NOTES ON THE MILNOR CONJECTURES

DANIEL DUGGER

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Introduction

These lectures concern the two Milnor conjectures and their proofs: from [V3], [OVV], and [M2]. Voevodsky's proof of the norm residue symbol conjecture—which is now eight years old—came with an explosion of ideas. The aim of these notes is to make this explosion a little more accessible to topologists. My intention is not to give a completely rigorous treatment of this material, but just to outline the main ideas and point the reader in directions where he can learn more. I've tried to make the lectures accessible to topologists with no specialized knowledge in this area, at least to the extent that such a person can come away with a general sense of how homotopy theory enters into the picture.

Let me apologize for two aspects of these notes. Foremost, they reflect only my own limited understanding of this material. Secondly, I have made certain expository decisions about which parts of the proofs to present in detail and which parts to keep in a "black box"—and the reader may well be disappointed in my choices. I hope that in spite of these shortcomings the notes are still useful.

Sections 1, 2, and 3 each depend heavily on the previous one. Section 4 could almost be read independently of 2 and 3, except for the need of Remark 2.10.

1. The Milnor conjectures

The Milnor conjectures are two purely algebraic statements in the theory of fields, having to do with the classification of quadratic forms. In this section we'll review the basic theory and summarize the conjectures. Appendix A contains some supplementary material, where several examples are discussed.

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1.1. **Background.** Let F be a field. In some sense our goal is to completely classify symmetric bilinear forms over F. To give such a form $(-,-)$ on F^n is the same as giving a symmetric $n \times n$ matrix A, where $a_{ij} = (e_i, e_j)$. Two matrices A_1 and A_2 represent the same form up to a change of basis if and only if $A_1 = PA_2P^T$ for some invertible matrix P. The main classical theorem on this topic says that if $\mathrm{char}(F) \neq 2$ then every symmetric bilinear form can be diagonalized by a change of basis. The question remains to decide when two given diagonal matrices D_1 and D_2 represent equivalent bilinear forms. For instance, do $\begin{bmatrix} 2 & 0 \\ 0 & 11 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ represent the same form over $\mathbb{Q}$?

To pursue this question one looks for invariants. The most obvious of these is the rank of the matrix A. This is in fact the unique invariant when the field is algebraically closed. For suppose a form is represented by a diagonal matrix D, and let λ be a nonzero scalar. Construct a new basis by replacing the ith basis element e_i by λe_i. The matrix of the form with respect to this new basis is the same as D, but with the ith diagonal entry multiplied by λ^2. The conclusion is that multiplying the entries of D by squares does not change the isomorphism class of the underlying form. This leads immediately to the classical theorem saying that if every element of F is a square (which we'll write as $F = F^2$) then a symmetric bilinear form is completely classified by its rank.

We now restrict to nondegenerate forms, in which case the matrix A is nonsingular. The element $\det(A) \in F^*$ is not quite an invariant of the bilinear form, since after a change of basis the determinant of the new matrix will be $\det(P)\det(A)\det(P^T) = \det(P)^2\det(A)$. However, the determinant is a well-defined invariant if we regard it as an element of $F^*/(F^*)^2$. Since $\frac{22}{3}$ is not a square in $\mathbb{Q}$, for instance, this tells us that the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 11 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ don't represent isomorphic forms over $\mathbb{Q}$.

The rank and determinant are by far the simplest invariants to write down, but they are not very strong. They don't even suffice to distinguish forms over $\mathbb{R}$. This case is actually a good example to look at. For $a_1, \dots, a_n \in \mathbb{R}^*$, let $\langle a_1, \dots, a_n \rangle$ denote the form on $\mathbb{R}^n$ defined by $(e_i, e_j) = \delta_{i,j} a_i$. Since every element of $\mathbb{R}$ is either a square or the negative of a square, it follows that every nondegenerate real form is isomorphic to an $\langle a_1, \dots, a_n \rangle$ where each $a_i \in \{1, -1\}$. When are two such forms isomorphic? Of course one knows the answer, but let's think through it. The Witt Cancellation Theorem (true over any field) says that if $\langle x_1, \dots, x_n, y_1, \dots, y_k \rangle \cong \langle x_1, \dots, x_n, z_1, \dots, z_k \rangle$ then $\langle y_1, \dots, y_k \rangle \cong \langle z_1, \dots, z_k \rangle$. So our problem reduces to deciding whether the n-dimensional forms $\langle 1, 1, \dots, 1 \rangle$ and $\langle -1, \dots, -1 \rangle$ are isomorphic. When n is odd the determinant distinguishes them, but when n is even it doesn't. Of course the thing to say is that the associated quadratic form takes only positive values in the first case, and only negative values in the second—but this is not exactly an 'algebraic' way of distinguishing the forms, in that it uses the ordering on $\mathbb{R}$ in an essential way. By the end of this section we will indeed have purely algebraic invariants we can use here.

1.2. **The Grothendieck-Witt ring.** In a moment we'll return to the problem of finding invariants more sophisticated than the rank and determinant, but first we need a little more machinery. From now on $\mathrm{char}(F) \neq 2$. By a **quadratic space**

I mean a pair (V, μ) consisting of a finite-dimensional vector space and a non-degenerate bilinear form μ. To systemize their study one defines the *Grothendieck-Witt ring* $GW(F)$. This is the free abelian group generated by isomorphism classes of pairs (V, μ), with the usual relation identifying the direct sum of quadratic spaces with the sum in the group. The multiplication is given by tensor product of vector spaces.

The classical theory of bilinear forms allows us to give a complete description of the abelian group $GW(F)$ in terms of generators and relations. Recall that $\langle a_1, \ldots, a_n\rangle$ denotes the n-dimensional space F^n with $(e_i, e_j) = \delta_{ij} a_i$. So $\langle a_1, \ldots, a_n\rangle = \langle a_1\rangle + \cdots + \langle a_n\rangle$ in $GW(F)$. The fact that every symmetric bilinear form is diagonalizable tells us that $GW(F)$ is generated by the elements $\langle a\rangle$ for $a \in F^*$, and we have already observed the relation $\langle ab^2\rangle = \langle a\rangle$ for any $a, b \in F^*$. As an easy exercise, one can also give a complete description for when *two-dimensional* forms are isomorphic: one must be able to pass from one to the other via the two relations

$$\langle ab^2\rangle = \langle a\rangle \quad \text{and} \quad \langle a, b\rangle = \langle a + b, ab(a + b)\rangle \tag{1.3}$$

where in the second we assume $a, b \in F^*$ and $a + b \neq 0$. As an example, working over $\mathbb{Q}$ we have

$$\langle 3, -2\rangle = \langle 12, -2\rangle = \langle 10, -240\rangle = \langle 90, -15\rangle.$$

To completely determine all relations in $GW(F)$, one shows that if two forms $\langle a_1, \ldots, a_n\rangle$ and $\langle b_1, \ldots, b_n\rangle$ are isomorphic then there is a chain of isomorphic diagonal forms connecting one to the other, where each link of the chain differs in exactly two elements. Thus, (1.3) is a complete set of relations for $GW(F)$. The reader may consult [S1, 2.9.4] for complete details here.

The multiplication in $GW(F)$ can be described compactly by

$$\langle a_1, \ldots, a_n\rangle \cdot \langle b_1, \ldots, b_k\rangle = \sum_{i,j} \langle a_i b_j\rangle.$$

1.4. **The Witt ring.** The *Witt ring* $W(F)$ is the quotient of $GW(F)$ by the ideal generated by the so-called 'hyperbolic plane' $\langle 1, -1\rangle$. Historically $W(F)$ was studied long before $GW(F)$, probably because it can be defined without formally adjoining additive inverses as was done for $GW(F)$. One can check that the forms $\langle a, -a\rangle$ and $\langle 1, -1\rangle$ are isomorphic, and therefore if one regards hyperbolic forms as being zero then $\langle a_1, \ldots, a_n\rangle$ and $\langle -a_1, \ldots, -a_n\rangle$ are additive inverses. So $W(F)$ can be described as a set of equivalence classes of quadratic spaces, and doesn't require working with 'virtual' objects.

Because $\langle a, -a\rangle \cong \langle 1, -1\rangle$ for any a, it follows that the ideal $\big(\langle 1, -1\rangle\big)$ is precisely the additive subgroup of $GW(F)$ generated by $\langle 1, -1\rangle$. As an abelian group, it is just a copy of $\mathbb{Z}$. So we have the exact sequence $0 \to \mathbb{Z} \to GW(F) \to W(F) \to 0$.

Let $GI(F)$ be the kernel of the dimension function $\dim\colon GW(F) \to \mathbb{Z}$, usually called the augmentation ideal. Let $I(F)$ be the image of the composite $GI(F) \hookrightarrow GW(F) \twoheadrightarrow W(F)$; one can check that $I(F)$ consists precisely of equivalence classes of even-dimensional quadratic spaces. Note that I is additively generated by forms $\langle 1, a\rangle$, and therefore I^n is additively generated by n-fold products $\langle 1, a_1\rangle\langle 1, a_2\rangle \cdots \langle 1, a_n\rangle$.

The dimension function gives an isomorphism $W/I \to \mathbb{Z}/2$. The determinant gives us a group homomorphism $GW(F) \to F^*/(F^*)^2$, but it does not extend to the Witt ring because $\det\langle 1,-1\rangle = -1$. One defines the *discriminant* of $\langle a_1,\ldots,a_n\rangle$ to be $(-1)^{\frac{n(n-1)}{2}}\cdot(a_1\cdots a_n)$, and with this definition the discriminant gives a map of sets $W(F)\to F^*/(F^*)^2$. It is not a homomorphism, but if we restrict to $I(F)\to F^*/(F^*)^2$ then it *is* a homomorphism. As the discriminant of $\langle 1,a\rangle\langle 1,b\rangle$ is a square, the elements of I^2 all map to 1. So we get an induced map $I/I^2\to F^*/(F^*)^2$, which is obviously surjective. It is actually an isomorphism—to see this, note that

$$\langle x,y\rangle\langle -1,y\rangle = \langle -x,xy,-y,y^2\rangle = \langle 1,-x,-y,xy\rangle$$

and so $\langle x,y\rangle \equiv \langle 1,xy\rangle \pmod{I^2}$. It follows inductively that $\langle a_1,\ldots,a_{2n}\rangle \equiv \langle 1,1,\ldots,1,a_1a_2\cdots a_{2n}\rangle \pmod{I^2}$. So if $\langle a_1,\ldots,a_{2n}\rangle$ is a form whose discriminant is a square, it is equivalent mod I^2 to either $\langle 1,1,\ldots,1\rangle = 2n\langle 1\rangle$ (if n is even) or $\langle 1,1,\ldots,1,-1\rangle = (2n-2)\langle 1\rangle$ (if n is odd). In the former case $2n\langle 1\rangle = 2\langle 1\rangle\cdot n\langle 1\rangle\in I^2$, and in the latter case $(2n-2)\langle 1\rangle = 2\langle 1\rangle\cdot(n-1)\langle 1\rangle\in I^2$. In either case we have $\langle a_1,\ldots,a_{2n}\rangle\in I^2$, and this proves injectivity.

The examples in the previous paragraph are very special, but they suggest why one might hope for 'higher' invariants which give isomorphisms between the groups I^n/I^{n+1} and something more explicitly defined in terms of the field F. This is what the Milnor conjecture is about.

Remark 1.5. For future reference, note that $2\langle 1\rangle = \langle 1,1\rangle\in I$, and therefore the groups I^n/I^{n+1} are $\mathbb{Z}/2$-vector spaces. Also observe that $GI(F)$ does not intersect the kernel of $GW(F)\to W(F)$, and so $GI(F)\to I(F)$ is an isomorphism. It follows that $(GI)^n/(GI)^{n+1}\cong I^n/I^{n+1}$, for all n.

1.6. **More invariants.** Recall that the **Brauer group** $\mathrm{Br}(F)$ is a set of equivalence classes of central, simple F-algebras, with the group structure coming from tensor product. The inverse of such an algebra is its opposite algebra, where the order of multiplication has been reversed.

From a quadratic space (V,μ) one can construct the associated Clifford algebra $C(\mu)$: this is the quotient of the tensor algebra $T_F(V)$ by the relations generated by $v\otimes v = \mu(v,v)$. Clifford algebras are $\mathbb{Z}/2$-graded by tensor length. If μ is even-dimensional then $C(\mu)$ is a central simple algebra, and if μ is odd-dimensional then the even part $C_0(q)$ is a central simple algebra. So we get an invariant of quadratic spaces taking its values in $\mathrm{Br}(F)$ (see [S1, 9.2.12] for more detail). This is usually called the **Clifford invariant**, or sometimes the **Witt invariant**. Since any Clifford algebra is isomorphic to its opposite, the invariant always produces a 2-torsion class.

Now we need to recall some Galois cohomology. Let $\bar{F}$ be a separable closure of F, and let $G = \mathrm{Gal}(\bar{F}/F)$. Consider the short exact sequence of G-modules $0\to\mathbb{Z}/2\to\bar{F}^*\to\bar{F}^*\to 0$, where the second map is squaring. Hilbert's Theorem 90 implies that $H^1(G;\bar{F}^*) = 0$, which means that the induced long exact sequence in Galois cohomology splits up into

$$0\to H^0(G;\mathbb{Z}/2)\to F^*\xrightarrow{2} F^*\to H^1(G;\mathbb{Z}/2)\to 0$$

and

$$0\to H^2(G;\mathbb{Z}/2)\to H^2(G;\bar{F}^*)\xrightarrow{2} H^2(G;\bar{F}^*).$$

The group $H^2(G;\bar{F}^*)$ is known to be isomorphic to $\mathrm{Br}(F)$, so we have $H^0(G;\mathbb{Z}/2) = \mathbb{Z}/2$, $H^1(G;\mathbb{Z}/2) = F^*/(F^*)^2$, and the 2-torsion in the Brauer group is precisely $H^2(G;\mathbb{Z}/2)$. From now on we will write $H^*(F;\mathbb{Z}/2) = H^*(G;\mathbb{Z}/2)$.

