MULTIPLICATIVE STRUCTURES ON HOMOTOPY SPECTRAL SEQUENCES, PART I

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1. Introduction

A tower of homotopy fiber sequences gives rise to a spectral sequence on homotopy groups. In modern times such towers are ubiquitous, and most of the familiar spectral sequences in topology can be constructed in this way. A **pairing of towers** $W_* \wedge X_* \to Y_*$ consists of maps $W_m \wedge X_n \to Y_{m+n}$ which commute (on-the-nose) with the maps in the towers. It is a piece of folklore that a pairing of towers gives rise to a pairing of the associated homotopy spectral sequences. This paper gives a careful proof of this general fact, for towers of spaces and towers of spectra.

Of course the main result is well-known, and in one form or another has been used continuously for the past forty years; the paper is therefore mostly expository. The only thing which might possibly be considered 'new' is the adaptation of the results to the modern category of symmetric spectra, given in section 6. The reason the paper exists at all is that I was trying to understand these spectrum-level results, and found the existing literature extremely frustrating. After a long time I finally gave up and decided to rebuild everything from scratch, and that is what the present paper does. After all the machinery is laid out and the sign conventions in place, the actual results are fairly simple.

When producing a pairing of spectral sequences, the work often divides into two parts. One part is completely formal, and says that in a certain kind of situation there is automatically a pairing. The non-formal part involves either getting into such a situation to begin with, or else interpreting what the formal machinery actually produced. The present paper deals only with the formal part; its companion [D] works through the non-formal part in a few standard examples.

Other references for multiplicative structures on spectral sequences are: [FS, Appendix A], [BK2], [GM, p. 162], [K], [MS, Thm. 4.2], [V], [Sp, Chap. 9.4], [Wh, XIII.8].

1.1. **Summary.** The main difficulty with this material is the need to be careful about details. Among other things, one has to get the signs right. This entails, for instance, having an explicit choice in mind for the differential in a long exact sequence on homotopy groups. One also has to be careful about keeping track of orientations on spheres. Several sections of the paper are devoted to details like this, as well as to recalling some basic material: sections 2 and 3 handle the case of spaces, and then Appendix C deals with the case of spectra.

Sections 4 deals with products. The ultimate reason homotopy spectral sequences are multiplicative is Proposition 4.1—everything else is just elaboration. I find this easier to understand than other discussions in the literature, but the reader should check those out for himself. A treatment of products similar to the one given here is in [Ad, pp. 236–243].

Sections 5 and 6 give the basic multiplicativity results for towers of spaces and spectra. For spectra, our category of choice is that of symmetric spectra [HSS], mostly because of the simplicity and elegance of the basic definitions. The results of [MMSS, S] show that theorems proven in this category will work in any of the other modern categories of spectra. It is also true that our proofs are generic enough that they should work in most other categories, with only slight modifications.

When dealing with pairings of towers, it's important that everything commute on-the-nose. A pairing that commutes only up to homotopy will not necessarily induce a pairing of spectral sequences—section 7 gives an example. Moreover, the conditions that would have to be checked to know there *is* an induced pairing are unwieldy in practice. This causes certain difficulties associated with the fact that not every symmetric spectrum is cofibrant and fibrant. If one has an on-the-nose pairing between towers and then applies a fibrant- (or cofibrant-) replacement functor to all the objects, there is not necessarily a pairing between the new towers. The bulk of the work in section 6 is to get around this problem via a small trick, but it's a trick that ends up being useful in many situations.

There are three appendices. Appendix A is a reference for certain conventions, but is not used for anything else in this paper. Appendix B deals with some technical issues needed for the trick in section 6. Finally, Appendix C gives a careful background treatment for basic results about the category of spectra, in particular about sign conventions for the boundaries in long exact sequences.

1.2. **Notation and terminology.** Here we list some things from the paper which might cause confusion. First, the **cofiber** of a map $A \to B$ refers to the pushout of $* \leftarrow A \to B$, and is denoted B/A; note that $A \to B$ need not be a monomorphism. The phrase 'homotopy cofiber' is never abbreviated as 'cofiber'.

If one has a construction on a model category—for instance, a monoidal product $\wedge$—then we denote its derived functor on the homotopy category by an underline—for instance, $\underline{\wedge}$. The derived functor is unique up to unique isomorphism, and so we always assume a specific one has been chosen. The notation Ho$(-,-)$ refers to the set of maps in the homotopy category.

The symbol $W_\perp$ denotes an augmented tower, described in the beginning of section 6.

The difference between 'homotopy cofiber sequence' and 'rigid homotopy cofiber sequence' is explained in Appendix C.4. In essence, the former refers to something going on in a homotopy category, whereas the latter refers to something in a model category. This difference is non-vacuous.

If the pieces of a filtration get smaller as the indices gets bigger, we write the filtration as F^p. If the pieces get bigger as the indices get bigger, we write F_p. So $F^{p+1} \subseteq F^p$, but $F_p \subseteq F_{p+1}$. Note that a cellular filtration of a space is of the first type if the pieces are indexed by codimension (hence the superscript), and of the second type if the pieces are indexed by dimension (hence the subscript).

2. Preliminaries

2.1. **Orientations.** We give $\mathbb{R}$ the usual orientation, and $\mathbb{R}^n$ the product orientation. The interval $I = [0,1]$ has 0 as basepoint, and is oriented as a subspace of $\mathbb{R}$. Let S^{n-1} and D^n denote the unit sphere and unit ball in $\mathbb{R}^n$, with $(-1, 0, \ldots, 0)$ as their basepoint. D^n inherits an orientation from $\mathbb{R}^n$, and S^{n-1} inherits the boundary orientation from D^n (see remark below). We fix once and for all a family of orientation-preserving, basepoint-preserving homeomorphisms $D^n/S^{n-1} \to S^n$ and $I \to D^1$. Since all such homeomorphisms are homotopic, the particular choice will not influence anything we do.

Remark 2.2. *Boundary orientations.* Recall that if M is an oriented manifold-with-boundary then there exists an embedding $I \times \partial M \to M$ which is the identity on $\{0\} \times \partial M$. The orientation of ∂M is chosen so that any such embedding is order-reversing, where $I \times \partial M$ has the product orientation. This is the convention forced on us if we want (1) ∂I to have the usual orientation, and (2) $\partial(M \times N) = (\partial M) \times N \cup (-1)^{\dim M} M \times \partial N$, where the sign indicates how the orientation on the second component compares to the product orientation on $M \times \partial N$.

Remark 2.3. We've chosen $(-1, 0, \ldots, 0)$ for the basepoint of D^n because this makes D^1 homeomorphic to I as a pointed, oriented space. It also ensures that S^0, oriented as ∂D^1, has the non-basepoint oriented postively—this is the 'correct' convention, necessary for $S^0 \wedge X$ to have the same orientation as X.

For any pointed space A, let $CA = A \wedge I$ and $\Sigma A = A \wedge (I/\partial I) \cong A \wedge S^1$. The suspension coordinate has been placed on the *right* because this will work better with the adjointness formula $\mathrm{Hom}(X \wedge S^1, Y) = \mathrm{Hom}(X, F(S^1, Y))$, particularly when we start working with spectra.

We will occasionally deal with orientations on spaces which are not manifolds (like $S^p \vee S^p$), or at least are not *a priori* manifolds (like $S^p \wedge S^q$). These will always be spaces X with a finite number of singular points, and we really mean an orientation on the deleted space $X - \{\text{singular points}\}$.

Exercise 2.4. CS^{p-1} inherits an orientation as a quotient of $S^{p-1} \times I$. Check that the induced orientation on $\partial(CS^{p-1})$ is $(-1)^{p-1}$ times the original orientation of $S^{p-1} \times \{1\}$—for short, $\partial(CS^{p-1}) = (-1)^{p-1} S^{p-1}$.

2.5. **Relative homotopy groups.** Let $f\colon A \to B$ be a map between pointed spaces. For $p \geq 1$ define $\pi_p(B,A)$ to be the set of equivalence classes of diagrams $\mathcal{D}$ of the form

$$\begin{array}{ccc} S^{p-1} & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^p & \longrightarrow & B \end{array} \tag{2.1}$$

(where the horizontal maps preserve the basepoint, of course). Two diagrams $\mathcal{D}$ and $\mathcal{D}'$ are regarded as equivalent if there is a diagram

$$\begin{array}{ccc} S^{p-1}\times I & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^p \times I & \longrightarrow & B \end{array}$$

of basepoint-preserving homotopies which restricts to $\mathcal{D}$ under the inclusion $\{0\} \hookrightarrow I$ and to $\mathcal{D}'$ under $\{1\} \hookrightarrow I$. Note that $\pi_p(B,A)$ depends on the map f; we are leaving this out of the notation only because the map is usually clear from context.

When $p \geq 2$, we pinch the equatorial disk D^{p-1} in D^p and choose an orientation-preserving homeomorphism $D^p/D^{p-1} \to D^p \vee D^p$. This gives a diagram

$$\begin{array}{ccc} S^{p-1} & \longrightarrow & S^{p-1} \vee S^{p-1} \\ \downarrow & & \downarrow \\ D^p & \longrightarrow & D^p \vee D^p \end{array}$$

and allows us to define a product on $\pi_p(B,A)$ in the usual way. One can check that this makes $\pi_p(B,A)$ into a group for $p \geq 2$ and an abelian group for $p > 2$.

The set $\pi_p(B,*)$ will be abbreviated as $\pi_p B$. Note that this is canonically isomorphic to $[D^p/S^{p-1}, B]_*$, and therefore to $[S^p, B]_*$ via our fixed homeomorphism $D^p/S^{p-1} \cong S^p$. We will use this identification freely in what follows. Note also that functoriality gives a natural map $\pi_p(B,A) \to \pi_p(B/A,*)$, for any map $A \to B$.

Remark 2.6. If $f\colon A \to B$ is a map of pointed spaces, the homotopy fiber of f is defined to be the pullback

$$\begin{array}{ccc} \operatorname{hofib}(f) & \longrightarrow & F(I,B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

where $F(I,B)$ is the space of basepoint-preserving maps $I \to B$, and $F(I,B) \to B$ sends a path γ to $\gamma(1)$. Observe that $\pi_p(B,A)$ is isomorphic to $\pi_{p-1}\operatorname{hofib}(f)$—we are choosing to use the former notation because it works better with respect to products.

We can define a canonical long exact sequence

$$\cdots \longrightarrow \pi_p(A,*) \xrightarrow{f_*} \pi_p(B,*) \xrightarrow{j} \pi_p(B,A) \xrightarrow{\kappa} \pi_{p-1}(A,*) \longrightarrow \cdots$$

which terminates as

$$\cdots \to \pi_0(A,*) \to \pi_0(B,*).$$

If $y \in \pi_p(B,A)$ is represented by a diagram $\mathcal{D}$ as above, then $\kappa(y)$ is the element of $\pi_{p-1}(A,*)$ represented by the map $S^{p-1} \to A$. Likewise, if $x \in \pi_p(B,*)$ is represented by a map $x\colon D^p/S^{p-1} \to B$ then $j(x)$ is the equivalence class of the diagram

$$\begin{array}{ccc} S^{p-1} & \xrightarrow{\;*\;} & A \\ \downarrow & & \downarrow \\ D^p & \xrightarrow{\;\bar{x}\;} & B. \end{array}$$

Note that everything we have done is functorial in the map f: there were no choices made in writing down either the groups or the maps in the exact sequences.

Remark 2.7. Sometimes it is tempting to abandon the disks D^n altogether, and instead work only with the spaces CS^{n-1}. For instance we could have defined $\pi_p(B,A)$ as equivalence classes of diagrams

$$\begin{array}{ccc} S^{p-1} & \xrightarrow{\;f\;} & A \\ \downarrow & & \downarrow \\ CS^{p-1} & \xrightarrow{\;g\;} & B. \end{array}$$

In this case one defines κ by sending an element represented by the above square to $(-1)^{p-1}[f]$. Ultimately this is because κ is a 'boundary' map, and by Exercise 2.4 $\partial(CS^{p-1}) = (-1)^{p-1}S^{p-1}$. If you leave out the sign in the definition of κ, you run into unpleasant-looking formulas for products later on.

3. The homotopy spectral sequence

By a **tower** we will simply mean a sequence of pointed spaces W_n with (basepoint-preserving) maps between them:

$$\cdots \longrightarrow W_3 \longrightarrow W_2 \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W_{-1} \longrightarrow \cdots .$$

In many cases one has either that $W_n = *$ for $n < 0$, or that $W_n \to W_{n-1}$ is the identity for $n < 0$; none of the basic ideas will be lost by thinking only of these simpler cases, if the reader desires.

The long exact sequences for each map in the tower of course patch together to form an exact couple (except for a truncation—see Remark 3.2 below). We are free to index the exact couple in any way we want, and our choice will depend on the kind of tower we're looking at. There are two basic situations which are most common:

- lim-towers: $\operatorname{colim}_n \pi_* W_n = 0$, and the spectral sequence is used to give information about $\lim_n \pi_* W_n$. In this situation it often turns out that $\lim_n \pi_* W_n$ is actually the same as $\pi_*(\operatorname{holim}_n W_n)$, and the latter is what we're really interested in.
- colim-towers: $\lim_n \pi_* W_n = 0$, and the spectral sequence is used to give information about $\operatorname{colim}_n \pi_* W_n$. Often one also has that $W_n \to W_{n-1}$ is the identity for $n \leq 0$, in which case we are getting information about $\pi_* W_0$.

