MULTIPLICATIVE STRUCTURES ON HOMOTOPY SPECTRAL SEQUENCES II

DANIEL DUGGER

Contents

1. Introduction

This short paper is a companion to [D1]. Here the main results of that paper are used to establish multiplicative structures on a few standard spectral sequences. The applications consist of (a) applying [D1, Theorem 6.1] to obtain a pairing of spectral sequences, and (b) identifying the pairing on the E_1- or E_2-term with something familiar, like a pairing of singular cohomology groups. Most of the arguments are straightforward, but there are subtleties that appear from time to time.

Originally the aim was just to record a careful treatment of pairings on Postnikov/Whitehead towers, but in the end other examples have been included because it made sense to do so. These examples are just the ones that I personally have needed to use at some point over the years, and so of course it is a very limited selection.

In this paper all the notation and conventions of [D1] remain in force. In particular, the reader is referred to [D1, Appendix C] for our standing assumptions about the category of spaces and spectra, and for basic results about signs for boundary maps. The symbol $\underline{\wedge}$ denotes the derived functor of $\wedge$, and $W_\perp$ denotes an 'augmented tower' as in [D1, Section 6]. $\text{Ho}(-,-)$ denotes maps in the homotopy category of $\mathcal{S}pectra$. If A is a pointed space we will write $\mathcal{F}(A,X)$ as an abbreviation for $\mathcal{F}(\Sigma^\infty A, X)$. Finally, the phrase 'globally isomorphic' is often used in the identification of E_2-terms of spectral sequences. It is explained in Remark 3.5.

2. Sign conventions in singular cohomology

If E is a ring spectrum and X is a space, then the Atiyah-Hirzebruch spectral sequence $E_2^{p,q} = H^p(X; E^q) \Rightarrow E^{p+q}(X)$ is multiplicative. The naive guess about what this means is that there is an isomorphism of bi-graded rings

$$\oplus_{p,q} E_2^{p,q}(X) \cong \oplus_{p,q} H^p(X; E^q),$$

where the products on the right-hand-side are the usual ones

$$\mu\colon H^p(X; E^q) \otimes H^s(X; E^t) \to H^{p+s}(X; E^{q+t})$$

induced by the pairings $E^q \otimes E^t \to E^{q+t}$. Unfortunately, this statement is just not true in general—one has to add an appropriate sign into the definition of μ, and these signs cannot be made to go away. In order to keep track of such signs in a simple way, it's useful to re-evaluate the 'standard' conventions about singular cohomology. I'm grateful to Jim McClure for conversations about these sign issues.

Let X be a CW-complex and $C_*(X)$ be its associated cellular chain complex. In most algebraic topology textbooks the corresponding cellular cochain complex is defined by $C^p(X) = \mathrm{Hom}(C_p(X), \mathbb{Z})$ and

$$(\delta\alpha)(c) = \alpha(\partial c), \quad \text{for any } \alpha \in C^p(X).$$

The cup-product of a p-cochain α and a q-cochain β is defined by the formula

$$(\alpha \cup \beta)(c \otimes d) = \alpha(c) \cdot \beta(d)$$

where c is a p-chain and d is a q-chain. (Note that we have written the above formula as if it were an external cup-product, so we technically need to throw in a diagonal map somewhere—we omit this to simplify the typography). Both of these formulas obviously violate the Koszul sign rule: we will abandon them and instead define

$$(\delta\alpha)(c) = -(-1)^p \alpha(\partial c) \qquad \text{and} \qquad (\alpha \cup \beta)(c \otimes d) = (-1)^{qp} \alpha(c) \cdot \beta(d). \tag{2.1}$$

The first equation may seem to have an unexpected minus sign, but here is the explanation. Recall that if A_* and B_* are chain complexes then there is an associated chain complex $\mathrm{Hom}(A, B)$. Our definition of δ corresponds to the differential on the chain complex $\mathrm{Hom}(C_*(X), \mathbb{Z}[0])$, where $\mathbb{Z}[0]$ is the complex with $\mathbb{Z}$ concentrated in dimension 0.

The sign conventions from (2.1) appear in [Do]. We'll of course use these same conventions for cohomology with coefficients, external cup products, and any similar construction we encounter.

Exercise 2.1. Check that δ is a derivation with respect to the cup-product, and that the dga $C^*(X; \mathbb{Z})$ defined via our new formulas is isomorphic to the dga $C^*_{\text{classical}}(X; \mathbb{Z})$ defined via the old formulas. In particular, our singular cohomology ring $H^*(X; \mathbb{Z})$ is isomorphic to the classical one.

2.2. **Cohomology with graded coefficients.** If A_* is a graded ring we next want to define the *singular cohomology ring with graded coefficients* $H^*_{grd}(X; A)$, making use of the natural sign conventions. It would be nice to just use the internal hom for chain complexes $\mathrm{Hom}(C_*(X), A)$, where A is interpreted as having zero differential, but unfortunately this might give us infinite products in places we don't really want them. Instead we'll consider a certain subcomplex. We set $C^{p,q}(X; A) = \mathrm{Hom}(C_p(X); A_q)$ and $C^n_{grd}(X; A) = \oplus_{p-q=n} C^{p,q}(X; A)$—that is, elements of $C^{p,q}(X; A)$ are regarded as having total degree $p - q$. For $\alpha \in C^n_{grd}(X; A)$ we define $\delta\alpha$ by the formula

$$(\delta\alpha)(c) = -(-1)^n \alpha(\partial c). \tag{2.2}$$

The homology of this complex will be denoted $H^*_{grd}(X;A)$; it has a natural direct sum decomposition into groups $H^{p,q}(X;A)$.

The *graded cup-product* will be defined on the chain complex $C^*_{grd}(X;A)$ as follows: if $\alpha \in C^{p,q}(X;A)$ and $\beta \in C^{s,t}(X;A)$ then

$$(\alpha \cup \beta)(c \otimes d) = (-1)^{(s-t)p}\alpha(c) \cdot \beta(d)$$

(where c is a p-chain and d is a q-chain). The sign is again just the one dictated by the usual Koszul convention, and δ becomes a derivation with respect to this product.

If $C_* \otimes D_* \to E_*$ is a pairing of graded abelian groups, one also has an external graded cup-product $H^*_{grd}(X;C) \otimes H^*_{grd}(Y;D) \to H^*_{grd}(X \times Y;E)$ defined in a similar fashion. It is this product which arises naturally in pairings of spectral sequences.

Exercise 2.3. Construct a bi-graded family of isomorphisms $\eta_{p,q} \colon H^{p,q}(Z;A) \to H^p(Z;A_q)$, natural in both Z and A, which makes the diagrams

$$\begin{tikzcd}
H^{p,q}(X;C) \otimes H^{s,t}(Y;D) \arrow[r] \arrow[d, "\eta\otimes\eta"'] & H^{p+s,q+t}(X \times Y;E) \arrow[d, "\eta"'] \\
H^p(X;C_q) \otimes H^s(Y;D_t) \arrow[r] & H^{p+s}(X \times Y;E_{q+t})
\end{tikzcd}$$

commute up to the sign $(-1)^{sq}$. Here $H^n(Z;A_m)$ denotes singular cohomology with coefficients in A_m as defined via the formulas in (2.1), and the bottom map is the cup-product pairing associated to $C_q \otimes D_t \to E_{q+t}$ (again with the signs from (2.1)).

Exercise 2.4. Repeat the above exercise, but this time show that the isomorphisms $\eta_{p,q}$ can be chosen to make the squares commute up to the sign $(-1)^{pt}$. Convince yourself that it is not possible to choose the $\eta_{p,q}$'s so that the squares commute on the nose.

3. Spectral sequences for filtered spaces

In this section we treat the Atiyah-Hirzebruch spectral sequence, the Serre spectral sequence, and spectral sequences coming from geometric realizations. Some other references for the former are [K], [GM, Appendix B], [V]. For the Serre spectral sequence see [K], [Mc, Chap. 5], [Sp, Chap. 9.4], and [Wh, XIII.8].

3.1. **Generalities.** Suppose given a sequence of cofibrations $\emptyset \rightarrowtail A_0 \rightarrowtail A_1 \rightarrowtail \cdots$ and let A denote the colimit. If $\emptyset \rightarrowtail B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B$ is another sequence of cofibrations, we may form the product sequence whose nth term is

$$(A \times B)_n = \bigcup_{i+j=n} (A_i \times B_j).$$

This is a sequence of cofibrations whose colimit is $A \times B$. Given a fibrant spectrum $\mathcal{E}$, one can look at the induced tower

$$\cdots \to \mathcal{F}(A_{2+}, \mathcal{E}) \to \mathcal{F}(A_{1+}, \mathcal{E}) \to \mathcal{F}(A_{0+}, \mathcal{E})$$

and identify the homotopy fibers as $\mathcal{F}(A_n/A_{n-1}, \mathcal{E})$. This is a lim-tower rather than a colim-tower, and is not convenient for seeing multiplicative structures; one doesn't have reasonable pairings $\mathcal{F}(A_k, \mathcal{E}) \wedge \mathcal{F}(B_n, \mathcal{E}) \to \mathcal{F}((A \times B)_{k+n}, \mathcal{E})$, for instance. Instead we have to use a slightly different tower.

