Then there exists a finite CW-complex $\widetilde{X}$, containing $\widetilde{A}$ as a subcomplex, together with a weak equivalence $\widetilde{X} \to X$ such that the square*

$$\begin{array}{ccc} \widetilde{A} & \rightarrowtail & \widetilde{X} \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_X} \\ A & \xrightarrow{f} & X \end{array}$$

commutes.

Proof. Let $\gamma_X\colon \widetilde{X} \to X$ be any finite model X. Since $[\widetilde{A}, \widetilde{X}] \to [\widetilde{A}, X]$ is a bijection, there is a map $\tilde{f}\colon \widetilde{A} \to \widetilde{X}$ such that $\gamma_X \tilde{f} \simeq f\gamma_A$. By cellular approximation, we may assume that $\tilde{f}$ is cellular. Choose such a homotopy. Let C_X denote the mapping cylinder of $\tilde{f}$, and let $\gamma_C\colon C_X \to X$ be the evident map. Note that γ_C gives a finite model for X, and that the diagram

$$\begin{array}{ccc} \widetilde{A} & \rightarrowtail & C_X \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ A & \xrightarrow{f} & X \end{array}$$

commutes. Here $\widetilde{A} \hookrightarrow C_X$ is the canonical inclusion into the top of the mapping cylinder. Since $(C_X, \tilde{A})$ is a finite CW-pair, the lemma is proven. □

Proposition C.3. *Let (X, A) be homotopically compact. If $f_0\colon (X_0, A_0) \to (X, A)$ and $f_1\colon (X_1, A_1) \to (X, A)$ are finite models for (X, A), then there exists a map*

$(X_0, A_0) \to (X_1, A_1)$ *such that the triangle*

$$\begin{array}{ccc} (X_0, A_0) & \xrightarrow{\simeq} & (X, A) \\ {\scriptstyle\simeq}\downarrow & \nearrow{\scriptstyle\simeq} & \\ (X_1, A_1) & & \end{array}$$

commutes up to homotopy. Additionally, there exists a zig-zag of finite models

$$\begin{array}{ccccccccc} (Y_0, B_0) & \longrightarrow & (Y_1, B_1) & \longleftarrow & (Y_2, B_2) & \longrightarrow & \cdots & \longleftarrow & (Y_r, B_r) \\ & \searrow^{\simeq} & & \swarrow_{\simeq} & \downarrow{\scriptstyle\simeq} & & & \swarrow_{\simeq} & \\ & & & & (X, A) & & & & \end{array}$$

such that $(Y_0, B_0) \to (X, A)$ *equals* $(X_0, A_0) \to (X, A)$ *and* $(Y_r, B_r) \to (X, A)$ *equals* $(X_1, A_1) \to (X, A)$*. That is, the category of finite models of* (X, A) *is connected.*

Proof. First use that $[A_0, A_1] \to [A_0, A]$ is a bijection to produce a map $g\colon A_0 \to A_1$ whose image under the bijection is $f_0|_A$. We may assume that g is cellular. Choose a homotopy from $f_0|_A$ to $f_1|_A \circ g$, and using the Homotopy Extension Property extend this to a homotopy $H\colon X_0 \times I \to X$ such that $H_0 = f_0$. Let $f' = H_1$. We now have a commutative diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle\simeq} \\ X_0 & \xrightarrow{f'} & X \end{array}$$

and so by the Relative Homotopy Lifting Property there exists a map $h\colon X_0 \to X_1$ such that the upper triangle commutes and the lower triangle commutes up to a homotopy relative to A_0. And again, we may assume that h is cellular. Putting our two homotopies together, we get the required homotopy-commutative triangle.

For the final statement of the proposition we can use a four-step zig-zag as follows:

$$\begin{array}{ccccccc} (X_0, A_0) & \xrightarrow{i_0} & (X_0 \times I, A_0 \times I) & \xleftarrow{i_1} & (X_0, A_0) & \xrightarrow{h} & (X_1, A_1) \\ & \searrow^{\simeq}_{f_0} & \downarrow{\scriptstyle J}{\scriptstyle\simeq} & \swarrow_{\simeq}^{f_1 h} & & \searrow_{\simeq}^{f_1} & \\ & & (X, A). & & & & \end{array}$$

The map labelled J is a homotopy for the triangle in the first part of the proposition. We leave the details for the reader. □

For us what is very useful about the class of homotopically compact spaces is that it includes all algebraic varieties:

Theorem C.4. *If X is an algebraic variety over $\mathbb{C}$ then X is homotopically compact.*

Proof. When X is a subvariety of some $\mathbb{C}^n$ this is a consequence of [Hironaka2, Theorem on page 170 and Remark 1.10]. For the general case we do an induction

on the size of an affine cover for X. Suppose that $\{U_1, \dots, U_n\}$ is an affine cover, and let $A = U_2 \cup \cdots \cup U_n$. Then we have the pushout diagram

$$\begin{array}{ccc} U_1 \cap A & \longrightarrow & A \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & X, \end{array}$$

which is also a homotopy pushout by [DI1, Corollary 1.6]. By induction we know that A is homotopically compact. Moreover, since U_1 is affine it is a subvariety of some $\mathbb{C}^n$, and therefore the same is true of $U_1 \cap A$. So both U_1 and $U_1 \cap A$ are homotopically compact by the base case. The result then follows by Lemma C.5 below. □

Lemma C.5. *Let A, X, and Y be homotopically compact spaces. Then the homotopy pushout of any diagram $X \leftarrow A \rightarrow Y$ is also homotopically compact.*

Proof. Let $f\colon A \to X$ and $g\colon A \to Y$ denote the maps, and choose a finite model $\widetilde{A} \to A$. By Lemma C.2 (applied twice) there exists a diagram

$$\begin{array}{ccccc} \widetilde{X} & \longleftarrow & \widetilde{A} & \longrightarrow & \widetilde{Y} \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow\simeq \\ X & \longleftarrow & A & \longrightarrow & Y. \end{array}$$

where $(\widetilde{X}, \widetilde{A})$ and $(\widetilde{Y}, \widetilde{A})$ are finite CW-pairs. The homotopy pushout of $X \leftarrow A \rightarrow Y$ is therefore weakly equivalent to that of $\widetilde{X} \leftarrow \widetilde{A} \rightarrow \widetilde{Y}$, and the latter clearly has the homotopy type of a finite CW-complex (in fact, in the latter case the pushout is itself a model for the homotopy pushout). □

Corollary C.6. *If (X, A) is a pair of algebraic varieties over $\mathbb{C}$ then (X, A) is homotopically compact.*

Proof. This follows from Theorem C.4 and Proposition C.1. □

Appendix D. Abelian categories and exact categories

The first part of this book developed K-theory in the context of modules over a ring R. This can be generalized in various ways, e.g. to abelian categories, to exact categories, or (perhaps) to triangulated categories. While not strictly necessary for most of our present purposes, it is good to have a basic sense of how some of these generalizations work.

This section discusses the foundations of abelian categories, Serre quotients, and exact categories. We outline the theory via a series of exercises. Most readers will probably not want to do *all* of these exercises; many of them are things one can just accept and move on. But doing some portion of them gives a nice introduction to the ins and outs of working with these objects.

D.1. **Abelian categories.** A **zero object** in a category $\mathcal{C}$ is an object $*$ that is both initial and terminal. This means that $\mathcal{C}(*, X)$ and $\mathcal{C}(X, *)$ are both singletons, for every object X. Most often we will be in settings where $\mathcal{C}$ is enriched over abelian groups, meaning that the hom-sets have abelian group structures with respect to

which composition is bi-additive. In such settings a zero object is usually denoted by 0.

In a category with a zero object a **kernel** of a map $f\colon A \to B$ is a map $K \to A$ that makes a pullback diagram

$$\begin{array}{ccc} K & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & B. \end{array}$$

Dually, a **cokernel** of $A \to B$ is a map $B \to C$ that makes an analogous pushout diagram. Note that by abuse of terminology we often refer to the object K as the kernel of f, but in fact it is really the map $K \to A$ that is the kernel. We write $\ker f$ for the domain of the kernel and $\operatorname{coker} f$ for the codomain of the cokernel. When we say that "the kernel of f is zero" we mean that it is the unique map $0 \to A$, and dually for the cokernel.

