NON-CLOSED CURVES IN $\mathbb{R}^n$ WITH FINITE TOTAL FIRST CURVATURE ARISING FROM THE SOLUTIONS OF AN ODE

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Abstract. The solution space of a constant coefficient ODE gives rise to a natural real analytic curve in Euclidean space. We give necessary and sufficient conditions on the ODE to ensure that this curve is a proper embedding of infinite length or has finite total first curvature. If all the roots of the associated characteristic polynomial are simple, we give a uniform upper bound for the total first curvature and show the optimal uniform upper bound must grow at least linearly with the order n of the ODE. We then examine the case where multiple roots are permitted. We present several examples illustrating that a curve can have finite total first curvature for positive/negative time and infinite total first curvature for negative/positive time as well as illustrating that other possibilities may occur.

1. Introduction

Throughout this paper, in the interests of notational simplicity, the word "curvature" will refer to the "first curvature". It is defined as follows. If $t \to \sigma(t)$ is an immersion of $\mathbb{R}$ into $\mathbb{R}^n$, then the *curvature* κ and the *total curvatures* $\kappa_\pm[\sigma]$ are given, respectively, by setting:

$$\kappa := \frac{||\dot\sigma \wedge \ddot\sigma||}{||\dot\sigma||^3}, \quad \kappa_-[\sigma] := \int_{-\infty}^0 \kappa||\dot\sigma||dt, \quad \kappa_+[\sigma] := \int_0^\infty \kappa||\dot\sigma||dt\,. \tag{1.a}$$

The *total curvature* is then given by $\kappa[\sigma] := \kappa_+[\sigma] + \kappa_-[\sigma]$. In this paper we shall construct a real analytic curve σ in Euclidean space which arises as the solution space of a constant coefficient ODE. We examine when σ is a proper immersion with finite total curvature. In the C^∞ context, one could start with a straight line, perturb it by putting a small bump in it, and get thereby a proper curve with finite total curvature. Thus working in the real analytic context is crucial when considering questions of this sort.

The curvature κ of Equation (1.a) is a local invariant of the curve which does not depend on the parametrization. If $\rho(t)$ is the radius of the best circle approximating σ at t, then $\kappa = \rho^{-1}$. One can extend the definition from the Euclidean setting to the Riemannian setting. Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g). If σ is a curve which is parametrized by arc length, then the *geodesic curvature* is defined by setting $\kappa_g(\sigma) := \|\nabla_{\dot\sigma}\dot\sigma\|$; $\kappa_g = 0$ if and only if σ is a geodesic. We have $\kappa_g = \kappa$ if $M = \mathbb{R}^m$ with the usual flat metric.

1.1. **History.** Let $\kappa[\sigma] := \kappa_+[\sigma]+\kappa_-[\sigma]$ be the total curvature. Fenchel [13] showed that a simple closed curve in $\mathbb{R}^3$ had $\kappa[\sigma] \geq 2\pi$. Fáry [12] and Milnor [15] showed the total curvature of any knot (i.e. of a circle which is embedded in $\mathbb{R}^3$) is greater than 4π. Castrillón López and Fernández Mateos [3], and Kondo and Tanaka [14] have examined the global properties of the total curvature of a curve in an arbitrary

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Riemannian manifold. The total curvature of open plane curves of fixed length in $\mathbb{R}^2$ was studied by Enomoto [7]. The analogous question for S^2 was examined by Enomoto and Itoh [8, 9]. Enomoto, Itoh, and Sinclair [11] examined curves in $\mathbb{R}^3$. We also refer to related work of Sullivan [16]. Buck and Simon [2] and Diao and Ernst [5] studied this invariant in the context of knot theory, and Ekholm [6] used this invariant in the context of algebraic topology. Alexander, Bishop, and Ghrist [1] extended these notions to more general spaces than smooth manifolds. The total curvature also appears in the study of Plateau's problem – see the discussion in Desideri and Jakob [4]. The total absolute torsion has also been examined analogously by Enomoto and Itoh [10]; we shall not touch on this. The literature on the subject is a vast one and we have only cited a few representative papers to give a flavor for the subject.

1.2. **Curves given by constant coefficient ODE's.** Let P be a real constant coefficient ordinary differential operator of degree $n = n_P \geq 2$ of the form:

$$P(\phi) := \phi^{(n)} + c_{n-1}\phi^{(n-1)} + \cdots + c_0\phi$$

where $\phi^{(k)} := \frac{d^k\phi}{dt^k}$ for $1 \leq k \leq n$ and $\phi = \phi(t)$. Let $\mathcal{S} = \mathcal{S}_P$ be the solution space, let $\mathcal{P} = \mathcal{P}_P$ be the associated characteristic polynomial, and let $\mathcal{R} = \mathcal{R}_P$ be the roots of $\mathcal{P}$, respectively:

$$\begin{aligned}
&\mathcal{S} := \{\phi : P(\phi) = 0\},\\
&\mathcal{P}(\lambda) := \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0,\\
&\mathcal{R} := \{\lambda \in \mathbb{C} : \mathcal{P}(\lambda) = 0\}\,.
\end{aligned}$$

We suppose for the moment that all the roots of $\mathcal{P}$ are simple (i.e. have multiplicity 1) and enumerate the roots of $\mathcal{P}$ in the form:

$$\mathcal{R} = \{s_1, \ldots, s_k, \mu_1, \bar{\mu}_1, \ldots, \mu_\ell, \bar{\mu}_\ell\} \text{ for } k + 2\ell = n$$

where $s_i \in \mathbb{R}$ for $1 \leq i \leq k$ and where $\mu_j = a_j + \sqrt{-1}b_j$ with $b_j > 0$ for $1 \leq j \leq \ell$. Since we have assumed that all the roots are simple, the standard basis for $\mathcal{S}$ is given by the functions

$$\begin{array}{lll}
\phi_1 := e^{s_1 t}, & \ldots, & \phi_k := e^{s_k t},\\
\phi_{k+1} := e^{a_1 t}\cos(b_1 t), & \phi_{k+2} := e^{a_1 t}\sin(b_1 t), & \ldots,\\
\phi_{n-1} := e^{a_\ell t}\cos(b_\ell t), & \phi_n := e^{a_\ell t}\sin(b_\ell t)\,. &
\end{array} \tag{1.b}$$

Of course, if all the roots are real, then $k = n$ and we omit the functions involving $\cos(\cdot)$ and $\sin(\cdot)$. Similarly, if all the roots are complex, then $k = 0$ and we omit the pure exponential functions. We define the associated curve $\sigma_P : \mathbb{R} \to \mathbb{R}^n$ by setting:

$$\sigma_P(t) := (\phi_1(t), \ldots, \phi_n(t))\,.$$

1.3. **The length of the curve σ_P.** Let $\Re(\lambda)$ denote the real part of a complex number λ. Define:

$$\begin{aligned}
r_+(P) &:= \max_{\lambda\in\mathcal{R}} \Re(\lambda) = \max(s_1, \ldots, s_k, a_1, \ldots, a_\ell),\\
r_-(P) &:= \min_{\lambda\in\mathcal{R}} \Re(\lambda) = \min(s_1, \ldots, s_k, a_1, \ldots, a_\ell)\,.
\end{aligned}$$