The cofiber sequences $A_n/A_{n-1} \hookrightarrow A/A_{n-1} \to A/A_n$ induce rigid homotopy fiber sequences $\mathcal{F}(A/A_n, \mathcal{E}) \to \mathcal{F}(A/A_{n-1}, \mathcal{E}) \to \mathcal{F}(A_n/A_{n-1}, \mathcal{E})$. We define an augmented colim-tower by setting $\mathbb{W}(A, \mathcal{E})_n = \mathcal{F}(A/A_{n-1}, \mathcal{E})$ and $\mathbb{B}(A, \mathcal{E})_n = \mathcal{F}(A_n/A_{n-1}, \mathcal{E})$. The associated spectral sequence $E_*(A, \mathcal{E})$ might be called the **$\mathcal{E}$-spectral sequence for the filtered space A**.

Exercise 3.2. Verify that the tower $\mathbb{W}(A, \mathcal{E})$ is weakly equivalent to the tower $\Omega\,\mathcal{F}(A_*, \mathcal{E})$ via a canonical zig-zag of towers. So the homotopy spectral sequences can be identified.

Now assume that $\mathcal{E}$ had a multiplication $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$. Then for any two pointed spaces X and Y we have the map

$$\mathcal{F}(X, \mathcal{E}) \wedge \mathcal{F}(Y, \mathcal{E}) \to \mathcal{F}(X \wedge Y, \mathcal{E} \wedge \mathcal{E}) \to \mathcal{F}(X \wedge Y, \mathcal{E}).$$

Using this, the obvious maps of spaces

$$(A \times B)/(A \times B)_{q+t-1} \to A/A_{q-1} \wedge B/B_{t-1}, \quad \text{and}$$
$$(A \times B)_{q+t}/(A \times B)_{q+t-1} \to A_q/A_{q-1} \wedge B_t/B_{t-1}$$

give pairings $\mathbb{W}(A, \mathcal{E}) \wedge \mathbb{W}(B, \mathcal{E}) \to \mathbb{W}(A \times B, \mathcal{E})$ and $\mathbb{B}(A, \mathcal{E}) \wedge \mathbb{B}(B, \mathcal{E}) \to \mathbb{B}(A \times B, \mathcal{E})$ which are compatible with the maps in the towers. So [D1, Thm 6.1] gives us a pairing of spectral sequences $E_*(A, \mathcal{E}) \otimes E_*(B, \mathcal{E}) \to E_*(A \times B, \mathcal{E})$. This is the 'formal' part of the construction.

3.3. **The Atiyah-Hirzebruch spectral sequence.** Now we specialize to where A and B are CW-complexes which are filtered by their skeleta. In this case we can identify the E_1- and E_2-terms, and we will need to be very explicit about how we do this. For convenience we take A and B to be *labelled CW-complexes*, meaning that they come with a chosen indexing of their cells. Let I_q be the indexing set for the q-cells in A, and let $C_q(A) = \oplus_{\sigma \in I_q} \mathbb{Z}$.

Recall that

$$E_1^{p,q} = \pi_p\,\mathcal{F}(A_q/A_{q-1}, \mathcal{E}) \cong \mathrm{Ho}(S^p \wedge A_q/A_{q-1}, \mathcal{E}).$$

An element $\sigma \in I_q$ specifies a map $S^q \to A_q/A_{q-1}$ which we will also call σ. Given an element $f \in E_1^{p,q} = \mathrm{Ho}(S^p \wedge A_q/A_{q-1}, \mathcal{E})$, restricting to each σ specifies an element in $\mathrm{Ho}(S^p \wedge S^q, \mathcal{E})$. We therefore get a cochain in $\mathrm{Hom}(C_q(A), \pi_{p+q}\mathcal{E}) = C_{grd}^{q,p+q}(A; \mathcal{E}_*)$, and we'll choose this assignment for our isomorphism

$$E_1^{p,q} \cong C_{grd}^{q,p+q}(A; \mathcal{E}_*).$$

Note that this isomorphism is completely natural with respect to maps of labelled CW-complexes.

We claim that the d_1-differential corresponds under this isomorphism to the differential on C^*_{grd} defined in Section 2. By naturality (applied a couple of times) it suffices to check this when A is the CW-complex with $A_{q-1} = *$, $A_q = S^q$, and $A_k = D^{q+1}$ for $k \geq q+1$. In this case our d_1 is the boundary map in the long exact homotopy sequence of $\mathcal{F}(S^{q+1}, \mathcal{E}) \to \mathcal{F}(D^{q+1}, \mathcal{E}) \to \mathcal{F}(S^q, \mathcal{E})$, which takes the form

$$\mathrm{Ho}(S^p \wedge S^q, \mathcal{E}) = \mathrm{Ho}(S^p, \mathcal{F}(S^q, \mathcal{E})) \xrightarrow{\partial} \mathrm{Ho}\left(S^{p-1}, \mathcal{F}(S^{q+1}, \mathcal{E})\right) = \mathrm{Ho}(S^{p-1} \wedge S^{q+1}, \mathcal{E}).$$

We know from [D1, C.6(d)] that the composite is $(-1)^{p-1}$ times the canonical map. Via our identification with cochains, we are looking at a map $C^q(A; \pi_{p+q}\mathcal{E}) \to C^{q+1}(A; \pi_{p+q}\mathcal{E})$, and the sign $(-1)^{p-1}$ is precisely the one for the coboundary δ defined in (2.2).

In a moment we will identify the pairing on E_2-terms, but before that we make a brief remark on the case $A = B$. The diagonal map $A \to A \times A$ is homotopic to a map Δ' which preserves the cellular filtration, and so Δ' induces a map of towers $\mathbb{W}_\perp(A \times A, \mathcal{E}) \to \mathbb{W}_\perp(A, \mathcal{E})$. Composing this with our above pairing gives $\mathbb{W}_\perp(A, \mathcal{E}) \wedge \mathbb{W}_\perp(A, \mathcal{E}) \to \mathbb{W}_\perp(A, \mathcal{E})$, and so we get a multiplicative structure on the spectral sequence $E_*(A, \mathcal{E})$.

Theorem 3.4 (Multiplicativity of the Atiyah-Hirzebruch spectral sequence)**.** *There is a natural pairing of spectral sequences $E_*(A, \mathcal{E}) \otimes E_*(B, \mathcal{E}) \to E_*(A \times B, \mathcal{E})$ together with natural isomorphisms $\oplus_{p,q} E_2^{p,q}(?, \mathcal{E}) \cong \oplus_{p,q} H^q(?, \mathcal{E}^{-p-q})$ (for $? = A, B, A \times B$) which make the diagrams*

$$\begin{CD}
E_2^{p,q}(A, \mathcal{E}) \otimes E_2^{s,t}(B, \mathcal{E}) @>>> E_2^{p+s,q+t}(A \times B, \mathcal{E}) \\
@VVV @VVV \\
H^q(A; \mathcal{E}^{-p-q}) \otimes H^t(B; \mathcal{E}^{-s-t}) @>>> H^{q+t}(A \times B; \mathcal{E}^{-p-q-s-t})
\end{CD}$$

commute, where the bottom map is the graded cup product from Section 2.

In the diagonal case, there is a natural isomorphism of rings $\oplus_{p,q} E_2^{p,q}(A, \mathcal{E}) \cong \oplus_{p,q} H^q(A; \mathcal{E}^{-p-q})$, where the latter is again given the graded cup product.

Remark 3.5. Rather than repeat the above statement for every multiplicative spectral sequence we come across, we'll just say that the E_2-term is **globally isomorphic** to the graded cup product (that is, they are naturally isomorphic as pairings of bigraded abelian groups).

Proof. We have done everything except identify the product. The pairing on E_1-terms is the map

$$\pi_p \mathcal{F}(A_q/A_{q-1}, \mathcal{E}) \wedge \pi_s \mathcal{F}(B_t/B_{t-1}, \mathcal{E}) \to \pi_{p+s} \mathcal{F}(A_q/A_{q-1} \wedge B_t/B_{t-1}, \mathcal{E}).$$

Recall that this sends $\alpha \colon S^p \wedge A_q/A_{q-1} \to \mathcal{E}$ and $\beta \colon S^s \wedge B_t/B_{t-1} \to \mathcal{E}$ to the composite

$$\alpha\beta \colon S^p \wedge S^s \wedge A_q/A_{q-1} \wedge B_t/B_{t-1} \to S^p \wedge A_q/A_{q-1} \wedge S^s \wedge B_t/B_{t-1} \to \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}.$$

Choosing a q-cell σ of A yields a map $S^q \to A_q/A_{q-1}$, and a t-cell θ of B gives a map $S^t \to B_t/B_{t-1}$. Under our identification with cochains, the 'value' of $\alpha\beta$ on the cell $\sigma \wedge \theta$ is the restriction of $\alpha\beta$ to $S^p \wedge S^s \wedge S^q \wedge S^t$.

If, on the other hand, we compute $\alpha(\sigma) \cdot \beta(\theta)$ in the ring $\pi_* \mathcal{E}$, we get the composite

$$[S^p \wedge S^q] \wedge [S^s \wedge S^t] \to [S^p \wedge A_q/A_{q-1}] \wedge [S^s \wedge B_t/B_{t-1}] \to \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}.$$

By inspection, this differs from $(\alpha\beta)(\sigma \wedge \theta)$ by the sign $(-1)^{sq}$, which is the same sign that was used in defining the graded cup product from section 2.2 (remember that under our isomorphism α lies in $C^{q,p+q}(A; \mathcal{E}_*)$ and β lies in $C^{t,s+t}(B; \mathcal{E}_*)$). $\square$

Remark 3.6. In the square from the statement of the theorem, the bottom map is $(-1)^{t(p+q)}$ times the 'un-graded' cup product on cohomology induced by the pairing $\mathcal{E}^{-p-q} \otimes \mathcal{E}^{-s-t} \to \mathcal{E}^{-p-q-s-t}$. This follows from Exercise 2.3. The signs are easy to remember, because they follow the usual conventions: The index 't' is commuted across the index '$-(p+q)$', and as a result the sign $(-1)^{-t(p+q)}$ is picked up (the minus sign can of course be left off the exponent). Note that for most of the familiar cohomology theories, like K-theory or complex cobordism, the signs end up being irrelevant because the coefficient groups are concentrated in even dimensions.