We will use categorical notions of monomorphisms and epimorphisms. The former will be denoted by $\rightarrowtail$ and the latter by $\twoheadrightarrow$.

Exercise D.2.

(a) Prove that a pullback of a monomorphism is a monomorphism, and dually for epimorphisms. Deduce that a kernel of a map is always a monomorphism and the cokernel of a map is always an epimorphism.

(b) Suppose that $\mathcal{C}$ is enriched over abelian groups and has a zero object. Prove that $f\colon A \to B$ is a monomorphism (resp. epimorphism) if and only if $\mathcal{C}(X, A) \to \mathcal{C}(X, B)$ (resp. $\mathcal{C}(B, X) \to \mathcal{C}(A, X)$) is a monomorphism for every object X.

(c) In the setting of (b) prove that a map is a monomorphism if and only if its kernel is zero, and dually for epimorphisms and cokernels.

Definition D.3. *An* **abelian category** *is a category enriched over abelian groups that has a zero object, finite coproducts and products which are equal, all kernels and cokernels, and where every monomorphism is the kernel of a map and every epimorphism is the cokernel of a map.*

Abelian categories were first introduced in [Buchsbaum] under the name "exact category" (a phrase that now means something different). The concept had been independently developed by Grothendieck and was used as the foundation for his treatment of homological algebra in [Grothen1]. Gabriel later developed the foundations of abelian categories extensively in [Gabriel], and this remains a very good reference. Note that the definition of abelian category as we have given it above is not identical to the one given in [Grothen1] and [Gabriel], but they are equivalent.

Here are a few examples:

- The category of modules over a ring (this is really the canonical example).
- The category of torsion abelian groups. More generally, if R is a commutative ring and S is a multiplicative system then the category of S-torsion modules (modules M where $S^{-1}M = 0$) is an abelian category.
- If R is Noetherian, the category of finitely-generated R-modules is abelian. Note that without the Noetherian hypothesis this category would not be guaranteed to have kernels.
- If $\mathcal{A}$ is an abelian category and I is a small category then the category of functors $\operatorname{Func}(I, \mathcal{A})$ is again an abelian category.

- If X is a topological space then the category of sheaves of abelian groups on X is an abelian category.

Exercise D.4. Let $\mathcal{A}$ be an abelian category.

(a) If $f\colon A \to B$ is a monomorphism prove that f is the kernel of the map $B \to \operatorname{coker}(f)$ ("every monomorphism is the kernel of its cokernel"). Note the dual result for epimorphisms.

(b) Prove that if a map is both a monomorphism and an epimorphism then it is an isomorphism.

(c) Prove that $\mathcal{A}$ has pullbacks and pushouts (hint: construct these as certain kernels and cokernels). If

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{g} & D \end{array}$$

is a pullback square, prove that $\ker f \to \ker g$ is an isomorphism; then prove the dual result about pushout squares and cokernels. Give an example in $\mathcal{A}b$ of a pullback square that is not a pushout, and also vice versa.

Prove that if the above is a pullback square and g is an epimorphism, then it is also a pushout square and f is an epimorphism. Dually, if the diagram is a pushout square and f is a monomorphism, then the diagram is also a pullback square and g is a monomorphism.

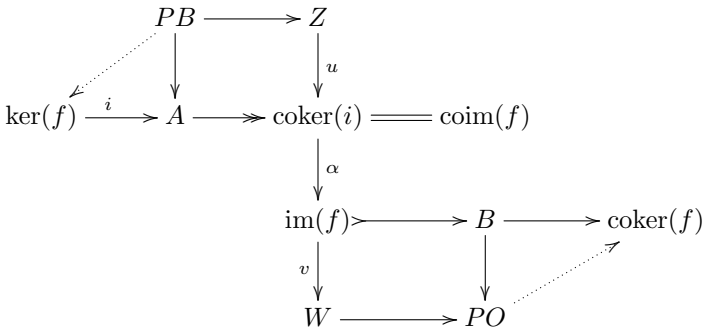
(d) Prove that the following conditions are equivalent:

(i) $0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$ is such that j is the kernel of p and p is the cokernel of j;

(ii) $0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$ yields left-exact sequences of abelian groups after applying $\operatorname{Hom}(X, -)$ and after applying $\operatorname{Hom}(-, X)$, for every X.

These equivalent conditions provide the notion of "short exact sequence" in an abelian category.

(e) The **image** of a map $f\colon A \to B$ is defined to be the kernel of the map $B \to \operatorname{coker}(f)$. Likewise, the **coimage** of f is the cokernel of $\ker(f) \to A$. There is a canonical map from the coimage to the image; prove that this is an isomorphism. Consequently, the canonical map $A \to \operatorname{im}(f)$ is an epimorphism and $\operatorname{coim}(f) \to B$ is a monomorphism. [Hint: This is a little tricky, so refer to the diagram and outline below:



We are trying to show that α is an isomorphism, so first assume $u\colon Z \to \operatorname{coker}(i)$ is such that $\alpha u = 0$. Form the pullback PB and argue that the map $PB \to A$

factors through $\ker(f)$ and hence $PB \to \operatorname{coker}(i)$ is zero. Then conclude $u = 0$ since $PB \to Z$ is an epimorphism. A dual argument shows that if $v\colon \operatorname{im}(f) \to W$ is such that $v\alpha = 0$ then $v = 0$.]

(f) Given a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ such that $gf = 0$ there is a canonical map $\operatorname{im}(f) \to \ker(g)$. Say that the sequence is exact at B if this canonical map is an isomorphism. Prove that a sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact in the sense of (e) if and only if it is exact at each spot. Show that a long exact sequence breaks up into a collection of interlocking short exact sequences as usual.

(g) Prove the Snake Lemma: Given a diagram with exact rows

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

there is an exact sequence

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \xrightarrow{\partial} \operatorname{coker}(a) \longrightarrow \operatorname{coker}(b) \longrightarrow \operatorname{coker}(c)$$

where all of the maps except for ∂ are the evident ones. If $A \to B$ is injective then so is $\ker(a) \to \ker(b)$, and if $B' \to C'$ is surjective then so is $\operatorname{coker}(b) \to \operatorname{coker}(c)$. [Hint: Construct the pullback PB

$$\begin{array}{ccccc} & & PB & \longrightarrow & \ker(c) \\ & \nearrow & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

and the induced map $A \to PB$, and prove that the cokernel of $A \to PB$ is $\ker(c)$ (use that the pullback of an epimorphism is an epimorphism). The map $PB \to B'$ factors through A', and therefore one gets an induced map $\operatorname{coker}(A \to PB) \to \operatorname{coker}(a)$. This constructs ∂, and everything else is tedious but mostly routine.]

Suppose given an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories. The kernel of F is defined to be the full subcategory of $\mathcal{A}$ consisting of objects X for which $F(X) \cong 0$. Such a subcategory has the property given in the following definition:

Definition D.5. *A full subcategory $\mathcal{S} \subseteq \mathcal{A}$ is called a **Serre subcategory** if it has the property that whenever $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$ then B is in $\mathcal{S}$ if and only if A and C are in $\mathcal{S}$. That is, a full subcategory is Serre if and only if it is closed under subobjects, quotients, and extensions.*

Is every Serre subcategory the kernel of an exact functor? It turns out the answer is yes. One can construct a quotient abelian category $\mathcal{A}/\mathcal{S}$ where the objects in $\mathcal{S}$ are all identified with 0 and where $\mathcal{A}/\mathcal{S}$ has the expected universal property. The projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{S}$ is exact and has kernel equal to $\mathcal{S}$.

The next couple of exercises will take the reader through the basic theory behind all of this.

Exercise D.6.