The numbers $r_\pm(P)$ control the growth of $||\sigma_P||$ as $t \to \pm\infty$. Section 2 is devoted to the proof of the following result:

Theorem 1.1. *Assume that all the roots of $\mathcal{P}$ are simple. If $r_+(P) > 0$, then σ_P is a proper embedding of $[0, \infty)$ into $\mathbb{R}^n$ with infinite length. If $r_-(P) < 0$, then σ_P is a proper embedding of $(-\infty, 0]$ into $\mathbb{R}^n$ with infinite length.*

1.4. **The total curvature.** We order the roots to ensure that:

$$s_1 > s_2 > \cdots > s_k \text{ and } a_1 \geq \cdots \geq a_\ell\,.$$

We then have $r_+(P) = \max(s_1, a_1)$ and $r_-(P) = \min(s_k, a_\ell)$. Section 3 is devoted to the proof of the following result:

Theorem 1.2. *Assume that all the roots of $\mathcal{P}$ are simple, that $r_+(P) > 0$, and that $r_-(P) < 0$.*

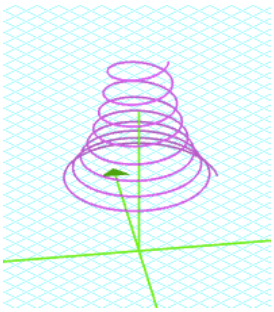
(1) *If $s_1 > a_1$, then $\kappa_+[\sigma_P] < \infty$; otherwise, $\kappa_+[\sigma_P] = \infty$.*
(2) *If $s_k < a_\ell$, then $\kappa_-[\sigma_P] < \infty$; otherwise $\kappa_-[\sigma_P] = \infty$.*

We note that if there are no complex roots, then $s_1 > 0$ and $s_k < 0$ and we may conclude that $\kappa_+[\sigma_P]$ and $\kappa_-[\sigma_P]$ are finite. This is quite striking as these curves are, obviously, not straight lines. On the other hand, if there are no real roots, then $a_1 > 0$ and $a_\ell < 0$ and we may conclude that $\kappa_+[\sigma_P]$ and $\kappa_-[\sigma_P]$ are infinite.

1.5. **Uniform bounds on the total curvature.** Theorem 1.2 shows $\kappa_+[\sigma_P]$ is finite if $s_1 > 0$, if all the roots are simple, and if $s_1 > \Re(\mu)$ for any complex root μ. In fact, one can give a uniform upper bound for $k_+[\sigma]$ if there are no complex roots and if all the real roots are simple without the assumption that $s_1 > 0$ where the uniform bound depends only on the dimension. If $s_1 > \cdots > s_n$, let $\sigma_{s_1,\ldots,s_n} := (e^{s_1 t}, \ldots, e^{s_n t})$. We will establish the following result in Section 4.

Theorem 1.3. $\kappa_+[\sigma_{s_1,\ldots,s_n}] \leq 2n(n-1)$.

Remark 1.4. Let $\sigma_n(t) := (e^t, \cos(nt)e^{-t}, \sin(nt)e^{-t}, e^{-2t})$. Since we have that $\lim_{n\to\infty} \kappa_\pm[\sigma_n] = \infty$, no uniform upper bound on the curvature is possible if complex roots are permitted. We picture below a 3-dimensional projection of such a curve:



Theorem 1.3 shows that there exists a dimension dependent uniform upper bound for the total curvature of a curve defined by an ODE of order n with simple real roots. We now show the optimal uniform upper bound must grow at least linearly in n. Let

$$u_{k,\theta} := e^{k\theta} \quad \text{and} \quad \sigma_{n,\theta}(t) := (e^{-u_{1,\theta}t}, e^{-u_{2,\theta}t}, \ldots, e^{-u_{n,\theta}t})\,.$$

We will establish the following result in Section 5:

Theorem 1.5. *Let $\epsilon > 0$ be given. There exists $\theta(\epsilon)$ so that if $\theta > \theta(\epsilon)$, then $\kappa_+[\sigma_{n,\theta}] \geq \frac{1}{3}(n-1) - \epsilon$.*

1.6. **Examples.** Section 6 treats several families of examples. We construct examples where $\kappa_+[\sigma_P]$ and $\kappa_-[\sigma_P]$ are both finite, where $\kappa_+[\sigma_P]$ is finite but $\kappa_-[\sigma_P]$ is infinite, where $\kappa_+[\sigma_P]$ is infinite but $\kappa_-[\sigma_P]$ is finite, and where both $\kappa_+[\sigma_P]$ and $\kappa_-[\sigma_P]$ are infinite.

1.7. **Changing the basis.** We took the standard basis for $\mathcal{S}$ to define the curve σ_P. More generally, let $\Psi := \{\psi_1, \dots, \psi_n\}$ be an arbitrary basis for $\mathcal{S}$. We define:

$$\sigma_{\Psi,P}(t) := (\psi_1(t), \dots, \psi_n(t))\,.$$

In Section 7, we extend Theorem 1.1 and Theorem 1.2 to this setting and verify that the properties we have been discussing are properties of the solution space $\mathcal{S}$ and not of the particular basis chosen:

Theorem 1.6. *Assume that all the roots of $\mathcal{P}$ are simple, that $r_+(P) > 0$, and that $r_-(P) < 0$. Then $\sigma_{\Psi,P}$ is a proper embedding of $[0,\infty)$ and of $(-\infty,0]$ into $\mathbb{R}^n$ with infinite length.*

1. *If $s_1 > a_1$, then $\kappa_+[\sigma_{\Psi,P}] < \infty$; otherwise, $\kappa_+[\sigma_{\Psi,P}] = \infty$.*
2. *If $s_k < a_\ell$, then $\kappa_-[\sigma_{\Psi,P}] < \infty$; otherwise $\kappa_-[\sigma_{\Psi,P}] = \infty$.*