In this paper we will use the indexing conventions that are most useful for colim-towers, because that turns out to be where pairings work best. (See Appendix A for the lim-tower conventions, however.) We set

$$D_1^{p,q} = \pi_p(W_q, *) \quad \text{and} \quad E_1^{p,q} = \pi_p(W_q, W_{q+1}) \qquad p \geq 1, q \in \mathbb{Z},$$

and the maps $j\colon D_1^{p,q} \to E_1^{p,q}$ and $\kappa\colon E_1^{p,q} \to D_1^{p-1,q+1}$ are as defined in the last section. The differential d_r has the form $E_r^{p,q} \to E_r^{p-1,q+r}$, and the spectral sequence is drawn on a grid with $E_r^{p,q}$ in the (p,q)-spot. This is usually called **Adams indexing**, and is designed so that the E_∞-term can be read along the vertical lines, with the group in the (p,q)-spot contributing to the qth filtration piece of a pth homotopy group.

Remark 3.1. There are many ways to index a spectral sequence, and the 'best' way depends both on personal taste and the situation at hand. Sometimes, for instance, it is convenient to 'flip' the above indexing and draw our $E_r^{p,q}$ in the $(-p,q)$ spot—this is nice if our pth homotopy groups are secretly $(-p)$th cohomology groups. We have settled on the indexing scheme which seems easiest to remember, and easiest to draw; but the reader is welcome to re-index things however he wants. For a Serre cohomology spectral sequence one would set $E_1^{p,q} = \pi_{-p-q}(W_p, W_{p+1})$, for example.

Remark 3.2. Note that we don't really have a spectral sequence, or even an exact couple; this is because π_0 and π_1 need not be abelian groups, and the long exact homotopy sequences need not be exact at the final π_0. One can either choose not to define the $E_r^{p,q}$ for $r > 1$, $p \leq 2$ because of these difficulties—in which case we have a 'fringed' spectral sequence—or else one follows [BK, IX.4] and obtains an 'extended' spectral sequence in which those $E_r^{p,q}$ are defined but may only be pointed sets or non-abelian groups. Either way, there is stuff to worry about.

One way we could avoid these issues is to state results under the restriction that the W_n's are connected, with abelian fundamental groups. Another way is to state results only in the range $p \geq 3$. We won't dwell on this; in this paper we'll implicitly assume the reader is in a situation where these fringe problems aren't there.

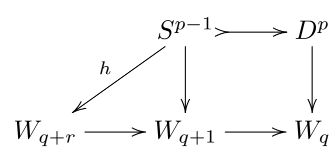
As with any spectral sequence, one defines a nested sequence of subgroups

$$0 \subseteq B_2^{p,q} \subseteq \cdots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \cdots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \cdots \subseteq Z_2^{p,q} \subseteq E_1^{p,q},$$

so that $Z_r^{p,q}$ consists of all elements which are killed by d_1 through d_{r-1}, and $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. We will need the following elementary result:

Lemma 3.3. *Let $\alpha \in E_1^{p,q} = \pi_p(W_q, W_{q+1})$.*

(a) α lies in $Z_r^{p,q}$ if and only if α can be represented by a diagram $\mathcal{D}$ as in (2.1) in which the map $S^{p-1} \to W_{q+1}$ factors (on-the-nose) through W_{q+r}. In other words, there is a commutative diagram



in which the right-hand-square represents the element α.

(b) If α and h are as above then $d_r(\alpha)$ can be represented by the diagram

$$\begin{array}{ccc} S^{p-2} & \rightarrowtail & D^{p-1} \\ {\scriptstyle *}\downarrow & & \downarrow \\ W_{q+r+1} & \longrightarrow & W_{q+r} \end{array}$$

where the left vertical arrow collapses everything to the basepoint, and the right vertical arrow is the composite $D^{p-1} \to D^{p-1}/S^{p-2} \cong S^{p-1} \xrightarrow{h} W_{q+r}$.

Proof. Part (a) is an application of the homotopy-extension-property. Choose a diagram as in (2.1) representing α, and let α_0 denote the 'top' map $S^{p-1} \to W_{q+1}$. From analyzing the exact couple, the fact that α lies in Z_r means that the (pointed) homotopy class $\alpha_0\colon S^{p-1} \to W_{q+1}$ lifts to W_{q+r}. So there is a map $h\colon S^{p-1} \to W_{q+r}$ for which the composite $S^{p-1} \to W_{q+r} \to W_{q+1}$ is homotopic to α_0. Choose a basepoint-preserving homotopy $H\colon S^{p-1} \times I \to W_{q+1}$ from α_0 to this composite. Projecting further down into W_q, we can glue this to the original map $D^p \to W_q$ to get $H'\colon (D^p \times \{0\}) \cup (S^{p-1} \times I) \to W_q$. The homotopy-extension-property for the pair (D^p, S^{p-1}) lets us extend H' over $D^p \times I$. Restricting H' to time $t = 0$ gives the original diagram representing a, whereas restricting to $t = 1$ gives a diagram where we have the required lifting to W_{q+r}.

Part (b) is a simple thought exercise. □

3.4. **Convergence.**

Let $\pi_p(W_-)$ denote $\operatorname{colim}_n \pi_p W_n$. Note that the terminology is misleading, because W_- isn't a space whose homotopy group we're looking at. Define a filtration on $\pi_p(W_-)$ by letting $F^q\pi_p(W_-)$ denote those elements in the image of $\pi_p(W_q)$. Let $\operatorname{Gr}^q \pi_p(W_-) = F^q/F^{q+1}$. Given $\alpha \in F^q\pi_p(W_-)$, choose a map $\beta\colon D^p/S^{p-1} \to W_q$ which lifts α (up to homotopy). Then $j(\beta)$ is an element in $\pi_p(W_q, W_{q+1}) = E_1^{p,q}$.

Exercise 3.5.

(a) Check that $j(\beta)$ is an infinite cycle, and that its class in E_∞ doesn't depend on the lifting β—in other words, verify that we have a well-defined map $F^q\pi_p(W_-) \to E_\infty^{p,q}$.

(b) Check that F^{q+1} maps to zero under this map, and so there is an induced map $\Gamma\colon \operatorname{Gr}^q \pi_p(W_-) \to E_\infty^{p,q}$.

(c) Verify that Γ is an injection.

Here is one basic convergence result:

Proposition 3.6. *Suppose W_* is a colim-tower (either of spaces or spectra).*

(a) If $RE_\infty = 0$ then Γ is an isomorphism.

(b) If $\lim^1_n \pi_* W_n = 0$, the spectral sequence converges conditionally.

(c) If both conditions from (a) and (b) hold, and also $\bigcap_q F^q\pi_p(W_-) = 0$, the spectral sequence converges strongly.

(d) Suppose both conditions from (a) and (b) hold, and also that for each $\alpha \in \pi_p(W_-)$ there exists an N such that α has at most one pre-image in $\pi_p(W_N)$. Then the spectral sequence converges strongly. (Note in particular that the condition is satisfied if $W_n \to W_{n-1}$ is the identity for all $n \ll 0$).

The reader should refer to [Bd] for an explanation of the $RE_\infty = 0$ condition. The most common situation in which it is satisfied is when for each p, q there exists an N such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq N$.

Proof. The statement in (b) is really just the definition of conditional convergence [Bd, Defn. 5.10]. The statement in (a) follows from [Bd, Lemma 5.6, Lemma 5.9(a)]. Part (c) is essentially [Bd, Theorem 8.10]—Boardman's group W is $\cap_q F^q(\pi_*(W_-))$ in this case.

For (d) we will show that the given condition implies $\cap_q F^q \pi_p(W_-) = 0$. Suppose α is in this intersection, and pick an N such that α has at most one pre-image in $\pi_p(W_N)$. The condition that α be in the intersection shows that it has exactly one pre-image, which we'll denote x. Then x is in $\cap_q Im(\pi_p(W_q) \to \pi_p(W_n))$. By [Bd, Lemma 5.9(a)], the $RE_\infty = 0$ condition implies that this intersection is zero. So $x = 0$, and therefore $\alpha = 0$. □

4. Products in $\pi_*(-,-)$

We choose once and for all a family of orientation-preserving, basepoint-preserving homeomorphisms $D^{p+q} \to D^p \times D^q$. These will of course carry the boundary homeomorphically to the boundary, in an orientation-preserving sense.

Let $f\colon A \to B$ and $g\colon C \to D$ be two maps between pointed spaces. Let P be the pushout of $A \wedge D \leftarrow A \wedge C \to B \wedge C$, and note that there is a canonical map $P \to B \wedge D$. One can construct a natural pairing $\pi_p(B, A) \otimes \pi_q(D, C) \to \pi_{p+q}(B \wedge D, P)$ in the following way. Suppose given two diagrams

$$\begin{array}{ccc} S^{p-1} & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^p & \longrightarrow & B \end{array} \qquad\qquad \begin{array}{ccc} S^{q-1} & \longrightarrow & C \\ \downarrow & & \downarrow \\ D^q & \longrightarrow & D. \end{array}$$

From these we form the new diagram

$$\begin{array}{ccccc} S^{p+q-1} & \xrightarrow{\cong} & (S^{p-1} \times D^q) \coprod_{S^{p-1} \times S^{q-1}} (D^p \times S^{q-1}) & \longrightarrow & (A \wedge D) \coprod_{(A \wedge C)} (B \wedge C) \\ \downarrow & & \downarrow & & \downarrow \\ D^{p+q} & \xrightarrow{\cong} & D^p \times D^q & \longrightarrow & B \wedge D \end{array}$$

which defines an element in $\pi_{p+q}(B \wedge D, P)$. One easily checks that this product is well-defined and bilinear.

Now suppose that $x \in \pi_p(B, A)$ and $y \in \pi_q(D, C)$ are represented by the diagrams above. Then $\kappa(x \cdot y)$ is an element of $\pi_{p+q-1}(P, *)$, and the inclusion $j\colon (P, *) \to (P, A \wedge C)$ gives us $j_*(\kappa(xy)) \in \pi_{p+q-1}(P, A \wedge C)$.

Likewise, κx is represented by the diagram

$$\begin{array}{ccccc} S^{p-2} & \longrightarrow & * & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ D^{p-1} & \longrightarrow & D^{p-1}/S^{p-2} & \longrightarrow & A, \end{array}$$

and multiplying by the element y in a way similar to the above yields

$$\begin{array}{ccccccc}
\partial(D^{p-1}\times D^q) & \longrightarrow & (D^{p-1}/\partial D^{p-1})\wedge S^{q-1} & \longrightarrow & A\wedge C & = & A\wedge C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D^{p-1}\times D^q & \longrightarrow & (D^{p-1}/S^{p-2})\wedge D^q & \longrightarrow & A\wedge D & \longrightarrow & P.
\end{array}$$

This diagram represents an element of $\pi_{p+q-1}(P, A\wedge C)$ which we will call $(\kappa x)\cdot y$ (by abuse of notation). In a similar manner, one constructs an element $x\cdot(\kappa y)$.

Proposition 4.1. *For x and y as above, $j_*\kappa(x\cdot y) = (\kappa x)y + (-1)^p x(\kappa y)$ as elements in $\pi_{p+q-1}(P, A\wedge C)$.*

Proof. By naturality, one reduces to the case where $A\to B$ is $S^{p-1}\hookrightarrow D^p$ and $C\to D$ is $S^{q-1}\hookrightarrow D^q$. The result becomes a geometric calculation, essentially boiling down to the identity of oriented manifolds $\partial(D^p\times D^q) = (S^{p-1}\times D^q)\cup(-1)^p(D^p\times S^{q-1})$; the sign indicates the appropriate change in orientation.

For complete details, let $T = S^{p-1}\wedge S^{q-1}$ and $U = (S^{p-1}\wedge D^q)\amalg_T(D^p\wedge S^{q-1})$. Then $T\subseteq U$, and we are dealing with the three homotopy elements $j_*\kappa(xy)$, $(\kappa x)y$, and $x(\kappa y)$ in $\pi_{p+q-1}(U,T)$. The idea will be to produce an injection $D\colon \pi_{p+q-1}(U,T)\hookrightarrow \mathbb{Z}\oplus\mathbb{Z}$, and then verify the identity by checking it in $\mathbb{Z}\oplus\mathbb{Z}$.

Note that T is a $(p+q-2)$-sphere, U is a $(p+q-1)$-sphere, and the inclusion $T\hookrightarrow U$ is basically the inclusion of the equator. So the quotient U/T is a wedge of two $(p+q-1)$-spheres. Everything carries a natural orientation determined by our chosen orientations of spheres and disks. In particular, U is oriented as $\partial(D^p\wedge D^q)$, and this may be written as $U = (S^{p-1}\wedge D^q)\cup(-1)^p(D^p\wedge S^{q-1})$. This implies $U/T\cong (S^{p-1}\wedge[D^q/S^{q-1}])\vee(-1)^p([D^p/S^{p-1}]\wedge S^{q-1})$ (as always, the $(-1)^p$ describes how the second sphere in the wedge is oriented with respect to the product orientation on $[D^p/S^{p-1}]\wedge S^{q-1}$.)

