Expectations, Stagnation and Fiscal Policy*

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Abstract:

Persistent stagnation and fiscal policy are examined in a New Keynesian model with adaptive learning determining expectations. We impose inflation and consumption lower bounds, which can be relevant when agents are pessimistic. The inflation target is locally stable under learning. Pessimistic initial expectations may sink the economy into steady-state stagnation with deflation. The deflation rate can be near zero for discount factors near one or if credit frictions are present. Following a severe pessimistic expectations shock a large temporary fiscal stimulus is needed to avoid or emerge from stagnation. A modest stimulus is sufficient if implemented early.

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I. Introduction

The sluggish macroeconomic performance of advanced market economies in the seven years after the Great Recession has raised interest in the possibility of the economy becoming stuck for long periods in a distinct stagnation state and that this stagnation might be associated with the zero lower bound (ZLB) for the policy interest rate.[1] One possible explanation for the stagnation state is that it is caused by a wide-spread lack of confidence on the part of economic agents. Specifically, a stagnation state with low output, deflation and interest rates constrained by the ZLB may be a self-fulfilling equilibrium of the economy. We develop an extension of a standard new Keynesian (NK) model to account for existence of a stagnation steady state. Our analysis assumes that economic agents make forecasts using adaptive learning (AL) and we impose the requirement that the stagnation steady state be (locally) stable under adaptive learning. Existence of a stagnation steady state is consistent with the observation that under the ZLB constraint, real economic performance of the US, Japanese and the euro area economies appears to be clearly worse than in the earlier period before the ZLB became binding.

Within the context of the standard NK model and rational expectations (RE), the implications of the ZLB have been approached from several angles. First, there is the possibility of exogenous shocks to demand that push the economy to the ZLB. Exogenous discount rate or, more plausibly, credit-spread shocks have been emphasized by Eggertsson and Woodford (2003), Christiano, Eichenbaum, and Rebelo (2011), Corsetti, Kuester, Meier, and Muller (2010) and Woodford (2011). These shocks are often assumed to follow a two-state Markov process in which the credit-spread shock disappears each period with a fixed probability, with aggregate output and inflation recovering as soon as the exogenous shock stops operating.

While this approach has been fruitful in suggesting suitable monetary and fiscal policy responses to such shocks, it has several somewhat unattractive features. It relies heavily on the persistence of a shock that evaporates according to an exogenous process, and recession ends as soon as the exogenous negative shock ends.[2] Furthermore, this approach does not do justice to an independent role for expectations.

A second approach, emphasized by Benhabib, Schmitt-Grohe, and Uribe (2001b), focuses squarely on the existence of multiple rational expectations equilibria (REE) when the interest-rate rule is subject to the ZLB. In particular, in addition to the intended steady state at the inflation rate targeted by monetary policy, there is a second, unintended steady state at a low inflation or modest deflation rate, as well as perfect foresight paths converging to the unintended steady state. This multiplicity was emphasized in Bullard (2010). Figure 1 gives a scatter plot of core inflation vs. the policy interest rate, as originally done in Bullard (2010) for Japan and US data and extended by Honkapohja (2016) using also euro area data.

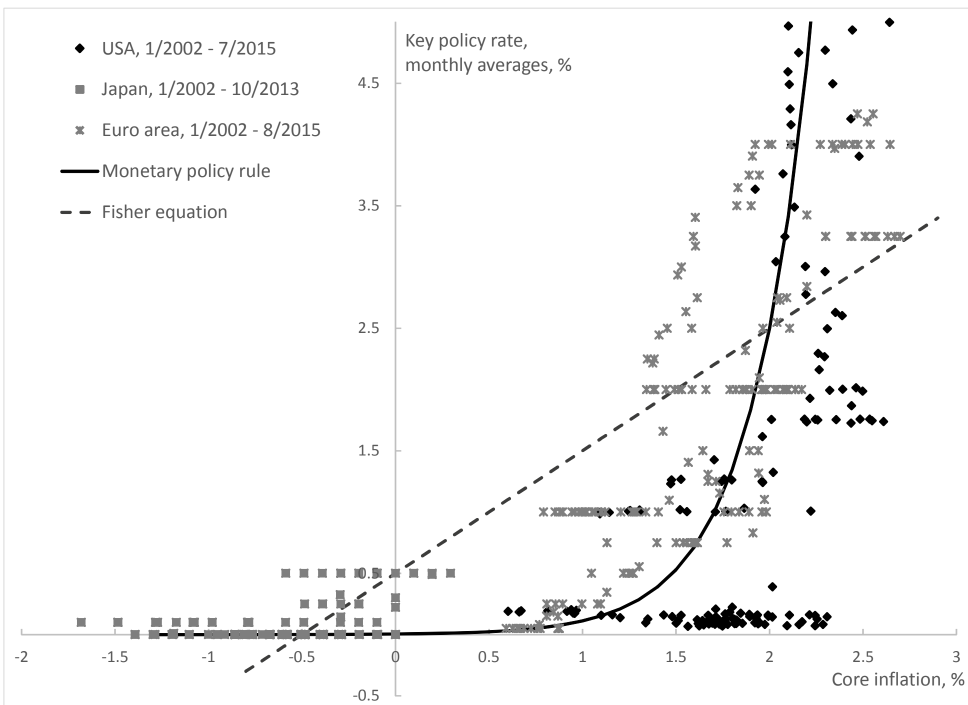

Figure 1: Interest rate vs inflation in Japan, US and euro area

Figure 1 uses monthly data, over 1/2002 to 1/2015 for euro area and US and to 10/2013 for Japan, and combines them in one figure.[3] The illustrated policy rule is drawn with a two-percent inflation target and is merely used to provide a common reference since the two percent target does not exactly match either U.S. or euro area practice. Inflation and interest rates at the two steady states in Figure 1 correspond to the two intersections of the Fisher equation and a Taylor-type interest rate rule. The Japanese data from this period is essentially entirely within the liquidity trap, while the US and euro area data show a mixture of liquidity-trap and non-liquidity trap periods. Both the US and the euro area had brief periods of deflation during 2009 and the Great Recession, followed by a period of inflation. However more recently, since 2013, inflation in both the euro area and the US has often been below target and sometimes shown signs of decline. Figure 1 thus suggests some possibility of convergence to an unintended low inflation steady state.

A major problem with this second approach is its neglect of the association of the ZLB with periods of recession, low output and stagnation. Although there is a long-run trade-off in the NK model between output and inflation, the extent of this trade-off is quite minor: at the unintended low inflation steady state the level of aggregate output is only very slightly below that of the intended steady state in Figure 1.

Figure 2, which gives real GDP per capita since 2001 for the US, Japan and the euro area, illustrates the association of depressed output levels in these countries with the ZLB. This is inconsistent with the view of two steady-states in the second approach. Taken together with Figure 1, there appears also to be the possibility of stagnation, i.e. persistently depressed levels of output, at low inflation or deflation steady states. For the US, the decrease from 2007Q4 to 2009Q2 was about 6.0%. Given an underlying trend growth in the US of real GDP per capita of 2% per year, one would have expected 3% total growth over this period, so one could argue this corresponds to a 9% GDP gap. For Japan, the decrease in GDP per capita from 1997Q1 to 1999Q1 was 3.5% and from 2008Q1 to 2009Q2 was 7.5%. For the euro area the drop in GDP per capita from 2008Q1 to 2009Q2 was 5.5%. Again, allowing for usual trend growth in GDP per capita, the resulting GDP gaps would be larger.

Another objection to the two-steady state view of recent events is that the unintended low-inflation steady state is not stable under adaptive learning. This point was emphasized in Evans, Guse, and Honkapohja (2008) and Benhabib, Evans, and Honkapohja (2014). We expand on this at length below, but the key point is that this makes it implausible that the economy will converge to the unintended steady state.

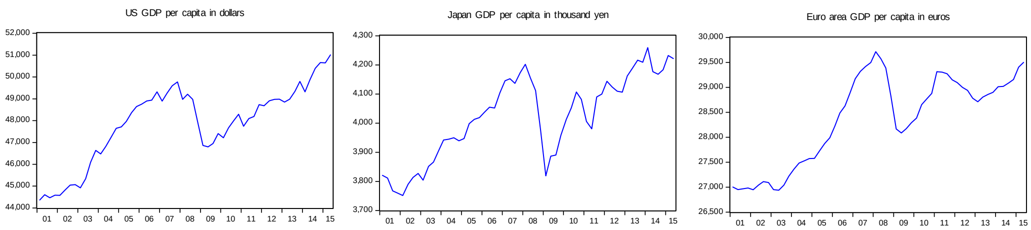

Figure 2: Real GDP per capita in local currency

A third approach relies on sunspot equilibria that can also be shown to exist when there are two steady states. A sunspot is modelled as a two-state Markov process with fixed transition probabilities. This can either be a stationary 2-state sunspot equilibrium, as in Aruoba, Cuba-Borda, and Schorfheide (2014) or a 2-state sunspot equilibrium with an absorbing state at the targeted steady state, as in Mertens and Ravn (2014). In this approach the state corresponding to deflation and recession is not due to a fundamental shock, but to a pure confidence shock.

This approach is attractive in that it gives full weight to the multiple equilibria issue. However it also has disadvantages. There is the practical question of exactly what variable is used to coordinate expectations, and there is again the issue of stability under learning. Two-state sunspot equilibria are not locally stable under learning when they are close to two steady states, one of which is not locally stable under learning as in the present case; e.g. see Evans and Honkapohja (2001), Chapter 12.

There is also an issue concerning the relatively small magnitude of recessions on this approach. The size of recessions appears to be greatest in the case of a Markov sunspot equilibrium with an absorbing state. However, even in this case the size of the recession is relatively mild: in the illustrations given in Mertens and Ravn (2014) the impact on output is -1.6%. This is a magnitude well below those in the Great Recession, which in turn were relatively small compared to the Great Depression during which substantial deflation and the ZLB was also attained; real GDP figures for the US show a 26.5% drop between 1929 and 1933.

This discussion motivates the approach that we take in the current paper. The dynamics of private-sector expectations are modeled using the AL approach rather than standard RE. RE assumes a great deal of knowledge on the part of agents and also implicitly assumes coordination of agents on those expectations. These criticisms of RE are particularly forceful when the economy is in an unusual situation, i.e. outside the usual regime of positive inflation and interest rates. In such circumstances the government may also need to consider policies outside the usual range of experience.

In Evans, Guse, and Honkapohja (2008) and Benhabib, Evans, and Honkapohja (2014) AL was introduced into the NK model with two steady states arising from the ZLB. These papers showed that while the unintended steady state is not locally stable under learning, it is on the edge of a deflation trap region in which inflation and output fall without bound. In the current paper we add lower bounds to inflation and consumption into a NK model. As discussed in Section III., we think such bounds are both plausible and more consistent with observed data. Although in normal times the inflation and consumption lower bounds are not relevant, they can play an important role during times of deep recession.

Depending on the magnitude of the inflation lower bound there are then one or three steady states. The critical level is a net deflation rate equal to the net discount rate. If the inflation lower bound is higher than this critical rate then the deflation trap region does not exist. However, if the inflation bound is below this critical rate, then there are three steady states, including a stagnation steady state at the inflation lower bound. This stagnation or "trap" steady state can have very low output accompanied by moderate deflation.

The three steady states all satisfy RE, and the model is therefore indeterminate. AL resolves the indeterminacy issue in the sense that, given initial expectations and the learning rule, the time path of the economy is pinned down. Thus AL explains how deep recessions accompanied by deflation and zero interest rates can emerge. We show that the usual targeted steady state is locally stable under learning, whereas the unintended steady state emphasized by Benhabib, Schmitt-Grohe, and Uribe (2001b) is unstable. There will either be convergence to the intended steady state or expectations will evolve toward the stagnation steady state, which is also locally stable under learning.[4]

The key point of our approach is that the low output and inflation during the period of exogenous discount rate, credit or other shocks, may have made agents generally more pessimistic about the future, and that these pessimistic expectations may well continue for a time after the exogenous shocks have ceased and the economy may have gone out of the basin of attraction of the targeted steady state.

The possibility of a stagnation steady state raises the question of whether policy can return the economy to normal levels. Can fiscal policy prevent the economy from converging to stagnation? If the economy has settled into stagnation, can fiscal policy return the economy to the targeted steady state? Earlier work has shown that the fiscal policy effects under AL can sometimes be significantly different from those based on the RE assumption.[5]

The AL approach used in the current paper is implemented as follows. We use the anticipated-utility approach advocated by Preston (2005) and Eusepi and Preston (2010), but extended for policy changes as discussed in Evans, Honkapohja, and Mitra (2009) and Mitra, Evans, and Honkapohja (2013). Agents are assumed to incorporate the announced path of future government spending and taxes into their intertemporal budget constraint, and thus take into account the known direct impact of the policy. At the same time, agents are assumed not to know the general equilibrium effects of the temporary change in fiscal policy, and to use adaptive learning to forecast future values of output and inflation. Under AL agents update each period their estimates of the coefficients in their forecast model, and the evolution of these parameters over time modulates the impact of fiscal policy under learning vis-a-vis the impacts under RE.

The structure of our paper is as follows. In Section II. we present the basic Rotemberg adjustment-cost version of the NK model with AL.

In Section III. we extend the model to include lower bounds for interest-rates, inflation and consumption. This section obtains the key existence and learning stability results for the different steady states, demonstrating in particular the possibility of a locally learnable stagnation steady state.

In Section IV. we first compare fiscal policy under RE and AL in normal times: the overall size of the output multipliers for government spending under AL and RE are about the same, but under AL the impact is front-loaded. We then provide numerical results for fiscal policy when expectations are sufficiently pessimistic that there is a high likelihood under unchanged policy of the economy converging to stagnation. We examine the impact of a temporary fiscal stimulus for a stated period of time.

In the latter situation the impact of fiscal policy is nonlinear: for a given duration, a small stimulus can fail to prevent convergence to the stagnation state, while a sufficiently large temporary stimulus can be very effective in returning the economy to the targeted steady state. The results are also stochastic, since convergence to the targeted steady state depends in part on the sequence of stochastic shocks, and the proportion of times stagnation is avoided depends on the magnitude and length of fiscal stimulus.

Section V. considers several important extensions. We consider worst-case situations in which the economy has converged to and fully adapted to a stagnation steady state. Even in this case there are fiscal policies that will return it over time to the targeted steady state. The section also discusses the connection between the discount factor and the magnitude of the deflation rate in the stagnation state and considers the implications of financial frictions. We show that with a high discount rate and financial frictions, the inflation rate in the stagnation steady state can be zero or even a low positive rate.

II. New Keynesian Model

We use a NK model following the approach developed in Eusepi and Preston (2010). The model uses a Rotemberg adjustment cost version of the pricing friction, which is convenient for solving under AL. We index households by i and firms by j, but in the temporary equilibrium dynamics that we study all households and firms will make identical decisions. We start with the households. We assume a cashless limit and that households are Ricardian. In this section we assume the ZLB on interest rates is never binding. The ZLB and other bounds are introduced in Section III..

The model incorporates random markup and productivity shocks, and our approach is to linearize the model around the targeted steady state, which is the usual procedure for local analysis. In Section III. we consider more global aspects of the economy. As the model is stochastic, we continue to use the linearized model in the analysis except in Figure 4, where the global learning dynamics incorporate quadratic adjustment costs in the market-clearing condition. Global nonlinear analysis in the fully nonlinear stochastic setup would be challenging, though approximations based on the assumption of point expectations could be used. See e.g. Benhabib, Evans, and Honkapohja (2014). Our results are consistent with that paper, so the qualitative results seem to be robust to this issue.

A. Households

The objective for agent i is to maximize expected, discounted utility subject to a standard flow budget constraint:

$$(1) \qquad Max\ \hat{E}_{0,i} \sum_{t=0}^{\infty} \beta^t U_{t,i}\left(C_{t,i}, h_{t,i}\right)$$

$$(2) \qquad \text{s.t. } C_{t,i} + b_{t,i} + \Upsilon_{t,i} = R_{t-1}\pi_t^{-1} b_{t-1,i} + Y_{t,i}.$$

Here $\hat{E}_{0,i}$ denotes the subjective expectation of i at $t = 0$, $C_{t,i}$ is the Dixit-Stiglitz consumption aggregator, $h_{t,i}$ is the labour input into production, $b_{t,i}$ denotes the real quantity of risk-free one-period nominal bonds held by the agent at the end of period t, $\Upsilon_{t,i}$ is the lump-sum tax collected by the government, R_{t-1} is the nominal interest rate factor between periods $t-1$ and t, P_t is the aggregate price level and the inflation rate is $\pi_t = P_t/P_{t-1}$. The utility function is assumed to take the parametric form $U_{t,i} = \log C_{t,i} - \gamma \frac{h_{t,i}^{1+\varepsilon}}{1+\varepsilon}$, where $\varepsilon > 0$. Household income $Y_{t,i}$ is given by $Y_{t,i} = \frac{W_t}{P_t} h_{t,i} + \Omega_t^i$, where W_t is the nominal wage and Ω_t^i denotes profits from holding shares in equal part of each firm. The subjective discount factor is denoted by $0 < \beta < 1$. The household decision problem is subject to the usual 'no Ponzi game' condition.

We assume households use a decision rule based on a linearization around the targeted steady state. Details of the solution to the household optimization problem are discussed in Online Appendix B (more details may also be found in the earlier working paper version Evans, Honkapohja, and Mitra (2016)). We assume consumers are Ricardian: they know the government runs a balanced budget policy.[6] Using the representative agent assumption $\hat{C}_{t,i} = \hat{C}_t$, $\hat{Y}_{t,i} = \hat{Y}_t$ and $\hat{E}_{t,i} = \hat{E}_t$ yields the consumption function[7]

$$\hat{C}_t = (1-\beta)\left[\sum_{s=0}^{\infty} \beta^s \hat{E}_t \left(\frac{\hat{Y}_{t+s}}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_{t+s}}{(\bar{C}/\bar{G})})\right)\right] - \hat{E}_t \sum_{s=1}^{\infty} \beta^s \hat{r}_{t+s}, \tag{3}$$

where $\hat{E}_t \hat{Y}_t = \hat{Y}_t$ and $\hat{E}_t \hat{G}_t = \hat{G}_t$ and a hatted variable indicates proportional deviations from its mean. e.g. $\hat{C}_{t,i} = \frac{C_{t,i} - \bar{C}}{\bar{C}}$ and $\hat{r}_{t+1} = \frac{r_{t+1} - \bar{r}}{\bar{r}}$.

B. Firms

The production function for each firm, producing good j, is given by $Y_{t,j} = A_t h_{t,j}^{\alpha}$, where $0 < \alpha \leq 1$ and A_t is a random productivity shock with mean $\bar{A}$. Output is differentiated and firms operate under monopolistic competition. Each firm faces the downward-sloping demand curve

$$P_{t,j} = \left(\frac{Y_{t,j}}{Y_t}\right)^{-1/\theta_t} P_t. \tag{4}$$

Here $P_{t,j}$ is the profit maximizing price set by firm j consistent with its production $Y_{t,j}$. The elasticity of substitution between two goods $\theta_t > 1$ is assumed to be a random stationary process with mean θ. Y_t is aggregate output, which is exogenous to the firm. The firms' problem is

$$\max \hat{E}_{T,j} \sum_{T=t}^{\infty} Q_{t,T} P_T \Omega_{T,j},$$

where, due to log utility, $Q_{t,T} = \beta^{T-t}\frac{P_t C_t}{P_T C_T}$, for $T \geq t$, where

$$\Omega_{t,j} = (1-\tau)\frac{P_{t,j}}{P_t}Y_{t,j} - \frac{W_t}{P_t}h_{t,j} - \frac{\psi}{2}(\frac{P_{t,j}}{P_{t-1,j}} - \pi^*)^2,$$

and where τ is the revenue tax rate to eliminate the steady state distortion in output caused by monopolistic competition. Here π^* is the (gross) rate of inflation $\pi_t = P_t/P_{t-1}$ targeted by policymakers. Thus firms view it as costly to change prices by an amount that differs from the policymaker target π^*. We interpret the quadratic term as reflecting the costs of justifying to consumers price increases at a rate higher than the target rate and the additional marketing costs of making customers aware of price increases below the target rate.[8]

We assume firms use a decision-rule for price setting based on a linearization around the targeted steady state. Online Appendix B shows how to obtain the infinite horizon linearized New Keynesian Phillips curve

$$\begin{aligned}(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t \quad &= \quad a_1\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{\pi}_{t+s} + a_2\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{Y}_{t+s} \\ &\quad + \sum_{s=0}^{\infty}(\beta\gamma_1)^s\left(-a_3\hat{E}_t\hat{A}_{t+s} - a_4\hat{E}_t\hat{G}_{t+s} + a_5\hat{E}_t\hat{\mu}_{t+s}\right). \qquad (5)\end{aligned}$$

where $\mu_t = \theta_t/(1-\theta_t)$, and coefficients $a_1, ..., a_5$ and γ_1 are defined in Online Appendix B. Interpretation of (5) is standard; see Appendix B.

C. Temporary equilibrium and learning

We can now define the temporary equilibrium which is given by the Phillips curve (5), the NK IS curve and an interest rate rule. The interest rate rule is, for example, $R_t = \beta^{-1}\left(\pi^* + \chi_\pi(\pi_t - \pi^*) + \check{\chi}_Y(Y_t - \bar{Y})\right)$, which in log-linearized form becomes

$$\hat{R}_t = \chi_\pi\hat{\pi}_t + \chi_Y\hat{Y}_t, \qquad (6)$$

where $\hat{R}_t = (R_t - \bar{R})/\bar{R}$, and $\chi_Y = \frac{\check{\chi}_Y\bar{Y}}{\pi^*}$. We also assume a government fiscal policy in which government spending is financed by lump-sum taxes.

In Appendix B it is shown that the IS equation for $\hat{Y}_t$ which combines the consumption function and the market clearing condition can be written as

$$
\begin{aligned}
\hat{Y}_t &= \bar{g}\hat{G}_t + (1-\beta)\beta^{-1}\sum_{s=1}^{\infty}\beta^s \hat{E}_t\hat{Y}_{t+s} - (1-\beta)\bar{g}\beta^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \\
&\quad -(1-\bar{g})(\hat{R}_t - \hat{E}_t\hat{\pi}_{t+1}) - (1-\bar{g})\sum_{s=2}^{\infty}\beta^{s-1}\hat{E}_t\hat{r}_{t+s}, \qquad (7)
\end{aligned}
$$

where $\bar{g} = \frac{\bar{G}}{\bar{Y}}$. This assumes the ZLB is not violated. The next section shows how to allow for cases in which the ZLB may bind.

The shocks to A_t and μ_t are assumed to follow exogenous AR(1) processes given by

$$
\hat{A}_t = \rho_A\hat{A}_{t-1} + \upsilon_{At} \text{ and } \hat{\mu}_t = \rho_\mu\hat{\mu}_{t-1} + \upsilon_{\mu t}, \qquad (8)
$$

where $0 < \rho_A, \rho_\mu < 1$ and the shocks υ_{At} and $\upsilon_{\mu t}$ are iid normal variables with zero mean and constant variances σ_A^2 and σ_μ^2.

This completes the description of the model apart from a specification of how expectations are formed. Under RE the solution technique is standard. See Online Appendix B for details. Under AL agents need to form forecasts of future inflation and output and, when a fiscal policy change occurs, of government spending and taxes. We assume that agents know the interest rate rule.

Under AL agents have perceived laws of motion (PLMs) of the same form as the RE solution of the economy under standard policy, but they update the coefficients using constant gain least-squares to allow for the indirect general-equilibrium impact of a policy change on future output and inflation. There is a stochastic steady state of the form

$$
\hat{\pi}_t = f_\pi + d_{\pi A}\hat{A}_t + d_{\pi\mu}\hat{\mu}_t \text{ and } \hat{Y}_t = f_Y + d_{YA}\hat{A}_t + d_{Y\mu}\hat{\mu}_t \qquad (9)
$$

where $\hat{A}_t, \hat{\mu}_t$ are observable processes (with known coefficients) given by (8). Under AL agents estimate the coefficients of (9) using constant-gain least squares, which discounts past data. (Online Appendix B gives details).

Given their time t estimates of the coefficients f_π, $d_{\pi A}$, $d_{\pi\mu}$, f_Y, d_{YA}, $d_{Y\mu}$ forecasts $\hat{E}_t\pi_{t+s}$ and $\hat{E}_t\hat{Y}_{t+s}$ are given by

$$\hat{E}_t\hat{\pi}_{t+s} = f_\pi + d_{\pi A}\rho_A^s\hat{A}_t + d_{\pi\mu}\rho_\mu^s\hat{\mu}_t \text{ and } \hat{E}_t\hat{Y}_{t+s} = f_Y + d_{YA}\rho_A^s\hat{A}_t + d_{Y\mu}\rho_\mu^s\hat{\mu}_t.$$

These forecasts can then be inserted into the model (5)-(7), and the infinite series summed, to determine the temporary equilibrium at time t. When there is no fiscal policy change, government spending is constant and $\hat{G}_t = \hat{E}_t\hat{G}_{t+s} = 0$ and the corresponding terms in (5)-(7) are zero.

III. Model with Lower Bounds

We now extend the temporary equilibrium framework of the model under learning to allow for the ZLB and other lower bounds. Most of the RE literature, following Eggertsson and Woodford (2003), has assumed that low inflation and output at the ZLB are triggered by exogenous preference or credit shocks that shift the RE equilibrium in such a way that the ZLB becomes a constraint.[9] Under RE the path of the economy is largely determined by these exogenous preference shocks.

In contrast, the approach followed here focuses directly on a pessimistic shock to expectations. An initial pessimistic expectations shock, under learning, has the capacity to drive the economy to low levels of output and inflation and become self-sustaining.[10] As noted in the Introduction, it is known from earlier work on AL in the NK model that there is the possibility of deflation traps that cannot be overcome by interest rate policy, due to the ZLB, and which push the economy along divergent deflationary trajectories. We think that in these circumstances other bounds may also be important, which will act to stabilize the economy along an otherwise divergent trajectory.

A. Lower bounds on R, π and C

A zero lower bound on net nominal interest rates corresponds to a bound $R_t \geq 1$. Usually, central banks prefer not to reduce net interest rates below a small positive number $\eta > 0$ and we thus impose the lower bound $R_t \geq 1 + \eta$. At the global level we also introduce two other lower bounds that will plausibly arise in extreme circumstances: an inflation lower bound $\underline{\pi}$ and a consumption lower bound $\underline{C}$.

An inflation lower bound is natural because it seems implausible that an output level slightly but persistently below the steady state level will eventually lead to deflation rates that intensify without bound. A lower bound can be motivated by downward wage rigidity or money illusion, e.g. see Akerlof and Dickens (2007) and Akerlof and Shiller (2009). This may also be empirically appealing because the extent of deflation appears bounded even at very low levels of aggregate output (see e.g. Ball and Mazumder (2011) and IMF (2013)).[11] We capture these factors through the simple device of an inflation lower bound $\underline{\pi}$, which we usually take to correspond to a modest rate of deflation.[12] The value of $\underline{\pi}$ may vary over time and across countries.

We assume $\underline{\pi} < \pi^*$, where π^* is the inflation rate targeted by monetary policy. A consumption lower bound would plausibly arise when consumption approaches the (perhaps socially determined) subsistence level. Below we assume that $\underline{C}$ is significantly below the targeted steady state. The spirit of this bound is similar to the subsistence level parameter used in Stone-Geary preferences; see, for example King and Rebelo (1993) and Ravn, Schmitt-Grohe, and Uribe (2008).[13] Although in normal times these bounds would not be apparent, they can play a role in stabilizing the economy at low levels of output at the ZLB.