1.8. **Roots with multiplicity greater than 1.** Powers of t arise in this setting. For example, if we consider the equation $\phi^{(n)} = 0$, then

$$\mathcal{S} = \operatorname{Span}\{1,\ t,\ \dots,\ t^{n-1}\}\,.$$

More generally, if s is a real eigenvalue of multiplicity $\nu \geq 2$, then we must consider the family of functions:

$$\{\phi_{s,0} := e^{st},\ \phi_{s,1} := te^{st},\ \dots,\ \phi_{s,\nu-1} := t^{\nu-1}e^{st}\} \tag{1.c}$$

while if $\mu = a + \sqrt{-1}b$ for $b > 0$ is a complex root of multiplicity $\nu \geq 2$, then we must consider the family of functions:

$$\begin{aligned}\{\phi_{\mu,0} &:= e^{at}\cos(bt), \phi_{\mu,1} := te^{at}\cos(bt), \dots, \phi_{\mu,\nu-1} := t^{\nu-1}e^{at}\cos(bt),\\ \tilde\phi_{\mu,0} &:= e^{at}\sin(bt), \tilde\phi_{\mu,1} := te^{at}\sin(bt), \dots, \tilde\phi_{\mu,\nu-1} := t^{\nu-1}e^{at}\sin(bt)\}\,.\end{aligned} \tag{1.d}$$

We will establish the following result in Section 8:

Theorem 1.7. *Assume that $r_+(P) > 0$ and that $r_-(P) < 0$.*

1. *If $s_1 = r_+(P)$ and if the multiplicity of s_1 as a root of $\mathcal{P}$ is larger than the corresponding multiplicity of any complex root μ of $\mathcal{P}$ with $\Re(\mu) = s_1$, then $\kappa_+[\sigma_{\Psi,P}] < \infty$; otherwise $\kappa_+[\sigma_{\Psi,P}] = \infty$.*
2. *If $s_k = r_-(P)$ and if the multiplicity of s_k as a root of $\mathcal{P}$ is larger than the corresponding multiplicity of any complex root μ of $\mathcal{P}$ with $\Re(\mu) = s_k$, then $\kappa_-[\sigma_{\Psi,P}] < \infty$; otherwise $\kappa_-[\sigma_{\Psi,P}] = \infty$.*

2. The proof of Theorem 1.1

Assume all the roots of $\mathcal{P}$ are simple. It then follows from the definition that

$$||\sigma_P||^2 = \sum_{i=1}^{k} e^{2s_it} + \sum_{j=1}^{\ell} e^{2a_jt}\,.$$

Thus $||\sigma_P||^2$ tends to infinity as $t \to \infty$ if and only if some s_i or some a_j is positive or, equivalently, if $r_+(P) > 0$. This implies that σ_P is a proper map from $[0,\infty)$ to $\mathbb{R}^n$ and that the length is infinite. If $s_1 > 0$, then $\phi_1 = e^{s_1t}$ is an injective map from $\mathbb{R}$ to $\mathbb{R}$ and consequently σ_P is an embedding of $\mathbb{R}$ into $\mathbb{R}^n$. If $a_1 > 0$, then $e^{a_1t}(\cos(b_1t), \sin(b_1t))$ is an injective map from $\mathbb{R}$ to $\mathbb{R}^2$ and again we may conclude that σ_P is an embedding. The analysis on $(-\infty,0]$ is similar if $r_-(P) < 0$ and is therefore omitted in the interests of brevity. $\square$

3. The proof of Theorem 1.2

Throughout our proof, we will let $C_i = C_i(P)$ denote a generic positive constant; we clear the notation after each case under consideration and after the end of any given proof; thus C_i can have different meanings in different proofs or in different sections of the same proof. We shall examine σ_P on $[0,\infty)$; the analysis on $(-\infty,0]$ is similar and will therefore be omitted. We suppose $r_+(P)>0$ or, equivalently, that $\max(s_1,a_1)>0$. We also assume that all the roots of $\mathcal{P}$ are simple. Suppose first that $s_1>a_1$ or that there are no complex roots. Let

$$\epsilon := \min_{\lambda\in\mathcal{R},\lambda\neq s_1}(s_1-\Re(\lambda)) = \min_{i>1,j\geq 1}(s_1-s_i,s_1-a_j)>0\,.$$

This measures the difference between the exponential growth rate of ϕ_1 and the growth (or decay) rates of the functions ϕ_i of Equation (1.b) for $i>1$ as $t\to\infty$. We have

$$||\dot\sigma_P\wedge\ddot\sigma_P||^2=\sum_{i<j}(\dot\phi_i\ddot\phi_j-\dot\phi_j\ddot\phi_i)^2\,. \tag{3.a}$$

Consequently, we may estimate:

$$\begin{aligned}&||\dot\sigma_P\wedge\ddot\sigma_P||\leq C_1e^{(2s_1-\epsilon)t}, \qquad ||\dot\sigma_P||^2\geq C_2e^{2s_1t}\text{ for } t\geq 0,\\ &\frac{||\dot\sigma_P\wedge\ddot\sigma_P||}{||\dot\sigma_P||^2}\leq C_3e^{-\epsilon t}\text{ for } t\geq 0\,.\end{aligned} \tag{3.b}$$

We integrate the estimate of Equation (3.b) to see $\kappa_+[\sigma_P]<\infty$.

Next suppose that $a_1>0$ and that $a_1\geq s_1$ (or that there are no real roots). Then e^{a_1t} is the dominant term and we have

$$||\dot\sigma_P||^2\leq C_1e^{2a_1t}\,. \tag{3.c}$$

The term $(\dot\phi_i\ddot\phi_j-\dot\phi_j\ddot\phi_i)^2$ in Equation (3.a) is maximized for $t\geq 0$ when we take $\phi_i=e^{a_1t}\cos(b_1t)$ and $\phi_j=e^{a_1t}\sin(b_1t)$. We have:

$$\begin{aligned}\dot\phi_i&=e^{a_1t}(a_1\cos(b_1t)-b_1\sin(b_1t))\\ \ddot\phi_i&=e^{a_1t}\{(a_1^2-b_1^2)\cos(b_1t)-2a_1b_1\sin(b_1t)\}\\ \dot\phi_j&=e^{a_1t}(a_1\sin(b_1t)+b_1\cos(b_1t)),\\ \ddot\phi_j&=e^{a_1t}\{(a_1^2-b_1^2)\sin(b_1t)+2a_1b_1\cos(b_1t)\},\\ \dot\phi_i^2+\dot\phi_j^2&=(a_1^2+b_1^2)e^{2a_1t},\\ (\dot\phi_i\ddot\phi_j-\dot\phi_j\ddot\phi_i)^2&=b_1^2(a_1^2+b_1)^2e^{4a_1t}\,.\end{aligned}$$

Since $b_1\neq 0$, we may estimate:

$$||\dot\sigma_P\wedge\ddot\sigma_P||\geq C_2e^{2a_1t}\,. \tag{3.d}$$

We use Equation (3.c) and Equation (3.d) to see

$$\frac{||\dot\sigma_P\wedge\ddot\sigma_P||}{||\dot\sigma_P||^2}\geq\frac{C_2}{C_1}>0\,. \tag{3.e}$$

We integrate the uniform estimate of Equation (3.e) to see $\kappa_+[\sigma_P]=\infty$. □