At this point we have the rank map $e_0\colon W(F) \to \mathbb{Z}/2 = H^0(F;\mathbb{Z}/2)$, which gives an isomorphism $W/I \to \mathbb{Z}/2$. We have the discriminant $e_1\colon I(F) \to F^*/(F^*)^2 = H^1(F;\mathbb{Z}/2)$ which gives an isomorphism $I/I^2 \to F^*/(F^*)^2$, and we have the Clifford invariant $e_2\colon I^2 \to H^2(F;\mathbb{Z}/2)$. With a little work one can check that e_2 is a homomorphism, and it kills I^3. The question of whether $I^2/I^3 \to H^2(F;\mathbb{Z}/2)$ is an isomorphism is difficult, and wasn't proven until the early 80s by Merkurjev [M] (neither surjectivity nor injectivity is obvious). The maps e_0, e_1, e_2 are usually called the *classical invariants* of quadratic forms.

The above isomorphisms can be rephrased as follows. The ideal I consists of all elements where $e_0 = 0$; I^2 consists of all elements such that $e_0 = 0$ and $e_1 = 1$; and by Merkujev's theorem I^3 is precisely the set of elements for which e_0, e_1, and e_2 are all trivial. Quadratic forms will be completely classified by these invariants if $I^3 = 0$, but unfortunately this is usually not the case. This brings us to the search for higher invariants. One early result along these lines is due to Delzant [De], who defined Stiefel-Whitney invariants with values in Galois cohomology. Unfortunately these are not the 'right' invariants, as they do not lead to complete classifications for elements in I^n, $n \geq 3$.

1.7. **Milnor's work.** At this point we find ourselves looking at the two rings $\mathrm{Gr}_I\, W(F)$ and $H^*(F;\mathbb{Z}/2)$, and we have maps between them in dimensions 0, 1, and 2. I think Milnor, inspired by his work on algebraic K-theory, wrote down the best ring he could find which would map to both rings above. In [Mi] he defined what is now called 'Milnor K-theory' as

$$K_*^M(F) = T_{\mathbb{Z}}(F^*)/\langle a \otimes (1-a) | a \in F - \{0,1\}\rangle$$

where $T_{\mathbb{Z}}(V)$ denotes the tensor algebra over $\mathbb{Z}$ on the abelian group V. The grading comes from the grading on the tensor algebra, in terms of word length. I will write $\{a_1, \dots, a_n\}$ for the element $a_1 \otimes \cdots \otimes a_n \in K_n^M(F)$.

Note that when dealing with $K_*^M(F)$ one must be careful not to confuse the addition—which comes from multiplication in F^*—with the multiplication. So for instance $\{a\}+\{b\} = \{ab\}$ but $\{a\}\cdot\{b\} = \{a, b\}$. This is in contrast to the operations in $GW(F)$, where one has $\langle a\rangle + \langle b\rangle = \langle a, b\rangle$ and $\langle a\rangle \otimes \langle b\rangle = \langle ab\rangle$. Unfortunately it is very easy to get these confused. Note that $\{a^2\} = 2\{a\}$, and more generally $\{a^2, b_1, \dots, b_n\} = 2\{a, b_1, \dots, b_n\}$.

Remark 1.8. From a modern perspective the name 'K-theory' applied to $K_*^M(F)$ is somewhat of a misnomer; one should not take it too seriously. The construction turns out to be more closely tied to algebraic cycles than to algebraic K-theory, and so I personally like the term 'Milnor cycle groups'. I doubt this will ever catch on, however.

Milnor produced two ring homomorphisms $\eta\colon K_*^M(F)/2 \to H^*(F;\mathbb{Z}/2)$ and $\nu\colon K_*^M(F)/2 \to \mathrm{Gr}_I\, W(F)$. To define the map ν, note first that we have already established an isomorphism $F^*/(F^*)^2 \to I/I^2$ sending $\{a\}$ to $\langle a, -1\rangle = \langle a\rangle - \langle 1\rangle$ (this is the inverse of the discriminant). This tells us what ν does to elements in degree 1. Since these elements generate $K_*^M(F)$ multiplicatively, to construct ν it suffices to verify that the appropriate relations are satisfied in the image. So we first need to check that

$$0 = \Big(\langle a\rangle - \langle 1\rangle\Big)\cdot\Big(\langle 1-a\rangle - \langle 1\rangle\Big) = \langle a(1-a)\rangle - \langle a\rangle - \langle 1-a\rangle + \langle 1\rangle = \langle a(1-a), 1\rangle - \langle a, 1-a\rangle,$$

but this follows directly from the second relation in (1.3). We also must check that $2\{a\}$ maps to 0, but $2\{a\} = \{a^2\} \mapsto \langle a^2 \rangle - \langle 1 \rangle$ and the latter vanishes by the first relation in (1.3). For future reference, note that $\nu(\{a\})$ is equal to both $\langle a, -1 \rangle$ and $\langle -a, 1 \rangle$ in I/I^2, since this group is 2-torsion.

Defining η is similar. We have already noticed that there is a natural isomorphism $H^1(F; \mathbb{Z}/2) \cong F^*/(F^*)^2$, and so it is clear where the element $\{a\}$ in $K_1^M(F) = F^*$ must be sent. The verification that $a \cup (1-a) = 0$ in $H^2(F; \mathbb{Z}/2)$ is in [Mi, 6.1].

Milnor observed that both η and ν were isomorphisms in all the cases he could compute. The claim that η is an isomorphism is nowadays known as *the Milnor conjecture*, and was proven by Voevodsky in 1996 [V1]. The claim that ν is an isomorphism goes under the name *Milnor's conjecture on quadratic forms*. For characteristic 0 it was proven in 1996 by Orlov, Vishik, and Voevodsky [OVV], who deduced it as a consequence of the work in [V1]. I believe the proof now works in characteristic p, based on the improved results of [V3]. A second proof, also in characteristic 0, was outlined by Morel [M2] using the motivic Adams spectral sequence, and again depended on results from [V1]; unfortunately complete details of Morel's proof have yet to appear.

It is interesting that the conjecture on quadratic forms doesn't have an independent proof, and is the less primary of the two. Note that both $K_*^M(F)/2$ and $GW(F)$ can be completely described in terms of generators and relations (although the latter does not quite imply that we know all the relations in $\operatorname{Gr}_I W(F)$, which is largely the problem). The map ν is easily seen to be surjective, and so the only question is injectivity. Given this, it is in some ways surprising that the conjecture is as hard as it is.

Remark 1.9. The map η is called the *norm residue symbol*, and can be defined for primes other than 2. The *Bloch-Kato conjecture* is the statement that $\eta \colon K_i^M(F)/l \to H^i(F; \mu_l^{\otimes i})$ is an isomorphism for l a prime different from $\operatorname{char}(F)$. This is a direct generalization of the Milnor conjecture to the case of odd primes. A proof was released by Voevodsky in 2003 [V5] (although certain auxiliary results required for the proof remain unwritten). I'm not sure anyone has ever considered an odd-primary analog of Milnor's conjecture on quadratic forms—what could replace the Grothendieck-Witt ring here?

At this point it might be useful to think through the Milnor conjectures in a few concrete examples. For these we refer the reader to Appendix A. Let's at least note here that through the work of Milnor, Bass, and Tate (cf. [Mi]) the conjectures could be verified for all finite fields and for all finite extensions of $\mathbb{Q}$ (in fact for all global fields).

Finally, let's briefly return to the classification of forms over $\mathbb{R}$. We saw earlier that this reduces to proving that the n-dimensional forms $\langle 1, 1, \ldots, 1 \rangle$ and $\langle -1, -1, \ldots, -1 \rangle$ are not isomorphic. Can we now do this algebraically? If they were isomorphic, they would represent the same element of $W(\mathbb{R})$. It would follow that $(2n)\langle 1 \rangle = 0$ in $W(\mathbb{R})$. Can this happen? The isomorphisms $\mathbb{Z}/2[a] \cong H^*(\mathbb{Z}/2; \mathbb{Z}/2) \cong K_*^M(\mathbb{R})/2 \cong \operatorname{Gr}_I W(\mathbb{R})$ show that $\operatorname{Gr}_I W(\mathbb{R})$ is a polynomial algebra on the class $\langle -1, -1 \rangle$ (the generator a corresponds to the generator -1 of $\mathbb{R}^*/(\mathbb{R}^*)^2$, and $\nu(-1) = \langle -1, -1 \rangle$). It follows that $2^k \langle 1 \rangle = \pm \langle -1, -1 \rangle^k$ is a generator for the group $I^k/I^{k+1} \cong \mathbb{Z}/2$. If $m = 2^i r$ where r is odd, then $m\langle 1 \rangle = 2^i \langle 1 \rangle \cdot r\langle 1 \rangle$. Since $r\langle 1 \rangle$ is the generator for W/I and $2^i\langle 1 \rangle$ is a generator for I^i/I^{i+1}, it follows that $m\langle 1\rangle$ is also a generator for I^i/I^{i+1}. In particular, $m\langle 1\rangle$ is nonzero. So we have proven via algebraic methods (although in this case also somewhat pathological ones) that $\langle 1, 1, \ldots, 1\rangle \ncong \langle -1, -1, \ldots, -1\rangle$.

1.10. **Further background reading.** There are several good expository papers on the theory of quadratic forms, for example [Pf1] and [S2]. The book [S1] is a very thorough and readable resource as well. For the Milnor conjectures themselves there is [Pf2], which in particular gives several applications of the conjectures; it also gives detailed references to original papers. The beginning sections of [AEJ] offer a nice survey concerning the search for 'higher' invariants of quadratic forms. It's worth pointing out that after Milnor's work definitions of e_3, e_4, and e_5 were eventually given—with a lot of hard work—but this was the state of the art until 1996. Finally, the introduction of [V3] gives a history of work on the Milnor conjecture.

2. Proof of the conjecture on the norm residue symbol

This section outlines Voevodsky's proof of the Milnor conjecture on the norm residue symbol [V1, V3]. Detailed, step-by-step summaries have been given in [M1] and [Su]. My intention here is not to give a complete, mathematically rigorous presentation, but rather just to give the flavor of what is involved.

Several steps in the proof involve manipulations with motivic cohomology based on techniques that were developed in [VSF]. I have avoided giving any details about these steps, in an attempt to help the exposition. Most of these details are not hard to understand, however—there are only a few basic techniques to keep track of, and one can read about them in [VSF] or [MVW]. But I hope that by keeping some of this stuff in a black box the overall structure of the argument will become clearer.

2.1. **Initial observations.** The aim is to show that $\eta\colon K_*^M(F)/2 \to H^*(F;\mathbb{Z}/2)$ is an isomorphism. To do this, one of the first things one might try to figure out is what kind of extra structure $K_*^M(F)/2$ and $H^*(F;\mathbb{Z}/2)$ have in common. For instance, they are both covariant functors in F, and the covariance is compatible with the norm residue symbol. It turns out they both have transfer maps for finite separable extensions (which, for those who like to think geometrically, are the analogs of covering spaces). That is, if $j\colon F \hookrightarrow F'$ is a separable extension of degree n then there is a map $j^!\colon K_*^M(F') \to K_*^M(F)$ such that $j^! j_*$ is multiplication by n, and similarly for $H^*(F;\mathbb{Z}/2)$. (Note that the construction of transfer maps for Milnor K-theory is not at all trivial—some ideas were given in [BT, Sec. 5.9], but the full construction is due to Kato [Ka, Sec. 1.7]). It follows that if n is odd then $K_*^M(F)/2 \to H^*(F;\mathbb{Z}/2)$ is a retract of the map $K_*^M(F')/2 \to H^*(F';\mathbb{Z}/2)$. So if one had a counterexample to the Milnor conjecture, field extensions of all odd degrees would still be counterexamples. This is often referred to as "the transfer argument".

Another observation is that both functors can be extended to rings other than fields, and if R is a discrete valuation ring then both functors have a 'localization sequence' relating their values on R, the residue field, and the quotient field. I will not go into details here, but if F is a field of characteristic p then by using the Witt vectors over F and the corresponding localization sequence, one can reduce the Milnor conjecture to the case of characteristic 0 fields. The argument is in [V1, Lemma 5.2]. In Voevodsky's updated proof of the Milnor conjecture [V3] this step is not necessary, but I think it's useful to realize that the Milnor conjecture is not hard because of 'crazy' things that might happen in characteristic p—it is hard even in characteristic 0.

2.2. **A first look at the proof.** The proof goes by induction. We assume the norm residue map $\eta\colon K_*^M(F)/2 \to H^*(F;\mathbb{Z}/2)$ is an isomorphism for all fields F and all $* < n$, and then prove it is also an isomorphism for $* = n$. The basic theme of the proof, which goes back to Merkurjev, involves two steps:

(1) Verify that η_n is an isomorphism for certain 'big' fields—in our case, those which have no extensions of odd degree and also satisfy $K_n(F) = 2K_n(F)$ (so that one must prove $H^n(F;\mathbb{Z}/2) = 0$). Notice that when $n = 1$ the condition $K_1 = 2K_1$ says that $F = F^2$.

(2) Prove that if F were a field for which η_n is *not* an isomorphism then one could expand F to make a 'bigger' counterexample, and could keep doing this until

you're in the range covered by step (1). This would show that no such F could exist.

In more detail one shows that for any $\{a_1, \dots, a_n\} \in K_n(F)$ one can construct an extension $F \hookrightarrow F'$ with the property that $\{a_1, \dots, a_n\} \in 2K_n(F')$ and $\eta_n \colon K_n(F')/2 \to H^n(F'; \mathbb{Z}/2)$ still fails to be an isomorphism. By doing this over and over and taking a big colimit, one gets a counterexample where $K_n^M = 2K_n^M$.

