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Characteristic Classes

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Vector Bundles

Let $\mathrm{Vect}_k(M, \mathbb{F})$ be the set of isomorphism classes of real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) vector bundles of rank k over a smooth connected m-dimensional manifold M. Let

$$\mathrm{Vect}(M, \mathbb{F}) = \bigcup_k \mathrm{Vect}_k(M, \mathbb{F})$$

Principal Bundles – Examples

Let H be a Lie group. A fiber bundle

$$\rho : \mathcal{P} \to M$$

with fiber H is said to be a principal bundle if there is a right action of H on $\mathcal{P}$ which acts transitively on the fibers, that is, if $\mathcal{P}/H = M$. If H is a closed subgroup of a Lie group G, then the natural projection $G \to G/H$ is a principal H bundle over the homogeneous space G/H. Let $\mathrm{O}(k)$ and $\mathrm{U}(k)$ denote the orthogonal and unitary groups, respectively. Let S^k denote the unit sphere in $\mathbb{R}^{k+1}$. Then we have natural principal bundles:

$$\mathrm{O}(k) \subset \mathrm{O}(k+1) \to S^k$$
$$\mathrm{U}(k) \subset \mathrm{U}(k+1) \to S^{2k+1}$$

Let $\mathbb{RP}^k$ and $\mathbb{CP}^k$ denote the real and complex projective spaces of lines through the origin in $\mathbb{R}^{k+1}$ and $\mathbb{C}^{k+1}$, respectively. Let

$$\mathbb{Z}_2 = \{\pm \mathrm{Id}\} \subset \mathrm{O}(k)$$
$$S^1 = \{\lambda \cdot \mathrm{Id} : |\lambda| = 1\} \subset \mathrm{U}(k)$$

One has $\mathbb{Z}_2$ and S^1 principal bundles:

$$\mathbb{Z}_2 \to S^{k-1} \to \mathbb{RP}^{k-1}$$
$$S^1 \to S^{2k-1} \to \mathbb{CP}^{k-1}$$

Frames

A frame $\boldsymbol{s} := (s_1, \ldots, s_k)$ for $V \in \mathrm{Vect}_k(M, \mathbb{F})$ over an open set $\mathcal{O} \subset M$ is a collection of k smooth sections to $V|_{\mathcal{O}}$ so that $\{s_1(P), \ldots, s_k(P)\}$ is a basis for the fiber V_P of V over any point $P \in \mathcal{O}$. Given such a frame $\boldsymbol{s}$, we can construct a local trivialization which identifies $\mathcal{O} \times \mathbb{F}^k$ with $V|_{\mathcal{O}}$ by the mapping

$$(P; \lambda_1, \ldots, \lambda_k) \to \lambda_1 s_1(P) + \cdots + \lambda_k s_k(P)$$

Conversely, given a local trivialization of V, we can take the coordinate frame

$$s_i(P) = P \times (0, \ldots, 0, 1, 0, \ldots, 0)$$

Thus, frames and local trivializations of V are equivalent notions.

Simple Covers

An open cover $\{\mathcal{O}_\alpha\}$ of M, where α ranges over some indexing set A, is said to be a simple cover if any finite intersection $\mathcal{O}_{\alpha_1} \cap \cdots \cap \mathcal{O}_{\alpha_k}$ is either empty or contractible.

Simple covers always exist. Put a Riemannian metric on M. If M is compact, then there exists a uniform $\delta > 0$ so that any geodesic ball of radius δ is geodesically convex. The intersection of geodesically convex sets is either geodesically convex (and hence contractible) or empty. Thus, covering M by a finite number of balls of radius δ yields a simple cover. The argument is similar even if M is not compact where an infinite number of geodesic balls is used and the radii are allowed to shrink near ∞.

Transition Cocycles

Let $\mathrm{Hom}(\mathbb{F}, k)$ be the set of linear transformations of $\mathbb{F}^k$ and let $\mathrm{GL}(\mathbb{F}, k) \subset \mathrm{Hom}(\mathbb{F}, k)$ be the group of all invertible linear transformations.