4. The proof of Theorem 1.3

Let $\sigma_{s_1,\dots,s_n}(t) := (e^{s_1t},\dots,e^{s_nt})$ for $s_1 > \dots > s_n$ and $n \geq 2$. We may express

$$\begin{aligned}
\kappa_+[\sigma_{s_1,\dots,s_n}] &= \int_0^\infty \sqrt{\sum_{i<j}\left\{s_is_j(s_i-s_j)e^{(s_i+s_j)t}\right\}^2}\left\{\sum_k s_k^2e^{2s_kt}\right\}^{-1}dt\\
&\leq \int_0^\infty \sum_{i<j}|s_is_j(s_i-s_j)|e^{(s_i+s_j)t}\left\{\sum_k s_k^2e^{2s_kt}\right\}^{-1}dt\\
&\leq \int_0^\infty \sum_{i<j}|s_is_j(s_i-s_j)|e^{(s_i+s_j)t}\left\{s_i^2e^{2s_it}+s_j^2e^{2s_jt}\right\}^{-1}dt\\
&= \sum_{i<j}\kappa_+[\sigma_{s_i,s_j}]\,.
\end{aligned}$$

Thus estimate $\kappa_+[\sigma_{s_1,\dots,s_n}] \leq n(n-1)$ for $n \geq 3$ will follow if we can establish the corresponding estimate $\kappa_+[\sigma_{s_i,s_j}] < 2$ for $n = 2$. We set $n = 2$ and consider 2 cases:

Case I: $s_1^2 \geq s_2^2$. Since $s_1 > s_2$, we must have $s_1 > 0$. We compute:

$$\begin{aligned}
\kappa_+[\sigma_{s_1,s_2}] &= \int_0^\infty |s_1s_2(s_1-s_2)|\,e^{(s_1+s_2)t}\left\{s_1^2e^{2s_1t}+s_2^2e^{2s_2t}\right\}^{-1}dt\\
&< \int_0^\infty |s_1s_2(s_1-s_2)|\,e^{(s_1+s_2)t}\left\{s_1^2e^{2s_1t}\right\}^{-1}dt\\
&= \int_0^\infty \left|s_1^{-1}s_2(s_1-s_2)\right|e^{(s_2-s_1)t}dt = \left|s_1^{-1}s_2\right| \leq 1\,.
\end{aligned}$$

Case II: $s_1^2 < s_2^2$. Since $s_1 > s_2$, either $s_1 > 0 > s_2$ or $0 > s_1 > s_2$. When t is small, $s_1^2e^{2s_1t} < s_2^2e^{2s_2t}$ while if t is large, $s_1^2e^{2s_1t} > s_2^2e^{2s_2t}$. Choose T so that $s_1^2e^{2s_1T} = s_2^2e^{2s_2T}$. Then

$$s_1^2e^{2s_1t} < s_2^2e^{2s_2t} \quad \text{if} \quad t < T \quad \text{and} \quad s_1^2e^{2s_1t} > s_2^2e^{2s_2t} \quad \text{if} \quad t > T\,.$$

We may decompose $\kappa_+[\sigma_{s_1,s_2}] = \mathcal{I}_1 + \mathcal{I}_2$ for

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^T |s_1s_2(s_1-s_2)|\,e^{(s_1+s_2)t}\left\{s_1^2e^{2s_1t}+s_2^2e^{2s_2t}\right\}^{-1}dt\\
\mathcal{I}_2 &= \int_T^\infty |s_1s_2(s_1-s_2)|\,e^{(s_1+s_2)t}\left\{s_1^2e^{2s_1t}+s_2^2e^{2s_2t}\right\}^{-1}dt\,.
\end{aligned}$$

Note that $e^{(s_1-s_2)T} = \left|s_2s_1^{-1}\right|$ and $e^{(s_2-s_1)T} = \left|s_2^{-1}s_1\right|$. We complete the proof by estimating:

$$\begin{aligned}
\mathcal{I}_1 &\leq \int_0^T |s_1s_2(s_1-s_2)|\,e^{(s_1+s_2)t}\left\{s_2^2e^{2s_2t}\right\}^{-1}dt\\
&= \left|s_1s_2^{-1}(s_1-s_2)\right|\int_0^T e^{(s_1-s_2)t}dt = \left|s_1s_2^{-1}\right|e^{(s_1-s_2)t}\Big|_0^T\\
&= \left|s_1s_2^{-1}\right|\left\{e^{(s_1-s_2)T}-1\right\} = \left|s_1s_2^{-1}\right|\left\{\left|s_2s_1^{-1}\right|-1\right\}\\
&= 1-\left|s_2s_1^{-1}\right| < 1,\\
\mathcal{I}_2 &\leq \int_T^\infty |s_1s_2(s_1-s_2)|\,e^{(s_1+s_2)t}\left\{s_1^2e^{2s_1t}\right\}^{-1}dt\\
&= \left|s_1^{-1}s_2(s_1-s_2)\right|\int_T^\infty e^{(s_2-s_1)t}dt = -\left|s_1^{-1}s_2\right|e^{(s_2-s_1)t}\Big|_{t=T}^\infty = 1. \qquad \square
\end{aligned}$$

5. The proof of Theorem 1.5

Let $\theta >> 1$. We set

$$u_{k,\theta} := e^{k\theta} \text{ and } \sigma_{n,\theta}(t) := (e^{-u_{1,\theta}t}, \dots, e^{-u_{n,\theta}t})\,.$$

We have:

$$\kappa_+[\sigma_{n,\theta}] = \int_0^\infty \frac{\left(\sum_{i<j}\left\{(u_{i,\theta}-u_{j,\theta})u_{i,\theta}u_{j,\theta}e^{-(u_{i,\theta}+u_{j,\theta})t}\right\}^2\right)^{\frac12}}{\sum_\ell u_{\ell,\theta}^2 e^{-2u_{\ell,\theta}t}}dt\,. \tag{5.a}$$

To obtain a lower estimate for $\kappa_+[\sigma_{n,\theta}]$, we must obtain an upper estimate for the denominator $D(t) := \sum_\ell u_{\ell,\theta}^2 e^{-2u_{\ell,\theta}t}$ in Equation (5.a). We determine the maximal term in $D(t)$ on various intervals and complete the proof of Theorem 1.5:

Lemma 5.1. *Set $f_{k,\theta}(t) := u_{k,\theta}e^{-u_{k,\theta}t}$.*

1. *There exists a unique positive real number $T_{k,\theta}$ so $f_{k,\theta}(T_{k,\theta}) = f_{k+1,\theta}(T_{k,\theta})$.*
   (a) *$T_{k,\theta} = \theta e^{-(k+1)\theta}(1-e^{-\theta})^{-1}$.*
   (b) *If $t < T_{k,\theta}$, then $f_{k,\theta}(t) < f_{k+1,\theta}(t)$.*
   (c) *If $t > T_{k,\theta}$, then $f_{k,\theta}(t) > f_{k+1,\theta}(t)$.*
2. *If $j \in \{k, k+1, k+2\}$ and if $t \in [T_{k+1,\theta}, T_{k,\theta}]$, then $f_{j,\theta}(t) \le f_{k+1,\theta}(t)$.*
3. *If $0 < \delta < 1$, there exists $\theta(\delta) > 1$ so that if $\theta \ge \theta(\delta)$, if $j \notin \{k, k+1, k+2\}$, and if $t \in [T_{k+1,\theta}, T_{k,\theta}]$, then $f_{j,\theta}(t) \le \delta f_{k+1,\theta}(t)$.*
4. *If $0 < \delta < 1$, there exists $\theta(\delta) > 1$ so that if $\theta \ge \theta(\delta)$, then*
$$\int_{T_{k+1,\theta}}^{T_{k,\theta}} \frac{u_{k,\theta}u_{k+1,\theta}(u_{k+1,\theta}-u_{k,\theta})e^{-(u_{k,\theta}+u_{k+1,\theta})t}}{u_{k+1,\theta}^2 e^{-2u_{k+1,\theta}t}}dt \ge 1-\delta\,.$$
5. *If $0 < \epsilon < 1$, there exists $\theta(\epsilon) > 1$ so $\theta \ge \theta(\epsilon)$ implies:*
   (a) $\displaystyle\int_{T_{k+1,\theta}}^{T_{k,\theta}} \kappa(\sigma_{n,\theta})ds \ge \tfrac13 - \tfrac1n\epsilon$ *for* $1 \le k \le n-1$.
   (b) $\kappa_+[\sigma_{n,\theta}] \ge \frac13(n-1) - \epsilon$.

Proof. Since $0 < u_{k,\theta} < u_{k+1,\theta}$, $f_{k,\theta}(t) - f_{k+1,\theta}(t)$ is negative for $t = 0$ and positive for t large. Thus there exists $T_{k,\theta} \in \mathbb{R}^+$ so $f_{k,\theta}(T_{k,\theta}) = f_{k+1,\theta}(T_{k,\theta})$. We show $T_{k,\theta}$ is unique by determining its value. We have:

$$\begin{aligned}
&f_{k,\theta}(T_{k,\theta}) = f_{k+1,\theta}(T_{k,\theta}) \quad \Leftrightarrow\\
&\log(u_{k,\theta}) - u_{k,\theta}T_{k,\theta} = \log(u_{k+1,\theta}) - u_{k+1,\theta}T_{k,\theta} \quad \Leftrightarrow\\
&k\theta - e^{k\theta}T_{k,\theta} = (k+1)\theta - e^{(k+1)\theta}T_{k,\theta} \quad \Leftrightarrow\\
&T_{k,\theta} = \theta(e^{(k+1)\theta} - e^{k\theta})^{-1} = \theta e^{-(k+1)\theta}(1-e^{-\theta})^{-1}\,.
\end{aligned}$$

Assertion 1 follows from this computation and the Intermediate Value Theorem.

Note that $T_{n,\theta} < T_{n-1,\theta} < \cdots < T_{2,\theta} < T_{1,\theta}$. Let $t \in [T_{k+1,\theta}, T_{k,\theta}]$. The inequality of Assertion 2 is immediate if $j = k+1$. Since $t \le T_{k,\theta}$, $f_{k,\theta}(t) \le f_{k+1,\theta}(t)$ by Assertion 1b. Since $t \ge T_{k+1,\theta}$, $f_{k+1,\theta}(t) \ge f_{k+2,\theta}(t)$ by Assertion 1c. This proves Assertion 2.

Assertion 3 estimates $f_{j,\theta}(t)$ for $t \in [T_{k+1,\theta}, T_{k,\theta}]$ for the remaining values of j which are distinct from k, $k+1$, and $k+2$. Let $1 \le k \le n-1$. Given $0 < \delta < 1$, choose $\theta(\delta) >> 1$ so $\theta \ge \theta(\delta)$ implies

$$\begin{aligned}
&(1-e^{-\theta})^{-1} \le 1+\delta \text{ and}\\
&u_{j,\theta} - u_{k+1,\theta} \ge (1-\delta)u_{j,\theta} \text{ if } 3 \le k+2 < j \le n\,.
\end{aligned} \tag{5.b}$$

By Equation (5.b), we have that:

$$T_{k,\theta} = \theta e^{-(k+1)\theta}(1-e^{-\theta})^{-1} \le (1+\delta)\theta e^{-(k+1)\theta}\,.$$

Let $j<k$ and $t\in[T_{k+1,\theta},T_{k,\theta}]$. Thus, in particular, $t\le T_{k,\theta}$. As $u_{k+1,\theta}-u_{j,\theta}>0$,

$$\begin{aligned}&f_{j,\theta}(t)f_{k+1,\theta}(t)^{-1}=e^{(j-k-1)\theta}e^{(u_{k+1,\theta}-u_{j,\theta})t}\\ &\le e^{(j-k-1)\theta}e^{u_{k+1,\theta}T_{k,\theta}}\le e^{(j-k-1)\theta}e^{e^{(k+1)\theta}\cdot(1+\delta)\theta e^{-(k+1)\theta}}\\ &=e^{(j-k+\delta)\theta}\,.\end{aligned}$$

This can be made arbitrarily small if θ is large since $j-k+\delta<0$. This proves Assertion 3 if $j<k$. Next suppose $j>k+2$. Since $t\in[T_{k+1,\theta},T_{k,\theta}]$,

$$t\ge T_{k+1,\theta}=\theta e^{-(k+2)\theta}(1-e^{-\theta})^{-1}\ge\theta e^{-(k+2)\theta}\,.$$

As $u_{k+1,\theta}-u_{j,\theta}<0$, Equation (5.b) implies:

$$\begin{aligned}&f_{j,\theta}(t)f_{k+1,\theta}(t)^{-1}=e^{\theta(j-k-1)}e^{(u_{k+1,\theta}-u_{j,\theta})t}\\ \le\;& e^{\theta(j-k-1)}e^{(u_{k+1,\theta}-u_{j,\theta})\theta e^{-(k+2)\theta}}\\ \le\;& e^{\theta(j-k-1)}e^{-(1-\delta)e^{j\theta}\theta e^{-(k+2)\theta}}=e^{\theta(j-k-1-(1-\delta)e^{(j-k-2)\theta})}\,.\end{aligned}$$

This term goes to zero as $\theta\to\infty$ since $j-k-2>0$. This establishes Assertion 3.