We begin with a discussion of the steady states that may arise when the lower bounds may be binding. In this section it is convenient to simplify the monetary policy rule, so that the Taylor-type rule responds only to inflation. Together with the interest-rate lower bound we have

$$\begin{aligned} R_t^* &= \beta^{-1}\chi_\pi(\pi_t - \pi^*) + \beta^{-1}\pi^*, \text{ with } \chi_\pi > 1, \text{ and} \\ R_t &= \max(R_t^*, 1 + \eta). \end{aligned}$$

This parameterization is consistent with our earlier linearization $\hat{R}_t = \chi_\pi \hat{\pi}_t$ at the intended steady state. In this section for convenience we abstract from the intrinsic random productivity and mark-up shocks.

To analyze the possible non-stochastic steady states we can focus attention on the Euler equations for consumption and price setting. These will hold with equality unless constrained by the consumption or inflation lower bounds. Setting $C_t = C_{t+1} = C$ and $\pi_t = \pi$, the consumption Euler equation implies the Fisher equation

$$R/\pi = \beta^{-1}$$

unless consumption is at its lower bound. Figure 3, which shows this relationship together with the steady state interest rate rule

$$R = \max\left(\beta^{-1}\chi_\pi(\pi - \pi^*) + \beta^{-1}\pi^*, 1 + \eta\right),$$

illustrates the usual indeterminacy result that in addition to the intended steady state at $\pi = \pi^*$ there is an unintended steady state at $\pi = \pi_L \equiv \beta(1+\eta)$. Figure 3 also shows the additional stagnation steady state arising when inflation and consumption are constrained at their lower bounds.

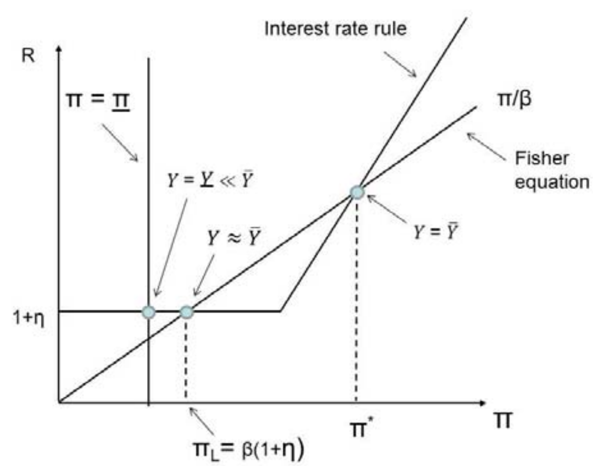

Figure 3: Existence of multiple steady states.

We assume $0 < \eta < \beta^{-1}\pi^* - 1$ and $\chi_\pi > 1$ so that $\pi_L < 1$ exists.

This multiplicity issue was analyzed in detail, under the RE assumption, in Benhabib, Schmitt-Grohe, and Uribe (2001b) and Benhabib, Schmitt-Grohe, and Uribe (2001a). Bullard (2010) gave a forceful argument that the pattern of inflation and interest rates in Japan and the US was cause for concern that the US experience might converge to a Japanese style stagnation with steady mild deflation. The remaining steady state equation is obtained from combining the price-setting Euler equation, the household static first-order condition and the mark-up equation. Online Appendix B shows these yield

$$(\pi - \pi^*)\pi(1-\beta) = \frac{\theta}{\alpha\psi}\left[C\gamma h^{1+\varepsilon} - \alpha(1-\tau)(1-\theta^{-1})Ah^{\alpha}\right]. \tag{10}$$

This is the steady-state NK Phillips curve equation, which must hold unless inflation is constrained by its lower bound. We will also need the GDP steady state accounting identity

$$Ah^{\alpha} = C + G + \frac{\psi}{2}(\pi - \pi^*)^2. \tag{11}$$

The above steady-state Phillips curve and Fisher equations hold unless inflation or consumption are constrained by their lower bounds. The inflation lower bound $\underline{\pi}$ holds if (10) would otherwise lead to an inflation rate lower than this bound, and similarly the consumption lower bound holds if otherwise we would have $\beta R > \pi$. Taking these into account, the Euler equation thus leads to the inequality

$$R/\pi \geq \beta^{-1} \text{ and } C \geq \underline{C}, \text{ with c.s.,} \tag{12}$$

which one could call the Fisher inequality, and the Phillips curve inequality

$$\begin{aligned}(\pi - \pi^*)\pi(1-\beta) &\geq \frac{\theta}{\alpha\psi}\left[C\gamma h^{1+\varepsilon} - \alpha(1-\tau)(1-\theta^{-1})Ah^{\alpha}\right] \\ \text{and } \pi &\geq \underline{\pi}, \text{ with c.s.}\end{aligned} \tag{13}$$

Here c.s. denotes that these inequalities hold with complementary slackness, i.e. if either inequality holds strictly then the other holds with equality. We can also write the interest-rate rule subject to its lower bound as

$$R \geq \beta^{-1}\chi_{\pi}(\pi - \pi^{*}) + \beta^{-1}\pi^{*} \text{ and } R \geq 1 + \eta, \text{ with c.s.} \tag{14}$$

Using the three inequalities (12), (13), (14) we can examine the possible steady states. We assume throughout that $G > 0$ and it is convenient to strengthen this slightly and assume that $G > \underline{G} > 0$ where $\underline{G}$ is specified below. In addition we assume that the consumption lower bound $\underline{C}$ is not too large, as further specified below.

B. *Existence and local stability of steady states under learning*

The number of steady states in the economy will depend critically on the inflation lower bound $\underline{\pi} < \pi^{*}$, specifically on whether $\underline{\pi} < \pi_L, \underline{\pi} = \pi_L$ or $\underline{\pi} > \pi_L$. Full analytical results are available for cases in which price adjustment costs are small. The existence results are given in the following proposition:

Proposition 1. *(i) Suppose that $\underline{\pi} < \pi_L$. For $\psi > 0$ sufficiently small, there are exactly three steady states, with $\pi \in \{\pi^{*}, \pi_L, \underline{\pi}\}$. (ii) If $\pi_L < \underline{\pi} < \pi^{*}$ then there is a unique steady state at $\pi = \pi^{*}$. (iii) If $\underline{\pi} = \pi_L$ then for $\psi > 0$ sufficiently small there is a steady state at $\pi = \pi^{*}$ and a continuum of steady states at $\pi = \pi_L$.*

Proofs of Propositions are given in Online Appendix C. We now consider the stability under AL of the steady states just described. As is well known, the local stability of an RE solution under least-squares learning, of the type outlined in Section II.C. is determined by expectational stability, or "E-stability" conditions, as discussed, for example in Evans and Honkapohja (2001). Although one could allow for the inclusion of exogenous productivity and mark-up shocks in this analysis, local stability in the current setting is governed by the intercepts of the forecast rules. We therefore simplify the theoretical stability results by assuming that the PLM for both output and inflation takes the form of an unknown constant plus a perceived white noise disturbance. Furthermore, for theoretical convenience in this section, we assume a forward-looking interest-rate rule $\hat{R}_t = \max[\chi_\pi \hat{E}_t \hat{\pi}_{t+1}, 1+\eta]$ where $\chi_\pi > 1$. (Proposition 2 also holds for the analogous contemporaneous-data rule.). We have:

Proposition 2. *If $\underline{\pi} < \pi_L$ then the steady state at π^* is locally E-stable and the steady state at $\underline{\pi}$ is locally E-stable, while the steady state at π_L is locally E-unstable for ψ sufficiently small. If $\underline{\pi} > \pi_L$ then the (unique) steady state at π^* is locally E-stable.*

For the case $\underline{\pi} < \pi_L$, Figure 4 illustrates the global E-stability dynamics that give the mean dynamics of expectations under AL, based on linearized decision rules subject to the lower-bound constraints, but incorporating the nonlinear market clearing condition (11). Here π^e is expected inflation, Y^e is expected output, E is the targeted steady state, S is the stagnation steady state and U is the unstable unintended steady state. The dashed line is the border between the basin of attractions for E and S. See Online Appendix C for a numerical example and details.

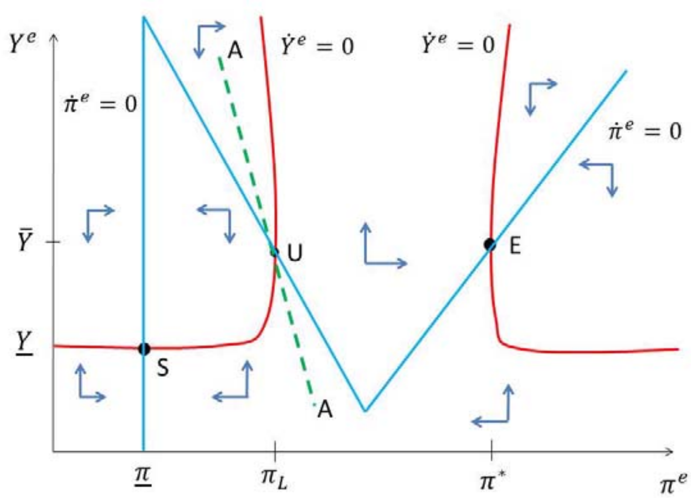

Figure 4: E-stability dynamics with forward-looking Taylor rule for the case of three steady states. π^e and Y^e are expected inflation and output.

The targeted and stagnation steady states have relatively large basins of attraction in the state space while the middle steady state creates a separating curve for these basins. The Figure shows that pessimistic expectations from negative shocks can create dynamics towards the stagnation state. Under AL, when $\underline{\pi} < \pi_L$, there is a stable stagnation steady state with deflation at $\pi = \underline{\pi}$ and a low level of output underpinned by the consumption lower bound. This possibility is potentially a major policy concern.

This theoretical analysis gives an explanation for the considerable concern among US and European policymakers about deflation and the possibility of their economies, following the financial crisis of 2007-9, becoming enmeshed in a long period of stagnation with mild deflation, similar to that experienced by Japan since the mid 1990s. This concern has been a large part of the motivation for setting and keeping policy interest rates near zero, and for innovative monetary policies like "quantitative easing" and "forward guidance." Because, in the stagnation-deflation trap, steady-state interest rates are at the ZLB, conventional monetary policy cannot move the economy back to the targeted steady state. The effectiveness of fiscal policy in this setting is then of particular interest.

In turning to an examination of fiscal policy we do not mean to suggest that monetary policy is not crucial in the face of large pessimistic shocks. For example the speed with which the policy rate is reduced can be critical. In addition, quantitative easing arising from purchases of a broad range of assets can be effective,[14] affecting a spectrum of interest rates. Finally, both forward guidance concerning future interest rates and explicit inflation targets may be important in affecting how household and firm expectations respond to observed data. We study fiscal policy in this setting primarily in order to examine its effectiveness as an alternative or supplement to unconventional monetary and financial policy when conventional monetary policy appears insufficient to guarantee avoiding convergence to stagnation.

IV. Fiscal Policy

We turn now to fiscal policy. A growing literature has been reconsidering the effects of fiscal policy in light of the relatively large fiscal stimuli adopted in various countries in the aftermath of the Great Recession. For example, Christiano, Eichenbaum, and Rebelo (2011), Corsetti, Kuester, Meier, and Muller (2010) and Woodford (2011) demonstrate the effectiveness of fiscal policy in models with monetary policy when the ZLB on the interest rate is reached. For a contrary view see Mertens and Ravn (2014). Most of this literature makes the RE assumption. The AL literature has shown that quite different results can arise both in NK and Real Business Cycle models; see Evans, Guse, and Honkapohja (2008), Benhabib, Evans, and Honkapohja (2014), Mitra, Evans, and Honkapohja (2013), Gasteiger and Zhang (2014) and Mitra, Evans, and Honkapohja (2016).

In this section we examine fiscal policy under AL, and it is convenient to study its impact first in normal times, when the economy is near the targeted steady state. We then turn to cases in which the economy would otherwise be at risk of falling into the stagnation steady state or even has already converged to the stagnation steady state.

Because we assume Ricardian households, we examine the impact of changes in the level of government purchases, and we focus on temporary increases in the level of government spending on goods and services.[15] When there is a change in fiscal policy, agents will take account of the tax effects of the announced path of policy. Given the Ricardian assumption, we can assume balanced budget increases in spending so that the path of taxes matches the path of government spending. We assume that initially, at $t = 0$, we are in the stochastic steady state corresponding to $G = \bar{G}$, and that at $t = 1$ the government announces an increase in government spending for T periods, i.e.

$$G_t = \tau_t = \begin{cases} \bar{G}', \; t = 1, ..., T \\ \bar{G}, \; t \geq T + 1. \end{cases}$$

Thus government spending and taxes are changed in period $t = 1$ and this change is reversed at a later period $T + 1$. We assume that the announcement is fully credible and the policy is implemented as announced. These assumptions could, of course, be relaxed.

Using simulations we study the impact on endogenous variables and also the distributed lag and discounted cumulative output multipliers for the fiscal policy (defined in Online Appendix E). The baseline parameters used in the simulations in both this and the following section are

$$\begin{aligned}\alpha &= 0.66, \beta = 0.99, \theta = 7.67, \tau = 0, \epsilon = 2, \gamma = 1, \psi = 128.8,\\ \bar{G} &= 0.2, \rho_A = \rho_\mu = 0.8, \sigma_A = \sigma_\mu = 0.0033.\end{aligned}$$

$A = 1.085$ is chosen so that output is approximately one; precisely $Y = 1.0081$. We interpret the parameters as corresponding to a quarterly calibration. For the inflation target we set $\pi^* = 1.005$. For quarterly data this corresponds to an annual rate of inflation of 2% which is a frequently used target for monetary policymakers. The value for ψ is based on a 15% markup of prices over marginal cost suggested in Leeper, Traum, and Walker (2011) (see their Table 2) and the price adjustment costs estimated from the average frequency of price reoptimization at intervals of 15 months (see Table 1 in Keen and Wang (2007)).

The numerical simulations in this section use the linearized system of equations given in Section II.. It would be possible to combine the linearized decision rules for consumption and price setting with the non-linear equations for market clearing, production, factor prices and labor supply, but this raises computational complexity.[16] The notation $\hat{Y}, \hat{C}, \hat{\pi}$, etc., continues to denote deviations from the targeted steady state values.

As a reference point we first briefly discuss multipliers for normal times when the ZLB does not bind. For our policy, there is not a large quantitative difference between the output multipliers under RE and AL, but they do differ in the time profile. Suppose the economy is initially in a steady state and consider a 10% increase in G for $T = 10$ periods. Under both RE and AL the maximum multiplier is about 0.8, but this is reached under RE at $t = 10$, while under AL this occurs at the start of the policy. Under AL the impact of policy is front-loaded, and is partially offset following the end of policy. Details and discussion are provided in Online Appendix E (see also figure E1 there).

We now turn to fiscal policy when the economy is at low levels of output and inflation due to pessimistic expectations following earlier large adverse shocks. In such cases the ZLB will often bind and from the earlier literature multipliers can now be expected to be large. However, when the stagnation steady state is a possible outcome without policy, there is the possibility of very large multipliers if policy can avoid stagnation.

A zero lower bound on net nominal interest rates corresponds to a bound on the gross nominal one-period interest rate $R_t \geq 1$. Recall that the steady state real interest rate factor is $\bar{r} = \beta^{-1}$. When the target inflation rate is $\pi^* \geq 1$, the steady state nominal interest-rate factor is $\bar{R} = \beta^{-1}\pi^*$. Because $\hat{R}_t = (R_t - \bar{R})/\bar{R} = (\beta R_t)/\pi^* - 1$, it follows that at the ZLB we have $\hat{R}_t = \beta/\pi^* - 1$. In practice, in our numerical simulations, much like the actual monetary policy followed in the US and the UK in the 2008 - 2015 period, we will assume net interest rates are bounded by some small value $\eta > 0$. Thus the ZLB is defined by the bound $\hat{R}_t \geq \frac{\beta(1+\eta)}{\pi^*} - 1$. We continue to assume that in normal times the interest-rate policy is given by the Taylor rule. With the ZLB added policy takes the form

$$\hat{R}_t = \max\left\{\hat{R}_t^*, \frac{\beta(1+\eta)}{\pi^*} - 1\right\}, \text{ where } \hat{R}_t^* = \chi_\pi \hat{\pi}_t + \chi_Y \hat{Y}_t.$$

The New Keynesian Phillips curve (5) is unaffected because it does not depend on the interest rate. However, the New Keynesian IS equation (7) is altered because expected future interest rates and the current interest rate may be subject to the ZLB.

Whether agents expect the ZLB to bind in the future depends both on the fundamental shocks and the beliefs of agents as measured by their estimates of the parameters of the PLM. There are actually four cases to consider depending on whether the ZLB is expected to bind in all future periods, no future periods, after a finite number of periods or up to some date. These cases are discussed in Online Appendix D. The condition determining the applicable case depends in part on the parameters ρ_A and ρ_μ, and for analytical convenience we restrict attention to the case $\rho_A = \rho_\mu$. For each case we must also allow for the possibility of the ZLB binding in the temporary equilibrium.

Inflation and consumption are also subject to lower bounds, and the modified IS and Phillips curve equations are described in Sections C and

D of the Online Appendix. We will take $\underline{\pi}$ to correspond to a modest rate of deflation. The lower bounds $\underline{C}$ and $\underline{\pi}$ come into play when expectations are very pessimistic and can become binding in the stagnation trap regions. We now use simulations to study the possible paths of the economy, under AL, that can arise from a pessimistic expectation shock, and examine the potential role for fiscal policy to prevent stagnation or ameliorate bad outcomes. We emphasize that these simulation results are designed to be illustrative, i.e. to exhibit the range of possible results that can be obtained in our model. Using the model to fit actual historical episodes is reserved for future research.

The impact of fiscal policy will depend sensitively on the values of $\underline{\pi}$ and $\underline{C}$ and the nonstochastic component of $\hat{\pi}^e(0) = f_\pi(0)$ and $\hat{Y}^e(0) = f_Y(0)$. There are cases in which without policy the economy will converge to a stagnation steady state rather than to the targeted steady state. If initial expectations are close to the edge of the stagnation trap region, fiscal policy may be able to shift the path to the targeted steady state. In cases involving possible convergence to the stagnation regime, the impact of fiscal policy may depend critically on the size and length of fiscal policy.

Before turning to simulations, recall that there are three possible steady states when the inflation lower bound $\underline{\pi}$ is below $\pi_L = \beta(1+\eta)$. In proportional deviation form this corresponds to $\hat{\underline{\pi}} < \hat{\pi}_L$ where $\hat{\underline{\pi}} = \underline{\pi}/\pi^* - 1$ and $\hat{\pi}_L = \beta(1+\eta)/\pi^* - 1$. The first steady state is the targeted steady state at $\pi = \pi^*$ i.e. at $\hat{\pi} = 0$. There is a second steady state at $\hat{\pi} = \hat{\pi}_L$ with $\hat{C}_L > \underline{\hat{C}}$. This steady state, however, is unstable under learning. Finally, there is the stagnation steady state at $\hat{\pi} = \hat{\underline{\pi}}$ and $\hat{C} = \underline{\hat{C}}$. If instead $\hat{\underline{\pi}} > \hat{\pi}_L$ then the usual targeted steady state is the unique steady state, and if $\hat{\underline{\pi}} = \hat{\pi}_L$ there will also be a continuum of steady states at $\hat{\pi} = \hat{\underline{\pi}}$ with $\hat{C} > \underline{\hat{C}}$.

For our parameterization with $\beta = 0.99$ and $\eta = 0.0001$, the critical value π_L for the inflation bound is approximately -0.0099. This is a deflation rate of 0.99% per period, which we take to be a quarter, i.e. just under 4% per annum. In our simulations below we set $\hat{\underline{\pi}} = -0.017$ or $\hat{\underline{\pi}} = -0.01475$. The inflation lower bound $\hat{\underline{\pi}} = -0.017$, which corresponds to about -1.21% per quarter, leads to three steady states, while the lower bound $\underline{\hat{\pi}} = -0.01475$, corresponding to about -0.98% per quarter, leads to a unique steady state.

For the consumption lower bound we set $\underline{\hat{C}} = -0.3$, which corresponds to a 30% reduction in consumption from the targeted steady state and which in turn corresponds to a drop of roughly 24% in aggregate output. Thus the level of output in the stagnation state corresponds roughly to the output drop in the Great Depression in the United States in the 1930s.[17] This is a fairly extreme assumption, and it would straightforward to examine calibrations consistent with stagnation steady states more in line with the Great Recession. We choose the setting $\underline{\hat{C}} = -0.3$ in order to consider the effectiveness of fiscal policy even in extreme cases in which the economy has settled into a persistent stagnation with output far below normal levels.[18] We set the gain parameter of agents at $\kappa = 0.10$. This relatively high value reflects the fact that we consider expectations that are sometimes far from rational values. In these circumstances agents have an incentive to adjust expectations relatively quickly to eliminate systematic forecast errors.[19]

To study the effectiveness of fiscal policy we choose initial expectations sufficiently pessimistic so that without policy change stagnation is likely: $f_\pi = -0.0148$ and $f_Y = -0.015$. With an annual inflation target of $\pi^* = 2\%$, and since steady state output is approximately $\bar{Y} = 1.0081$, these values for f_π and f_Y correspond to initial inflation expectations of just under -1.0% per quarter (an annual rate of -3.9%) and output expectations 1.5% below the level of the targeted steady state (assuming the exogenous shocks are at their mean values). This setting can be thought of as follows. At the beginning of time $t = 0$ the economy suffers a pessimistic expectation shock, which resets mean expectations to levels below the targeted steady state, specifically $\hat{\pi}^e = -0.0148$ and $\hat{Y}^e = -0.015$. The $t = 0$ values of inflation and output are also at these same pessimistic values. We then contrast the evolution of the economy under learning when fiscal policy is unchanged with the evolution of the economy under learning when at $t = 1$ a temporary fiscal stimulus is initiated of known duration. One natural interpretation of the pessimistic expectations at $t = 0$ is that they arose from the impact of recent adverse discount rate or credit friction shocks that had dissipated by $t = 1$.[20]

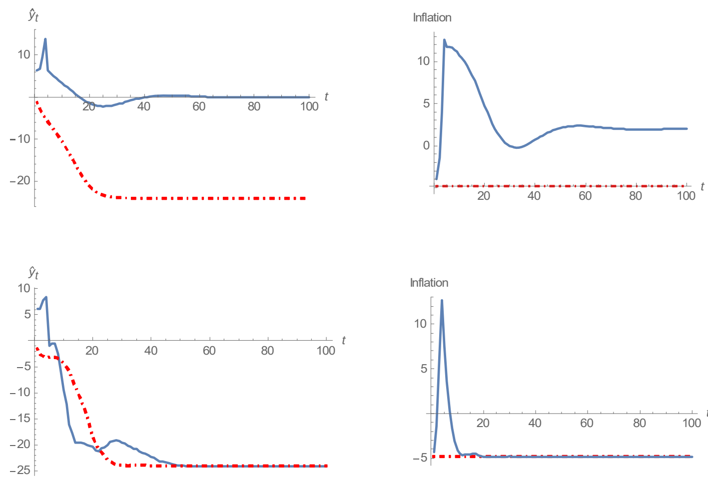

Figure 5: Large policy change. Paths for output gap $\hat{y}$ (in percent) and inflation (annualized percent rate) under learning with policy (solid line) and without policy (dashed line). Top: means of paths with convergence to targeted steady state under policy. Lower: means of paths with convergence to stagnation trap despite policy.

Without fiscal policy these initial expectations are sufficiently pessimistic so that inflation at $t = 1$ falls immediately to the lower bound $\underline{\hat{\pi}} = -0.017$. This is accompanied by small reductions in consumption and output, and the interest rate falls to the ZLB. Because of the ZLB, and with inflation at its (negative) lower bound, current and expected future real interest rates are positive and approximately equal to the deflation rate. Using the temporary equilibrium consumption function and market clearing equations, with inflation and expected inflation at the lower bound, it is shown at the end of Online Appendix D that $\hat{Y}_t = f_{Y,t} - \Delta$, where $\Delta = \frac{(1-\bar{g})(\pi_L - \underline{\pi})}{(1-\beta)\pi^*} > 0$, and $\hat{C}_t = (1-\bar{g})^{-1}\hat{Y}_t$. Thus at each time t output is lower than expected output resulting in expected output falling over time. More specifically, it can be shown that the RLS updating equation for f_Y in Section II.C. can be approximated by $f_{Y,t+1} = f_{Y,t} + \kappa\left(\hat{Y}_t - f_{Y,t}\right) = f_{Y,t} - \kappa\Delta$, with also $d_{YA,t}, d_{Y\mu,t} \to 0$. It follows that $\hat{E}_t Y_{t+s} = f_{Y,t}, \hat{Y}_t$ and $\hat{C}_t$ will steadily fall over time until $\hat{C}_t$ is constrained by the consumption lower bound, at which point the economy reaches and stays in the stagnation steady state. With no fiscal policy change the initial pessimistic expectations lead the economy into a deflation trap at $\underline{\pi}$ with output approximately 24% below the targeted steady state.

As shown in Online Appendix E, Figure E2, a small fiscal stimulus fails to prevent stagnation. In contrast, if G is changed by a sufficiently large amount the economy can be shifted to the targeted steady state. In Figure 5 we consider a policy that increases government spending from $G = 0.2$ to $G = 0.28$ for $T = 4$ periods. Output and inflation increase monotonically during the period of the stimulus, $t = 1, \ldots, 4$. In approximately 99.61% of the simulations there is then convergence to the intended steady state under this fiscal policy. Figure 5 illustrates the results based on $10,000$ simulations. The top panel shows the mean paths of output and inflation for the paths for which fiscal policy is effective in the sense that it leads to convergence to the intended steady state. The lower panel shows that in the relatively few simulations (approximately 0.39%) in which the economy fails to avoid convergence to stagnation, fiscal policy still has substantial positive effects on output. In sharp contrast, without policy change, *all* simulations converge to the stagnation state. As seen in the Online Appendix, Figure E3, the multipliers are very large.

The intuition for the results in which there is convergence to the intended steady state under policy is as follows. In period $t = 1$ pessimistic expectations for inflation and output are predetermined. The increase in G has a large effect on output because there is only a small crowding out effect on consumption. Although there is deflation and interest rates are near the ZLB, the high level of output in period $t = 1$ increases inflation above its expected level. Consequently in the following period inflation and output expectations are both revised upward. The higher inflation and output expectations and continued high G lead in period $t = 2$ to even higher output and to inflation close to the target level. Beginning in period $t = 3$, inflation has risen above target. The high output and inflation levels in period $t = 3, 4$, result in inflation and output expectations being large enough so that when the government stimulus is removed in $t = 5$, output falls back close to normal levels, though inflation remains above target for a sustained period of time. Because expectations of inflation are above target levels, it still takes significant time for the economy to converge to the targeted steady state, but expectations are now within the basin of attraction of the targeted steady state and there is asymptotic convergence to the target.

In summary, temporary increases in G are effective in raising output. Small temporary increases in G lead only to temporary increases in Y, but large temporary increases in G can shift the economy back to the targeted steady state resulting in a permanent increase in output. It is important to note that whether or not the fiscal policy is successful in pushing the economy back to the targeted steady state depends in part on the sequence of stochastic shocks hitting the economy over time.

Table 1 shows the results for the same initial expectations and for alternative values of G and T. Table 1 shows the probability that the fiscal stimulus results in eventual convergence to the targeted steady state.[21] It is not surprising that many entries in Table 1 are neither 100 or 0. Starting from the given initial pessimistic expectations, the sequence of serially correlated random productivity and mark-up shocks affect realized inflation and output over time, which under learning affects the expected inflation and output.[22] For a fiscal policy that is usually successful, a particularly unfavorable sequence of shocks can adversely affect expectations enough to prevent the policy from working. Similarly, under a fiscal policy that will normally be unsuccessful, a particularly favorable sequence of shocks can in some cases be sufficient to lead to convergence to the targeted steady state. Table 1 shows, however, that for a substantial range of policies, in particular for G between 0.27 and 0.40 with T between 2 and 5 quarters, a fiscal stimulus is successful at least 99% of the time. In these cases the cumulative multipliers are very large, reflecting the fact that the policies prevent the economy from descending into stagnation and push it back permanently (or almost permanently) to the targeted steady state, even though the fiscal stimulus is temporary.