(a) Given a Serre subcategory $\mathcal{S} \subseteq \mathcal{A}$, define a map $f\colon A \to B$ to be an **$\mathcal{S}$-equivalence** if $\ker f$ and $\operatorname{coker} f$ are in $\mathcal{S}$. Prove that the $\mathcal{S}$-equivalences have the two-out-of-three property: if $A \xrightarrow{f} B \xrightarrow{g} C$ are composable maps in $\mathcal{A}$ and two of f, g, and gf are $\mathcal{S}$-equivalences, then so is the third. [Hint: Produce a long exact sequence of the form $0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$.]

(b) If f is an $\mathcal{S}$-equivalence and a monomorphism prove that any pushout of f is also those things. Dually, if g is an $\mathcal{S}$-equivalence and an epimorphism prove than any pullback is as well.

(c) Prove that if $f\colon A \to B$ is an $\mathcal{S}$-equivalence and a monomorphism then any pullback of f is also those things, and dually for epimorphisms and pushouts. [This one is tricky, so use the following outline. Starting with the pullback diagram on the left form the pushout diagram on the right:

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array} \qquad\qquad \begin{array}{ccc} A \oplus C & \xrightarrow{\pi_2} & C \\ {\scriptstyle -f\oplus p}\downarrow & & \downarrow \\ B & \longrightarrow & Q. \end{array}$$

Prove that $B \to Q$ is an epimorphism and is equal to the cokernel of f, hence Q is in $\mathcal{S}$. Then prove that the square on the right is also a pullback, so that the kernels of the two vertical maps are isomorphic. Use this to prove that the cokernel of $P \to C$ is a suboject of Q and hence lies in $\mathcal{S}$.]

(d) Prove that the $\mathcal{S}$-equivalences are closed under pullbacks and pushouts. [Hint: First show that any $\mathcal{S}$-equivalence can be factored as an epimorphism followed by a monomorphism, both of which are themselves $\mathcal{S}$-equivalences.]

(e) Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Recall that an R-module M is said to be S-torsion if $S^{-1}M = 0$. Check that the S-torsion modules form a Serre subcategory, and that the S-equivalences are the maps $f\colon M \to N$ such that $S^{-1}f$ is an isomorphism.

Exercise D.7. Let $\mathcal{A}$ be an abelian category and $\mathcal{S} \subseteq \mathcal{A}$ a Serre subcategory. Define a new category $\mathcal{A}/\mathcal{S}$ as follows: the objects are the same as those of $\mathcal{A}$, and morphisms from A to B consist of equivalence classes of certain zig-zags:

$$\mathcal{A}/\mathcal{S}(A,B) = \{\ A \xleftarrow[\sim_{\mathcal{S}}]{} A' \longrightarrow B' \xleftarrow[\sim_{\mathcal{S}}]{} B\ \}/\sim .$$

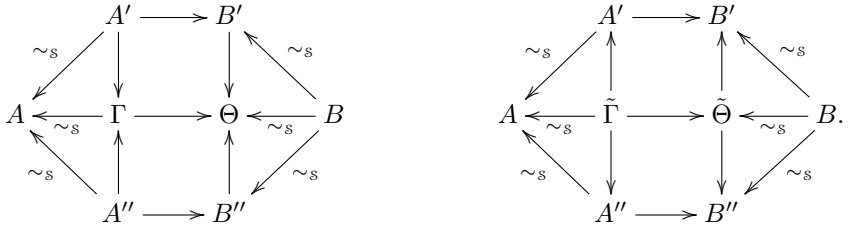
Here the indicated maps are $\mathcal{S}$-equivalences, and two zig-zags are equivalent if there is a commutative diagram

$$\begin{array}{ccccccc} & & A' & \longrightarrow & B' & & \\ & \swarrow^{\sim_{\mathcal{S}}} & \uparrow & & \uparrow & \searrow^{\sim_{\mathcal{S}}} & \\ A & \xleftarrow[\sim_{\mathcal{S}}]{} & \Gamma & \longrightarrow & \Theta & \xleftarrow[\sim_{\mathcal{S}}]{} & B \\ & \nwarrow^{\sim_{\mathcal{S}}} & \downarrow & & \downarrow & \nearrow^{\sim_{\mathcal{S}}} & \\ & & A'' & \longrightarrow & B'' & & \end{array}$$

with the original two zig-zags forming the upper and lower edges. Note that it is not immediately clear that this relation is transitive, and it is also not clear how to define composition in $\mathcal{A}/\mathcal{S}$—these will both be explained shortly. [Warning: If $\mathcal{A}$ is not small then there are potential set-theoretic issues in the above definition,

as $\mathcal{A}/\mathcal{S}(A,B)$ is not clearly a set. One can bypass this issue by using Grothendieck universes, but see also the comment in (b) below.]

(a) Prove that given a commutative diagram as on the left (below) there exists a commutative diagram on the right having the same upper and lower edges.



Use this to prove that the relation used to define $\mathcal{A}/\mathcal{S}(A,B)$ is transitive, hence an equivalence relation. [Hint: Exercise D.6(d).]

(b) Prove that every zig-zag in $\mathcal{A}/\mathcal{S}(A,B)$ is equivalent to a zig-zag where $A' \to A$ is a monomorphism and $B \to B'$ is an epimorphism—call these "special" zig-zags. Prove additionally that if two special zig-zags are equivalent then there is a diagram as above where the middle zig-zag is also special. [Note: This only requires Exercise D.6(c)]. [Discussion: This implies that we can redefine $\mathcal{A}/\mathcal{S}(A,B)$ via the analogous definitions that only involve special zig-zags. In abelian categories where the objects have underlying sets, this provides a constraint that allows us to avoid the set-theoretic issues mentioned above.]

(c) The set $\mathcal{A}/\mathcal{S}(A,B)$ is sometimes described as

$$\operatorname*{colim}_{A' \hookrightarrow A, B \twoheadrightarrow B'} \operatorname{Hom}_{\mathcal{A}}(A', B')$$

where $A' \hookrightarrow A$ is a monomorphism that is an $\mathcal{S}$-equivalence and $B \twoheadrightarrow B'$ is an epimorphism that is an $\mathcal{S}$-equivalence. The colimit is taken over a certain category that can be easily reconstructed by looking at the definition of $\mathcal{A}/\mathcal{S}(A,B)$ above. Do this reconstruction and prove that the two descriptions of the morphism sets are the same. [Also, check that when we fix A the category whose objects are monomorphisms $A' \hookrightarrow A$ that are $\mathcal{S}$-equivalences is cofiltered.]

(d) Prove that every zig-zag in $\mathcal{A}/\mathcal{S}(A,B)$ is equivalent to one in which $A' \to A$ is the identity map, and also to one in which $B \to B'$ is the identity map (note that we are not claiming that both maps can be made to be the identity at once).

(e) Define composition in $\mathcal{A}/\mathcal{S}$ as follows. Given classes $U \in \mathcal{A}/\mathcal{S}(A,B)$ and $V \in \mathcal{A}/\mathcal{S}(B,C)$ represent U by a zig-zag where the final map is the identity and represent V by a zig-zag where the initial map is the identity. Then take the evident concatenation of zig-zags. Check that this is well-defined.

(f) Given $U, V \in \mathcal{A}/\mathcal{S}(A,B)$ define $U+V$ as follows. First represent U and V by special zig-zags, then prove that one can in fact represent them by zig-zags where the first and third map in U are equal to the first and third map in V (respectively). Define $U+V$ to be the zig-zag obtained by adding the two middle maps (it might be useful to also think about the description of the hom-sets from (c)). Verify that this is well-defined and makes $\mathcal{A}/\mathcal{S}$ into a category enriched over abelian groups.

(g) Verify that $\mathcal{A}/\mathcal{S}$ is an abelian category (this is somewhat long and tedious).

(h) Prove that all objects in $\mathcal{S}$ are isomorphic to 0 in $\mathcal{A}/\mathcal{S}$.

