Handbook of Global Analysis
Demeter Krupka and David Saunders

Abstracts

Global aspects of Finsler geometry
Tadashi Aikou and László Kozma

Finsler geometry originates from the calculus of variations, started in the twenties of the last century. The first essential movement towards global aspects of Finsler metrics on manifolds was due to L. Auslander in 1955. Then, the development of global (principal) connection theory, applied for Finslerian structures by M. Matsumoto in the sixties, opened the door to discuss the generalization of Riemannian global results for Finsler geometry. This was getting through by Chern's flourishing school in the last 15 years.

In this report we intend to sketch the state of today in this respect without the purpose of completeness. First the basics in Finsler Geometry are described: the fundamental function, the Chern connection, torsions and curvatures, the flag curvature. Then the properties of geodesics, the exponential map, the minimality of geodesics, and the Hopf-Rinow theorem are given. We discuss the first and second variation formulae, Jacobi fields, conjugate points, injectivity radius and related topics, such as Cartan-Hadamard and Bonnet-Myers theorems. We quote the fundamental comparison theorems of Finsler geometry: some depends on bounds of flag curvature, while others depend on bounds of the Ricci scalar curvature. Several rigidity theorems are presented, which give assumptions on the Finsler structure for the reduction to a locally Minkowski or Riemannian space. Then, applying Morse's theory for the energy functional, some results on the length and multiplicity of closed geodesics of Finsler manifolds, and the sphere theorem are reviewed. Finally, we report how the Gauss-Bonnet theorem have been extended for Finsler manifolds.

Morse theory and nonlinear differential equations
Thomas Bartsch, Andrzej Szulkin and Michel Willem

In this survey we treat Morse theory on Hilbert manifolds for functions with degenerate critical points. We describe the global and the local aspects of the theory, in particular the Morse inequalities, the Morse Lemma and critical groups. We consider applications to semi-linear elliptic problems and to closed geodesics on a riemannian manifold. The theory is extended to strongly indefinite functionals. Applications are given to periodic solutions of first order hamiltonian systems.

Index theory
David Bleecker

Early history of index theory is given, covering events leading to the discovery of the Atiyah-Singer Index Theorem. Basic facts are proven about Fredholm operators and the behavior of the index under perturbations, composition, etc.. The connection between families of Fredholm operators and K-theory is made in the Atiyah-Janich Theorem. Elliptic pseudo-differential operators on compact manifolds are shown to yield Fredholm operators. Outlines are given of the embedding proof of the Atiyah-Singer Index Theorem and the heat kernel proof for twisted Dirac operators. Statements are provided for twisted versions of the classical index theorems; e.g., the Hirzebruch-Signature, Chern-Gauss-Bonnet, and Hirzebruch-Riemann-Roch Theorems. Brief treatments of G-index theory and the Atiyah-Patodi-Singer Theorem are included.

Partial differential equations on closed and open manifolds
Jürgen Eichhorn

We present a survey of some important classes of partial differential equations on manifolds and of methods for solving them. This concerns questions of spectral theory, the heat equation and the heat kernel, the wave equation, Huygens' principle and the Hamiltonian approach, index theory on open manifolds, the continuity method and a choice of nonlinear equations important in geometry and mathematical physics. The spaces under consideration are linear and non-linear Sobolev structures which we briefly define at the beginning of our contribution.

The spectral geometry of operators of Dirac and Laplace type
Peter Gilkey

We survey results concerning asymptotic formulae in spectral geometry. We give explicit combinatorial formulas for both the heat trace asymptotics and for the heat content asymptotics in the context of smooth manifolds with boundary for the realization with respect to a variety of elliptic boundary conditions of an operator of Laplace type. We relate these formulaes to questions in spectral geometry and to the index theorem.

Lagrangian formalism on Grassmann manifolds
D. R. Grigore

The Lagrangian formalism on a arbitrary non-fibrating manifold is described from the kinematical point of view by (higher-order) Grassmann manifolds; such manifolds are obtained by factorization of the regular velocity manifold to the action of the differential group. Here we present the basic concepts of the Lagrangian formalism, as Lagrange, Euler-Lagrange and Helmholtz-Sonin forms, relevant in this context. These objects come in pairs, namely we have homogeneous objects (defined on the regular velocity manifold) and non-homogeneous objects (defined on the Grassmann manifold) and we give the connection between them. As a result the generic expressions for a variationally trivial Lagrangian and for a locally variational differential equation remain the same as in the fibrating case. Finally, we concentrate on the case of second order Grassmann bundles which are relevant for many physically interesting cases.

Sobolev spaces on manifolds
Emmanuel Hebey and Frédéric Robert

Sobolev spaces are important tools in several branches of mathematics. We discuss various aspects of Sobolev spaces on manifolds. While Sobolev spaces are well understood in Euclidean space, surprises occur in the context of Riemannian manifolds. Starting from the very first definition of such spaces, we discuss existence and nonexistence of Sobolev embeddings, different types of Sobolev inequalities, including the isoperimetric and the Nash inequality, and the difficult question of getting sharp constants in Sobolev inequalities.