$G\backslash T$	1	2	3	4	5	10	20	40
0.22	0	0	0	0	0	0	0	0
0.24	2	12	16	22	21	13	3	0
0.27	88	100	100	100	99	86	99	10
0.28	98	100	100	100	100	96	70	14
0.30	100	100	100	100	100	100	100	39
0.35	100	100	100	100	100	100	97	99
0.40	100	100	100	100	100	98	92	100
0.50	100	100	100	99	98	86	39	49
0.60	100	100	100	99	95	59	21	2
1.0	100	98	89	56	42	9	2	2

Table 1: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from pessimistic expectations.

It can also be seen that in many cases a fiscal stimulus that is too long can be counterproductive. For example, for $G = 0.28$ the effectiveness of the stimulus decreases if T is increased to $T = 10$ quarters or longer. This is a reflection of the negative effect on consumption of the tax burden associated with higher government spending, which we assume is correctly foreseen by households. In particular, in the first period when a fiscal policy of a given magnitude ΔG, for T periods, is initiated, the impact on aggregate output is largest if $T = 1$. In this case the present value of the tax burden is simply ΔG and the direct impact of this on consumption is $-(1-\beta)\Delta G$, which is very small compared to the increase in aggregate demand for output from government spending ΔG. For larger T the present value of the tax burden is larger; consequently the reduction in consumption in the initial period is greater, leading to a smaller initial increase in aggregate output and inflation. Against this, of course, a larger T means that the increase in demand continues for a longer period of time, which means under learning that expectations will adjust to a greater extent to the higher values of output and inflation realized during the policy. These offsetting factors account for the complicated patterns seen in Table 1.

It is also of interest to investigate fiscal policy for cases of initial pessimistic expectations when $\underline{\hat{\pi}}$ is high enough that there is a unique steady state. Figure E4 in the Online Appendix gives results for the case in which $\hat{\pi}$ is just above the level needed to avoid the low-level trap. Although without the fiscal stimulus there is no longer the possibility of convergence to a stagnation steady state, there would still be a deep recession. The depth and severity of the recession in this case can be greatly reduced by a large fiscal stimulus: multipliers are large, reaching levels over 3, with a long-run cumulative multiplier of over 5. Thus even in cases in which there is a unique steady state, fiscal policy can be important when there is a sufficiently large pessimistic expectations shock that drives the economy into recession and deflation and monetary policy to the ZLB.

V. Further Results and Discussion

Our results raise two key questions. In the preceding section we looked at the effectiveness of fiscal policy when expectations were subject to a pessimistic shock that would lead to convergence to the stagnation trap equilibrium in the absence of fiscal policy. Suppose, however, that fiscal policy is not contemplated until the economy has already converged to the trap. Can a fiscal stimulus still be effective is extracting the economy from the stagnation trap and returning it to the targeted steady state? The second question we consider is the size of the critical deflation rate below which a stagnation trap exists. Under our calibration this corresponds to an annual deflation rate of about 4% per year. Are there circumstances in which milder deflation can result in a stagnation trap?

A. Escape from stagnation

Suppose now that the economy has been allowed to converge to the stagnation steady state as a result of a large pessimistic shock. This is clearly a worst-case setting since we are assuming that expectations have fully adapted to the stagnation steady state. To examine this question we use numerical simulations, with the calibration of the previous section, but now set the intercepts of the forecast rules so that mean forecasts correspond to mean inflation and output rates at the stagnation steady states. Table 2 gives the results for 100 simulations. As in Table 1 we consider combinations of increased government spending levels G and policy length T.

$G\backslash T$	1	2	3	4	5	6	7	8	12	16	20
0.4	0	0	0	0	0	0	0	0	0	0	0
0.7	0	0	0	1	6	19	47	60	82	60	37
0.8	0	0	1	13	53	77	86	89	63	33	25
0.9	0	1	29	68	85	90	88	81	36	17	11
1.0	0	14	66	87	90	91	80	60	20	9	5
1.2	0	73	91	88	79	61	44	31	4	4	3
1.5	58	89	84	62	33	22	11	9	1	2	1
1.7	76	88	72	32	16	10	6	6	2	1	0
2	83	66	30	13	6	5	4	2	3	0	0

Table 2: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from stagnation expectations.

From Table 2 it can be seen that a fiscal stimulus can be successful in extracting the economy from the stagnation trap even if expectations have settled into levels consistent with the trap. However, the size of the stimulus is now very large – much larger than was required in Table 1, when expectations were less pessimistic – and it has a lower chance of full success. The policies with the highest probability of success, between 89% and 91%, are fiscal expansions that are both relatively short and big, e.g. a six-quarter stimulus at $G = 1.0$, a five-fold increase in G. A less huge, but still very large, stimulus of $G = 0.7$ for twelve quarters, has a 82% chance of success. Table 2 shows a general trade-off between magnitude and duration, with some intriguing nonlinearities that reflect the balance of factors discussed earlier in connection with Table 1.[23] Of course our numerical results will be sensitive to the parameterization used. For example, a smaller stimulus might have a higher chance of success if the consumption lower bound corresponds to a less drastically reduced level of output in the stagnation steady state.

While we find from these results that a sufficiently large stimulus of appropriate duration can have a high probability of extracting the economy from a stagnation trap steady state, an equally important conclusion is that a higher probability of avoiding the stagnation trap can be achieved, with a much smaller stimulus, if the policy is implemented earlier, when expectations are less pessimistic. Following a large adverse shock to expectations, in which there is risk of the economy descending into the stagnation trap, a fiscal stimulus should be implemented as early as possible.

B. Credit frictions and calibration of discount factor

The results of the previous section emphasize the importance of the level of the inflation lower bound for the existence of a stagnation steady state. This brings up two delicate but important issues. The first is appropriate calibration of the discount factor β. Our numerical results have used the quarterly calibration of $\beta = 0.99$. While this is fairly standard, there are good reasons to consider alternative, higher, values. The historical average realized net real interest rate on US Treasuries bills is not more than 1% per annum. In an economy without growth this corresponds to a discount factor of about $\beta = 0.9975$. In Online Appendix E, Figure E5, we provide simulation results for $\beta = 0.9975$.[24] With pessimistic initial expectations we again get a very high likelihood of convergence to stagnation, but now, in the stagnation state, there is deflation at around 1.4% per annum. Again a large fiscal stimulus can avoid stagnation in a very high proportion of cases.

A second factor that can lead to a higher level of the critical inflation rate is the existence of credit frictions. Various models have been proposed that generate a spread between different interest rates on loans. A prominent example within a NK setting is described in Curdia and Woodford (2010) and developed at length in Curdia and Woodford (2015). Their framework posits a heterogenous agents set-up with two types of household, at any given time, experiencing different realizations of taste shocks. This leads to lending from agents who are currently more patient to those who are currently more impatient. Frictions in the financial intermediation sector result in a borrowing rate above the lending rate.

Embedding a heterogeneous agents framework into our model is beyond the scope of the current paper. However, it is natural to incorporate a shortcut, motivated by Woodford (2011), which is to assume that the market interest rate relevant in household Euler equations for the "intertemporal allocation of expenditure is not the same as the central bank's policy rate" (p. 16). Woodford (2011) and Curdia and Woodford (2015) focus on the implications of the time variation in this spread, while for our purposes the key implication is a positive steady state spread $\varphi = R - i > 0$, where i is the policy rate and R is the interest rate relevant for household decision-making. The benchmark calibration in Curdia and Woodford (2015) corresponds to a value $\varphi = 0.0025$, i.e. to 1% per annum.

Online Appendix G shows that incorporating the spread φ and the policy rate ZLB into a Taylor rule, with inflation target π^*, leads to the market interest rate equation $R = \max\left(\beta^{-1}\chi_\pi(\pi - \pi^*) + \beta^{-1}\pi^*, 1 + \eta'\right)$, where $\eta' = \eta + \varphi$ and η is the policy rate ZLB which we set near zero. To capture the impact of a credit friction $\varphi > 0$ we can thus simply replace η by $\eta' = \eta + \varphi$. A higher spread φ increases $\pi_L = \beta(1 + \eta')$. This has a number of implications, one of which is particularly relevant for policy: if $\beta(1 + \eta') > 1$ then it is possible to have $1 < \underline{\pi} < \pi_L < \pi^*$, so that the stagnation steady state has a zero or low positive inflation rate.

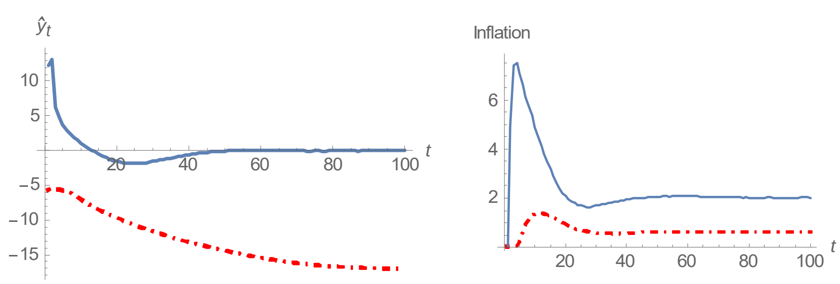

Figure 6: High β and credit spread. Paths under learning with policy change (solid line) and without policy change (dashed line). Means of paths with eventual convergence to targeted steady state under policy.

Figure 6 gives simulation results using the higher discount factor of $\beta = 0.9975$ and adding a credit friction corresponding to 1.08% annually (very slightly above the the benchmark rate used in Curdia and Woodford (2015)). Here again $\hat{y}$ denotes the output gap in percent and inflation is at annual rates. For these simulations we set $\hat{\underline{\pi}} = -0.0049$, somewhat below the critical value, so that there are three steady states, including a low-level stagnation trap. This value of $\hat{\underline{\pi}}$ corresponds to $\underline{\pi} = 1.000755$, yielding an inflation lower bound 0.03%, i.e. slightly above zero. Initial expectations following a presumed large pessimistic shock are set at $\hat{\pi}^e = -0.0048$ (expected inflation rate just under 0.1% per annum) and $\hat{Y}^e = -0.05$. For these initial expectations, there is a high likelihood, without a fiscal stimulus, of converging to the stagnation steady state.

We consider fiscal policies that increase G from 0.20 to 0.38 for $T = 2$ periods, i.e. a very large, but short, fiscal stimulus. Based on $10,000$ simulations, almost 86% of the with-policy simulations converged to the intended steady state, 11% eventually sank to the stagnation steady state, while approximately 3% had not yet converged. In sharp contrast, without policy change, 73.66% of the simulations sink to the stagnation steady state, 25.33% converge to the targeted steady state while the remaining 1.1% had not yet converged.[25] Figure 6 shows the mean paths of the 86% of simulations that under the policy converge to the intended steady state.[26] Online Appendix G, Figure E6, discusses the remaining paths and also the multipliers for the case of Figure 6.

In summary, credit frictions increase the critical inflation rate. This indicates an additional reason for concern if inflation and inflation expectations are persistently below the central bank target, especially if they are low and falling. Such circumstances raise the possibility of a path to stagnation and the potential need for aggressive macroeconomic policy.

VI. Conclusions

Sluggish real economic performance at a long-lasting ZLB has made the possibility of secular stagnation a prominent topic of discussion. Our first objective in this paper was to extend a standard NK model in a way that makes stagnation at the ZLB, with a low level of aggregate output, a possible steady state outcome for the economy. The model can have three steady states, with stagnation arising when economic agents have pessimistic expectations concerning future inflation and aggregate output. We show that both the targeted and stagnation steady states are locally stable under AL (adaptive learning), while the third steady state is unstable under learning.

A second objective of our paper has been to consider the impact of fiscal policy. Under AL agents take account of the direct effects of the announced policy, but use learning rules to forecast future values of inflation and aggregate output. A large pessimistic expectation shock (due, say, to a recent financial crisis) can push the economy to the ZLB and along a path to steady state stagnation and deflation. In this setting a fiscal stimulus can be particularly potent. A sufficiently large temporary increase in government spending can increase output and inflation enough to prevent the economy being pulled into deflation and stagnation. The chances of policy success are significantly greater if the policy is implemented early, before expectations deteriorate greatly. However, even if expectations have adapted to the stagnation trap, a large temporary fiscal stimulus can dislodge the economy from a stagnation trap.

The existence of a stagnation steady state arises if the inflation lower bound is below a critical rate. If the discount factor is near one and there are significant credit frictions then the critical rate can be high enough so that the stagnation state has zero or even positive inflation. Thus positive but low and declining inflation and inflation expectations raise the possibility of the economy entering a stagnation trap. The speed with which the economy returns to the targeted steady state can depend on the size and duration of a fiscal stimulus and whether the stimulus is implemented early or later.

From these observations it can be seen that the framework of this paper can encompass a wide range of outcomes arising from a large pessimistic shock to expectations. Using this framework to explain recent (and future) events for the different major economies in the wake of the 2007-9 financial crisis is reserved for future research.[27]

Notes

[1]For different arguments and explanations for long-lasting stagnation see, for example, Summers (2013), Teulings and Baldwin (2014), Eggertsson and Mehrotra (2014) and Benigno and Fornaro (2015).

[2]There is also an issue with existence of a rational expectations solution when the probability of the shock ending is too small. A related issue for calibrated models is the length of time that Japan has been at the ZLB.

[3]See Online Appendix A for details of data used in Figures 1 and 2.

[4]Our stability results bear some similarities to other macroeconomic learning models with multiple steady states, e.g. Marcet and Nicolini (2003) and Evans, Honkapohja, and Romer (1998). The former examines policies to avoid hyperinflation in seigniorage models of inflation. The latter demonstrates the possibilitity of self-fulfilling cycles between high and low growth rates.

[5]The Great Recession and the ZLB have led to renewed interest in fiscal policy and a fairly voluminous recent literature; see, e.g., Ramey (2011), Leeper, Traum, and Walker (2011) and Coenen et al. (2012).

[6]Thus agents immediately know the full tax impact of changes in fiscal policy. One could instead assume agents forecast future taxes using AL; in an RBC model Mitra, Evans, and Honkapohja (2016) found this slightly strengthened fiscal multipliers.

[7]We are assuming that households forecast their own future incomes when making consumption decisions, as in Eusepi and Preston (2010) and Eusepi and Preston (2012). An advantage of our approach is that it yields a consumption function close to traditional formulations based on the permanent income and life-cycle models. An alternative approach is that agents forecast wage rates and their share of profits. For this approach see Online Appendix F.

[8]Benhabib, Schmitt-Grohe, and Uribe (2001b) and Benhabib and Eusepi (2005) use this formulation in the context of the utility loss of household-firms. Eusepi and Preston (2010) also use this formulation with $\pi^* = 1$.

[9]See Christiano, Eichenbaum, and Rebelo (2011) and Woodford (2011). On this approach global indeterminacy isssues are often not addressed.

[10]Arias, Erceg, and Trabandt (2016) and Milani (2011) consider expectation shocks in other settings.

[11]An alternative empirical explanation in the Great Recession, e.g. Coibion and Gorodnichenko (2015), is that inflation expectations stayed above actual experience. We want to allow for expected inflation to adapt to observed inflation in the long run. For the Great Depression, Eggertsson (2008) argues that New Deal policies generated positive wage and price inflation wedges.

[12]One can justify $\underline{\pi}$ formally by introducing an asymmetry into the inflation adjustment cost term: see Benhabib, Evans, and Honkapohja (2014). If nominal wage rigidity were explicitly modeled, the bound could be on wage deflation.

[13]Our procedure for incorporating the consumption lower bound differs somewhat from using Stone-Geary preferences, but is convenient given our treatment of the two other lower bounds. Because the key property is a positive lower bound to consumption, it is clear that changing to Stone-Geary preferences would yield very similar qualitative results.

[14]See Honkapohja (2016) for an example in a variant of the current model.

[15]In further work it would be of interest to introduce alternative fiscal frameworks with distortionary taxes and/or public debt.

[16]Mitra, Evans, and Honkapohja (2016) found in an RBC model that including nonlinear temporary equilibrium equations made little difference, even when the shocks were large or steady states were changed. However, the computations were 150 times slower.

[17]In our simulations we continue to use the linearized model, for the reasons given earlier, but now the relevant equations are subject to the lower bounds on the interest rate, inflation and consumption.

[18]In the US Great Depression, deflation rates of about 10% per year were observed during 1931-2, but from 1933 deflation became less severe or nonexistent. Several explanations are possible. New Deal policies were introduced specifically to limit wage and price decreases. In addition, a version of our model with a lower bound on wage inflation is consistent with temporarily high price deflation rates associated with reductions in

aggregate output that ease bottlenecks.

[19]The qualitative features are fairly robust to the value of the learning gain parameter κ, but quantitative results may be affected by κ.

[20]In order to focus on the impact of pessimistic expectations we have not explicitly included these shocks. Introducing them would clearly allow us to generate suitable initial expectations.

[21]These results are based on 100 simulations for each cell. The extreme value $G = 1.0$ and extended lengths of $T = 20$ and 40 are included only for purposes of comparison. In the Online Appendix Table E1 gives the corresponding cumulative multipliers as of $t = 40$ i.e. 10 years after the policy has been implemented.

[22]We remark that for our calibration of exogenous shocks, with expectations initially at the targeted steady state, deflation is rarely observed – less than once every two hundred periods. Increasing the variances of the shocks, or increasing their serial correlation holding the unconditional variances constant, makes the results more stochastic.

[23]There is no reason a priori to restrict the fiscal stimulus to be of fixed size over time until termination. We have not studied other time profiles.

[24]We note that Eggertsson (2010) uses a calibration of $\beta = 0.997$ in a model of the US economy during the Great Depression. During the Great Depression deflation reached 10% per year during the trough.

[25]The smaller difference between π^* and π_L, when β and φ are high, increases the importance of the sequence of random shocks in determining the asymptotic path.

[26]Note that the mean without policy path shown in Figure 6 includes some paths that do not converge to the stagnation state.

[27]As we have emphasized, our results have been obtained through an extension of the basic standard New Keynesian model; this facilitates understanding of the key forces at work. For serious applied work it would, of course, be important to incorporate many features found in more elaborate models, including investment and capital, separate wage and price frictions, habit persistence, distortionary taxes, and an explicit financial sector.

REFERENCES

Akerlof, G. A., and W. T. Dickens (2007): "Unfinished Business in the Macroeconomics of Low Inflation: A Tribute to George and Bill by Bill and George," *Brookings Papers on Economic Activity*, pp. 31–48.

Akerlof, G. A., and R. J. Shiller (2009): *Animal Spirits*. Princeton University Press, Princeton, NJ.

Arias, J. E., C. Erceg, and M. Trabandt (2016): "The Macroeconomic Risks of Undesirably Low Inflation," *European Economic Review, in press.*

Aruoba, S. B., P. Cuba-Borda, and F. Schorfheide (2014): "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries," Working paper.

Ball, L., and S. Mazumder (2011): "Inflation Dynamics and the Great Recession," *Brookings Papers on Economic Activity*, pp. 337–381.

Benhabib, J., and S. Eusepi (2005): "The Design of Monetary and Fiscal Policies: A Global Perspective," *Journal of Economic Theory*, 123, 40–73.

Benhabib, J., G. W. Evans, and S. Honkapohja (2014): "Liquidity Traps and Expectation Dynamics: Fiscal Stimulus or Fiscal Austerity?," *Journal of Economic Dynamics and Control*, 45, 220–238.

Benhabib, J., S. Schmitt-Grohe, and M. Uribe (2001a): "Monetary Policy and Multiple Equilibria," *American Economic Review*, 91, 167–186.

——— (2001b): "The Perils of Taylor Rules," *Journal of Economic Theory*, 96, 40–69.

Benigno, G., and L. Fornaro (2015): "Stagnation Traps," mimeo., London School of Economics.

Bullard, J. (2010): "Seven Faces of The Peril," *Federal Reserve Bank of St. Louis Review*, 92, 339–352.

Christiano, L., M. Eichenbaum, and S. Rebelo (2011): "When is the Government Spending Multiplier Large?," *Journal of Political Economy*, 119, 78–121.

Coenen, G., et al. (2012): "Effects of Fiscal Stimulus in Structural Models," *American Economic Journal: Macroeconomics*, 4, 22–68.

Coibion, O., and Y. Gorodnichenko (2015): "Is the Phillips Curve Alive and Well After All? Inflation Expectations and the Missing Disinflation," *American Economic Journal: Macroeconomics*, 7, 197–232.

Corsetti, G., K. Kuester, A. Meier, and G. J. Muller (2010): "Debt Consolidation and Fiscal Stabilization of Deep Recessions," *American Economic Review: Papers and Proceedings*, 100, 41–45.

Curdia, V., and M. Woodford (2010): "Credit Spreads and Monetary Policy," *Journal of Money, Credit and Banking*, 42, S3–S35.

——— (2015): "Credit Frictions and Optimal Monetary Policy," Working paper, Federal Reserve Bank of San Francisco.

Eggertsson, G. B. (2008): "Was the New Deal Contractionary?," Working paper.

——— (2010): "What Fiscal Policy is Effective at Zero Interest Rates," *NBER Macroeconomics Annual 2010*, 25, 59–112.

Eggertsson, G. B., and N. R. Mehrotra (2014): "A Model of Secular Stagnation," Working paper nr. 20574, NBER.

Eggertsson, G. B., and M. Woodford (2003): "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity*, (1), 139–233.

Eusepi, S., and B. Preston (2010): "Central Bank Communication and Expectations Stabilization," *American Economic Journal: Macroeconomics*, 2, 235–271.

——— (2012): "Debt, Policy Uncertainty and Expectations Stabilization," *Journal of the European Economic Association*, 10, 860–886.

Evans, G. W., E. Guse, and S. Honkapohja (2008): "Liquidity Traps, Learning and Stagnation," *European Economic Review*, 52, 1438–1463.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

Evans, G. W., S. Honkapohja, and K. Mitra (2009): "Anticipated Fiscal Policy and Learning," *Journal of Monetary Economics*, 56, 930–953.

——— (2016): "Expectations, Stagnation and Fiscal Policy," Discussion paper 11428, CEPR.

Evans, G. W., S. Honkapohja, and P. Romer (1998): "Growth Cycles," *American Economic Review*, 88, 495–515.

Gasteiger, E., and S. Zhang (2014): "Anticipation, Learning and Welfare: the Case of Distortionary Taxation," *Journal of Economic Dynamics and Control*, 49, 113–126.

Honkapohja, S. (2016): "Monetary Policy to Counter the Zero Interest Rate: an Overview of Research," *Empirica*, 43, 235–256.

IMF (2013): *World Economic Outlook, April 2013*. IMF Publication Services, Washington, D.C.

Keen, B., and Y. Wang (2007): "What Is a Realistic Value for Price Adjustment Costs in New Keynesian Models?," *Applied Economics Letters*, 11, 789–793.

King, R. G., and S. T. Rebelo (1993): "Transitional Dynamics and Economic Growth in the Neoclassical Model," *American Economic Review*, 83, 908–931.

Leeper, E. M., N. Traum, and T. B. Walker (2011): "Clearing Up the Fiscal Multiplier Morass," NBER working paper 17444.

Marcet, A., and J. P. Nicolini (2003): "Recurrent Hyperinflations and Learning," *American Economic Review*, 93, 1476–1498.

Mertens, K., and M. O. Ravn (2014): "Fiscal Policy in an Expectation Driven Liquidity Trap," *Review of Economic Studies*, 81, 1637–1667.

Milani, F. (2011): "Expectation Shocks and Learning as Drivers of the Business Cycle," *Economic Journal*, 121, 379–401.

Mitra, K., G. W. Evans, and S. Honkapohja (2013): "Policy Change and Learning in the RBC Model," *Journal of Economic Dynamics and Control*, 37, 1947–1971.

——— (2016): "Fiscal Policy Multipliers in an RBC Model with Learning," *Macroeconomic Dynamics,* forthcoming.

Preston, B. (2005): "Learning about Monetary Policy Rules when Long-Horizon Expectations Matter," *International Journal of Central Banking*, 1, 81–126.

Ramey, V. A. (2011): "Can Government Purchases Stimulate the Economy?," *Journal of Economic Literature*, 49, 673–685.

Ravn, Morten, O., S. Schmitt-Grohe, and M. Uribe (2008): "Macroeconomics of Subsistence Points," *Macroeconomic Dynamics* , Supplement 1, 12, 136–147.

Summers, L. (2013): "Why stagnation might prove to be the new normal," *Financial Times*.

Teulings, C., and R. Baldwin (eds.) (2014): *Secular Stagnation: Facts, Causes and Cures*. CEPR Press, London.

Woodford, M. (2011): "Simple Analytics of the Government Expenditure Multiplier," *American Economic Journal: Macroeconomics*, 3, 1–35.

Appendix for Online Publication

A Data

Figure 1: The interest-rate rule curve takes the form $\mathcal{I} = A * \exp(B\Pi)$, where Π denotes net inflation and $\mathcal{I}$ denotes the net interest rate. Japan switched the policy target in 2013 to monetary base.

Figure 2: Data are from Macrobond data base which in turn utilizes standard data sources. GDP data is volume data with 2010 as reference year and in local currency. GDP data is annualized. This was specifically done for the Euro area by multiplying quarterly data by 4. Population data is total population and it is interpolated for quarters.

B Model Details

We develop here the model outlined in Section II. following the analysis of Eusepi and Preston (2010).

Consumers

There is a static FOC for the household concerning labor-leisure choice, which is

$$\frac{W_t}{P_t} = \gamma h_{t,i}^{\varepsilon} C_{t,i}. \tag{B1}$$

To derive the linearized consumption function, we first linearize the Euler equation

$$C_{t,i}^{-1} = \beta R_t \hat{E}_{t,i} \left(\pi_{t+1}^{-1} C_{t+1,i}^{-1} \right) \tag{B2}$$

to get

$$\tilde{C}_{t,i} = \hat{E}_{t,i} \tilde{C}_{t+1,i} - \beta \bar{C} \hat{E}_{t,i} \tilde{r}_{t+1}, \tag{B3}$$

where tilde indicates deviation from the steady state, e.g. $\tilde{C}_{t,i} = C_{t,i} - \bar{C}$, and the bar denotes the deterministic steady state. Here

$$r_{t+1} \equiv \frac{R_t}{\pi_{t+1}}.$$

The next step is iterate the linearized Euler equation forward. We have

$$\tilde{C}_{t,i} = \hat{E}_{t,i}\tilde{C}_{t+s,i} - \beta\bar{C}\hat{E}_{t,i}\sum_{i=1}^{s}\tilde{r}_{t+i},$$

where $\tilde{C}_{t,i} = C_{t,i} - \bar{C}$ and $\tilde{r}_{t+i} = r_{t+i} - \bar{r}$. This can also be written as

$$\hat{C}_{t,i} = \hat{E}_{t,i}\hat{C}_{t+1,i} - \hat{E}_{t,i}\hat{r}_{t+1},$$

$$\hat{C}_{t,i} = \frac{C_{t,i} - \bar{C}}{\bar{C}} \text{ and } \hat{r}_{t+1} = \frac{r_{t+1} - \bar{r}}{\bar{r}},$$

which yields

(B4) $$\hat{C}_{t,i} = \hat{E}_{t,i}\hat{C}_{t+s,i} - \hat{E}_{t,i}\sum_{i=1}^{s}\hat{r}_{t+i},$$

where we have used $\beta^{-1} = \bar{r}$.