(i) Let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ be the identity on objects and send a map $f\colon A \to B$ to the zig-zag $A \xleftarrow{\mathrm{id}} A \xrightarrow{f} B \xleftarrow{\mathrm{id}} B$. Prove the following:
  (i) A zig-zag $A \xleftarrow{\sim_{\mathcal{S}}} A' \longrightarrow B' \xleftarrow{\sim_{\mathcal{S}}} B$ represents 0 in $\mathcal{A}/\mathcal{S}(A,B)$ if and only if $A' \to B'$ factors through an object in $\mathcal{S}$. In particular, if $f \in \mathcal{A}(A,B)$ then $\pi(f) = 0$ if and only if f factors through an object in $\mathcal{S}$.
  (ii) π sends monomorphisms to monomorphisms, and (dually) epimorphisms to epimorphisms.
  (iii) π is an exact functor.
  (iv) π sends $\mathcal{S}$-equivalences to isomorphisms.

(j) Given A in $\mathcal{A}$, $\pi(A) \cong 0$ if and only if A is in $\mathcal{S}$. Likewise, given $f\colon A \to B$ in $\mathcal{A}$ one has that $\pi(f)$ is an isomorphism if and only if f is an $\mathcal{S}$-equivalence.

(k) Prove that if $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories and $F(X) \cong 0$ for every X in $\mathcal{S}$, then there is a unique functor $\bar{F}\colon \mathcal{A}/\mathcal{S} \to \mathcal{B}$ making the evident triangle commute.

(l) For every pair of objects X, Y in $\mathcal{A}$ and a map $X \xrightarrow{u} Y$ in $\mathcal{A}/\mathcal{S}$, there exist maps $X \to \hat{Y}$ and $\hat{X} \to Y$ in $\mathcal{A}$ together with commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & \hat{Y} \\ \| & & \downarrow \cong \\ X & \xrightarrow{u} & Y \end{array} \qquad \begin{array}{ccc} \hat{X} & \longrightarrow & Y \\ \cong \downarrow & & \| \\ X & \xrightarrow{u} & Y \end{array}$$

in $\mathcal{A}/\mathcal{S}$.

(m) Given a monomorphism $W \rightarrowtail Z$ in $\mathcal{A}/\mathcal{S}$ there exists a monomorphism $\hat{W} \rightarrowtail \hat{Z}$ in $\mathcal{A}$ and a commutative diagram

$$\begin{array}{ccc} \hat{W} & \rightarrowtail & \hat{Z} \\ \cong \downarrow & & \downarrow \cong \\ W & \rightarrowtail & Z \end{array}$$

in $\mathcal{A}/\mathcal{S}$ where the vertical maps are isomorphisms.

(n) Every exact sequence in $\mathcal{A}/\mathcal{S}$ is isomorphic to the image under π of an exact sequence in $\mathcal{A}$.
[Compare parts (l) and (m) to Corollary 5.11. The connection will be clearer after Exercise D.11 below.]

Remark D.8. The category $\mathcal{A}/\mathcal{S}$ from the above exercise is called the **Gabriel quotient** or **Serre quotient** of abelian categories. Often it is just called the **quotient**. It was first constructed by Gabriel [Gabriel], and in that reference Serre categories were called *épaisse* ("thick") subcategories.

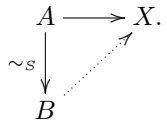
The construction of $\mathcal{A}/\mathcal{S}$ is elegant from a theoretical perspective, but it can be hard to work with in practice. In many nice situations, though, $\mathcal{A}/\mathcal{S}$ can be identified with a subcategory of $\mathcal{A}$. For this to work out, we need to have enough objects in $\mathcal{A}/\mathcal{S}$ that "see" all of the objects in $\mathcal{S}$ as zero and also "see" all of the $\mathcal{S}$-equivalences as if they were isomorphisms. We explore this next.

Exercise D.9. Let $\mathcal{S} \subseteq \mathcal{A}$ be a Serre subcategory.

(a) Say that an object X of $\mathcal{A}$ is **$\mathcal{S}$-null** if $\mathrm{Hom}_{\mathcal{A}}(A, X) = 0$ for all A in $\mathcal{S}$. Likewise, say that X is **$\mathcal{S}$-local** if for every $\mathcal{S}$-equivalence $A \to B$ the induced map

$\mathrm{Hom}_{\mathcal{A}}(B, X) \to \mathrm{Hom}_{\mathcal{A}}(A, X)$ is an isomorphism. Prove that if X is $\mathcal{S}$-local then it is $\mathcal{S}$-null, and give a counterexample to the converse.

(b) Verify that if X is $\mathcal{S}$-local then any solid-arrow diagram below has a unique lifting as shown:



(c) If X and Y are $\mathcal{S}$-local and $f\colon X \to Y$ is an $\mathcal{S}$-equivalence, prove that f is in fact an isomorphism.

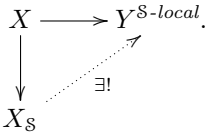
(d) Prove that the following are equivalent:
   (i) X is $\mathcal{S}$-local,
   (ii) $\mathrm{Hom}_{\mathcal{A}}(A, X) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(A, X)$ for all A in $\mathcal{S}$,
   (iii) For every A in $\mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(A, X) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(\pi A, \pi X)$ is an isomorphism.

   [Hint: For (i)⇒(iii) construct the inverse to the given map.]

(e) Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Let $\mathcal{S}$ be the Serre subcategory of S-torsion modules. Verify that a module M is $\mathcal{S}$-local if and only if it is S-local (meaning that every element of S acts invertibly on M).

The above exercise suggests the idea that we try to embed $\mathcal{A}/\mathcal{S}$ into $\mathcal{A}$ via the $\mathcal{S}$-local objects. So for every object X in $\mathcal{A}/\mathcal{S}$ (equivalently, an object of $\mathcal{A}$) we need to attach a corresponding $\mathcal{S}$-local object. The case of S-torsion R-modules suggests that this sometimes can be done in a universal way, as in the following definition:

Definition D.10. *Let $\mathcal{S} \subseteq \mathcal{A}$ be a Serre subcategory. An **$\mathcal{S}$-localization** of an object X in $\mathcal{A}$ is an object $X_{\mathcal{S}}$ and an $\mathcal{S}$-equivalence $X \to X_S$ satisfying the following universal property:*



*That is, $X \to X_{\mathcal{S}}$ is univeral among maps to $\mathcal{S}$-local objects. The Serre subcategory $\mathcal{S}$ is called **localizing** if every object has an $\mathcal{S}$-localization.*

Exercise D.11. Suppose $\mathcal{S} \subseteq \mathcal{A}$ is a Serre subcategory.

(a) If $\mathcal{S}$ is localizing, show that there exists a functor $T\colon \mathcal{A}/\mathcal{S} \to \mathcal{A}$ sending each X to its $\mathcal{S}$-localization $X_{\mathcal{S}}$ and that this functor is right adjoint to π.

(b) Conversely to the situation in (a), assume given a functor $T\colon \mathcal{A}/\mathcal{S} \to \mathcal{A}$ that is right adjoint to π (but without assuming that $\mathcal{S}$ is localizing). Prove the following:
   (i) The image of T lies in the $\mathcal{S}$-local objects.
   (ii) The co-unit $\epsilon_{\pi X}\colon \pi T(\pi X) \to \pi X$ is an isomorphism, for every X. Since π is surjective on objects, this implies that every co-unit is an isomorphism (one then says that T is a "section" of π).
   (iii) Applying the functor π to the unit $X \to T\pi X$ yields an isomorphism, for every X.

(iv) The unit map $X \to T\pi X$ is an $\mathcal{S}$-localization, for every X. Hence $\mathcal{S}$ is localizing.

[Note that many texts define a localizing subcategory to be a Serre subcategory where π admits a right adjoint. The above shows this definition to be equivalent to ours.]

(c) Continue to assume that T is the right adjoint to π. Prove that T is fully-faithful, thereby identifying $\mathcal{A}/\mathcal{S}$ with the full subcategory of $\mathcal{A}$ consisting of the $\mathcal{S}$-local objects.

(d) Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Prove that the quotient category R-Mod $/\mathcal{S}$ is equivalent to the category of $S^{-1}R$-modules.

D.12. **Exact categories.** The notion of an "exact category" was introduced in [Quillen5]. It is a weakening of the notion of abelian category in which one does not require the existence of kernels and cokernels for *all* maps, but only for certain select classes. A good example to keep in mind is the category of finitely-generated projectives over a ring R, where one is only guaranteed kernels (resp. cokernels) for *split* surjections (resp. split injections). The basic tools of K-theory can be developed in this weakened context, and this allows (for example) for tools that apply to the groups $K(R)$ and $G(R)$ simultaneously.

