Applications of Affine and Weyl Geometry

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Eduardo García-Río, Peter Gilkey, Stana Nikčević, and Ramón Vázquez-Lorenzo
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ABSTRACT

Pseudo-Riemannian geometry is, to a large extent, the study of the Levi-Civita connection, which is the unique torsion-free connection compatible with the metric structure. There are, however, other affine connections which arise in different contexts, such as conformal geometry, contact structures, Weyl structures, and almost Hermitian geometry. In this book, we reverse this point of view and instead associate an auxiliary pseudo-Riemannian structure of neutral signature to certain affine connections and use this correspondence to study both geometries. We examine Walker structures, Riemannian extensions, and Kähler–Weyl geometry from this viewpoint. This book is intended to be accessible to mathematicians who are not expert in the subject and to students with a basic grounding in differential geometry. Consequently, the first chapter contains a comprehensive introduction to the basic results and definitions we shall need—proofs are included of many of these results to make it as self-contained as possible. Para-complex geometry plays an important role throughout the book and consequently is treated carefully in various chapters, as is the representation theory underlying various results. It is a feature of this book that, rather than as regarding para-complex geometry as an adjunct to complex geometry, instead, we shall often introduce the para-complex concepts first and only later pass to the complex setting.

The second and third chapters are devoted to the study of various kinds of Riemannian extensions that associate to an affine structure on a manifold a corresponding metric of neutral signature on its cotangent bundle. These play a role in various questions involving the spectral geometry of the curvature operator and homogeneous connections on surfaces. The fourth chapter deals with Kähler–Weyl geometry, which lies, in a certain sense, midway between affine geometry and Kähler geometry. Another feature of the book is that we have tried wherever possible to find the original references in the subject for possible historical interest. Thus, we have cited the seminal papers of Levi-Civita, Ricci, Schouten, and Weyl, to name but a few exemplars. We have also given different proofs of various results than those that are given in the literature, to take advantage of the unified treatment of the area given herein.

KEYWORDS

curvature decomposition, deformed Riemannian extension, Kähler–Weyl geometry, modified Riemannian extension, Riemannian extension, spectral geometry of the curvature operator

This book is dedicated to
Carmen, Emily, George, Hugo, Luis, Manuel, Montse, and Susana.

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Preface

The fundamental theorem of pseudo-Riemannian geometry associates to each pseudo-Riemannian metric g a unique affine connection,

$$\nabla = {}^{g}\nabla ,$$

called the Levi-Civita connection (we refer to Levi-Civita [151] and to Ricci and Levi-Civita [188]), and pseudo-Riemannian geometry focuses, to a large extent, on the geometry of this connection. There are, however, other natural connections that play a role when considering different geometric structures, such as almost Hermitian structures, almost contact structures, and Weyl structures. Affine connections also arise naturally in conformal geometry. In all these cases, the affine connections under consideration are adapted to the structures under investigation. One can reverse this point of view and associate an auxiliary pseudo-Riemannian structure to a given affine connection and then use this correspondence to examine the geometry of both objects. As an exemplar, the Riemannian extension is a natural, neutral signature, pseudo-Riemannian metric on the cotangent bundle of an underlying affine manifold. This construction, which goes back to the work of Patterson and Walker [177], has played an important role in many investigations.

This book examines a number of different areas of differential geometry which are related to affine differential geometry—Walker structures, Riemannian extensions, and (para)-Kähler–Weyl geometry. It is intended to be accessible to graduate students who have had a basic course in differential geometry, as well as to mathematicians who are not necessarily experts in the subject. For that reason, Chapter 1 contains a basic introduction to the matters under consideration. The Lie derivative and bracket, connections, Kähler geometry, and curvature are discussed. The notion of a curvature model is treated and the basic decomposition theorems of Singer–Thorpe [194], Higa [125, 126], and Tricerri–Vanhecke [200] are given, not only in the Riemannian, but in the pseudo-Riemannian and in the para-complex settings as well. Almost para-Hermitian and almost Hermitian structures are presented both in the Riemannian and pseudo-Riemannian categories. The Gray–Hervella [118] classification of almost Hermitian structures is extended to the almost para-Hermitian and to the almost pseudo-Hermitian contexts. The geometry of the cotangent bundle is outlined (tautological 1-form, evaluation map, complete lift) and the various natural metrics on the cotangent bundle (Riemannian extensions, deformed Riemannian extensions, modified Riemannian extensions) are defined – these will be examined in further detail in Chapter 2 and Chapter 3. A short introduction to Walker geometry and recurrent curvature is included. Self-dual Walker metrics are discussed and it is shown that any such metric is locally isometric to the metric of a Riemannian extension. The Jacobi operator,