Next use the flow budget constraint and the NPG (no Ponzi game) condition to obtain an intertemporal budget constraint. Write

(B5) $$b_{t,i} = r_t b_{t-1,i} + \zeta_{t,i},$$

where $r_t = R_{t-1}/\pi_t$ and

$$\zeta_{t,i} = Y_{t,i} - C_{t,i} - G_{t,i}.$$

In (B5) consumers are assumed to know that the government will run a balanced budget policy. Iterate (B5) forward and impose

(B6) $$\lim_{s\to\infty}\hat{E}_{t,i}(D_{t,t+s})^{-1}b_{t+s,i} = 0,$$

where

$$D_{t,t+s} = \prod_{i=1}^{s} r_{t+i},$$

with $r_{t+s} = R_{t+s-1}/\pi_{t+s}$. We obtain the life-time budget constraint of the household

$$0 = \Delta_{t-1,i} + \zeta_{t,i} + \sum_{s=1}^{\infty} \hat{E}_t (D_{t,t+s})^{-1} \zeta_{t+s,i},$$

where $\Delta_{t-1,i} = r_t b_{t-1,i}$ and

$$\zeta_{t+s,i} = Y_{t+s,i} - C_{t+s,i} - G_{t+s,i}. \tag{B7}$$

Because there is zero net government debt and we have representative agents, it follows that $\Delta_{t-1,i} = b_{t-1,i} = 0$ for all agents. Thus

$$0 = \zeta_{t,i} + \sum_{s=1}^{\infty} \hat{E}_t (D_{t,t+s})^{-1} \zeta_{t+s,i}, \tag{B8}$$

Linearizing (B8) we have

$$\begin{aligned} 0 &= \tilde{\zeta}_{t,i} + \sum_{s=1}^{\infty} \beta^s \tilde{\zeta}_{t+s,i} - \bar{\zeta} \sum_{s=1}^{\infty} \beta^{s+1} \sum_{i=1}^{s} \tilde{r}_{t+i}, \\ \tilde{\zeta}_{t+s,i} &= \tilde{Y}_{t+s,i} - \tilde{C}_{t+s,i} - \tilde{G}_{t+s,i}, \end{aligned}$$

where tilde denotes deviation from the non-stochastic steady state for each variable, for example, $\tilde{Y}_{t+s,i} = Y_{t+s,i} - \bar{Y}$. Note that here $\bar{\zeta} = 0$ by market clearing. Thus, using also $\tilde{G}_{t+s,i} = \tilde{G}_{t+s}$ for all i, the linearized lifetime budget constraint of the household is

$$\sum_{s=0}^{\infty} \beta^s \tilde{C}_{t+s,i} = \sum_{s=0}^{\infty} \beta^s (\tilde{Y}_{t+s,i} - \tilde{G}_{t+s}). \tag{B9}$$

Here G_t is the level of government purchases, assumed exogenous, and we are assuming Ricardian households with identical taxes so that for each agent we may set $\Upsilon_t = G_t$. (For explanation of the terms Ricardian and non-Ricardian households in this context see Benhabib, Evans, and Honkapohja

(2014).)

Combining (B4) with the linearized budget constraint in expectational form we get:

$$
\begin{aligned}
0 &= \tilde{Y}_{t,i} - \tilde{C}_{t,i} - \tilde{G}_{t,i} + \sum_{s=1}^{\infty} \beta^s \hat{E}_{t,i}(\tilde{Y}_{t+s,i} - \tilde{G}_{t+s,i}) \\
&\quad - \sum_{s=1}^{\infty} \beta^s \tilde{C}_{t,i} - \beta \bar{C} \sum_{s=1}^{\infty} \beta^s \hat{E}_{t,i} \sum_{i=1}^{s} \tilde{r}_{t+i}.
\end{aligned}
$$

This yields the consumption function for consumer i

$$
\begin{aligned}
\tilde{C}_{t,i} &= (1-\beta)[\tilde{Y}_{t,i} - \tilde{G}_{t,i} + \sum_{s=1}^{\infty} \beta^s \hat{E}_{t,i}(\tilde{Y}_{t+s,i} - \tilde{G}_{t+s,i})] \\
&\quad - \beta \bar{C} \hat{E}_{t,i} \sum_{s=1}^{\infty} \beta^s \tilde{r}_{t+s}.
\end{aligned}
$$

This can also be written in proportional form yielding

$$
\begin{aligned}
\hat{C}_t &= (1-\beta)\left[\frac{\hat{Y}_t}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_t}{(\bar{C}/\bar{G})} + \sum_{s=1}^{\infty} \beta^s \hat{E}_t \left(\frac{\hat{Y}_{t+s}}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_{t+s}}{(\bar{C}/\bar{G})})\right)\right] \\
&\quad - \hat{E}_t \sum_{s=1}^{\infty} \beta^s \hat{r}_{t+s},
\end{aligned}
$$

which is (3) making use of the representative agent assumption.

Firms

The first-order condition for the firm's choice of $P_{t,j}$ is given by

$$
\begin{aligned}
\text{(B10)} \qquad 0 &= (1-\tau)(1-\theta_t)\left(\frac{P_{t,j}}{P_t}\right)^{-\theta_t} Y_t \\
&\quad + S_{t,j} Y_t \theta_t \left(\frac{P_{t,j}}{P_t}\right)^{-\theta_t - 1} - \psi \frac{P_t}{P_{t-1,j}}\left(\frac{P_{t,j}}{P_{t-1,j}} - \pi^*\right) \\
&\quad + \hat{E}_{t,j} Q_{t,t+1} \frac{P_t}{P_{t,j}} \psi \frac{P_{t+1,j}}{P_{t,j}}\left(\frac{P_{t+1,j}}{P_{t,j}} - \pi^*\right).
\end{aligned}
$$

Here we use $Q_{t,t} = 1$ and

$$\text{(B11)} \qquad S_{t,j} = \frac{W_t/P_t}{\partial Y_{t,j}/\partial h_{t,j}} = \frac{W_t/P_t}{\alpha A_t h_{t,j}^{\alpha-1}}$$

is the real marginal cost. It's useful to define the mark-up μ by

$$\text{(B12)} \qquad \mu_t = \frac{\theta_t}{\theta_t - 1}.$$

The steady state at π^* satisfies

$$(1-\tau)(1-\theta) + \bar{S}\theta = 0.$$

In the steady state, of course, $\mu = \theta(\theta-1)^{-1}$. From above steady state real marginal cost is

$$\text{(B13)} \qquad \bar{S} = \frac{(\theta-1)(1-\tau)}{\theta} = (1-\tau)\mu^{-1}.$$

The steady-state NK Phillips curve (10) is obtained from the relationship (B10), setting $P_{t,j} = P_t$, $S_{t,j} = S$, $Y_t = Y$, $h_t = h, \theta_t = \theta, P_{t+1}/P_t = \pi, A_t = A$ and $Q_{t,t+1} = \beta/\pi$. This gives

$$0 = (1-\tau)(1-\theta)Ah^\alpha + SAh^\alpha\theta - \psi\pi(\pi - \pi^*) + \beta\psi\pi(\pi - \pi^*)$$

Using (B1) and (B11) gives $S = \alpha^{-1}\gamma A^{-1}h^{1+\varepsilon-\alpha}C$, which leads to (10).

We remark that the steady-state Phillips curve equation here differs somewhat from the one in Evans, Guse, and Honkapohja (2008). The latter paper uses a representative household-firm in which the price-adjustment costs are quadratic in utility. In the current set-up households and firms are distinct. With utility $\log(C)$ this leads to a multiplicative factor C on the right-hand side of (10) not present in Evans, Guse, and Honkapohja (2008).

We make the assumption that the target inflation rate is $P_t/P_{t-1} = \pi^* \geq 1$, i.e. the net inflation rate may be positive. As discussed above, price adjustment costs are assumed to be quadratic in terms of the deviation from the target inflation rate and this is also analytically convenient. The market clearing condition is

$$\text{(B14)} \qquad Y_t = C_t + G_t + \frac{1}{2}\psi(\pi_t - \pi^*)^2.$$

We need to linearize around the steady state $\pi^*, \bar{Y}, \bar{S}, \bar{C}, \bar{Q}, \bar{h}$. Clearly $\bar{Q} = \beta/\pi^*$ is the steady state value of $Q_{t,t+1}$ and $\bar{S}$ is given above. From (B14) with $\pi_t = \pi^*$ we have $\bar{Y} = \bar{C} + \bar{G}$. Finally, in a steady state (B11) and (B1) can be combined to give

$$\text{(B15)} \qquad \bar{S} = \gamma\alpha^{-1}A^{-1}\bar{h}^{1+\varepsilon-\alpha}\bar{C}.$$

Equation (B15) together with the steady-state production function $\bar{Y} = A\bar{h}^{\alpha}$, market-clearing $\bar{Y} = \bar{C} + \bar{G}$ and (B13) determines steady values of $\bar{Y}, \bar{S}, \bar{C}, \bar{h}$ at the targeted steady state π^*.

We need to linearize around steady state $\bar{Y}, \bar{S}, \pi^*, \bar{Q}$ where $\bar{Q}$ is the steady state value of $Q_{t,t+1}$. Note that $\bar{Q} = \beta/\pi^*$. It is useful to define the mark-up μ by equation (B12). In the steady state $\mu = \theta(\theta-1)^{-1}$. Log-linearizing (B12) gives

$$\hat{\mu}_t = -(\theta - 1)^{-1}\hat{\theta}_t.$$

From above the mean real marginal cost is

$$\bar{S} = \frac{(\theta-1)(1-\tau)}{\theta} = (1-\tau)\mu^{-1}.$$

Next, we linearize (B10) and obtain

$$\begin{aligned}
0 \;=\; & (1-\tau)(1-\theta)\tilde{Y}_t - (1-\tau)(1-\theta)\theta\bar{Y}\left(\frac{P_{t,j}}{P_t} - 1\right) - (1-\tau)\bar{Y}\tilde{\theta}_t + \\
& \theta\bar{Y}\tilde{S}_{t,j} + \bar{S}\theta\tilde{Y}_t - \bar{S}\bar{Y}\theta(1+\theta)\left(\frac{P_{t,j}}{P_t} - 1\right) + \bar{S}\bar{Y}\tilde{\theta}_t - \\
& \pi^*\psi\left(\frac{P_{t,j}}{P_{t-1,j}} - \pi^*\right) + (\pi^*)^2\,\hat{E}_{t,j}\left[\bar{Q}\psi\left(\frac{P_{t+1,j}}{P_{t,j}} - \pi^*\right)\right].
\end{aligned}$$

where $\tilde{\theta}_t = \theta_t - \theta$, $\tilde{Y}_t = Y_t - \bar{Y}$, etc. Also, to the first order

$$\frac{P_{t+1,j}}{P_{t,j}} - 1 = \tilde{p}_{t+1,j} - \tilde{p}_{t,j},$$

where $\tilde{p}_t \equiv \ln P_t - (\pi^* - 1)t$ and $\tilde{p}_{t,j} \equiv \ln P_{t,j} - (\pi^* - 1)t$. Note that $(1-\tau)(1-\theta)\tilde{Y}_t = \bar{S}\theta\tilde{Y}_t$, so these terms cancel. We must also take into account that $S_{t,j}$ depends on endogenous variables. Write (B11) in the form

$$\alpha A_t^{1/\alpha} Y_t^{(\alpha-1)/\alpha} \left(\frac{P_{t,j}}{P_t}\right)^{-\theta_t(a-1)/\alpha} S_{t,j} = w_t$$

which is linearized:

$$\begin{aligned} \tilde{w}_t \;=\; & \alpha A^{1/\alpha} Y^{(\alpha-1)/\alpha} \tilde{S}_{t,j} + A^{1/\alpha-1} Y^{(\alpha-1)/\alpha} \bar{S} \tilde{A}_t + \alpha A^{1/\alpha} Y^{-1/\alpha} \left(\frac{\alpha-1}{\alpha}\right) \bar{S}\tilde{Y}_t \\ & + \alpha A^{1/\alpha} Y^{(\alpha-1)/\alpha} \bar{S} \left(\frac{-\theta(\alpha-1)}{\alpha}\right) \left(\frac{P_{t,j}}{P_t} - 1\right). \end{aligned}$$

Solve for $\tilde{S}_{t,j}$ and use the approximation $\frac{P_{t,j}}{P_t} - 1 = \tilde{p}_{t,j} - \tilde{p}_t$ from above to get

$$\text{(B16)} \quad \begin{aligned} \tilde{S}_{t,j} \;=\; & \tilde{w}_t / (\alpha A^{1/\alpha} Y^{(\alpha-1)/\alpha}) - A^{-1} \bar{S} \alpha^{-1} \tilde{A}_t \\ & - \left(\frac{\alpha-1}{\alpha}\right) \bar{S} Y^{-1} \tilde{Y}_t - \bar{S} \left(\frac{-\theta(\alpha-1)}{\alpha}\right) (\tilde{p}_{t,j} - \tilde{p}_t). \end{aligned}$$

It follows that

$$\begin{aligned} 0 \;=\; & -(1-\tau)\tilde{\theta}_t + \theta \left[\bar{S}\hat{w}_t - \bar{S}\alpha^{-1}\hat{A}_t - \left(\frac{\alpha-1}{\alpha}\right) \bar{S}\hat{Y}_t\right] \\ & -\bar{S}\theta[1 - \frac{\theta(\alpha-1)}{\alpha}] (\tilde{p}_{t,j} - \tilde{p}_t) + \bar{S}\tilde{\theta}_t - \\ & \pi^* \bar{Y}^{-1} \psi (\tilde{p}_{t,j} - \tilde{p}_{t-1,j}) + \hat{E}_{t,j} \left[\bar{Q}\bar{Y}^{-1} (\pi^*)^2 \psi (\tilde{p}_{t+1,j} - \tilde{p}_{t,j})\right]. \end{aligned}$$

Then combine the terms involving $\tilde{p}_{t,j} - \tilde{p}_t$ and rearrange the coefficients using the steady state relations. Also combine the terms involving $\tilde{\theta}_t$ and use the log-linearization between $\hat{\mu}_t$ and $\hat{\theta}_t$. This yields the result

$$\text{(B17)}\qquad \begin{aligned} \tilde{p}_{t,j}-\tilde{p}_{t-1,j} &= \beta\hat{E}_{t,j}(\tilde{p}_{t+1,j}-\tilde{p}_{t,j})+\frac{\bar{\omega}}{\bar{\psi}}(\tilde{p}_t-\tilde{p}_{t,j}) \\ &\quad +\frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}[\hat{\mu}_t-\alpha^{-1}\hat{A}_t-\left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t+\hat{w}_t]. \end{aligned}$$

where $\bar{\omega}=\theta\bar{Y}\bar{S}(1-\theta(\alpha-1)/\alpha)$ and $\bar{\psi}=\psi\pi^*$. Here

$$\hat{\mu}_t=\frac{\mu_t-\mu}{\mu}.$$

Next, we use the back-shift operator technique on (B17); see pp. 393-5 of Sargent (1987). Here $B^{-1}E_{t-1}x_{t+j}=E_{t-1}x_{t+j+1}$. As emphasized by Sargent, it is legitimate to operate on both sides of an equation by polynomials involving non-positive powers of B. Taking expectations $\hat{E}_{t,j}$ of (B17) and rearranging we get

$$\begin{aligned} &\left[1-\left(1+\beta^{-1}+\frac{\bar{\omega}}{\beta\bar{\psi}}\right)B+\beta^{-1}B^2\right]\hat{E}_{t,j}\tilde{p}_{t+1,j} \\ =\ &\hat{E}_{t,j}\left[-\frac{\bar{\omega}}{\beta\bar{\psi}}\tilde{p}_t-\frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{\mu}_t-\alpha^{-1}\hat{A}_t-\left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t+\hat{w}_t)\right]. \end{aligned}$$

The quadratic in B can be factored into the product $(1-\gamma_1 B)(1-\gamma_2 B)$ with roots $0<\gamma_1<1<\gamma_2$ satisfying

$$\gamma_1\gamma_2=\beta^{-1}\text{ and }\gamma_1+\gamma_2=\beta^{-1}(1+\beta+\bar{\omega}\bar{\psi}^{-1}).$$

We write

$$\begin{aligned} &(1-\gamma_1 B)(1-\gamma_2 B)E_{t,j}\tilde{p}_{t+1,j} \\ =\ &\hat{E}_{t,j}\left[-\frac{\bar{\omega}}{\beta\bar{\psi}}\tilde{p}_t-\frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{\mu}_t-\alpha^{-1}\hat{A}_t-\left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t+\hat{w}_t)\right] \end{aligned}$$

or

$$\begin{aligned}
& (B^{-1}-\gamma_1)(B^{-1}-\gamma_2)E_{t,j}\tilde{p}_{t-1,j} \\
= \; & \hat{E}_{t,j}\left[-\frac{\bar{\omega}}{\beta\bar{\psi}}\tilde{p}_t - \frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{\mu}_t - \alpha^{-1}\hat{A}_t - \left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t + \hat{w}_t)\right].
\end{aligned}$$

Operating on both sides by $(B^{-1}-\gamma_2)^{-1}$ we get

$$\begin{aligned}
& (B^{-1}-\gamma_1)E_{t,j}\tilde{p}_{t-1,j} \\
= \; & \frac{1}{(B^{-1}-\gamma_2)}\hat{E}_{t,j}\left[-\frac{\bar{\omega}}{\beta\bar{\psi}}\tilde{p}_t - \frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{\mu}_t - \alpha^{-1}\hat{A}_t - \left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t + \hat{w}_t)\right] \\
= \; & \frac{\gamma_2^{-1}}{(1-\gamma_2^{-1}B^{-1})}\hat{E}_{t,j}\left[\frac{\omega}{\beta\bar{\psi}}\tilde{p}_t + \frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{\mu}_t - \alpha^{-1}\hat{A}_t - \left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t + \hat{w}_t)\right].
\end{aligned}$$

Writing $(1-\gamma_2^{-1}B^{-1})^{-1} = 1+\gamma_2^{-1}B^{-1}+\gamma_2^{-2}B^{-2}+\ldots$ and using $\gamma_1\gamma_2 = \beta^{-1}$ we obtain

$$\begin{aligned}
\tilde{p}_{t,j} \; = \; & \gamma_1\tilde{p}_{t-1,j} + \frac{\gamma_1}{\bar{\psi}} \times \\
& \left(\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\left[\bar{\omega}\tilde{p}_{t+s} + \theta\bar{Y}\bar{S}(\hat{\mu}_t - \alpha^{-1}\hat{A}_t - \left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_t + \hat{w}_t)\right]\right). \qquad \text{(B18)}
\end{aligned}$$

as the evolution of the optimal price of firm j.

We now define $\tilde{\pi}_t = \tilde{p}_t - \tilde{p}_{t-1}$. Note that $\tilde{\pi}_t$ is the rate of inflation net of the target rate π^*. Using

$$\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{\pi}_{t+s} = \sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s} - \sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s-1}$$

and

$$\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s-1} = \tilde{p}_{t-1} + \gamma_2^{-1}\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s},$$

we obtain

$$\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s} = (1-\gamma_2^{-1})^{-1}\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{\pi}_{t+s} + (1-\gamma_2^{-1})^{-1}\tilde{p}_{t-1}.$$

Plugging into (B18) we obtain

$$\begin{aligned}\tilde{p}_{t,j} &= \gamma_1\tilde{p}_{t-1,j} + \frac{\gamma_1\bar{\omega}}{\bar{\psi}(1-\beta\gamma_1)}\tilde{p}_{t-1} + \frac{\gamma_1\bar{\omega}}{\bar{\psi}(1-\beta\gamma_1)}\sum_{s=0}^{\infty}(\beta\gamma_1)^s\hat{E}_{t,j}\tilde{\pi}_{t+s} + \\ &\quad \frac{\gamma_1\theta\bar{Y}\bar{S}}{\bar{\psi}}\sum_{s=0}^{\infty}(\beta\gamma_1)^s\hat{E}_{t,j}(\hat{\mu}_{t+s} - \alpha^{-1}\hat{A}_{t+s} - \left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_{t+s} + \hat{w}_{t+s}).\end{aligned}$$

Subtracting $\tilde{p}_{t-1,j}$ from both sides and collecting terms, and imposing the representative agent assumption, the coefficient of $\tilde{p}_{t-1,j}$ becomes

$$\begin{aligned}(\gamma_1 - 1) + \frac{\gamma_1\bar{\omega}}{\bar{\psi}(1-\beta\gamma_1)} &= \frac{(\gamma_1-1)(1-\beta\gamma_1) + (\bar{\omega}/\bar{\psi})\gamma_1}{1-\beta\gamma_1} \\ &= \frac{\gamma_1(\bar{\omega}/\bar{\psi} + \beta + 1 - \beta\gamma_1) - 1}{1-\beta\gamma_1} = \frac{\gamma_1\beta\gamma_2 - 1}{1-\beta\gamma_1} = 0.\end{aligned}$$

Note that

$$\gamma_1 = 1 - \frac{\gamma_1\bar{\omega}}{\bar{\psi}(1-\beta\gamma_1)}. \tag{B19}$$

Using the representative agent assumption the resulting equation becomes

$$\begin{aligned}\tilde{\pi}_t &= \frac{\gamma_1\bar{\omega}}{\bar{\psi}(1-\beta\gamma_1)}\sum_{s=0}^{\infty}(\beta\gamma_1)^s\hat{E}_t\tilde{\pi}_{t+s} + \\ &\quad \frac{\gamma_1\theta\bar{Y}\bar{S}}{\bar{\psi}}\sum_{s=0}^{\infty}(\beta\gamma_1)^s\hat{E}_t(\hat{\mu}_{t+s} - \alpha^{-1}\hat{A}_{t+s} - \left(\frac{\alpha-1}{\alpha}\right)\hat{Y}_{t+s} + \hat{w}_{t+s}).\end{aligned} \tag{B20}$$

Next use the equation

$$w_t = \gamma\left(\frac{Y_t}{A_t}\right)^{\varepsilon/\alpha} C_t$$

to obtain

$$\hat{w}_t = \frac{\varepsilon}{\alpha}(\hat{Y}_t - \hat{A}_t) + (1-\bar{g})^{-1}\hat{Y}_t - \frac{\bar{g}}{1-\bar{g}}\hat{G}_t. \tag{B21}$$

At the targeted steady state $\bar{Y} = \bar{C} + \bar{G}$ has the linear approximation

$$\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t,$$

where $\bar{g} \equiv \frac{\bar{G}}{\bar{Y}}$ and $\frac{\bar{C}}{\bar{Y}} = 1 - \bar{g}$. We get

$$\begin{aligned}
\tilde{\pi}_t \quad = \quad & \frac{\gamma_1 \bar{\omega}}{\bar{\psi}(1-\beta\gamma_1)} \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \tilde{\pi}_{t+s} + \frac{\gamma_1 \theta \bar{Y} \bar{S}}{\bar{\psi}} \left[\sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t [\hat{\mu}_{t+s} + \right. \\
\text{(B22)} \qquad & \left. (\frac{-1-\varepsilon}{\alpha})\hat{A}_{t+s} + \left(\frac{1-\alpha}{\alpha} + \frac{\varepsilon}{\alpha} + (1-\bar{g})^{-1} \right) \hat{Y}_{t+s} - \frac{\bar{g}}{1-\bar{g}} \hat{G}_{t+s}] \right].
\end{aligned}$$

Letting $\hat{\pi}_t \equiv \tilde{\pi}_t / \pi^*$ and substituting into (B20) we finally obtain the Phillips curve

$$\begin{aligned}
\hat{\pi}_t \quad = \quad & a_1 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\pi}_{t+s} + a_2 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{Y}_{t+s} - a_3 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{A}_{t+s} \\
\text{(B23)} \qquad & + a_4 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{G}_{t+s} + a_5 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\mu}_{t+s}.
\end{aligned}$$

where the coefficients a_i are defined as:

$$\begin{aligned}
a_1 \quad &= \quad \frac{\bar{\omega}\gamma_1}{\bar{\psi}(1-\beta\gamma_1)}; a_2 = \frac{\gamma_1 \theta \bar{Y} \bar{S}}{\bar{\psi}\pi^*} \left(\frac{1-\alpha}{\alpha} + \frac{\varepsilon}{\alpha} + (1-\bar{g})^{-1} \right); \\
a_3 \quad &= \quad \frac{-\gamma_1 \theta \bar{Y} \bar{S}}{\bar{\psi}\pi^*} (\frac{1+\varepsilon}{\alpha}); a_4 = \frac{-\gamma_1 \theta \bar{Y} \bar{S}}{\bar{\psi}\pi^*} \frac{\bar{g}}{1-\bar{g}}; a_5 = \frac{\gamma_1 \theta \bar{Y} \bar{S}}{\bar{\psi}\pi^*}.
\end{aligned}$$

Note that by (B19) we have $a_1 = 1 - \gamma_1$. Rearrange the above equation to get (5). The interpretation of equation (5) is as follows. Higher expected future inflation, and higher current and expected future aggregate output lead to higher current inflation. Current inflation is also increased when future monopoly power is expected to be higher. Higher expected future productivity lowers expected future marginal costs and hence reduces inflation. Finally, conditional on expected future output, higher current and expected future government spending is associated with lower consumption, higher labor supply (conditional on real wages) and hence lower real wages, which leads to lower inflation.

Temporary equilibrium and learning

To get the IS curve, we combine the consumption function (3) with the linearized market clearing condition $\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{G}}{\bar{Y}}\hat{G}_t$, or

$$\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t, \tag{B24}$$

where $\bar{g} = \frac{\bar{G}}{\bar{Y}}$. (As in the Appendix to Eusepi and Preston (2010), the adjustment costs drop out from the log-linearized market clearing equation.) This yields

$$\hat{Y}_t = \bar{g}\hat{G}_t + (1-\beta)\left[\hat{Y}_t - \bar{g}\hat{G}_t + \sum_{s=1}^{\infty}\beta^s\hat{E}_t\left(\hat{Y}_{t+s} - \bar{g}\hat{G}_{t+s}\right)\right] - (1-\bar{g})\hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{r}_{t+s}.$$

Note that from $r_{t+1} \equiv \frac{R_t}{\pi_{t+1}}$ we have

$$\hat{r}_{t+1} = \hat{R}_t - \hat{\pi}_{t+1}. \tag{B25}$$

The market clearing condition is (B14) which at the targeted steady state $\pi_t = \pi^*$ is

$$\bar{Y} = \bar{C} + \bar{G}$$

It follows that the linear approximation around the targeted steady state is

$$\begin{aligned}
\hat{Y}_t &= (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t, \\
\bar{g} &\equiv \frac{\bar{G}}{\bar{Y}} \text{ and } \frac{\bar{C}}{\bar{Y}} = 1-\bar{g}.
\end{aligned}$$

We thus have

$$\begin{aligned}
\hat{C}_t = {} & (1-\beta)(1-\bar{g})^{-1}\hat{Y}_t + (1-\beta)(1-\bar{g})^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} \\
& -(1-\beta)\bar{g}(1-\bar{g})^{-1}\hat{G}_t - (1-\beta)\bar{g}(1-\bar{g})^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \\
& -\beta\hat{R}_t - \beta\hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{R}_{t+s} + \hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{\pi}_{t+s}.
\end{aligned} \tag{B26}$$

For the contemporaneous interest rate rule $\hat{R}_t = \chi_\pi \hat{\pi}_t + \chi_Y \hat{Y}_t$ we have

$$\begin{aligned}
\hat{C}_t &= \left((1-\beta)(1-\bar{g})^{-1} - \beta\chi_Y\right)\hat{Y}_t \\
&\quad + \left((1-\beta)(1-\bar{g})^{-1} - \beta\chi_Y\right)\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{Y}_{t+s} \\
&\quad -\beta\chi_\pi\hat{\pi}_t + (1-\beta\chi_\pi)\sum_{s=1}^{\infty}\beta^s \hat{E}_t\hat{\pi}_{t+s} - (1-\beta)\bar{g}(1-\bar{g})^{-1}\hat{G}_t \\
&\quad -(1-\beta)\bar{g}(1-\bar{g})^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \qquad \text{(B27)}
\end{aligned}$$

Next, we describe the least-squares updating rule for the forecast rule coefficients of $\hat{\pi}_t$ and $\hat{Y}_t$. Agents are assumed to use constant gain recursive least squares (RLS). The parameter estimates based on data through time t are

$$\phi_{\pi,t} = \begin{pmatrix} f_{\pi,t} \\ d_{\pi A,t} \\ d_{\pi\mu,t} \end{pmatrix}, \phi_{Y,t} = \begin{pmatrix} f_{Y,t} \\ d_{YA,t} \\ d_{Y\mu,t} \end{pmatrix}, z_t = \begin{pmatrix} 1 \\ \hat{A}_t \\ \hat{\mu}_t \end{pmatrix}.$$

The RLS formulae corresponding to estimates of equations (9) are

$$\begin{aligned}
\phi_{\pi,t} &= \phi_{\pi,t-1} + \kappa\mathcal{R}_t^{-1}z_t(\pi_t - \phi_{\pi,t-1}z_t), \\
\phi_{Y,t} &= \phi_{Y,t-1} + \kappa\mathcal{R}_t^{-1}z_t(Y_t - \phi_{Y,t-1}z_t), \\
\mathcal{R}_t &= \mathcal{R}_{t-1} + \kappa(z_t z_t' - \mathcal{R}_{t-1}).
\end{aligned}$$

Here $0 < \kappa < 1$ is the "gain" parameter that discounts old data at rate $1-\kappa$ per period (taken to be one quarter), to allow for adaptation of parameters to structural changes like policy changes. We assume that parameter estimates under learning are updated at the end of the period. Thus in time t, when expectations are formed, agents observe the current value of the exogenous variables $\hat{A}_t$ and $\hat{\mu}_t$ but use estimates $\phi_{\pi,t-1}$, $\phi_{y,t-1}$ in making forecasts. The initial values of all parameter estimates ϕ and $\mathcal{R}$ are set to the initial steady state values under RE.