Let $\mathcal{A}$ be a category enriched over abelian groups that has finite products and coproducts, which are equal. Suppose given a subclass of monomomorphisms and a subclass of epimorphisms, called "admissible" monomomorphisms and epimorphisms (respectively). In this situation we say that $\mathcal{A}$ is an **exact category** if the following criteria are satisfied:

(1) Admissible monomorphisms are closed under composition and isomorphism, and dually for admissible epimorphisms.
(2) Pullbacks of admissible epimorphisms exist and are again admissible epimorphisms, and dually for pushouts of admissible monomorphisms.
(3) Admissible monomorphisms have cokernels, and those are admissible epimorphisms; moreover, every admissible monomorphism is the kernel of its cokernel. Dually for admissible epimorphisms.
(4) For every pair of objects X and Y, the inclusion $X \hookrightarrow X \oplus Y$ is an admissible monomorphism (the projection $X \oplus Y \to Y$ is its cokernel and therefore an admissible epimorphism).

Given an exact category, define a sequence $A \xrightarrow{i} B \xrightarrow{p} C$ to be **exact** if i is an admissible monomorphism and p is its cokernel (and note that this is equivalent to saying that p is an admissible epimorphism and i is its kernel).

Here are a few examples:

- Every abelian category can be considered an exact category by declaring all monomorphisms and epimorphisms to be admissible.
- Let R be a ring and let $\mathcal{P}(R)$ be the full subcategory of R-Mod consisting of the finitely-generated projectives. Define a monomorphism in $\mathcal{P}(R)$ to be admissible if its cokernel is a projective R-module, and likewise define an epimorphism to be admissible if its kernel is a projective R-module. Then $\mathcal{P}(R)$ is an exact category.
- Let R be a ring and $\mathcal{A} \subseteq R$-Mod any full subcategory that is closed under direct sums. Define an epimorphism $f\colon A \to B$ to be admissible if it is split

and $\ker f$ is in $\mathcal{A}$, and dually for admissible epimorphisms. Then $\mathcal{A}$ is an exact category.

- Suppose that $\mathcal{A}$ is an abelian category and $\mathcal{B} \subseteq \mathcal{A}$ is a full subcategory with the property that if $X \in \mathrm{ob}(\mathcal{B})$ and $X \cong Y$ in $\mathcal{A}$ then $Y \in \mathrm{ob}(\mathcal{B})$. Suppose also that $\mathcal{B}$ is closed under extensions: if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{A}$ and $X', X'' \in \mathrm{ob}(\mathcal{B})$ then $X \in \mathrm{ob}(\mathcal{B})$. Define a monomorphism in $\mathcal{B}$ to be admissible if it is a monomorphism in $\mathcal{A}$ and its cokernel (as a map in $\mathcal{A}$) lies in $\mathcal{B}$, and dually for admissible epimorphisms. Then $\mathcal{B}$ is an exact category.

Appendix E. Bernoulli numbers

There are different conventions for naming the Bernoulli numbers, especially when one enters the topology literature. We adopt what seems to be the most common definition, which is the following:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \cdot \frac{x^k}{k!}.$$

Expanding the power series yields

$$\begin{aligned}\frac{x}{e^x-1} &= 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots \\ &= 1 - \Big(\frac{1}{2}\Big)x + \Big(\frac{1}{6}\Big)\cdot\frac{x^2}{2} - \Big(\frac{1}{30}\Big)\cdot\frac{x^4}{4!} + \Big(\frac{1}{42}\Big)\cdot\frac{x^6}{6!} - \Big(\frac{1}{30}\Big)\cdot\frac{x^8}{8!}\cdots\end{aligned}$$

So we have

k	0	1	2	3	4	5	6	7	8	9
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0

From the table one guesses that $B_{2n+1} = 0$ for $n > 0$. This is easy to prove: if we set $f(x) = \frac{x}{e^x-1}$ then we can isolate the odd powers of x by examining $f(x) - f(-x)$. But algebra yields

$$\Big(\frac{x}{e^x-1}\Big) - \Big(\frac{-x}{e^{-x}-1}\Big) = -x.$$

Computing the coefficients of $\frac{x}{e^x-1}$ is not the most efficient way of computing Bernoulli numbers, as one can deduce from the large denominators in the above formula. A better method is via a certain recursive formula, and this is best remembered by a "mnemonic":

$$(B+1)^n = B^n. \tag{E.1}$$

Do not take this formula literally! It is shorthand for the following procedure. First expand the left-hand-side via the binomial formula, treating B as a formal variable. Then rewrite the formula by "lowering all indices", meaning changing every B^i to a B_i. This gives the desired recursive formula.

For example: $(B+1)^2 = B^2$ yields $B^2 + 2B + 1 = B^2$, which in turn gives $B_2 + 2B_1 + 1 = B_2$. Cancelling the B_2's we obtain $2B_1 + 1 = 0$, or $B_1 = -\frac{1}{2}$. Likewise, $(B+1)^3 = B^3$ yields $B_3 + 3B_2 + 3B_1 + 1 = B_3$, thereby giving

$$B_2 = -\tfrac{1}{3}(1 + 3B_1) = -\tfrac{1}{3}\cdot -\tfrac{1}{2} = \tfrac{1}{6}.$$

And so on. For the record here are a few more of the Bernoulli numbers, computed in this way:

k	0	1	2	4	6	8	10	12	14	16	18
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$

Note that we have not yet justified the recursive formula (E.1). We will do this after a short interlude.

E.2. **Sums of powers.** The Bernoulli numbers first arose in work of Jakob Bernoulli on computing formulas for the power sums

$$1^t + 2^t + 3^t + \cdots + n^t.$$

Most modern students have seen the formulas

$$1 + 2 + \cdots + n = \tfrac{n(n+1)}{2} \quad \text{and} \quad 1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6}.$$

The Bernoulli formulas generalize these to give

$$1^t + 2^t + \cdots + n^t = P_t(n)$$

where P_t is a degree $t+1$ polynomial in n with rational coefficients. It is somewhat surprising that the formulas for the P_t's can be given using a single set of coefficients, the Bernoulli numbers.

The Bernoulli formulas are most succinctly written using our mnemonic device of lowering indices. We write

$$1^t + 2^t + \cdots + n^t = \tfrac{1}{t+1}\Big[\big(B + (n+1)\big)^{t+1} - B^{t+1}\Big]. \tag{E.3}$$

Let us work through the first few examples of this. For $t = 1$ we have

$$\begin{aligned} 1 + 2 + \cdots + n = \tfrac{1}{2}\Big[(B + (n+1))^2 - B^2\Big] &= \tfrac{1}{2}\Big[B_2 + 2B_1(n+1) + (n+1)^2 - B_2\Big] \\ &= \tfrac{1}{2}\Big[(n+1)^2 - (n+1)\Big] \\ &= \tfrac{1}{2}(n+1)n. \end{aligned}$$

For $t = 2$ we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 &= \tfrac{1}{3}\Big[(B + (n+1))^3 - B^3\Big] \\ &= \tfrac{1}{3}\Big[3B_2(n+1) + 3B_1(n+1)^2 + (n+1)^3\Big] \\ &= \tfrac{1}{3}(n+1)\Big[\tfrac{1}{2} - \tfrac{3}{2}(n+1) + (n+1)^2\Big] \\ &= \tfrac{1}{3}(n+1)\Big[\tfrac{1}{2}n + n^2\Big] \\ &= \tfrac{1}{6}(n+1)n(2n+1). \end{aligned}$$

We leave it to the reader to derive the $t = 3$ formula:

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Proof of the Bernoulli formula (E.3). Start with the identity of power series

$$1 + e^x + e^{2x} + \cdots + e^{nx} = \frac{e^{(n+1)x} - 1}{e^x - 1} = \Big(\frac{x}{e^x - 1}\Big) \cdot \Big(\frac{e^{(n+1)x} - 1}{x}\Big).$$

The coefficient of x^t on the left-hand-side is

$$\frac{1}{t!}(1^t + 2^t + \cdots + n^t).$$

The coefficient of x^t on the right-hand-side is

$$\sum_{k=0}^{t} \frac{B_k}{k!} \cdot \frac{(n+1)^{t+1-k}}{(t+1-k)!}.$$

Equating coefficients and rearranging yields the Bernoulli formula immediately. □

Now let us return to our recursive formula (E.1) for computing the Bernoulli numbers. Note that this is an immediate consequence of (E.3) by taking $n = 0$ to get $0 = (B+1)^{t+1} - B^{t+1}$.