Let $\{\boldsymbol{s}_\alpha\}$ be frames for a vector bundle V over some open cover $\{\mathcal{O}_\alpha\}$ of M. On the intersection $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, one may express $\boldsymbol{s}_\alpha = \psi_{\alpha\beta} \boldsymbol{s}_\beta$, that is

$$s_{\alpha,i}(P) = \sum_{1 \le j \le k} \psi_{\alpha\beta,i}{}^{j}(P) s_{\beta,j}(P)$$

The maps $\psi_{\alpha\beta}: \mathcal{O}_\alpha \cap \mathcal{O}_\beta \to \mathrm{GL}(\mathbb{F}, k)$ satisfy

$$\begin{aligned} &\psi_{\alpha\alpha} = \mathrm{Id} \quad \text{on } \mathcal{O}_\alpha \\ &\psi_{\alpha\beta} = \psi_{\alpha\gamma}\psi_{\gamma\beta} \quad \text{on } \mathcal{O}_\alpha \cap \mathcal{O}_\beta \cap \mathcal{O}_\gamma \end{aligned} \qquad [1]$$

Let G be a Lie group. Maps belonging to a collection $\{\psi_{\alpha\beta}\}$ of smooth maps from $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ to G which satisfy eqn [1] are said to be transition cocycles with values in G; if $G \subset \mathrm{GL}(\mathbb{F}, k)$, they can be used to define a vector bundle by making appropriate identifications.

Reducing the Structure Group

If G is a subgroup of $\mathrm{GL}(\mathbb{F}, k)$, then V is said to have a G-structure if we can choose frames so the transition cocycles belong to G; that is, we can reduce the structure group to G.

Denote the subgroup of orientation-preserving linear maps by

$$\mathrm{GL}^+(\mathbb{R}, k) := \{\psi \in \mathrm{GL}(\mathbb{R}, k): \det(\psi) > 0\}$$

If $V \in \mathrm{Vect}_k(M, \mathbb{R})$, then V is said to be orientable if we can choose the frames so that

$$\psi_{\alpha\beta} \in \mathrm{GL}^+(\mathbb{R}, k)$$

Not every real vector bundle is orientable; the first Stiefel–Whitney class $sw_1(V) \in H^1(M; \mathbb{Z}_2)$, which is defined later, vanishes if and only if V is orientable. In particular, the Möbius line bundle over the circle is not orientable.

Similarly, a real (resp. complex) bundle V is said to be Riemannian (resp. Hermitian) if we can reduce the structure group to the orthogonal group $\mathrm{O}(k) \subset \mathrm{GL}(\mathbb{R}, k)$ (resp. to the unitary group $\mathrm{U}(k) \subset \mathrm{GL}(\mathbb{C}, k)$).

We can use a partition of unity to put a positive-definite symmetric (resp. Hermitian symmetric) fiber metric on V. Applying the Gram–Schmidt process then constructs orthonormal frames and shows that the structure group can always be reduced to $\mathrm{O}(k)$ (resp. to $\mathrm{U}(k)$); if V is a real vector bundle, then the structure group can be reduced to the special orthogonal group $\mathrm{SO}(k)$ if and only if V is orientable.

Lifting the Structure Group

Let τ be a representation of a Lie group H to $\mathrm{GL}(\mathbb{F}, k)$. One says that the structure group of V can be lifted to H if there exist frames $\{s_\alpha\}$ for V and smooth maps $\phi_{\alpha\beta}: \mathcal{O}_\alpha \cap \mathcal{O}_\beta \to H$, so $\tau\phi_{\alpha\beta} = \psi_{\alpha\beta}$ where eqn [1] holds for ϕ.