Neither of the above two steps is trivial, but step (1) involves nothing very fancy—it is a calculation in Galois cohomology which takes a few pages, but is not especially hard. See [V3, Section 5]. Step (2) is the more subtle and interesting step. Note that if $\underline{a} = \{a_1, \dots, a_n\} \notin 2K_n^M(F)$ then none of the a_i's can be in F^2. There are several ways one can extend F to a field F' such that $\underline{a} \in 2K_n^M(F')$: one can adjoin a square root of any a_i, for instance. The problem is to find such an extension where you have enough control over the horizontal maps in the diagram

$$\begin{array}{ccc} K_n^M(F)/2 & \longrightarrow & K_n^M(F')/2 \\ \downarrow{\scriptstyle \eta_F} & & \downarrow{\scriptstyle \eta_{F'}} \\ H^n(F; \mathbb{Z}/2) & \longrightarrow & H^n(F'; \mathbb{Z}/2) \end{array}$$

to show that if η_F fails to be an isomorphism then so does $\eta_{F'}$. The selection of the 'right' F' is delicate.

We will alter our language at this point, because we will want to bring more geometry into the picture. Any finitely-generated separable extension $F \hookrightarrow F'$ is the function field of a smooth F-variety. A **splitting variety** for an element $\underline{a} \in K_n^M(F)$ is a smooth variety X, of finite type over F, with the property that $\underline{a} \in 2K_n^M(F(X))$. Here $F(X)$ denotes the function field of X. As we just remarked, there are many such varieties: $X = \operatorname{Spec} F[u]/(u^2 - a_1)$ is an example. The particular choice we'll be interested in is more complicated.

Given $b_1, \dots, b_k \in F$, let $q_{\underline{b}}$ be the quadratic form in 2^k variables corresponding to the element

$$\langle 1, -b_1 \rangle \otimes \langle 1, -b_2 \rangle \otimes \cdots \otimes \langle 1, -b_k \rangle \in GW(F).$$

For example, $q_{b_1,b_2}(x_1, \dots, x_4) = x_1^2 - b_1 x_2^2 - b_2 x_3^2 + b_1 b_2 x_4^2$. Such q's are called **Pfister forms**, and they have a central role in the modern theory of quadratic forms (see [S1, Chapter 4], for instance).

For $a_1, \dots, a_n \in F$, define $Q_{\underline{a}}$ to be the projective quadric in $\mathbb{P}^{2^{n-1}}$ given by the equation

$$q_{a_1,\dots,a_{n-1}}(x_0, \dots, x_{[2^{n-1}-1]}) - a_n x_{2^{n-1}}^2 = 0.$$

In [V3] these are called **norm quadrics**. A routine argument [V3, Prop. 4.1] shows that $Q_{\underline{a}}$ is a splitting variety for $\underline{a}$. The reason for choosing to study this particular splitting variety will not be clear until later; isolating this object is one of the key aspects of the proof.

The name of the game will be to understand enough about the difference between $K_n^M(F)/2$ and $K_n^M(F(Q_{\underline{a}}))/2$ (as well as the corresponding Galois cohomology groups) to show that $K_n^M(F(Q_{\underline{a}}))/2 \to H^n(F(Q_{\underline{a}}); \mathbb{Z}/2)$ still fails to be an isomorphism. Voevodsky's argument uses motivic cohomology—of the quadrics $Q_{\underline{a}}$ and other objects—to 'bridge the gap' between $K_n^M(F)/2$ and $K_n^M(F(Q_{\underline{a}}))/2$.

2.3. **Motivic cohomology enters the picture.** Motivic cohomology is a bigraded functor $X \mapsto H^{p,q}(X;\mathbb{Z})$ defined on the category of smooth F-schemes. Actually it is defined for all simplicial smooth schemes, as well as for more general objects. One of the lessons of the last ten years is that one can set up a model category which contains all these objects, and then a homotopy theorist can deal with them in much the same ways he deals with ordinary topological spaces. From now on I will do this implicitly (without ever referring to the machinery involved).

The coefficient groups $H^{p,q}(\operatorname{Spec} F;\mathbb{Z})$ vanish for $q<0$ and for $p>q\geq 0$. For us an important point is that the groups $H^{n,n}(\operatorname{Spec} F;\mathbb{Z})$ are canonically isomorphic to $K_n^M(F)$. Proving this is not simple! An account is given in [MVW, Lecture 5]. Finally, we note that one can talk about motivic cohomology with finite coefficients $H^{p,q}(X;\mathbb{Z}/n)$, related to integral cohomology via the exact sequence

$$\cdots \to H^{p,q}(X;\mathbb{Z}) \xrightarrow{\times n} H^{p,q}(X;\mathbb{Z}) \to H^{p,q}(X;\mathbb{Z}/n) \to H^{p+1,q}(X;\mathbb{Z}) \to \cdots$$

The sequence shows $H^{n,n}(\operatorname{Spec} F;\mathbb{Z}/2) \cong K_n^M(F)/2$ and $H^{p,q}(\operatorname{Spec} F;\mathbb{Z}/2)=0$ for $p>q\geq 0$.

Now, there is also an analagous theory $H_L^{p,q}(X;\mathbb{Z})$ which is called **Lichtenbaum (or étale) motivic cohomology**. There is a natural transformation $H^{p,q}(X;\mathbb{Z}) \to H_L^{p,q}(X;\mathbb{Z})$. The theory $H_L^{*,*}$ is the closest theory to $H^{*,*}$ which satisfies descent for the étale topology (essentially meaning that when $E \to B$ is an étale map there is a spectral sequence starting with $H_L^{*,*}(E)$ and converging to $H_L^{*,*}(B)$). The relation between $H^{*,*}$ and $H_L^{*,*}$ is formally analagous to that between a cohomology theory and a certain Bousfield localization of it. It is known that $H_L^{p,q}(X;\mathbb{Z}/n)$ is canonically isomorphic to étale cohomology $H^p_{et}(X;\mu_n^{\otimes q})$, if n is prime to $\operatorname{char}(F)$. From this it follows that $H_L^{p,q}(\operatorname{Spec} F;\mathbb{Z}/2)$ is the Galois cohomology group $H^p(F;\mathbb{Z}/2)$, for all q. At this point we can re-phrase the Milnor conjecture as the statement that the maps $H^{p,p}(\operatorname{Spec} F;\mathbb{Z}/2) \to H_L^{p,p}(\operatorname{Spec} F;\mathbb{Z}/2)$ are isomorphisms.

There are other conjectures about the relation between $H^{*,*}$ and $H_L^{*,*}$ as well. A conjecture of Lichtenbaum says that $H^{p,q}(X;\mathbb{Z}) \to H_L^{p,q}(X;\mathbb{Z})$ should be an isomorphism whenever $p \leq q+1$. Note that this would imply a corresponding statement for $\mathbb{Z}/n$-coefficients, and in particular would imply the Milnor conjecture. Also, since one knows $H^{n+1,n}(\operatorname{Spec} F;\mathbb{Z})=0$ Lichtenbaum's conjecture would imply that $H_L^{n+1,n}(\operatorname{Spec} F;\mathbb{Z})$ also vanishes. This latter statement was conjectured independently by both Beilinson and Lichtenbaum, and is known as a the **Generalized Hilbert's Theorem 90** (the case $n=1$ is a translation of the statement that $H^1_{Gal}(F;\bar{F}^*)=0$, which follows from the classical Hilbert's Theorem 90).

By knowing enough about how to work with motivic cohomology, Voevodsky was able to prove the following relation among these conjectures (as well as other relations which we won't need):

Proposition 2.4. *Fix an $n \geq 0$. Assume that $H_L^{k+1,k}(\operatorname{Spec} F;\mathbb{Z}_{(2)})=0$ for all fields F and all $0 \leq k \leq n$. Then for any smooth simplicial scheme X over a field F, the maps $H^{p,q}(X;\mathbb{Z}/2) \to H_L^{p,q}(X;\mathbb{Z}/2)$ are isomorphisms when $q \geq 0$ and $p \leq q \leq n$; and they are monomorphisms for $p-1=q\leq n$. In particular, applying this when $p=q$ and $X=\operatorname{Spec} F$ verifies the Milnor conjecture in dimensions $\leq n$.*

It's worth pointing out that the proof uses nothing special about the prime 2, and so the statement is valid for all other primes as well.

For us, the importance of the above proposition is two-fold. First, it says that to prove the Milnor conjecture one only has to worry about the vanishing of one set of groups (the $H_L^{n+1,n}$'s) rather than two sets (the kernel and cokernel of η). Secondly, inductively assuming that the Generalized Hilbert's Theorem 90 holds up through dimension n is going to give us a lot more to work with than inductively assuming the Milnor conjecture up through dimension n. Instead of just knowing stuff about $H^{n,n}$ of fields, we know stuff about $H^{p,q}$ of any smooth simplicial scheme. The need for this extra information is a key feature of the proof.

2.5. **Čech complexes.** We only need one more piece of machinery before returning to the proof of the Milnor conjecture. This piece is hard to motivate, and its introduction is one of the more ingenious aspects of the proof. The reader will just have to wait and see how it arises in section 2.6 (see also Remark 3.10).

Let X be any scheme. The **Čech complex $\check{C}X$** is the simplicial scheme with $(\check{C}X)_n = X \times X \times \cdots \times X$ ($n+1$ factors) and the obvious face and degeneracies. This simplicial scheme can be regarded as augmented by the map $X \to \operatorname{Spec} F$.

For a topological space the realization of the associated Čech complex is always contractible—in fact, choosing any point of X allows one to write down a contracting homotopy for the simplicial space $\check{C}X$. But in algebraic geometry the scheme X may not have rational points; i.e., there may not exist any maps $\operatorname{Spec} F \to X$ at all! If X does have a rational point then the same trick lets one write down a contracting homotopy, and therefore $\check{C}X$ behaves as if it were $\operatorname{Spec} F$ in all computations. (More formally, $\check{C}X$ is homotopy equivalent to $\operatorname{Spec} F$ in the motivic homotopy category).

Working in the motivic homotopy category, one finds that for any smooth scheme Y the set of homotopy classes $[Y, \check{C}X]$ is either empty or a singleton. The latter holds precisely if Y admits a Zariski cover $\{U_\alpha\}$ such that there exist scheme maps $U_\alpha \to X$ (not necessarily compatible on the intersections). The object $\check{C}X$ has no 'higher homotopy information', only this very simple discrete information about whether or not certain maps exist. One should think of $\check{C}X$ as very close to being contractible. I point out again that in topology there is always at least one map between nonempty spaces, and so $\check{C}X$ is not very interesting.

If $E \to B$ is an étale cover, then there is a spectral sequence whose input is $H_L^{*,*}(E;\mathbb{Z})$ and which converges to $H_L^{*,*}(B;\mathbb{Z})$ (this is the étale descent property). In particular, if X is a smooth scheme and we let $F' = F(X)$, $X' = X \times_F F'$, then $X' \to X$ and $\operatorname{Spec} F' \to \operatorname{Spec} F$ are both étale covers. The scheme X' necessarily has a rational point over F', so $\check{C}X'$ and $\operatorname{Spec} F'$ look the same to H_L. The étale descent property then shows that $\check{C}X$ and $\operatorname{Spec} F$ also look the same: in other words, the maps $H_L^{p,q}(\operatorname{Spec} F;\mathbb{Z}) \to H_L^{p,q}(\check{C}X;\mathbb{Z})$ are all isomorphisms (and the same for finite coefficients). This is *not* true for $H^{*,*}$ in place of $H_L^{*,*}$. One might paraphrase all this by saying that in the étale world $\check{C}X$ is contractible, just as it is in topology.

2.6. **The proof.** Now I am going to give a complete summary of the proof as it appears in [V1, V3]. Instead of proving the Milnor conjecture in its original form one instead concentrates on the more manageable conjecture that $H_L^{i+1,i}(\operatorname{Spec} F;\mathbb{Z}_{(2)}) = 0$ for all i and all fields F. One assumes this has been proven in the range $0 \le i < n$, and then shows that it also follows for $i = n$.

Suppose that F is a field with $H_L^{n+1,n}(F;\mathbb{Z}_{(2)}) \neq 0$. The transfer argument shows that any extension field of odd degree would still be a counterexample, so we can assume F has no extensions of odd degree. One checks via some Galois cohomology computations—see [V3, section 5]—that if such a field has $K_n^M(F) = 2K_n^M(F)$ then $H_L^{n+1,n}(\operatorname{Spec} F;\mathbb{Z}_{(2)}) = 0$. So our counterexample cannot have $K_n^M(F) = 2K_n^M(F)$. By the reasoning from section 2.2, it will suffice to show that for every $a_1,\dots,a_n \in F$ the field $F(Q_{\underline{a}})$ is still a counterexample. We will in fact show that $H_L^{n+1,n}(F;\mathbb{Z}_{(2)}) \to H_L^{n+1,n}(F(Q_{\underline{a}});\mathbb{Z}_{(2)})$ is injective.

Suppose u is in the kernel of the above map, and consider the diagram

$$\begin{array}{ccccc} & & H_L^{n+1,n}(\operatorname{Spec} F;\mathbb{Z}_{(2)}) & \longrightarrow & H_L^{n+1,n}(\operatorname{Spec} F(Q_{\underline{a}});\mathbb{Z}_{(2)}) \\ & & \downarrow \cong & & \\ H^{n+1,n}(\check{C}Q_{\underline{a}};\mathbb{Z}_{(2)}) & \longrightarrow & H_L^{n+1,n}(\check{C}Q_{\underline{a}};\mathbb{Z}_{(2)}). & & \end{array}$$

Let u' denote the image of u in $H_L^{n+1,n}(\check{C}Q_{\underline{a}};\mathbb{Z}_{(2)})$. One can show (after some extensive manipulations with motivic cohomology) that the hypothesis on u implies that u' is the image of an element in $H^{n+1,n}(\check{C}Q_{\underline{a}};\mathbb{Z}_{(2)})$. It will therefore be sufficient to show that this group is zero.

