

Complete k -Curvature Homogeneous Pseudo-Riemannian Manifolds

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(Received: 3 May 2004; accepted: 21 September 2004)

Abstract. For $k \geq 2$, we exhibit complete k -curvature homogeneous neutral signature pseudo-Riemannian manifolds which are not locally affine homogeneous (and hence not locally homogeneous). All the local scalar Weyl invariants of these manifolds vanish. These manifolds are Ricci flat, Osserman, and Ivanov–Petrova.

Mathematics Subject Classification (2000): 53B20.

Key words: affine k -curvature homogeneous, Ivanov–Petrova manifold, k -curvature homogeneous, locally affine homogeneous, Osserman manifold, Ricci flat

1. Introduction

We consider a pair $\mathcal{M} := (M, g_M)$ where g_M is a pseudo-Riemannian metric of signature (p, q) on a smooth manifold M of dimension $m := p + q$. Let R be the associated Riemann curvature tensor and let $\nabla^k R$ denote the k th covariant derivative of the curvature tensor. We say that \mathcal{M} is k -curvature homogeneous if given any two points $P, Q \in M$, there exists an isomorphism $\phi_{P,Q}$ from $T_P M$ to $T_Q M$ so that

$$\phi^* g_Q = g_P, \phi^* R_Q = R_P, \dots, \phi^* \nabla^k R_Q = \nabla^k R_P.$$

This means that the metric, curvature tensor, and covariant derivatives up to order k of the curvature tensor ‘look the same’ at each point.

There is an equivalent algebraic formalism. Consider

$$\mathcal{U}_m^k := (V, g, A^0, A^1, \dots, A^k)$$

where g is an inner product on a m -dimensional real vector space V and where $A^i \in \otimes^{4+i} V^*$ for $0 \leq i \leq k$. We say that \mathcal{U}_m^k is a k -model for \mathcal{M} if for every point $P \in M$, there is an isomorphism $\phi_P : T_P M \rightarrow V$ so that

$$\phi_P^* g = g, \phi_P^* A^0 = R_P, \dots, \phi_P^* A^k = \nabla^k R_P.$$

If \mathcal{M} is k -curvature homogeneous, then $\mathcal{U}_{m,P}^k := (T_P M, g_P, R_P, \dots, \nabla^k R_P)$ is a k -model for \mathcal{M} for any point $P \in M$; conversely, if \mathcal{M} admits a k -model, then \mathcal{M} is k -curvature homogeneous. Let $\mathcal{N} := (N, h)$ be a homogeneous pseudo-Riemannian manifold. We say that \mathcal{M} is *modeled on* \mathcal{N} if $(T_Q N, h_Q, R_{h,Q})$ is a 0-model for \mathcal{M} for any (and hence for all) $Q \in N$.

There are a number of important results in this area in the Riemannian setting ($p = 0$). Takagi [32] was the first to exhibit 0-curvature homogeneous manifolds which are not locally homogeneous; his examples were noncompact. Subsequently, compact examples were exhibited by Ferus *et al.* [11]. Tomassini [33] studied principal fiber bundles with one-dimensional fiber over a 0-curvature homogeneous base. Other examples may be found in [22, 23, 34, 37]. Tsukada [35] classified 0-curvature homogeneous hypersurfaces of dimension $m \geq 4$ in complete and simply connected Riemannian space forms; the case $m = 3$ was subsequently treated (but not completely solved) by Calvaruso *et al.* [8]. Kowalski and Prüfer [21] exhibited four-dimensional algebraic curvature tensors which are not realizable by any 0-curvature homogeneous space.

Scalar invariants can be obtained by using the Weyl calculus to contract indices in pairs in a polynomial expression involving the curvature and its higher covariant derivatives. For example, the scalar curvature is given by

$$\tau := \sum_{ijkl} g^{ij} g^{kl} R_{ijkl}.$$

Clearly, if \mathcal{M} is locally homogeneous, then all such scalar invariants are necessarily constant.

We summarize some important results in this field in the Riemannian setting.

THEOREM 1.1. *Let \mathcal{M} be a Riemannian manifold. Then*

- (1) *If \mathcal{M} is modeled on an irreducible Riemannian symmetric space \mathcal{N} , then \mathcal{M} is locally symmetric and hence locally isometric to \mathcal{N} (Tricerri and Vanhecke [36]).*
- (2) *There exists an integer k_m so that if \mathcal{M} is a complete simply connected manifold of dimension m which is k_m -curvature homogeneous, then \mathcal{M} is homogeneous (Singer [29]).*
- (3) *If all local scalar Weyl invariants up to order $\frac{1}{2}m(m-1)$ are constant on a Riemannian manifold \mathcal{M} , then \mathcal{M} is locally homogeneous and \mathcal{M} is determined up to local isometry by these invariants (Prüfer *et al.* [27]).*

We refer to Berger [1, 2] for a further discussion, a proof of Theorem 1.1 (1) being given on page 46 of [1] for example. We also note that we have changed the definition of k_m slightly; the number k_m used here corresponds to $k_m + 1$ in the notation of Singer and other authors.