To prove Assertion 4, we compute:

$$\begin{aligned}&\int_{T_{k+1,\theta}}^{T_{k,\theta}}\frac{u_{k,\theta}u_{k+1,\theta}(u_{k+1,\theta}-u_{k,\theta})e^{-(u_{k,\theta}+u_{k+1,\theta})t}}{u_{k+1,\theta}^2e^{-2u_{k+1,\theta}t}}dt\\ &=u_{k,\theta}u_{k+1,\theta}^{-1}e^{(u_{k+1,\theta}-u_{k,\theta})t}\Big|_{t=T_{k+1,\theta}}^{T_{k,\theta}}\\ &=u_{k,\theta}u_{k+1,\theta}^{-1}e^{(u_{k+1,\theta}-u_{k,\theta})T_{k,\theta}}\{1-e^{(u_{k+1,\theta}-u_{k,\theta})(T_{k+1,\theta}-T_{k,\theta})}\}\\ &=1-e^{(u_{k+1,\theta}-u_{k,\theta})(T_{k+1,\theta}-T_{k,\theta})}\\ &=1-e^{\{e^{\theta k}(e^{\theta}-1)\}\cdot\{\theta(e^{\theta}-1)^{-1}\{e^{-(k+1)\theta}-e^{-k\theta}\}}\\ &=1-e^{\theta(e^{-\theta}-1)}.\end{aligned}$$

Assertion 4 follows as $\theta(e^{-\theta}-1)$ tends to $-\infty$ as θ tends to ∞.

We use Assertion 2 and Assertion 3 to see that if $t\in[T_{k+1,\theta},T_{k,\theta}]$, then

$$\begin{aligned}&\sum_\ell u_{\ell,\theta}^2e^{-2u_{\ell,\theta}t}\le(3+n\delta)u_{k+1,\theta}^2e^{-2u_{k+1,\theta}t},\\ &\int_{T_{k+1,\theta}}^{T_{k,\theta}}\kappa(\sigma_{n,\theta})ds\\ &=\int_{T_{k+1,\theta}}^{T_{k,\theta}}\frac{(\sum_{i<j}\{(u_{i,\theta}-u_{j,\theta})u_{i,\theta}u_{j,\theta}e^{-(u_{i,\theta}+u_{j,\theta})t}\}^2)^{\frac12}}{\sum_\ell u_{\ell,\theta}^2e^{-2u_{\ell,\theta}t}}dt\\ &\ge\int_{T_{k+1,\theta}}^{T_{k,\theta}}\frac{(u_{k+1,\theta}-u_{k,\theta})u_{k+1,\theta}u_{k,\theta}e^{-(u_{k,\theta}+u_{k+1,\theta})t}}{\sum_\ell u_{\ell,\theta}^2e^{-2u_{\ell,\theta}t}}dt\\ &\ge\int_{T_{k+1,\theta}}^{T_{k,\theta}}\frac{(u_{k+1,\theta}-u_{k,\theta})u_{k+1,\theta}u_{k,\theta}e^{-(u_{k,\theta}+u_{k+1,\theta})t}}{(3+n\delta)u_{k+1,\theta}^2e^{-2u_{k+1,\theta}t}}dt\\ &\ge(1-\delta)(3+n\delta)^{-1}.\end{aligned}$$

Assertion 5a now follows by choosing $\delta=\delta(\epsilon)$ appropriately. We sum this estimate for $1\le k\le n-1$ to establish Assertion 5b and thereby complete the proof of Theorem 1.5. □

6. Examples

We now examine several specific cases. Since the eigenvalues are to be simple, we can just specify $\mathcal{P}$ or equivalently $\mathcal{R}$; the corresponding operator P is then:

$$P = \mathcal{P}\left(\frac{d}{dt}\right) = \prod_{\lambda\in\mathcal{R}} \left\{\frac{d}{dt} - \lambda\right\}.$$

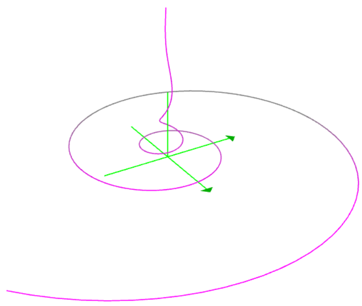
Example 6.1. Let $\mathcal{P}(\lambda) = \lambda^n - 1$. The roots of $\mathcal{P}$ are the n^{th} roots of unity and all the roots have multiplicity 1. Since $\mathcal{P}(1) = 0$, 1 is always a root.

Case I: Suppose that n is odd. Then 1 is the only real root of $\mathcal{P}$. The remaining roots are all complex. Thus $k = 1$ and it follows that σ_P is a proper embedding of infinite length from $[0,\infty)$ to $\mathbb{R}^n$. If $\lambda^n = 1$ and $\lambda \neq 1$, then necessarily $\Re(\lambda) < 1$. It now follows that $\kappa_+[\sigma_P]$ is finite. There exists a complex n^{th} root of unity with $\Re(\lambda) < 0$. Consequently, σ_P is also a proper embedding of infinite length from $(-\infty, 0]$ to $\mathbb{R}^n$. Since there are no real roots with $s_i < 0$, we conclude $\kappa_-[\sigma_P]$ is infinite.

Case II: Suppose that n is even. Then ± 1 are the two real roots of $\mathcal{P}$. It now follows that σ_P is a proper embedding of infinite length from $[0,\infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$. If $\lambda^n = 1$ and λ is not real, then $-1 < \Re(\lambda) < 1$. Consequently, $\kappa_+[\sigma_P]$ and $\kappa_-[\sigma_P]$ are both finite.

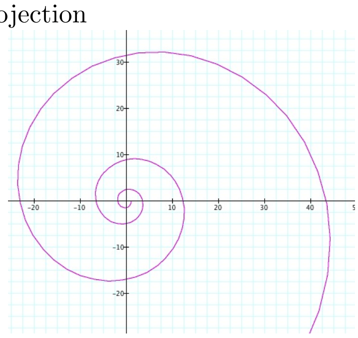
Example 6.2. Let $n \geq 3$. Let $\{1, \dots, n-2, -1 \pm \sqrt{-1}\}$ be the roots of $\mathcal{P}$. Then σ_P is a proper embedding of infinite length from $[0,\infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$, $\kappa_+[\sigma_P]$ is finite, and $\kappa_-[\sigma_P]$ is infinite. We adjust the angular parameter to emphasize the radial revolution and let the roots be $\{1, -1 \pm 5\sqrt{-1}\}$. This yields the curve:

$$C(t) = (\cos(5t)e^{-t}, \sin(5t)e^{-t}, e^t)$$



This curve curve hugs the z axis for $t > 0$ and becomes a spiral in the xy plane for $t < 0$. It has exponentially decaying curvature as $t \to \infty$ and infinite curvature as $t \to -\infty$. We draw the 2-dimensional projection

$$D(t) = (\cos(5t)e^{-t}, \sin(5t)e^{-t})$$

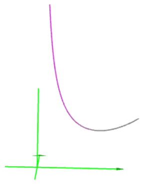


Example 6.3. Let $n \geq 3$. Let $\{-1, \dots, 2-n, 1 \pm \sqrt{-1}\}$ be the roots of $\mathcal{P}$. Then σ_P is a proper embedding of infinite length from $[0,\infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$, $\kappa_+[\sigma_P]$ is infinite, and $\kappa_-[\sigma_P]$ is finite.