Spin Structures

For $k \geq 3$, the fundamental group of $\mathrm{SO}(k)$ is $\mathbb{Z}_2$. Let $\mathrm{Spin}(k)$ be the universal cover of $\mathrm{SO}(k)$ and let

$$\tau : \mathrm{Spin}(k) \to \mathrm{SO}(k)$$

be the associated double cover; set $\mathrm{Spin}(2) = S^1$ and let $\tau(\lambda) = \lambda^2$. An oriented bundle V is said to be spin if the transition functions can be lifted from $\mathrm{SO}(k)$ to $\mathrm{Spin}(k)$; this is possible if and only if the second Stiefel–Whitney class of V, which is defined later, vanishes. There can be inequivalent spin structures, which are parametrized by the cohomology group $H^1(M; \mathbb{Z}_2)$.

The Tangent Bundle of Projective Space

The tangent bundle $T\mathbb{RP}^m$ of real projective space is orientable if and only if m is odd; $T\mathbb{RP}^m$ is spin if and only if $m \equiv 3 \bmod 4$. If $m \equiv 3 \bmod 4$, there are two inequivalent spin structures on this bundle as $H^1(\mathbb{RP}^m; \mathbb{Z}_2) = \mathbb{Z}_2$.

The tangent bundle $T\mathbb{CP}^m$ of complex projective space is always orientable; $T\mathbb{CP}^m$ is spin if and only if m is odd.

Principal and Associated Bundles

Let H be a Lie group and let

$$\phi_{\alpha\beta} : \mathcal{O}_\alpha \cap \mathcal{O}_\beta \to H$$

be a collection of smooth functions satisfying the compatibility conditions given in eqn [1]. We define a principal bundle $\mathcal{P}$ by gluing $\mathcal{O}_\alpha \times H$ to $\mathcal{O}_\beta \times H$ using ϕ:

$$(P, h)_\alpha \sim (P, \phi_{\alpha\beta}(P)h)_\beta \quad \text{for } P \in \mathcal{O}_\alpha \cap \mathcal{O}_\beta$$

Because right multiplication and left multiplication commute, right multiplication gives a natural action of H on $\mathcal{P}$:

$$(P, h)_\alpha \cdot \tilde{h} := (P, h \cdot \tilde{h})_\alpha$$

The natural projection $\mathcal{P} \to \mathcal{P}/H = M$ is an H fiber bundle.

Let τ be a representation of H to $\mathrm{GL}(\mathbb{F}, k)$. For $\xi \in \mathcal{P}, \lambda \in \mathbb{F}^k$, and $h \in H$, define a gluing

$$(\xi, \lambda) \sim (\xi \cdot h^{-1}, \tau(h)\lambda)$$

The associated vector bundle is then given by

$$\mathcal{P} \times_\tau \mathbb{F}^k := \mathcal{P} \times \mathbb{F}^k / \sim$$

Clearly, $\{\tau\phi_{\alpha\beta}\}$ are the transition cocycles of the vector bundle $\mathcal{P} \times_\tau \mathbb{F}^k$.

Frame Bundles

If V is a vector bundle, the associated principal $\mathrm{GL}(\mathbb{F}, k)$ bundle is the bundle of all frames; if V is given an inner product on each fiber, then the associated principal $\mathrm{O}(k)$ or $\mathrm{U}(k)$ bundle is the bundle of orthonormal frames. If V is an oriented Riemannian vector bundle, the associated principal $\mathrm{SO}(k)$ bundle is the bundle of oriented orthonormal frames.

Direct Sum and Tensor Product

Fiber-wise direct sum (resp. tensor product) defines the direct sum (resp. tensor product) of vector bundles:

$$\begin{aligned}\oplus : &\mathrm{Vect}_k(M, \mathbb{F}) \times \mathrm{Vect}_n(M, \mathbb{F})\\ &\to \mathrm{Vect}_{k+n}(M, \mathbb{F})\\ \otimes : &\mathrm{Vect}_k(M, \mathbb{F}) \times \mathrm{Vect}_n(M, \mathbb{F})\\ &\to \mathrm{Vect}_{kn}(M, \mathbb{F})\end{aligned}$$

The transition cocycles of the direct sum (resp. tensor product) of two vector bundles are the direct sum (resp. tensor product) of the transition cocycles of the respective bundles.

