is given by the real analytic functions

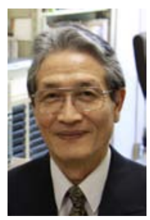
$$\begin{array}{lll} \phi_1 := e^{r_1 t}, & \dots, \phi_k := e^{r_k t}, & \\ \phi_{k+1} := e^{a_1 t}\cos(b_1 t), & \phi_{k+2} := e^{a_1 t}\sin(b_1 t), & \dots, \\ \phi_{n-1} := e^{a_u t}\cos(b_u t), & \phi_n := e^{a_u t}\sin(b_u t). & \end{array}$$

The associated curve $\sigma_P : \mathbb{R} \to \mathbb{R}^n$, the curvature κ of σ_P, the element of arc length ds, and the total first curvature $\kappa[\sigma_P]$ are given by:

$$\sigma_P(t) := (\phi_1(t), \dots, \phi_n(t)), \quad \kappa := \frac{\|\dot\sigma_P \wedge \ddot\sigma_P\|}{\|\dot\sigma_P\|^3},$$
$$ds := \|\dot\sigma_P\| dt, \quad \kappa[\sigma_P] := \int_{\sigma_P} \kappa ds = \int_{-\infty}^{\infty} \frac{\|\dot\sigma_P \wedge \ddot\sigma_P\|}{\|\dot\sigma_P\|^2} dt .$$

Let $\Re(\lambda)$ (resp. $\Im(\lambda)$) be the real (resp. imaginary) part of a complex number λ. We say the *real roots are dominant* if $r_1 > 0 > r_k$ and if $r_1 > \Re(\lambda) > r_k$ for any $\lambda \in \mathcal{R} - \{r_1, r_k\}$. This means that $\lim_{t\to\infty} \|e^{-r_1 t}\sigma_P\| = 1$, $\lim_{t\to-\infty} \|e^{-r_k t}\sigma_P\| = 1$, and $\lim_{t\to\pm\infty} \|\sigma_P\| = \infty$. The remaining roots are then said to be *subdominant*. We refer to P. Gilkey, C. Y. Kim, H. Matsuda, J. H. Park, and S. Yorozu [21] for the proof of the following result.


C. Y. Kim


H. Matsuda


S. Yorozu

Theorem 4.1 Let σ_P be the curve defined by a real constant coefficient ordinary differential equation P of order n with simple roots and with $r_1 > 0 > r_k$.

1. The curve σ_P is a proper embedding of $\mathbb{R}$ into $\mathbb{R}^n$ with infinite length. The total first curvature $\kappa[\sigma_P]$ is finite if and only if the real roots are dominant.

2. If there are no complex roots, then $\kappa[\sigma_P] \le \frac{\pi}{4} n(n-1)$. Given $\epsilon > 0$, there exists an ODE P_ϵ of order n with no complex roots so that $\kappa[\sigma_{P_\epsilon}] \ge \frac{1}{3}(n-1) - \epsilon$ and so that the real roots are dominant. Consequently, any universal upper bound must grow at least linearly with n and at worst quadratically. If subdominant complex roots are allowed, then there is no uniform upper bound.

If σ_P is defined by some other basis for the solution space of P than the standard basis, then Assertion 1 continues to hold and can be generalized to the case when the roots have multiplicities greater than 1.

Now if the indices $\{i, j, \ell\}$ are distinct, then

$$c(e^i)c(e^j)c(e^\ell) = c(e^j)c(e^\ell)c(e^i) = c(e^\ell)c(e^i)c(e^j)\,.$$

We have $R_{ij}{}^k{}_\ell = -R_{ij\ell}{}^k$. The Bianchi identity yields $R_{ij\ell}{}^k + R_{j\ell i}{}^k + R_{\ell ij}{}^k = 0$. Consequently, we may assume that the indices $\{i, j, \ell\}$ are not distinct in Equation (5.3.b). We cannot have $i = j$ as $R_{ij\ell}{}^k + R_{ji\ell}{}^k = 0$. Consequently, either $i = \ell$ or $j = \ell$; this yields the same thing. We suppose $i = \ell \neq j$. Then $c(e^i)c(e^j)c(e^\ell) = c(e^j)$. We may derive Assertion 3 from Assertion 2 by computing:

$$\tfrac{1}{2}c(e^i)c(e^j)R(e_i, e_j)e^k = R_{ij}{}^k{}_i c(e^j)1 = \rho_j{}^k e^j\,.$$