We remark that Cahen *et al.* [7] used a classification result of Berger to show that if \mathcal{M} is a Lorentzian ($p = 1$) manifold which is modeled on an irreducible Lorentzian symmetric space, then \mathcal{M} has constant sectional curvature. Thus, Assertion (1) has a natural, and even stronger, extension to the Lorentzian setting.

Singer established the bound $k_m < \frac{1}{2}m(m - 1)$. Bounds of $3m - 5$ and $\frac{3}{2}m - 1$ for k_m have been established by Yamato [38] and Gromov [17]. In the low-dimensional setting, Sekigawa *et al.* [30, 31] showed that $k_3 = k_4 = 1$. We refer to the discussion by Boeckx *et al.* [4] for further details concerning k -curvature homogeneous manifolds in the Riemannian setting.

Theorem 1.1 (2) extends to the pseudo-Riemannian setting:

THEOREM 1.2 (Podesta and Spiro [28]). *There exists an integer $k_{p,q}$ so that if (M, g) is a complete simply connected pseudo-Riemannian manifold of signature (p, q) which is $k_{p,q}$ -curvature homogeneous, then (M, g) is homogeneous.*

Opozda [25] has established an analogue of this result in the affine setting.

In the Lorentzian setting, examples of curvature homogeneous manifolds which are not locally homogeneous were constructed by Cahen *et al.* [7]. Subsequently, 1-curvature homogeneous manifolds which are not locally homogeneous have been constructed by Bueken and Vanhecke [6]; we also refer to related work of Bueken and Djorić [5]. These examples are important since they show that the results of [30, 31] do not extend to the indefinite setting. Pravda *et al.* [26] exhibited Lorentz manifolds all of whose scalar Weyl invariants vanish and which are not locally homogeneous; thus, Theorem 1.1 (3) is false in the Lorentzian case.

Not as much is known if $p \geq 2$ and if $q \geq 2$, i.e. if one is in neither the Riemannian or Lorentzian contexts. The authors [16] exhibited a family of complete 1-curvature homogeneous pseudo-Riemannian manifolds of signature $(r + 1, r + 1)$ on \mathbb{R}^{2r+2} for $r \geq 2$ which were 0-modeled on an irreducible symmetric space and which were not 2-curvature homogeneous (and hence not homogeneous); two other families of 0-curvature pseudo-Riemannian manifolds were also exhibited that are 0-modeled on irreducible symmetric spaces. Thus, Theorem 1.1 (1) fails in this context. We also refer to [15] for other examples of 0-curvature homogeneous pseudo-Riemannian manifolds.

Let $k = p + 2 \geq 2$ be given. In this paper, we will exhibit a family of complete neutral signature metrics $g_{2p+6, f}$ on \mathbb{R}^{2p+6} which are k -curvature homogeneous but not locally homogeneous for generic values of f . We shall be defining a number of tensors. To simplify the discussion, we shall only give the nonzero entries in these tensors up to the usual \mathbb{Z}_2 symmetries.

Introduce coordinates $(x, y, z_0, \dots, z_p, \bar{x}, \bar{y}, \bar{z}_0, \dots, \bar{z}_p)$ on \mathbb{R}^{2p+6} . Let $f = f(y)$ be a smooth function on \mathbb{R} . Let $\mathcal{M}_{2p+6, f} := (\mathbb{R}^{2p+6}, g_{2p+6, f})$ be the pseudo-Riemannian manifold of balanced (i.e. neutral) signature $(p + 3, p + 3)$ where the

nonzero metric components are:

$$g_{2p+6,f}(\partial_{z_i}, \partial_{\bar{z}_j}) = \delta_{ij}, \quad g_{2p+6,f}(\partial_x, \partial_{\bar{x}}) = 1, \\ g_{2p+6,f}(\partial_y, \partial_{\bar{y}}) = 1, \quad \text{and} \quad g_{2p+6,f}(\partial_x, \partial_x) = -2F_{2p+6,f},$$

where

$$F_{2p+6,f} = F_{2p+6,f}(y, z_0, \dots, z_p) := f(y) + yz_0 + y^2z_1 + \dots + y^{p+1}z_p.$$

Choose a basis \mathcal{B} for \mathbb{R}^{2p+6} of the form

$$\mathcal{B} := \{X, Y, Z_0, \dots, Z_p, \bar{X}, \bar{Y}, \bar{Z}_0, \dots, \bar{Z}_p\}.$$

Consider the models $\mathcal{U}_{2p+6}^i := (\mathbb{R}^{2p+6}, g_{2p+6}, A_{2p+6}^0, \dots, A_{2p+6}^i)$ for $0 \leq i \leq p+2$ where the inner product g_{2p+6} and the tensors $A_{2p+6}^i \in \otimes^{4+i}(\mathbb{R}^{2p+6})^*$ have nonzero components