Let $\tilde{C}$ be defined by the cofiber sequence $(\check{C}Q_{\underline{a}})_+ \to (\operatorname{Spec} F)_+ \to \tilde{C}$. This means $\tilde{H}^{*,*}(\tilde{C})$ fits in an exact sequence

$$\to H^{p-1,q}(\check{C}Q_{\underline{a}}) \to \tilde{H}^{p,q}(\tilde{C}) \to H^{p,q}(\operatorname{Spec} F) \to H^{p,q}(\check{C}Q_{\underline{a}}) \to \tilde{H}^{p+1,q}(\tilde{C}) \to \cdots$$

So the reduced motivic cohomology of $\tilde{C}$ detects the 'difference' between the motivic cohomology of $\check{C}Q_{\underline{a}}$ and $\operatorname{Spec} F$. The fact that $H^{i,n}(\operatorname{Spec} F;\mathbb{Z}) = 0$ for $i > n$ shows that $H^{n+1,n}(\check{C}Q_{\underline{a}};\mathbb{Z}_{(2)}) \cong \tilde{H}^{n+2,n}(\tilde{C};\mathbb{Z}_{(2)})$. Since $Q_{\underline{a}}$ has a rational point (and therefore $\check{C}Q_{\underline{a}}$ is contractible) over a degree 2 extension of F, it follows from the transfer argument that the above group is killed by 2. To show that the group is zero it is therefore sufficient to prove that the image of $\tilde{H}^{n+2,n}(\tilde{C};\mathbb{Z}_{(2)}) \to \tilde{H}^{n+2,n}(\tilde{C};\mathbb{Z}/2)$ is zero. This is the same as the image of $\tilde{H}^{n+2,n}(\tilde{C};\mathbb{Z}) \to \tilde{H}^{n+2,n}(\tilde{C};\mathbb{Z}/2)$, which I'll denote by $\tilde{H}^{n+2,n}_{int}(\tilde{C};\mathbb{Z}/2)$.

So far most of what we have done is formal; but now we come to the crux of the argument. For any smooth scheme X one has cohomology operations acting on $H^{*,*}(X;\mathbb{Z}/2)$. In particular, one can produce analogs of the Steenrod operations: the Bockstein acts with bi-degree $(1,0)$, and Sq^{2^i} acts with bi-degree $(2^i, 2^{i-1})$. From these one defines the Milnor Q_i's, which have bi-degree $(2^{i+1}-1, 2^i-1)$. In ordinary topology these are defined inductively by $Q_0 = \beta$ and $Q_i = [Q_{i-1}, Sq^{2^i}]$, whereas motivically one has to add some extra terms to this commutator (these arise because the motivic cohomology of a point is nontrivial). One shows that $Q_i \circ Q_i = 0$, and that $Q_i = \beta q + q\beta$ for a certain operation q. It follows from the latter formula that Q_i maps elements in $\tilde{H}_{int}$ to elements in $\tilde{H}_{int}$. All of these facts also work in ordinary topology, it's just that the proofs here are a little more complex.

The next result is [V3, Cor. 3.8]. It is the first of two main ingredients needed to complete the proof.

Proposition 2.7. *Let X be a smooth quadric in $\mathbb{P}^{2^n}$, and let $\tilde{C}X$ be defined by the cofiber sequence $(\check{C}X)_+ \to (\operatorname{Spec} F)_+ \to \tilde{C}X$. Then for $i \leq n$, every element of $\tilde{H}^{*,*}(\tilde{C}X;\mathbb{Z}/2)$ that is killed by Q_i is also in the image of Q_i.*

This is a purely 'topological' result, in that its proof uses no algebraic geometry. It follows from the most basic properties of the Steenrod operations, motivic cohomology (like Thom isomorphism), and elementary facts about the characteristic numbers of quadrics. The argument is purely homotopy-theoretic.

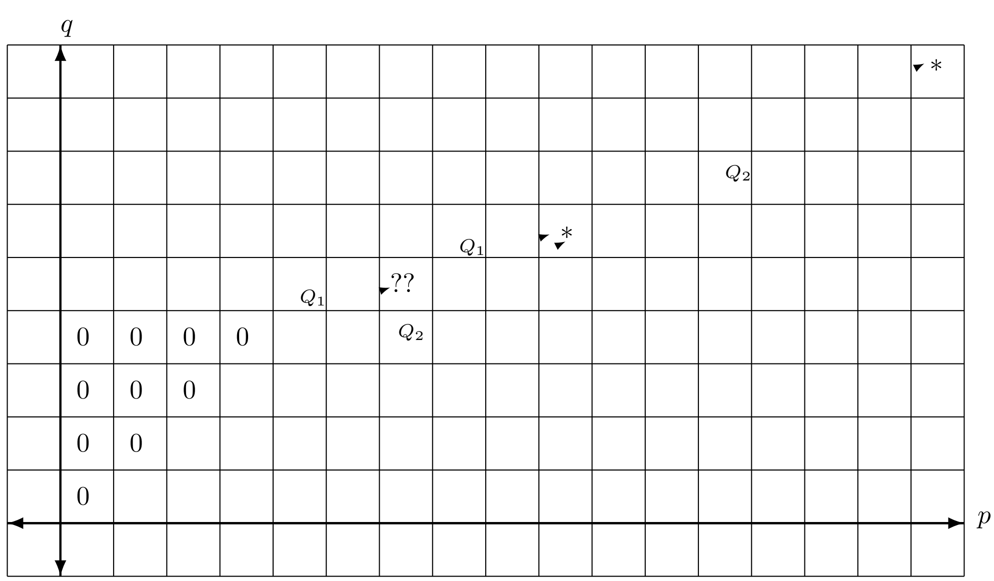
The second main result we will need is where all the algebraic geometry enters the picture. Voevodsky deduces it from results of Rost, who showed that the motive of $Q_{\underline{a}}$ splits off a certain direct summand. See [V3, Th. 4.9].

Proposition 2.8. $\tilde{H}^{2^n,2^{n-1}}(\tilde{C};\mathbb{Z}_{(2)}) = 0$.

Using the above two propositions we can complete the proof of the Milnor conjecture. In order to draw a concrete picture, let us just assume $n = 4$ for the moment. We are trying to show that $\tilde{H}^{6,4}_{int}(\tilde{C};\mathbb{Z}/2) = 0$. Consider the diagram

$$\begin{array}{ccc} H^{p,q}(\operatorname{Spec} F;\mathbb{Z}/2) & \longrightarrow & H^{p,q}(\check{C}Q_{\underline{a}};\mathbb{Z}/2) \\ \downarrow & & \downarrow \\ H^{p,q}_L(\operatorname{Spec} F;\mathbb{Z}/2) & \xrightarrow{\cong} & H^{p,q}_L(\check{C}Q_{\underline{a}};\mathbb{Z}/2). \end{array}$$

Our inductive assumption together with Proposition 2.4 implies that the vertical maps are isomorphisms for $p \leq q \leq n-1$, and monomorphisms for $p-1 = q \leq n-1$. So the top horizontal map is an isomorphism in the first range and a monomorphism in the second. The long exact sequence in motivic cohomology then shows that $\tilde{H}^{p,q}(\tilde{C};\mathbb{Z}/2) = 0$ for $p \leq q \leq n-1$. This is where our induction hypothesis has gotten us. The following diagram depicts what we now know about $\tilde{H}^{p,q}(\tilde{C};\mathbb{Z}/2)$ (the group marked ?? is $\tilde{H}^{6,4}$, the one we care about):



At this point Proposition 2.7 shows that $Q_1\colon H^{6,4} \to H^{9,5}$ is injective, and that $Q_2\colon H^{9,5} \to H^{16,8}$ is injective. Since the Q_i's take integral elements to integral elements, we have an inclusion

$$Q_2Q_1\colon \tilde{H}^{6,4}_{int}(\tilde{C};\mathbb{Z}/2) \hookrightarrow \tilde{H}^{16,8}_{int}(\tilde{C};\mathbb{Z}/2).$$

But it follows directly from Proposition 2.8 that $\tilde{H}^{16,8}_{int}(\tilde{C};\mathbb{Z}/2) = 0$, and so we are done.

The argument for general n follows exactly this pattern: one uses the composite of the operations $Q_1, Q_2, \dots, Q_{n-2}$, but everything else is the same.

2.9. **Summary.** Here is a list of some of the key elements of the proof:

(1) The re-interpretation of the Milnor conjecture as a comparison of different bi-graded motivic cohomology theories. An extensive knowledge about such theories allows one to deduce statements for any smooth simplicial scheme from statements only about fields (cf. Proposition 2.4).
(2) Choice of the splitting variety $Q_{\underline{a}}$ (needed for Propositions 2.7 and 2.8).
(3) The introduction and use of Čech complexes.
(4) The construction of Steenrod operations on motivic cohomology and development of their basic properties, leading to the proof of Proposition 2.7.
(5) The 'geometric' results of Rost on motives of quadrics, which lead to Proposition 2.8.

2.10. **A notable consequence.** The integral motivic cohomology groups of a point $H^{p,q}(\operatorname{Spec} F)$ are largely unknown—the exception is when $q = 0, 1$. However, the proof of the Milnor conjecture tells us exactly what $H^{p,q}(\operatorname{Spec} F;\mathbb{Z}/2)$ is. First of all, independently of the Milnor conjecture it can be shown to vanish when $p \geq q$ and when $q < 0$. By Proposition 2.4 (noting that we now know the hypothesis to be satisfied for all n), it follows that

$$H^{p,q}(\operatorname{Spec} F;\mathbb{Z}/2) \to H^p_{et}(\operatorname{Spec} F;\mu_2^{\otimes q})$$

is an isomorphism when $p \leq q$ and $q \geq 0$. As $\mu_2^{\otimes q} \cong \mu_2$, the étale cohomology groups are periodic in q; that is, $H^*_{et}(\operatorname{Spec} F;\mu_2^{\otimes *}) \cong H^*_{Gal}(F;\mathbb{Z}/2)[\tau,\tau^{-1}]$ where τ has degree $(0,1)$.

The conclusion is that $H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2) \cong H^*_{Gal}(F;\mathbb{Z}/2)[\tau]$, where τ is the canonical class in $H^{0,1}$ and the Galois cohomology is regarded as the subalgebra lying in degrees (k,k). Of course the Milnor conjecture tells us that the Galois cohomology is the same as mod 2 Milnor K-theory, and so we can also write $H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2) \cong \big(K^M_*(F)/2\big)[\tau]$.

2.11. **Further reading.** Both the original papers of Voevodsky [V1, V3] are very readable, and remain the best sources for the proof. Summaries have also been given in [M1] and [Su]. A proof of the general Bloch-Kato conjecture was recently given in [V5]—the proof is similar in broad outline to the 2-primary case we described here, but with several important differences. See the introduction to [V5].

Of course in this section I have completely avoided discussing the two main elements of the proof, namely Propositions 2.7 and 2.8. The proof of Proposition 2.7 is in [V1, V3] and is written in a way that can be understood by most homotopy theorists. Proposition 2.8 depends on results of Rost, which seem to be largely unpublished. See [R1, R2] for summaries.

For more about why Čech complexes arise in the proof, see Proposition 3.9 in the next section.

3. Proof of the conjecture on quadratic forms

In this section and the next I will discuss two proofs of Milnor's conjecture on quadratic forms. The first is from [OVV], the second was announced in [M2]. Both depend on Voevodsky's proof of the norm residue conjecture. As I keep saying, I'm only going to give a vague outline of how the proofs go, but with references for where to find more information on various aspects. The present section deals with the [OVV] proof.

3.1. **Preliminaries.** Recall that we are concerned with the map $\nu\colon K_*^M(F)/2 \to \mathrm{Gr}_I\, W(F)$ defined by $\nu(\{a_1,\dots,a_n\}) = \langle 1,-a_1\rangle\cdots\langle 1,-a_n\rangle$. The fact that I is additively generated by the forms $\langle 1,x\rangle$ shows that ν is obviously surjective; so our task is to prove injectivity. In general, the product $\langle 1,b_1\rangle\cdots\langle 1,b_n\rangle$ is called an ***n*-fold Pfister form**, and denoted $\langle\langle b_1,\dots,b_n\rangle\rangle$. Note that it has dimension 2^n. The proof is intimately tied up with the study of such forms.

Milnor proved that the map $\nu\colon K_2^M(F)/2 \to I^2/I^3$ is an isomorphism. He used ideas of Delzant [De] to define Stiefel-Whitney invariants for quadratic forms, which in dimension 2 give a map $I^2/I^3 \to K_2^M(F)/2$. One could explicitly check that this was an inverse to ν. Unfortunately, this last statement generally fails in larger dimensions; the Stiefel-Whitney invariants don't carry enough information. See [Mi, 4.1, 4.2].

3.2. **The Orlov-Vishik-Voevodsky proof.** We first need to recall some results about Pfister forms proven in the 70's. The first is an easy corollary of the so-called Main Theorem of Arason-Pfister (cf. [S1, 4.5.6]). For a proof, see [EL, pp. 192-193].