3.7. **The Serre spectral sequence.** Let $p\colon X \to B$ be a fibration with fiber F, where B is a pointed, connected CW-complex. Let $B_0 \subseteq B_1 \subseteq \cdots$ be the skeletal filtration of B, and define $X_i = p^{-1}B_i$. We'll assume that the inclusions $X_i \hookrightarrow X_{i+1}$ are cofibrations between cofibrant objects, and consider the augmented tower $\mathbb{W}_n = \mathcal{F}(X/X_{n-1}, H\mathbb{Z})$, $\mathbb{B}_n = \mathcal{F}(X_n/X_{n-1}, H\mathbb{Z})$. The associated homotopy spectral sequence $E_*(X)$ is the **Serre spectral sequence** for the fibration.

It is easy to see that there is a natural identification

$$X_n/X_{n-1} \cong \bigvee_\alpha \Big[p^{-1}e^n_\alpha/p^{-1}\partial(e^n_\alpha)\Big],$$

where the wedge ranges over the n-cells e^n_α of B. The interior of a cell e^n is just the interior of D^n, so we can take a closed disk around the origin with radius $\frac{1}{2}$—call this smaller disk U. Then $\tilde{H}^*(p^{-1}e^n/p^{-1}\partial(e^n))$ may be canonically identified with $\tilde{H}^*(p^{-1}U/p^{-1}\partial U)$, and we are better off than before because ∂U is actually a sphere (rather than just the image of one). The diagram

$$\begin{tikzcd}
p^{-1}(0) \arrow[r, "\sim"] \arrow[d, tail, "\sim"'] & p^{-1}U \arrow[d, two heads] \\
U \times p^{-1}(0) \arrow[r, two heads] \arrow[ur, dotted, "\lambda"] & U
\end{tikzcd}$$

has a lifting as shown, and this lifting will be a weak equivalence. It restricts to a weak equivalence $\partial U \times p^{-1}(0) \to p^{-1}(\partial U)$ (because this is a map of fibrations over ∂U, and it is a weak equivalence on all fibers). Therefore we have the diagram

$$\begin{tikzcd}
* \arrow[d, equal] & p^{-1}(\partial U) \arrow[l] \arrow[r, tail] & p^{-1}(U) \\
* & \partial U \times p^{-1}(0) \arrow[l] \arrow[r, tail] \arrow[u, "\sim"'] & U \times p^{-1}(0) \arrow[u, "\sim"']
\end{tikzcd}$$

and this necessarily induces a weak equivalence on the pushouts. In this way we get an identification

$$\tilde{H}^k(p^{-1}U/p^{-1}(\partial U)) \cong \tilde{H}^k([U/\partial U]\wedge p^{-1}(0)_+) \cong \tilde{H}^k(S^n \wedge p^{-1}(0)_+) \cong H^{k-n}(p^{-1}(0)).$$

Of course the first isomorphism depended on the lifting λ, and so is not canonical.

We refer to [Mc, Chap. 5] for a detailed discussion of local coefficient systems and their use in this particular context. But once the right definitions are in place the argument we gave for the Atiyah-Hirzebruch spectral sequence in the last section adapts verbatim to naturally identify the (E_1, d_1)-complex as

$$E_1^{p,q} \cong \pi_p\, \mathcal{F}(X_q/X_{q-1}, H\mathbb{Z}) \cong C^q(B; \mathcal{H}^{p+q}(F))$$

where $\mathcal{H}^*(F)$ denotes the appropriate system of coefficients. The differential on the cochain complex is still the one from section 2, appropriate for cellular cohomology with graded coefficients.

Now suppose $X' \to B'$ is another fibration satisfying the same basic assumptions as $X \to B$. We give $B \times B'$ the product cellular filtration, and then pull it back to get a corresponding filtration of $X \times X'$. This coincides with the product of the filtrations on X and X', and so we get a pairing of spectral sequences by the discussion in section 3.1. The identification of the pairing with the graded cup product again follows exactly as for the Atiyah-Hirzebruch spectral sequence.

Theorem 3.8 (Multiplicativity of the Serre spectral sequence)**.** *There is a natural pairing of Serre spectral sequences $E_*(X) \otimes E_*(X') \to E_*(X \times X')$ such that the pairing of E_2-terms is globally isomorphic to the graded cup product on singular cohomology with local coefficients.*

3.9. **Spectral sequences for simplicial objects.** Filtered spaces also arise in the context of geometric realizations. Let X_* be a Reedy cofibrant simplicial space, in which case the skeletal filtration of the realization $|X|$ is a sequence of cofibrations. There is a resulting tower of rigid homotopy cofiber sequences with $W_q(X_*, \mathcal{E}) = \mathcal{F}(|X|/\operatorname{sk}_{q-1}|X|, \mathcal{E})$ and $B_q(X_*, \mathcal{E}) = \mathcal{F}(\operatorname{sk}_q|X|/\operatorname{sk}_{q-1}|X|, \mathcal{E})$.

If Y_* is another Reedy cofibrant simplicial space, we can equip $|X| \times |Y|$ with the product filtration. We also have the product simplicial space $X_* \times Y_*$, equipped with its skeletal filtration. There is a natural map $\eta\colon |X \times Y| \to |X| \times |Y|$, and this is actually an isomorphism (using adjointness arguments one reduces to the case where X and Y are the simplicial sets Δ^m and Δ^n, where it is (T2) from [D1, Appendix C]). Unfortunately η does *not* preserve the filtrations, as can be seen by taking X and Y both to be the simplicial set Δ^1 (regarded as a discrete simplicial space). The product filtration on $|\Delta^1| \times |\Delta^1|$ is smaller than the skeletal filtration coming from $|\Delta^1 \times \Delta^1|$.

The formal machinery of section 3.1 gives a pairing from $E_*(|X|, \mathcal{E})$ and $E_*(|Y|, \mathcal{E})$ to the spectral sequence for the product filtration on $|X| \times |Y|$—let's call this $E_*(|X| \times |Y|, \mathcal{E})$. Often one would like to have a pairing into $E_*(|X \times Y|, \mathcal{E})$, but this doesn't seem to follow from our basic results. Here are two ways around this. One can replace $E_*(|X|, \mathcal{E})$ with the homotopy spectral sequence for the cosimplicial spectrum $[n] \mapsto \mathcal{F}(X_n, \mathcal{E})$, and similarly for $E_*(|Y|, \mathcal{E})$ and $E_*(|X \times Y|, \mathcal{E})$. The paper [BK] proves that if $M^* \wedge N^* \to Q^*$ is a (level-wise) pairing of cosimplicial spaces, then there is an associated pairing of spectral sequences—this gives us what we wanted. Having not checked the details in [BK], I can say nothing more about this approach; their results clearly depend on more than the formal theorems of [D1], but I couldn't tell from their paper exactly what the important ingredient is.

Here is another approach which sometimes works. While $\eta\colon |X \times Y| \to |X| \times |Y|$ does not preserve the filtrations, η^{-1} *is* filtration-preserving (by functoriality and adjointness arguments it suffices to check this when X and Y are the simplicial sets Δ^n and Δ^m). So η^{-1} induces a map of spectral sequences $E_*(|X \times Y|, \mathcal{E}) \to E_*(|X| \times |Y|, \mathcal{E})$. We have the following:

Proposition 3.10. *If X_* and Y_* are simplicial sets, the natural map of spectral sequences $E_*(|X \times Y|, \mathcal{E}) \to E_*(|X| \times |Y|, \mathcal{E})$ is an isomorphism on E_2-terms.*

Proof. This follows from the work in section 3.3, since it identifies both E_1-terms as cellular chain complexes computing $H^*(|X \times Y|, \mathcal{E}^*)$, but for different CW-decompositions. $\square$

It follows that when X_* and Y_* are simplicial sets we get our desired pairing $E_*(|X|, \mathcal{E}) \otimes E_*(|Y|, \mathcal{E}) \to E_*(|X \times Y|, \mathcal{E})$ from the E_2-terms onward. This observation will be used in section 7.

Exercise 3.11. Is Proposition 3.10 true for simplicial spaces? I haven't worked out the answer to this.

4. The Postnikov/Whitehead spectral sequence

For each spectrum E and each $n \in \mathbb{Z}$, let P_nE denote the nth Postnikov section of E; this is a spectrum obtained from E by attaching cells to kill off all homotopy groups from dimension $n+1$ and up. The construction can be set up so that if E is fibrant then all the P_nE are also fibrant, and there are natural maps $E \to P_nE$ and $P_nE \to P_{n-1}E$ making the obvious triangle commute. So we have a tower of fibrant spectra

$$\cdots \to P_2E \to P_1E \to P_0E \to \cdots$$

and the homotopy cofiber of $P_{n+1}E \to P_nE$ is an Eilenberg-MacLane spectrum of type $\Sigma^{n+2}H(\pi_{n+1}E)$.