Temporary equilibrium, further computations

The IS curve under the contemporaneous interest-rate rule is obtained from combining the consumption function (B27) with the market-clearing equation $\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t$. This yields

$$\begin{aligned} \beta(1-\bar{g})\chi_\pi \hat{\pi}_t + (\beta + \beta(1-\bar{g})\chi_Y)\,\hat{Y}_t &= [(1-\bar{g})(1-\beta\chi_\pi)]\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{\pi}_{t+s} \\ &\quad + [(1-\beta)-(1-\bar{g})\beta\chi_Y]\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{Y}_{t+s} \\ &\quad + \bar{g}\hat{G}_t - (1-\beta)\bar{g}\sum_{s=0}^{\infty}\beta^s \hat{E}_t \hat{G}_{t+s} \end{aligned} \tag{B28}$$

Henceforth, we use the following short-hand notation

$$\begin{aligned} b_1 &\equiv (1-\bar{g})(1-\beta\chi_\pi), \\ b_2 &\equiv (1-\beta)-(1-\bar{g})\beta\chi_Y. \end{aligned}$$

Let us now write the price setting equation (5) and the demand equation (B28) (under subjective expectations) in matrix form. Let

$$M = \begin{pmatrix} 1-a_1 & -a_2 \\ \beta(1-\bar{g})\chi_\pi & \beta+\beta(1-\bar{g})\chi_Y \end{pmatrix} \tag{B29}$$

Then

$$M\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = \sum_{s=1}^{\infty}\begin{pmatrix} a_1(\beta\gamma_1)^s & a_2(\beta\gamma_1)^s \\ b_1\beta^s & b_2\beta^s \end{pmatrix}\begin{pmatrix} \hat{E}_t\hat{\pi}_{t+s} \\ \hat{E}_t\hat{Y}_{t+s} \end{pmatrix} + \begin{pmatrix} -a_4 \\ \beta\bar{g} \end{pmatrix}\hat{G}_t +$$
$$\sum_{s=0}^{\infty}\begin{pmatrix} -a_3(\beta\gamma_1)^s & a_5(\beta\gamma_1)^s \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{E}_t\hat{A}_{t+s} \\ \hat{E}_t\hat{\mu}_{t+s} \end{pmatrix} + \sum_{s=1}^{\infty}\begin{pmatrix} -a_4(\beta\gamma_1)^s \\ -(1-\beta)\bar{g}\beta^s \end{pmatrix}\hat{E}_t\hat{G}_{t+s}.$$

For the shock terms above we get

$$\sum_{s=0}^{\infty}\begin{pmatrix} -a_3(\beta\gamma_1)^s & a_5(\beta\gamma_1)^s \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \rho_A^s\hat{A}_t \\ \rho_\mu^s\hat{\mu}_t \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{s=0}^{\infty}\left(-a_3(\rho_A\beta\gamma_1)^s\hat{A}_t + a_5(\rho_\mu\beta\gamma_1)^s\hat{\mu}_t\right) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{s=0}^{\infty}\left(-a_3(\rho_A\beta\gamma_1)^s\hat{A}_t + a_5(\rho_\mu\beta\gamma_1)^s\hat{\mu}_t\right) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t + a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t \\ 0 \end{pmatrix}.$$

Consider a change in government spending that is known to be temporary. We assume that initially, at $t = 0$, we are in the steady state corresponding to $G = \bar{G}$, and consider the following policy experiment, assumed fully credible and announced at the start of period 1:

(B30)
$$G_t = \tau_t = \begin{cases} \bar{G}', \ t = 1, ..., T \\ \bar{G}, \ t \geq T+1, \end{cases}$$

i.e., government spending and taxes are changed in period $t = 1$ and this change is reversed at a later period $T+1$. Thus, the experiment is one where the policy change is announced in period 1 to take place in the future for a fixed number T of periods. Denote the change in government spending by ΔG $(= \bar{G}' - \bar{G}$) so that

$$\hat{G}_t = \begin{cases} \frac{\Delta G}{\bar{G}}, \ t = 1, ..., T \\ 0, \ t \geq T+1. \end{cases}$$

We first consider the evolution of the learning economy during the period when the policy increase is in effect i.e. for periods $t = 1, ..., T$. Then we have

$$\sum_{s=1}^{\infty}\begin{pmatrix} -a_4(\beta\gamma_1)^s \\ -(1-\beta)\bar{g}\beta^s \end{pmatrix}\hat{E}_t\hat{G}_{t+s} = \sum_{s=1}^{T-t}\begin{pmatrix} -a_4(\beta\gamma_1)^s \\ -(1-\beta)\bar{g}\beta^s \end{pmatrix}\frac{\Delta G}{\bar{G}} =$$

$$\begin{pmatrix} -a_4\sum_{s=1}^{T-t}(\beta\gamma_1)^s \\ -(1-\beta)\bar{g}\sum_{s=1}^{T-t}\beta^s \end{pmatrix}\frac{\Delta G}{\bar{G}} = \begin{pmatrix} -a_4\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1} \\ -\beta(1-\beta)\bar{g}\frac{1-\beta^{T-t}}{1-\beta} \end{pmatrix}\frac{\Delta G}{\bar{G}}.$$

Write the final form of the model when agents are learning in the following matrix form (which is true for $1 \le t \le T$)

$$M\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = \sum_{s=1}^{\infty}\begin{pmatrix} a_1(\beta\gamma_1)^s & a_2(\beta\gamma_1)^s \\ b_1\beta^s & b_2\beta^s \end{pmatrix}\begin{pmatrix} \hat{E}_t\hat{\pi}_{t+s} \\ \hat{E}_t\hat{Y}_{t+s} \end{pmatrix} +$$

$$\begin{pmatrix} -a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t + a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} -a_4\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1} \\ -\beta(1-\beta)\bar{g}\frac{1-\beta^{T-t}}{1-\beta} \end{pmatrix}\frac{\Delta G}{\bar{G}} + \begin{pmatrix} -a_4 \\ \beta\bar{g} \end{pmatrix}\frac{\Delta G}{\bar{G}}. \tag{B31}$$

Note that when $t > T$, the model evolution under learning is governed by

$$M\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = \sum_{s=1}^{\infty}\begin{pmatrix} a_1(\beta\gamma_1)^s & a_2(\beta\gamma_1)^s \\ b_1\beta^s & b_2\beta^s \end{pmatrix}\begin{pmatrix} \hat{E}_t\hat{\pi}_{t+s} \\ \hat{E}_t\hat{Y}_{t+s} \end{pmatrix} +$$

$$\begin{pmatrix} -a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t + a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t \\ 0 \end{pmatrix}. \tag{B32}$$

since $\hat{G}_t = 0$ when $t > T$.

We consider PLMs of the same form as the standard minimal state variable (MSV) solution of the economy. One can solve the model under RE with fixed G_t to get a stochastic steady state of the form

$$\hat{\pi}_t = f_\pi + d_{\pi A}\hat{A}_t + d_{\pi\mu}\hat{\mu}_t,$$
$$\hat{Y}_t = f_Y + d_{YA}\hat{A}_t + d_{Y\mu}\hat{\mu}_t,$$

where $\hat{A}_t, \hat{\mu}_t$ are observable processes (with known coefficients) given by (8). These can be used to construct forecasts $\hat{E}_t \pi_{t+s}$ and $\hat{E}_t \hat{Y}_{t+s}$ which are then inserted into the model (B31) to govern the evolution of the economy for the first T periods (and by (B32) for periods after T).

Using the MSV form of the PLM we get

$$\begin{aligned}\hat{E}_t \hat{\pi}_{t+s} &= f_\pi + d_{\pi A} \hat{E}_t \hat{A}_{t+s} + d_{\pi\mu} \hat{E}_t \hat{\mu}_{t+s} \\ &= f_\pi + d_{\pi A} \rho_A^s \hat{A}_t + d_{\pi\mu} \rho_\mu^s \hat{\mu}_t.\end{aligned}$$

Similarly,

$$\hat{E}_t \hat{Y}_{t+s} = f_Y + d_{YA} \rho_A^s \hat{A}_t + d_{Y\mu} \rho_\mu^s \hat{\mu}_t.$$

Consider the term below that needs to be evaluated in the first row of (B31)

$$\begin{aligned}& a_1 \sum_{s=1}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\pi}_{t+s} + a_2 \sum_{s=1}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{Y}_{t+s} \\ =\ & a_1 \sum_{s=1}^{\infty} (\beta\gamma_1)^s (f_\pi + d_{\pi A} \hat{E}_t \hat{A}_{t+s} + d_{\pi\mu} \hat{E}_t \hat{\mu}_{t+s}) \\ & + a_2 \sum_{s=1}^{\infty} (\beta\gamma_1)^s (f_Y + d_{YA} \rho_A^s \hat{A}_t + d_{Y\mu} \rho_\mu^s \hat{\mu}_t) \\ =\ & (a_1 f_\pi + a_2 f_Y) \frac{\beta\gamma_1}{1-\beta\gamma_1} + (a_1 d_{\pi A} + a_2 d_{YA}) \frac{\rho_A \beta\gamma_1}{1-\rho_A \beta\gamma_1} \hat{A}_t \\ & + (a_1 d_{\pi\mu} + a_2 d_{Y\mu}) \frac{\rho_\mu \beta\gamma_1}{1-\rho_\mu \beta\gamma_1} \hat{\mu}_t.\end{aligned}$$

Similarly consider the term below that is required to be evaluated in the second row of (B31)

$$
\begin{aligned}
& b_1 \sum_{s=1}^{\infty} \beta^s \hat{E}_t \hat{\pi}_{t+s} + b_2 \sum_{s=1}^{\infty} \beta^s \hat{E}_t \hat{Y}_{t+s} \\
= \; & b_1 \sum_{s=1}^{\infty} \beta^s (f_\pi + d_{\pi A} \hat{E}_t \hat{A}_{t+s} + d_{\pi\mu} \hat{E}_t \hat{\mu}_{t+s}) \\
& + a_2 \sum_{s=1}^{\infty} (\beta\gamma_1)^s (f_Y + d_{YA} \rho_A^s \hat{A}_t + d_{Y\mu} \rho_\mu^s \hat{\mu}_t) \\
= \; & (b_1 f_\pi + b_2 f_Y) \frac{\beta}{1-\beta} + (b_1 d_{\pi A} + b_2 d_{YA}) \frac{\rho_A \beta}{1 - \rho_A \beta} \hat{A}_t \\
& + (b_1 d_{\pi\mu} + b_2 d_{Y\mu}) \frac{\rho_\mu \beta}{1 - \rho_\mu \beta} \hat{\mu}_t.
\end{aligned}
$$

We can obtain a mapping from the PLM to the ALM from (B31) for the first T periods (and from (B32) for periods after T).

We now combine terms of the right hand side of (B31). The first row on the right hand side of (B31) is given by

$$
\begin{aligned}
& (a_1 f_\pi + a_2 f_Y) \frac{\beta\gamma_1}{1 - \beta\gamma_1} + (a_1 d_{\pi A} + a_2 d_{YA}) \frac{\rho_A \beta\gamma_1}{1 - \rho_A \beta\gamma_1} \hat{A}_t \\
& + (a_1 d_{\pi\mu} + a_2 d_{Y\mu}) \frac{\rho_\mu \beta\gamma_1}{1 - \rho_\mu \beta\gamma_1} \hat{\mu}_t - a_3 (1 - \rho_A \beta\gamma_1)^{-1} \hat{A}_t \\
& + a_5 (1 - \rho_\mu \beta\gamma_1)^{-1} \hat{\mu}_t - a_4 (\beta\gamma_1 \frac{1 - (\beta\gamma_1)^{T-t}}{1 - \beta\gamma_1} + 1) \frac{\Delta G}{\bar{G}}.
\end{aligned}
$$

The second row on the right hand side of (B31) is given by

$$
\begin{aligned}
& (b_1 f_\pi + b_2 f_Y) \frac{\beta}{1-\beta} + (b_1 d_{\pi A} + b_2 d_{YA}) \frac{\rho_A \beta}{1 - \rho_A \beta} \hat{A}_t \\
& + (b_1 d_{\pi\mu} + b_2 d_{Y\mu}) \frac{\rho_\mu \beta}{1 - \rho_\mu \beta} \hat{\mu}_t - \beta \bar{g} [(1-\beta) \frac{1 - \beta^{T-t}}{1 - \beta} - 1] \frac{\Delta G}{\bar{G}}.
\end{aligned}
$$

This process gives us the mapping for the T-map as below.

We collect the terms for the intercept in preceding two equations. This gives two equations (B33) and (B34) to solve for the T-map for the intercept terms T_π, T_Y :

$$\begin{aligned}(1-a_1)T_\pi - a_2 T_Y &= (a_1 f_\pi + a_2 f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1} \\ &\quad -a_4(\beta\gamma_1 \frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1} + 1)\frac{\Delta G}{\bar{G}},\end{aligned} \tag{B33}$$

$$\begin{aligned}\beta(1-\bar{g})\chi_\pi T_\pi + (\beta + (1-\bar{g})\chi_Y)T_Y &= (b_1 f_\pi + b_2 f_Y)\frac{\beta}{1-\beta} \\ &\quad -\beta\bar{g}[(1-\beta)\frac{1-\beta^{T-t}}{1-\beta} - 1]\frac{\Delta G}{\bar{G}}.\end{aligned} \tag{B34}$$

Similarly, consider the terms involving $\hat{A}_t$

$$(1-a_1)T_{\pi A} - a_2 T_{YA} = (a_1 d_{\pi A} + a_2 d_{YA})\frac{\rho_A\beta\gamma_1}{1-\rho_A\beta\gamma_1} - a_3(1-\rho_A\beta\gamma_1)^{-1}, \tag{B35}$$

$$\beta(1-\bar{g})\chi_\pi T_{\pi A} + (\beta + (1-\bar{g})\chi_Y)T_{YA} = (b_1 d_{\pi A} + b_2 d_{YA})\frac{\rho_A\beta}{1-\rho_A\beta}. \tag{B36}$$

Equations (B35) and (B36) are solved for the coefficients $T_{\pi A}$ and T_{YA}. Finally, consider terms involving $\hat{\mu}_t$

$$(1-a_1)T_{\pi\mu} - a_2 T_{Y\mu} = (a_1 d_{\pi\mu} + a_2 d_{Y\mu})\frac{\rho_\mu\beta\gamma_1}{1-\rho_\mu\beta\gamma_1} + a_5(1-\rho_\mu\beta\gamma_1)^{-1}, \tag{B37}$$

$$\beta(1-\bar{g})\chi_\pi T_{\pi\mu} + (\beta + (1-\bar{g})\chi_Y)T_{Y\mu} = (b_1 d_{\pi\mu} + b_2 d_{Y\mu})\frac{\rho_\mu\beta}{1-\rho_\mu\beta}. \tag{B38}$$

and equations (B37)-(B38) are solved for the coefficients $T_{\pi\mu}$ and $T_{Y\mu}$.

These six equations (B33), (B34), (B35), (B36), (B37), and (B38) yield the mapping

$$(f_\pi, f_Y, d_{\pi A}, d_{YA}, d_{\pi\mu}, d_{Y\mu}) \to (T_\pi, T_Y, T_{\pi A}, T_{YA}, T_{\pi\mu}, T_{Y\mu})$$

from the PLM to the ALM in parameter space and the fixed points of the map correspond to the MSV REE solution. This is the T-mapping for periods $1, \ldots, T$. For periods $t > T$ the same equations, together with the requirement $\Delta G = 0$, give the T-map.

RE Solution with policy change

We need to compute the RE solution when the fiscal policy changes. Computing the effect of policy changes under RE is somewhat simpler using the Euler equation approach. We first consider the IS curve equation. Imposing symmetry in equation (B3), we obtain

$$\frac{C_t - \bar{C}}{\bar{C}} = E_t(\frac{C_{t+1} - \bar{C}}{\bar{C}}) - E_t(\frac{r_{t+1} - \bar{r}}{\beta^{-1}}),$$

or in proportional deviation form

$$\hat{C}_t = E_t \hat{C}_{t+1} - E_t \hat{r}_{t+1}$$

where

$$\hat{r}_{t+1} = \frac{r_{t+1} - \bar{r}}{\bar{r}}; \bar{r} = \beta^{-1}.$$

Then using (B24), we obtain

$$\text{(B39)} \qquad \hat{Y}_t = E_t \hat{Y}_{t+1} - \frac{\bar{C}}{\bar{Y}} E_t \hat{r}_{t+1} + \frac{\bar{G}}{\bar{Y}} \left(\hat{G}_t - E_t \hat{G}_{t+1} \right)$$

as the IS curve in (proportional) deviation form. If we use the interest rate rule (6) in (B39) above we obtain

$$\text{(B40)} \quad (1 + \frac{\bar{C}}{\bar{Y}} \chi_Y) \hat{Y}_t + \frac{\bar{C}}{\bar{Y}} \chi_\pi \hat{\pi}_t = E_t \hat{Y}_{t+1} + \frac{\bar{C}}{\bar{Y}} E_t \hat{\pi}_{t+1} + \frac{\bar{G}}{\bar{Y}} \left(\hat{G}_t - E_t \hat{G}_{t+1} \right)$$

Since

$$\frac{\bar{C}}{\bar{Y}} = 1 - \frac{\bar{G}}{\bar{Y}} \equiv 1 - \bar{g},$$

we can rewrite (B40) as

(B41)

$$(1 + (1 - \bar{g})\chi_Y) \hat{Y}_t + (1 - \bar{g})\chi_\pi \hat{\pi}_t = E_t \hat{Y}_{t+1} + (1 - \bar{g}) E_t \hat{\pi}_{t+1} + \bar{g} \left(\hat{G}_t - E_t \hat{G}_{t+1} \right)$$

We now compute the one-step forward looking Phillips curve. Imposing symmetry, we obtain from (B17)

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} (\hat{\mu}_t - \alpha^{-1} \hat{A}_t - \left(\frac{\alpha - 1}{\alpha} \right) \hat{Y}_t + \hat{w}_t).$$

Substituting in (B21) $\hat{w}_t = \frac{\varepsilon}{\alpha}(\hat{Y}_t - \hat{A}_t) + (1-\bar{g})^{-1}\hat{Y}_t - \frac{\bar{g}}{1-\bar{g}}\hat{G}_t$ this becomes (B42)

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \left[\left(\frac{1+\varepsilon}{\alpha} + \frac{\bar{g}}{1-\bar{g}} \right) \hat{Y}_t + (\hat{\mu}_t - (\frac{1+\varepsilon}{\alpha}) \hat{A}_t) - \frac{\bar{g}}{1-\bar{g}} \hat{G}_t \right].$$

Writing (B41) and (B42) in matrix form we get

$$\begin{pmatrix} 1 & -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \left(\frac{1+\varepsilon}{\alpha} + \frac{\bar{g}}{1-\bar{g}} \right) \\ (1-\bar{g})\chi_\pi & 1 + (1-\bar{g})\chi_Y \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix}$$
$$= \begin{pmatrix} \beta & 0 \\ 1-\bar{g} & 1 \end{pmatrix} \begin{pmatrix} E_t \hat{\pi}_{t+1} \\ E_t \hat{Y}_{t+1} \end{pmatrix} + \begin{pmatrix} \frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} & -\frac{(1+\varepsilon)\theta \bar{Y} \bar{S}}{\alpha \bar{\psi}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\mu}_t \\ \hat{A}_t \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\bar{g}}{1-\bar{g}} \hat{G}_t \\ \bar{g} \left(\hat{G}_t - E_t \hat{G}_{t+1} \right) \end{pmatrix}.$$

Note that $\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} = \frac{(\theta-1)\bar{Y}(1-\tau)}{\bar{\psi}}$ which is denoted by ξ (with $\tau = 0$) in Eusepi-Preston (2010) and is set equal to 0.06 in footnote 11, p. 243, of their paper.

Inverting the matrix on the left hand side of the above system we can obtain the system

$$\text{(B43)} \quad \begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = \Psi \begin{pmatrix} E_t \hat{\pi}_{t+1} \\ E_t \hat{Y}_{t+1} \end{pmatrix} + \mathit{\Gamma} \begin{pmatrix} \hat{\mu}_t \\ \hat{A}_t \end{pmatrix} + \Gamma \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\bar{g}}{1-\bar{g}} \hat{G}_t \\ \bar{g} \left(\hat{G}_t - E_t \hat{G}_{t+1} \right) \end{pmatrix},$$

which can used to compute numerically the RE solution with the fiscal policy change. Thus, (B43) gives the system under RE when the Taylor rule (6) is followed. We will be using this system to compute the RE solution when there is a change in government purchases (and a balanced budget).

Solving this equation forward yields

$$\begin{aligned}
\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} &= \sum_{i=0}^{\infty} \Psi^i \left(F E_t \begin{pmatrix} \hat{\mu}_{t+i} \\ \hat{A}_{t+i} \end{pmatrix} + \Gamma H_{t+i} \right), \text{ where} \\
H_t &= \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\bar{g}}{1-\bar{g}} \hat{G}_t \\ \bar{g} \left(\hat{G}_t - E_t \hat{G}_{t+1} \right) \end{pmatrix},
\end{aligned}$$

which can be written as

$$\text{(B44)} \qquad \begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = \sum_{i=0}^{\infty} \Psi^i F \begin{pmatrix} \rho_\mu & 0 \\ 0 & \rho_A \end{pmatrix}^i \begin{pmatrix} \hat{\mu}_t \\ \hat{A}_t \end{pmatrix} + \sum_{i=0}^{\infty} \Psi^i \Gamma H_{t+i}.$$

The first term on the right-hand side of (B44) is the MSV solution when government spending is constant. This takes the form

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix}_{MSV} = \begin{pmatrix} \bar{d}_{\pi A} \\ \bar{d}_{YA} \end{pmatrix} \hat{A}_t + \begin{pmatrix} \bar{d}_{\pi\mu} \\ \bar{d}_{Y\mu} \end{pmatrix} \hat{\mu}_t,$$

where

$$\begin{aligned}
\begin{pmatrix} \bar{d}_{\pi A} \\ \bar{d}_{YA} \end{pmatrix} &= (M - N_A)^{-1} cons_A, \\
\begin{pmatrix} \bar{d}_{\pi\mu} \\ \bar{d}_{Y\mu} \end{pmatrix} &= (M - N_\mu)^{-1} cons_\mu,
\end{aligned}$$

and

$$\begin{aligned}
N_A &= \begin{pmatrix} a_1 \frac{\rho_A \beta \gamma_1}{1-\rho_A \beta \gamma_1} & a_2 \frac{\rho_A \beta \gamma_1}{1-\rho_A \beta \gamma_1} \\ b_1 \frac{\rho_A \beta}{1-\rho_A \beta} & b_2 \frac{\rho_A \beta}{1-\rho_A \beta} \end{pmatrix}, \ cons_A = \begin{pmatrix} -a_3 (1 - \rho_A \beta \gamma_1)^{-1} \\ 0 \end{pmatrix}, \\
N_\mu &= \begin{pmatrix} a_1 \frac{\rho_\mu \beta \gamma_1}{1-\rho_\mu \beta \gamma_1} & a_2 \frac{\rho_\mu \beta \gamma_1}{1-\rho_\mu \beta \gamma_1} \\ b_1 \frac{\rho_\mu \beta}{1-\rho_\mu \beta} & b_2 \frac{\rho_\mu \beta}{1-\rho_\mu \beta} \end{pmatrix}, \ cons_\mu = \begin{pmatrix} a_5 (1 - \rho_\mu \beta \gamma_1)^{-1} \\ 0 \end{pmatrix}.
\end{aligned}$$

The second term gives the modification due to changes in government spending and it is calculated as follows. For $t = 1$ we have

$$\begin{aligned}
HRE_1 &\equiv \sum_{s=0}^{\infty} \Psi^i \Gamma H_{1+i} = \sum_{s=0}^{T-2} \Psi^i \Gamma \bar{H}_1 + \Psi^{T-1} \Gamma \bar{H}_2 \\
&= (1 - \Psi^{T-1})(1 - \Psi)^{-1} \bar{H}_1 + \Psi^{T-1} \Gamma \bar{H}_2, \text{ where} \\
\bar{H}_1 &= \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\Delta G}{1-\bar{g}} \\ 0 \end{pmatrix} \text{ and } \bar{H}_2 = \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\Delta G}{1-\bar{g}} \\ -\Delta G \end{pmatrix}.
\end{aligned}$$

In general,

$$HRE_t = (1 - \Psi^{T-t})(1 - \Psi)^{-1} \bar{H}_1 + \Psi^{T-t} \Gamma \bar{H}_2$$

for $t = 1, \ldots, T-1$. For $t = T$ we have

$$HRE_T = \Gamma \bar{H}_2$$

and

$$HRE_t = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $t \geq T + 1$.

In total, the RE solution is the sum of the MSV solution with constant government spending plus the term HRE_t.

C Details of model with lower bounds

Proofs of Propositions

The more complete version of Proposition 1 is the following:

Proposition 3. *Suppose that $\underline{\pi} < \pi_L$. Then for $\psi > 0$ sufficiently small, there are exactly three steady states*
(i) $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ and C uniquely determined by (10) and (11),
(ii) $\pi = \pi_L$, with $R = 1+\eta$ and C uniquely determined by (10) and (11),
(iii) $\pi = \underline{\pi}$, with $R = 1+\eta$ and $C = \underline{C}$.
If $\pi_L < \underline{\pi} < \pi^*$ then there is a unique steady state at $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ and C uniquely determined by (10)-(11).
If $\underline{\pi} = \pi_L$ then for $\psi > 0$ sufficiently small there is a steady state at $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ and C uniquely determined by (10)-(11) and a continuum of steady states at $\pi = \pi_L$, with $R = 1+\eta$ and with C satisfying $\underline{C} \le C \le C_L$, where C_L is uniquely determined by (10)-(11).