Proposition 3.3 (Elman-Lam)**.** *$\langle\langle a_1,\dots,a_n\rangle\rangle \equiv \langle\langle b_1,\dots,b_n\rangle\rangle$ (mod I^{n+1}) if and only if $\langle\langle a_1,\dots,a_n\rangle\rangle = \langle\langle b_1,\dots,b_n\rangle\rangle$ in $GW(F)$.*

Combining the result for $n=2$ with Milnor's theorem that $K_2^M(F)\to I^2/I^3$ is an isomorphism, we get the following (note that the minus signs are there because $\nu(\{a_1,\dots,a_n\}) = \langle\langle -a_1,\dots,-a_n\rangle\rangle$):

Corollary 3.4. *$\langle\langle a_1,a_2\rangle\rangle = \langle\langle b_1,b_2\rangle\rangle$ in $GW(F)$ if and only if $\{-a_1,-a_2\} = \{-b_1,-b_2\}$ in $K_*^M(F)/2$.*

Say that two n-fold Pfister forms $A=\langle\langle a_1,\dots,a_n\rangle\rangle$ and $B=\langle\langle b_1,\dots,b_n\rangle\rangle$ are **simply-*p*-equivalent** if there are two indices i, j where $\langle\langle a_i,a_j\rangle\rangle = \langle\langle b_i,b_j\rangle\rangle$ and $a_k=b_k$ for all $k\notin\{i,j\}$. The forms A and B are **chain-*p*-equivalent** if there is a chain of forms starting with A and ending with B in which every link of the chain is a simple-p-equivalence. Note that it follows immediately from the previous corollary that if A and B are chain-p-equivalent then $\{-a_1,\dots,-a_n\}=\{-b_1,\dots,-b_n\}$.

The following result is [EL, Main Theorem 3.2]:

Proposition 3.5. *Let $A=\langle\langle a_1,\dots,a_n\rangle\rangle$ and $B=\langle\langle b_1,\dots,b_n\rangle\rangle$. The following are equivalent:*

(a) A and B are chain-p-equivalent.
(b) $\{-a_1,\dots,-a_n\}=\{-b_1,\dots,-b_n\}$ in $K_*^M(F)/2$.
(c) $A\equiv B$ (mod I^{n+1}).
(d) $A=B$ in $GW(F)$.

Note that $(a)\Rightarrow(b)\Rightarrow(c)$ is trivial, and $(c)\Rightarrow(d)$ was mentioned above. So the new content is in $(d)\Rightarrow(a)$. I will not give the proof, but refer the reader to [S1, 4.1.2]. The result below is a restatement of $(c)\Rightarrow(b)$:

Corollary 3.6. *The equality* $\nu(\{a_1, \dots, a_n\}) = \nu(\{b_1, \dots, b_n\})$ *can only occur if* $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$.

Unfortunately the above corollary does not show that ν is injective, as a typical element $x \in K_*^M(F)/2$ is a *sum* of terms $\{a_1, \dots, a_n\}$. A term $\{a_1, \dots, a_n\}$ is called a **pure symbol**, whereas a general $x \in K_*^M(F)$ is just a **symbol**. The key ingredient needed from [OVV] is the following:

Proposition 3.7. *If* $x \in K_*^M(F)/2$ *is a nonzero element then there is a field extension* $F \hookrightarrow F'$ *such that the image of* x *in* $K_*^M(F')/2$ *is a nonzero pure symbol.*

It is easy to see that the previous two results prove the injectivity of ν. If $x \in K_n^M(F)/2$ is a nonzero element in the kernel of ν, then by passing to F' we find a nonzero pure symbol which is also in the kernel. Corollary 3.6 shows this to be impossible, however.

We are therefore reduced to proving Proposition 3.7. If we write $x = \underline{a}_1 + \ldots + \underline{a}_k$, where each $\underline{a}_i$ is a pure symbol, then we know we can make $\underline{a}_i$ vanish by passing to the function field $F(Q_{\underline{a}_i})$ (where $Q_{\underline{a}_i}$ is the splitting variety produced in the last section). Our goal will be to show that $\underline{a}_i$ is the *only* term that vanishes:

Proposition 3.8 (Orlov-Vishik-Voevodsky)**.** *If* $\underline{a} = \{a_1, \dots, a_n\}$ *is nonzero in* $K_n^M(F)/2$*, then the kernel of* $K_n^M(F)/2 \to K_n^M(F(Q_{\underline{a}}))/2$ *is precisely* $\mathbb{Z}/2$ *(generated by* $\underline{a}$*).*

Granting this for the moment, let i be the largest index for which x is nonzero in $K_n^M(F')/2$, where $F' = F(Q_{\underline{a}_1} \times \cdots \times Q_{\underline{a}_i})$. Since x will become zero over $F'(Q_{\underline{a}_{i+1}})$, the above result says that $x = \underline{a}_{i+1}$ in $K_n^M(F')/2$. This is precisely what we wanted.

So finally we have reduced to the same kind of problem we tackled in the last section, namely controlling the map $K_n^M(F)/2 \to K_n^M(F(Q_{\underline{a}}))/2$. The techniques needed to prove Proposition 3.8 are exactly the same as those from the last section. There is a again a homotopical ingredient and a geometric ingredient.

Proposition 3.9. *If* X *is a smooth scheme over* F*, then for every* $n \geq 0$ *there is an exact sequence of the form*

$$0 \to H^{n,n-1}(\check{C}X; \mathbb{Z}/2) \to H^{n,n}(\operatorname{Spec} F; \mathbb{Z}/2) \to H^{n,n}(\operatorname{Spec} F(X); \mathbb{Z}/2).$$

Recall that $H^{n,n}(\operatorname{Spec} E; \mathbb{Z}/2) \cong K_n^M(E)/2$ for any field E. So the above sequence is giving us control over the kernel of $K_*^M(F)/2 \to K_*^M(F(Q_{\underline{a}}))/2$. The proof uses the conclusion from Proposition 2.4 (which is known by Voevodsky's proof of the Milnor conjecture) and some standard manipulations with motivic cohomology. See [OVV, Prop. 2.3].

Remark 3.10. In some sense Proposition 3.9 explains why Čech complexes are destined to come up in the proofs of these conjectures.

If the above proposition is thought of as a 'homotopical' part of the proof, the geometric part is the following. It is deduced using Rost's results on the motive of $Q_{\underline{a}}$; see [OVV, Prop. 2.5].

Proposition 3.11. *There is a surjection* $\mathbb{Z}/2 \to H^{2^n-1, 2^{n-1}-1}(\check{C}Q_{\underline{a}}; \mathbb{Z}/2)$.

The previous two results immediately yield a proof of 3.8. By Proposition 3.9 we must show that $H^{n,n-1}(\check{C}Q_{\underline{a}}; \mathbb{Z}/2) \cong \mathbb{Z}/2$ (and we know the group is nontrivial).

But we saw in the last section that $H^{n,n-1}(\check{C}Q_{\underline{a}};\mathbb{Z}/2) \cong H^{n+1,n-1}(\tilde{C}Q_{\underline{a}};\mathbb{Z}/2)$, where $\tilde{C}Q_{\underline{a}}$ is the homotopy cofiber of $(\check{C}Q_{\underline{a}})_+ \to (\operatorname{Spec} F)_+$. We also saw that the operation $Q_{n-2}\cdots Q_2Q_1$ gives a monomorphism $H^{n+1,n-1}(\tilde{C}Q_{\underline{a}};\mathbb{Z}/2) \hookrightarrow H^{2^n-1,2^{n-1}-1}(\tilde{C}Q_{\underline{a}};\mathbb{Z}/2)$. But now we are done, since by 3.11 the latter group has at most two elements.

This completes the proof of the injectivity of ν.

4. Quadratic forms and the Adams spectral sequence

In [M2] Morel announced a proof of the quadratic form conjecture over characteristic zero fields, using the motivic Adams spectral sequence. The approach depends on having computed the motivic Steenrod algebra, but I'm not sure what the status of this is—certainly no written account is presently available. Despite this frustrating point, Morel's proof is very exciting; while it uses Voevodsky's computation of $H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2)$—see Remark 2.10—it somehow avoids using any other deep results about quadratic forms! So I'd like to attempt a sketch.

The arguments below take place in the motivic stable homotopy category. All the reader needs to know as background is that it formally behaves much as the usual stable homotopy category, and that there is a bigraded family of spheres $S^{p,q}$. The suspension (in the triangulated category sense) of $S^{p,q}$ is $S^{p+1,q}$, and $S^{2,1}$ is the suspension spectrum of the variety $\mathbb{P}^1$.

4.1. **Outline.** We have our maps $\nu_n\colon K_n^M(F)/2 \to I^n/I^{n+1}$, and need to prove that they are injective. We will see that the Adams spectral sequence machinery gives us, more or less for free, maps $s_n\colon I^n/I^{n+1} \to K_n^M(F)/(2,J)$ where J is a subgroup of boundaries from the spectral sequence. The composite $s_n\nu_n$ is the natural projection, and so the whole game is to show that J is zero. That is, one needs to prove the vanishing of a line of differentials. Using the multiplicative structure of the spectral sequence and the algebra of the E_2-term, this reduces just to proving that the differentials on a certain 'generic' element vanish. This allows one to reduce to the case of the prime field $\mathbb{Q}$, then to $\mathbb{R}$, and ultimately to a purely topological problem.

4.2. **Basic setup.** Now I'll expand on this general outline. The first step is to produce a map $q\colon GW(F) \to \{S^{0,0}, S^{0,0}\}$ where $\{-,-\}$ denotes maps in the motivic stable homotopy category. Recall from Section 1.2 that one knows a complete description of $GW(F)$ in terms of generators and relations. For $a \in F^*$ we let $q(\langle a\rangle)$ be the map $\mathbb{P}^1 \to \mathbb{P}^1$ defined in homogeneous coordinates by $[x,y] \to [x,ay]$. By writing down explicit $\mathbb{A}^1$-homotopies one can verify that the relations in $GW(F)$ are satisfied in $\{S^{0,0}, S^{0,0}\}$, and so q extends to a well-defined map of abelian groups. It is actually a ring map. Further details about all this are given in [M3].

Now we build an Adams tower for $S^{0,0}$ based on the motivic cohomology spectrum $H\mathbb{Z}/2$. Set $W_0 = S^{0,0}$, and define W_1 by the homotopy fiber sequence $W_1 \to S^{0,0} \to H\mathbb{Z}/2$. Then consider the map $W_1 \cong S^{0,0} \wedge W_1 \to H\mathbb{Z}/2 \wedge W_1$, and let W_2 be the homotopy fiber. Repeat the process to define W_3, W_4, etc. This gives us a tower of cofibrations

$$\begin{array}{ccccccc} & & H\wedge W_2 & & H\wedge W_1 & & H\wedge W_0 \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & W_0, \end{array}$$

where we have written H for $H\mathbb{Z}/2$. For any Y the tower yields a filtration on $\{Y, S^{0,0}\}$ by letting $\mathcal{F}^n$ be the subgroup of all elements in the image of $\{Y, W_n\}$ (note that there is no *a priori* guarantee that the filtration is Hausdorff.) The tower yields a homotopy spectral sequence whose abutment has something to do with the associated graded of the groups $\{S^{*,0} \wedge Y, S^{0,0}\}$. If the filtration is not Hausdorff these associated graded groups may not be telling us much about $\{S^{*,0} \wedge Y, S^{0,0}\}$, but this will not matter for our application. We will be interested in the case $Y = S^{0,0}$.

Set $E_1^{a,b} = \{S^{a,0}, H \wedge W_b\}$, so that $d_r\colon E_r^{a,b} \to E_r^{a-1,b+r}$. My indexing has been chosen so that the picture of the spectral sequence has $E_1^{a,b}$ in spot (a,b) on a grid, rather than at spot $(b-a,a)$ as is more typical for the Adams spectral sequence—but the picture itself is the same in the end. Formal considerations give inclusions

$$\mathcal{F}^k\{S^{n,0}, S^{0,0}\}/\mathcal{F}^{k+1}\{S^{n,0}, S^{0,0}\} \hookrightarrow E_\infty^{n,k}$$

(however, there is no a priori reason to believe the map is surjective). In particular, if $\mathcal{F}^*$ is the filtration on $\{S^{0,0}, S^{0,0}\}$ then we have inclusions $\mathcal{F}^k/\mathcal{F}^{k+1} \hookrightarrow E_\infty^{0,k}$.

Let $G\mathbb{I}(F)$ be the kernel of the mod 2 dimension function $\dim\colon GW(F) \to \mathbb{Z}/2$. The powers $G\mathbb{I}(F)^n$ define a filtration on $GW(F)$. One can check that q maps $G\mathbb{I}^1$ into $\mathcal{F}^1$. Since the Adams filtration $\mathcal{F}^n$ on $\pi_{0,0}(S^{0,0})$ will be multiplicative, one finds that q maps $G\mathbb{I}^n$ into $\mathcal{F}^n$. So we get maps $(G\mathbb{I})^n/(G\mathbb{I})^{n+1} \to \mathcal{F}^n/\mathcal{F}^{n+1} \to E_\infty^{0,n}$.

In a moment I'll say more about what the Adams spectral sequence looks like in this case, but first let's relate $G\mathbb{I}$ to what we really care about. One easily checks that $G\mathbb{I} = GI \oplus \mathbb{Z}$, where the $\mathbb{Z}$ is the subgroup generated by $\langle 1,1\rangle = 2\langle 1\rangle$. So $G\mathbb{I}^n = GI^n \oplus \mathbb{Z}$, where the $\mathbb{Z}$ is generated by $2^n\langle 1\rangle$. It follows that $G\mathbb{I}^n/G\mathbb{I}^{n+1} \cong [GI^n/GI^{n+1}] \oplus \mathbb{Z}/2$. Finally, recall from Remark 1.5 that the natural map $GI \to I$ is an isomorphism. Putting everything together, we have produced invariants $[I^n/I^{n+1}] \oplus \mathbb{Z}/2 \to E_\infty^{0,n}$.

4.3. **Analysis of the spectral sequence.** So far the discussion has been mostly formal. We have produced a spectral sequence, but not said anything concrete about it. The usefulness of the above invariants hinges on what $E_\infty^{0,n}$ looks like. If things work as in ordinary topology, then the E_2 term will turn out to be $E_2^{a,b} = \operatorname{Ext}^b_{H^{**}H}(\Sigma^{b+a,0}H^{**}, H^{**})$ where I've again written $H = H\mathbb{Z}/2$ and $\Sigma^{k,0}$ denotes a grading shift on the bi-graded module H^{**}. So we need to know the algebra $H^{**}H$, but unfortunately there is no published source for this calculation. In [V2] Voevodsky defines Steenrod operations and shows that they satisfy analogs of the usual Adem relations; he doesn't show that these generate all of $H^{**}H$, though. However, let's assume we knew this—so we are assuming $H^{**}H$ is the algebra Voevodsky denotes A^{**} and calls the motivic Steenrod algebra [V2, Section 11].