If A is a cofibrant, pointed space, we can map $\Sigma^\infty A$ into this tower and thereby get a tower of function spectra

$$\cdots \to \mathcal{F}(A, P_2E) \to \mathcal{F}(A, P_1E) \to \mathcal{F}(A, P_0E) \to \cdots$$

The homotopy cofiber of $\mathcal{F}(A, P_{n+1}E) \to \mathcal{F}(A, P_nE)$ is weakly equivalent to $\underline{\mathcal{F}}(A, \Sigma^{n+2}H(\pi_{n+1}E))$, and the resulting homotopy spectral sequence has

$$E_1^{p,q} \cong \pi_p\, \underline{\mathcal{F}}(A, \Sigma^{q+2}H(\pi_{q+1}E)) \cong H^{q-p+2}(A; \pi_{q+1}E).$$

The spectral sequence abuts to $\pi_{p-1}\, \underline{\mathcal{F}}(A, E) = \tilde{E}^{1-p}(A)$. This turns out to be another construction of the Atiyah-Hirzebruch spectral sequence—see [GM, Appendix] for some information about how the two spectral sequences are related.

Assume that E, F, and G are fibrant spectra, and that there is a pairing $E \wedge F \to G$. There do not exist reasonable pairings $P_nE \wedge P_mF \to P_{n+m}G$, and so Postnikov towers are not convenient for seeing multiplicative structures on spectral sequences. This is related to the Postnikov tower being a lim-tower rather than a colim-tower. Instead we will use the 'reverse' of the Postnikov tower, sometimes called the Whitehead tower. If W_nE denotes the homotopy fiber of $E \to P_{n-1}E$, then there are natural maps $W_nE \to W_{n-1}E$ and so we get a new tower. The homotopy cofiber of $W_{n+1}E \to W_nE$ is weakly equivalent to $\Sigma^nH(\pi_nE)$. We will modify these towers in an attempt to produce a pairing $W_*E \wedge W_*F \to W_*G$.

To explain the idea, let's forget about cofibrancy/fibrancy issues for just a moment. Consider the following maps:

$$\begin{tikzcd}
W_mE \wedge W_nF \arrow[r] \arrow[d, dotted, "\lambda"'] & E \wedge F \arrow[d] & \\
W_{m+n}G \arrow[r, "p"] & G \arrow[r, "j"] & P_{m+n-1}G.
\end{tikzcd}$$

The horizontal row is a homotopy fiber sequence. The spectrum $W_mE \wedge W_nF$ is $(m+n-1)$-connected, and so the composite $W_mE \wedge W_nF \to P_{m+n-1}G$ is null-homotopic. Choosing a null-homotopy lets us construct a lifting $\lambda\colon W_mE \wedge W_nF \to W_{m+n}G$. If we had two different liftings λ and λ', their difference would lift to a map $W_mE \wedge W_nF \to \Omega P_{m+n-1}G$ and so would be null-homotopic (again, because the domain is $(m+n-1)$-connected). So the lift λ is unique up to homotopy.

The situation, then, is that we can produce pairings $W_mE \wedge W_nF \to W_{m+n}G$, but so far they don't necessarily commute with the structure maps in the towers. They certainly commute up to homotopy—this follows from the 'uniqueness' considerations in the above paragraph— but we need them to commute on the nose.

By using obstruction theory we will be able to alter these maps so that the relevant diagrams do indeed commute. The argument proceeds in a few steps.

Lemma 4.1. *For each fibrant spectrum E there is a natural tower of rigid homotopy cofiber sequences $(\widetilde{W}_*E, \widetilde{C}_*E)$ such that every $\widetilde{W}_nE$ and $\widetilde{C}_nE$ is cofibrant-fibrant, together with a natural zig-zag of weak equivalences from $\widetilde{W}_*E$ to W_*E.*

Proof. First take W_*E and apply a cofibrant-replacement functor Q to all the levels: this produces QW_*E, a tower of cofibrant spectra. Then perform the telescope construction from [D1, B.4] to get a tower of cofibrations between cofibrant objects TW_*E and a weak equivalence $TW_*E \to QW_*E$. Let C_nE denote the cofiber of $TW_{n+1}E \to TW_nE$. Finally, let F be the fibrant-replacement functor for $\mathcal{S}pectra$ such that $F(*) = *$ given in [D1, C.3(c)]. Applying F to the rigid tower (TWE, CE) gives a new rigid tower which has the desired properties. □

At this point we have towers where everything is cofibrant-fibrant, so the argument we have already explained will construct maps $\widetilde{W}_mE \wedge \widetilde{W}_nF \to \widetilde{W}_{m+n}G$ which commute up to homotopy with the maps in the towers. By considering the diagram

$$\begin{tikzcd}
\widetilde{W}_mE \wedge \widetilde{W}_nF \arrow[r] \arrow[d] & \widetilde{W}_{m+n}G \arrow[d] \\
\widetilde{C}_mE \wedge \widetilde{C}_nF & \widetilde{C}_{m+n}G
\end{tikzcd}$$

one can see that there is a unique homotopy class $\widetilde{C}_mE \wedge \widetilde{C}_nF \to \widetilde{C}_{m+n}G$ which makes the square commute. This is because $\widetilde{C}_{m+n}G$ is an Eilenberg-MacLane spectrum of type $\Sigma^{n+m}H(\pi_{n+m}G)$, and $\widetilde{W}_mE \wedge \widetilde{W}_nF \to \widetilde{C}_mE \wedge \widetilde{C}_nF$ induces an isomorphism on the corresponding cohomology group (since both the domain and codomain are $(m+n-1)$-connected). So at this point we have produced a *homotopy-pairing* $(\widetilde{W}E, \widetilde{C}E) \underline{\wedge} (\widetilde{W}F, \widetilde{C}F) \to (\widetilde{W}G, \widetilde{C}G)$ (see [D1, 6.3]). We will prove:

Proposition 4.2. *The homotopy-pairing $(\widetilde{W}E, \widetilde{C}E) \underline{\wedge} (\widetilde{W}F, \widetilde{C}F) \to (\widetilde{W}G, \widetilde{C}G)$ is locally realizable.*

The following lemma encapsulates the basic facts we will need. The proof will be left to the reader.

Lemma 4.3 (Obstruction theory)**.** *Suppose that $X \to Y$ is a fibration of spectra which induces isomorphisms on π_k for $k \geq n$. Let $A \rightarrowtail B$ be a cofibration which induces isomorphism on π_k for $k < n$. Then any diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

has a lifting as shown.

Proof of Proposition 4.2. First we truncate the towers, and we might as well assume we are dealing with truncations $\tau_{0\leq k}(\widetilde{W}E, \widetilde{C}E)$ and $\tau_{0\leq l}(\widetilde{W}F, \widetilde{C}F)$ because the argument will be the same no matter what the lower bounds are. For the rest of the argument we will only be dealing with these finite towers, and will omit the τ's from the notation.

We replace $(\widetilde{W}E, \widetilde{C}E)$ and $(\widetilde{W}F, \widetilde{C}F)$ by the equivalent towers (TWE, CE) and (TWF, CF) constructed in the proof of Lemma 4.1, because these consist of cofibrations between cofibrant spectra. It is easy to see that one can also find a tower $\widehat{W}_*G$ consisting of fibrant spectra and *fibrations*, together with a weak equivalence $\widetilde{W}_*G \to \widehat{W}_*G$ (remember that all our towers are finite). We will construct a pairing of towers $TW_*E \wedge TW_*F \to \widehat{W}_*G$ which realizes the homotopy-pairing. For $TW_0E \wedge TW_0F \to \widehat{W}_0G$ we choose any map in the correct homotopy class. Next consider the diagram

$$\begin{tikzcd}
 & & \widehat{W}_1G \arrow[d] \\
TW_0E \wedge TW_1F \arrow[r] & TW_0E \wedge TW_0F \arrow[r] & \widehat{W}_0G.
\end{tikzcd}$$

The vertical map is a fibration inducing isomorphisms on π_1 and higher, and the spectrum $TW_0E \wedge TW_1F$ is 0-connected; so there is a lifting $\mu_{(0,1)}$. Next, look at the diagram

$$\begin{tikzcd}
TW_1E \wedge TW_1F \arrow[r] \arrow[d] & TW_0E \wedge TW_1F \arrow[r] & \widehat{W}_1G \arrow[d] \\
TW_1E \wedge TW_0F \arrow[r] & TW_0E \wedge TW_0F \arrow[r] & \widehat{W}_0G.
\end{tikzcd}$$

This diagram commutes (because the missing vertical arrow in the middle may be filled in). The right vertical map is a fibration which induces isomorphisms on π_1 and higher. The left vertical map induces isomorphisms on π_0 and lower (because both domain and range are 0-connected), and is a cofibration. So there is a lifting $\mu_{(1,0)}\colon TW_1E \wedge TW_0F \to \widehat{W}_1G$.

This process may be continued to inductively define $\mu_{(0,2)}$, $\mu_{(1,1)}$, and $\mu_{(2,0)}$, and then onward from level three. In this way, we construct the required pairing of towers. This pairing agrees with the original homotopy-pairing because of the 'uniqueness' of the liftings λ in our original discussion; the details are left for the reader.