E.4. **Miscellaneous facts.** Here are a couple more useful facts about the Bernoulli numbers. We outline the proofs in Exercise E.8 below.

Theorem E.5. *The Bernoulli numbers are given by the formulas*

$$B_n = \sum_{j=0}^{n} \tfrac{1}{j+1} \sum_{s=0}^{j} (-1)^s \tbinom{j}{s} s^n = \sum_{j=0}^{n} \tfrac{j!}{j+1} (-1)^j S(n,j)$$

where $S(n,j)$ is the Stirling number of the second kind.

Theorem E.6 (Claussen/von Staudt)**.**

(a) $B_{2n} \equiv -\sum_p \frac{1}{p} \bmod \mathbb{Z}$, where the sum is taken over all primes p such that $p-1$ divides $2n$.

(b) When expressed as a fraction in lowest terms, B_{2n} has square-free denominator consisting of the product of all primes p such that $p-1$ divides $2n$.

Note that (b) follows immediately from (a). As an application of the power of this result, observe that

$$B_{12} \equiv_{\mathbb{Z}} -\sum_{p-1|12} \tfrac{1}{p} = -(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{13}) = -\tfrac{3421}{2730} \equiv_{\mathbb{Z}} -\tfrac{691}{2730}.$$

In fact the last fraction is the exact value of B_{12}. Similarly, we can now immediately predict that the denominator of B_{20} will be $2 \cdot 3 \cdot 5 \cdot 11 = 330$. Note that the primes 2 and 3 will always appear in the denominators of Bernoulli numbers.

Proposition E.7. *For any even n and any $k \in \mathbb{Z}$, $\frac{k^n(k^n-1)B_n}{n} \in \mathbb{Z}$.*

Proof. We follow Milnor and Stasheff here [MS, Lemma B.2]. Write

$$f(x) = 1 + e^x + e^{2x} + \cdots + e^{(k-1)x} = \frac{e^{kx} - 1}{e^x - 1}.$$

Note that $f^{(r)}(0) = 1^r + 2^r + \cdots + (k-1)^r$. In particular, the derivatives of f evaluated at 0 are all integers.

Next consider the logarithmic derivative

$$\begin{aligned}\frac{f'(x)}{f(x)} = D(\log(f(x))) &= \frac{ke^{kx}}{e^{kx}-1} - \frac{e^x}{e^x-1}\\ &= k\Big[\frac{1}{1-e^{-kx}}\Big] - \Big[\frac{1}{1-e^{-x}}\Big]\\ &= \frac{1}{x}\Big[\frac{-kx}{e^{-kx}-1} - \frac{-x}{e^{-x}-1}\Big]\\ &= \frac{1}{x}\Big[\sum \tfrac{B_i}{i!}(-kx)^i - \sum \tfrac{B_i}{i!}(-x)^i\Big]\\ &= \sum_i (-1)^i \tfrac{B_i}{i!}(k^i-1)x^{i-1}\\ &= \tfrac{k-1}{2} + \tfrac{B_2}{2!}(k^2-1)x + \tfrac{B_4}{4!}(k^4-1)x^3 + \cdots\end{aligned}$$

The $(2t-1)$st derivative of this expression, evaluated at 0, is $\frac{B_{2t}}{2t}(k^{2t}-1)$.

However, iterated use of the quotient rule shows that the $(2t-1)$st derivative of $f'(x)/f(x)$, evaluated at 0, can be written as an integral linear combination of $f(0), f'(0), f''(0), \ldots$ divided by $f(0)^{2t}$. Since $f(0) = k$ and all the deriviates of f have integral values at 0, this gives $\frac{B_{2t}}{2t}(k^{2t}-1)\cdot k^{2t} \in \mathbb{Z}$. □

Exercise E.8. Fill out the following steps to prove Theorems E.5 and E.6:

(a) Start with the identity of formal power series $x = \log(1+(e^x-1)) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(e^x-1)^j$. Use the binomial theorem to expand $(e^x-1)^j$ and then group terms.

(b) Define the Stirling numbers of the second kind $S(N,k)$ to be the number of ways of dividing an N-element set into k unlabelled nonempty pieces. Show that $S(N,k)\cdot k!$ is the number of surjections from $\underline{N}$ to $\underline{k}$. In the set of all functions from $\underline{N}$ to $\underline{k}$, let U_a be the subset that do not contain a in the image. Use an inclusion-exclusion argument based on the sets $\{U_i\}_{1\le i\le k}$ to prove that

$$S(N,k)k! = k^N - k\cdot(k-1)^N + \tbinom{k}{2}(k-2)^N - \cdots$$

Deduce that $B_{2n} = \sum_{j=0}^{n} \frac{j!}{j+1}(-1)^j S(2n,j)$.

(c) Prove that if p is prime then $\binom{p-1}{s} \equiv (-1)^s \pmod p$ for all $0 \le s \le p-1$ and that

$$1^n + 2^n + \cdots + (p-1)^n \equiv_p \begin{cases} -1 & \text{if } p-1 | n,\\ 0 & \text{otherwise.}\end{cases}$$

(For the first, use Pascal's identity and $\binom{p}{s} \equiv_p 0$ for $1 \le s \le p-1$. For the second, let g be a multiplicative generator for $\mathbb{F}_p^*$ and then rewrite the given sum as a partial geometric sum. Use $g^{p-1} = 1$.)

(d) Consider the formula from Theorem E.5:

$$B_{2n} = \sum_{j=0}^{2n} \tfrac{1}{j+1} \sum_{s=0}^{j} (-1)^s \tbinom{j}{s} s^{2n}.$$

For convenience let $\beta(j)$ benote the term of the sum indexed by j, which we know by (b) can be written as $\beta(j) = \frac{j!}{j+1}(-1)^j \cdot S(2n,j)$. Prove that

(i) If $j > 3$ and $j+1$ is composite then $j+1 | j!$ and so $\beta(j)$ is an integer.

(ii) If $j = 3$ then $\beta(j)$ is an integer.

(iii) If $j+1$ is prime then $\sum_{s=0}(-1)^s\binom{j}{s}s^{2n} \equiv (-1 \text{ or } 0) \pmod{j+1}$ depending on whether $j+1$ divides $2n$ or not. In the latter case, conclude $\beta(j)$ is an integer.

Then deduce the theorem of Claussen and von Staudt.

Exercise E.9. The Bernoulli numbers arise in a variety of Taylor expansions. For example, prove that

$$\tfrac{1-e^{-x}}{x} = \sum_{m\geq 1} \tfrac{B_m}{m\cdot m!}x^m = -\tfrac{x}{2} + \sum_{m\geq 1}\tfrac{B_{2m}}{2m\cdot(2m)!}x^{2m}.$$

[Hint: Let $f(x)$ be the left-hand side. Express $f'(x)$ in terms of $\frac{x}{e^x-1}$, use expansion of this series in terms of Bernoulli numbers, then integrate term by term.]

Appendix F. The algebra of symmetric functions

Let $S = \mathbb{Z}[x_1,\dots,x_n]$ be equipped with the evident Σ_n-action that permutes the indices. It is a well-known theorem that the ring of invariants is a polynomial ring on the elementary symmetric functions:

$$S^{\Sigma_n} \cong \mathbb{Z}[\sigma_1,\dots,\sigma_n].$$

Let $s_k = x_1^k + x_2^k + \cdots + x_n^k$, the ***k*th power sum** of the variables x_i. Since s_k is a Σ_n-invariant we have

$$s_k = S_k(\sigma_1,\dots,\sigma_n)$$

for a unique polynomial S_k in n variables (with integer coefficients). The polynomial S_k is called the kth **Newton polynomial**.