We have a natural map $\pi_{p+q-1}(U,T)\to\pi_{p+q-1}(U/T)$: an element represented by

$$\begin{array}{ccc}
S^{p+q-2} & \xrightarrow{\alpha_0} & T \\
\downarrow & & \downarrow \\
D^{p+q-1} & \xrightarrow{\alpha_1} & U.
\end{array}$$

is sent to the map $\alpha_2\colon D^{p+q-1}/S^{q+q-2}\to U/T$, which is a map from a $(p+q-1)$-sphere to a wedge of two $(p+q-1)$-spheres. Such a map has two degrees $\deg_+\alpha_2$ and $\deg_-\alpha_2$, obtained by projecting away either of the two spheres making up U/T. In this way we obtain a map $D\colon\pi_{p+q-1}(U,T)\to\mathbb{Z}\oplus\mathbb{Z}$ sending α to $(\deg_+\alpha_2, \deg_-\alpha_2)$. One can check geometrically that

$$D\Big(j_*\big(\kappa(xy)\big)\Big) = (1,1), \qquad D(\kappa(x)y) = (1,0), \quad \text{and} \quad D(x\kappa(y)) = (0,(-1)^p).$$

To understand these, check that for the element $j_*(\kappa(xy))$ the corresponding α_2 is the map $D^{p+q-1}/S^{p+q-2}\to U/T$ which pinches an equatorial $(p+q-2)$-sphere to a point; so the degree is $(1,1)$. For $\kappa(x)y$, the map α_2 is basically the inclusion of the first wedge-summand in U/T. And finally, for $x\kappa(y)$ the corresponding α_2 is the inclusion of the second wedge-summand. This summand is not oriented in the standard way, however, and that's why the degree of the map is $(-1)^p$ rather than

1. In a minute we will see that D is an injection, but if you accept that then we have verified the identity $j_*(\kappa(xy)) = \kappa(x)y + (-1)^p x\kappa(y)$.

The map $T \hookrightarrow U$ is null, so $\text{hofib}(T \to U) \simeq T \times \Omega U$. From this it's easy to compute that $\pi_{p+q-1}(U,T) \cong \pi_{p+q-2}(T \times \Omega U) \cong \mathbb{Z} \oplus \mathbb{Z}$. But we know two elements $(\kappa x)y$ and $x(\kappa y)$ in $\pi_{p+q-1}(U,T)$, and we have already calculated that their images under D are $(1,0)$ and $(0,(-1)^p)$. So the image of D has rank 2, therefore D is an injection (in fact, an isomorphism). □

Exercise 4.2. Let B/A denote the pushout of $* \leftarrow A \to B$, and recall that there is a map $\pi_p(B,A) \to \pi_p(B/A,*)$ induced by the map of pairs $(B,A) \to (B/A,*)$.

Check that there is a natural map $\phi\colon (B \wedge D)/P \to [B/A] \wedge [D/C]$ and that the following diagram commutes:

$$\begin{array}{ccc}
\pi_k(B,A) \otimes \pi_k(D,C) & \longrightarrow & \pi_{k+l}(B \wedge D, P) \\
\downarrow{\scriptstyle pr\otimes pr} & & \downarrow{\scriptstyle pr} \\
\pi_k(B/A) \otimes \pi_l(D/C) & & \pi_{k+l}((B \wedge D)/P) \\
& \searrow \qquad \swarrow{\scriptstyle \phi} & \\
& \pi_{k+l}(B/A \wedge D/C). &
\end{array}$$

(Note that, because of naturality, it suffices to check this in the universal case).

5. Pairings of spectral sequences

Now suppose that we have three towers W_*, X_*, and Y_*, with the resulting homotopy spectral sequences denoted by $E_*(W)$, $E_*(X)$, and $E_*(Y)$. Assume that there are pairings $W_m \wedge X_n \to Y_{m+n}$ such that the following squares are commutative (not just homotopy-commutative!):

$$\begin{array}{ccccc}
W_{m+1} \wedge X_n & \longrightarrow & Y_{m+n+1} & \longleftarrow & W_m \wedge X_{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
W_m \wedge X_n & \longrightarrow & Y_{m+n} & \longleftarrow & W_m \wedge X_n.
\end{array}$$

Our first claim is that there is an induced pairing

$$\pi_k(W_m, W_{m+1}) \otimes \pi_l(X_n, X_{n+1}) \to \pi_{k+l}(Y_{m+n}, Y_{m+n+1}).$$

This follows from the construction of products in section 4, together with naturality. In terms of our spectral sequences we have produced a multiplication

$$E_1^{p,q}(W) \otimes E_1^{s,t}(X) \to E_1^{p+s,q+t}(Y).$$

It follows from Proposition 4.1 and naturality that the differential $d_1 = j\kappa$ is a derivation with respect to this product. This immediately implies that the pairing on E_1-terms descends to a well-defined pairing on E_2-terms.

We must next show that the d_2 differentials behave as derivations with respect to the product on E_2, but this is a similar argument. By Lemma 3.3, elements

$x \in E_2^{p,q}(W)$ and $y \in E_2^{s,t}(X)$ can be represented by squares

$$\begin{array}{ccccc} & & S^{p-1} & \rightarrowtail & D^p \\ & \overset{f}{\swarrow} & \downarrow & & \downarrow \\ W_{q+2} & \longrightarrow & W_{q+1} & \longrightarrow & W_q \end{array} \qquad \begin{array}{ccccc} & & S^{s-1} & \rightarrowtail & D^s \\ & \overset{g}{\swarrow} & \downarrow & & \downarrow \\ X_{t+2} & \longrightarrow & X_{t+1} & \longrightarrow & X_t \end{array}$$

in which the indicated lifts exist. The outer 'squares' define elements $\bar{x} \in \pi_p(W_q, W_{q+2})$ and $\bar{y} \in \pi_s(X_t, X_{t+2})$, and Proposition 4.1 gives us an identity

$$j\kappa(\bar{x}\bar{y}) = \kappa(\bar{x})\bar{y} + (-1)^p \bar{x}\kappa(\bar{y})$$

in the group $\pi_{p+s-1}\Big((W_q \wedge X_{t+2}) \amalg_{(W_{q+2}\wedge X_{t+2})} (W_{q+2} \wedge X_t), W_{q+2} \wedge X_{t+2}\Big)$. By naturality we get an identity in $\pi_{p+s-1}(Y_{q+t+2}, Y_{q+t+3})$ (you could actually put Y_{q+t+4} in the second spot for a stronger identity). A little thought shows that this is the derivation property that we asked for.

Since d_2 is a derivation, the multiplication on E_2 descends to E_3. The same argument as above shows that d_3 is a derivation, and we continue. We have proven:

Proposition 5.1. *The product $E_1(W) \otimes E_1(X) \to E_1(Y)$ descends to pairings of the E_r-terms, satisfying the Leibniz rule $d_r(a \cdot b) = d_r(a) \cdot b + (-1)^p a \cdot d_r(b)$ for $a \in E_r^{p,q}(W)$ and $b \in E_r^{s,t}(X)$. (As always, we are ignoring behavior 'near the fringe').*

Remark 5.2. Massey [M] has given a general algebraic criterion for checking when a product on the D- and E-terms of an exact couple gives rise to a pairing of spectral sequences. We could have arranged the above argument in terms of those criteria, but personally I find that more distracting than helpful. Massey's criteria are direct translations of what it means for each d_r to be a derivation, and in practice I find it easier just to remember the derivation condition.

Recall that $E_\infty^{p,q} = \bigcap_r Z_r^{p,q} / \bigcup B_r^{p,q}$. From Proposition 5.1 it follows that E_∞ has an induced product. We of course want to know that this product has something to do with what the spectral sequence is converging to. The pairing of towers $W_* \wedge X_* \to Y_*$ induces a pairing $\pi_*(W_-) \otimes \pi_*(X_-) \to \pi_*(Y_-)$. This respects filtrations, and so descends to a pairing of the associated graded groups $\mathrm{Gr}_* \pi_p(W_-) \otimes \mathrm{Gr}_* \pi_s(X_-) \to \mathrm{Gr}_* \pi_{p+s}(Y_-)$.

Proposition 5.3. *The following diagram is commutative (where Γ is the map from Exercise 3.5):*

$$\begin{array}{ccc} \mathrm{Gr}_q \pi_p W_- \otimes \mathrm{Gr}_t \pi_s X_- & \longrightarrow & \mathrm{Gr}_{q+t} \pi_{p+s} Y_- \\ \downarrow{\scriptstyle \Gamma\otimes\Gamma} & & \downarrow{\scriptstyle \Gamma} \\ E_\infty^{p,q}(W) \otimes E_\infty^{s,t}(X) & \longrightarrow & E_\infty^{p+s,q+t}(Y) \end{array}$$

Proof. This is a simple matter of chasing through the definitions. □

5.4. **Augmented towers of spaces.** Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of pointed spaces, where the composite is null (not just null-homotopic). Then there is an induced map $A \to \mathrm{hofib}(B \to C)$ which sends a point a to the pair consisting of $f(a)$ and the constant path from the basepoint to $gf(a)$. The sequence will be called a **rigid homotopy fiber sequence** if the composite is null and this induced map $A \to \mathrm{hofib}(B \to C)$ is a weak equivalence. In this case we have a map $B/A \to C$ and we consider the composite

$$\pi_k(B,A) \xrightarrow{pr} \pi_k(B/A) \longrightarrow \pi_k C.$$

It can be checked that the composite is an isomorphism for $k > 0$. These isomorphisms allow us to (canonically) rewrite the long exact sequence for $A \to B$ as

$$\cdots \to \pi_k A \to \pi_k B \to \pi_k C \to \pi_{k-1} A \to \cdots$$

terminating in $\pi_0 A \to \pi_0 B \to \pi_0 C$ (note that the sequence extends one term further to the right than the sequence from Section 2).

Now suppose given a diagram of pointed spaces

$$\begin{array}{ccccccccc} & & B_2 & & B_1 & & B_0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & \cdots \end{array}$$

where each $W_{n+1} \to W_n \to B_n$ is a rigid homotopy fiber sequence. We will refer to this as an **augmented tower**. The long exact sequences from each level patch together, and the homotopy spectral sequence takes on the form $E_1^{p,q}(W,B) = \pi_p B_q$, where $p \geq 0$ and $q \in \mathbb{Z}$. (Note that the spectral sequence is now defined when $p = 0$, and that as always we are ignoring 'fringe' behavior when it is unpleasant).

Assume given three such towers (W,B), (X,C), and (Y,D), together with a pairing of towers $W \wedge X \to Y$. Suppose also that we have maps $B_m \wedge C_n \to D_{m+n}$ such that the obvious diagrams

$$\begin{array}{ccc} W_m \wedge X_n & \longrightarrow & Y_{m+n} \\ \downarrow & & \downarrow \\ B_m \wedge C_n & \longrightarrow & D_{m+n} \end{array}$$

all commute. We will call this data **a pairing $(W,B) \wedge (X,C) \to (Y,D)$**. Note that the pairings $B \wedge C \to D$ give induced pairings of homotopy groups $\pi_r B \otimes \pi_s C \to \pi_{r+s} D$.

Proposition 5.5. *There is a pairing of spectral sequences $E_*(W,B) \otimes E_*(X,C) \to E_*(Y,D)$ which on E_1-terms is the obvious multiplication $\pi_p B_q \otimes \pi_s C_r \to \pi_{p+s} D_{q+r}$.*

This result is also proven in [FS, Appendix A].

Proof. The pairing is the one produced by Proposition 5.1. Checking that the multiplication on E_1-terms coincides with the above description consists of chasing through how things are defined, together with Exercise 4.2. □

5.6. **The simplicial setting.** Up until now we have always worked with topological spaces, and have benefited from the fact that there are so many useful isomorphisms around: we have used $D^p/S^{p-1} \cong S^p$ repeatedly, for instance. For pairings between towers of simplicial sets the treatment becomes more complicated, because such isomorphisms are no longer available. With enough trouble one could carry out all our arguments purely in the simplicial setting, but there is also an easier way out using geometric realization.

By assumptions (T1)–(T4) in Appendix C, it follows that geometric realization preserves products. So there are natural isomorphisms $|K| \wedge |L| \to |K \wedge L|$ for pointed simplicial sets K and L. Suppose that W, X, and Y are towers of pointed simplicial sets, and that we have a pairing $W \wedge X \to Y$. Applying geometric realization, one obtains maps $|W_m| \wedge |X_n| \to |Y_{m+n}|$ commuting with the maps in the towers $|W|$, $|X|$, and $|Y|$. We can now apply all the results which have been developed already.

6. Towers of spectra

This section extends the previous results to the case of spectra. In order to accomplish this without getting lost in category-specific constructions, we will assume certain generic properties about our category of spectra; these are outlined in Appendix C. The reader will want to review the notion of *rigid homotopy fiber sequence* given there.

Here is some terminology. Suppose given a diagram of spectra

$$\begin{array}{ccccccccc} & & B_2 & & B_1 & & B_0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & \cdots \end{array}$$

The spectra W_* and the maps between them form a **tower**. The whole diagram, consisting of the W's and B's, is an **augmented tower**, which will be denoted (W, B) or just $W_\perp$ (the subscript is to remind us there is an augmentation). If each sequence $W_{n+1} \to W_n \to B_n$ is a rigid homotopy fiber sequence (defined in section C.4), we will say that $W_\perp$ is a **rigid tower**, or a **tower of rigid homotopy fiber sequences**. A rigid tower gives rise to an exact couple and a homotopy spectral sequence with $E_1^{p,q}(W, B) = \pi_p B_q$; the boundary of the long exact homotopy sequence is the one from C.9(a).