At this point we have a pairing of towers, but we need a pairing of augmented rigid towers. We can't just take cofibers in $\widehat{W}_*G$ because the maps are fibrations, not cofibrations. So let $Q(\widehat{W}_*G) \xrightarrow{\sim} \widehat{W}_*G$ be the cofibrant approximation guaranteed in [D1, Lemma B.2]. We have a diagram

$$\begin{tikzcd}
 & Q(\widehat{W}_*G) \arrow[d, two heads, "\sim"] \\
TWE \wedge TWF \arrow[r] & \widehat{W}_*G
\end{tikzcd}$$

and by [D1, Lemma B.3] the lower left corner is a cofibrant tower; so there is a lifting. The tower $Q(\widehat{W}G)$ consists of cofibrations, and so augmenting by the cofibers gives a rigid tower. The new pairing automatically passes to cofibers to give $(TWE, CE) \wedge (TWF, CF) \to (Q(\widehat{W}G), CQ(\widehat{W}G))$.

Finally, consider

$$\begin{tikzcd} & Q(\widehat{W}_*G) \arrow[d, two heads, "\sim"] \\ \widetilde{W}_*G \arrow[r, "\sim"] & \widehat{W}_*G. \end{tikzcd}$$

The tower $\widetilde{W}_*G$ was cofibrant, so there is a lifting. This will be a levelwise equivalence, and therefore induces an equivalence on the cofibers. So we get an equivalence of augmented towers $(\widetilde{W}_*G, \widetilde{C}_*G) \to (Q(\widehat{W}G), CQ(\widehat{W}G))$. Thus, we have constructed the required realization of our homotopy-pairing. $\square$

For each cofibrant space X consider the tower of function spectra $\mathcal{F}(X_+, \widetilde{W}E_\perp)$ (recall that the $\widetilde{W}_*E$ are all fibrant). This is a tower of rigid homotopy cofiber sequences, and we will call the associated spectral sequence $E_*(X, E)$ the **Whitehead spectral sequence for X based on E**. The homotopy-pairing $\widetilde{W}E_\perp \,\triangle\, \widetilde{W}F_\perp \to \widetilde{W}G_\perp$ induces a homotopy-pairing on towers of function spectra, and by [D1, Prop. 6.10] this is locally-realizable and so induces a pairing of spectral sequences: for any cofibrant spaces X and Y we have $E_*(X, E) \otimes E_*(Y, F) \to E_*(X \times Y, G)$. What is left is to identify the pairing on E_1-terms with the pairing on singular cohomology (up to the correct sign).

If X is a spectrum with a single non-vanishing homotopy group in dimension m, there is a unique isomorphism in the homotopy category $\mathbb{S}^m \,\triangle\, H(\pi_m X) \to X$ with the property that the composite

$$\pi_m X \to \pi_0 H(\pi_m X) \xrightarrow{\sigma_l} \pi_m(\mathbb{S}^m \wedge H(\pi_m X)) \to \pi_m X$$

is the identity map (the first map in the composite is the one provided by [D1, Section C.7]). If $X \wedge Y \to Z$ is a pairing of spectra where X, Y, and Z each have a single non-vanishing homotopy group in dimensions m, n and $m+n$, then the diagram in $\mathrm{Ho}(\mathcal{S}pectra)$

$$\begin{tikzcd} \mathbb{S}^m \,\triangle\, H(\pi_m X) \,\triangle\, \mathbb{S}^n \wedge H(\pi_n Y) \arrow[r] \arrow[d] & \mathbb{S}^{m+n} \,\triangle\, H(\pi_{m+n} Z) \arrow[d] \\ X \wedge Y \arrow[r] & Z \end{tikzcd}$$

is commutative. Here the top map interchanges the $\mathbb{S}^n$ and the $H(\pi_m X)$ and then uses the map $H(\pi_m X) \otimes H(\pi_n Y) \to H(\pi_{m+n} Z)$ induced by the pairing $\pi_m X \otimes \pi_n Y \to \pi_{m+n} Z$ (cf. [D1, Section C.7]). The above observations are simple calculations in the homotopy category of spectra.

In our situation we have specific isomorphisms $\pi_m \widetilde{C}_m E \cong \pi_m E$, and the same for F and G. This is because $\widetilde{W}_m E \to \widetilde{C}_m E$ induces an isomorphism on π_m, $\widetilde{W}_m E$ is connected by a chosen zig-zag of weak equivalences to $W_m E$, and the map $W_m E \to E$ induces an isomorphism on π_m as well. The pairing $\widetilde{C}_m E \wedge \widetilde{C}_n F \to \widetilde{C}_{m+n} G$ induces a pairing on homotopy groups which corresponds to the expected pairing $\pi_m E \otimes \pi_n F \to \pi_{m+n} G$ under these isomorphisms. Putting all the above statements together, we have proven:

Lemma 4.4. *In Ho(Spectra) there exist isomorphisms* $\mathbb{S}^n \wedge H(\pi_n E) \to \widetilde{C}_n E$, $\mathbb{S}^n \wedge H(\pi_n F) \to \widetilde{C}_n F$, *and* $\mathbb{S}^n \wedge H(\pi_n G) \to \widetilde{C}_n G$ *for all* $n \in \mathbb{Z}$, *such that the diagrams*

$$\begin{tikzcd}
\mathbb{S}^m \wedge H(\pi_m E) \wedge \mathbb{S}^n \wedge H(\pi_n F) \arrow[r] \arrow[d] & \mathbb{S}^{m+n} \wedge H(\pi_{m+n} G) \arrow[d] \\
\widetilde{C}_m E \wedge \widetilde{C}_n F \arrow[r] & \widetilde{C}_{m+n} G
\end{tikzcd}$$

all commute (in the homotopy category).

The lemma tells us that if A and B are spectra the pairing

$$\left[\oplus_{p,q}\mathrm{Ho}(\mathbb{S}^p \mathbin{\underline{\wedge}} A, \widetilde{C}_q E)\right] \otimes \left[\oplus_{s,t}\mathrm{Ho}(\mathbb{S}^s \mathbin{\underline{\wedge}} B, \widetilde{C}_t F)\right] \to \oplus_{u,v}\mathrm{Ho}(\mathbb{S}^u \mathbin{\underline{\wedge}} A \mathbin{\underline{\wedge}} B, \widetilde{C}_v G)$$

is globally isomorphic to the pairing obtained from the maps

$$\begin{tikzcd}
\mathrm{Ho}(\mathbb{S}^p \mathbin{\underline{\wedge}} A, \mathbb{S}^q \mathbin{\underline{\wedge}} H(\pi_q E)) \otimes \mathrm{Ho}(\mathbb{S}^s \mathbin{\underline{\wedge}} B, \mathbb{S}^t \mathbin{\underline{\wedge}} H(\pi_t F)) \arrow[d] \\
\mathrm{Ho}(\mathbb{S}^{p+s} \mathbin{\underline{\wedge}} A \mathbin{\underline{\wedge}} B, \mathbb{S}^{q+t} \mathbin{\underline{\wedge}} H(\pi_{q+t} G)).
\end{tikzcd}$$

Now, the left-suspension map gives an isomorphism

$$\mathrm{Ho}(\mathbb{S}^p \mathbin{\underline{\wedge}} B, \mathbb{S}^q \mathbin{\underline{\wedge}} H(\pi_q E)) \cong \mathrm{Ho}(\mathbb{S}^{p-q} \mathbin{\underline{\wedge}} B, H(\pi_q E)) = H^{q-p}(B; \pi_q E),$$

and similarly for the F and G terms. This allows us to rewrite the above pairing as a pairing of singular cohomology groups, but the suspension maps introduce signs. For any spectra M and N, the diagram

$$\begin{tikzcd}
\mathrm{Ho}\,(\mathbb{S}^a \mathbin{\underline{\wedge}} A, M) \otimes \mathrm{Ho}\,(\mathbb{S}^b \mathbin{\underline{\wedge}} B, N) \arrow[r, "\sigma_l^q \wedge \sigma_l^t"] \arrow[d] & \mathrm{Ho}\,(\mathbb{S}^{q+a} \mathbin{\underline{\wedge}} A, \mathbb{S}^q \mathbin{\underline{\wedge}} M) \otimes \mathrm{Ho}\,(\mathbb{S}^{t+b} \mathbin{\underline{\wedge}} B, \mathbb{S}^t \mathbin{\underline{\wedge}} N) \arrow[d] \\
\mathrm{Ho}\,(\mathbb{S}^{a+b} \mathbin{\underline{\wedge}} A \wedge B, M \mathbin{\underline{\wedge}} N) \arrow[r, "\sigma_l^{q+t}"] & \mathrm{Ho}\,(\mathbb{S}^{q+t+a+b} \wedge A \wedge B, \mathbb{S}^{q+t} \wedge M \wedge N)
\end{tikzcd}$$

commutes up to the sign $(-1)^{ta}$ (one compares the string '$qatb$' to the string '$qtab$' and sees that the t and a must be commuted). Taking $A = \Sigma^\infty X_+$ and $B = \Sigma^\infty Y_+$, we now conclude:

Theorem 4.5 (Multiplicativity of the Postnikov/Whitehead spectral sequence)**.** *For cofibrant spaces X and Y there is a pairing of Whitehead spectral sequences in which the E_1-term $E_1^{p,q} \otimes E_1^{s,t} \to E_1^{p+s,q+t}$ is globally isomorphic to the pairing*

$$H^{q-p}(X; \pi_q E) \otimes H^{t-s}(Y; \pi_t F) \to H^{q+t-p-s}(X \times Y; \pi_{q+t} G)$$

up to a sign of $(-1)^{t(p-q)}$.

Remark 4.6. At first glance the sign given here doesn't agree with the sign we obtained in Theorem 3.4: if we were to re-index the Atiyah-Hirzebruch spectral sequence in the above form, the sign would be $(-1)^{q(t-s)}$. While this is not the same as the above sign, the two are consistent. Using the family of isomorphisms $(-1)^{pq}\colon H^{q-p}(X; \pi_q E) \cong H^{q-p}(X; \pi_q E)$ and similarly for the F and G terms, the sign $(-1)^{t(p-q)}$ transforms into $(-1)^{q(t-s)}$. See Exercise 2.4.