The form of $H^{**}H$ is very close to that of the usual Steenrod algebra, and so one has a chance at doing some of the Ext computations. In fact, it is not very hard. Some hints about this are given in Appendix B, but for now let me just tell you the important points:

(1) $E_2^{p,q} = 0$ if $p < 0$.
(2) $E_2^{0,0} = \mathbb{Z}/2$.
(3) For $n \geq 1$, $E_2^{0,n} = H^{n,n} \oplus \mathbb{Z}/2$. The inclusion $\oplus_n H^{n,n} \hookrightarrow \oplus_n E_2^{0,n}$ is a ring homomorphism, where the domain is regarded as a subring of H^{**}.

Most of these computations make essential use of Remark 2.10, and therefore depend on Voevodsky's proof of the norm residue conjecture. Also note the connection between (3) and Milnor K-theory, given by the isomorphism $H^{n,n} \cong K_n^M(F)/2$.

The above two facts show that everything in $E_2^{0,n}$ is a permanent cycle and thus $E_\infty^{0,n} = (\mathbb{Z}/2 \oplus K_n^M(F)/2)/J$ where J is the subgroup of all boundaries. Recall that one has maps

$$K_n^M(F)/2 \xrightarrow{\nu_n} I^n/I^{n+1} \to E_\infty^{0,n} \cong [K_n^M(F)/2 \oplus \mathbb{Z}/2]/J.$$

The composition can be checked to be the obvious one. To prove that ν_n is injective, we need to prove that $J = 0$. That is, we need to prove the vanishing of all differentials landing in $E^{0,*}$ (which necessarily come from $E^{1,*}$). As for the computation of the $E^{1,*}$ column, here are the additional facts we need:

(4) $E_2^{1,0} = 0$.
(5) $E_2^{1,1} = H^{0,1} \oplus H^{2,2} \cong \mathbb{Z}/2 \oplus H^{2,2}$.
(6) The images of the two maps

$$E_2^{0,1} \otimes E_2^{1,n-1} \to E_2^{1,n} \qquad E_2^{1,n-1} \otimes E_2^{0,1} \to E_2^{1,n}$$

generate $E_2^{1,n}$ as an abelian group.
(7) The composite $H^{1,1} \otimes H^{2,2} \hookrightarrow E_2^{0,1} \otimes E_2^{1,1} \to E_2^{1,2}$ is zero.

Again, let me say that none of these computations is particularly difficult, and the reader can find some hints in Appendix B. Portions of columns 0 and 1 of our E_2-term are shown below:

$H^{4,4} \oplus \mathbb{Z}/2$	??		
$H^{3,3} \oplus \mathbb{Z}/2$	??		
$H^{2,2} \oplus \mathbb{Z}/2$	??		
$H^{1,1} \oplus \mathbb{Z}/2$	$H^{2,2} \oplus \mathbb{Z}/2$		
$\mathbb{Z}/2$	0		

Remark 4.4. If one only looks at the $\mathbb{Z}/2$'s appearing in the above diagram, the picture looks just like the ordinary topological Adams spectral sequence. The $\mathbb{Z}/2$'s in our 0th column indeed turn out to be "h_0^n's", just as in topology. The $\mathbb{Z}/2$ in $E_2^{1,1}$ is a little more complicated, though—it doesn't just come from Sq^2, like the usual h_1 does (see Appendix B for what it *does* come from).

We need to prove that all the differentials leaving the $E^{1,*}$ column vanish. By fact (6) and the multiplicative structure of the spectral sequence, it is sufficient to prove that all differentials leaving $E_2^{1,1}$ vanish (starting with $d_2\colon E_2^{1,1} \to E_2^{0,3}$). We will do this in several steps.

The following result basically shows that, just as in ordinary topology, all the $\mathbb{Z}/2$'s in column 0 survive to E_∞.

Lemma 4.5. *The image of $d_r\colon E_r^{1,1} \to E_r^{0,r+1}$ lies in the subgroup $H^{r+1,r+1}$, for every $r \geq 2$.*

Proof. Suppose there is an element $x \in E_r^{1,1}$ such that $d_r(x)$ does not lie in $H^{r+1,r+1}$ (or rather its image in E_r). We can write $x = \underline{a} + y$ where $\underline{a} \in H^{2,2} = K_2^M(F)/2$ and $y \in H^{0,1} \cong \mathbb{Z}/2$. In expressing $\underline{a}$ as a sum of pure symbols, one notes that only a finite number of elements of F are involved. By naturality of the spectral sequence, we can therefore assume F is a finitely-generated extension of $\mathbb{Q}$.

But now we can choose an embedding $F \hookrightarrow \mathbb{C}$, and again use naturality. The groups $K_n^M(\mathbb{C})/2$ are all zero, and therefore our assumption implies that over $\mathbb{C}$ we have $E_{r+1}^{0,r+1} = 0$ (in other words, the $\mathbb{Z}/2$ in $E_2^{0,r+1}$ dies in the spectral sequence). But there is a 'topological realization map' from our spectral sequence over $\mathbb{C}$ to the usual Adams spectral sequence in topology, where we know that none of the $\mathbb{Z}/2$'s in $E^{0,*}$ ever die. □

Remark 4.6. There is also a purely algebraic proof of the above result. One reduces via naturality to the case of algebraically closed fields, where all the $H^{n,n}$'s are zero. Then one shows that the $\mathbb{Z}/2$'s in the 0th column form a polynomial algebra, and that the composite $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \hookrightarrow E_2^{1,1} \otimes E_2^{0,1} \to E_2^{1,2}$ is zero (just as in ordinary topology). The fact that the spectral sequences is multiplicative takes care of the rest.

Lemma 4.7. *For $\underline{a} \in H^{2,2}$ one has $d_r(\underline{a}) = 0$, for every r.*

Proof. It follows from facts (3) and (7), together with the multiplicative structure of the spectral sequence, that everything in the image of $d_r \colon H^{2,2} \to H^{r+1,r+1}$ is killed by $H^{1,1}$. This is the key to the proof.

Let $z = d_r(\underline{a})$. Consider the naturality of the spectral sequence for the map $j \colon F \to F(t)$. It follows from the previous paragraph that $j(z) = d_r(j\underline{a})$ is killed by $F(t)^*$. In particular, $\{t\} \cdot j(z) = 0$ in $K_{r+2}^M(F(t))/2$. But by [Mi, Lem. 2.1] there is a map $\partial_t \colon K_{r+2}^M(F(t))/2 \to K_{r+1}^M(F)/2$ with the property that $\partial_t(\{t\} \cdot j(z)) = z$. So we conclude that $z = 0$, as desired. □

Proposition 4.8. *All differentials leaving $E^{1,1}$ are zero.*

Proof. Recall $E_2^{1,1} \cong H^{0,1} \oplus H^{2,2} \cong \mathbb{Z}/2 \oplus H^{2,2}$. By the previous lemma we are reduced to analyzing the maps $d_r \colon H^{0,1} \to H^{r+1,r+1}$. Since $H^{0,1}(\mathbb{Q}) \to H^{0,1}(F)$ is an isomorphism, it suffices to prove the result in the case $F = \mathbb{Q}$.

Now use naturality with respect to the field extension $\mathbb{Q} \hookrightarrow \mathbb{R}$. The maps $K_n^M(\mathbb{Q})/2 \to K_n^M(\mathbb{R})/2$ are isomorphisms for $n \geq 3$ (see Appendix A), so now we've reduced to $F = \mathbb{R}$. But here we can again use a 'topological realization' map to compare our Adams spectral sequence to the corresponding one in the context of $\mathbb{Z}/2$-equivariant homotopy theory. This map is readily seen to be an isomorphism on the $E^{0,*}$ column: the point is that the $\mathbb{Z}/2$-equivariant cohomology groups $H^{n,n}$ are isomorphic to the corresponding mod 2 motivic cohomology groups over $\mathbb{R}$ (see [Du, 2.8, 2.11], for instance). We are essentially seeing a reflection of the fact that $GW(\mathbb{R})$ may be identified with the Burnside ring of $\mathbb{Z}/2$, which coincides with $\{S^{0,0}, S^{0,0}\}$ in the $\mathbb{Z}/2$-equivariant stable homotopy category. In any case, we are finally reduced to showing the vanishing of certain differentials in a topological Adams spectral sequence: the paper [LZ] seems to essentially do this (but I haven't thought about this part carefully—I'm relying on remarks from [M2]). □

This completes Morel's proof of the quadratic form conjecture for characteristic zero fields (modulo the identification of $H^{**}H$, which we assumed).

Remark 4.9. We restricted to characteristic zero fields because the identification of $H^{**}H$ has never been claimed in characteristic p. If we make the wild guess that in positive characteristic $H^{**}H$ still has the same form, most of the argument goes through verbatim. There are two exceptions, where we used topological realization functors. The first place was to show that the image of the d_r's didn't touch the $\mathbb{Z}/2$'s in $E_2^{0,*}$, but Remark 4.6 mentioned that this could be done another way. The second place we used topological realization was at the final stage of the argument, to analyze the differentials $d_r \colon H^{0,1} \to H^{r+1,r+1}$. As before, this reduces to the case of a prime field. But for F a finite field one has $K_n^M(F) = 0$ for $n \geq 2$, so for prime fields there is in fact nothing to check.

In summary, the same general argument would work in characteristic p if one knew that $H^{**}H$ had the same form.

4.10. **Further reading.** There is very little completed literature on the subjects discussed in this section. Several documents are available on Morel's website, however; the draft [M5] is particularly relevant, although it only slightly expands on [M2]. For information on the motivic Steenrod algebra, see [V2]. Finally, Morel recently released another proof of Milnor's quadratic form conjecture, using very different methods. See [M4].

Appendix A. Some examples of the Milnor conjectures

This is a supplement to Section 1. We examine the Milnor conjectures in the cases of certain special fields F.

(a) F is algebraically closed. Since $F = F^2$, every nondegenerate form is isomorphic to one of the form $\langle 1, 1, \ldots, 1\rangle$. So $GW(F) \cong \mathbb{Z}$, and $W(F) \cong \mathbb{Z}/2$ with $I(F) = 0$. Thus, $\mathrm{Gr}_I\, W(F) \cong \mathbb{Z}/2$.

The absolute Galois group is trivial, so $H^*(F; \mathbb{Z}/2) = \mathbb{Z}/2$.

Finally, the fact that $F = F^2$ implies that $K^M_*(F)/2 = 0$ for $* \geq 1$. This is because the generators all lie in $K^M_1(F)$, and if $a = x^2$ then $\{a\} = \{x^2\} = 2\{x\} = 0 \in K^M_1(F)/2$.

(b) $F = F^2$. This case is suggested by the previous one. We only need to check that the hypothesis implies $H^*(F; \mathbb{Z}/2) = 0$ for $* \geq 1$. Strangely, I haven't been able to find an easy proof of this.

(c) $F = \mathbb{R}$. In this case we know forms are classified by their rank and signature, and it follows that $GW(\mathbb{R})$ is the free abelian group generated by $\langle 1\rangle$ and $\langle -1\rangle$. Also, $\langle -1\rangle^2 = \langle 1\rangle$. So $GW(\mathbb{R}) \cong \mathbb{Z}[x]/(x^2 - 1)$, and $W(\mathbb{R}) \cong \mathbb{Z}$ with $I(\mathbb{R}) = 2\mathbb{Z}$. Hence $\mathrm{Gr}_I\, W(\mathbb{R}) \cong \mathbb{Z}/2[a]$.

The absolute Galois group of $\mathbb{R}$ is $\mathbb{Z}/2$, so $H^*(\mathbb{R}; \mathbb{Z}/2) = H^*(\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[a]$.

Finally we consider $K^M_*(\mathbb{R})/2$. The group $K^M_1(\mathbb{R})/2 = \mathbb{R}^*/(\mathbb{R}^*)^2 \cong \{1, -1\}$ (the set consisting of 1 and -1). A similar calculation, based on the fact that every element of $\mathbb{R}$ is a square up to sign, shows that $K^M_i(\mathbb{R})/2 \cong \mathbb{Z}/2$ for every i, with the nonzero element being $\{-1, -1, \ldots, -1\}$. So $K^M_*(\mathbb{R})/2 \cong \mathbb{Z}/2[a]$ as well.

(d) $F = \mathbb{F}_q$, q odd. Here $F^* \cong \mathbb{Z}/(q-1)$ and so $K^M_1/2 = F^*/(F^*)^2 \cong \mathbb{Z}/2$. If g is the generator, then $\{g, g, \ldots, g\}$ generates $K^M_n/2$ (but may be zero). In fact one can show (cf. [Mi, Ex. 1.5]) that $\{g, g\} = 0$ in K^M_2, from which it follows that $K^M_* = 0$ for $* \geq 2$. So $K^M_*(F)/2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, in degrees 0 and 1.

For a finite field the absolute Galois group is $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. The Galois cohomology $H^*(\hat{\mathbb{Z}}; \mathbb{Z}/2)$ is just the mod 2 cohomology of $B\mathbb{Z} \simeq S^1$; so it is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, with the generators in degrees 0 and 1.

Again noting that $F^*/(F^*)^2 \cong \mathbb{Z}/2$, it follows that the Grothendieck-Witt group is generated by $\langle 1\rangle$ and $\langle g\rangle$. A simple counting argument (cf. [S1, Lem. 2.3.7]) shows that every element of $\mathbb{F}_q^*$ is a sum of two squares. Writing $g = a^2 + b^2$ one finds that

$$\langle 1, 1\rangle = \langle a^2, b^2\rangle = \langle a^2 + b^2, a^2b^2(a^2 + b^2)\rangle = \langle a^2 + b^2, a^2 + b^2\rangle = \langle g, g\rangle.$$

That is, $2(\langle 1\rangle - \langle g\rangle) = 0$. It follows that $GW(F) = \mathbb{Z} \oplus \mathbb{Z}/2$, with corresponding generators $\langle 1\rangle$ and $\langle 1\rangle - \langle g\rangle$.