Harmonic maps
Frédéric Hélein and John C. Wood

Harmonic maps are maps between Riemannian manifolds which extremize a natural energy functional or 'Dirichlet integral'. They include harmonic functions between Euclidean spaces, geodesics, minimal immersions, and harmonic morphisms (maps which preserve Laplace's equation). The Euler-Lagrange equations satisfied by a harmonic map form a semi-linear elliptic system of partial differential equations of second order.

We concentrate on the key questions of *existence*, *uniqueness* and *regularity* of harmonic maps between given manifolds. We survey some of the main methods of global analysis for answering these questions, together with the approach using twistor theory and integrable systems.

Topology of differentiable mappings
Kevin Houston

To study the topology of a differentiable mapping one can consider its image or its fibres. A proportion of this survey paper looks at how the latter can be studied in the case of singular complex analytic maps. An important aspect of this is study of the local case, the primary object of interest of which is the Milnor Fibre. More generally, Stratified Morse Theory is used to investigate the topology of singular spaces. In the complex case we can use rectified homotopical depth to generalize the Lefschetz Hyperplane Theorem. In the less studied case of images of maps we describe a powerful spectral sequence that can be used to investigate the homology of the image of a finite and proper map using the alternating homology of the multiple point spaces of the map.

Group actions and Hilbert's fifth problem
Sören Illman

In the first section of the article we consider Hilbert's fifth problem concerning Lie's theory of transformation groups. In his fifth problem Hilbert asks the following. Given a continuous action of a locally euclidean group G on a locally euclidean space M, can one choose coordinates in G and M so that the action is real analytic? We discuss the affirmative solutions given in Theorems 1.1 and 1.2, and also present known counterexamples to the general question posed by Hilbert. Theorem 1.1 is the celebrated result from 1952, due to Gleason, Montgomery and Zippin, which says that every locally euclidean group is a Lie group. Theorem 1.2 is a more recent result, due to the author, which says that every Cartan (thus in particular, every proper) C^s differentiable action, $1 \leq s \leq \infty$, of a Lie group G is equivalent to a real analytic action.

The remaining part of the article, Sections 2-18, is then used to give a complete, and to a very large extent selfcontained, proof of Theorem 1.2. This tour brings us into many different topics within the theory of transformation groups.

Exterior differential systems
Niky Kamran

We review the main existence theorems for integral manifolds of exterior differential sytems, with a special emphasis the Cartan-Kähler Theorem for involutive analytic exterior differential systems. These theorems are illustrated on a number of classical problems in differential geometry and contact geometry of differential equations. We also give an introduction to the Cartan-Kuranishi Prolongation Theorem, and to the characteristic cohomology of exterior differential systems.

Weil bundles as generalized jet spaces
Ivan Kolář

We generalize the concept of higher order velocity and we interpret a Weil bundle as the space of A-velocities for an arbitrary Weil algebra A. Using a recent identification of every product preserving bundle functor on manifolds with a Weil functor T^A, we deduce geometric results on T^A-prolongations of various geometric objects. We characterize every fiber product preserving bundle functor F on fibered manifolds in a jet-like manner and we study F-prolongations of several geometric structures.

Distributions, vector distributions, and immersions of manifolds in Euclidean spaces
Július Korbaš

Our main topics are *Schwartzian distributions* (including generalized sections of vector bundles), *vector distributions* (including the vector field problem), and *immersions* of smooth manifolds in Euclidean spaces (including isometric immersions). The Atiyah-Singer index theorem (covered by D. Bleecker's contribution in this Handbook) is a kind of node at which the three topics are joined.

Geometry of differential equations
Boris Kruglikov and Valentin Lychagin

We review geometric and algebraic methods of investigations of systems of partial differential equations. Classical and modern approaches are reported.

Global variational theory in fibred spaces
D. Krupka

We survey recent developments in the general theory of higher order, global integral variational functionals on fibred manifolds. First we study differential forms on jet prolongations of fibred manifolds. Then we introduce basic global concepts of the theory of Lagrange structures, such as the Lagrangian, the Lepage form, the Euler-Lagrange form, and characterize their properties in terms of differentiation and integration theory on manifolds. We study properties of the Euler-Lagrange mapping, and variational functionals, invariant with respect to automorphisms of underlying fibred manifolds. We finally discuss selected topics: examples of Lepage forms, the variational sequence and its consequences, possible formulations of the Hamilton theory, lifting functors and natural bundles, and properties of natural variational functionals on natural bundles. Remarks on the proofs of basic assertions are presented.

Second Order Ordinary Differential Equations in Jet Bundles and the Inverse Problem of the Calculus of Variations
O. Krupková and G. E. Prince

This article is ostensibly concerned with the inverse problem in the calculus of variations for a single independent variable: "when does a system of second order ordinary differential equations admit an equivalent variational formulation as a set of Euler-Lagrange equations?"