$$\mathcal{J}(x) : y \to \mathcal{R}(y, x)x ,$$

is introduced and classical results concerning symmetric spaces, spaces of constant sectional curvature, and constant holomorphic sectional curvature are presented using Jacobi vector fields. Chapter 1 concludes with a brief review of the spectral geometry of the curvature operator. Complete proofs of a number of results are presented to keep the treatment as self-contained as possible.

Chapter 2 examines the geometry of Riemannian extensions in more detail and extends the discussion of Chapter 1 in that regard. Riemannian extensions form a natural family of examples and provide a first link between affine and pseudo-Riemannian geometry. The relevant facts concerning the metric of the classical Riemannian extension are developed in some detail. Riemannian extensions are then used to study the spectral geometry of the curvature tensor in the affine setting; affine Osserman surfaces and affine Ivanov–Petrova surfaces are examined in some detail. Chapter 2 concludes with a fairly lengthy treatment of homogeneous affine connections. Homogeneous connections on surfaces that are not the Levi-Civita connection of a metric of constant curvature form two natural classes which are not disjoint, and the intersection of the two non-metric classes is studied using the corresponding Riemannian extensions.

Chapter 3 presents generalizations of Riemannian extensions. While Riemannian extensions are useful in constructing self-dual Ricci flat metrics, the modified Riemannian extensions are a source of non-Ricci flat Einstein metrics. Four-dimensional geometry is explored in some detail. A new approach, based on the generalized Goldberg–Sachs theorem, is used to obtain some previously known results on the classification of 4-dimensional Osserman metrics from this point of view. The usefulness of modified Riemannian extensions is made clear when discussing 4-dimensional Osserman metrics whose Jacobi operators have non-degenerate Jordan normal form. Para-Kähler manifolds of constant para-holomorphic sectional curvature are treated in this fashion, as are a variety of higher-dimensional Osserman metrics – it is shown that any such manifold is locally isometric to the modified Riemannian extension metric of a flat affine manifold. By considering non-flat affine Osserman connections, one obtains a family of *deformed* para-Kähler metrics with the same spectrum of the Jacobi operator. This provides a useful family of Osserman metrics whose Jacobi operators have non-trivial Jordan normal form. The related property of being (semi) para-complex Osserman is also considered, and it is shown that any modified Riemannian extension is a semi para-complex Osserman metric. Here, the curvature identities induced by the almost para-complex structure play an essential role; note that the para-holomorphic sectional curvature does not necessarily determine the curvature in the general setting.

Chapter 4 treats (para)-Kähler–Weyl geometry. Classical Weyl geometry is, in a certain sense, midway between affine and Riemannian geometry, and (para)-Kähler–Weyl geometry is a natural generalization of Kähler geometry. Results are presented both in the complex and para-complex settings. Only the 4-dimensional setting is of interest in this context – any (para)-Kähler–Weyl structure is trivial in dimension $m \geq 6$. By contrast, any para-Hermitian manifold or any pseudo-Hermitian manifold admits a unique (para)-Kähler–Weyl structure if m=4. The alternating Ricci tensor ρ_a carries essential information about the structure in dimension four – the (para)-Kähler–Weyl structure is trivial if and only if $\rho_a = 0$. All the possible algebraic possibilities for the values of ρ_a

can be realized by left-invariant structures on 4-dimensional Lie groups for Hermitian manifolds or para-Hermitian manifolds – the case of pseudo-Hermitian manifolds of signature (2, 2) is excluded, as quite different techniques would be needed to study that case that are tangential to the thrust of our main discussion. This result is used to examine the space of algebraic (para)-Kähler–Weyl curvature tensors as a module over the appropriate structure groups and to show that any algebraic possibility can be realized geometrically.

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