$$\begin{aligned} g_{2p+6}(X, \bar{X}) &= g_{2p+6}(Y, \bar{Y}) = g_{2p+6}(Z_i, \bar{Z}_i) = 1, \quad A_{2p+6}^0(X, Y, Z_0, X) = 1, \\ A_{2p+6}^1(X, Y, Z_1, X; Y) &= A_{2p+6}^1(X, Y, Y, X; Z_1) = 1, \\ A_{2p+6}^2(X, Y, Z_2, X; Y, Y) &= A_{2p+6}^2(X, Y, Y, X; Z_2, Y) \\ &= A_{2p+6}^2(X, Y, Y, X; Y, Z_2) = 1, \dots \tag{1a} \\ A_{2p+6}^p(X, Y, Z_p, X; Y, \dots, Y) &= A_{2p+6}^p(X, Y, Y, X; Z_p, Y, \dots, Y) \\ &= \dots = A_{2p+6}^p(X, Y, Y, X; Y, \dots, Y, Z_p) = 1, \\ A_{2p+6}^{p+1}(X, Y, Y, X; Y, \dots, Y) &= 1, \quad \text{and} \quad A_{2p+6}^{p+2}(X, Y, Y, X; Y, \dots, Y) = 1. \end{aligned}$$

THEOREM 1.3.

- (1) All geodesics in $\mathcal{M}_{2p+6,f}$ extend for infinite time.
- (2) $\exp_P : T_P \mathbb{R}^{2p+6} \rightarrow \mathbb{R}^{2p+6}$ is a diffeomorphism for any $P \in \mathbb{R}^{2p+6}$.
- (3) \mathcal{U}_{2p+6}^p is a p -model for $\mathcal{M}_{2p+6,f}$.
- (4) If the derivatives $f^{(p+3)}(y)$ and $f^{(p+4)}(y)$ are positive on the whole real line, then \mathcal{U}_{2p+6}^{p+2} is a $p+2$ -model for $\mathcal{M}_{2p+6,f}$.

It is convenient to work in the affine setting. Let

$$\mathcal{R}(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

be the curvature operator defined by a torsion free connection ∇ on the tangent bundle of a smooth manifold M . Following Opzoda [24], we say that (M, ∇) is affine k -curvature homogeneous if given any two points P and Q of M , there is a linear isomorphism $\phi: T_P M \rightarrow T_Q M$ so that $\phi^* \nabla^i \mathcal{R}_Q = \nabla^i \mathcal{R}_P$ for $0 \leq i \leq k$. Taking ∇ to be the Levi-Civita connection of a pseudo-Riemannian metric then yields that any k -curvature homogeneous manifold is necessarily affine k -curvature

homogeneous by simply forgetting the requirement that ϕ be an isometry; there is no metric present in the affine setting.

We say that (M, ∇) is *locally affine homogeneous* if given any points P and Q in M , there is a diffeomorphism Φ from a neighborhood of P to a neighborhood of Q so that $\Phi(P) = Q$ and so that $\Phi^*\nabla = \nabla$. If (M, ∇) is locally affine homogeneous, necessarily (M, ∇) is affine k -curvature homogeneous for any k . Examples of 2-curvature homogeneous affine manifolds which are not locally affine homogeneous are known; we refer to the discussion in [12, 18–20, 24] for this and related results.

We will show that all the scalar Weyl invariants of $\mathcal{M}_{2p+6,f}$ vanish; these manifolds provide additional examples showing Theorem 1.1 (3) can fail outside the Riemannian context. To show that $\mathcal{M}_{2p+6,f}$ is not locally homogeneous, we must define a suitable invariant. We assume $f^{(p+4)} > 0$ and set

$$\alpha_{2p+6,f} := \frac{f^{(p+3)} f^{(p+5)}}{f^{(p+4)} f^{(p+4)}}.$$

Let ∇ be the Levi-Civita connection of $g_{2p+6,f}$. We will show that $\alpha_{2p+6,f}$ is a local affine invariant of $(\mathbb{R}^{2p+6}, \nabla)$; it is not of Weyl type. For generic f , the zero set of the derivative $\alpha'_{2p+6,f}$ is discrete and, hence, $\alpha_{2p+6,f}$ is not constant on any open set; thus, for generic f , $\mathcal{M}_{2p+6,f}$ is not locally affine homogeneous and hence not locally homogeneous; furthermore, the scalar Weyl invariants do not determine $\mathcal{M}_{2p+6,f}$ up to local isometry.

THEOREM 1.4. *Assume that the derivatives $f^{(p+3)}(y)$ and $f^{(p+4)}(y)$ are positive on the whole real line. Then*

- (1) *All scalar Weyl invariants of $\mathcal{M}_{2p+6,f}$ vanish.*
- (2) *If $\mathcal{M}_{2p+6,f}$ is affine $p + 3$ curvature homogeneous, then $\alpha_{2p+6,f}$ is constant.*
- (3) *If ϕ is a local diffeomorphism of $\mathcal{M}_{2p+6,f}$ such that $\phi^*\nabla = \nabla$, then we have that $\phi^*\alpha_{2p+6,f} = \alpha_{2p+6,f}$.*
- (4) *If $\alpha_{2p+6,f}$ is nonconstant, then $\mathcal{M}_{2p+6,f}$ is not locally affine curvature homogeneous.*

We have

COROLLARY 1.5. *Let $s = \min(p, q)$. If $s \geq 3$, then $k_{p,q} \geq s$.*

There are two special cases which are important. Set

$$\mathcal{M}_{2p+6}^1 := \mathcal{M}_{2p+6,e^y} \quad \text{and} \quad \mathcal{M}_{2p+6}^2 := \mathcal{M}_{2p+6,e^y+e^{2y}}.$$

THEOREM 1.6.