Because efforts to solve this famous 120 year old problem over the last 3 decades have involved the development of many entirely new geometric frameworks for second order ordinary differential equations, we firstly describe this underlying geometry, covering topics such as calculus on jet bundle prolongations of fibred manifolds, the geometric description of second order ordinary differential equations by both forms and vector fields, and of variational structures, globally and locally. We then turn to the inverse problem in both its covariant and contravariant forms and derive and discuss the famous Helmholtz conditions, being the necessary and sufficient conditions for the existence of a Lagrangian. We give the various geometric versions of the renowned work of Douglas who solved the problem for two degrees of freedom and review the latest progress on the problem for arbitrary degrees of freedom using exterior differential systems theory. The article is intended to provide a comprehensive introduction to the various aspects of current research.

Elements of noncommutative geometry
Giovanni Landi

We give an introduction to noncommutative geometry and its use. In the presented approach, a geometric space is given a spectral description as a triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of a *-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ together with an unbounded self-adjoint operator D interacting with the algebra in a bounded manner. The aim is to carry geometrical concepts over to a new class of spaces for which the algebra of functions $\mathcal{A}$ is noncommutative in general. We supplement the general theory with examples which include toric noncommutative spaces and spaces coming from quantum groups.

De Rham cohomology
M. A. Malakhaltsev

The aim of the paper is to give a brief introduction to the de Rham cohomology theory and to expose some relevant results in differential geometry. It includes the following topics: 1) De Rham complex. De Rham cohomology; 2) Integration and de Rham cohomology. De Rham currents. Harmonic forms; 3) Generalizations of the de Rham complex; 4) Equivariant de Rham cohomology; 5) Complexes of differential forms associated to differential geometric structures. No proofs are given, however the main statements are supplied with references to literature where the reader can find detailed exposition including proofs. The bibliography: 97 titles.

Topology of manifolds with corners
J. Margalef-Roig and E. Outerelo Domínguez

The study of manifolds with corners was originally developed by J. Cerf and A. Douady as a natural generalization of the concept of finite-dimensional manifold with smooth boundary, and applications of this type of manifolds in differential topology arose immediately after its definition. In the setting of Global Analysis a very natural task is to extend the results of finite-dimensional manifolds with corners to infinite dimensional manifolds. Thus, in this article, we survey the main features of the manifolds with corners modeled on Banach spaces or on larger categories of spaces as can be the normed spaces, the locally convex vector spaces and the convenient vector spaces, that have arisen as very important, in the last years.

Jet manifolds and natural bundles
D. J. Saunders

We introduce manifolds of jets, including jets of sections, jets of immersed submanifolds and higher-order velocities, and describe some of the geometrical structures which are canonically associated with these manifolds. As applications, we give brief introductions to the integrability theory for differential equations using Spencer cohomology, and to the calculus of variations and the associated variational complexes. We finally describe how jets may be used to characterise those operators having chart-independent coordinate representations by using the concepts of natural bundle and natural operator.

Some aspects of differential theories
József Szilasi and Rezső L. Lovas

As Serge Lang wrote, it is possible to lay down the foundations (and more beyond) for manifolds modeled on Banach or Hilbert spaces rather than finite dimensional spaces *at no extra cost*. In this article we briefly outline how the theory works if the model space is a more general locally convex (real) topological vector space, and the underlying differential calculus is the infinite-dimensional calculus initiated by A. D. Michel and A. Bastiani. In order to present at least one essential application of basic techniques of functional analysis, we discuss in detail a coordinate-free characterization of differential operators due to J. Peetre. In the last part we consider the covariant derivative operator discovered by S. S. Chern and H. Rund (independently), and which became an indispensable tool for present day Finsler geometry. We show that the Chern-Rund derivative can be interpreted as a differential operator on the *base manifold*.

Variational sequences
R. Vitolo

Variational sequences are complexes of modules or sheaf sequences in which one of the operations is the Euler-Lagrange operator, *i.e.*, the differential operator taking a Lagrangian into its Euler-Lagrange form, whose kernel is the Euler-Lagrange equation.

In this paper we present the most common approaches to variational sequences and discuss some directions of the current research on the topic.

The Oka-Grauert-Gromov principle for holomorphic bundles
Pit-Mann Wong

The Oka-Grauert-Gromov principle is a very powerful tool in the theory of holomorphic fiber bundles, more generally subelliptic bundles, over Stein spaces. The basic form asserts that such bundles must be holomorphically trivial if it is topologically trivial. This is particular useful in the study of holomorphic mappings from a Stein space into a projective variety. The situation is particularly nice in the case of hyperbolic geometry where the domain is the complex Euclidean space. The Oka-Grauert-Gromov principle in this case says that every subelliptic bundle is trivial. For example, the pull-back of the arc spaces or the parametrized jet bundles of any order are trivial and information of derivatives of any order can be dealt with using only classical function theory. The theory can even be extended to the case of Cartesian spaces defined over p-adic number fields. The basic form of Oka-Grauert-Gromov principle now is equivalent to the assertion that every projective module, over a ring p-adic convergent power series, is free. This is an analogue of the classical Serre's problem that every projective module, over a polynomial ring with coefficients in a field, is free. In the complex case, the principle can be applied to many important problems: submersion, immersion, embedding problems; extension problems; theory of complete intersections; homotopy theory, to name a few. The analogue of these more sophisticated problems of the principle are, as yet, to be explored in the p-adic case.