Suppose given rigid towers (W, B), (X, C), (Y, D) together with a pairing $(W, B) \wedge (X, C) \to (Y, D)$ (this means the same thing as in section 5.4). One gets a pairing on E_1-terms of the spectral sequence by using the composites

$$\begin{array}{ccc} \mathrm{Ho}(\mathbb{S}^k, B_m) \otimes \mathrm{Ho}(\mathbb{S}^l, C_n) \longrightarrow \mathrm{Ho}(\mathbb{S}^{k+l}, B_m \triangle C_n) & & \\ \downarrow & & \\ \mathrm{Ho}(\mathbb{S}^{k+l}, B_m \wedge C_n) & \longrightarrow & \mathrm{Ho}(\mathbb{S}^{k+l}, D_{m+n}). \end{array} \tag{6.1}$$

Theorem 6.1. *Given a pairing between towers of rigid homotopy fiber sequences, the above pairing on E_1-terms descends to a pairing of spectral sequences.*

It's useful to have a result which applies to unaugmented towers. Recall that given a map $f\colon A \to B$ between cofibrant spectra there is a cofibrant spectrum Cf called the *canonical homotopy cofiber* of f, together with a long exact sequence of homotopy groups (cf. C.9(k)). Both Cf and the long exact sequence are functorial in the map f. If W_* is a tower of cofibrant spectra, then there is a resulting spectral sequence with $E_1^{p,q}(W) = \mathrm{Ho}\,(\mathbb{S}^p, C(W_{q+1} \to W_q))$.

Suppose given towers of cofibrant spectra W_*, X_*, and Y_*, together with a pairing $W \wedge X \to Y$. Let B_n denote the canonical homotopy cofiber of $W_{n+1} \to W_n$, and define C_n and D_n similarly. There is an induced pairing $B_m \wedge C_n \to D_{m+n}$, which we explain as follows. Heuristically, a 'point' in B_m is specified either by the data $[s \in I, w \in W_{m+1}]$ or the data $[w \in W_m]$, with the relations that $[0, w \in W_{m+1}] = *$ and $[1 \in I, w \in W_{m+1}] = [pw \in W_m]$ (where p is the map $W_{m+1} \to W_m$). Given $[s \in I, w \in W_{m+1}] \in B_m$ and $[t \in I, x \in X_{n+1}] \in C_n$ we define the product to be the point $[s+t-1 \in I, p(wx) \in Y_{m+n+1}] \in D_{m+n}$ if $s+t \geq 1$ (where $wx \in Y_{m+n+2}$ and $p(wx)$ is the image of this point in Y_{m+n+1}), and to be the basepoint if $s+t \leq 1$. The reader will be left to check that this respects the identifications, and to see that this heuristic description can be translated into a purely category-theoretic construction of $B_m \wedge C_n \to D_{m+n}$. The following will be deduced as a corollary of Theorem 6.1.

Theorem 6.2. *Given a pairing $W \wedge X \to Y$ between towers of cofibrant spectra, there is a pairing of spectral sequences $E_*(W) \otimes E_*(X) \to E_*(Y)$ which on E_1-terms is the pairing induced by $B_m \wedge C_n \to D_{m+n}$.*

6.3. **Homotopy-pairings.** Proving that spectral sequences are multiplicative is just a matter of checking the derivation formulas. In the case where our spectra are fibrant and we are dealing with homotopy groups π_k where $k \geq 1$, the proof is exactly the same as the one for towers of spaces. Ultimately, things work for spectra because we can reduce to this case by suspending enough times. In order to make this work, we need to navigate through some annoying issues surrounding cofibrancy and fibrancy. We now develop the tools for doing this. For this section, the reader should familiarize himself with Appendix B.

Suppose that we have three rigid towers (W, B), (X, C), and (Y, D), but we only have a **homotopy-pairing**, meaning that there are maps $W_m \vartriangle X_n \to Y_{m+n}$ and $B_m \vartriangle C_n \to D_{m+n}$ in $\mathrm{Ho}(\mathcal{S}pectra)$ making the usual squares commute (in $\mathrm{Ho}(\mathcal{S}pectra)$). We will say that this pairing is **realizable** if there are rigid towers (W', B'), (X', C'), and (Y', D') such that

(i) Each pair consists of cofibrant-fibrant spectra,
(ii) There are isomorphisms in $\mathrm{Ho}(\mathcal{R}gdTow)$ of the form $(W, B) \to (W', B')$, $(X, C) \to (X', C')$, and $(Y, D) \to (Y', D')$, and
(iii) There is a pairing of towers $(W', B') \wedge (X', C') \to (Y', D')$ such that the following diagrams in $\mathrm{Ho}(\mathcal{S}pectra)$ are commutative:

$$\begin{array}{ccccc} W_m \vartriangle X_n & \to & W'_m \vartriangle X'_n & \to & W'_m \wedge X'_n \\ \downarrow & & & & \downarrow \\ Y_{m+n} & & \longrightarrow & & Y'_{m+n} \end{array} \qquad \begin{array}{ccccc} B_m \vartriangle C_n & \to & B'_m \vartriangle C'_n & \to & B'_m \wedge C'_n \\ \downarrow & & & & \downarrow \\ D_{m+n} & & \longrightarrow & & D'_{m+n}. \end{array}$$

Given a tower (W, B) and integers $j < k$, we let $\tau_{j \leq k}(W, B)$ denote the finite tower where we have removed W_i and B_i for $i > k$ and $i < j$. We will say that the pairing is **locally realizable** if for any four integers $j < k$ and $l < m$ the homotopy-pairing between finite towers $\tau_{j \leq k}(W, B) \vartriangle \tau_{l \leq m}(X, C) \to \tau_{j+l \leq k+m}(Y, D)$ is realizable. When checking that a spectral sequence is multiplicative, one must check all the derivation formulas—but these only depend on finite sections of the towers. Using this observation, we will eventually prove the following result.

Proposition 6.4. *If the homotopy-pairing $W_\perp \vartriangle X_\perp \to Y_\perp$ is locally realizable then there is an induced pairing of spectral sequences $E_*(W, B) \otimes E_*(X, C) \to E_*(Y, D)$ which on E_1-terms is the map*

$$Ho\,(\mathbb{S}^k, B_m) \otimes Ho\,(\mathbb{S}^l, C_n) \to Ho\,(\mathbb{S}^k \vartriangle \mathbb{S}^l, B_m \vartriangle C_n) \to Ho\,(\mathbb{S}^{k+l}, D_{m+n}).$$

It is not true that every homotopy-pairing is locally realizable—see section 7 for a counterexample. However, every 'honest' pairing of towers is also a homotopy-pairing, and we can show that all of these are locally realizable. This is not a tautology because of the cofibrant-fibrant condition in our notion of 'realizable'.

Proposition 6.5. *If $(W, B) \wedge (X, C) \to (Y, D)$ is a pairing of rigid towers (without any cofibrancy/fibrancy conditions on the spectra) then the resulting homotopy-pairing is locally realizable.*

Proof. Since we are only concerned with *local* realizability, we can assume all the towers are finite. Using Lemma B.2 there are towers QW_*, FW_* and maps

$$FW_* \overset{\sim}{\hookleftarrow} QW_* \overset{\sim}{\twoheadrightarrow} W_*$$

such that

(1) QW_* is a tower of cofibrations between cofibrant spectra, and
(2) FW_* is a tower of cofibrations between cofibrant-fibrant spectra.

Let QB_n denote the cofiber of $QW_{n+1} \to QW_n$, and define FB_n similarly. We apply the same lemma to X_* and Y_* to get QX_*, QY_*, FX_*, FY_*, QC_*, etc. Note that there are induced weak equivalences of rigid towers

$$(FW, FB) \overset{\sim}{\longleftarrow} (QW, QB) \overset{\sim}{\longrightarrow} (W, B) \text{ and } (FX, FC) \overset{\sim}{\longleftarrow} (QX, QC) \overset{\sim}{\longrightarrow} (X, C).$$

Now by Lemma B.3 the tower $QW \wedge QX$ is cofibrant. So we can get a lifting in the diagram

$$\begin{array}{ccccc} & & & & QY_* \\ & & & \nearrow & \downarrow\!\!\!\!\downarrow \sim \\ QW \wedge QX & \longrightarrow & W \wedge X & \longrightarrow & Y. \end{array}$$

Also by Lemma B.3, the map $QW \wedge QX \to FW \wedge FX$ is a trivial cofibration (since it is the composite $QW \wedge QX \to QW \wedge FX \to FW \wedge FX$ and QW, QX, FW, and FX are all cofibrant towers). So we get a lifting in the diagram

$$\begin{array}{ccccc} QW \wedge QX & \longrightarrow & QY & \longrightarrow & FY \\ \downarrow & & & \nearrow & \\ FW \wedge FX. & & & & \end{array}$$

The pairing passes to cofibers to give $(FW, FB) \wedge (FX, FC) \to (FY, FD)$, and a routine diagram chase shows that this is compatible with the original pairing under the various weak equivalences. So we have produced the desired realization. □

6.6. **Proofs of the main results.** We start with several lemmas. If (W, B) is a tower of spectra, let $(\mathbb{S}^1 \wedge W, \mathbb{S}^1 \wedge B)$ be the tower whose nth level is $\mathbb{S}^1 \wedge W_n$, with the obvious structure maps. One defines $W \wedge \mathbb{S}^1$ similarly. If the objects W_* and B_* are cofibrant then these are still towers of rigid homotopy cofiber sequences.

Lemma 6.7. *When the spectra in W_* and B_* are cofibrant there is a canonical 'right suspension' isomorphism of spectral sequences $E_*^{p,q}(W) \to E_*^{p+1,q}(W \wedge \mathbb{S}^1)$ which on E_1-terms is the map $x \mapsto x\sigma$ defined in section C.3(b).*
Likewise, there is a 'left suspension' isomorphism $E_*^{p,q}(W) \to E_*^{p+1,q}(\mathbb{S}^1 \wedge W)$ which on E_1-terms is $x \mapsto (-1)^p \sigma x$.

Proof. One has to check that the suspension isomorphisms commute with the differentials in the spectral sequences, but this follows from C.9(f). Note that the signs are in the left-suspension isomorphism because of the formula $d_r(\sigma x) = -d_r(x)$, for $x \in E_r^{p,q}$. □

Lemma 6.8. *Suppose that the spectra W_* and B_* are fibrant, and let $w \in E_1^{p,q}(W,B)$ where $p > 0$. If w survives to E_r then there is a commutative diagram*

$$\begin{array}{ccccccccc} W_{q+r} & \longrightarrow & W_{q+r-1} & \longrightarrow & \cdots & \longrightarrow & W_{q+1} & \longrightarrow & W_q \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ \Sigma^\infty(S^{p-1}) & = & \Sigma^\infty(S^{p-1}) & = & \cdots & = & \Sigma^\infty(S^{p-1}) & \longrightarrow & \Sigma^\infty D^p \end{array}$$

such that the induced map $\Sigma^\infty(D^p/S^{p-1}) \to B_q$ represents w. Also, given any such diagram the composite $\Sigma^\infty S^{p-1} \to W_{q+r} \to B_{q+r}$ represents $d_r(w)$.

Proof. Given a map $X \to Y$ between fibrant spectra one defines $\pi_k(Y,X)$ to be equivalence classes of diagrams, analogously to what was done in section 2. It is a formal exercise to check that one gets an induced long exact sequence—the proof is exactly the same as the unstable case. If $F \to E \to B$ is a rigid homotopy fiber sequence one compares the long exact sequences:

$$\begin{array}{ccccccccc} \pi_k F & \longrightarrow & \pi_k E & \longrightarrow & \pi_k(E,F) & \longrightarrow & \pi_{k-1}F & \longrightarrow & \pi_{k-1}E \\ \| & & \| & & \downarrow & & \| & & \| \\ \pi_k F & \longrightarrow & \pi_k E & \longrightarrow & \pi_k B & \longrightarrow & \pi_{k-1}F & \longrightarrow & \pi_{k-1}E. \end{array}$$

The second square obviously commutes, and the third square commutes by the first part of Remark C.10. So the map $\pi_k(E,F) \to \pi_k B$ is an isomorphism, and this proves the $r = 1$ case. The proof for general r is the same as for the unstable case, using the homotopy extension property. □

Proof of Theorem 6.1. We assume that d_1 through d_{r-1} have been checked to be derivations, and we verify the identity $d_r(wx) = d_r(w)x + (-1)^p w(dx)$. The first case to consider is where all the spectra are fibrant and we have $w \in E_r^{p,q}(W,B)$ and $x \in E_r^{s,t}(X,C)$ where $p, s \geq 0$. Here we can use exactly the same method as for $\mathcal{T}op_*$: the above lemma lets us reduce to a universal case. We will not write out the details again because they are the same as in Propositions 4.1 and 5.5. Note, however, that the argument uses (S2) and (S4) from section C.1.

Now assume we are in the general case: we have $w \in E_r^{p,q}(W,B)$ and $x \in E_r^{s,t}(X,C)$ and must verify that $d_r(wx) = d_r(w)x + (-1)^p w(dx)$. This equation only depends on finite sections of the towers. Using the method of Proposition 6.5, we can replace finite sections of the towers (W,B), (X,C), and (Y,D) by towers of cofibrant objects; therefore we have reduced to this case.

Choose M and N large enough so that $\sigma^M x$ and $y\sigma^N$ have positive dimension. There is the obvious pairing of towers

$$(\mathbb{S}^M \wedge W, \mathbb{S}^M \wedge B) \wedge (X \wedge \mathbb{S}^N, C \wedge \mathbb{S}^N) \to (\mathbb{S}^M \wedge Y \wedge \mathbb{S}^N, \mathbb{S}^M \wedge D \wedge \mathbb{S}^N)$$

(note that we needed the spectra to be cofibrant to know these are towers of rigid homotopy cofiber sequences). The derivation condition for this new pairing, if we knew it, would say that

$$d_r(\sigma^M x \cdot y\sigma^N) = d_r(\sigma^M x) \cdot y\sigma^N + (-1)^{p+M}\sigma^M x \cdot d_r(y\sigma^N).$$

By Lemma 6.7 (applied repeatedly) we can re-write the two sides as

$$\begin{aligned}(-1)^M\sigma^M d_r(xy)\sigma^N &= (-1)^M\sigma^M(d_rx)y\sigma^N + (-1)^{p+M}\sigma^M x(d_ry)\sigma^N\\ &= (-1)^M\sigma^M\Big((d_rx)y+(-1)^p x(d_ry)\Big)\sigma^N.\end{aligned}$$

By cancelling the signs and the σ's (which are isomorphisms), we obtain the desired relation.