The proof of the Proposition 3 uses the following result:

Lemma 4. *Let*

$$\begin{aligned}
\xi(C;\pi,\psi,G) &= C\gamma A^{-(1+\varepsilon)/\alpha}\left(C+G+\frac{\psi}{2}(\pi-\pi^*)^2\right)^{(1+\varepsilon)/\alpha} \\
&\quad -\alpha(1-\tau)(1-\theta^{-1})\left(C+G+\frac{\psi}{2}(\pi-\pi^*)^2\right).
\end{aligned}$$

Let $\underline{G} = \left((\alpha A/\gamma)(1-\tau)(1-\theta^{-1})\right)^{\frac{\alpha}{1+\varepsilon}}$. Then provided $G > \underline{G}$, there exists $\check{\psi} > 0$ such that for all $0 < \psi < \check{\psi}$ the function $\xi(C;\pi,\psi,G)$ is strictly monotonically increasing in C and for given $\pi, \psi < \check{\psi}$ and $G > \underline{G}$ we have $\lim_{C\to\infty}\xi(C;\pi,\psi,G) = +\infty$.

Proof of Lemma 4: Computing the derivative we have

$$\begin{aligned}
\left.\frac{d\xi}{dC}\right|_{\psi=0} &= \gamma C\frac{1+\varepsilon}{\alpha A}\left(A^{-1}(C+G)\right)^{\frac{1+\varepsilon}{\alpha}-1} + \gamma A^{-1}(C+G)^{\frac{1+\varepsilon}{\alpha}} \\
&\quad -\alpha(1-\tau)(1-\theta^{-1}).
\end{aligned}$$

If $G > \underline{G}$ then $\frac{d\xi}{dC}\big|_{\psi=0} > 0$ for all $C \geq 0$. Since $d\xi/dC$ is continuous in ψ, then result follows. ■

Proof of Proposition 3: First suppose the inflation lower bound satisfies $\underline{\pi} < \pi_L$. (i) $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ satisfy (12) and (14). From (10) and (11) we have $0 = \xi(C; \pi^*, \psi, G)$. We have $\xi(0; \pi^*, \psi, G) = -\alpha(1-\tau)(1-\theta^{-1})G < 0$. Since we are assuming $G > \underline{G}$ it follows from the Lemma that there is a unique $C = \bar{C}$ that solves (10) and (11).

(ii) $\pi = \pi_L$, with $R = 1 + \eta$ satisfy (12) and (14). From (10) and (11) we have $(\pi_L - 1)\pi_L(1-\beta)\alpha\psi\theta^{-1} = \xi(C; \pi_L, \psi, G)$. For ψ sufficiently small the term $(\pi_L - 1)\pi_L(1-\beta)\alpha\psi\theta^{-1}$ can be made arbitrarily close to zero. Since $\xi(0; \pi_L, \psi, G) = -\alpha(1-\tau)(1-\theta^{-1})\left(G + \frac{\psi}{2}(\pi_L - \pi^*)^2\right) < 0$, the Lemma again applies and there is a unique $C = C_L$ that solves (10) and (11).

(iii) $\pi = \underline{\pi}$ with $R = 1 + \eta$ satisfy (12) and (14) provided $C = \underline{C}$. We thus need to establish that (13) holds with strict inequality, i.e. that

$$(\underline{\pi} - 1)\underline{\pi}(1-\beta)\alpha\psi\theta^{-1} > \xi(\underline{C}; \underline{\pi}, \psi, G).$$

As in part (ii) the Lemma implies that, given $G > \underline{G}$ and ψ sufficiently small, there exists $\tilde{C} > 0$ such that

$$(\underline{\pi} - 1)\underline{\pi}(1-\beta)\alpha\psi\theta^{-1} = \xi(\tilde{C}; \underline{\pi}, \psi, G).$$

Thus for consumption lower bound $\underline{C} < \tilde{C}$ (13) holds with strict inequality.

It is straightforward to see that there is no other steady state. Suppose first that there is a steady state at π with $\underline{\pi} < \pi < \pi_L$ or $\pi > \pi^*$. Then $R > \beta^{-1}\pi$. By (12) this implies $C = \underline{C}$. But there exists C_π such that (10) is satisfied and since we can assume $\underline{C} < C_\pi$ it follows from the Lemma that (13) holds with strict inequality. However this implies $\pi = \underline{\pi}$, which contradicts our assumption. If instead $\pi_L < \pi < \pi^*$. Then $R < \beta^{-1}\pi$. But this contradicts (12).

Next, suppose $\pi_L < \underline{\pi} < \pi^*$. By (13) there cannot be a steady state at $\pi < \underline{\pi}$. Clearly there is again a steady state at $\pi = \pi^* \geq 1$. This is the unique steady state since steady states π with $\underline{\pi} < \pi < \pi^*$ or $\pi > \pi^*$ can be ruled out using the above arguments.

If $\pi_L = \underline{\pi} < \pi^*$ the steady state at π^* again exists. Any other steady state satisfies $C \geq \underline{C}$ and by (iii) $\underline{C}$ is a steady state. We also have $(\underline{\pi} - 1)\underline{\pi}(1-\beta)\alpha\psi\theta^{-1} = \xi(C_L; \underline{\pi}, \psi, G)$ from part (ii) above. Then select $\check{C} \in (\underline{C}, C_L)$. As the function $\xi(C; \underline{\pi}, \psi, G)$ is strictly increasing in C according to the Lemma, we have $(\underline{\pi} - 1)\underline{\pi}(1-\beta)\alpha\psi\theta^{-1} > \xi(\check{C}; \underline{\pi}, \psi, G)$ so that (13) holds with strict inequality. It follows that $\check{C}$ is a steady state. ■

Proof of Proposition 2: For the consumption function we employ equation (B26), which with G_{t+s}, held constant can be written in the form

$$\hat{C}_t = \left(\frac{1-\beta}{1-\bar{g}}\right)\left[\hat{Y}_t + \sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{Y}_{t+s}\right] - \left[\beta\hat{R}_t + \beta\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{R}_{t+s}\right] + \left[\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{\pi}_{t+s}\right].$$

Assuming steady-state learning, we have $\hat{E}_t\hat{Y}_{t+s} = f_Y$ and $\hat{E}_t\hat{\pi}_{t+s} = f_\pi$. With the forward looking interest rate rule $\hat{R}_t = \chi_\pi \hat{E}_t \hat{\pi}_{t+1}$ we also have $\hat{R}_t = \hat{E}_t\hat{R}_{t+s} = \chi_\pi f_\pi$ for all s locally near the targeted steady state. Thus near the targeted steady state

$$\text{(C1)} \quad \hat{C}_t = \left(\frac{1-\beta}{1-\bar{g}}\right)\left(\hat{Y}_t + \frac{\beta}{1-\beta}f_Y\right) - \beta\chi_\pi f_\pi - \frac{\beta^2}{1-\beta}\chi_\pi f_\pi + \frac{\beta}{1-\beta}f_\pi.$$

Assuming that government spending and the shocks are constant in equation (5), the Phillips curve is

$$\begin{aligned}(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t &= a_1\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{\pi}_{t+s} + a_2\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{Y}_{t+s} \\ &= a_1 f_\pi \frac{\beta\gamma_1}{1-\beta\gamma_1} + a_2 f_Y \frac{\beta\gamma_1}{1-\beta\gamma_1}.\end{aligned}$$

The third equation is the linearized market-clearing condition (B24). It can be noted that at the steady state $a_1 = 1-\gamma_1$. For steady state learning, the system giving the temporary equilibrium map $(\hat{\pi}, \hat{Y}) = T(f_\pi, f_Y)$ takes the linearized form with coefficient matrix DT at the targeted steady state given by (Mathematica routine for the details is available on request)

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = DT \begin{pmatrix} f_\pi \\ f_Y \end{pmatrix}, \text{ where}$$

$$DT = \begin{pmatrix} \frac{\beta(1-\gamma_1)}{1-\beta\gamma_1} + \frac{a_2(\chi_\pi-1)(\bar{g}-1)}{\gamma_1(1-\beta)} & \frac{a_2}{\gamma_1(1-\beta\gamma_1)}) \\ \frac{(\chi_\pi-1)(\bar{g}-1)}{1-\beta} & 1 \end{pmatrix}.$$

E-stability is determined by the eigenvalues of $DT-I$ and holds if these have negative real parts. It is easily verified that $\text{tr}(DT-I)<0$ and $\det(DT-I)>0$ when $\chi_\pi>1$ and $\bar{g}<1$, implying E-stability of the targeted steady state.

Next, consider the steady state π_L. The lower bound on the interest rate is binding locally near π_L, so we impose the constraint $\chi_\pi=0$ and evaluate the other variables at their low steady state values and impose $\psi=0$. Local stability of the low steady state is determined by the eigenvalues of DT at the steady state. It can be computed that

$$DT = \begin{pmatrix} \frac{a_2(1-\bar{g})}{(1-\beta)(1-a_1)} + \frac{\beta\gamma_1 a_1}{(1-\beta\gamma_1)(1-a_1)} & \frac{a_2}{(1-\beta\gamma_1)(1-a_1)} \\ \frac{1-\bar{g}}{1-\beta} & 1 \end{pmatrix}$$

and that $\det(DT-I)<0$ since $a_2>0$ and $0<\beta, a_1, \gamma_1, \bar{g}<1$. By continuity of eigenvalues, it follows that the low steady state π_L is unstable also for sufficiently small $\psi>0$.

Next, consider the trap steady state, where the bounds $\underline{\pi}$ and $\underline{C}$ are strictly binding as described in part (iii) of Proposition 3. Then, at the trap steady state values for f_π and f_Y, we have that the temporary equilibrium values for $\hat{\pi}_t$ and $\hat{Y}_t$ are equal to their bounds. Moreover, for sufficiently small variations in expectations f_π and f_Y the temporary equilibrium for $\hat{\pi}_t$ and $\hat{Y}_t$ remains at the bound values. It follows that $\partial\hat{\pi}_t/\partial f_\pi = \partial\hat{\pi}_t/\partial f_Y = 0$ and $\partial\hat{Y}_t/\partial f_\pi = \partial\hat{Y}_t/\partial f_Y = 0$ at the trap steady state which implies E-stability of the steady state. ■

We remark that $\det(DT-I)<0$ implies that the π_L steady state has local dynamics under learning that take the form of a saddle. Stability of the targeted steady state and instability of the π_L steady state have also been observed for the version of the model in which price adjustment costs are formulated in terms of utility losses. See, for example, Benhabib, Evans, and Honkapohja (2014).

Construction of Phase Diagram of Global E-stability Dynamics

In Figure C1 we use standard calibrated values for the structural parameters given below in Section E, and we set the interest rate rule parameters at $\chi_\pi = 1.5, \eta = 0.0001$. For convenience we set $\pi^* = 1$. Finally, we set the lower bound for consumption at 10% below the intended steady state and the lower bound for (net) inflation at -1.3%,i.e. $\underline{\hat{\pi}} = -0.013$. We also impose an upper bound to inflation to ensure existence of a temporary equilibrium. This is not needed in the linearized model with market clearing linearized around the targeted steady state.

The origin of Figure C1 represents the targeted steady state $\hat{y} = \hat{\pi} = 0$, i.e. y, π are in proportional deviation from targeted steady state form. The unintended low steady state has an output level very close to the targeted steady state; specifically, it is only -0.00040% below the value of output at the targeted steady state. The corresponding (net) inflation rate at the unintended steady state is -0.9901% i.e. $\hat{\pi}_L = -0.01008$. Finally the stagnation trap steady state, corresponding to $\underline{\hat{\pi}} = -0.013$, has an output level equal to 6.93% below the value of output at the targeted steady state.

It can be seen that the intended steady state at $\hat{\pi} = \hat{C} = 0$ is locally stable under learning (with the dynamics locally cyclical). The unintended steady state created by the ZLB is locally unstable (the dynamics are a saddle) and the stagnation steady state is locally stable. The downward sloping (almost straight line) curve through the middle steady state is the line separating the basins of attraction of the target and stagnation steady states. We refer to the latter domain as the "stagnation trap" or the "deflation trap" region. In Benhabib, Evans, and Honkapohja (2014) the basin of attraction of the targeted steady state was called the "corridor of stability" and the complement region containing explosive paths was called the "deflation trap."

We remark that if random productivity and mark-up shocks are introduced, real-time learning can converge to an ergodic distribution around the two stable steady states. This requires a sufficiently large support for the random shocks. See Sections 14.3.1 - 14.3.2 of Evans and Honkapohja (2001) for this kind of phenomenon in a simple model.

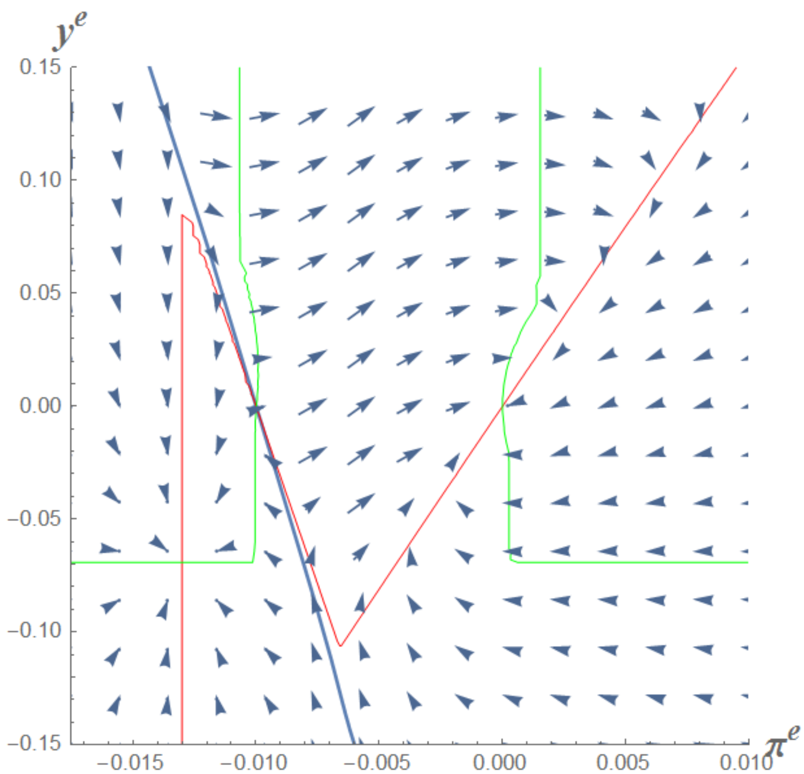

Figure C1: E-stability dynamics with forward looking Taylor rule in the case of three steady states. Here y^e and π^e denote expectations as proportional deviations from the targeted steady state, i.e. $\hat{Y}^e$ and $\hat{\pi}^e$.

We here give the additional details for constructing and interpreting Figure C1. In constructing this Figure we ignore the impact of exogenous shocks, so that we set $\hat{A}_t = \hat{\mu}_t = 0$. Consequently, the forecast rule coefficients χ_π and χ_Y consist only of the two intercepts f_π and f_Y, which allows us to illustrate global learning dynamics using a 2-dimensional figure. Under real-time learning the least-square updating equations at the end of Section C. simplify and are replaced by

$$
\begin{aligned}
f_{\pi,t} &= f_{\pi,t-1} + \kappa\left(\pi_t - f_{\pi,t-1}\right) \\
f_{Y,t} &= f_{Y,t-1} + \kappa\left(Y_t - f_{Y,t-1}\right).
\end{aligned}
$$

It is know that these real-time learning dynamics are, for small gains $\kappa > 0$, approximated by the E-stability equations given below.

The nonlinear market-clearing equation, where variables are expressed in terms of proportional deviations from the targeted steady state, is given by

$$
\hat{Y}_t = (1-\bar{g})\hat{C}_t + \frac{\psi}{2\bar{Y}}\hat{\pi}_t^2. \tag{C2}
$$

We use this rather than the linearized market-clearing equation because we are looking at global dynamics that include regions around all three of the steady states. In Figure C1 we set $\pi^* = 1$ and thus $\hat{\pi}_t = \pi_t - 1$. In the absence of lower bound constraints the temporary equilibrium equation for the Phillips curve is given by

$$
(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t = (a_1 f_\pi + a_2 f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1}, \tag{C3}
$$

where $\hat{\pi}^e = f_\pi$ and $\hat{Y}^e = f_y$. The temporary equilibrium equation for consumption is given by (C1). We modify (C1) by incorporating the ZLB and the nonlinear market-clearing equation (C2) into it. This gives the aggregate demand function

$$
\begin{aligned}
\hat{Y}_t &= \frac{\psi}{2\bar{Y}\beta}\hat{\pi}_t^2 + f_Y - \frac{1-\bar{g}}{1-\beta}\max[\chi_\pi f_\pi, \beta - 1] + \frac{1-\bar{g}}{1-\beta}f_\pi \\
&\equiv \frac{\psi}{2\bar{Y}\beta}\hat{\pi}_t^2 + F(f_Y, f_\pi).
\end{aligned} \tag{C4}
$$

The temporary equilibrium for $(\hat{Y}_t, \hat{\pi}_t)$ is given by equation (C3) and (C4), where the Phillips curve and the consumption function underlying the aggregate demand curve are interpreted as inequalities subject to lower bound constraints and holding with complementary slackness. That is, (C3) holds unless $\hat{\pi}_t < \underline{\pi}$, in which case $\hat{\pi}_t = \underline{\pi}$, and (C4) holds unless $\hat{C}_t < \underline{\hat{C}}$, in which case $\hat{C}_t = \underline{\hat{C}}$.

Substituting (C4) into (C3) gives

$$\hat{\pi}_t = \gamma_1^{-1} a_2 \left[\frac{\psi}{2\bar{Y}\beta} \hat{\pi}_t^2 + F(f_Y, f_\pi) \right] + \gamma_1^{-1} G(f_Y, f_\pi),$$

where

$$G(f_Y, f_\pi) = (a_1 f_\pi + a_2 f_Y) \frac{\beta\gamma_1}{1 - \beta\gamma_1}.$$

This can be rearranged to

$$\text{(C5)} \qquad \mathfrak{A}\hat{\pi}_t^2 - \hat{\pi}_t + \gamma_1^{-1}[a_2 F(f_Y, f_\pi) + G(f_Y, f_\pi)] = 0,$$

where

$$\mathfrak{A} = \frac{\gamma_1^{-1} a_2 \psi}{2\bar{Y}\beta}.$$

This shows that for given $\hat{\pi}^e$ there are two solutions to the quadratic (C5), provided $\hat{Y}^e = f_Y$ is not too large, and we choose the one with the smaller inflation rate, which is the economically relevant solution: this is the solution in which higher $\hat{Y}^e$ gives higher $\hat{Y}_t$ and $\hat{\pi}_t$. If $\hat{Y}^e$ is sufficiently large no temporary equilibrium solution exists to our equations. (We omit the formal details concerning existence of temporary equilibrium as they are not central to our analysis. To cover this case we replace (C3) with the common real part of the complex roots to (C5). This procedure means that the vector field in Figure C1 is continuous. This real part is in effect a maximum inflation rate, i.e. an upper bound to inflation. (Assuming instead for this case that inflation is given by a suitable fixed inflation upper bound gives similar results.)

This procedure defines the temporary equilibrium map

$$\left(\hat{\pi}, \hat{Y}\right) = T\left(\hat{\pi}^e, \hat{Y}^e\right)$$

giving the realized values of $\hat{\pi}_t$ and $\hat{Y}_t$ for given expectations $\hat{\pi}^e$ and $\hat{Y}^e$. The three steady states correspond to the fixed points of this map. E-stability dynamics are given by

$$\frac{d}{d\tau}T\left(\hat{\pi}^e,\hat{Y}^e\right)=T\left(\hat{\pi}^e,\hat{Y}^e\right)-\left(\hat{\pi}^e,\hat{Y}^e\right),$$

where τ represents "notional" time, which can, however, be linked to real time t according to the equation $\tau \approx \kappa t$. Figure C1 plots the vector field generated by $T\left(\hat{\pi}^e,\hat{Y}^e\right)-\left(\hat{\pi}^e,\hat{Y}^e\right)$. This vector field shows the paths of expectations $\left(\hat{\pi}^e,\hat{Y}^e\right)=(f_\pi,f_Y)$ under the simple learning rule given above.

To compute the curve separating the basins of attraction of the target and trap steady states one recalls that middle steady state is a saddle point, so that its one-dimensional stable manifold under the E-stability differential equation gives the boundary.

D Temporary equilibria with lower bounds and fiscal policy

We here develop the model details when the economy with exogenous shocks is subject to interest rate, inflation and output lower bounds, and fiscal policy is included. We start by focusing on the interest rate lower bound (ZLB). The Phillips curve is unaffected by the ZLB. Using the calculations after (B32) in Appendix B we have, using the first row of (B31), the Phillips Curve

$$\begin{aligned}(1-a_1)\hat{\pi}_t-a_2\hat{Y}_t \quad &= \quad (a_1f_\pi+a_2f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1}\\ &+\left[(a_1d_{\pi A}+a_2d_{YA})\frac{\rho_A\beta\gamma_1}{1-\rho_A\beta\gamma_1}-\frac{a_3}{1-\rho_A\beta\gamma_1}\right]\hat{A}_t\\ &+\left[(a_1d_{\pi\mu}+a_2d_{Y\mu})\frac{\rho_\mu\beta\gamma_1}{1-\rho_\mu\beta\gamma_1}+\frac{a_5}{1-\rho_\mu\beta\gamma_1}\right]\hat{\mu}_t \qquad \text{(D1)}\\ &-a_4(\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}+1)\frac{\Delta G}{\bar{G}},\end{aligned}$$

where the term in $\frac{\Delta G}{G}$ is set to zero for $t > T$. It is convenient to write this as

$$(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t = TPCfn(d, f, \hat{A}_t, \hat{\mu}_t, t)$$

where $d' = (d_{\pi A}, d_{YA}, d_{\pi\mu}, d_{Y\mu})$ and $f' = (f_\pi, f_Y)$.

For the IS curve we start by combining (B26) with $\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t$, which yields

$$\begin{aligned}\beta\hat{Y}_t &= \bar{g}\hat{G}_t + (1-\beta)\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} - (1-\beta)\bar{g}\sum_{s=0}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \\ &\quad -(1-\bar{g})\sum_{s=1}^{\infty}\beta^s\hat{E}_t(\hat{R}_{t+s-1} - \hat{\pi}_{t+s}).\end{aligned}$$

We write this as

$$\text{(D2)} \qquad \beta\hat{Y}_t + (1-\bar{g})\beta\hat{R}_t = ISG_t + ISPY_t + ISR_t,$$

where

$$\begin{aligned}ISPY_t &= (1-\beta)\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} + (1-\bar{g})\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\pi}_{t+s}, \\ ISG_t &= \bar{g}\hat{G}_t - (1-\beta)\bar{g}\sum_{s=0}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s}, \\ ISR_t &= -(1-\bar{g})\sum_{s=2}^{\infty}\beta^s\hat{E}_t\hat{R}_{t+s-1}.\end{aligned}$$

$ISPY_t$ and ISG_t are given by

$$\begin{aligned}ISPY_t &= \frac{\beta(1-\bar{g})}{1-\beta}f_\pi + \beta f_Y + ((1-\bar{g})d_{\pi A} + (1-\beta)d_{YA})\frac{\rho_A\beta}{1-\rho_A\beta}\hat{A}_t \\ &\quad +((1-\bar{g})d_{\pi\mu} + (1-\beta)d_{Y\mu})\frac{\rho_\mu\beta}{1-\rho_\mu\beta}\hat{\mu}_t\end{aligned}$$

and

$$ISG_t = \begin{cases} \beta\Delta G - \beta(1-\beta^{T-t})\Delta G \text{ if } t \leq T \\ 0 \text{ if } t \geq T+1. \end{cases}$$

For ISR_t we need to determine when agents expect the ZLB to apply in the future. For simplicity we assume $\rho_A = \rho_\mu = \rho$ and $0 \leq \rho < 1$. It then can be shown that there are four cases, depending on $\hat{A}_t$ and $\hat{\mu}_t$. Let

$$\begin{aligned} K &= \hat{A}_t(\chi_\pi d_{\pi A} + \chi_Y d_{YA}) + \hat{\mu}_t(\chi_\pi d_{\pi\mu} + \chi_Y d_{Y\mu}) \\ L &= \frac{\beta(1+\eta)}{\pi^*} - 1 - \chi_\pi f_\pi - \chi_Y f_Y. \end{aligned}$$

Note that K depends on $\hat{A}_t$ and $\hat{\mu}_t$ and that under learning both K and L depend on the PLM parameter estimates, which are evolving over time.

The ZLB $R \geq 1+\eta$ binds if and only if the forecasted interest rate based on the Taylor rule satisfies

$$\hat{E}_t \hat{R}^*_{t+s} = \chi_\pi \hat{E}_t \hat{\pi}_{t+s} + \chi_Y \hat{E}_t \hat{Y}_{t+s} \leq \frac{\beta(1+\eta)}{\pi^*} - 1.$$

It is then easy to see that this holds when $K\rho^s \leq L$. There are four cases:

1. The ZLB never binds (in anticipation) if $K\rho^s > L$. This happens when $K \geq 0, L < 0$ or if $K, L < 0$ and $\rho K > L$ or if $K = 0, L \neq 0$ or if $K > 0$ and $L = 0$.

2. The ZLB holds for all $s \geq 1$ if $K \leq 0, L \geq 0$ or $K, L > 0, \rho K \leq L$.

3. The ZLB holds for all $s \geq \hat{s}$, where $\hat{s}$ is the smallest integer below $\hat{s}^* = \ln(L/K)/\ln(\rho)$, if $K, L > 0, \rho K > L$.

4. The ZLB holds for all $1 \leq s \leq \hat{s}$ if $K, L < 0, \rho K \leq L$.

The value of ISR_t depends on the case. Let

$$ISR_t = -(1-\bar{g}) \times ISRi_t \text{ in case } i = 1, 2, 3, 4.$$

In case 1 we have

$$\begin{aligned} ISR1_t &= \frac{\beta\rho}{1-\beta\rho}\left[\hat{A}_t(\chi_\pi d_{\pi A} + \chi_Y d_{YA}) + \hat{\mu}_t(\chi_\pi d_{\pi\mu} + \chi_Y d_{Y\mu})\right] \\ &\quad + (\chi_\pi f_\pi + \chi_Y f_Y)\frac{\beta^2}{1-\beta}. \end{aligned}$$

In case 2 we have

$$ISR2_t = \frac{\beta^2}{1-\beta}\left(\beta\frac{(1+\eta)}{\pi^*} - 1\right).$$

In case 3 we have

$$\begin{aligned} ISR3_t &= \frac{\beta^2\rho\left(1-(\beta\rho)^{\hat{s}-1}\right)}{1-\beta\rho}\left[\hat{A}_t(\chi_\pi d_{\pi A} + \chi_Y d_{YA}) + \hat{\mu}_t(\chi_\pi d_{\pi\mu} + \chi_Y d_{Y\mu})\right] \\ &\quad +(\chi_\pi f_\pi + \chi_Y f_Y)\frac{\beta(1-\beta^{\hat{s}-1})}{1-\beta} + \left(\beta\frac{(1+\eta)}{\pi^*} - 1\right)\frac{\beta^{\hat{s}+1}}{1-\beta}. \end{aligned}$$

In case 4 we have

$$\begin{aligned} ISR4_t &= \frac{\beta(\beta\rho)^{\hat{s}}}{1-\beta\rho}\left[\hat{A}_t(\chi_\pi d_{\pi A} + \chi_Y d_{YA}) + \hat{\mu}_t(\chi_\pi d_{\pi\mu} + \chi_Y d_{Y\mu})\right] \\ &\quad +(\chi_\pi f_\pi + \chi_Y f_Y)\frac{\beta^{\hat{s}+1}}{1-\beta} + \frac{\beta^2}{1-\beta}\left(\beta\frac{(1+\eta)}{\pi^*} - 1\right)(1-\beta^{\hat{s}-1}). \end{aligned}$$

We can now solve for the tentative temporary equilibrium values $\hat{\pi}_t^{tent}, \hat{Y}_t^{tent}$ for $\hat{\pi}_t, \hat{Y}_t$ that would obtain if none of the lower bounds at time t apply. These are given by

$$\begin{aligned} (1-a_1)\hat{\pi}_t^{tent} - a_2\hat{Y}_t^{tent} &= TPC(d, f, \hat{A}_t, \hat{\mu}_t, t) \\ (1-\bar{g})\beta\chi_\pi\hat{\pi}_t^{tent} + \beta(1+\chi_Y(1-\bar{g}))\hat{Y}_t^{tent} &= TIS(d, f, \hat{A}_t, \hat{\mu}_t, t) \end{aligned}$$

where $TIS(d, f, \hat{A}_t, \hat{\mu}_t, t) = ISG_t + ISPY_t + ISR_t$ and $ISR_t = ISRn_t$ for case $n = 1, 2, 3, 4$. Under real-time learning we use the time t estimates of d, f.