5. Bockstein spectral sequences

In this section we consider two spectral sequences: one is the classical Bockstein spectral sequence for the homotopy cofiber sequence $H\mathbb{Z} \xrightarrow{\times n} H\mathbb{Z} \to H\mathbb{Z}/n$. The other is the Bockstein spectral sequence for inverting the Bott element in connective K-theory.

5.1. **The Bockstein spectral sequence for $H\mathbb{Z}$.** Consider the following tower $(W_*, B_*)_{*\geq 0}$:

$$\begin{tikzcd} & H\mathbb{Z}/n & H\mathbb{Z}/n & H\mathbb{Z}/n \\ \cdots \arrow[r, "n"] & H\mathbb{Z} \arrow[u] \arrow[r, "n"] & H\mathbb{Z} \arrow[u] \arrow[r, "n"] & H\mathbb{Z}. \arrow[u] \end{tikzcd}$$

We extend this to negative degrees by taking $W_q = H\mathbb{Z}$, $B_q = *$, and $W_{q+1} \to W_q$ to be the identity map. This is a tower of rigid homotopy cofiber sequences, and there is an obvious pairing $(W, B) \wedge (W, B) \to (W, B)$ which comes from the multiplications on $H\mathbb{Z}$ and $H\mathbb{Z}/n$ (cf. [D1, Appendix C.7]).

For any cofibrant space X, let $W_\perp X$ denote the augmented tower whose levels are $\mathcal{F}(X_+, W_{n+1}) \to \mathcal{F}(X_+, W_n) \to \mathcal{F}(X_+, B_n)$; these are rigid homotopy cofiber sequences, since $H\mathbb{Z}$ and $H\mathbb{Z}/n$ are fibrant. The homotopy spectral sequence for $W_\perp X$ is called the **mod n Bockstein spectral sequence**, and has the form

$$E_1^{p,q} = H^{-p}(X; \mathbb{Z}/n) \Rightarrow H^{-p}(X; \mathbb{Z}).$$

The d_1-differential is the usual Bockstein homomorphism. The multiplication on (W, B) gives rise to pairings of towers $W_\perp X \wedge W_\perp Y \to W_\perp(X \times Y)$, and therefore to pairings of spectral sequences by [D1, Thm 6.1]. The following result is immediate, and unlike the examples in sections 3 and 4 there are no extra signs floating around.

Theorem 5.2 (Multiplicativity of the Bockstein spectral sequence)**.** *For cofibrant spaces X and Y there is a pairing of Bockstein spectral sequences whose E_1-term is isomorphic to the usual pairing $H^{-p}(X; \mathbb{Z}/n) \otimes H^{-s}(Y; \mathbb{Z}/n) \to H^{-p-s}(X \times Y; \mathbb{Z}/n)$ of singular cohomology groups. The spectral sequence converges to the usual pairing on $H^*(-; \mathbb{Z})$.*

5.3. **The Bockstein spectral sequence for bu.** Let bu denote a commutative ring spectrum representing connective K-theory, and assume we have a map of ring spectra $bu \to H\mathbb{Z}$. Assume there is a map $\mathbb{S}^2 \to bu$ which represents a generator in $\mathrm{Ho}(\mathbb{S}^2, bu) \cong \mathbb{Z}$ (this is automatic if bu is a fibrant spectrum). Consider the induced map

$$\beta \colon \mathbb{S}^2 \wedge bu \xrightarrow{\beta \wedge id} bu \wedge bu \xrightarrow{\mu} bu.$$

It can be shown that $\mathbb{S}^2 \wedge bu \to bu \to H\mathbb{Z}$ is a homotopy cofiber sequence. If we let (W, B) be the tower

$$\begin{tikzcd} & \mathbb{S}^4 \wedge H\mathbb{Z} & \mathbb{S}^2 \wedge H\mathbb{Z} & H\mathbb{Z} & \mathbb{S}^{-2} \wedge H\mathbb{Z} & \\ \cdots \arrow[r, "\mathbb{S}^4 \wedge \beta"] & \mathbb{S}^4 \wedge bu \arrow[u] \arrow[r, "\mathbb{S}^2 \wedge \beta"] & \mathbb{S}^2 \wedge bu \arrow[u] \arrow[r, "\beta"] & bu \arrow[u] \arrow[r, "\mathbb{S}^{-2} \wedge \beta"] & \mathbb{S}^{-2} \wedge bu \arrow[u] \arrow[r, "\mathbb{S}^{-4} \wedge \beta"] & \cdots \end{tikzcd}$$

then one sees that there is a pairing $(W,B)\wedge(W,B)\to(W,B)$ (this uses that bu is commutative). Unfortunately we are not yet in a position to apply [D1, Thm 6.1]: (W,B) is not a rigid tower, because we don't know that $\mathbb{S}^2\wedge bu\to bu\to H\mathbb{Z}$ is null rather than just null-homotopic. We don't get a long exact sequence on homotopy groups until we choose null-homotopies for each layer, and these must be accounted for. In this particular case any two null-homotopies are themselves homotopic, and so there should be no problems with compatibility, but it's awkward to formulate results along these lines. The best way I know to proceed is actually to discard the $H\mathbb{Z}$'s and consider just the tower W_* consisting of suspensions of bu. We are then in a position to apply [D1, Thm 6.2], but we must work a little harder to identify what's happening on the E_1-terms.

First, one can replace bu by a cofibrant commutative ring spectrum, and this will be cofibrant as a symmetric spectrum [SS, Thm. 4.1(3)]. Because of this, we will just assume that our bu was cofibrant in the first place. By [D1, B.4] there is an augmented tower TW_* consisting of cofibrations between cofibrant spectra, a weak equivalence $TW_*\to W_*$, and a pairing $TW\wedge TW\to TW$ which lifts the pairing $W\wedge W\to W$. We let C_n denote the cofiber of $TW_{n+1}\to TW_n$ and note that our pairing $TW\wedge TW\to TW$ extends to $(TW,C)\wedge(TW,C)\to(TW,C)$.

For any cofibrant space X, we consider the derived tower of function spectra $\mathcal{F}_{der}(X_+,TW_\perp)$ from [D1, B.7]. By [D1, Theorems 6.1,6.10] there is an induced pairing of spectral sequences. Note that we have $E_1^{p,q}=\pi_p\,\underline{\mathcal{F}}(X_+,C_q)\cong\pi_p\,\mathcal{F}(X,\Sigma^{2q}H\mathbb{Z})\cong H^{2q-p}(X;\mathbb{Z})$. As usual, what we want is to identify the pairing on E_1-terms with a pairing on singular cohomology. The following does this:

Lemma 5.4. *In $Ho(\mathcal{S}pectra)$ it is possible to choose a collection of isomorphisms $C_n\to\mathbb{S}^{2n}\,\underline{\wedge}\,H\mathbb{Z}$ such that the following diagrams commute:*

$$\begin{tikzcd}
C_m\,\underline{\wedge}\,C_n \arrow[rrr] \arrow[d] & & & C_{m+n} \arrow[d] \\
\mathbb{S}^{2m}\,\underline{\wedge}\,H\mathbb{Z}\,\underline{\wedge}\,\mathbb{S}^{2n}\,\underline{\wedge}\,H\mathbb{Z} \arrow[r, "1\,\underline{\wedge}\,t\,\underline{\wedge}\,1"] & \mathbb{S}^{2m}\,\underline{\wedge}\,\mathbb{S}^{2n}\,\underline{\wedge}\,H\mathbb{Z}\,\underline{\wedge}\,H\mathbb{Z} \arrow[rr, "1\,\underline{\wedge}\,\mu"] & & \mathbb{S}^{2(m+n)}\,\underline{\wedge}\,H\mathbb{Z}.
\end{tikzcd}$$

Proof. This is very similar to the proof of Lemma 4.4, making use of the fact that the graded ring $\oplus_n\mathrm{Ho}\,(\mathbb{S}^{2n},C_n)$ is isomorphic to the ring of Laurent polynomials $\mathbb{Z}[t,t^{-1}]$. Details are left to the reader. □

The spectral sequence converges to

$$\operatorname{colim}\big[\pi_*\,\underline{\mathcal{F}}(X_+,bu)\xrightarrow{\beta\cdot}\pi_*\,\underline{\mathcal{F}}(X_+,bu)\xrightarrow{\beta\cdot}\cdots\big]$$

which is $\beta^{-1}[\pi_*\,\underline{\mathcal{F}}(X_+,bu)]=\beta^{-1}bu^*(X)$. If we write $\beta^{-1}bu^p(X)$ for the pth graded piece of $\beta^{-1}(bu^*X)$ then we can state the final result as follows:

Theorem 5.5. *For any space X there is a conditionally convergent spectral sequence*

$$E_1^{p,q}=H^{2q-p}(X;\mathbb{Z})\Rightarrow\beta^{-1}bu^{-p}(X).$$

If X and Y are two cofibrant spaces there is a pairing of spectral sequences whose E_1-term is globally isomorphic to the usual pairing on singular cohomology, and which converges to the pairing $\beta^{-1}bu^*(X)\otimes\beta^{-1}bu^*(Y)\to\beta^{-1}bu^*(X\times Y)$.