So at this point we have reduced to the case where $p, s \geq 1$. Once again, using the method of Proposition 6.5 we can 'locally' replace the towers $W_\perp$, $X_\perp$, $Y_\perp$ by towers of fibrant spectra. But now we are back in the case handled in the first paragraph, so we are done. □

Proof of Theorem 6.2. Recall that W, X, and Y are towers of cofibrant objects, and B, C, and D denote the canonical homotopy cofibers for the respective towers. Each of the towers W, X, and Y can be replaced by the corresponding telescopic tower TW, TX, or TY from section B.4. Proposition B.5 shows these are towers consisting of cofibrations between cofibrant objects, and come with weak equivalences to W, X, and Y. We augment them with the cofibers TB, TC, and TD in each level. One readily checks that the spectra TB and B are in fact canonically isomorphic (and the same for C and D). So we can identify the spectral sequences $E_*(TW, TB) \cong E_*(W, B)$, etc.

By the discussion in section B.4 there is a pairing $TW \wedge TX \to TY$ compatible with $X \wedge Y \to Z$. On cofibers this induces maps $TB \wedge TC \to TD$, which exactly coincide with the maps $B \wedge C \to D$ we started with. Finally, Theorem 6.1 gives us a pairing $E_*(TW, TB) \otimes E_*(TX, TC) \to E_*(TY, TD)$, and using the isomorphisms from above this gets translated to $E_*(W) \otimes E_*(X) \to E_*(Y)$. □

Proof of Proposition 6.4. The proof of Theorem 6.1 works verbatim. □

6.9. **Towers of function spectra.** We close this section with one last result which is sometimes useful. Recall from section B.7 that if $W_\perp$ is a rigid tower and A is a cofibrant spectrum, then there is a 'derived tower' of function spectra $\mathcal{F}_{\text{der}}(A, W_\perp)$ and a resulting homotopy spectral sequence which we'll denote $E_*(A, W_\perp)$.

Suppose that (W, B), (X, C), and (Y, D) are rigid towers with a homotopy pairing $W_\perp \,\underline{\wedge}\, X_\perp \to Y_\perp$. It is immediate that if M and N are cofibrant spectra then there is an induced homotopy-pairing $\mathcal{F}_{\text{der}}(M, W_\perp) \,\underline{\wedge}\, \mathcal{F}_{\text{der}}(N, X_\perp) \to \mathcal{F}_{\text{der}}(M \wedge N, Y_\perp)$.

Proposition 6.10. *If the original homotopy-pairing $W_\perp \,\underline{\wedge}\, X_\perp \to Y_\perp$ is locally realizable, so is the induced pairing on towers of function spectra. So for cofibrant spectra M and N there is a naturally defined pairing of spectral sequences $E_*(M, W_\perp) \otimes E_*(N, X_\perp) \to E_*(M \wedge N, Y_\perp)$ which on E_1-terms is induced by*

$$\underline{\mathcal{F}}(M, B) \,\underline{\wedge}\, \underline{\mathcal{F}}(N, C) \to \underline{\mathcal{F}}(M \,\underline{\wedge}\, N, B \,\underline{\wedge}\, C) \to \underline{\mathcal{F}}(M \,\underline{\wedge}\, N, D).$$

Proof. Only the first statement requires justification, but it is a routine exercise in abstract homotopy theory—one just has to chase through certain diagrams. □

7. A counterexample

We give an example showing that a homotopy-pairing between towers (defined in 6.3) does not necessarily induce a pairing of spectral sequences. A related problem arises when the pairings commute on-the-nose but where the homotopy fiber sequences are not rigid.

Let (W, B) be the following tower of rigid homotopy cofiber sequences:

$$\begin{array}{ccccc} & & S^{k-1} & & S^{k} \\ & & \uparrow & & \uparrow \\ * & \longrightarrow & S^{k-1} & \longrightarrow & D^{k}. \end{array}$$

Here the maps are the obvious ones and the indexing is so that $W_0 = D^k$. Actually we want to regard W as a tower of spectra, so we mean to apply Σ^∞ to everything. Let (X, C) be the similar tower

$$\begin{array}{ccccc} & & S^{l-1} & & S^{l} \\ & & \uparrow & & \uparrow \\ * & \longrightarrow & S^{l-1} & \longrightarrow & D^{l} \end{array}$$

and let (Y, D) be the tower

$$\begin{array}{ccccccc} & & S^{k+l-2} & & S^{k+l-1} & & * \\ & & \uparrow & & \uparrow & & \uparrow \\ * & \longrightarrow & S^{k+l-2} & \longrightarrow & D^{k+l-1} & \longrightarrow & D^{k+l-1}. \end{array}$$

If desired, we could extend all of these to infinite towers in the obvious way. We will give a homotopy-pairing $W_\perp \wedge X_\perp \to Y_\perp$ which does not give rise to a pairing of spectral sequences, and is therefore not locally realizable.

When either m or n is zero let $W_m \wedge X_n \to Y_{m+n}$ be the trivial map (collapsing everything to the basepoint), and let it be the canonical identification $S^{k-1} \wedge S^{l-1} \to S^{k+l-2}$ when $m = n = 1$. Similarly we let $B_m \wedge C_n \to D_{m+n}$ be the trivial map for $m = n = 0$ and the canonical identification when either $m = 1$ or $n = 1$. This defines a homotopy-pairing of towers.

Let w be the obvious element in $E_1^{k,0}(W) = \pi_k(\Sigma^\infty S^k)$, and similarly for $x \in E_1^{l,0}(X)$. Note that dw and dx are the obvious generators as well, by C.9(b). Then $wx = 0$, but $(dw)x + (-1)^k w(dx)$ is twice the generator in $E_1^{k+l-1,1}(Y)$ when l is even. So we do not have $d(wx) = (dw)x + (-1)^l w(dx)$.

If one modifies the above towers by changing all the D^n's to $*$'s, then one gets a similar example where the pairing is on-the-nose (not just a homotopy-pairing) but the layers of the towers are not *rigid* homotopy cofiber sequences.

Appendix A. lim-towers

Pairings do not work especially well in the case of lim-towers, because it is hard to relate the pairing on the tower to whatever the spectral sequence is converging to. However, in the interest of providing a useful reference we will set down the usual indexing conventions and properties of lim-towers. The material in this section most naturally follows that of Section 3.

So assume W_* is a tower of spaces (or spectra) with the property that $\operatorname{colim}_n \pi_* W_n = 0$. Let $\pi_p(W)$ denote $\lim_n \pi_p W_n$—we'll call this the ***p*th homotopy group of the tower**. The spectral sequence will be used to give information about $\pi_p W$, therefore we want to choose our indexing conventions so that $E_\infty^{p,q}$

contributes to this group. For lim-towers, we set

$$D_1^{p,q} = \pi_{p+1}W_{q-1} \quad \text{and} \quad E_1^{p,q} = \pi_{p+1}(W_{q-1}, W_q).$$

The maps i, j, and k in the exact couple are the same as always, and the differentials still have the form $d_r \colon E_r^{p,q} \to E_r^{p-1,q+r}$.

The group $\pi_p W$ comes with a natural filtration, defined by setting

$$F^q\pi_p(W) = \ker\Big(\pi_p W \to \pi_p W_{q-1}\Big).$$

That is, F^q contains all the elements which die at level $q-1$. Set $\mathrm{Gr}^q = F^q/F^{q+1}$.

Suppose given an element $\alpha \in F^q\pi_p W$. Then α gives us a homotopy class in $[S^p, W_q]_*$ which becomes zero in W_{q-1}. Choose a specific representative $a \colon S^p \to W_q$, and choose a specific null homotopy $D^{p+1} \to W_{q-1}$. This data defines an element in $\pi_{p+1}(W_{q-1}, W_q) = E_1^{p,q}$, which by construction is an infinite cycle; so it represents a class in $E_\infty^{p,q}$.

Exercise A.1.

(a) Check that the class in $E_\infty^{p,q}$ does not depend on the choices made in the construction, so we have a well-defined map $F^q\pi_p W \to E_\infty^{p,q}$. (You will have to use the assumption that $\mathrm{colim}_n\, \pi_* W_n = 0$.)
(b) Observe that $F^{q+1}\pi_p W$ maps to zero, so induces $\mathrm{Gr}^q\, \pi_p W \to E_\infty^{p,q}$.
(c) Verify that the map in (b) is an inclusion.

The reason our map $\mathrm{Gr}^q\, \pi_p W \to E_\infty^{p,q}$ is not a surjection is easy to understand, and worth remembering. An element of $E_\infty^{p,q}$ gives us a homotopy class in $[S^p, W_q]$ which can be lifted arbitrarily far up the tower: it can be lifted to W_{q+10}, W_{q+100}, W_{q+1000}, etc. However, this is *not* the same as saying that it can be lifted to an element of $\lim_n \pi_p W_n$.

Here is a summary of some useful convergence properties:

Proposition A.2. *Assume that W_* is a lim-tower (of spaces or spectra).*

(i) If $RE_\infty = 0$ then the map $\mathrm{Gr}^q\, \pi_p W \to E_\infty^{p,q}$ is an isomorphism and the spectral sequence converges strongly to $\pi_* W$.
(ii) If $\lim^1_n \pi_* W_n = 0$, then the natural map $\pi_p(\mathrm{holim}_n\, W_n) \to \pi_p W$ is an isomorphism.
(iii) If $W_n = *$ for $n \ll 0$ and $RE_\infty = 0$, then $\lim^1_n \pi_*(W_n) = 0$ as well.

Proof. Part (a) follows as in the proof of [Bd, Thm 8.13]. Part (b) follows from the Milnor exact sequence

$$0 \to \lim{}^1_n\, \pi_p W_n \to \pi_p(\operatorname*{holim}_n W_n) \to \lim_n \pi_p(W_n) \to 0.$$

Part (c) follows from [Bd, Lemma 5.9(b)]. □

Appendix B. Manipulating towers

This section contains some basic observations that are helpful when manipulating towers. They are used in section 6, and in the applications from [D].

B.1. **Finite towers and smash products.** Let J_n denote the indexing category $n \to (n-1) \to \cdots \to 1 \to 0$. We will call an element of $\mathcal{S}pectra^{J_n}$ an n-tower. Note that any n-tower X_* may be regarded as an infinite tower by setting $X_k = *$ for $k > n$ and $X_k = X_0$ for $k < 0$.

There is a so-called *Reedy* model category structure on $\mathcal{S}pectra^{J_n}$ such that a map of n-towers $X_* \to Y_*$ is a

(1) weak equivalence iff each $X_i \to Y_i$ is a weak equivalence;
(2) fibration iff each $X_i \to Y_i$ is a fibration;
(3) cofibration iff each $X_i \amalg_{X_{i+1}} Y_{i+1} \to Y_i$ is a cofibration.

Note that the cofibrant objects are towers in which X_n is cofibrant and every $X_i \to X_{i-1}$ is a cofibration. The model structure gives us the following, in particular:

Lemma B.2. *If X_* is an n-tower, then there exist n-towers QX_* and FX_* together with maps $FX_* \overset{\sim}{\hookleftarrow} QX_* \overset{\sim}{\twoheadrightarrow} X_*$ such that every object of FX_* is cofibrant-fibrant, every object of QX_* is cofibrant, and every map in QX_* and FX_* is a cofibration.*

Proof. QX_* is a cofibrant-replacement for X_*, and FX_* is a fibrant-replacement for QX_*. □

Suppose that X_* is an m-tower and Y_* is an n-tower. We define an $(n+m)$-tower $X \wedge Y$ by setting

$$(X \wedge Y)_k = \operatorname*{colim}_{i+j \geq k} X_i \wedge Y_j.$$

The colimit is over the obvious indexing category. There is a pushout diagram

$$\begin{array}{ccc} \coprod_{i+j=k}\Big[(X_{i+1} \wedge Y_j) \amalg_{(X_{i+1}\wedge Y_{j+1})} (X_i \wedge Y_{j+1})\Big] & \longrightarrow & (X \wedge Y)_{k+1} \\ \downarrow & & \downarrow \\ \coprod_{i+j=k}(X_i \wedge Y_j) & \longrightarrow & (X \wedge Y)_k. \end{array}$$

From this one can deduce the following lemma (we will actually only need the case where X_* or Y_* is the trivial tower $*$, which is a little easier to prove):

Lemma B.3. *If $f\colon X_* \rightarrowtail X'_*$ and $g\colon Y_* \rightarrowtail Y'_*$ are cofibrations of m-towers and n-towers, respectively, then the map $f \Box g\colon (X \wedge Y') \amalg_{(X\wedge Y)} (X' \wedge Y) \to X' \wedge Y'$ is a cofibration of $(m+n)$-towers. If either of the maps f and g is also a weak equivalence, then so is $f \Box g$.*

B.4. **Telescopic replacements for infinite towers.** We describe a construction which replaces an infinite tower by a 'nicer' one, in a way that preserves pairings.