- (1) \mathcal{M}_{2p+6}^1 *is a homogeneous space.*

- (2) \mathcal{M}_{2p+6}^2 is $p + 2$ -modeled on \mathcal{M}_{2p+6}^1 .
- (3) \mathcal{M}_{2p+6}^2 is not locally $p + 3$ -affine curvature homogeneous.

The Jacobi operator is the self-adjoint operator characterized by the property $g(J(X)Y, Z) = R(Y, X, X, Z)$. One says that \mathcal{M} is *nilpotent Osserman* if 0 is the only eigenvalue of the Jacobi operator $J(X)$ for any tangent vector X . If $\{e_1, e_2\}$ is an oriented orthonormal basis for a nondegenerate 2-plane π , then the skew-symmetric endomorphism $\mathcal{R}(\pi) := \mathcal{R}(e_1, e_2)$ is independent of the particular basis chosen. One says that \mathcal{M} is *nilpotent Ivanov–Petrova* if 0 is the only eigenvalue of $\mathcal{R}(\pi)$ for any such π . We refer to [13, 14] for a further discussion of these operators in this context.

THEOREM 1.7. $\mathcal{M}_{2p+6, f}$ is Ricci flat, nilpotent Osserman, and nilpotent Ivanov–Petrova.

Theorem 1.1 (1) fails in this setting. We refer to [16] for a further discussion of this phenomena and here content ourselves with showing:

THEOREM 1.8. Assume that the derivatives $f^{(p+3)}(y)$ and $f^{(p+4)}(y)$ are positive on the whole real line. Then:

- (1) $\mathcal{M}_{6, f}$ is a six-dimensional neutral signature manifold.
- (2) $\mathcal{M}_{6, f}$ is 2-curvature homogeneous and complete.
- (3) $\mathcal{M}_{6, f}$ is modeled on an irreducible neutral signature symmetric space.
- (4) All the local scalar Weyl invariants of $\mathcal{M}_{6, f}$ vanish identically.
- (5) $\mathcal{M}_{6, f}$ is not affine 3-curvature homogeneous for generic f .

There is a four-dimensional example $\mathcal{M}_{4, f} := (\mathbb{R}^4, g_{4, f})$ where the nonzero metric components are:

$$g_{4, f}(\partial_x, \partial_x) = -2f(y) \quad \text{and} \quad g_{4, f}(\partial_x, \partial_{\bar{x}}) = g_{4, f}(\partial_y, \partial_{\bar{y}}) = 1.$$

This example is defined, at least in a formal sense, by setting $p = -1$ in the discussion given earlier. Assume $f^{(2)} > 0$ and $f^{(3)} > 0$. Dunn [9] showed that $\mathcal{M}_{4, f}$ is a 1-curvature homogeneous complete manifold which is 0-modeled on an irreducible symmetric space and which is not locally homogeneous for generic f .

The remainder of this paper is devoted to the proving Theorems 1.3–1.8. In Section 2, we determine the Christoffel symbols of the connection ∇ relative to the coordinate frame and establish Assertions (1) and (2) of Theorem 1.3. In Section 3, we compute the curvature of the metric $g_{2p+6, f}$; Theorem 1.4 (1) and Theorem 1.7 follow from this computation. In Section 4, we prove Assertions (3) and (4) of Theorem 1.3. In Section 5, we complete the proof of Theorem 1.4; Corollary 1.5 and Theorem 1.6 follow as scholiums to these computations. We conclude the paper in Section 6 with the proof of Theorem 1.8.

2. The Geodesics of $\mathcal{M}_{2p+6,f}$

The non-zero covariant derivatives can be easily calculated as

$$\begin{aligned}\nabla_{\partial_x} \partial_y &= \nabla_{\partial_y} \partial_x = -(\partial_y F_{2p+6,f}) \partial_{\bar{x}}, \\ \nabla_{\partial_x} \partial_x &= (\partial_y F_{2p+6,f}) \partial_{\bar{y}} + \sum_i y^{i+1} \partial_{\bar{z}_i}, \\ \nabla_{\partial_x} \partial_{z_i} &= \nabla_{\partial_{z_i}} \partial_x = -y^{i+1} \partial_{\bar{x}}.\end{aligned}$$

This exhibits a crucial feature of this metric:

$$\nabla\{\partial_x, \partial_y, \partial_{z_i}\} \in \text{Span}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_i}\}, \text{ and } \nabla\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_i}\} = \{0\}. \quad (2a)$$

Assertions (1) and (2) of Theorem 1.3 will follow from the following technical Lemma by setting

$$\begin{aligned}u_1 &= x, & u_2 &= y, & u_3 &= z_0, \dots, & u_{p+3} &= z_p, \\ u_{p+4} &= \bar{x}, & u_{p+5} &= \bar{y}, & u_{p+6} &= \bar{z}_0, \dots, & u_{2p+6} &= \bar{z}_p.\end{aligned}$$

LEMMA 2.1. *Let (u_1, \dots, u_n) be coordinates on \mathbb{R}^n . Let g be a pseudo-Riemannian metric on \mathbb{R}^n so that $\nabla_{\partial_{u_a}} \partial_{u_b} = \sum_{\{c \text{ so } c > a, b\}} \Gamma_{ab}^c(u_1, \dots, u_{c-1}) \partial_{u_c}$. Then*

- (1) (\mathbb{R}^n, g) is a complete pseudo-Riemannian manifold.
- (2) $\exp_P : T_P \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism for all P in \mathbb{R}^n .