The above spectral sequence of course has the same form (up to re-indexing) as the Atiyah-Hirzebruch spectral sequence for complex K-theory. One finds in the end that $\beta^{-1}bu^*(X)\cong K^*(X)$.

6. The homotopy-fixed-point spectral sequence

Suppose $\mathcal{E}$ is a fibrant spectrum with an action of a discrete group G. The homotopy fixed set $\mathcal{E}^{hG}$ is defined to be $\mathcal{F}(EG_+, \mathcal{E})^G$.

Theorem 6.1. *If $\mathcal{E}$ is a fibrant ring spectrum and G is a discrete group acting via ring automorphisms, then there is a multiplicative spectral sequence of the form $E_2^{p,q} = H^q(G; \pi_{p+q}\mathcal{E}) \Rightarrow \pi_p(\mathcal{E}^{hG})$. Here the pairing on E_2-terms is the pairing on group cohomology with graded coefficients, defined analagously to what was done in section 2.*

Remark 6.2. The above means that the pairing on E_2-terms $H^q(G; \pi_{p+q}\mathcal{E}) \otimes H^t(G; \pi_{s+t}\mathcal{E}) \to H^{q+t}(G; \pi_{p+q+s+t}\mathcal{E})$ is $(-1)^{t(p+q)}$ times the 'standard' pairing on group cohomology induced by $\pi_*\mathcal{E} \otimes \pi_*\mathcal{E} \to \pi_*\mathcal{E}$.

Proof. Take any model for EG which is an equivariant G-CW-complex (with G acting freely on the set of cells in each dimension). Write EG_k for $\mathrm{sk}_k\, EG$, and let $\mathbb{W}_n = \mathcal{F}(EG/EG_{n-1}, \mathcal{E})$, $\mathbb{B}_n = \mathcal{F}(EG_n/EG_{n-1}, \mathcal{E})$. The spectral sequence for this tower is just the Atiyah-Hirzebruch spectral sequence based on EG and $\mathcal{E}$, from section 3.3. The sequences $\mathbb{W}_{n+1} \to \mathbb{W}_n \to \mathbb{B}_n$ are actually G-equivariant homotopy fiber sequences, which implies that they give homotopy fiber sequences on H-fixed sets for any H. We'll consider the associated tower $(\mathbb{W}_*^G, \mathbb{B}_*^G)$.

The E_1-term of the spectral sequence is $E_1^{p,q} = \pi_p[\mathcal{F}(EG_q/EG_{q-1}, \mathcal{E})^G]$. The map

$$\pi_p[\mathcal{F}(EG_q/EG_{q-1}, \mathcal{E})^G] \to \pi_p\, \mathcal{F}(EG_q/EG_{q-1}, \mathcal{E}) = \mathrm{Ho}\,(S^p \wedge [EG_q/EG_{q-1}], \mathcal{E})$$

has its image in the G-fixed set of the right-hand-side, and one can check that this gives an isomorphism $E_1^{p,q} \cong \mathrm{Ho}\,(S^p \wedge [EG_q/EG_{q-1}], \mathcal{E})^G$. Since EG_q/EG_{q-1} is a wedge of q-spheres indexed by the set of q-cells, with G action induced by that on the indexing set, one obtains a natural isomorphism with the group $\mathrm{Hom}_G(C_*(EG), \pi_{p+q}\mathcal{E})$. The identification works the same as in section 3.3, and the description of the differential carries over as well. In fact the map of towers $(\mathbb{W}_*^G, \mathbb{B}_*^G) \to (\mathbb{W}_*, \mathbb{B}_*)$ lets us compare the differential with the one on the Atiyah-Hirzebruch spectral sequence, which we have already analyzed. So the identification of the E_2-term follows.

The pairing of augmented towers $\mathbb{W}(EG, \mathcal{E}) \wedge \mathbb{W}(EG, \mathcal{E}) \to \mathbb{W}(EG \times EG, \mathcal{E})$ from section 3 obviously restricts to fixed sets, giving $\mathbb{W}(EG, \mathcal{E})^G \wedge \mathbb{W}(EG, \mathcal{E})^G \to \mathbb{W}(EG \times EG, \mathcal{E})^G$. This uses that $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$ is G-equivariant. If one accepts that the diagonal map $EG \to EG \times EG$ is homotopic to a map Δ' which is both cellular and G-equivariant, then we get an equivariant map of towers $\mathbb{W}(EG \times EG, \mathcal{E}) \to \mathbb{W}(EG, \mathcal{E})$ and can restrict to fixed sets. The existence of Δ' follows from the G-equivariant cellular approximation theorem. Therefore we have an induced pairing of augmented towers $\mathbb{W}_\perp^G \wedge \mathbb{W}_\perp^G \to \mathbb{W}_\perp^G$. The identification of the product on E_2-terms again can be done by comparing with the Atiyah-Hirzebruch spectral sequence for the tower $(\mathbb{W}_n, \mathbb{B}_n)$. Everything follows exactly as in section 3.3. $\square$

Remark 6.3. Instead of filtering EG by skeleta, the spectral sequence can also be constructed via a Postnikov tower on $\mathcal{E}$, just as in section 4 (by functoriality, the Postnikov sections of $\mathcal{E}$ inherit G-actions). Instead of needing a G-equivariant cellular approximation for the diagonal map, one instead needs to carry out G-equivariant obstruction theory.

7. Spectral sequences from open coverings

In this section we give a second treatment of the Atiyah-Hirzebruch spectral sequence, together with a similar approach to the Leray-Serre spectral sequence. Our towers are obtained by using open coverings and their generalization to *hypercovers*. We will assume a basic knowledge of hypercovers, for which the reader can consult [DI]. Basically, one starts with an open cover $\{U_a\}$ of a space X and then chooses another open cover for every double intersection $U_a \cap U_b$; for every resulting 'triple intersection' another covering is chosen, and so on. All of this data is compiled into a simplicial space U_*, called a hypercover.

The discussion in this section is much sloppier than the previous ones, and should probably be improved at some point...

7.1. **The descent spectral sequence.** Given a hypercover U_* of a space X and a sheaf of abelian groups F on X, we let $F(U_*)$ denote the cochain complex one gets by applying F to the open sets in U_*. If all the pieces of the hypercover U_n are such that $H^k_{\text{shf}}(U_n, F) = 0$ for $k > 0$, then Verdier's Hypercovering Theorem gives an isomorphism $H^p(F(U_*)) \to H^p_{\text{shf}}(X, F)$ which is functorial for X, U, and F. It is easy to explain how to get this. First, choose a functorial, injective resolution I_* for F, and look at the double complex of sections $D_{mn} = I_n(U_m)$. This double complex has two edge homomorphisms $I_*(X) \to \operatorname{Tot} D_{**}$ and $F(U_*) \to \operatorname{Tot} D_{**}$; the two spectral sequences for the homology of a double complex immediately show that these maps give isomorphisms on homology. Composing them appropriately gives our natural isomorphism $H^p(F(U_*)) \to H^p(I_*(X)) = H^p_{\text{shf}}(X, F)$.

Given a fibrant spectrum $\mathcal{E}$ and a hypercover U_* of a X, we use the simplicial space U_* to set up a tower as in section 3.9. We let $W_n = \mathcal{F}(|U|/\operatorname{sk}_{n-1}|U|, \mathcal{E})$, $B_n = \mathcal{F}(\operatorname{sk}_n|U|/sk_{n-1}|U|, \mathcal{E})$, and denote the resulting spectral sequence by $E_*(U, \mathcal{E})$. It is a theorem of [DI] that the natural map $|U_*| \to X$ is a weak equivalence, and so the spectral sequence converges to $\mathcal{E}^*(X)$. We'll call this the **descent spectral sequence** based on U and $\mathcal{E}$.

Note that the spectral sequence is functorial in several ways. It is clearly functorial in $\mathcal{E}$, and if $V_* \to U_*$ is a map of hypercovers of X then there is a natural map $E_*(U, \mathcal{E}) \to E_*(V, \mathcal{E})$. Also, if $f\colon Y \to X$ is a map then there is a pullback hypercover $f^{-1}U_* \to Y$, and a map of spectral sequences $E_*(U, \mathcal{E}) \to E_*(f^{-1}U, \mathcal{E})$.

So far we have said nothing about the E_2-term. A space X is **locally contractible** if given any point x and any open neighborhood $x \in V$, there is an open neighborhood $x \in W \subseteq V$ such that W is contractible. Given such a space X, one can build a hypercover U_* of X in which every level is a disjoint union of contractible opens. We'll call U_* a **contractible hypercover**.

If $U_* \to X$ is a hypercover there is a natural isomorphism $E_2^{p,q}(U, \mathcal{E}) \cong H^q(\mathcal{E}^{-p-q}(U_*))$. If we assume X is locally contractible and U_* is a contractible hypercover, we can simplify this further. Let $\tilde{\mathcal{E}}^q$ denote the sheafification of the presheaf $V \mapsto \mathcal{E}^q(V)$, and note that this is a locally constant sheaf on X. It follows that $\mathcal{E}^q(V) \to \tilde{\mathcal{E}}^q(V)$ is an isomorphism for every contractible V. In particular, $H^q(\mathcal{E}^{p-q}(U_*)) \to H^q(\tilde{\mathcal{E}}^{p-q}(U_*))$ is an isomorphism. But since the sheaves $\tilde{\mathcal{E}}^*$ are locally constant, it follows that if V is a contractible open set in X then $H^p_{\text{shf}}(V, \tilde{\mathcal{E}}^q) = 0$ for $p > 0$ (see [Br, II.11.12]). The fact that U_* is a contractible hypercover now shows that

$$E_2^{p,q}(U,\mathcal{E}) \cong H^q(\mathcal{E}^{p-q}(U_*)) \cong H^q(\tilde{\mathcal{E}}^{p-q}(U_*)) \cong H^q_{\mathrm{shf}}(X,\tilde{\mathcal{E}}^{-p-q})$$

(and all these isomorphisms are natural).