We next need to incorporate the lower bounds on inflation, consumption and interest rate in the time t temporary equilibrium. The consumption lower bound gives an output lower bound. In proportional terms the lower bound $\underline{C}$ is $\underline{\hat{C}} = \frac{\underline{C}-\bar{C}}{\bar{C}}$. From $\hat{Y} = (1-\bar{g})\hat{C} + \bar{g}\hat{G}$ this gives the lower bound on $\hat{Y}$ of

$$\begin{aligned}
\underline{\hat{Y}}_t &= (1-\bar{g})\underline{\hat{C}} + \bar{g}\hat{G}_t, \text{ or} \\
\underline{\hat{Y}}_t &= (1-\bar{g})\underline{\hat{C}} + \frac{G_t - \bar{G}}{\bar{Y}}.
\end{aligned}$$

Note that $\underline{\hat{Y}}_t = (1-\bar{g})\underline{\hat{C}}$ after the fiscal policy stimulus has ended.

First we check for the ZLB at time t. Assuming $\chi_\pi \hat{\pi}_t^{tent} + \chi_Y \hat{Y}_t^{tent} \geq \beta(1+\eta) - 1$ so that the ZLB at t does not bind then set $\pi\pi_t = \hat{\pi}_t^{tent}$, $YY_t = \hat{Y}_t^{tent}$ and $RR_t = \chi_\pi \hat{\pi}_t^{tent} + \chi_Y \hat{Y}_t^{tent}$. If instead $\chi_\pi \hat{\pi}_t^{tent} + \chi_Y \hat{Y}_t^{tent} < \beta(1+\eta) - 1$ then we set $RR_t = \beta(1+\eta) - 1$ and set YY_t and $\pi\pi_t$ to solve

$$\begin{aligned}
&\text{(D3)} & (1-a_1)\pi\pi_t - a_2 YY_t &= TPC(d, f, \hat{A}_t, \hat{\mu}_t, t) \\
&\text{(D4)} & (1-\bar{g})\beta\left(\beta(1+\eta)-1\right) + \beta YY_t &= TIS(d, f, \hat{A}_t, \hat{\mu}_t, t).
\end{aligned}$$

Next, if $\pi\pi_t < \underline{\pi}$ ("situation 1") then we calculate $YYnew_t$ and $RRnew_t$ by simultaneously solving (D2) with $\hat{Y}_t = YYnew_t$ and $RRnew_t = \chi_\pi \underline{\pi} + \chi_Y YYnew_t$. If $RRnew_t > \beta(1+\eta) - 1$ then the situation 1 step has ended and we set $\pi\pi_t = \underline{\pi}$, $YY_t = YYnew_t$ and $RR_t = RRnew_t$. If instead $RRnew_t \leq \beta(1+\eta) - 1$ then set $\pi\pi_t = \underline{\pi}$, YY_t is set to solve (D4), and $RR_t = \beta(1+\eta) - 1$. If now $YY_t < \underline{\hat{Y}}_t$ then $\pi\pi_t = \underline{\pi}$, $YY_t = \underline{\hat{Y}}_t$ and $RR_t = \max(\chi_\pi \underline{\pi} + \chi_Y \underline{\hat{Y}}_t, \beta(1+\eta) - 1)$.

It remains to consider situation 2 in which $\pi\pi_t \geq \underline{\pi}$ and $YY_t < \underline{\hat{Y}}_t$. It is assumed below that $a_2 > 0$ and $0 < a_1 < 1$. This is satisfied in the calibrated cases and ensures that $\pi\pi_t > \underline{\pi}$. We set $YY_t = \underline{\hat{Y}}_t$, $\pi\pi_t$ to solve (D3) with $YY_t = \underline{\hat{Y}}_t$, and $RR_t = \max(\chi_\pi \pi\pi_t + \chi_Y \underline{\hat{Y}}_t, \beta(1+\eta) - 1)$.

The resulting values for YY_t, $\pi\pi_t$ and RR_t are the temporary equilibrium values for $\hat{Y}_t, \hat{\pi}_t$ and $\hat{R}_t$. We remark that we have not assumed that firms and households restrict forecasts to obey the consumption and inflation lower bounds. This seems natural since households may not be aware of these aggregate lower bound constraints. Under adaptive learning expectations of future inflation and output will have to eventually obey these lower bounds.

Consumption and output at the inflation lower-bound: If inflation and inflation expectations are at the inflation lower bound, i.e. $\hat{\pi}_t = \hat{E}_t\hat{\pi}_{t+s} = \hat{\underline{\pi}}$, where $\underline{\pi} < \pi_L$, then interest rates and expected interest rates are also at their lower bound, i.e. $\hat{R}_t = \hat{E}_t\hat{R}_{t+s} = \frac{\beta(1+\eta)}{\pi^*} - 1 \equiv \hat{R}_{ZLB}$. Inserting these into the consumption function (B26) we obtain

$$(1-g)\hat{C}_t = (1-\beta)\hat{Y}_t + \beta f_{Y,t} - \frac{\beta(1-\bar{g})}{1-\beta}(\hat{R}_{ZLB} - \hat{\underline{\pi}}).$$

Here we have simplified by ignoring the impact on expected output of the exogenous shocks $\hat{A}_t, \hat{\mu}_t$, i.e. we are setting $\hat{E}_t Y_{t+s} = f_{Y,t}$. This approximation is reasonable since in the temporary equilibrium at the inflation lower bound the shocks do not affect output. Combining this equation with the linearized market-clearing equation $\hat{Y}_t = (1-\bar{g})\hat{C}_t$ gives

$$\hat{Y}_t = f_{Y,t} - \frac{1-\bar{g}}{1-\beta}(\hat{R}_{ZLB} - \hat{\underline{\pi}}) \text{ where}$$

$$\hat{R}_{ZLB} - \hat{\underline{\pi}} = \frac{\beta(1+\eta) - \underline{\pi}}{\pi^*} = \frac{\pi_L - \underline{\pi}}{\pi^*} > 0$$

since we assume $\underline{\pi} < \pi_L$.

E Fiscal Policy Details and Further Simulations

Denoting the change in government spending by ΔG $(= \bar{G}' - \bar{G}\ )$ we have

$$\hat{G}_t = \begin{cases} \frac{\Delta G}{\bar{G}}, \ t = 1, ..., T \\ 0, \ t \geq T+1. \end{cases}$$

It is straightforward to compute $\sum_{s=0}^{\infty}(\beta\gamma_1)^s \hat{E}_t\hat{G}_{t+s}$ and $\sum_{s=0}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s}$, which will depend on calendar time, and include these terms in (5) and (7) when determining the temporary equilibrium.

It is useful to begin with looking at the fiscal multiplier in normal times, when the ZLB does not bind, and then move on to the more general case when the ZLB and the inflation and consumption lower bounds may be binding. In both cases we will provide information on the output multipliers for changes in government spending, and we show both the multiplier viewed as a distributed lag response and the cumulative multiplier over time. The cumulative multipliers are computed as a discounted sum using the discount factor β. Specifically, we compute

$$ym_t = \frac{Y_t - Y_t^{np}}{(\bar{G}' - \bar{G})} \text{ and } ycm_t = \frac{\sum_{i=1}^{t} \beta^{i-1}(Y_i - Y_i^{np})}{(\bar{G}' - \bar{G}) \sum_{i=1}^{T} \beta^{i-1}}, \text{ for } t = 1, 2, 3, \ldots$$

Because of discounting the cumulative multiplier will be finite even in those cases considered below in which policy leads to a permanent change in the level of output. In the formula above, Y_i^{np} denotes the level of output in period i in the absence of a policy change.

Fiscal policy in normal times

Here we use the set-up of Section II. and compute numerically government spending multipliers during normal times when the ZLB does not bind. To illustrate we consider the temporary policy change discussed above with $T = 10$.

In the examples we set $\chi_Y = 0$ to prevent monetary policy from directly acting against the output effects of fiscal policy. However we set $\chi_\pi > 1$ in line with the Taylor principle, in order to ensure both that the economy is determinate and that it is stable under least-squares learning. We set the values of ρ_A, ρ_μ and σ_A, σ_μ so that deflation is a very infrequent phenomenon in normal times and the ZLB is almost never reached. The gain parameter κ of agents is set equal to 0.04. Figure E1 shows the output and inflation paths under learning (solid line) and RE (dotted line) and the output multipliers (impact and cumulative) for a surprise temporary policy change with $T = 10$. Initial beliefs of agents and the values of the exogenous variables are at the steady state. For this example we set $\chi_\pi = 1.5$ and $\chi_Y = 0$ and consider an increase in G of 5%. The Figure shows the mean values of percent deviations of inflation and output from the steady state over $10,000$ simulations. For this setting the ZLB is never violated.

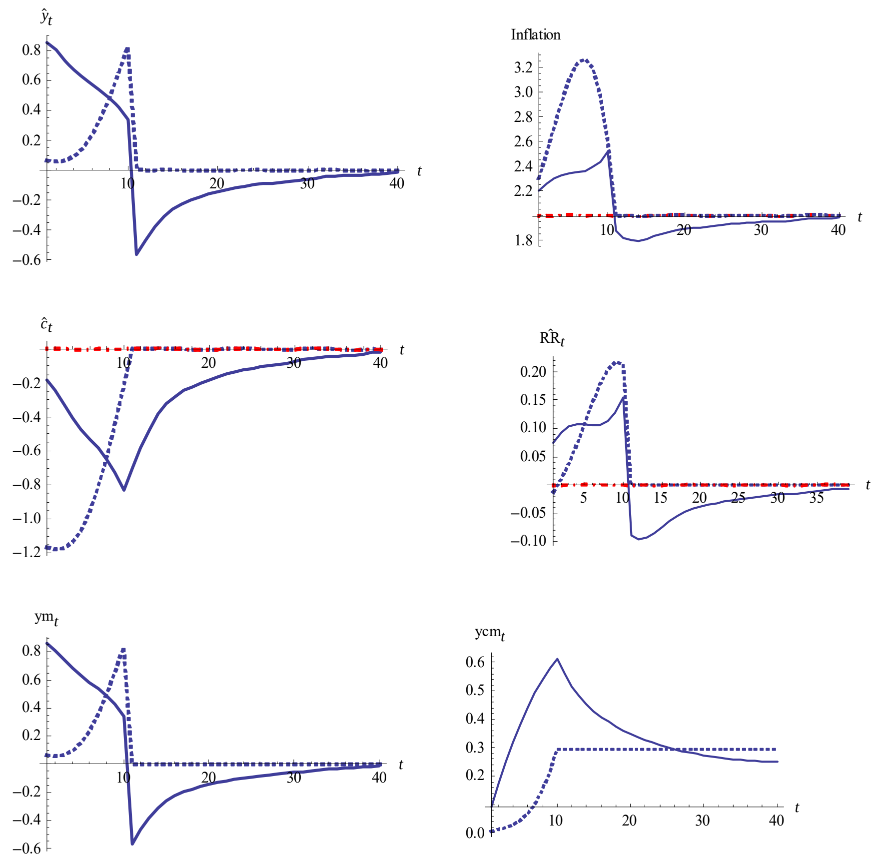

Figure E1: The upper panel shows the output and inflation paths under RE (dotted line), learning (solid line) for a temporary policy change with $T = 10$ (inflation in this and all figures is the actual annualized inflation rate). The middle panel shows the paths of the corresponding consumption and *ex ante* one period real rate of interest ($\hat{R}_t - \hat{E}_t \pi_{t+1}$). The lower panel shows the distributed lag and cumulative output multipliers. Here and in subsequent figures $\hat{y}_t$ is used for $\hat{Y}_t$.

The most notable results are that the output and multiplier effects are larger under learning in early periods of policy, compared to RE. Under learning the maximum positive output effect is at the beginning of the policy, while under RE the maximum effect is in the last period of policy. Once the policy ends, the output effects are reversed under learning, with negative deviations for several periods after the stimulus ends. This contraction is the result of the higher expected inflation of agents, developed during the policy implementation, which leads agents to anticipate higher future real interest rates in accordance with the active Taylor rule.

To understand these results, we first examine the path under RE, which is fairly complex, and best analyzed starting from the last period of the policy. From $t = T + 1 = 11$, because there are no endogenous predetermined state variables in the NK model, the economy will return to the initial RE stochastic steady state. Consider next the economy at $t = T = 10$. The extra government spending ΔG at $T = 10$ has an impact on aggregate demand that is much larger than the small reduction in consumption resulting from the corresponding one-period tax increase. Because of consumption smoothing the reduction in consumption at $T = 10$ turns out to be relatively small. The high level of output and employment at $t = 10$ leads to higher real wages, and thus higher marginal costs and higher inflation through the Phillips curve. This in turn leads to high nominal and real interest rates through the Taylor rule. Now consider the economy at earlier dates $t < 10$. The reduction in consumption is greater in earlier periods and largest at $t = 1$. This is because households anticipate both a longer period of higher taxes and a longer period of higher real interest rates. It follows that under RE the increase in output is smallest at $t = 1$, due to the crowding out, and also that the impact on inflation is low in early periods. Under RE the largest impact of fiscal policy is at the *end* of the period of increased government spending.

Consider in contrast the path under learning. This path is best understood beginning with the impact effect at $t = 1$. Households reduce consumption because of the foreseen period of temporary tax increases, but they do not foresee the sustained period of high real interest rates. The reduction in consumption is thus much smaller than under RE and there is a large increase in output and employment due to the additional government spending. Through the Phillips curve there is also an increase in inflation and interest rates. At $t = 2$ expectations of future output and inflation will both be revised upward. Because higher expected inflation translates, into higher expected future nominal and real interest rates (since $\chi_\pi > 1$ in the Taylor rule), consumption falls. For later periods with $t < T$ increasing expected inflation and real interest rates leads to further reductions in consumption and output, with continued moderate inflation. Finally, at $t = 11$, when the policy ends, there is a substantial drop in output because the reduction in government spending is not offset by an increase in consumption, which remains low due to continued high expected inflation and real interest rates under adaptive learning. The low output levels for $t > 10$ continue for a period of time until inflation expectations, in response to observed low inflation rates, return to the steady state level. Thus under learning the largest impact of fiscal policy is at the *start* of the policy, and is partially offset following the end of the policy.

Similar results are obtained if, continuing to assume that the exogenous variables are initially at their steady state levels, one now assumes that initial beliefs of inflation and output are lower than steady state values, but not so low that the ZLB will ever be obtained. Simulation results (not reported) show that this alters the path of the economy, both with and without the change in fiscal policy, but the distributed lag and cumulative output multipliers are broadly similar. The cumulative output multipliers are higher under AL than under RE during the policy implementation period, with the impact, relative to RE, concentrated in the early part of the policy. The maximum output effect of the fiscal policy under learning is in the early part of the policy, while the maximum output effect under RE occurs as the policy ends. The additional output increase under learning during the policy period is offset by lower levels of output after the policy ends.

Fiscal policy when lower bounds may be binding

We now use the framework of Sections III. and IV.. Consider now the effectiveness fiscal policy when the ZLB can be binding. For the same initial expectations as used in Figure 5, the following Figure E2 shows the impact of a small fiscal stimulus with a duration of $T = 40$ periods. Under a fiscal policy that increases G by 10%, from $G = 0.2$ to $G = 0.22$, there are positive multipliers during the policy period, with a cumulative multiplier of around 1.1, after 100 periods, which is mostly reached by period 40. However, the economy sinks back into the deflation trap around period 40.

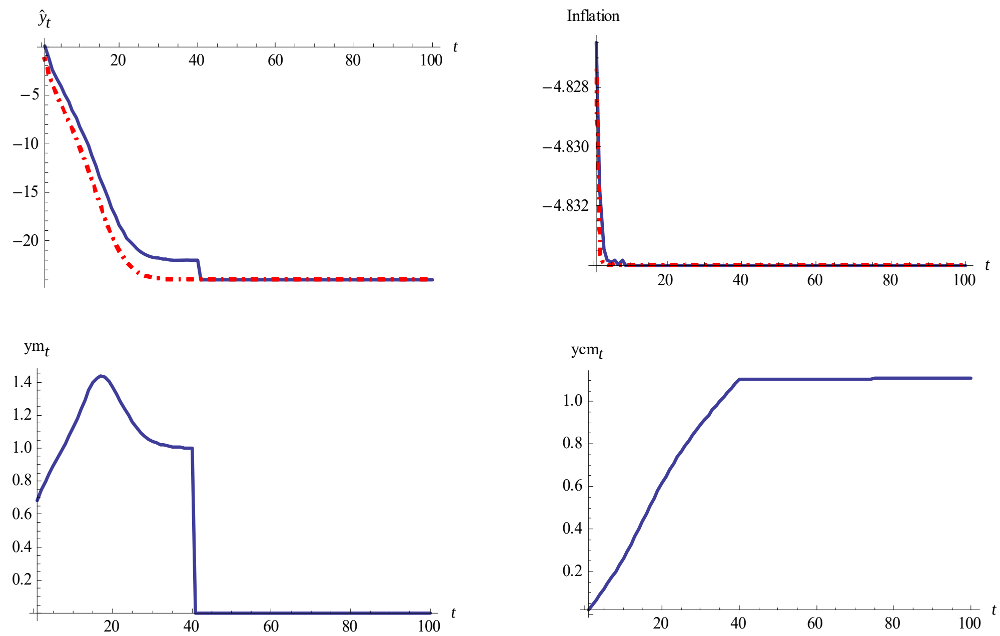

Figure E2: Small policy change. The upper panel shows the output and inflation paths under learning with policy change (solid line) and learning without policy change (dashed line) for a policy change with $T = 40$. The lower panel shows the distributed lag and cumulative output multipliers. All except one of the $10,000$ replications converge to the stagnation state with policy change (all replications converge to the stagnation state without policy change).

Results for large fiscal stimulus including multipliers were discussed in the main paper. See Figure 5. Here Figure E3 the has an added bottom panel showing the distributed lag and cumulative output multipliers averaged over all $10,000$ simulations.

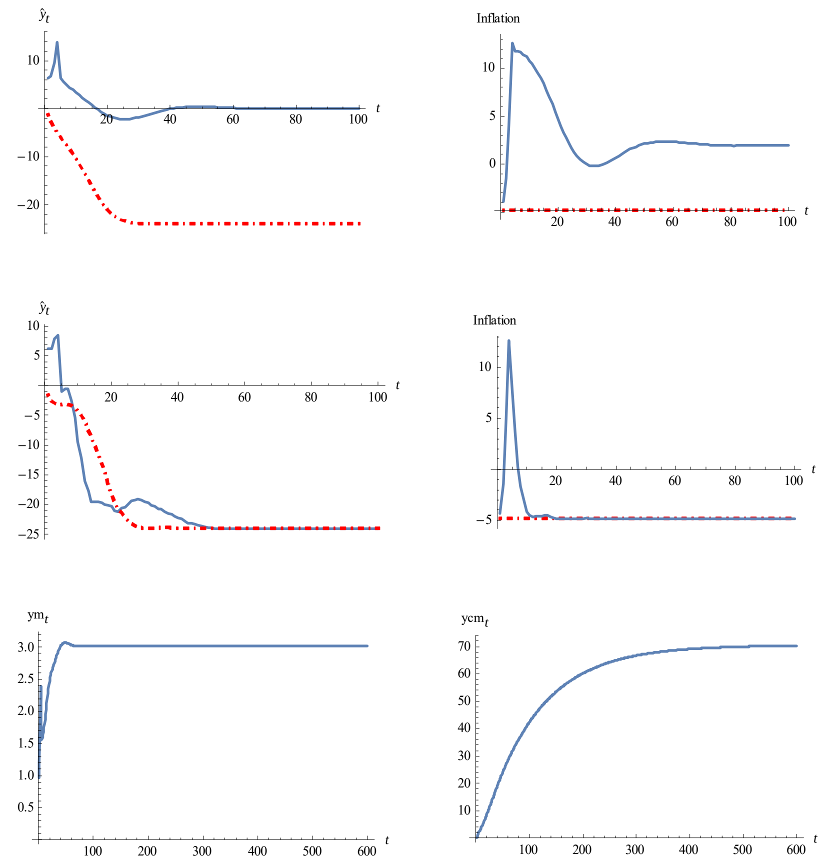

Figure E3: Large policy change. The top two panels show the output and inflation paths under learning with policy change (solid line) and without policy change (dashed line) for $T = 4$. Top panel: means of paths with convergence to targeted steady state under policy. Middle panel: means of paths with convergence to deflation trap despite policy. Bottom panel: distributed lag and cumulative output multipliers across all paths.

The following Table E1 gives the cumulative multipliers for the pessimistic initial expectations $f_\pi = -0.0148$ and $f_Y = -0.015$ used in Figure 5 and Table 1 of main text.

$G \backslash T$	1	2	3	4	5	10	20	40
0.22	4.6	3.1	2.6	2.3	2.2	2.2	1.5	1.1
0.24	12.6	14.6	11.6	11.1	8.4	3.6	1.7	1.1
0.25	36.8	35.9	27.9	19.4	14.0	5.4	2.2	1.1
0.27	76.2	43.1	29.0	21.9	17.5	8.1	17.5	1.3
0.28	73.5	37.8	25.4	19.2	15.4	7.7	3.4	1.4
0.30	59.9	30.2	20.3	15.3	12.4	6.4	3.4	1.5
0.35	40.0	20.2	13.6	10.2	8.3	6.9	2.3	1.6
0.40	30.0	15.1	10.2	7.7	6.2	3.2	1.7	1.3
0.50	20.0	10.1	6.8	5.2	4.0	2.0	1.2	1.0
0.60	15.0	7.6	5.0	3.7	2.9	1.4	1.0	1.0
1.0	7.4	3.5	2.1	1.4	1.1	0.9	0.9	0.9
2.0	2.9	1.0	0.7	0.7	0.8	0.9	0.9	1.0

Table E1: Cumulative multipliers through $t = 40$ for fiscal policies starting from pessimistic expectations. Based on 100 simulations for each cell.

Case of unique steady state.

In Figure E4 we consider the case in which the inflation lower bound is high enough so that there is a unique steady state. In this case the low level trap does not exist and the targeted steady state is unique. Initial pessimistic expectations can still lead to a very long transition to the targeted steady state and effectiveness of fiscal policy is of interest.

Figure E4 shows the impact on output and inflation of an increase in G from 0.20 to 0.24 for $T = 40$ periods where agents continue to use a gain parameter of 10%. For these simulations we set $\underline{\hat{\pi}} = -0.01475$, about 0.98% per quarter, i.e. just above the level needed to avoid the low-level trap. Initial expectations following a large pessimistic shock are set at $\hat{\pi}^e = -0.0165$ (inflation around -1.2% per quarter) and $\hat{y}^e = -0.01$. The results are reported on the basis of 6000 simulations. The cumulative multipliers here are smaller than the more dramatic values given in Table E1. However, they are substantially larger than those seen in Figure E1. This is because, although there is a unique steady state, the economy is initially in a liquidity trap. Consequently the multipliers are higher than when fiscal policy is conducted in normal times and lower bounds are not present. The relatively high values for the multipliers also in large part reflect the impact of the fiscal policy on expectations under learning: the sustained increases in output and inflation during the policy implementation period have positive impacts on inflation and output after the policy has ended due to improved expectations during the policy.

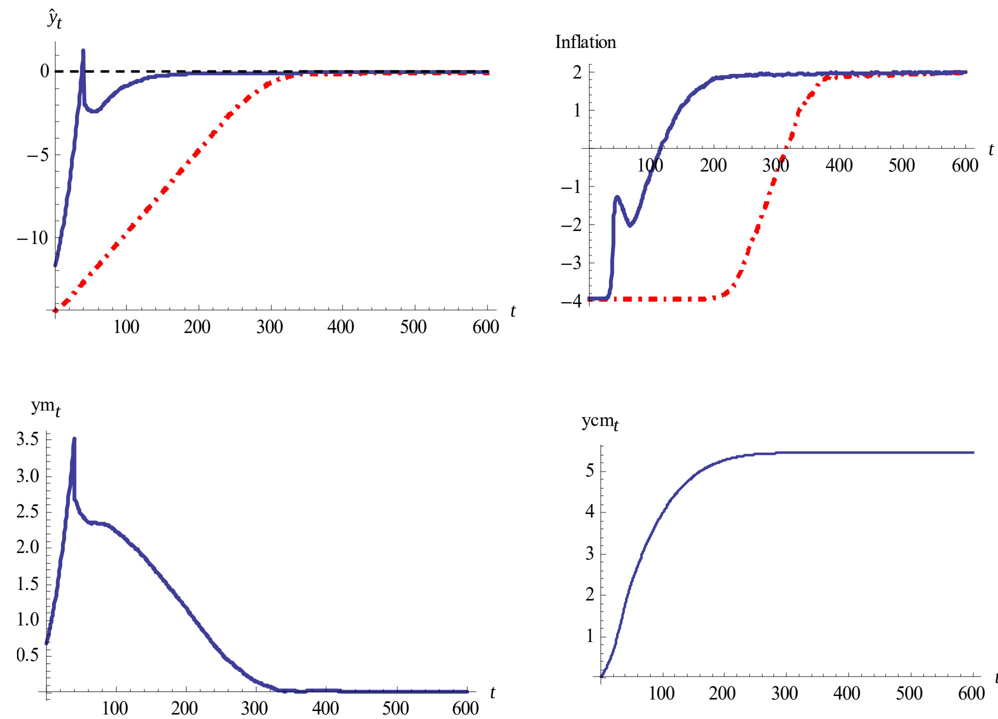

Figure E4: Policy change when there is a unique steady state with learning gain parameter equal to 10%. Top panel: output and inflation paths under learning with policy change (solid line) and without policy change (dashed line) for temporary policy change. Bottom panel: distributed lag and cumulative output multipliers.

Deflation and calibration of the discount factor

The results of Section V. emphasized the importance of the level of the inflation lower bound for the existence of a stagnation steady state and the how the critical inflation rate π_L depends upon both the discount factor β and the credit spread φ. The simulations there allowed simultaneously for both high β and $\varphi > 0$. To illustrate the separate effects we here look at the impact of high β with $\varphi = 0$.