The computation of the Witt group depends on whether or not -1 is a square; since $F^* = \mathbb{Z}/(q-1)$ and -1 has order 2, then -1 is a square precisely when $4|(q-1)$. So if $q \equiv 1 (\mathrm{mod}\ 4)$ then $\langle 1\rangle = \langle -1\rangle$ and $W(F) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$; in this case $I(F) = (\langle 1\rangle - \langle g\rangle) \cong \mathbb{Z}/2$. If $q \equiv 3 (\mathrm{mod}\ 4)$ then $\langle g\rangle = \langle -1\rangle$ and we have $W(F) \cong \mathbb{Z}/4$ with $I(F) = (2)$. In either case $\mathrm{Gr}_I\, W(F) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Remark A.1. Although Milnor's quadratic form conjecture says that $\mathrm{Gr}_I W(F)$ depends only on the absolute Galois group of F, this example makes it clear that the same cannot be said for $W(F)$ itself.

(e) $F = \mathbb{Q}$. This case is considerably harder, so we will only make a few observations. Note that as an abelian group one has

$$\mathbb{Q}^* \cong \mathbb{Z}/2 \times \Big(\oplus_p \mathbb{Z}\Big),$$

by the fundamental theorem of arithmetic; the direct sum is over the set of all primes. Here the isomorphism sends a fraction q to its sign (in the $\mathbb{Z}/2$ factor) together with the list of exponents in the prime factorization of q. So $K_1^M(\mathbb{Q})/2 \cong \mathbb{Z}/2 \oplus (\oplus_p \mathbb{Z}/2)$.

As the above isomorphism may suggest, to go further it becomes convenient to work with one completion at a time. The case $F = \mathbb{R}$ has already been discussed, so what is left is the p-adics. We will return to $F = \mathbb{Q}$ after discussing them.

(f) $F = \mathbb{Q}_p$. We will concentrate on the case where p is odd; the case $p = 2$ is similar, and can be left to the reader. We know $K_1^M(\mathbb{Q}_p)/2 \cong H^1(\mathbb{Q}_p; \mathbb{Z}/2) \cong \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$. A little thought (cf. [S1, 5.6.2]) shows this group is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, with elements represented by 1, g, p, and pg, where $1 < g < p$ is any integer which generates the multiplicative group $\mathbb{F}_p^*$. By [Se, Section II.5.2] one has $H^2(\mathbb{Q}_p; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and $H^i(\mathbb{Q}_p; \mathbb{Z}/2) = 0$ for $i \geq 3$.

The fact that $K_1^M(\mathbb{Q}_p)/2$ only has four elements tells us that $K_*^M(\mathbb{Q}_p)/2$ can't be too big. By finding the appropriate relations to write down, Calvin Moore proved that $K_*^M(\mathbb{Q}_p)/2 = 0$ for $* \geq 3$ [Mi, Ex. 1.7], and that $K_2^M(\mathbb{Q}_p)/2 = \mathbb{Z}/2$. This is an exercise for the reader.

The group $GW(\mathbb{Q}_p)$ will be generated by the four elements $\langle 1 \rangle$, $\langle g \rangle$, $\langle p \rangle$, and $\langle pg \rangle$. The theory again depends on whether or not -1 is a square, which is when $p \equiv 1 (\mathrm{mod}\ 4)$. When $p \equiv 1 (\mathrm{mod}\ 4)$ one has $\langle 1 \rangle = \langle -1 \rangle$ and so $\langle x \rangle = \langle -x \rangle$ for any x. As a result $\langle g, g \rangle = \langle g, -g \rangle = \langle 1, -1 \rangle = \langle 1, 1 \rangle$, and similarly $\langle p, p \rangle = \langle pg, pg \rangle = \langle 1, 1 \rangle$. One finds that $GW(\mathbb{Q}_p) = \mathbb{Z} \oplus (\mathbb{Z}/2)^3$ with corresponding generators $\langle 1 \rangle$, $\langle 1 \rangle - \langle p \rangle$, $\langle 1 \rangle - \langle g \rangle$, and $\langle 1 \rangle - \langle pg \rangle$. Since $\langle 1, -1 \rangle = 2\langle 1 \rangle$, $W(\mathbb{Q}_p) = (\mathbb{Z}/2)^4$ with the same generators. I is generated by $\langle 1, p \rangle$, $\langle 1, g \rangle$, and $\langle 1, pg \rangle$; I^2 is generated by $\langle 1, p, g, pg \rangle$, and $I^3 = 0$. So $\mathrm{Gr}_I W = \mathbb{Z}/2 \oplus (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/2$. Note that this is the first example we've seen where $I^2 \neq 2I$.

When $p \equiv 3 (\mathrm{mod}\ 4)$ we can take $g = -1$. One has $\langle 1, 1 \rangle = \langle -1, -1 \rangle$ by the same reasoning as for $\mathbb{F}_p$ (-1 is the sum of two squares), and so $\langle p, p \rangle = \langle -p, -p \rangle$. Note that

$$\langle p, p, p, p \rangle = \langle p, -p, -p, p \rangle = \langle 1, -1, -1, 1 \rangle = \langle 1, 1, 1, 1 \rangle$$

and so $4(\langle 1 \rangle - \langle p \rangle) = 0$. Also,

$$\langle p, p, p \rangle = \langle p, -p, -p \rangle = \langle 1, -1, -p \rangle \quad \text{and} \quad \langle 1, 1, 1 \rangle = \langle 1, -1, -1 \rangle.$$

So $3(\langle 1 \rangle - \langle p \rangle) = \langle -1 \rangle - \langle -p \rangle$. Of course $GW(\mathbb{Q}_p)$ is generated by $\langle 1 \rangle$, $\langle 1 \rangle - \langle -1 \rangle$, $\langle 1 \rangle - \langle p \rangle$, and $\langle 1 \rangle - \langle -p \rangle$, and the previous computation shows the last generator is not needed. So we have a surjective map $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \to GW(\mathbb{Q}_p)$ sending the standard generators to $\langle 1 \rangle$, $\langle 1 \rangle - \langle -1 \rangle$, and $\langle 1 \rangle - \langle p \rangle$. This is readily checked to be injective once one knows that $\langle 1, 1 \rangle \ncong \langle p, p \rangle$. If these forms were isomorphic it would follow by reduction mod some power of p that $\langle 1, 1 \rangle$ was isotropic over some $\mathbb{F}_{p^e}$; that is, we would have $\langle 1,1\rangle \cong \langle 1,-1\rangle$. But we've already computed $GW(\mathbb{F}_{p^e})$, and know this is not the case.

The Witt ring is $W(\mathbb{Q}_p) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$ with generators $\langle 1\rangle$ and $\langle 1\rangle - \langle p\rangle$. The ideal I is generated by $2\langle 1\rangle$ and $\langle 1\rangle - \langle p\rangle$; I^2 is generated by $2(\langle 1\rangle - \langle p\rangle)$; $I^3 = 0$. Again we have $\mathrm{Gr}_I\, W \cong \mathbb{Z}/2 \oplus (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/2$.

(g) Return to $F = \mathbb{Q}$. Our understanding of the higher Milnor K-groups of $\mathbb{Q}$ is based on passing to the various completions $\mathbb{Q}_p$ and $\mathbb{R}$. A computation of Bass and Tate [Mi, Lem. A.1] gives an exact sequence

$$0 \to K_2^M(\mathbb{Q})/2 \to K_2^M(\mathbb{R})/2 \oplus \Big(\oplus_p K_2^M(\mathbb{Q}_p)/2\Big) \to \mathbb{Z}/2 \to 0,$$

and we already know $K_2^M(\mathbb{Q}_p)/2 \cong K_2^M(\mathbb{R})/2 \cong \mathbb{Z}/2$. A computation of Tate [Mi, Th. A.2, Ex. 1.8] shows that for $* \geq 3$ one has

$$K_*^M(\mathbb{Q})/2 \cong \oplus_p K_*^M(\mathbb{Q}_p)/2 \oplus K_*^M(\mathbb{R})/2 \cong 0 \oplus \mathbb{Z}/2.$$

To compute $H^*(\mathbb{Q};\mathbb{Z}/2)$ we again work one completion at a time. A theorem of Tate [Se, Section II.6.3, Th. B] says that for $i \geq 3$ one has

$$H^i(\mathbb{Q};\mathbb{Z}/2) \cong H^i(\mathbb{R};\mathbb{Z}/2) \times \prod_p H^i(\mathbb{Q}_p;\mathbb{Z}/2) \cong H^i(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Our computation of $\mathbb{Q}^*/(\mathbb{Q}^*)^2 \cong H^1(\mathbb{Q};\mathbb{Z}/2)$ shows that the map $H^1(\mathbb{Q};\mathbb{Z}/2) \to H^1(\mathbb{R};\mathbb{Z}/2) \times \prod_p H^1(\mathbb{Q}_p;\mathbb{Z}/2)$ is injective. More of Tate's work [Se, Sec. II.6.3, Th. A] identifies the dual of the kernel with the kernel of $H^2(\mathbb{Q};\mathbb{Z}/2) \to H^2(\mathbb{R};\mathbb{Z}/2) \times (\oplus_p H^2(\mathbb{Q}_p;\mathbb{Z}/2))$—thus, this latter map is also injective. Using this, [Se, Sec. II.6.3, Th. C] gives a short exact sequence

$$0 \to H^2(\mathbb{Q};\mathbb{Z}/2) \to H^2(\mathbb{R};\mathbb{Z}/2) \oplus (\oplus_p H^2(\mathbb{Q}_p;\mathbb{Z}/2)) \to \mathbb{Z}/2 \to 0.$$

As we have already remarked that $H^2(\mathbb{Q}_p;\mathbb{Z}/2) = H^2(\mathbb{R};\mathbb{Z}/2) = \mathbb{Z}/2$, this completes the calculation of $H^*(\mathbb{Q};\mathbb{Z}/2)$.

The method for computing the Witt group $W(\mathbb{Q})$ proceeds similarly by working one prime at a time. See [S1, Section 5.3]. One has an isomorphism of groups $W(\mathbb{Q}) \cong \mathbb{Z} \oplus (\oplus_p W(\mathbb{F}_p))$ [S1, Thm. 5.3.4]. With enough trouble one can compute $\mathrm{Gr}_I\, W(\mathbb{Q})$, but we will leave this for the reader to consider.

Remark A.2. Note that the verification of the Milnor conjectures for $F = \mathbb{Q}$ tells us exactly how to classify quadratic forms over $\mathbb{Q}$ by invariants. First one needs the invariants over $\mathbb{R}$ (which are just rank and signature), and then one needs the invariants over each $\mathbb{Q}_p$—but for $\mathbb{Q}_p$ one has $I^3 = 0$, and so p-adic forms are classified by the three classical invariants e_0, e_1, and e_2. These observations are essentially the content of the classical Hasse-Minkowski theorem.

The method we've used above, of working one completion at a time, works for all global fields; this is due to Tate for Galois cohomology, and Bass and Tate for K_*^M. In this way one verifies the Milnor conjecture for this class of fields [Mi, Lemma 6.2]. Note in particular that the class includes all finite extensions of $\mathbb{Q}$.

Appendix B. More on the motivic Adams spectral sequence

This final section is a supplement to Section 4. I will give some hints on computing the E_2-term of the motivic Adams spectral sequence, for the reader who would like to try this at home. The computations are not hard, but there are several small issues that are worth mentioning.

B.1. **Setting things up.** $H^{**}H$ is the algebra of operations on mod 2 motivic cohomology. We will write this as $\mathcal{A}$ from now on. There is the Bockstein $\beta \in \mathcal{A}^{1,0}$ and there are squaring operations $Sq^{2i} \in \mathcal{A}^{2i,i}$. We set $Sq^{2i+1} = \beta Sq^{2i} \in \mathcal{A}^{2i+1,i}$. Finally, there is an inclusion of rings $H^{**} \to \mathcal{A}$ sending an element t to the operation left-multiplication-by-t. Under our standing assumptions about $\mathcal{A}$ (see Section 4), it is free as a left H^{**}-module with a basis consisting of the admissible sequences $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_k}$.

There are two main differences between what happens next and what happens in ordinary topology. These are:

(a) The vector space $H^{**} = H^{**}(pt)$, regarded as a left $\mathcal{A}$-module, is nontrivial.
(b) The image of $H^{**} \hookrightarrow \mathcal{A}$ is not central.

The above two facts are connected. Let $t \in H^{**}$ and let Sq denote some Steenrod operation. It is not true in general that $Sq(t\cdot x) = t\cdot Sq(x)$—instead there is a Cartan formula for the left-hand side [V2, 9.7], which involves Steenrod operations on t. So the operations $Sq\cdot t$ and $t\cdot Sq$ are not the same element of $\mathcal{A}$. There is one notable exception, which is when all the Steenrod squares vanish on t. This happens for elements in $H^{n,n}$, for dimension reasons. So we have

(c) Every element of $H^{n,n}$ is central in $\mathcal{A}$.

It is important that we can completely understand H^{**} as an $\mathcal{A}$-module. This will follow from (1) the fact that $H^{**} \cong \big(\oplus_n H^{n,n}\big)[\tau]$ (see Remark 2.10); (2) all Steenrod operations vanish on $H^{n,n}$ for dimension reasons; (3) all Sq^i's vanish on τ except for Sq^1, and $Sq^1(\tau) = \rho = \{-1\} \in H^{1,1}$; (4) the Cartan formula. In particular we note the following two facts about H^{**}, which are all that will be needed later (the second fact only needs Remark 2.10):

(d) The map $Sq^2\colon H^{n-1,n} \to H^{n+1,n+1}$ is zero for all $n \geq 1$.
(e) The map $H^{p,q} \otimes H^{i,j} \to H^{p+i,q+j}$ is surjective for $q \geq p \geq 0$ and $j \geq i \geq 0$.