First suppose given a sequence of spectra $\cdots \to E_2 \to E_1 \to E_0$. Let $T_0 = E_0$, and let T_1 be the pushout of

$$T_0 \longleftarrow E_1 \wedge \Sigma^\infty S^0 \xrightarrow{i_1} E_1 \wedge \Sigma^\infty I_+.$$

The right map is induced by the inclusion $\{1\} \hookrightarrow I$. Note that if E_1 is cofibrant then $E_1 \wedge \Sigma^\infty S^0 \to E_1 \wedge \Sigma^\infty I_+$ is a trivial cofibration, and so $T_0 \to T_1$ is a trivial cofibration (which admits a retraction $T_1 \to T_0$). There is a composite map $E_2 \to E_1 \xrightarrow{i_0} E_1 \wedge \Sigma^\infty I_+ \to T_1$, and we let T_2 be the pushout of

$$T_1 \longleftarrow E_2 \wedge \Sigma^\infty S^0 \xrightarrow{i_1} E_2 \wedge \Sigma^\infty I_+.$$

This gives us a sequence of maps $T_0 \to T_1 \to T_2 \to \cdots$, and we define TE to be the colimit. This is the **telescope** of the sequence E_*. It comes with a map $E_0 \to TE$, and if all the E_* are cofibrant this is a trivial cofibration; also, there is a retraction $TE \to E_0$.

If (W, B) is a rigid tower, denote the telescope of the sequence $\cdots \to W_{n+1} \to W_n$ by TW_n. Note that there are canonical maps $TW_{n+1} \to TW_n$, and let TB_n be the cofiber. We have a map of augmented towers $(TW, TB) \to (W, B)$. The proof of the following result is routine:

Proposition B.5. *If the W_n's were all cofibrant then (TW, TB) is a tower of rigid homotopy cofiber sequences, the maps $TW_{n+1} \to TW_n$ are all cofibrations, and the map of towers $(TW, TB) \to (W, B)$ is a weak equivalence.*

Now suppose that $W_\perp \wedge X_\perp \to Y_\perp$ is a pairing of rigid towers. We claim that there are pairings $TW_\perp \wedge TX_\perp \to TY_\perp$ making the diagram

$$\begin{array}{ccc} TW_\perp \wedge TX_\perp & \longrightarrow & TY_\perp \\ \downarrow & & \downarrow \\ W_\perp \wedge X_\perp & \longrightarrow & Y_\perp \end{array}$$

commute. We'll justify this by defining the product heuristically. Loosely speaking, a 'point' in TW_m may be specified by giving a 'level' $k \geq m$, a 'point' $w \in W_k$, and a parameter $t \in I$. If $t = 1$ this data is identified with the data $[k-1, pw, 0]$ where pw denotes the image of w in W_{m-1}. Given data $[k, w \in W_k, t \in I]$ and $[l, x \in X_l, s \in I]$, the product is defined to be the point specified by $[k+l-1, p(w \cdot x) \in Y_{k+l-1}, t+s-1]$ if $t + s \geq 1$, and the point $[k+l, w \cdot x \in Y_{k+l}, t+s]$ if $t + s \leq 1$. The reader may check that this respects the identifications, makes the above diagram commute, and the definition of the pairing can be translated into a purely category-theoretic construction (the latter is not very pleasant, but it can de done). The product extends to the cofibers $TB \wedge TC \to TD$ in the expected way.

B.6. **The homotopy category of rigid towers.** Infinite towers of rigid homotopy fiber sequences (W, B) form a category which we'll denote as $\mathcal{R}gdTow$. Let $\mathcal{W}$ denote the subcategory consisting of maps $(W, B) \to (X, C)$ such that $W_n \to X_n$ and $B_n \to C_n$ are weak equivalences for all n. Finally, let $\mathrm{Ho}(\mathcal{R}gdTow)$ denote the localization $\mathcal{W}^{-1}(\mathcal{R}gdTow)$—we will ignore the question of whether this localization actually exists in our universe, since we will use it only as a useful way of organizing certain ideas.

Every rigid tower gives rise to a spectral sequence $E_*(W, B)$, and this construction is functorial. Moreover, a weak equivalence of rigid towers induces an isomorphism on spectral sequences. So we actually have a functor from $\mathrm{Ho}(\mathcal{R}gdTow)$ to the category of spectral sequences. This observation is helpful in Section 6.3.

By an **objectwise-fibrant replacement** of a rigid tower (W, B) we mean a rigid tower (W', B') in which all the objects are fibrant, together with a chosen weak equivalence $(W, B) \to (W', B')$. The objectwise-fibrant replacements of (W, B) form a category in the obvious way, and this category is contractible—the functor F from section C.3(c) gives a natural zig-zag from any (W', B') to the distinguished object (FW, FB).

B.7. **Towers of function spectra.** If $W_\perp = (W, B)$ is a rigid tower and X is a cofibrant spectrum, we let $\mathcal{F}_{\mathrm{der}}(X, W_\perp)$ denote any tower obtained by choosing an objectwise-fibrant replacement $(W, B) \to (W', B')$ and then forming the rigid tower $(\mathcal{F}(X, W'), \mathcal{F}(X, B'))$. The fact that the category of choices is contractible may be interpreted as saying the function tower $\mathcal{F}_{\mathrm{der}}(X, W_\perp)$ is 'homotopically unique'. It implies that the homotopy spectral sequence of $\mathcal{F}_{\mathrm{der}}(X, W_\perp)$ is unique up to unique isomorphism—given two objectwise-fibrant replacements $W_\perp \to W'_\perp$ and $W_\perp \to W''_\perp$, there is a uniquely defined isomorphism between $E_*(\mathcal{F}(X, W'_\perp))$ and $E_*(\mathcal{F}(X, W''_\perp))$ obtained by zig-zagging through our category of objectwise-fibrant replacements. Another way of saying the same thing is to observe that the *homotopy category* of objectwise-fibrant replacements is a contractible groupoid.

Appendix C. Spectra

Let $\mathcal{T}op$ denote a subcategory of topological spaces which is complete and co-complete, contains every finite CW-complex and every cellular map between them, and has the structure of a closed symmetric monoidal category. We denote the tensor by $\times$. We also assume that:

(T1) On the subcategory of finite CW-complexes the functor $\times$ coincides with the 'usual' Cartesian product.
(T2) If $A \hookrightarrow X$ is a cellular inclusion between finite CW-complexes, the quotient X/A in $\mathcal{T}op$ coincides with the usual quotient of topological spaces.
(T3) For the geometric realization functor $|-|\colon s\mathcal{S}et \to \mathcal{T}op$, the natural maps $|\Delta^m \times \Delta^n| \to |\Delta^m| \times |\Delta^n|$ are isomorphisms, for all $m, n \geq 0$.
(T4) $\mathcal{T}op$ has a model category structure in which the weak equivalences are the usual ones, the fibrations are the Serre fibrations, and such that the monoidal product $\times$ satisfies the analogue of Quillen's SM7.

For example, one can take $\mathcal{T}op$ to be the category of compactly-generated spaces (cf. [Ho] for a good reference) or the category of Δ-generated spaces introduced by Jeff Smith. We let $\wedge$ and $\mathcal{F}(-,-)$ denote the associated symmetric monoidal structure on the pointed category $\mathcal{T}op_*$.

C.1. **Basic notions.** Our preferred model for spectra is the category of symmetric spectra based on topological spaces. Rather than assume the reader has any detailed knowledge of this category, however, we just list the basic properties we will need. We assume given a certain pointed category $\mathcal{S}pectra$ together with the following additional information:

(S1) A cofibrantly-generated, proper model category structure on $\mathcal{S}pectra$ which is Quillen-equivalent to the model category of Bousfield-Friedlander spectra.
(S2) A Quillen pair $\Sigma^\infty\colon \mathcal{T}op_* \rightleftarrows \mathcal{S}pectra\colon \Omega^\infty$, such that if X is a finite CW-complex then $\underline{\Sigma}^\infty X$ corresponds to the 'usual' stabilization of X under the Quillen equivalence from (a).
(S3) A symmetric monoidal smash product $\wedge$ on $\mathcal{S}pectra$ satisfying the analog of Quillen's SM7, whose unit is $\Sigma^\infty S^0$. We assume chosen a specific derived functor $\underline{\wedge}$ on $\mathrm{Ho}(\mathcal{S}pectra)$: this is a bifunctor with a natural transformation $X \underline{\wedge} Y \to X \wedge Y$ of functors $\mathcal{S}pectra \times \mathcal{S}pectra \to \mathrm{Ho}(\mathcal{S}pectra)$ which is an isomorphism when X and Y are cofibrant.
(S4) A natural isomorphism $\eta_{X,Y}\colon \Sigma^\infty(X \wedge Y) \to \Sigma^\infty X \wedge \Sigma^\infty Y$.

(S5) A bifunctor $\mathcal{F}(-,-)$ which together with $\wedge$ makes $\mathcal{S}pectra$ into a closed symmetric monoidal category. (It then follows that $\mathcal{F}(-,-)$ also satisfies the relevant analog of SM7). We assume a specific derived functor $\underline{\mathcal{F}}(-,-)$ has been chosen on the homotopy category.

(S6) A cofibrant object $\mathbb{S}^{-1}$ together with a chosen isomorphism $c\colon \Sigma^\infty S^1 \wedge \mathbb{S}^{-1} \to \Sigma^\infty S^0$ in $\mathrm{Ho}(\mathcal{S}pectra)$. We define $\mathbb{S}^k = \Sigma^\infty S^k$ for $k \geq 0$ and $\mathbb{S}^k = (\mathbb{S}^{-1})^{\wedge -k}$ for $k < 0$. Note that we have specific maps

$$a_k\colon \mathbb{S}^k = \mathbb{S}^1 \wedge \cdots \wedge \mathbb{S}^1 = \Sigma^\infty S^1 \wedge \cdots \wedge \Sigma^\infty S^1 \to \Sigma^\infty(S^1 \wedge \cdots \wedge S^1) \to \Sigma^\infty S^k$$

for $k \geq 1$, where the last map is obtained by choosing any orientation-preserving map of spaces $(S^1)^{\wedge k} \to S^k$. Also, in addition to the map c we have its twist: this is the composite $ct\colon \mathbb{S}^{-1} \wedge \mathbb{S}^1 \to \mathbb{S}^1 \wedge \mathbb{S}^{-1} \to \mathbb{S}^0$. Based on these we can define specific 'associativity' maps $a_{k,l}\colon \mathbb{S}^k \wedge \mathbb{S}^l \to \mathbb{S}^{k+l}$: if $k, l \geq 0$ or if $k, l \leq 0$ we use the associativity isomorphism for $\wedge$; if $k > 0$ and $l < 0$ we use the associativity isomorphisms and the map c (repeatedly); if $k < 0$ and $l > 0$ we use associativity and ct. It follows that for any $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ we have a chosen identification $\mathbb{S}^{n_1} \wedge \mathbb{S}^{n_2} \wedge \cdots \wedge \mathbb{S}^{n_k} \simeq \mathbb{S}^{n_1+\cdots+n_k}$ in $\mathrm{Ho}(\mathcal{S}pectra)$.

Remark C.2. For a spectrum E, we will sometimes write $\pi_k E$ for $\mathrm{Ho}\,(\mathbb{S}^k, E)$—however, with this abbreviated notation it is easy to forget that we are not really dealing with homotopy classes of maps unless E is fibrant.

The above properties are satisfied by the category of symmetric spectra based on topological spaces from [HSS, Section 6]. From them one can derive all of the expected properties of $\mathcal{S}pectra$. Some of the properties we develop below are needed for [D] rather than the present paper.

C.3. **Basic properties.** Here are the first three we will need:

(a) Suppose $X \twoheadrightarrow Y$ is a fibration between fibrant objects, with fiber F. Suppose also that

$$\begin{array}{ccc} F & \longrightarrow & X \\ {\scriptstyle\simeq}\uparrow & & {\scriptstyle\simeq}\uparrow \\ \tilde{F} & \rightarrowtail & \tilde{X} \end{array}$$

is a commutative square where $\tilde{F} \rightarrowtail \tilde{X}$ is a cofibration between cofibrant objects, and the vertical maps are weak equivalences. Then the induced map $\tilde{X}/\tilde{F} \to Y$ is a weak equivalence.

(b) The two suspension maps

$$\sigma_l\colon \mathrm{Ho}\,(\mathbb{S}^k, A) \to \mathrm{Ho}\,(\mathbb{S}^1 \underline{\wedge} \mathbb{S}^k, \mathbb{S}^1 \underline{\wedge} A) \xleftarrow{a_{1,k}} \mathrm{Ho}\,(\mathbb{S}^{k+1}, \mathbb{S}^1 \underline{\wedge} A)$$

and

$$\sigma_r\colon \mathrm{Ho}\,(\mathbb{S}^k, A) \to \mathrm{Ho}\,(\mathbb{S}^k \underline{\wedge} \mathbb{S}^1, A \underline{\wedge} \mathbb{S}^1) \xleftarrow{a_{k,1}} \mathrm{Ho}\,(\mathbb{S}^{k+1}, A \underline{\wedge} \mathbb{S}^1),$$

are isomorphisms. We will use the notation $\sigma_l(x) = \sigma x$ and $\sigma_r(x) = x\sigma$.

(c) There is a fibrant-replacement functor $X \overset{\sim}{\rightarrowtail} FX$ such that $F(*) = *$.