The set of line bundles $\mathrm{Vect}_1(M, \mathbb{F})$ is a group under $\otimes$. The unit in the group is the trivial line bundle $\mathbb{1} := M \times \mathbb{F}$; the inverse of a line bundle L is the dual line bundle $L^* := \mathrm{Hom}(L, \mathbb{F})$ since

$$L \otimes L^* = \mathbb{1}$$

Pullback Bundle

Let $\rho : V \to M$ be the projection associated with $V \in \mathrm{Vect}_k(M, \mathbb{F})$. If f is a smooth map from N to M, then the pullback bundle f^*V is the vector bundle over N which is defined by setting

$$f^*V := \{(P, v) \in N \times V : f(P) = \rho(v)\}$$

The fiber of f^*V over P is the fiber of V over $f(P)$.

Let $\{s_\alpha\}$ be local frames for V over an open cover $\{\mathcal{O}_\alpha\}$ of M. For $P \in f^{-1}(\mathcal{O}_\alpha)$, define

$$\{f^*s_\alpha\}(P) := (P, s_\alpha(f(P)))$$

This gives a collection of frames for f^*V over the open cover $\{f^{-1}(\mathcal{O}_\alpha)\}$ of N. Let

$$f^*\psi_{\alpha\beta} := \psi_{\alpha\beta} \circ f$$

be the pullback of the transition functions. Then

$$\begin{aligned}\{f^*s_\alpha\}(P) &= (P, \psi_{\alpha\beta}(f(P))s_\beta(f(P)))\\ &= \{(f^*\psi_{\alpha\beta})(f^*s_\beta)\}(P)\end{aligned}$$

This shows that the pullback of the transition functions for V are the transition functions of the pullback $f^*(V)$.

Homotopy

Two smooth maps f_0 and f_1 from N to M are said to be homotopic if there exists a smooth map $F : N \times I \to M$ so that $f_0(P) = F(P, 0)$ and so that $f_1(P) = F(P, 1)$. If f_0 and f_1 are homotopic maps from N to M, then f_1^*V is isomorphic to f_2^*V.

Let $[N, M]$ be the set of all homotopy classes of smooth maps from N to M. The association $V \to f^*V$ induces a natural map

$$[N, M] \times \mathrm{Vect}_k(M, \mathbb{F}) \to \mathrm{Vect}_k(N, \mathbb{F})$$

If M is contractible, then the identity map is homotopic to the constant map c. Consequently, $V = \mathrm{Id}^*V$ is isomorphic to $c^*V = M \times \mathbb{F}^k$. Thus, any vector bundle over a contractible manifold is trivial. In particular, if $\{\mathcal{O}_\alpha\}$ is a simple cover of M and if $V \in \mathrm{Vect}(M, \mathbb{F})$, then $V|_{\mathcal{O}_\alpha}$ is trivial for each α. This shows that a simple cover is a trivializing cover for every $V \in \mathrm{Vect}(M, \mathbb{F})$.

Stabilization

Let $\mathbb{1} \in \mathrm{Vect}_1(M, \mathbb{F})$ denote the isomorphism class of the trivial line bundle $M \times \mathbb{F}$ over an m-dimensional manifold M. The map $V \to V \oplus \mathbb{1}$ induces a stabilization map

$$s : \mathrm{Vect}_k(M, \mathbb{F}) \to \mathrm{Vect}_{k+1}(M, \mathbb{F})$$

which induces an isomorphism

$$\begin{aligned}&\mathrm{Vect}_k(M, \mathbb{R}) = \mathrm{Vect}_{k+1}(M, \mathbb{R}) \quad \text{for } k > m\\ &\mathrm{Vect}_k(M, \mathbb{C}) = \mathrm{Vect}_{k+1}(M, \mathbb{C}) \quad \text{for } 2k > m\end{aligned} \qquad [2]$$

These values of k comprise the stable range.

The *K*-Theory

The direct sum $\oplus$ and tensor product $\otimes$ make $\mathrm{Vect}(M, \mathbb{F})$ into a semiring; we denote the associated ring defined by the Grothendieck construction by $\mathrm{K}\mathbb{F}(M)$. If $V \in \mathrm{Vect}(M, \mathbb{F})$, let $[V] \in \mathrm{K}\mathbb{F}(M)$ be the corresponding element of K-theory; $\mathrm{K}\mathbb{F}(M)$ is generated by formal differences $[V_1] - [V_2]$; such formal differences are called virtual bundles.