Proof. We shall adopt the notational convention that the empty sum is 0. Let $\gamma(t) = (u_1(t), \dots, u_n(t))$ be a curve in \mathbb{R}^n ; γ is a geodesic if and only if

$$\ddot{u}_c(t) + \sum_{\{a,b \text{ so } a,b < c\}} \dot{u}_a(t) \dot{u}_b(t) \Gamma_{ab}^c(u_1, \dots, u_{c-1})(t) = 0 \quad \text{for all } c.$$

We solve this system of equations recursively. Let $\gamma(t; \vec{u}^0, \vec{u}^1)$ be defined by

$$u_c(t) = u_c^0 + u_c^1 t - \int_0^t \int_0^s \sum_{\{a,b \text{ so } a,b < c\}} \dot{u}_a(r) \dot{u}_b(r) \Gamma_{ab}^c(u_1, \dots, u_{c-1})(r) dr ds.$$

Then $\gamma(0; \vec{u}^0, \vec{u}^1)(0) = \vec{u}^0$ while $\dot{\gamma}(0; \vec{u}^0, \vec{u}^1)(0) = \vec{u}^1$. Thus, every geodesic arises in this way so all geodesics extend for infinite time. Furthermore, given $P, Q \in \mathbb{R}^n$, there is a unique geodesic $\gamma = \gamma_{P,Q}$ so that $\gamma(0) = P$ and $\gamma(1) = Q$ where

$$\begin{aligned}u_c^0 &= P_c, & u_c^1 &= Q_c - P_c + \\ &+ \int_0^1 \int_0^s \sum_{\{a,b \text{ so } a,b < c\}} \dot{u}_a(r) \dot{u}_b(r) \Gamma_{ab}^c(u_1, \dots, u_{c-1})(r) dr ds.\end{aligned}$$

This shows that \exp_P is a diffeomorphism from $T_P \mathbb{R}^n$ to \mathbb{R}^n . \square

We note that it is rather rare in Riemannian geometry that the equations for geodesics can be solved in explicit form in global coordinates. The assumption that $\nabla_{\partial_{u_a}} \partial_{u_b} = \sum_{\{c \text{ so } c > a, b\}} \Gamma_{ab}^c(u_1, \dots, u_{c-1}) \partial_{u_c}$ makes permits the use of the recursive formalism we have in fact used. This Lemma is also used in [16] to study certain manifolds of signatures $(s, 2s)$.

3. The Curvature of $\mathcal{M}_{2p+6, f}$

In view of Equation (2a), in computing curvatures and higher covariant derivatives, only derivatives of highest weight play a role; we never need to consider quadratic terms in Christoffel symbols. Thus, the nonzero curvatures are

$$R_{2p+6, f}(\partial_x, \partial_y, \partial_y, \partial_x) = (\partial_y)^2 F_{2p+6, f}$$

and

$$R_{2p+6, f}(\partial_x, \partial_y, \partial_{z_i}, \partial_x) = (i+1)y^i.$$

Proof of Theorem 1.7. Let $J_{2p+6, f}$ (resp. $\mathcal{R}_{2p+6, f}$) be the Jacobi operator (resp. the curvature operator) defined by the pseudo-Riemannian metric $g_{2p+6, f}$. Let ξ_i be arbitrary tangent vectors. Then

$$\text{Range } \{\mathcal{R}_{2p+6, f}(\xi_1, \xi_2)\} \subset \text{Span}_{C^\infty} \{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_0}, \dots, \partial_{\bar{z}_p}\}$$

and

$$\text{Span}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_0}, \dots, \partial_{\bar{z}_p}\} \subset \text{Ker}\{\mathcal{R}_{2p+6, f}(\xi_1, \xi_2)\}.$$

Thus, $\mathcal{R}_{2p+6, f}(\xi_1, \xi_2)\mathcal{R}_{2p+6, f}(\xi_3, \xi_4) = 0$ so $J_{2p+6, f}(\xi)^2 = 0$ and $\mathcal{R}_{2p+6, f}(\pi)^2 = 0$ for any tangent vector ξ and any nondegenerate 2-plane π . Consequently, $J_{2p+6, f}(\xi)$ and $\mathcal{R}_{2p+6, f}(\pi)$ have only the eigenvalue 0. \square

Similarly, the non-zero entries in $\nabla^k R$ for any $k \geq 0$ are given by

$$\nabla^k R_{2p+6, f}(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots, \partial_y) = (\partial_y)^{k+2} F_{2p+6, f},$$

$$\nabla^k R_{2p+6, f}(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots, \partial_y) = \partial_{z_i} (\partial_y)^{k+1} F_{2p+6, f},$$

and

$$\nabla^k R_{2p+6, f}(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots, \partial_{z_i}, \dots, \partial_y) = \partial_{z_i} (\partial_y)^{k+1} F_{2p+6, f}.$$