Suppose $\rho \geq 0$. Then

$$\begin{aligned}(\Delta^1\omega, \omega)_{L^2} &= (-\omega_{;ii}, \omega)_{L^2} + (\rho\omega, \omega)_{L^2} \\ &= (\nabla\omega, \nabla\omega)_{L^2} + (\rho\omega, \omega)_{L^2} \geq (\rho\omega, \omega)_{L^2} \geq 0\,.\end{aligned}$$

Consequently, if ω is a smooth 1-form with $\Delta^1\omega = 0$, then $\nabla\omega = 0$ so ω is parallel and hence $\|\omega\|$ is constant. But since $\rho > 0$ at some point, we have $(\rho\omega, \omega)(P) = 0$ so $\omega(P) = 0$. Since $\|\omega\|$ is constant, this implies ω vanishes identically which proves Assertion 4. □

5.3.3 POINCARÉ DUALITY. The following result is due to the French mathematician J. Poincaré in the topological setting.

Jules Henri Poincaré (1854–1912)

Let $\star$ be the Hodge operator and let $c(\text{dvol})$ be Clifford multiplication by the volume form as discussed in Section 5.2. We shall apply Lemma 5.9 and use the Hodge Decomposition Theorem to establish Poincaré duality [53]. If V and W are finite-dimensional vector spaces, then we say that a map f from $V \times W$ to $\mathbb{R}$ is a *perfect pairing* if f is bilinear, if given any v in V there exists w in W so that $f(v, w) \neq 0$, and if given any w in W there exists v in V so that $f(v, w) \neq 0$. Equivalently, f exhibits V as the dual of W and W as the dual of V. Let

$$\mathcal{I}(\omega_p, \theta_{m-p}) := \int_M \omega_p \wedge \theta_{m-p} \quad \text{for } \omega_p \in C^\infty(\Lambda^p M) \text{ and } \theta_{m-p} \in C^\infty(\Lambda^{m-p} M)\,. \tag{5.3.c}$$

The Fundamental Theorem of Ordinary Differential Equations is often called the Cauchy–Lipschitz Theorem or the Picard–Lindelöf Theorem and is named after Émile Picard, Ernst Lindelöf, Rudolf Lipschitz, and Augustin–Louis Cauchy. It deals with the existence and uniqueness of solutions to an ordinary differential equation.

A. Cauchy (1789–1857)

R. Lipschitz (1832–1903)

E. Picard (1856–1941)

For each $g \in G$, this result shows that there exists an open neighborhood $\mathcal{O}_g$ of g and there exists $\epsilon_g > 0$ so the flow Φ_t^X exists for $|t| \le \epsilon_g$ on $\mathcal{O}_g$. Since $(L_g)_* X = X$, L_g commutes with Φ_t^X, i.e., $L_g \Phi_t^X = \Phi_t^X L_g$. This shows that $\Phi_t^X = L_g \Phi_t^X L_{g^{-1}}$ is well-defined on $L_g(\mathcal{O}_e)$ for $|t| < \epsilon_e$. Consequently, we can choose ϵ uniformly on G. We use the semi-group property $\Phi_t^X \Phi_s^X = \Phi_{s+t}^X$ to extend the flow for all t and prove Assertion 2. Since $\Phi_{ts}^X = \Phi_t^{sX}$,

$$\exp^{\mathfrak{g}}(tX) = \Phi_1^{tX} = \Phi_t^X\,;$$

Assertion 3 follows.

Let $F : G \to H$ be a group homomorphism. If $\xi \in T_{e_G} G$, choose $X \in \mathfrak{g}$ so $X(e_G) = \xi$. Let $\tilde{\xi} := F_* \xi \in T_{e_H} H$. Let $\tilde{X} \in \mathfrak{h}$ satisfy $\tilde{X}(e_H) = \tilde{\xi}$. Since $L_{Fh} F = F L_h$,

$$F_* X(h) = F_*(L_h)_* X(e) = (L_{F(h)})_* F_* X(e) = L_{F(h)} \tilde{\xi} = \tilde{X}(Fh)$$

and thus $F_* X = \tilde{X}$. Thus, by Lemma 6.1, F_* is a Lie algebra morphism. If $F_* = 0$, then

$$\partial_t F \exp^{\mathfrak{g}}(t\xi)|_{t=t_0} = X^H_{F_*\xi}(F \exp^{\mathfrak{g}}(t_0 \xi)) = 0$$

and, consequently, $F \exp^{\mathfrak{g}}(t\xi) = e_H$ for all t. Since $\exp^{\mathfrak{g}}$ is a local diffeomorphism from a neighborhood of 0 in $T_{e_G} G$ to G, this implies F is constant on a neighborhood of e_G. Since G is connected, F is constant; this shows Assertion 4. □

Suppose G is connected. We shall show in Corollary 6.20 that if G is compact, then $\exp^{\mathfrak{g}}$ is surjective. The exponential map need not be subjective if G is not compact; we will show in Lemma 6.25 that exponential map for $\mathrm{SL}(2, \mathbb{R})$ is not surjective.