We are aiming to compute $\mathrm{Ext}^a_{\mathcal{A}}(H^{**}, \Sigma^{b,0}H^{**})$. In ordinary topology we could use the normalized bar construction to do this, but one has to be careful here because H^{**}, as a left $\mathcal{A}$-module, is not the quotient of $\mathcal{A}$ by a two-sided ideal. One way to see this is to use the fact that $Sq^1(\tau) = \rho$. Under the quotient map $\mathcal{A} \to H^{**}$ sending θ to $\theta(1)$, Sq^1 maps to zero but $Sq^1\tau$ does not (it maps to ρ).

So instead of the normalized bar construction we must use the unnormalized one. This can be extremely annoying, but for the most part it turns out not to influence the "low-dimensional" calculations we're aiming for. It is almost certainly an issue when computing past column two of the Adams E_2 term, though. Anyway, let

$$B_n = \mathcal{A} \otimes_{H^{**}} \mathcal{A} \otimes_{H^{**}} \cdots \otimes_{H^{**}} \mathcal{A} \otimes_{H^{**}} H^{**}$$

($n+1$ copies of $\mathcal{A}$). The final H^{**} can be dropped off, of course, but it's useful to keep it there because the $\mathcal{A}$-module structure on H^{**} is nontrivial and enters into the definition of the boundary map. If we denote the generators of B_n as $x = a[\theta_1|\theta_2|\cdots|\theta_n]t$ then the differential is

$$d(x) = (a\theta_1)[\theta_2|\cdots|\theta_n]t + a[\theta_1\theta_2|\theta_3|\cdots|\theta_n]t + \cdots + a[\theta_1|\cdots|\theta_{n-1}]\theta_n(t).$$

The good news is that our coefficients have characteristic 2, and so we don't have to worry about signs. Note that B_n, as a left H^{**}-module, is free on generators $1[\theta_1|\cdots|\theta_n]1$ where each θ_i is an admissible sequence of Steenrod operations (and we must include the possibility of the null sequence $Sq^0 = 1$). We will often drop the 1's off of either end of the bar element, for convenience.

Generators of $\mathrm{Hom}_{\mathcal{A}}(B_n, H^{**})$ can be specified by giving a bar element $[\theta_1|\cdots|\theta_n]$ together with an element $t \in H^{**}$. This data defines a homomorphism $B_n \to H^{**}$ sending the generator $[\theta_1|\cdots|\theta_n]$ to t and all other generators of B_n to zero. Let's denote this homomorphism by $t[\theta_1|\cdots|\theta_n]^*$. These elements generate $\mathrm{Hom}_{\mathcal{A}}(B_n, H^{**})$ as an abelian group.

The last general point to make concerns the multiplicative structure in the cobar construction. If we were working with $\mathrm{Ext}_A(k,k)$ where k is commutative and A is an augmented k-algebra, multiplying two of the above generators in the cobar complex just amounts to concatenating the bar elements—the labels $t \in k$ commute with the θ's, and so can be grouped together: e.g. $t[\theta_1|\cdots|\theta_n] \cdot u[\alpha_1|\cdots|\alpha_k] = tu[\theta_1|\cdots|\theta_n|\alpha_1|\cdots|\alpha_k]$. In our case, the fact that H^{**} is not central in $\mathcal{A}$ immensely complicates the product on the cobar complex: very roughly, the u has to be commuted across each θ_i, and in each case a resulting Cartan formula will introduce new terms into the product. Luckily there is one case where these complications aren't there, which is when $u \in H^{n,n}$—for then u is in the center of $\mathcal{A}$, and the product works just as above. We record this observation for future use:

(f) $t[\theta_1|\cdots|\theta_n]^* \cdot u[\alpha_1|\cdots|\alpha_k]^* = tu[\theta_1|\cdots|\theta_n|\alpha_1|\cdots|\alpha_k]^*$ when $u \in H^{q,q}$.

B.2. **Computations.** We are trying to compute the groups $\mathrm{Ext}^a_{\mathcal{A}}(H^{**}, \Sigma^{b,0}H^{**})$, and from here on everything is fairly straightforward. As an example let's look at $b = 1$. Since $H^{p,q} \neq 0$ only when $0 \le p \le q$, one sees that $\mathrm{Hom}_{\mathcal{A}}(B_0, H^{**}) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(B_1, \Sigma^{1,0}H^{**}) \cong H^{0,0} \oplus H^{1,1}$. The generators for this group are elements of the form $s[Sq^1]^*$ and $t[Sq^2]^*$, where $s \in H^{0,0}$ and $t \in H^{1,1}$.

We likewise find that $\mathrm{Hom}_{\mathcal{A}}(B_2, \Sigma^{1,0}H^{**}) \cong H^{0,1} \oplus H^{0,1} \oplus H^{0,1} \oplus H^{0,1}$, generated by elements $s[Sq^1|1]^*$, $s[1|Sq^1]^*$, $t[Sq^2|1]^*$, and $t[1|Sq^2]^*$. A similar analysis shows that $\mathrm{Hom}_{\mathcal{A}}(B_n, \Sigma^{1,0}H^{**})$ only has such 'degenerate' terms for $n \ge 2$. No degenerate terms like these contribute elements to Ext (at worst they can contribute relations to Ext). So the Ext^n's vanish for $n \ge 2$. An analysis of the coboundary shows that everything in dimension 1 is a cycle. So we find that

$$0 = \mathrm{Ext}^0(H^{**}, \Sigma^{1,0}H^{**}) = \mathrm{Ext}^n(H^{**}, \Sigma^{1,0}H^{**}), \quad \text{for } n \ge 2$$

and

$$\mathrm{Ext}^1(H^{**}, \Sigma^{1,0}H^{**}) \cong H^{0,0} \oplus H^{1,1}$$

with a typical element in the latter group having the form $s[Sq^1]^* + t[Sq^2]^*$ (where $s \in H^{0,0}$ and $t \in H^{1,1}$).

In general, one sees for degree reasons that the 'non-degenerate' terms in $\mathrm{Hom}_{\mathcal{A}}(B_n, \Sigma^{n,0}H^{**})$ all have the form $t[\theta_1|\cdots|\theta_n]^*$ where each θ_i is either Sq^1

or Sq^2. In $\mathrm{Hom}_{\mathcal{A}}(B_{n-1}, \Sigma^{n,0}H^{**})$ one has non-degenerate terms $u[\theta_1|\cdots|\theta_{n-1}]^*$ of the following types:

(i) Each $\theta_i \in \{Sq^1, Sq^2\}$, and at least one Sq^2 occurs. Here $u \in H^{j-1,j}$ where j is the number of Sq^2's.
(ii) Each $\theta_i \in \{Sq^1, Sq^2, Sq^3\}$, and exactly one Sq^3 occurs. Here $u \in H^{j+1,j+1}$ where j is the number of Sq^2's.
(iii) Each $\theta_i \in \{Sq^1, Sq^2, Sq^2Sq^1\}$, and exactly one Sq^2Sq^1 occurs. Here one has $u \in H^{j+1,j+1}$ where j is the number of Sq^2's.
(iv) Each $\theta_i \in \{Sq^1, Sq^2, Sq^4\}$, and exactly one Sq^4 occurs. Here $u \in H^{j+2,j+2}$ where j is the number of Sq^2's.

To analyze the part of the boundary $B_n \to B_{n-1}$ that we care about, one only needs to know the Adem relations $Sq^1Sq^2 = Sq^3$ and $Sq^2Sq^2 = \tau Sq^3Sq^1$. (In fact, since Sq^3Sq^1 doesn't appear in any of the bar elements relevant to $\mathrm{Hom}(B_{n-1}, \Sigma^{n,0}H^{**})$, one may as well pretend $Sq^2Sq^2 = 0$.) From this it's easy to compute that $\mathrm{Ext}^n(H^{**}, \Sigma^{n,0}H^{**}) \cong H^{0,0} \oplus H^{n,n}$ where a typical element has the form $s[Sq^1|Sq^1|\cdots|Sq^1]^* + t[Sq^2|Sq^2|\cdots|Sq^2]^*$. The computation uses remark B.1(d). Also, one sees that all elements $s[Sq^1|Sq^2]^*$ and $s[Sq^2|Sq^1]^*$ are zero in Ext^2 (being the coboundaries of $s[Sq^3]^*$ and $s[Sq^2Sq^1]^*$, respectively). Using remark (f) from Section B.1, this completely determines $\oplus_n \mathrm{Ext}^n(H^{**}, \Sigma^n H^{**})$ as a subring of the whole Ext-algebra.

The next step is to compute $\mathrm{Ext}^0(H^{**}, \Sigma^{1,0}H^{*,*})$, $\mathrm{Ext}^1(H^{**}, \Sigma^{2,0}H^{*,*})$, and $\mathrm{Ext}^2(H^{**}, \Sigma^{3,0}H^{*,*})$ completely. The first group is readily seen to vanish. For the second group one has to grind out another term of the bar construction, but it's a very small term. One finds that

$$\mathrm{Ext}^1(H^{**}, \Sigma^{2,0}H^{*,*}) \cong H^{0,1} \oplus H^{2,2}$$

where the generators have the form $s[Sq^2]^* + (Sq^1s)[Sq^3]^*$ and $t[Sq^4]^*$. To get the Ext^2 group one will need three more Adem relations, namely

$$Sq^2Sq^3 = Sq^5 + Sq^4Sq^1, \quad Sq^2Sq^4 = Sq^6 + \tau Sq^5Sq^1, \quad \text{and} \quad Sq^3Sq^2 = \rho Sq^3Sq^1.$$

Then the same kind of coboundary calculations (but a few more of them) show that

$$\mathrm{Ext}^2(H^{**}, \Sigma^{3,0}H^{*,*}) \cong H^{1,2} \oplus H^{2,2}$$

where the generators are $s[Sq^2|Sq^2]^* + (Sq^1s)[Sq^3|Sq^2]^*$ and $t[Sq^1|Sq^4]^* = t[Sq^4|Sq^1]^*$ (these last two classes are the same in Ext). It is important to note that all elements $u[Sq^2|Sq^4]^*$ and $u[Sq^4|Sq^2]^*$ are coboundaries (of $u[Sq^6]^*$ and $u[Sq^4Sq^2]^*$, respectively). This justifies fact (7) on page 20. To justify fact (6) from that same page (for $n = 2$), one notices that the cycles $s[Sq^2|Sq^2]^* + (Sq^1s)[Sq^3|Sq^2]^*$ and $t[Sq^4|Sq^1]^*$ decompose as a products

$$\big(s_1[Sq^2]^* + (Sq^1s_1)[Sq^3]^*\big)\cdot\big(s_2[Sq^2]^*\big) \qquad \text{and} \qquad \big(t_1[Sq^4]^*\big)\cdot\big(t_2[Sq^1]^*\big)$$

for some $s_1 \in H^{0,1}$, $s_2 \in H^{1,1}$, $t_1 \in H^{2,2}$, and $t_2 \in H^{0,0}$. This uses remarks (e) and (f) from Section B.1, together with the fact that $(Sq^1s_1)s_2 = Sq^1(s_1s_2)$ for $s_2 \in H^{2,2}$ (by the Cartan formula).

The final step is to analyze the groups $\mathrm{Ext}^{n-1}(H^{**}, \Sigma^{n,0}H^{**})$ for $n \geq 4$; these complete the $E^{1,*}$ column of the Adams spectral sequence. One doesn't have to compute them explicitly, just enough to know that every element is decomposable as a sum of products from $\text{Ext}^{n-2}(H^{**}, \Sigma^{n-1,0}H^{**})$ and $\text{Ext}^1(H^{**}, \Sigma^{1,0}H^{**})$.

The calculations involve nothing more than what we've done so far, except for more sweat. It's fairly easy to write down all the cocycles made up from the classes of types (i)-(iv) listed previously. All bar elements which have a Sq^4 in them are cocycles, for instance. But note that such a bar element will either begin or end with a Sq^1 or a Sq^2, so that it decomposes as a product of smaller degree cocycles (this again depends on B.1(e,f)). One also finds cocycles of the form

$$s[Sq^1|Sq^1|\cdots|Sq^3|Sq^1|\cdots|Sq^1]^* + s[Sq^1|Sq^1|\cdots|Sq^2Sq^1|Sq^1|\cdots|Sq^1]^*,$$

but for each of these a common $[Sq^1]^*$ can be pulled off of either the left or right side—again showing it to be decomposable.

Certainly there are cocycles which are *not* decomposable, like ones of the form

$$s[Sq^2|Sq^1|\cdots|Sq^1|Sq^3]^* + s[Sq^2Sq^1|Sq^1|\cdots|Sq^1|Sq^2]^*.$$

But this is the coboundary of $s[Sq^2Sq^1|Sq^1|\cdots|Sq^1|Sq^3]$, and so vanishes in Ext.

Anyway, I am definitely not going to give all the details. But with enough diligence one can see that all elements of $\text{Ext}^{n-1}(H^{**}, \Sigma^{n,0}H^{**})$ for $n \geq 3$ do indeed decompose into products.

Remark B.3. A final note about Adem relations, for those who want to try their hand at further calculations. Every formula I've seen for the motivic Adem relations—in publications or preprints—seems to either contain typos or else is just plain wrong. A good test for a given formula is to see whether it gives $Sq^3Sq^2 = \rho Sq^3Sq^1$ (this formula follows from the smaller Adem relation $Sq^2Sq^2 = \tau Sq^3Sq^1$, the derivation property of the Bockstein, the fact that $\beta^2 = 0$, and the identity $Sq^3 = \beta Sq^2$).

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Department of Mathematics, University of Oregon, Eugene, OR 97403
Email address: `ddugger@math.uoregon.edu`