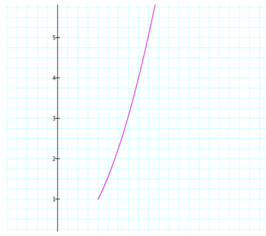
Example 6.4. Let $n \geq 2$. Let $\{1, \dots, n-1, -1\}$ be the roots of $\mathcal{P}$. Then σ_P is a proper embedding of infinite length from $[0,\infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$,

$\kappa_+[\sigma_P]$ is finite, and $\kappa_-[\sigma_P]$ is finite. The following curve hugs the z axis for $t < 0$ and hugs the curve $y = x^2$ in the xy plane for $t > 0$. The total curvature is finite. It has exponentially decaying curvature as $t \to \infty$ and infinite curvature as $t \to -\infty$.

$C(t) = (e^t, e^{2t}, e^{-t})$



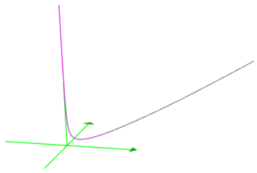
$D(t) = (e^t, e^{2t})$



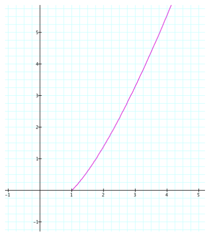
By considering the roots $\{1, a, -1\}$ for $a > 0$, one can construct curves which asymptotically approach the curve $y = x^a$ for $x > 0$ in the xy plane as $t \to \infty$.

Example 6.5. Let $n = 3$. Let $\{1, 1, -1\}$ be the roots of $\mathcal{P}$. Then σ_P is a proper embedding of infinite length from $[0, \infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$, $\kappa_+[\sigma_P]$ is finite, and $\kappa_-[\sigma_P]$ is finite. The following curve hugs the z axis for $t < 0$ and hugs the curve (e^t, te^t) for $t > 0$. Both have finite total curvature.

$C(t) = (e^t, te^t, e^{-t})$



$D(t) = (e^t, te^t)$



Example 6.6. Let $n = 4$. Let the roots of $\mathcal{P}$ be $\{1 \pm 5\sqrt{-1}, -1 \pm \sqrt{-1}\}$. Then σ_P is a proper embedding of infinite length from $[0, \infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$, $\kappa_+[\sigma_P]$ is infinite, and $\kappa_-[\sigma_P]$ is infinite. This yields

$$C(t) = (e^t \cos(5t), e^t \sin(5t), e^{-t} \cos(5t), e^{-t} \sin(5t))\,.$$

Example 6.7. Let $n = 2k + 1 \geq 5$ be odd. Let

$$\{0, 1 \pm \sqrt{-1}, -1 \pm \sqrt{-1}, \ldots, -(k-1) \pm \sqrt{-1}\}$$

be the roots of $\mathcal{P}$. Then σ_P is a proper embedding of infinite length from $[0, \infty)$ to $\mathbb{R}^n$ and from $(-\infty, 0]$ to $\mathbb{R}^n$, $\kappa_+[\sigma_P]$ is infinite, and $\kappa_-[\sigma_P]$ is infinite.

7. The proof of Theorem 1.6

Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be the standard basis for $\mathcal{S}$ given in Equation (1.b) and let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be any other basis for $\mathcal{S}$. Express

$$\psi_i = \Theta_i^j \phi_j$$

where we adopt the *Einstein* convention and sum over repeated indices. We use Θ_i^j to make a linear change of basis on $\mathbb{R}^n$ and to regard $\sigma_{\Psi,P} = \Theta \circ \sigma_P$; correspondingly, this defines a new inner product $\langle \cdot, \cdot \rangle := \Theta^*(\cdot, \cdot)$ on $\mathbb{R}^n$ so that

$$||\dot\sigma_{\Psi,P}|| = ||\dot\sigma_P||_\Theta \text{ and } ||\dot\sigma_{\Psi,P} \wedge \ddot\sigma_{\Psi,P}|| = ||\dot\sigma_P \wedge \ddot\sigma_P||_\Theta\,. \tag{7.a}$$

Any two norms on a finite dimensional real vector space are equivalent. Thus

$$C_1||v|| \leq ||v||_\Theta \leq C_2||v||\,. \tag{7.b}$$

The desired result now follows from Theorem 1.1, Theorem 1.2, Equation (7.a), and Equation (7.b). $\square$

8. The proof of Theorem 1.7

We will assume that Ψ is the standard basis for $\mathcal{S}$ as the methods discussed in Section 7 suffice to derive the general result from this specific example. We shall deal with $[0,\infty)$ as the situation for $(-\infty,0]$ is similar. The proof that $r_+(P)>0$ implies σ_P is a proper embedding of $[0,\infty)$ into $\mathbb{R}^n$ with infinite length is unchanged by any questions of multiplicity since e^{st} or $\{e^{at}\cos(bt),e^{at}\sin(bt)\}$ are still among the solutions of P for suitably chosen s or (a,b). We adopt the notation of Equation (1.c) to define the functions $\phi_{s,\ell}=t^\ell e^{st}$ for $s\in\mathbb{R}$ and we adopt the notation of Equation (1.d) to define the functions $\phi_{\mu,\ell}=t^\ell e^{at}\cos(bt)$ and $\tilde\phi_{\mu,\ell}=t^\ell e^{at}\sin(bt)$ for $\mu=a+b\sqrt{-1}$. We divide our discussion of $\kappa_+[\sigma_P]$ into several cases:

Case I: Suppose that $s_1>a_1$ and that s_1 is a real root of order ν. If $\nu=1$, the proof of Theorem 1.2 extends to show $\kappa_+[\sigma_P]$ is finite; the multiplicity of the other roots plays no role as the exponential decay $e^{-\epsilon t}$ swamps any powers of t. We suppose therefore that the multiplicity $\nu(s_1)>1$. We will show that there exists t_0 so that:

$$||\dot\sigma_P||^2\geq C_1t^{2\nu-2}e^{2s_1t}\text{ for }t\geq t_0, \tag{8.a}$$

$$||\dot\sigma_P\wedge\ddot\sigma_P||\leq C_2t^{2\nu-4}e^{2s_1t}\text{ for }t\geq t_0\,. \tag{8.b}$$

It will then follow that

$$\frac{||\dot\sigma_P\wedge\ddot\sigma_P||}{||\dot\sigma_P||^2}\leq C_3t^{-2}\text{ for }t\geq t_0\,.$$

Since this is integrable on $[0,\infty)$, we may conclude $\kappa_+[\sigma_P]$ is finite as desired.