6.2.9 THE ADJOINT REPRESENTATION. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. If $\xi \in \mathfrak{g}$, let $\mathrm{ad}(\xi) \in \mathrm{Hom}(\mathfrak{g})$ be defined by $\mathrm{ad}(\xi) : \eta \to [\eta, \xi]$. The Jacobi identity yields:

$$\begin{aligned} [\mathrm{ad}(\xi), \mathrm{ad}(\eta)]\gamma &= \mathrm{ad}(\xi)\,\mathrm{ad}(\eta)\gamma - \mathrm{ad}(\eta)\,\mathrm{ad}(\xi)\gamma = \mathrm{ad}(\xi)[\eta, \gamma] - \mathrm{ad}(\eta)[\xi, \gamma] \\ &= [\xi, [\eta, \gamma]] - [\eta, [\xi, \gamma]] = [[\xi, \eta], \gamma] = \mathrm{ad}[\xi, \eta]\gamma. \end{aligned}$$

Proof. Let ω be a p-form with $\Delta^p = 0$, and let $h \in G$. Because the metric is left-invariant, $L_h^* \Delta = \Delta L_h^*$. Therefore, $L_h^* \omega \in \ker\{\Delta^p\}$ as well. On the other hand, G is assumed to be connected. Consequently, there is a smooth curve $\gamma(t)$ connecting h to the identity and providing a homotopy between L_h and L_{Id}. The homotopy axiom then shows $[L_h^*\omega] = [\omega]$ in de Rham cohomology. The Hodge Decomposition Theorem (see Theorem 5.13) then shows $L_h^*\omega = \omega$. The argument is the same for right-invariance; Assertion 1 now follows.

Suppose ω is left-invariant and that $\omega = d\psi$ for some smooth $(p-1)$-form which is not necessarily left-invariant. We average over the group to see that:

$$\begin{aligned} \omega &= \int_G (L_h^*\omega)\,\mathrm{dvol}(h) = \int_G (L_h^* d\psi)\,\mathrm{dvol}(h) = d\int_G (L_h^*\psi)\,\mathrm{dvol}(h) \\ &= d\{\Psi\} \quad \text{where} \quad \Psi := \int_G (L_h^*\psi)\,\mathrm{dvol}(h)\,. \end{aligned}$$

Assertion 2 now follows by noting Ψ is left-invariant. We have used, of course, the fact that we can interchange the order of differentiation and integration. Assertion 1 shows that $[i]$ is surjective and Assertion 2 shows that $[i]$ is injective as a map from $H^p(\mathfrak{g}^*, d)$ to $H^p_{\mathrm{dR}}(G)$. Assertion 3 now follows. □

6.9.1 THE HOPF STRUCTURE THEOREM. The results in this section arise from work of H. Hopf [33].

H. Hopf (1894–1971)

We refer the reader to the discussion in Section 8.1.6 for the definition of a unital graded ring. We say that a connected unital graded ring R is a *co-ring* if we have a co-multiplication θ which is a graded ring morphism from R to $R \otimes R$ which is *co-associative*, i.e.,

$$(\mathrm{Id} \otimes \theta) \circ \theta = (\theta \otimes \mathrm{Id}) \circ \theta\,.$$

We do not assume the co-multiplication is co-commutative. We say that R is a *Hopf algebra* if, in addition, there is an augmentation $\varepsilon : R \to F$ so

$$(\mathrm{Id} \otimes \epsilon) \circ \theta = \mathrm{Id} \quad \text{and} \quad (\epsilon \otimes \mathrm{Id}) \circ \theta = \mathrm{Id}\,.$$

We use the existence of a co-unit to pin θ down slightly. We expand

$$\theta(a) = 1 \otimes b_n + \text{stuff} + a_n \otimes 1$$

where the deleted material "stuff" belongs to $\oplus_{i>0, j>0, i+j=n} R_i \otimes R_j$. We conclude