Proof of Theorem 1.4 (1). We may decompose $T\mathbb{R}^{2p+6} = \mathcal{V} \oplus \bar{\mathcal{V}}$, where

$$\mathcal{V} := \text{Span}\left\{\partial_x + \frac{1}{2}g_{2p+6, f}(\partial_x, \partial_x)\partial_{\bar{x}}, \partial_y, \partial_{z_0}, \dots, \partial_{z_p}\right\},$$

and

$$\bar{\mathcal{V}} := \text{Span}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_0}, \dots, \partial_{\bar{z}_p}\}.$$

Let π_1 denote projection on the first factor. There are tensors $A^k \in \otimes^{k+4}\mathcal{V}^*$ so that $\pi_1^* A^k = \nabla^k R$. Since \mathcal{V} is a totally isotropic subspace, this shows all scalar invariants formed using the Weyl calculus vanish. \square

4. A Model for $\mathcal{M}_{2p+6,f}$

We can now make a crucial observation. We have

$$\nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots, \partial_y) = \begin{cases} 0 & \text{if } i < k, \\ (k+1)! & \text{if } i = k. \end{cases} \quad (4a)$$

Proof of Theorem 1.3 (3, 4). We shall exploit the upper triangular form of Equation (4a). Let $a^i(y, \bar{z})$ and $b_i^j(y, \bar{z})$ be smooth functions to be chosen presently. Set

$$X = \partial_x - \frac{1}{2} g_{2p+6,f}(\partial_x, \partial_x) \partial_{\bar{x}}, \quad Y = \partial_y + \sum_j a^j \partial_{z_j}, \quad \text{and} \quad Z_i = \sum_j b_i^j \partial_{z_j}.$$

Assume the matrix (b_i^j) is invertible; let (\hat{b}_i^j) be the inverse matrix. Set dually

$$\bar{X} = \partial_{\bar{x}}, \quad \bar{Y} = \partial_{\bar{y}}, \quad \text{and} \quad \bar{Z}_i = - \sum_j a^j \hat{b}_j^i \partial_{\bar{y}} + \sum_j \hat{b}_j^i \partial_{\bar{z}_j}.$$

This is then a hyperbolic basis, i.e. the first relation of Equation (1a) holds.

We shall assume the matrix b_i^j is triangular:

$$Z_i = \sum_{j \leq i} b_i^j \partial_{z_j}.$$

The relation $\nabla^k R(X, Y, Y, X; Y, \dots, Y) = 0$ for $0 \leq k \leq p$ leads to the equations:

$$\begin{aligned} 0 &= \nabla^p R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots) + (p+1) a^p R(\partial_x, \partial_y, \partial_{z_p}, \partial_x; \partial_y, \dots), \\ 0 &= \nabla^{p-1} R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots) + p \sum_{p-1 \leq i \leq p} a^i R(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots), \dots \\ 0 &= R(\partial_x, \partial_y, \partial_y, \partial_x) + \sum_{0 \leq i \leq p} a^i R(\partial_x, \partial_y, \partial_{z_i}, \partial_x). \end{aligned}$$

By Equation (4a), $\nabla^k R(\partial_x, \partial_y, \partial_{z_k}, \partial_x, \partial_y, \dots) \neq 0$ and thus this triangular system of equations determines the coefficients a^i uniquely.

Similarly, the relations $\nabla^k R(X, Y, Z_j, X; Y, \dots) = \delta_{jk}$ leads to the equations:

$$\begin{aligned} 1 &= b_p^p \nabla^p R(\partial_x, \partial_y, \partial_{z_p}, \partial_x; \partial_y, \dots), \\ 1 &= b_{p-1}^{p-1} \nabla^p R(\partial_x, \partial_y, \partial_{z_{p-1}}, \partial_x; \partial_y, \dots), \\ 0 &= \sum_{p-1 \leq i \leq p} b_p^i \nabla^{p-1} R(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots), \dots, \\ 1 &= b_0^0 R(\partial_x, \partial_y, \partial_{z_0}, \partial_x), \\ 0 &= \sum_{0 \leq i \leq 1} b_1^i \nabla^{p-1} R(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots), \\ 0 &= \sum_{0 \leq i \leq p} b_p^i R(\partial_x, \partial_y, \partial_{z_i}, \partial_x). \end{aligned}$$

This system of equations is triangular. First solve for b_p^p , then for $\{b_{p-1}^{p-1}, b_p^{p-1}\}$, and finally for $\{b_0^0, \dots, b_p^0\}$. Again, the fact that $\nabla^k R(\partial_x, \partial_y, \partial_{z_k}, \partial_y; \partial_y, \dots) \neq 0$ is crucial.