For $\beta = 0.99$ the critical π_L, when $\eta = 0$, corresponds to 1% deflation per quarter, i.e. to 4% per year. Because the magnitude of deflation in Japan and Europe (as well as the US even in 2009-2010) has been below this value, this suggests either that the inflation lower bound $\underline{\pi}$ is above π_L or that policy has prevented inflation from falling below the critical level. On the other hand if $\beta = 0.995$ or $\beta = 0.9975$ is the appropriate value to use in the consumption Euler equation, then the critical deflation rate is only around 2% or 1% per year, in line with values that have occasionally been observed, e.g., in Japan in various periods since the 1990s. The possibility of deflation rates above these levels would appear to be a serious concern, for example, in at least some countries in Europe in 2016.

In Figure E5 we redo the simulations using the higher discount factor of $\beta = 0.9975$. We again assume agents use a gain parameter 10%. For these simulations we set $\hat{\underline{\pi}} = -0.0085$, which is somewhat below the critical value, so that there are three steady states including a low-level trap. For this value of $\hat{\underline{\pi}}$ steady state quarterly inflation is 0.9964575 which corresponds to a deflation rate of about 1.4% per annum. As in Figures E1 and E2 (and Figure 5) we set $\underline{\hat{C}} = -0.3$ which corresponds to a 30% reduction in consumption from the targeted steady state. Initial expectations following a presumed large pessimistic shock are set at $\hat{\pi}^e = -0.0074$ (about -1% per annum) and $\hat{Y}^e = -0.025$.

From Figure E5 one can see that in the absence of fiscal policy the economy would fall into a stagnation state with deflation. For these parameters a short but very aggressive fiscal stimulus is needed to avoid stagnation. In these simulations we consider an increase in G from 0.20 to 0.35 for $T = 2$ periods.

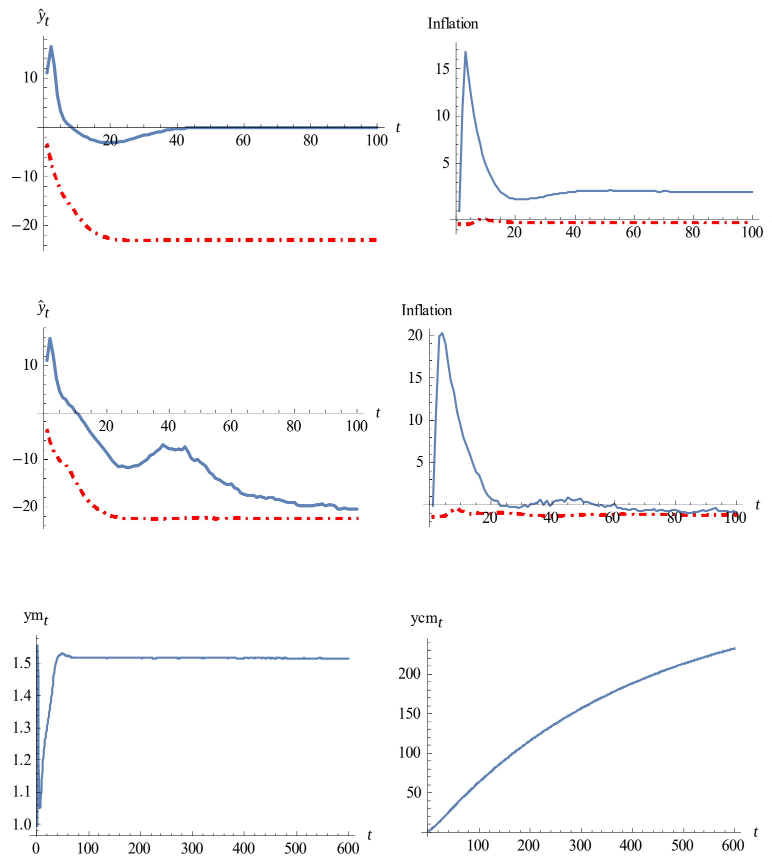

Figure E5: High discount factor, $\beta = 0.9975$. The top two panels show paths under learning with policy change (solid line) and without policy change (dashed line). Top panel: means of paths with eventual convergence to targeted steady state under policy. Middle panel: mean paths with eventual convergence to deflation trap despite policy. Bottom panel: distributed lag and cumulative output multipliers across all paths.

Our fiscal policy almost guarantees that the stagnation state will be avoided since for these parameters, based on $10,000$ simulations, 98.6% of the with-policy simulations converged to the intended steady state while 1.4% eventually sank to the stagnation steady state. In sharp contrast, without policy change, 95.3% of the simulations sink to the stagnation steady state and only the remaining 4.7% converge to the targeted steady state. In Figure E5 the top panel shows the mean paths of those simulations that under the policy converge to the intended steady state. The middle panel shows the mean paths of those few paths that, despite policy, eventually converge to the stagnation steady state. For these cases we note that the process is typically very slow. The bottom panel shows the multipliers averaged over all simulations. We remark that the cumulative multipliers are very high, even though the policy does not guarantee escape from eventual stagnation and deflation. Finding a mix of policies that maximizes the chance of avoiding the stagnation trap would be useful to consider in future work.

F Model with forecasting of wages and profits

We have assumed that households forecast their own income while making future decisions about their consumption as in Eusepi and Preston (2010) and Eusepi and Preston (2012). In this approach households directly forecast their period income which is the sum of wage income and profit (dividend) income. Since agents' wage income depends on their own labor supply choice, this approach has agents forecasting variables (income) that depend on endogenous variables like labor supply. This issue does not arise in the case of firms forecasting aggregate income since the latter is assumed to be exogenous to the firm's choice of price level while maximizing their own profits (however, see below). An advantage of our approach is that it yields a consumption function close to traditional formulations based on the permanent income and life-cycle models. Woodford (2013) also employs a consumption function based on current and expected income net of taxes and real interest rates.

Preston (2005), on the other hand, assumes that agents only forecast variables that are exogenous to their own decision problem. We now show that our results are robust when agents forecast in this fashion. This alternative approach is implemented by assuming that households forecast wage rates and profits, i.e. only variables that are exogenous to their decision problem (instead of their own income) as in Eusepi and Preston (2012b) and Eusepi, Giannoni and Preston (2012). To operationalize this approach households use a consumption function that depends on forecasts of wages and profits. The details of this approach are given below. Households use PLMs for wages and profits which take the same form as the minimal state variable solution and use constant gain learning of the same form used before. This affects the consumption function (see equation (F3) in this Online Appendix) and hence the aggregate demand equation of the model.

Firms take aggregate demand as exogenous when choosing their optimal price so their decision problem is unaffected and the Phillips curve stays the same as before. We remark, however, that there are potentially two ways of implementing the firms' forecasting problem. In one approach they are assumed to forecast future inflation, aggregate demand and wages (apart from the exogenous shocks); see equation (B20) in Online Appendix B. This approach is adopted in Eusepi and Preston (2012b) and Eusepi, Giannoni and Preston (2012). However, since these authors assume constant returns to scale in the production function, firms do not have to forecast future aggregate demand in their analysis. As we assume decreasing returns to scale, firms also have to forecast aggregate demand in equation (B20). Alternately, if firms make use of the labor supply schedule, the production function and the market clearing condition (i.e. use the three equations preceding equation (B22)) then they only need to forecast future inflation and aggregate demand as in equation (B22). We make use of this latter simplifying assumption in what follows.

We consider the case when fiscal policy changes in normal times i.e. in situations when the ZLB does not bind as in Section E. For illustrative purposes we consider the same policy change in Figure E1. We find that the qualitative dynamics illustrated in Figure E1 remain unchanged when households forecast wages and profits instead of income (including the dynamics of the distributed and cumulative lag output multipliers). The quantitative dynamics are also similar and only slightly different, e.g. consumption falls slightly more towards the end of the policy change when households forecast wages and profits compared to the situation in Figure E1 (for brevity, we do not present the figure here). This indicates that our results are robust to the alternative approach in which agents forecast wages and profits.

It can be noted that a similar phenomenon is observed in Kuang and Mitra (2015, 2016) in the context of the real business cycle model. The statistical results, impulse responses etc. are very similar in these two scenarios: compare Kuang and Mitra.

Formal details.

We now redo the consumption function when households forecast wages and profits. Refer to the linearized Euler equation and the lifetime budget constraint i.e. (B3) and (B9) and the definition of household period income which is

$$Y_{t,i} = w_t h_{t,i} + \Omega_t^i.$$

In deviation form, the previous equation is

$$\tilde{Y}_{t,i} = \bar{w}\tilde{h}_{t,i} + \tilde{w}_t\bar{h} + \tilde{\Omega}_t^i. \tag{F1}$$

Since all households earn the same profits, $\tilde{\Omega}_t^i = \tilde{\Omega}_t$,and using the static linearized first order condition of the household i.e.

$$\tilde{w}_t = \frac{\epsilon\bar{w}}{\bar{h}}\tilde{h}_{t,i} + \frac{\bar{w}}{\bar{c}}\tilde{c}_{t,i}$$

we obtain (recall bars over variables indicate their steady state values)

$$\bar{w}\tilde{h}_{t,i} = \bar{h}\epsilon^{-1}\tilde{w}_t - \bar{h}\epsilon^{-1}\bar{w}\bar{c}^{-1}\tilde{c}_{t,i}.$$

Substituting this in (F1) we obtain

$$\tilde{Y}_{t,i} = (1+\epsilon^{-1})\bar{h}\tilde{w}_t - \bar{h}\bar{w}\epsilon^{-1}\bar{c}^{-1}\tilde{c}_{t,i} + \tilde{\Omega}_t.$$

In turn substituting this equation into the linearized infinite horizon budget constraint we have

$$\sum_{s\geq 0}\beta^s\tilde{G}_{t+s} = \sum_{s\geq 0}\beta^s[(1+\epsilon^{-1})\bar{h}\tilde{w}_{t+s} + \tilde{\Omega}_{t+s} - (1+\bar{h}\epsilon^{-1}\bar{w}\bar{c}^{-1})\tilde{c}_{t+s,i}]. \tag{F2}$$

Defining

$$
\begin{aligned}
S\tilde{G}_t &= \hat{E}_t \sum_{s=1}^{\infty} \beta^s \tilde{G}_{t+s}, \\
S\tilde{w}_t &= \hat{E}_t \sum_{s=1}^{\infty} \beta^s \tilde{w}_{t+s}, \\
S\tilde{\Omega}_t &= \hat{E}_t \sum_{s=1}^{\infty} \beta^s \tilde{\Omega}_{t+s}.
\end{aligned}
$$

and taking expectations at time t of equation (F2) we obtain the following

$$
\tilde{G}_t + S\tilde{G}_t = (1+\epsilon^{-1})\bar{h}(\tilde{w}_t + S\tilde{w}_t) + \tilde{\Omega}_t + S\tilde{\Omega}_t - (1+\bar{h}\bar{w}\epsilon^{-1}\bar{c}^{-1}) \sum_{s=0}^{\infty} \hat{E}_{ti}\beta^s \tilde{c}_{t+s,i}.
$$

Using the one-step ahead consumption Euler equation we then obtain the following

$$
\begin{aligned}
(1+\bar{h}\bar{w}\epsilon^{-1}\bar{c}^{-1})(1-\beta)^{-1}\tilde{c}_{t,i} &= (1+\epsilon^{-1})\bar{h}(\tilde{w}_t + S\tilde{w}_t) + \tilde{\Omega}_t + S\tilde{\Omega}_t \\
&\quad -(1+\bar{h}\bar{w}\epsilon^{-1}\bar{c}^{-1})\beta\bar{c} \sum_{s=1}^{\infty} \beta^s \sum_{j=1}^{s} \hat{E}_{ti}\tilde{r}_{t+j}.
\end{aligned}
$$

Using

$$
\sum_{s=1}^{\infty} \beta^s \sum_{j=1}^{s} \hat{E}_{ti}\tilde{r}_{t+j} = (1-\beta)^{-1} \sum_{j=1}^{\infty} \beta^j \hat{E}_{ti}\tilde{r}_{t+j}
$$

we get finally

$$
\begin{aligned}
\tilde{c}_{t,i} &= \frac{(1-\beta)}{(1+\bar{h}\bar{w}\epsilon^{-1}\bar{c}^{-1})}[(1+\epsilon^{-1})\bar{h}(\tilde{w}_t + S\tilde{w}_t) + \tilde{\Omega}_t + S\tilde{\Omega}_t - \tilde{G}_t - S\tilde{G}_t] \\
&\quad -\beta(1-\beta)\bar{c}(1-\beta)^{-1} \sum_{j=1}^{\infty} \beta^j \hat{E}_{ti}\tilde{r}_{t+j}.
\end{aligned}
$$

In proportional deviation form, and dropping i (by symmetry),

$$\begin{aligned}\hat{C}_t &= \frac{(1-\beta)}{(\bar{c}+\bar{h}\bar{w}\epsilon^{-1})}[(1+\epsilon^{-1})\bar{w}\bar{h}(\hat{w}_t+S\hat{w}_t)+\bar{\Omega}(\hat{\Omega}_t+S\hat{\Omega}_t) \\ &\quad -\bar{G}(\hat{G}_t+S\hat{G}_t)]-\hat{E}_t\sum_{j=1}^{\infty}\beta^j\hat{r}_{t+j}.\end{aligned}$$

Here $\hat{r}_{t+j}=\beta\tilde{r}_{t+j}$. Finally we obtain the consumption function of the representative household in the case when they forecast future wages and profits

$$\begin{aligned}\hat{C}_t &= \frac{(1-\beta)}{(\bar{c}+\bar{h}\bar{w}\epsilon^{-1})}[(1+\epsilon^{-1})\bar{w}\bar{h}(\hat{w}_t+S\hat{w}_t)+\bar{\Omega}(\hat{\Omega}_t+S\hat{\Omega}_t) \\ &\quad -\bar{G}(\hat{G}_t+S\hat{G}_t)]-\beta\hat{R}_t-\beta S\hat{R}_t+S\hat{\pi}_t, \\ S\hat{R}_t &= \hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{R}_{t+s};\, S\hat{\pi}_t=\hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{\pi}_{t+s}.\end{aligned} \tag{F3}$$

Since we only consider the case when the ZLB never binds, the interest rate rule (6) is then plugged into the consumption function. $\hat{C}_t$ is thus determined based on $\hat{w}_t, \hat{\Omega}_t, \hat{G}_t, \hat{\pi}_t, \hat{Y}_t$(if $\chi_Y \neq 0$) and expectations of these variables.

In temporary equilibrium the following also hold

$$\begin{aligned}\hat{w}_t &= \epsilon\hat{h}_t+\hat{C}_t, \\ \hat{Y}_t &= (1-\bar{g})^{-1}\hat{C}_t+\bar{g}\hat{G}_t, \\ \hat{h}_t &= \alpha^{-1}(\hat{Y}_t-\hat{A}_t), \\ \bar{\Omega}\hat{\Omega}_t &= \bar{Y}\hat{Y}_t-\bar{w}\bar{h}(\hat{h}_t+\hat{w}_t)=\bar{Y}(\hat{Y}_t-(1-\bar{\Omega})(\hat{h}_t+\hat{w}_t).\end{aligned}$$

The above equations follow from the first order condition of the household, market clearing condition, the production function and (F1) respectively, all in proportional deviation from the targeted steady state. This gives the temporary equilibrium in terms of $\hat{C}_t$ from (F3), $\hat{w}_t, \hat{Y}_t$, $\hat{h}_t, \hat{\Omega}_t$ from the previous four equations and $\hat{\pi}_t$ from the original Phillips curve equation (which remains unchanged in this formulation).

To obtain $S\hat{w}_t$, $S\hat{\Omega}_t$ in (F3) we assume constant gain learning of $\hat{\Omega}^e$ and $\hat{w}^e$ by regression of $\hat{\Omega}_t$ and $\hat{w}_t$ on intercepts and on $\hat{A}_t$ and $\hat{\mu}_t$. For our policy change and assumed learning forms, the infinite sums simplify further. $S\hat{G}_t = 0$ for $t \geq T$ while for $1 \leq t \leq T-1$

$$S\hat{G}_t = \frac{\beta(1-\beta^{T-t})}{1-\beta}\frac{\Delta G}{\bar{G}}.$$

For $S\hat{w}_t$ and $S\hat{\Omega}_t$ we use the PLMs

$$\begin{aligned}
\hat{w}_t &= f_w + d_{wA}\hat{A}_t + d_{w\mu}\hat{\mu}_t, \\
\hat{\Omega}_t &= f_\Omega + d_{\Omega A}\hat{A}_t + d_{\Omega\mu}\hat{\mu}_t.
\end{aligned}$$

As before, under adaptive learning, agents estimate the coefficients of the previous equations and given their time t estimates of the parameter coefficients, the forecasts $\hat{E}_t\hat{w}_{t+s}$ and $\hat{E}_t\hat{\Omega}_{t+s}$ are given by

$$\begin{aligned}
\hat{E}_t\hat{w}_{t+s} &= f_w + d_{wA}\rho_A^s\hat{A}_t + d_{w\mu}\rho_\mu^s\hat{\mu}_t, \\
\hat{E}_t\hat{\Omega}_{t+s} &= f_\Omega + d_{\Omega A}\rho_A^s\hat{A}_t + d_{\Omega\mu}\rho_\mu^s\hat{\mu}_t.
\end{aligned}$$

Using these PLMs and continuing to use the same PLMs for $\hat{\pi}_t, \hat{Y}_t$ as before we obtain the following (the final line uses knowledge of the Taylor rule on the part of households)

$$\begin{aligned}
S\hat{w}_t &= \hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{w}_{t+s} = \frac{\beta}{1-\beta}f_w + d_{wA}\frac{\beta\rho_A}{1-\beta\rho_A}\hat{A}_t + d_{w\mu}\frac{\beta\rho_\mu}{1-\beta\rho_\mu}\hat{\mu}_t, \\
S\hat{\Omega}_t &= \hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\Omega}_{t+s} = \frac{\beta}{1-\beta}f_\Omega + d_{\Omega A}\frac{\beta\rho_A}{1-\beta\rho_A}\hat{A}_t + d_{\Omega\mu}\frac{\beta\rho_\mu}{1-\beta\rho_\mu}\hat{\mu}_t, \\
S\hat{\pi}_t &= \hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\pi}_{t+s} = \frac{\beta}{1-\beta}f_\pi + d_{\pi A}\frac{\beta\rho_A}{1-\beta\rho_A}\hat{A}_t + d_{\pi\mu}\frac{\beta\rho_\mu}{1-\beta\rho_\mu}\hat{\mu}_t, \\
S\hat{Y}_t &= \frac{\beta}{1-\beta}f_Y + d_{\pi A}\frac{\beta\rho_A}{1-\beta\rho_A}\hat{A}_t + d_{\pi\mu}\frac{\beta\rho_\mu}{1-\beta\rho_\mu}\hat{\mu}_t, \\
S\hat{R}_t &= \chi_\pi S\hat{\pi}_t + \chi_Y S\hat{Y}_t.
\end{aligned}$$

The Phillips curve continues to be given by equation (5) where the infinite sums simplify to

$$
\begin{aligned}
\sum_{s=1}^{\infty}(\beta\gamma_1)^s \hat{E}_t\hat{\pi}_{t+s} &= \frac{\beta\gamma_1}{1-\beta\gamma_1}f_\pi + d_{\pi A}\frac{\beta\gamma_1\rho_A}{1-\beta\gamma_1\rho_A}\hat{A}_t + d_{\pi\mu}\frac{\beta\gamma_1\rho_\mu}{1-\beta\gamma_1\rho_\mu}\hat{\mu}_t, \\
\sum_{s=1}^{\infty}(\beta\gamma_1)^s \hat{E}_t\hat{Y}_{t+s} &= \frac{\beta\gamma_1}{1-\beta\gamma_1}f_Y + d_{\pi A}\frac{\beta\gamma_1\rho_A}{1-\beta\gamma_1\rho_A}\hat{A}_t + d_{\pi\mu}\frac{\beta\gamma_1\rho_\mu}{1-\beta\gamma_1\rho_\mu}\hat{\mu}_t, \\
\sum_{s=0}^{\infty}(\beta\gamma_1)^s \hat{E}_t\hat{A}_{t+s} &= \sum_{s=0}^{\infty}(\beta\gamma_1\rho_A)^s\hat{A}_t = \frac{1}{1-\beta\gamma_1\rho_A}\hat{A}_t, \\
\sum_{s=0}^{\infty}(\beta\gamma_1)^s \hat{E}_t\hat{\mu}_{t+s} &= \sum_{s=0}^{\infty}(\beta\gamma_1\rho_\mu)^s\hat{\mu}_t = \frac{1}{1-\beta\gamma_1\rho_\mu}\hat{\mu}_t,
\end{aligned}
$$

and finally for $1 \leq t \leq T-1$,

$$
\sum_{s=0}^{\infty}(\beta\gamma_1)^s \hat{E}_t\hat{G}_{t+s} = \frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}\frac{\Delta G}{\bar{G}}
$$

and the same sum is zero for $t \geq T$.

G Model with Credit Frictions

Curdia and Woodford (2015) work explicitly through the aggregation problem and show that the aggregate implications correspond to interpreting φ as the average of i and the borrowing rate; the shortcut in our representative-agent setting is then simply to directly interpret the market interest rate for households as $R_t = i_t + \varphi$, where i_t is the interest rate set by policymakers. It is, of course, the policy interest rate that is subject to the lower bound. The benchmark calibration in Curdia and Woodford (2015) corresponds to a value $\varphi = 0.0025$, i.e. to 1% per annum.

We next discuss the implications of including credit frictions. Incorporating an interest rate spread $\varphi > 0$ is formally identical to our Section III. model without a credit friction, but in which the central bank places a floor to its policy rate at $1 + \eta'$ where $\eta' = \varphi > 0$. To see this, note first that, in our current setting and with inflation target π^*, the steady state market interest rate R satisfies $R = \beta^{-1}\pi^*$ and the corresponding policy rate with credit frictions is $i = \beta^{-1}\pi^* - \varphi$. The Taylor rule for the policy rate, subject to the $1 + \eta$ lower bound, is given by $i = \max\left(\beta^{-1}\chi_\pi(\pi - \pi^*) + \beta^{-1}\pi^* - \varphi, 1 + \eta\right)$. Equivalently the market interest rate satisfies $R = \max\left(\beta^{-1}\chi_\pi(\pi - \pi^*) + \beta^{-1}\pi^*, 1 + \eta'\right)$, where $\eta' = \eta + \varphi$. As in Section IV., under learning agents use knowledge of this relationship and forecasts of inflation to forecast future market interest rates. It follows that to capture the impact of a steady state credit friction $\varphi > 0$ in our model we simply replace η by $\eta' = \eta + \varphi$.

Before turning to numerical simulations it is useful to reconsider Figure 3, showing the existence of multiple steady states. The variable R on the vertical axis has a lower bound of $1+\eta'$ and is now interpreted as the market interest rate, not the policy rate. The corresponding unintended steady state is at $\pi_L = \beta(1 + \eta')$. Provided the inflation lower bound satisfies $\underline{\pi} < \pi_L$ there are three steady states as shown in Figure 3. For given $\underline{\pi}$ an increase in the credit spread φ increases η' leading to an increase in the (locally unstable) unintended steady state associated with π_L. This will increase the basin of attraction of the stagnation steady state and reduce the basin of attraction of the targeted steady state.

A new result of considerable practical significance arising from credit frictions is that if $\beta(1+\eta') > 1$ then it is possible to have $1 < \underline{\pi} < \pi_L < \pi^*$. Thus, not only does including a credit friction raise the critical inflation rate to a mild deflation level, but it is also possible for the critical inflation rate to be positive. In this case a stagnation steady state can correspond to a zero or low positive inflation rate. Another new phenomenon is that if φ is large enough then both the targeted steady state and the unintended steady state disappear and only the stagnation steady states remains.

Figure 6 of the main text looked at the case of high β and $\varphi > 0$, and showed the mean paths of those simulations that under the policy converge to the intended steady state. Figure E6 gives additional detail concerning the divergent paths and multipliers for this set of simulations. The top panel shows the mean dynamics for the convergent cases as in Figure 6. The middle panel shows the mean paths of those few paths that, despite policy, eventually converge to the stagnation steady state. For these cases we note that the process is typically very slow. The bottom panel shows the multipliers averaged over all simulations. We remark that the cumulative multipliers are very high, even though the policy does not guarantee escape from eventual stagnation and deflation. Finding a mix of policies that maximizes the chance of avoiding the stagnation trap would be useful to consider in future work.

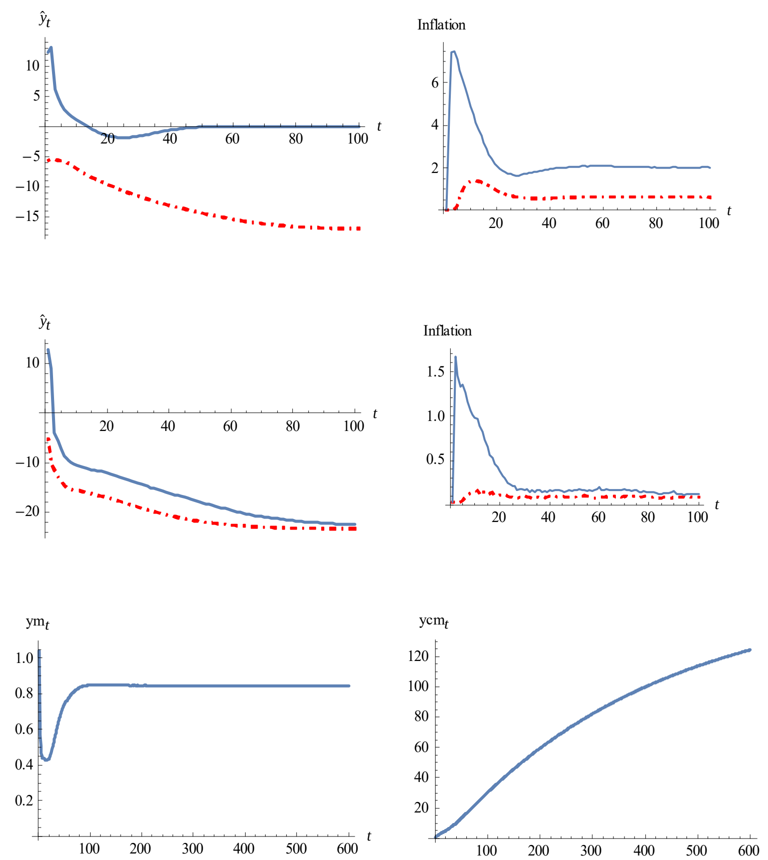

Figure E6: High β and credit spread. The top two panels show paths under learning with policy change (solid line) and without policy change (dashed line). Top panel: means of paths with eventual convergence to targeted steady state under policy. Middle panel: mean paths with eventual convergence to deflation trap despite policy. Bottom panel: distributed lag and cumulative output multipliers across all paths.

Additional References for Online Appendix

Eusepi, S. and B. Preston (2012b), "The Maturity Structure of Debt, Monetary Policy and Expectations Stabilization," mimeo.

Eusepi, S., M. Giannoni and B. Preston (2012), "Long-Tern Debt Pricing and Monetary Policy Transmission under Imperfect Knowledge," Staff report no. 547, Federal Reserve Bank of New York Staff Reports.

Kuang, P. and K. Mitra (2015), "Long-Run Growth Uncertainty," Discussion Paper no. 15-07, University of Birmingham.

Kuang, P. and K. Mitra (2016), "Long-Run Growth Uncertainty," *Journal of Monetary Economics,* 79, 67-80.

Sargent, T.J. (1987), *Macroeconomic Theory*, second edition, Academic Press, New York.

Woodford, M. (2013): "Macroeconomic Analysis without the Rational Expectations Hypothesis," *Annual Review of Economics*, 5, 303-346.