Let us calculate the simplest examples of the Newton polynomials. Clearly $s_1 = \sigma_1$, and so $S_1(\sigma_1,\dots,\sigma_n) = \sigma_1$. For s_2 we compute that

$$s_2 = x_1^2 + \cdots + x_n^2 = (x_1+\cdots+x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n) = \sigma_1^2 - 2\sigma_2.$$

These calculations get more difficult as the exponents get larger.

It is useful to adopt the following notation when working with the ring of invariants. If m is a monomial in the x_i's then $[m]$ denotes the sum of all elements in the Σ_n-orbit of m. For example,

$$[x_1x_2] = \sigma_2, \quad [x_1^k] = s_k, \quad \text{and} \quad [x_1^2x_2] = \sum_{i\neq j} x_i^2x_j.$$

If $H \leq \Sigma_n$ is the stabilizer of m then we can also write

$$[m] = \sum_{gH\in\Sigma_n/H} gm.$$

Let us use the above notation to help work out the third Newton polynomial. Elementary algebra easily yields the equation

$$s_3 = [x_1^3] = [x_1]^3 - 3[x_1^2x_2] - 6[x_1x_2x_3].$$

Here one considers the product $(x_1+\cdots+x_n)^3$ and reasons that a term like $x_1^2x_2$ appears three times in the expansion, and terms like $x_1x_2x_3$ appear six times. Via a similar process we work out that

$$[x_1^2x_2] = [x_1]\cdot[x_1x_2] - 3[x_1x_2x_3].$$

Putting everything together, we have found that

$$s_3 = \sigma_1^3 - 3(\sigma_1\sigma_2 - 3\sigma_3) - 6\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

This final expression is the third Newton polynomial S_3.

The next result gives a recursive formula for the Newton polynomials:

Proposition F.1 (The Newton identities)**.** *For $k \geq 2$ one has the identity*

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^k \sigma_{k-1} s_1 + (-1)^{k+1} k \sigma_k.$$

Consequently, there is an analogous inductive formula for the Newton polynomials:

$$S_k = \sigma_1 S_{k-1} - \sigma_2 S_{k-2} + \cdots + (-1)^k \sigma_{k-1} S_1 + (-1)^{k+1} k \sigma_k.$$

Proof. The key is the formula

$$[x_1 x_2 \cdots x_{j-1} x_j^k] = [x_1 \cdots x_j] \cdot [x_1^{k-1}] - [x_1 \cdots x_j x_{j+1}^{k-1}]$$

which holds for $k > 2$, whereas when $k = 2$ we have

$$[x_1 x_2 \cdots x_{j-1} x_j^2] = [x_1 \cdots x_j] \cdot [x_1] - (j+1)[x_1 \cdots x_j x_{j+1}].$$

In the latter case the point is that a term $x_1 \ldots x_{j+1}$ appears $j+1$ times in the product $[x_1 \ldots x_j] \cdot [x_1]$.

When $k = 2$ the identity from the statement of the lemma has already been verified by direct computation. For $k > 2$ start with the simple formula

$$s_k = [x_1^k] = [x_1] \cdot [x_1^{k-1}] - [x_1 x_2^{k-1}] = \sigma_1 s_{k-1} - [x_1 x_2^{k-1}].$$

Next observe that

$$[x_1 x_2^{k-1}] = \begin{cases} [x_1 x_2] \cdot [x_1^{k-2}] - [x_1 x_2 x_3^{k-2}] & \text{if } k > 3, \\ [x_1 x_2] \cdot [x_1^{k-2}] - 3[x_1 x_2 x_3] & \text{if } k = 3. \end{cases}$$

If $k = 3$ we are now done, otherwise repeat the above induction step. The details are left to the reader. □

As an application of Proposition F.1 observe that we have

$$S_3 = \sigma_1 S_2 - \sigma_2 S_1 + 3\sigma_3 = \sigma_1(\sigma_1^2 - 2\sigma_2) - \sigma_2 \sigma_1 + 3\sigma_3 = \sigma_1^3 - 3\sigma_1 \sigma_2 + 3\sigma_3,$$

agreeing with our earlier calculation. Here is a table showing the first few Newton polynomials:

TABLE 6.2. Newton polynomials

k	S_k
1	σ_1
2	$\sigma_1^2 - 2\sigma_2$
3	$\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$
4	$\sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4$
5	$\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1^2\sigma_3 + 5\sigma_1\sigma_2^2 - 5\sigma_2\sigma_3 - 5\sigma_1\sigma_4 + 5\sigma_5$

Here are two useful properties of the S_k:

Proposition F.3.

(a) $S_k(-\sigma_1, \sigma_2, -\sigma_3, \sigma_4, \ldots) = (-1)^k S_k(\sigma_1, \sigma_2, \ldots)$.
(b) When p is prime, $S_p \equiv \sigma_1^p$ modulo p.

Proof. For part (a), use the symmetry of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ that sends $x_i \mapsto -x_i$. The power sum s_k is mapped to $(-1)^k s_k$, and σ_j is mapped to $(-1)^j \sigma_j$.

For (b) just use that in characteristic p one has $x_1^p + \cdots + x_n^p = (x_1 + \cdots + x_n)^p$. □

The Newton polynomials also show up in the following:

Proposition F.4. *Let $\alpha = \alpha_1 t + \alpha_2 t^2 + \cdots \in R[[t]]$, where R is a commutative ring. Then*

$$\tfrac{d}{dt}\big(\log(1+\alpha)\big) = \tfrac{\alpha'}{1+\alpha} = \mu_1 + \mu_2 t + \mu_3 t^2 + \cdots$$

where $\mu_k = (-1)^{k-1} S_k(\alpha_1, \ldots, \alpha_k)$.

Proof. Equate coefficients in the identity

$$\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^3 + \cdots \; = \alpha' = (1 + \alpha_1 t + \alpha_2 t^2 + \cdots) \cdot (\mu_1 + \mu_2 t + \cdots).$$

This gives a series of identities for each μ_k that parallel the Newton identities. The result then follows by an easy induction. □

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Index of Common Symbols

$\mathcal{C}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$, maps from X to Y in category $\mathcal{C}$, 5
X_+, space X with an added disjoint basepoint, 5
$\mathrm{Rad}(I)$, radical of the ideal I, 5
$G(R)$, Grothendieck group of modules, 15
$QF(R)$, quotient field, 16
$K(R)$, Grothendieck group of projectives, 19
$V(I)$, vanishing set of ideal I, 22
$\mathbb{A}^n$, affine space, 22
$\mathrm{Spec}\, R$, Zariski prime ideal spectrum, 23
$\mathcal{A}ff$, category of affine schemes, 23
$\mathbb{Z}_p^\wedge$, p-completion of $\mathbb{Z}$, 24
$\mathrm{Ann}(x)$, annihilator, 26
$\ell(M)$, length of M, 27
$\mathrm{CH}_{d-1}(R)$, codimension one Chow homology group, 31
c_1, first homological Chern class, 32
$\mathrm{Ass}\, M$, set of associated primes, 36
$\mathrm{CH}_*(R)$, Chow homology groups, 37
$\mathrm{Cl}(R)$, ideal class group, 38
$\mathrm{Pic}(R)$, Picard group, 38
$\mathrm{rank}_q(P)$, rank of projective P at q, 39
$\mathrm{Sym}_R(M)$, symmetric algebra over R generated by R-module M, 43
$G(M \mid S^{-1}M = 0)$, Grothendieck group of modules with a condition, 52
$K^{cplx}(R)$, K-group of chain complexes, 59
ΣC, suspension of a chain complex, 60
$K(R,S)$, relative K-group, 62
$K(R)_I$, 62
$K(R,I)$, 62
$K(R,\leq n)$, 62
$\mathrm{Supp}\, M$, support of module M, 65
$K(R)_Z$, K-group supported on Z, 65
$K^{exct}(R)$, K-group of exact complexes, 67
$K^{n-exct}(R)$, K-group of n-exact complexes, 70
$K_1(R)$, degree-one K-group, 72
$GL(R)$, infinite general linear group, 72
$E(R)$, subgroup of elementary matrices in $GL(R)$, 73
$SK_1(R)$, 76
$\mathrm{Ho}\,(\mathcal{T}op) = \mathrm{Ho}\,\mathcal{T}$, homotopy category of topological spaces, 85
$\Sigma^\infty X$, suspension spectrum of the space X, 94
$\Omega^\infty W$, infinite loop space of the spectrum W, 95
$\lim^1$, first derived functor of inverse limit, 96
$\mathsf{FamVS}(X)$, category of families of vector spaces over X, 100
$\underline{n}$, trivial bundle of rank n, 101
$\mathsf{Vect}(X)$, category of vector bundles over X, 104
$\mathrm{Vect}_k(X)$, set of isomorphism classes of rank k vector bundles over X, 107
$\underline{\mathrm{Hom}}(E_1,E_2)$, Hom-bundle, 109
$\mathcal{O}(1)$, hyperplane line bundle, 113
$\mathcal{O}(n)$, nth power of $\mathcal{O}(1)$, 113