If $k > p$, then the only nonzero component of $\nabla^k R$ is given by

$$\nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y \dots \partial_y) = f^{(k+2)}(y).$$

There is still a bit of freedom left in the choice of basis. Let ε_0 and ε_1 be nonzero functions. We set

$$\begin{aligned} X^1 &= \varepsilon_0 X, & Y^1 &= \varepsilon_1 Y, & Z_0^1 &= \varepsilon_0^{-2} \varepsilon_1^{-1} Z_0, \dots, & Z_p^1 &= \varepsilon_0^{-2} \varepsilon_1^{-p-1} Z_p, \\ \bar{X}^1 &= \varepsilon_0^{-1} \bar{X}, & \bar{Y}^1 &= \varepsilon_1^{-1} \bar{Y}, & \bar{Z}_0^1 &= \varepsilon_0^2 \varepsilon_1 \bar{Z}_0, \dots, & \bar{Z}_p^1 &= \varepsilon_0^2 \varepsilon_1^{p+1} \bar{Z}_p. \end{aligned}$$

The normalizations of Equation (1a) are preserved for $\{g_{2p+6,f}, R, \dots, \nabla^p R\}$. Also,

$$\begin{aligned} \nabla^{p+1} R_{2p+6,f}(X^1, Y^1, Y^1, X^1; Y^1 \dots Y^1) &= \varepsilon_0^2 \varepsilon_1^{p+3} f^{(p+3)}, \\ \nabla^{p+2} R_{2p+6,f}(X^1, Y^1, Y^1, X^1; Y^1 \dots Y^1) &= \varepsilon_0^2 \varepsilon_1^{p+4} f^{(p+4)}. \end{aligned}$$

As $f^{(p+3)} > 0$ and $f^{(p+4)} > 0$, we may set

$$\varepsilon_1 := \frac{f^{(p+3)}}{f^{(p+4)}} \quad \text{and} \quad \varepsilon_0 := \frac{1}{\{\varepsilon_1^{p+3} f^{(p+3)}\}^{\frac{1}{2}}}.$$

This shows that \mathcal{U}_{2p+6}^{p+2} is a $p+2$ model for $\mathcal{M}_{2p+6,f}$. \square

Proof of Theorem 1.6 (1). Suppose we set $f(y) = e^y$, $\varepsilon_0 = e^{-y/2}$ and $\varepsilon_1 = 1$. Then $\nabla^i R_{2p+6,f}(X^1, Y^1, Y^1, X^1; Y^1 \dots Y^1) = 1$ for any i . Consequently, \mathcal{M}_{2p+6}^1 is a simply connected complete k -curvature homogeneous manifold for any k . Theorem 1.2 now implies \mathcal{M}_{2p+6}^1 is homogeneous. \square

Note that the full strength of Theorem 1.2 is not necessary. Results of Belger and Kowalski [3] show an analytic pseudo-Riemannian manifold which is k -curvature homogeneous for all k is locally homogeneous; in our setting the exponential coordinates are analytic diffeomorphisms so the qualifier ‘local’ can be removed.

5. A Local Invariant

Let $k \geq p+1$. Define a generalization of the classical Jacobi operator by setting

$$J_{k,2p+6,f}(Y) : X \rightarrow \nabla_{Y, \dots, Y}^k R_{2p+6,f}(X, Y)Y.$$

Expand $X = a\partial_x + b\partial_y$ and $Y = c\partial_x + d\partial_y$. Then

$$J_{k,2p+6,f}(Y)X = (ad - bc)d^k f^{(k+2)}(d\partial_{\bar{x}} - c\partial_{\bar{y}}).$$

Proof of Theorem 1.4 (2). Choose $\{Y, X\}$ so $J_{p+1,2p+6,f}(Y)X \neq 0$. Then necessarily $d \neq 0$ and $(ad - bc) \neq 0$. Let h be any Riemannian metric on $\mathcal{M}_{2p+6,f}$;

$$\begin{aligned} & \frac{h(J_{p+1,2p+6,f}(Y)X, J_{p+3,2p+6,f}(Y)X)}{h(J_{p+2,2p+6,f}(Y)X, J_{p+2,2p+6,f}(Y)X)} \\ &= \frac{(ad - bc)^2 d^{2p+4} f^{(p+3)} f^{(p+5)}}{(ad - bc)^2 d^{2p+4} f^{(p+4)} f^{(p+4)}} \frac{h(d\partial_{\bar{x}} - c\partial_{\bar{y}}, d\partial_{\bar{x}} - c\partial_{\bar{y}})}{h(d\partial_{\bar{x}} - c\partial_{\bar{y}}, d\partial_{\bar{x}} - c\partial_{\bar{y}})} \\ &= \alpha_{2p+6,f}. \end{aligned}$$

Thus, $\alpha_{2p+6,f}$ is an affine invariant of $\{\nabla^{p+1}\mathcal{R}, \nabla^{p+2}\mathcal{R}, \nabla^{p+3}\mathcal{R}\}$. \square

Proof of Theorem 1.6 (2, 3). If we set $f = e^y + e^{2y}$, then $\alpha_{2p+6,f}$ is not locally constant so \mathcal{M}_{2p+6}^2 is not locally $p + 3$ -affine curvature homogeneous. It is, however, $p + 2$ -curvature modeled on \mathcal{M}_{2p+6}^1 . \square

Proof of Corollary 1.5. Let $s = \min(\bar{p}, \bar{q})$. Assume $s \geq 3$. We show that $k_{\bar{p}, \bar{q}} \geq s$ by exhibiting a manifold \mathcal{M} of signature (\bar{p}, \bar{q}) which is $s - 1$ curvature homogeneous but not s affine curvature homogeneous (and hence not homogeneous).