Remark 7.2. Let us for a moment forget about contractible hypercovers, and look at all of them. For each map of hypercovers $V_* \to U_*$ there is an induced map of spectral sequences $E_*(U,\mathcal{E}) \to E_*(V,\mathcal{E})$. Two homotopic maps $V_* \rightrightarrows U_*$ induce the same map of spectral sequences from E_2 on. So if we forget about E_1-terms, we have a diagram of spectral sequences indexed by the homotopy category of hypercovers HC_X. We can take the colimit of these spectral sequences, and the Verdier Hypercovering Theorem identifies the E_2-term with sheaf cohomology. So the hypothesis that the base X be locally contractible is not really necessary for the development of our spectral sequence. It does make things easier to think about, though, so we will continue to specialize to that case.

Now suppose that $\mathcal{E}$ is a fibrant ring spectrum, Y is another locally contractible space, and $V_* \to Y$ is a contractible hypercovering. The product simplicial space $U_* \times V_*$ is a contractible hypercover of $X \times Y$. By the discussion in section 3.9 we get a pairing of spectral sequences $E_*(U,\mathcal{E}) \otimes E_*(V,\mathcal{E}) \to E_*(|U| \times |V|,\mathcal{E})$ and a map of spectral sequences $E_*(U \times V,\mathcal{E}) \to E_*(|U| \times |V|,\mathcal{E})$. It follows from Proposition 3.10 that the latter map is an isomorphism (from the hypercover U one defines a simplicial set $\pi_0 U$ by applying $\pi_0(-)$ in every dimension, and in our case $U \to \pi_0(U)$ is a levelwise weak equivalence; the same holds for V and $U \times V$, and so we are really just dealing with simplicial sets). Therefore we get the following:

Theorem 7.3 (Multiplicativity of the descent spectral sequence)**.** *Let X and Y be locally contractible spaces, with contractible hypercovers $U_* \to X$ and $V_* \to Y$. Then there is a pairing of descent spectral sequences $E_*(U,\mathcal{E}) \otimes E_*(V,\mathcal{E}) \to E_*(U \times V,\mathcal{E})$ in which the E_2-term is globally isomorphic to*

$$H^q_{shf}(X,\mathcal{E}^{-p-q}) \otimes H^t_{shf}(Y,\mathcal{E}^{-s-t}) \to H^{q+t}_{shf}(X \times Y,\mathcal{E}^{-p-q-s-t})$$

up to a sign difference of $(-1)^{t(p+q)}$.

Proof. To identify the product on E_2-terms we use the fact that if F and G are two sheaves on X and Y, and U and V contractible hypercovers of X and Y, then the pairing of cosimplicial abelian groups $F(U_*) \otimes G(V_*) \to (F \otimes G)(U_* \times V_*)$ induces the cup product on sheaf cohomology via an Eilenberg-Zilber map and the isomorphisms $H^p(F(U_*)) \cong H^p(X,F)$, etc. Someone should write down a careful proof of this someday, but I have proven enough 'trivial' things for one paper.

The sign comes from an implicit use of the suspension isomorphism. The E_2-term of $E_*(U,\mathcal{E})$ is most naturally identified with $H^q_{\mathrm{shf}}(X,\mathcal{G}^{p,q})$ where $\mathcal{G}^{p,q}$ denotes the sheafification of $V \mapsto \mathcal{E}^{-p}(S^q \wedge V_+)$. Clearly $\mathcal{G}^{p,q}$ is isomorphic to the sheaf $\tilde{\mathcal{E}}^{-p-q}$ via the left-suspension isomorphism, but this introduces signs into the pairings much like in Theorem 4.5. We leave the details to the reader. □

The assumption in the above theorem that X and Y be locally contractible is probably not really necessary, as in Remark 7.2; but I have not worked out the details. Also, when X and Y are both locally contractible and paracompact, sheaf cohomology with locally constant coefficients is isomorphic to singular cohomology;

the pairing on sheaf cohomology corresponds to the cup product under this isomorphism [Br, Theorem III.1.1]. So the descent spectral sequence takes the same form as the Atiyah-Hirzebruch spectral sequence in this case.

Exercise 7.4. If $U_* \to X$ is a contractible hypercover, let $\pi_0(U)$ be the simplicial set defined above—obtained by replacing each U_n by its set of path components. There is a natural map $U \to \pi_0(U)$, and this is a levelwise weak equivalence. Convince yourself that the descent spectral sequence $E_*(U, \mathcal{E})$ is canonically isomorphic to the Atiyah-Hirzebruch spectral sequence for the space $|\pi_0(U)|$ based on its skeletal filtration. Conclude that everything in this section is just a restatement of the results in section 3.3.

7.5. **The Leray spectral sequence.** Now assume that $\pi\colon E \to B$ is a fibration with fiber F, and that B is locally contractible. Choose a hypercover U_* of B and consider the pullback hypercover $\pi^{-1}U_*$ of E. Once again, we know by [DI] that $|\pi^{-1}U_*| \to E$ is a weak equivalence. Taking $\mathcal{E} = H\mathbb{Z}$, the skeletal filtration of $|\pi^{-1}U_*|$ gives a spectral sequence for computing the homotopy groups of the function space $\mathcal{F}(E, H\mathbb{Z})$—i.e., it computes the singular cohomology groups of E. This is the **Leray (or Leray-Serre) spectral sequence**.

Exercise 7.6. Let $H^n\pi^{-1}$ denote the sheaf on B obtained by sheafifying $V \mapsto H^n_{\text{sing}}(\pi^{-1}V)$. Check that $H^n\pi^{-1}$ is a locally constant sheaf on B whose stalks are isomorphic to $H^n_{\text{sing}}(F)$. So if B is simply-connected, it is a constant sheaf. We will abbreviate $H^n\pi^{-1}$ as $\mathcal{H}^n(F)$.

It follows that if U is a contractible hypercover then the E_2-term of the spectral sequence is isomorphic to $H^q_{\text{shf}}(B, \mathcal{H}^{-p-q}(F))$. When B is simply connected this is $H^q_{shf}(B, H^{-p-q}F)$, and when B is paracompact this can be identified with singular cohomology.

Once again, if $E' \to B'$ is a second fiber bundle with fiber F', there is a pairing of spectral sequences which on E_2-terms has the form

$$H^q_{\text{shf}}(B, \mathcal{H}^{-p-q}F) \otimes H^t_{\text{shf}}(B', \mathcal{H}^{-s-t}F') \to H^{q+t}_{\text{shf}}(B \times B', \mathcal{H}^{-p-q-s-t}(F \times F')).$$

One has the usual sign difference from the canonical pairing, for the same reasons as in Theorem 7.3. If $E' \to B'$ is the same as $E \to B$, then we can compose with the diagonal map to get a multiplicative structure on the Leray-Serre spectral sequence for $E \to B$.

Theorem 7.7 (Multiplicativity of Leray-Serre)**.** *Let $E \to B$ and $E' \to B'$ be fibrations where B and B' are locally contractible. Then there is a pairing of Leray-Serre spectral sequences for which the E_2-term is globally isomorphic to the pairing*

$$H^q_{shf}(B, \mathcal{H}^{-p-q}F) \otimes H^t_{shf}(B', \mathcal{H}^{-s-t}F') \to H^{q+t}_{shf}(B \times B', \mathcal{H}^{-p-q-s-t}(F \times F')).$$

except for a sign difference of $(-1)^{t(p+q)}$.

Exercise 7.8. Fill in the many missing details from this section.

Exercise 7.9. Of course we didn't need to take $\mathcal{E} = H\mathbb{Z}$ in the above discussion. We could have used any ring spectrum, in which case we would obtain the combination Atiyah-Hirzebruch-Leray-Serre spectral sequence. Think through the details of this.

References

[AH] M. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Symp. Pure Math. **3**, pp. 7–38.

[BK] A. Bousfield and D. Kan, *A second quadrant homology spectral sequence*, Trans. Amer. Math. Soc. **177** (1973), 305–318.

[Br] G. Bredon, *Sheaf Theory*, Springer-Verlag New York, 1997.

[Do] A. Dold, *Lectures on algebraic topology*, Springer-Verlag, Berlin-New York, 1972.

[D1] D. Dugger, *Multiplicative structures on homotopy spectral sequences I*, 2003, preprint.

[DI] D. Dugger and D. C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$-realizations*, to appear in Math. Zeit.

[GM] J. Greenlees and J. P. May, *Tate cohomology*, Memoirs Amer. Math. Soc. **113** (1995), no. 543.

[K] R. Kultze, *Über multiplicative Eigenschaften von spektralen Sequenzen*, Math. Ann. **158** (1965), 233–268.

[Mc] J. McCleary, *A user's guide to spectral sequences*, Cambridge University Press, 2001.

[SS] S. Schwede and B. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511.

[Sp] E.H. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1966.

[V] V. V. Vershinin, *Multiplication in spectral sequences*, Trudy. Inst. Mat. (Novosibirsk) **7** (1987), Issled. Geom. Mat. Anal. 61–70, 199.

[Wh] G.W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.

Department of Mathematics, University of Oregon, Eugene, OR 97403
Email address: ddugger@math.uoregon.edu