$\mathrm{Fr}(J)$, frame bundle of vector bundle J, 117
$\mathbb{P}(E)$, bundle of projective spaces associated to vector bundle E, 118
$C(X)$, ring of continuous functions on X, 128
$\Gamma(E)$, space of sections of E, 128
$\underline{\mathrm{Iso}}(E,F)$, bundle of fiber isomorphisms, 139
$\mathrm{Iso}_X(E,F)$, space of bundle isomorphisms, 139
O, infinite orthogonal group, 147
U, infinite unitary group, 148
$\mathrm{Vect}_n^{or}(X)$, isomorphism classes of rank n oriented bundles over X, 149
$VI_n(W)$, space of n-tuples of independent vectors inside of W, 151
$\mathrm{Gr}_n(W)$, Grassmannian of n-planes in W, 151
$\gamma_n(W)$, tautological n-bundle over $\mathrm{Gr}_n(W)$, 155
BG, classifying space of topological group G, 162
$KO(X)$, Grothendieck group of real vector bundles over the compact Hausdorff space X, 166
$KO^{st}(X)$, stable KO-group of X, 166
$KO_{Grt}(X)$, Grothendieck group of vector bundles on X, 172
$ko = bo$, connective real K-theory spectrum, 176
$ku = bu$, connective complex K-theory spectrum, 176
$\mathcal{K}(X,A)$, Grothendieck group of chain complexes of vector bundles that are exact on A, 191
$D_i(E)$, elementary complex consisting of E in degrees i and $i+1$, 193
$\mathcal{K}(X,A)_A$, Grothendieck group of A-relative exact chain complexes on X, 196
$\rho(n)$, Hurwitz-Radon number for n, 216
$[r,s,n]$-formula, sums-of-squares formula of type $[r,s,n]$, 216
Cl_k, real Clifford algebra on k generators, 218
$\mathrm{Cl}_n(R)$, Clifford algebra over R on n generators, 219
$\mathrm{Cl}(V,q)$, Clifford algebra for the vector space V equipped with quadratic form q, 225
ΓV, Dold construction for the chain complex V, 239
$\mathcal{U}_E$, Thom class for the bundle E, 256
$\mathrm{Th}\, E$, Thom space of the bundle E, 258
$N_{M/Z}$, normal bundle of Z in M, 265
$[Z] = [Z]_M$, fundamental class for submanifold $Z \hookrightarrow M$, 267
$[Z]_{rel}$, relative fundamental class for submanifold Z, 267
$J_{V,v}$, exterior (or dual Koszul) complex, 275
$J^*_{V,v}$, Koszul complex for vector space V and $v \in V$, 275
$K(x_1,\ldots,x_n;R)$, algebraic Koszul complex associated to $x_1,\ldots,x_n \in R$, 278
$H_i(x_1,\ldots,x_n;R)$, ith Koszul homology group, 279
$\mathcal{O}_X$, sheaf of algebraic functions on the scheme X, 287
$F_{(x)}$, stalk of sheaf F at point x, 287
$\mathcal{O}_{X,x}$, stalk of $\mathcal{O}_X$ at point x, 287
$\kappa(p)$, residue field of $\mathcal{O}_{X,p}$, for p a point in X, 290
$\mathrm{Bl}_A(X)$, blow-up of X as the closed subscheme A, 297
$R[It]$, Rees ring for ideal $I \subseteq R$, 299
$G(X)_Z$, Grothendieck group of coherent sheaves on X whose support is contained in Z, 316
$e(f_1,\ldots,f_r)$, Koszul multiplicity of the sequence $\underline{f}$, 329

$G_{grd}(R)$, Grothendieck group of graded modules, 337
Δf, finite difference operator, 343
Sf, finite sum operator, 343
$e^H(E)$, singular cohomology Euler class of bundle E, 370
$e^K(E)$, K-theoretic Euler class of bundle E, 374
$\lambda_t(E)$, generating function for the λ^i-operations, for vector bundle E, 377
$[x^k]p(x)$, coefficient of x^k in $p(x)$, 378
$c_i(E)$, ith singular cohomology Chern class of bundle E, 381
$c(E)$, total Chern class of bundle E, 381
$w_i(E)$, ith Stiefel-Whitney class of real bundle E, 384
$w(E)$, total Stiefel-Whitney class of E, 384
$\mathrm{ch}(E)$, topological Chern character of bundle E, 396
$\mathring{D}^{2n}$, interior of the disk $D^{2n} \subseteq \mathbb{R}^{2n}$, 406
$\dim f$, complex dimension of the map f, 406
$\mathrm{CO}(f)$, set of complex orientations on the map f, 408
$\mathrm{genus}_{\mathcal{E}}(X)$, $\mathcal{E}$-theory genus of the almost complex manifold X, 418
Td-genus(X), Todd genus of the almost complex manifold X, 419
$\int \alpha$, coefficient in the top-dimensional piece of cohomology class α, 419
$\mathrm{Td}(E)$, Todd class of E, 426
Td-num$_X(E)$, Todd number of the vector bundle E, 437
$H^i(X;\mathcal{F})$, sheaf cohomology group with coefficients in $\mathcal{F}$, 437
ω_X, canonical line bundle on X, 440
$\mathrm{div}(f)$, divisor zeros/poles of the meromorphic function f, 447
$L(D)$, space of meromorphic functions f with $\mathrm{div}(f) \geq -D$, 447
$\ell(D)$, dimension of $L(D)$, 447
$\deg(D)$, degree of the divisor D, 448
K_X, canonical class on Riemann surface X, 450
$\mathcal{O}_X(D)$, holomorphic line bundle associated to divisor D, 455
$x +_F y$, formal sum of x and y using the formal group law F, 460
$I(x)$, inverse function for a formal group law, 461
$\mathbb{G}_a$, the additive algebraic group $(\mathbb{A}^1, +)$, 463
$\mathbb{G}_m$, the multiplicative algebraic group $(\mathbb{A}^1 - 0, \cdot)$, 463
$\mathrm{FGL}(R)$, set of formal group laws over R, 463
$\log_F$, logarithm of theformal group law F, 466
$\exp_F$, the exponential of the formal group law F, 466
ω_F, the invariant differential for the formal group law F, 471
$h_k(x_1, \ldots, x_r)$, kth complete homogeneous symmetric polynomial, 474
MU, complex cobordism spectrum, 477
MO, real bordism spectrum, 479
$MO(n)$, Thom space of $\gamma_n \to BO(n)$, 479
$MU(n)$, Thom space of $\gamma_n \to BU(n)$, 479
$\tilde{\gamma}^k(E)$, kth reduced γ-operation on the bundle E, 482
$\gamma_t(x)$, generating function for γ^i-operations, 484
$F_n\mathcal{E}^p(X)$, Atiyah-Hirzebruch filtration on cohomology group $\mathcal{E}^p(X)$, 488
$F_n^\gamma K^0(X)$, γ-filtration on $K^0(X)$, 491

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