Let $(\bar{p}, \bar{q}) = (s + a, s + b)$ where a, b are non-negative integers. Let $\mathbb{R}^{(a,b)}$ denote \mathbb{R}^{a+b} with a flat metric of signature (a, b) ; if $(a, b) = (0, 0)$, then \mathbb{R}^{a+b} is just a single point. Let $s = p + 3$ for $p \geq 0$. Let $\mathcal{M}_{s,f,a,b} := \mathbb{R}^{(a,b)} \times \mathcal{M}_{2p+6,f}$ where $f = e^y + e^{2y}$. The same arguments as those used earlier extend to show that $\mathcal{M}_{s,f,a,b}$ is $p + 2 = s - 1$ curvature homogeneous and is not affine $p + 3 = s$ curvature homogeneous. \square

We note that a similar argument can be used to establish a corresponding lower bound in the affine setting for the Opozda constant [25]. The case $s = 2$ can be treated using an appropriate signature $(2, 2)$ example as the base case; see Dunn et al. [10] for details.

6. Irreducibility

We restrict to the case $p = 0$. Set $f = 0$ to define $\mathcal{M}_{6,0}$. The discussion in Section 2 then yields that $\mathcal{M}_{6,0}$ is complete. The computations of Section 3 show $\nabla R_{g_{6,0}} = 0$ so $\mathcal{M}_{6,0}$ is a symmetric space. Furthermore, the discussion of Section 4 shows that \mathcal{U}_6^0 is a 0-model for $\mathcal{M}_{6,0}$. Thus, $\mathcal{M}_{6,0}$ is a 0-model for $\mathcal{M}_{6,f}$. We complete the proof of Theorem 1.8 by showing that \mathcal{U}_6^0 is irreducible as the other assertions then follow.

Let $Z = Z_0$ and $\bar{Z} = \bar{Z}_0$. Let $\mathbb{R}^3 = \text{Span}\{X, Y, Z\}$. We consider an affine model $\mathcal{V} = (\mathbb{R}^3, B)$ where $B \in \otimes^4(\mathbb{R}^3)^*$ is defined by

$$B(X, Y, Z, X) = 1.$$

LEMMA 6.1. *The affine model \mathcal{V} is irreducible.*

Proof. Suppose a nontrivial decomposition $\mathbb{R}^3 = V_1 \oplus V_2$ induces a corresponding decomposition $B = B_1 \oplus B_2$. Assume the notation chosen so $\dim(V_1) = 2$ and $\dim(V_2) = 1$. Let $0 \neq \xi \in V_2$. Since $\dim(V_2) = 1$, $B_2 = 0$ so $B(\eta_1, \eta_2, \eta_3, \xi) = 0$ for all $\eta_i \in \mathbb{R}^3$. We expand $\xi = aX + bY + cZ$. We then have

$$\begin{aligned} a &= B(\xi, Y, Z, X) = 0, & b &= B(X, \xi, Z, X) = 0, & \text{and} \\ c &= B(X, Y, \xi, X) = 0. \end{aligned}$$

Thus, $\xi = 0$ which is false. This contradiction proves the Lemma. \square

Let π be the natural projection from \mathbb{R}^6 to $W := \mathbb{R}^6/\mathcal{K}$ where

$$\mathcal{K} := \{\xi \in \mathbb{R}^6 : A_6^0(\eta_1, \eta_2, \eta_3, \xi) = 0 \forall \eta_i \in \mathbb{R}^3\} = \text{Span}\{\bar{X}, \bar{Y}, \bar{Z}\}.$$

We suppose \mathcal{U}_6^0 is reducible and argue for a contradiction. Let $\mathbb{R}^6 = V_1 \oplus V_2$ be a non-trivial decomposition with a corresponding decomposition

$$g_{6,0} = g_{6,0,1} \oplus g_{6,0,2} \quad \text{and} \quad A_6^0 = A_{6,1}^0 \oplus A_{6,2}^0. \quad (6a)$$

This also induces a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. We set $W_i := V_i/\mathcal{K}_i$ to decompose $W = W_1 \oplus W_2$ and $B = B_1 \oplus B_2$. By Lemma 6.1, this decomposition is trivial; we choose the notation so $W_2 = \{0\}$ and hence $V_2 \subset \mathcal{K}_2 \subset \mathcal{K}$. Since \mathcal{K} is a null subspace, $g_{6,0,2}$ is trivial. This is a contradiction as $g_{6,0} = g_{6,0,1} \oplus g_{6,0,2}$ and $g_{6,0}$ is non-singular. This contradiction completes the proof of Theorem 1.8. \square

Acknowledgements

It is a pleasure to acknowledge helpful suggestions from the referee. The research of P. Gilkey was partially supported by the Max Planck Institute in the Mathematical Sciences (Leipzig). The research of S. Nikčević was partially supported by MM 1646 (Srbija). We are grateful to Professors O. Kowalski and L. Vanhecke for introducing us to this area in the first instance and to Professor García-Río for suggesting this as a fruitful area of enquiry. We also acknowledge with pleasure helpful conversations with Dra. S. López Ornat.

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