Proof of (c). This can be deduced from the small-object argument, and is the only place we need the full power of the cofibrantly-generated assumption from (S1). If

$\{A_a \rightarrowtail B_a\}$ is a set of generating trivial cofibrations, we let F_1X be the pushout of

$$\begin{array}{ccc} \coprod A_a & \longrightarrow & X \\ \downarrow & & \vdots \\ \coprod B_a & \cdots\to & F_1X \end{array}$$

where the coproduct ranges over all maps $A_a \to X$ *which do not factor through the initial object*. We continue with the usual constructions from the small-object argument to define F_2X, F_3X, etc., but at each stage we leave out any maps which factored through the initial object. The object $FX = \operatorname{colim}_k F_kX$ is the desired fibrant-replacement. □

C.4. **Homotopy cofiber sequences.** Given a map $f\colon A \to B$ between cofibrant spectra, we define the **canonical homotopy cofiber** of f, denoted Cf, to be the pushout of

$$\begin{array}{ccccc} A & \xrightarrow{\cong} & A \wedge \Sigma^\infty S^0 & \xrightarrow{i} & A \wedge \Sigma^\infty I \\ {\scriptstyle f}\downarrow & & & & \vdots \\ B & & \cdots\cdots\cdots & \to & Cf \end{array}$$

where $i\colon S^0 \to I$ is the boundary inclusion. Since A is cofibrant it follows that $A \wedge \Sigma^\infty I$ is contractible, and $A \wedge \Sigma^\infty S^0 \to A \wedge \Sigma^\infty I$ is a cofibration. So $B \to Cf$ is a cofibration, and since B is cofibrant so is Cf. Note that there is a canonical isomorphism from $A \wedge \Sigma^\infty(I/\partial I)$ to the cofiber of $B \to Cf$, and so a canonical map $Cf \to A \wedge \Sigma^\infty S^1$. This gives us the sequence $A \to B \to Cf \to A \underline{\wedge} \Sigma^\infty S^1$ in $\mathrm{Ho}(\mathcal{S}pectra)$, and we'll call such a sequence a *canonical triangle*.

We define a triangulation on $\mathrm{Ho}(\mathcal{S}pectra)$ by taking $X \mapsto X \underline{\wedge} \Sigma^\infty S^1$ to be the shift automorphism, and taking the distinguished triangles to be those which are isomorphic to a canonical triangle for some map $f\colon A \to B$ between cofibrant objects. Finally, a sequence $A \to B \to C$ in $\mathcal{S}pectra$ is called a **homotopy cofiber sequence** if it can be completed in $\mathrm{Ho}(\mathcal{S}pectra)$ to a distinguished triangle $A \to B \to C \to A \underline{\wedge} \mathbb{S}^1$.

A sequence $A \to B \to C$ in $\mathcal{S}pectra$ is a **rigid homotopy cofiber sequence** if the composite $A \to C$ is null (not just null-homotopic), and there exists a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle\simeq}\uparrow & & {\scriptstyle\simeq}\uparrow \\ \tilde{A} & \rightarrowtail & \tilde{B} \end{array}$$

such that $\tilde{f}\colon \tilde{A} \rightarrowtail \tilde{B}$ is a cofibration between cofibrant objects, the vertical maps are weak equivalences, and the induced map $B/A \to C$ is a weak equivalence as well. The sequence $\mathbb{S}^k \to * \to \mathbb{S}^{k+1}$ is an example of a homotopy cofiber sequence which is not rigid.

Remark C.5. The difference between 'homotopy cofiber sequence' and 'rigid homotopy cofiber sequence' is like the difference between diagrams in a homotopy category and the homotopy category of diagrams. To say that $M \to N \to Q$ is a homotopy cofiber sequence is to say that, working entirely in $\mathrm{Ho}(\mathcal{S}pectra)$, there is an isomorphism between $M \to N \to Q$ and a sequence of the form $A \rightarrowtail B \to B/A$

with A and B cofibrant. It is a *rigid* homotopy cofiber sequence if there is a zig-zag of weak equivalences from the diagram $M \to N \to Q$ to a diagram $A \rightarrowtail B \to B/A$, where the intermediate sequences have null composites; the zig-zag is a diagram in $\mathcal{S}pectra$ as opposed to a diagram in the homotopy category.

Hovey [Ho] defines a homotopy cofiber sequence to be a distinguished triangle in $\mathrm{Ho}(\mathcal{S}pectra)$—so in our language it is a homotopy cofiber sequence $M \to N \to Q$ with a chosen map $Q \to M \underline{\wedge} \mathbb{S}^1$. This is another way of dealing with the same issue, but when talking about towers and spectral sequences it becomes inconvenient; having to specify the connecting homomorphism for each layer of the tower is too much data to have to worry about.

Exercise C.6. Here is a series of claims, whose justifications we leave to the reader:

(a) Since $\tilde{f}\colon \tilde{A} \rightarrowtail \tilde{B}$ is a map between cofibrant objects, we have a natural map $C\tilde{f} \to \tilde{A} \wedge \mathbb{S}^1$. Putting this together with the weak equivalences $C\tilde{f} \to C$ and $\tilde{A} \to A$ gives a map $C \to A \wedge \mathbb{S}^1$ in $\mathrm{Ho}(\mathcal{S}pectra)$. The definition of this map does not depend on the choice of $\tilde{A}$, $\tilde{B}$, and $\tilde{f}$. We will refer to this as the 'map induced by the rigid homotopy cofiber sequence $A \to B \to C$'.

(b) For the rigid homotopy cofiber sequence $\Sigma^\infty S^{k-1} \hookrightarrow \Sigma^\infty D^k \to \Sigma^\infty S^k$, the induced map $\Sigma^\infty S^k \to \Sigma^\infty S^{k-1} \underline{\wedge} \mathbb{S}^1$ is $(-1)^k$ times the canonical identification.

(c) Suppose given the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

in which the composites $A \to C$ and $X \to Z$ are both null. The top row is a rigid homotopy cofiber sequence if and only if the bottom row is one.

(d) There are the obvious dual notions of **homotopy fiber sequence**, and **rigid homotopy fiber sequence**. The classes of rigid homotopy cofiber sequences and rigid homotopy fiber sequences are the same, by C.3(a) and its dual.

(e) Suppose that $X \to Y \to Z$ is a rigid homotopy fiber/cofiber sequence. If X, Y, and Z are all fibrant and A is a cofibrant spectrum, then the induced sequence $\mathcal{F}(A, X) \to \mathcal{F}(A, Y) \to \mathcal{F}(A, Z)$ is also a rigid homotopy cofiber sequence. Dually, if X, Y, and Z are all cofibrant and $\mathcal{E}$ is a fibrant spectrum, then $\mathcal{F}(Z, \mathcal{E}) \to \mathcal{F}(Y, \mathcal{E}) \to \mathcal{F}(X, \mathcal{E})$ is a rigid homotopy fiber sequence.

(f) By the dual of (a) it follows that if $X \to Y \to Z$ is a rigid homotopy fiber sequence then there is a canonically defined map in the homotopy category $\underline{\mathcal{F}}(\mathbb{S}^1, Z) \to X$.

(g) Suppose $A \to B \to C$ is a rigid homotopy cofiber sequence between cofibrant objects, with induced map $C \to A \wedge \mathbb{S}^1$ in $\mathrm{Ho}(\mathcal{S}pectra)$. Let $\mathcal{E}$ be a fibrant spectrum. For the rigid homotopy fiber sequence $\mathcal{F}(C, \mathcal{E}) \to \mathcal{F}(B, \mathcal{E}) \to \mathcal{F}(A, \mathcal{E})$, the induced map $\underline{\mathcal{F}}(\mathbb{S}^1, \mathcal{F}(A, \mathcal{E})) \to \mathcal{F}(C, \mathcal{E})$ coincides with the composite

$$\underline{\mathcal{F}}(\mathbb{S}^1, \mathcal{F}(A, \mathcal{E})) \cong \underline{\mathcal{F}}(\mathbb{S}^1 \underline{\wedge} A, \mathcal{E}) \to \underline{\mathcal{F}}(A \underline{\wedge} \mathbb{S}^1, \mathcal{E}) \to \underline{\mathcal{F}}(C, \mathcal{E}),$$

where the second map is induced by the twist.

(h) Let $A \to B \to C$ be a rigid homotopy cofiber sequence between cofibrant objects, with induced map $f\colon C \to A \underline{\wedge} \mathbb{S}^1$. If X is another cofibrant object, then both $A \wedge X \to B \wedge X \to C \wedge X$ and $X \wedge A \to X \wedge B \to X \wedge C$ are rigid homotopy cofiber sequences. The induced map for the first is the composite $C \wedge X \to A \wedge \mathbb{S}^1 \wedge X \to A \wedge X \wedge \mathbb{S}^1$, and for the second is $X \wedge C \to X \wedge A \wedge \mathbb{S}^1$.

C.7. **Eilenberg-MacLane spectra.** Let $\mathcal{A}b$ denote the category of abelian groups. We will assume

(S7) There is a functor $H\colon \mathcal{A}b \to \mathcal{S}pectra$ such that
- (i) $H(0) = *$;
- (ii) $\mathrm{Ho}\,(\mathbb{S}^k, HA) = 0$ if $k \neq 0$;
- (iii) Each HA is fibrant;
- (iv) There is a natural isomorphism $A \to \mathrm{Ho}\,(\mathbb{S}^0, HA)$;
- (v) There is a natural transformation $HA \wedge HB \to H(A \otimes B)$; and
- (vi) If $0 \to A \to B \to C \to 0$ is an exact sequence of abelian groups then $HA \to HB \to HC$ is a rigid homotopy fiber sequence.

In particular, note that if R is a ring then the multiplication $R \otimes R \to R$ gives rise to a multiplication $HR \wedge HR \to HR$.

For the category of symmetric spectra based on topological spaces, one can define HA to be the symmetric spectrum whose nth space is $\mathrm{AG}((S^1)^{\wedge n}; A)$—this is the space defined in [DT] consisting of configurations of points in $(S^1)^{\wedge n}$ labelled by elements of A. It can be checked that $H(-)$ satisfies the above properties.

C.8. **Cohomology theories.**

Given objects $E, X \in \mathrm{Ho}(\mathcal{S}pectra)$, one defines $E^p(X) = \mathrm{Ho}\,(\mathbb{S}^{-p} \underline{\wedge} X, E)$ and $E_p(X) = \pi_p(E \wedge X)$. Observe that a map $E \underline{\wedge} F \to G$ induces a corresponding pairing $E^p(X) \otimes F^q(Y) \to G^{p+q}(X \underline{\wedge} Y)$ in the expected way; this involves using the twist map $X \underline{\wedge} \mathbb{S}^{-q} \to \mathbb{S}^{-q} \underline{\wedge} X$.

The pairing $\underline{\mathcal{F}}(X, E) \underline{\wedge}\, \underline{\mathcal{F}}(Y, F) \to \underline{\mathcal{F}}(X \underline{\wedge} Y, E \underline{\wedge} F)$ also yields a pairing of graded abelian groups

$$\Big[\oplus_p \mathrm{Ho}\,(\mathbb{S}^{-p}, \underline{\mathcal{F}}(X, E))\Big] \otimes \Big[\oplus_q \mathrm{Ho}\,(\mathbb{S}^{-q}, \underline{\mathcal{F}}(Y, F))\Big] \to \oplus_r \mathrm{Ho}\,\Big(\mathbb{S}^{-r}, \underline{\mathcal{F}}(X \underline{\wedge} Y, G)\Big).$$

We will leave it to the reader to check that the adjunctions $\mathrm{Ho}\,(\mathbb{S}^{-p}, \underline{\mathcal{F}}(X, E)) \cong \mathrm{Ho}\,(\mathbb{S}^{-p} \underline{\wedge} X, E)$ induce isomorphisms between this graded pairing and the graded pairing $E^*(X) \otimes F^*(Y) \to G^*(X \wedge Y)$ (this is just a matter of keeping the signs straight). This is a general fact about closed symmetric monoidal categories, and doesn't use anything special about $\mathrm{Ho}(\mathcal{S}pectra)$.

C.9. **Boundary maps.** In the following list, parts (a), (g), and (h) define boundary homomorphisms for long exact sequences of homotopy groups, homology groups, and cohomology groups, respectively. The other parts gives basic corollaries of these definitions (some of the proofs are sketched below).

(a) If $F \to E \to B$ is a rigid homotopy fiber sequence, we define $\partial_k\colon \pi_k B \to \pi_{k-1} F$ to be $(-1)^k$ times the composite

$$\mathrm{Ho}(\mathbb{S}^k, B) = \mathrm{Ho}(\mathbb{S}^{k-1} \wedge \mathbb{S}^1, B) = \mathrm{Ho}(\mathbb{S}^{k-1}, \underline{\mathcal{F}}(\mathbb{S}^1, B)) \longrightarrow \mathrm{Ho}(\mathbb{S}^{k-1}, F).$$

(b) For the rigid homotopy fiber sequence $\Sigma^\infty S^{k-1} \to \Sigma^\infty D^k \to \Sigma^\infty S^k$, the boundary map ∂_k sends the canonical generator of $\pi_k(\Sigma^\infty S^k)$ to the canonical generator of $\pi_{k-1}(\Sigma^\infty S^{k-1})$.