We establish Equation (8.a) by noting that we have the following estimate:

$$\begin{aligned}||\dot\sigma_P||^2 &= \sum_{i=1}^n|\dot\phi_i|^2\geq|\dot\phi_{s_1,\nu-1}|^2=\{s_1t^{\nu-1}+(\nu-1)t^{\nu-2}\}^2e^{2s_1t}\\ &\geq s_1^2t^{2(\nu-1)}e^{2s_1t}\text{ for }t\text{ sufficiently large}\,.\end{aligned}$$

When dealing with $[0,\infty)$, we may take $t_0=1$. However, when dealing with $(-\infty,0]$, we must take $t_0<<0$ to ensure that the term $s_1t^{\nu-1}$ dominates the term $(\nu-1)t^{\nu-2}$ since these terms might have opposite signs and cancellation could occur.

We may compute that:

$$||\dot\sigma_P\wedge\ddot\sigma_P||^2=\sum_{i<j}(\dot\phi_i\ddot\phi_j-\dot\phi_j\ddot\phi_i)^2\,. \tag{8.c}$$

The assumption $s_1>a_1$ shows that the maximal term in this sum occurs when $\phi_i=\phi_{s_1,\nu-1}$ and $\phi_j=\phi_{s_1,\nu-2}$ and thus

$$||\dot\sigma_P\wedge\ddot\sigma_P||^2\leq\frac{n(n-1)}{2}\{\dot\phi_{s_1,\nu-1}\ddot\phi_{s_1,\nu-2}-\dot\phi_{s_1,\nu-2}\ddot\phi_{s_1,\nu-1}\}^2\text{ for }t\geq t_0\,.$$

We have:

$$\begin{aligned}&\dot\phi_{s_1,\nu-1}=(s_1t^{\nu-1}+(\nu-1)t^{\nu-2})e^{s_1t},\\ &\ddot\phi_{s_1,\nu-1}=(s_1^2t^{\nu-1}+2s_1(\nu-1)t^{\nu-2}+(\nu-1)(\nu-2)t^{\nu-3})e^{s_1t},\\ &\dot\phi_{s_1,\nu-2}=(s_1t^{\nu-2}+(\nu-2)t^{\nu-3})e^{s_1t},\\ &\ddot\phi_{s_1,\nu-2}=(s_1^2t^{\nu-2}+2s_1(\nu-2)t^{\nu-3}+(\nu-2)(\nu-3)t^{\nu-4})e^{s_1t},\end{aligned}$$

Consequently:

$$\begin{aligned}
&\dot\phi_{s_1,\nu-1}\ddot\phi_{s_1,\nu-2}-\dot\phi_{s_1,\nu-2}\ddot\phi_{s_1,\nu-1}\\
=\;&\Big\{(s_1t^{\nu-1}+(\nu-1)t^{\nu-2})\\
&\quad\times(s_1^2t^{\nu-2}+2s_1(\nu-2)t^{\nu-3}+(\nu-2)(\nu-3)t^{\nu-3})\Big\}e^{2s_1t}\\
-\;&\Big\{(s_1t^{\nu-2}+(\nu-2)t^{\nu-3})\\
&\quad\times(s_1^2t^{\nu-1}+2s_1(\nu-1)t^{\nu-2}+(\nu-1)(\nu-2)t^{\nu-3})\Big\}e^{2s_1t}
\end{aligned}$$

The leading terms cancel:

$$\{(s_1t^{\nu-1}s_1^2t^{\nu-2})-(s_1t^{\nu-2}s_1^2t^{\nu-1})\}e^{2s_1t}=0\,.$$

The remaining terms are $O(t^{2\nu-4}e^{2s_1t})$ as desired; Equation (8.b) now follows. This shows $\kappa_+[\sigma_P]$ is finite if $s_1>a_1$.

Case II: Suppose $a_1>s_1$. Choose the complex root $\mu_1=a_1+b_1\sqrt{-1}$ to have maximal multiplicity ν among all the complex roots $t\in\mathcal{R}$ with $\Re(t)=a_1$. The dominant term in Equation (8.c) occurs when $\phi_i=\phi_{\mu_1,\nu-1}$ and $\phi_j=\tilde\phi_{\mu_1,\nu-1}$. Differentiating powers of t lowers the order in t and give rise to lower order terms. Thus we may ignore these derivatives and use the computations performed in Section 3 to see:

$$\begin{aligned}
&C_1t^{2\nu-2}e^{2a_1t}\le||\dot\sigma_P||^2\le C_2t^{2\nu-2}e^{2a_1t}\text{ for }t\ge t_0,\\
&(\dot\phi_i\ddot\phi_j-\dot\phi_j\ddot\phi_i)^2\ge C_3t^{4(\nu-1)}e^{4a_1t}\text{ for }t\ge t_0\,.
\end{aligned}$$

We may now conclude that $\kappa_+[\sigma_P]=\infty$.

Case III: The difficulty comes when $a_1=s_1$. If μ_1 is a complex root of multiplicity at least as great as the multiplicity of s_1, the $\{\phi_{\mu_1,\nu-1},\tilde\phi_{\mu_1,\nu-1}\}$ terms dominate the computation and the argument given above in Case II implies $\kappa_+[\sigma_P]$ is infinite. On the other hand, if all the complex roots with $\Re(\lambda)=s_1$ have multiplicity less than the multiplicity of s_1, then the $\phi_{s_1,\nu-1}$ terms dominate the computation and the argument given above in Case I shows that $\kappa_+[\sigma_P]$ is finite. □

We conclude this section with an example where the multiplicity plays a crucial role and where our previous results are not applicable.

Example 8.1. Let $P(\phi)=\phi^{(n)}$ for $n\ge2$. Then $\mathcal{R}=\{0\}$ and 0 is a root of multiplicity n. We have $\mathcal{S}=\operatorname{Span}\{\phi_1:=1,\phi_2:=t,...,\phi_n:=t^{n-1}\}$. Since $t\in\mathcal{S}$, σ_P is a proper map of infinite length on $[0,\infty)$ and on $(-\infty,0]$. We have:

$$\begin{aligned}
&|\dot\sigma_P|^2\ge C_1t^{2n-2},\text{ and}\\
&\sum_{i<j}(\ddot\phi_i\dot\phi_j-\ddot\phi_j\dot\phi_i)^2\\
=\;&\sum_{i<j}((i-1)(i-2)(j-1)-(j-1)(j-2)(i-1))^2t^{2(i+j-3)}\\
&\le C_2t^{2(2n-4)}\,.
\end{aligned}$$

Consequently $|\kappa|\le C_3\frac{t^{2n-4}}{t^{2n-2}}$ for $|t|\ge1$. This is integrable so $\kappa_+[\sigma_P]<\infty$ and $\kappa_-[\sigma_P]<\infty$.

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