(c) Let $A \to B \to C$ be a rigid homotopy cofiber sequence between cofibrant objects, and let E be a fibrant spectrum. Then $\mathcal{F}(C, E) \to \mathcal{F}(B, E) \to \mathcal{F}(A, E)$

is a rigid homotopy fiber sequence and the associated boundary map ∂_k is equal to $(-1)^k$ times the composite

$$\begin{array}{ccc}
\operatorname{Ho}(\mathbb{S}^k \mathbin{\underline{\wedge}} A, E) \xrightarrow{\cong} \operatorname{Ho}(\mathbb{S}^{k-1} \mathbin{\underline{\wedge}} \mathbb{S}^1 \mathbin{\underline{\wedge}} A, E) & & \\
\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow t & & \\
\qquad\qquad\qquad\qquad\quad \operatorname{Ho}(\mathbb{S}^{k-1} \mathbin{\underline{\wedge}} A \mathbin{\underline{\wedge}} \mathbb{S}^1, E) \longrightarrow \operatorname{Ho}(\mathbb{S}^{k-1} \mathbin{\underline{\wedge}} C, E) & &
\end{array}$$

(using the canonical adjunctions $\operatorname{Ho}(\mathbb{S}^k, \mathcal{F}(A, E)) = \operatorname{Ho}(\mathbb{S}^k \wedge A, E)$, etc.)

(d) If E is a fibrant spectrum then $\mathcal{F}(\Sigma^\infty S^k, E) \to \mathcal{F}(\Sigma^\infty D^k, E) \to \mathcal{F}(\Sigma^\infty S^{k-1}, E)$ is a rigid homotopy fiber sequence. The diagram

$$\begin{array}{ccc}
\pi_t\, \underline{\mathcal{F}}(S^{k-1}, E) & \xrightarrow{\partial} & \pi_{t-1}\, \underline{\mathcal{F}}(S^k, E) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\operatorname{Ho}(\mathbb{S}^t \wedge S^{k-1}, E) & \longrightarrow & \operatorname{Ho}(\mathbb{S}^{t-1} \wedge S^k, E)
\end{array}$$

commutes up to $(-1)^{t-1}$, where the bottom map is induced by the canonical identification.

(e) If $A \to B \to C$ is a rigid homotopy fiber sequence, it is also a rigid homotopy cofiber sequence; so there is an induced map $C \to A \mathbin{\underline{\wedge}} \mathbb{S}^1$. The map ∂_k is equal to $(-1)^k$ times the composite

$$\operatorname{Ho}(\mathbb{S}^k, C) \to \operatorname{Ho}(\mathbb{S}^k, A \mathbin{\underline{\wedge}} \mathbb{S}^1) \cong \operatorname{Ho}(\mathbb{S}^{k-1} \mathbin{\underline{\wedge}} \mathbb{S}^1, A \mathbin{\underline{\wedge}} \mathbb{S}^1) \cong \operatorname{Ho}(\mathbb{S}^{k-1}, A)$$

where the final map is the inverse to the right-suspension map.

(f) If $A \to B \to C$ is a rigid homotopy cofiber sequence between cofibrant objects, then $\mathbb{S}^1 \wedge A \to \mathbb{S}^1 \wedge B \to \mathbb{S}^1 \wedge C$ and $A \wedge \mathbb{S}^1 \to B \wedge \mathbb{S}^1 \to C \wedge \mathbb{S}^1$ are both rigid homotopy cofiber sequences. For the diagrams

$$\begin{array}{ccc}
\operatorname{Ho}(\mathbb{S}^k, C) & \xrightarrow{\sigma_l} & \operatorname{Ho}(\mathbb{S}^{k+1}, \mathbb{S}^1 \wedge C) \\
\partial_C \Big\downarrow & & \Big\downarrow \partial_{\mathbb{S}^1 \wedge C} \\
\operatorname{Ho}(\mathbb{S}^{k-1}, A) & \xrightarrow{\sigma_l} & \operatorname{Ho}(\mathbb{S}^k, \mathbb{S}^1 \wedge A)
\end{array}
\qquad
\begin{array}{ccc}
\operatorname{Ho}(\mathbb{S}^k, C) & \xrightarrow{\sigma_r} & \operatorname{Ho}(\mathbb{S}^{k+1}, C \wedge \mathbb{S}^1) \\
\partial_C \Big\downarrow & & \Big\downarrow \partial_{C \wedge \mathbb{S}^1} \\
\operatorname{Ho}(\mathbb{S}^{k-1}, A) & \xrightarrow{\sigma_r} & \operatorname{Ho}(\mathbb{S}^k, A \wedge \mathbb{S}^1),
\end{array}$$

the first one anti-commutes and the second one commutes. This may be written as $\partial(\sigma x) = -\sigma(\partial x)$ and $\partial(x\sigma) = (\partial x)\sigma$.

(g) If E is a spectrum and $A \to B \to C$ is a rigid homotopy cofiber sequence, define the homology boundary map $d\colon E_k(C) \to E_{k-1}(A)$ as $(-1)^k$ times the composite

$$\pi_k(E \mathbin{\underline{\wedge}} C) \to \pi_k(E \mathbin{\underline{\wedge}} A \mathbin{\underline{\wedge}} \mathbb{S}^1) \cong \pi_{k-1}(E \mathbin{\underline{\wedge}} A)$$

where the second map is the inverse to the right-suspension map. When E, A, B, and C are all cofibrant then $E \wedge A \to E \wedge B \to E \wedge C$ is a rigid homotopy cofiber sequence, and the above map is just the associated ∂_k defined in (a).

(h) The diagram

$$\begin{array}{ccc}
E_n(C) & \xrightarrow{d} & E_{n-1}(A) \\
\Big\downarrow \sigma_r & & \Big\downarrow \sigma_r \\
E_{n+1}(C \mathbin{\underline{\wedge}} S^1) & \xrightarrow{d} & E_n(A \mathbin{\underline{\wedge}} S^1)
\end{array}$$

commutes; that is, $d(x\sigma) = d(x)\sigma$.

(i) In the situation of (g), define the cohomology boundary map $\delta^*\colon E^k(A) \to E^{k+1}(C)$ to be the boundary map ∂_{-k} for $\underline{\mathcal{F}}(C,E) \to \underline{\mathcal{F}}(B,E) \to \underline{\mathcal{F}}(A,E)$. So it is $(-1)^k$ times the composite

$$\begin{array}{ccc} \mathrm{Ho}\,(\mathbb{S}^{-k} \mathbin{\underline{\wedge}} A, E) & \xrightarrow{\cong} & \mathrm{Ho}\,(\mathbb{S}^{-k-1} \mathbin{\underline{\wedge}} \mathbb{S}^1 \mathbin{\underline{\wedge}} A, E) & & \\ & & \downarrow{\scriptstyle t} & & \\ & & \mathrm{Ho}\,(\mathbb{S}^{-k-1} \mathbin{\underline{\wedge}} A \mathbin{\underline{\wedge}} \mathbb{S}^1, E) & \longrightarrow & \mathrm{Ho}\,(\mathbb{S}^{-k-1} \mathbin{\underline{\wedge}} C, E). \end{array}$$

(j) If E is a multiplicative spectrum then there is a slant product $E^p(X) \otimes E_q(X \mathbin{\underline{\wedge}} Y) \to E_{-p+q}(Y)$ defined as follows. If $\alpha \in E^p(X)$ is represented by $\mathbb{S}^{-p} \mathbin{\underline{\wedge}} X \to E$ and $x \in E_q(X \mathbin{\underline{\wedge}} Y)$ is represented by $\mathbb{S}^q \to E \mathbin{\underline{\wedge}} X \mathbin{\underline{\wedge}} Y$, then $\alpha\backslash x$ is represented by $\mathbb{S}^{-p} \wedge (-)$ applied to the map

$$\begin{array}{ccccccc} \mathbb{S}^q & \longrightarrow & E \mathbin{\underline{\wedge}} X \mathbin{\underline{\wedge}} Y & \xrightarrow{1\wedge(\mathbb{S}^p\wedge\alpha)\wedge 1} & E \mathbin{\underline{\wedge}} (\mathbb{S}^p \mathbin{\underline{\wedge}} E) \mathbin{\underline{\wedge}} Y & & \\ & & & & \downarrow{\scriptstyle t\wedge 1\wedge 1} & & \\ & & & & \mathbb{S}^p \mathbin{\underline{\wedge}} E \mathbin{\underline{\wedge}} E \mathbin{\underline{\wedge}} Y & \xrightarrow{1\wedge\mu\wedge 1} & \mathbb{S}^p \mathbin{\underline{\wedge}} E \mathbin{\underline{\wedge}} Y \end{array}$$

One checks that $(\delta\alpha)\backslash x = (-1)^{|\alpha|}\alpha\backslash(dx)$.

(k) If $f\colon A \to B$ is a map between cofibrant objects, let Cf denote the canonical homotopy cofiber defined in C.4. If $\mathrm{Cyl}\, f$ denotes the pushout of $B \leftarrow A \to A \wedge \Sigma^\infty I_+$ then the cofiber of $A \to \mathrm{Cyl}\, f$ is canonically the same as Cf. There is a diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & Cf \\ \| & & \uparrow & & \| \\ A & \longrightarrow & \mathrm{Cyl}\, f & \longrightarrow & Cf \end{array}$$

in which the vertical maps are weak equivalences. On the bottom we have a rigid homotopy cofiber sequence (the top is not one, because the composite is not null). So we have the associated map $\partial_k\colon \mathrm{Ho}\,(S^k, Cf) \to \mathrm{Ho}\,(S^{k-1}, A)$ defined in (a). This gives a long exact sequence

$$\cdots \longrightarrow \mathrm{Ho}\,(\mathbb{S}^k, A) \longrightarrow \mathrm{Ho}\,(\mathbb{S}^k, B) \longrightarrow \mathrm{Ho}\,(\mathbb{S}^k, Cf) \xrightarrow{\partial} \mathrm{Ho}\,(\mathbb{S}^{k-1}, A) \longrightarrow \cdots$$

which is functorial in the map f.

Remark C.10. The sign in (a) was chosen to make (b) true. It follows by naturality that for a diagram of the form

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \uparrow{\scriptstyle f} & & \uparrow & & \uparrow{\scriptstyle g} \\ \Sigma^\infty S^{k-1} & \rightarrowtail & \Sigma^\infty D^k & \longrightarrow & \Sigma^\infty S^k \end{array}$$

in which $k \geq 1$ and the bottom row is the usual cofiber sequence, one has $\partial([g]) = [f]$. This makes sense in light of ∂ being a 'boundary' map. In part (g) the sign was chosen so that the diagram

$$\begin{array}{ccc} \mathrm{Ho}\,(S^n, E\wedge S^n) & \xrightarrow{d} & \mathrm{Ho}\,(S^{n-1}, E\wedge S^{n-1}) \\ {\scriptstyle \sigma_r^n}\uparrow & \nearrow{\scriptstyle \sigma_r^{n-1}} & \\ \mathrm{Ho}\,(S^0, E) & & \end{array}$$

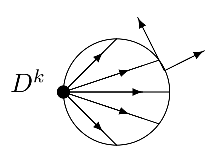
commutes, where the horizontal map is the homology boundary for the cofiber sequence $S^{n-1} \to D^n \to S^n$ and the vertical maps are iterated right-suspension maps. This agrees with the standard conventions for singular cohomology. Finally, the sign in (i) was chosen so as to make (j) true.

Proofs of the above claims. Part (c) is an immediate consequence of C.6(g). As a result of (c), one deduces that in the homotopy fiber sequence $\mathcal{F}(S^1,\mathcal{E}) \to \mathcal{F}(I,\mathcal{E}) \to \mathcal{F}(S^0,\mathcal{E}) = \mathcal{E}$, the boundary map $\partial_k\colon \mathrm{Ho}\,(S^k,\mathcal{E}) \to \mathrm{Ho}\,(S^{k-1},\mathcal{F}(S^1,\mathcal{E}))$ is $(-1)^{k-1}$ times the canonical adjunction. This follows from C.6(b), which says that the rigid homotopy cofiber sequence $S^0 \hookrightarrow I \to S^1$ has induced map $S^1 \to S^0 \wedge S^1$ equal to -1. We will apply this observation to prove (b).

First observe that for any point $x \in D^k$ we may consider the straight-line path from the basepoint to x, and we may project this path onto D^k/S^{k-1}. This gives us a map $D^k \to \mathcal{F}(I, S^k)$. (To be completely precise, we use the given description to write down a map of *spaces* $D^k \wedge I \to S^k$. Then we apply Σ^∞ and use the isomorphism from (S4) to get $\Sigma^\infty D^k \wedge \Sigma^\infty I \to \Sigma^\infty S^k$. Finally we take the adjoint of this map.) We in fact have a diagram

$$\begin{array}{ccccc} S^{k-1} & \longrightarrow & D^k & \longrightarrow & S^k \\ \downarrow & & \downarrow & & \| \\ F(S^1, S^k) & \longrightarrow & F(I, S^k) & \longrightarrow & S^k \end{array}$$

We have already remarked that for the bottom fiber sequence ∂_k is $(-1)^{k-1}$ times the canonical adjunction. The adjoint of the left vertical map may be checked to have degree $(-1)^{k-1}$, so these signs cancel and we have proven (b). An easy way to check that the map has the claimed degree is to just draw the following picture, showing the image of the map $S^{k-1} \wedge I \to D^k$ sending $(v,t) \mapsto tv$.



Orienting D^k as the image of $S^{k-1}\wedge I$ means orienting it as $\partial D^k \times$(outward normal), whereas the usual orientation of D^k is (outward normal) $\times\, \partial D^k$. These differ by the sign $(-1)^{k-1}$.

Part (d) is immediate from (c), C.6(b), and the definition in (a). By naturality it suffices to check part (e) when $A \to B \to C$ is $S^{k-1} \to D^k \to S^k$, but in this case the identification follows from (b) and C.6(b).

Part (f) follows from the description of the boundary map given in (e), together with C.6(h). The remaining parts are all routine. $\square$

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