The Grothendieck construction (*see* K-theory) introduces nontrivial relations. Let S^m denote the standard sphere in $\mathbb{R}^{m+1}$. Since

$$T(S^m) \oplus \mathbb{1} = (m + 1)\mathbb{1}$$

we can easily see that $[TS^m] = m[\mathbb{1}]$ in $\mathrm{K}\mathbb{R}(S^m)$, despite the fact that $T(S^m)$ is not isomorphic to $m\mathbb{1}$ for $m \neq 1, 3, 7$.

Let L denote the nontrivial real line bundle over $\mathbb{RP}^k$. Then $T\mathbb{RP}^k \oplus \mathbb{1} = (k + 1)L$, so

$$[T\mathbb{RP}^k] = (k + 1)[L] - [\mathbb{1}]$$

The map $V \to \mathrm{Rank}(V)$ extends to a surjective map from $\mathrm{K}\mathbb{F}(M)$ to $\mathbb{Z}$. We denote the associated ideal of virtual bundles of virtual rank 0 by

$$\widetilde{\mathrm{K}\mathbb{F}}(M) := \ker(\mathrm{Rank})$$

In the stable range, $V \to [V] - k[\mathbb{1}]$ identifies

$$\begin{aligned} \mathrm{Vect}_k(M, \mathbb{R}) &= \widetilde{\mathrm{K}\mathbb{R}}(M) \quad \text{if } k > m \\ \mathrm{Vect}_k(M, \mathbb{C}) &= \widetilde{\mathrm{K}\mathbb{C}}(M) \quad \text{if } 2k > m \end{aligned} \qquad [3]$$

These groups contain nontrivial torsion. Let L be the nontrivial real line bundle over $\mathbb{RP}^k$. Then

$$\widetilde{\mathrm{K}\mathbb{R}}(\mathbb{RP}^k) = \mathbb{Z} \cdot \{[L] - [\mathbb{1}]\}/2^{\nu(k)}\mathbb{Z}\{[L] - [\mathbb{1}]\}$$

where $\nu(k)$ is the Adams number.

Classifying Spaces

Let $\mathrm{Gr}_k(\mathbb{F}, n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{F}^n$. By mapping a k-plane π in $\mathbb{F}^n$ to the corresponding orthogonal projection on π, we can identify $\mathrm{Gr}_k(\mathbb{F}, n)$ with the set of orthogonal projections of rank k:

$$\{\xi \in \mathrm{Hom}(\mathbb{F}^n) : \xi^2 = \xi,\ \xi^* = \xi,\ \mathrm{tr}(\xi) = k\}$$

There is a natural associated tautological k-plane bundle

$$V_k(\mathbb{F}, n) \in \mathrm{Vect}_k(\mathrm{Gr}_k(\mathbb{F}, n), \mathbb{F})$$

whose fiber over a k-plane π is the k-plane itself:

$$V_k(\mathbb{F}, n) := \{(\xi, x) \in \mathrm{Hom}(\mathbb{F}^n) \times \mathbb{F}^n : \xi x = x\}$$

Let $[M, \mathrm{Gr}_k(\mathbb{F}, n)]$ denote the set of homotopy equivalence classes of smooth maps f from M to $\mathrm{Gr}_k(\mathbb{F}, n)$. Since $[f_1] = [f_2]$ implies that $f_1^* V$ is isomorphic to $f_2^* V$, the association

$$f \to f^* V_k(\mathbb{F}, n) \in \mathrm{Vect}_k(M, \mathbb{F})$$

induces a map

$$[M, \mathrm{Gr}_k(\mathbb{F}, n)] \to \mathrm{Vect}_k(M, \mathbb{F})$$

This map defines a natural equivalence of functors in the stable range:

$$\begin{aligned} [M, \mathrm{Gr}_k(\mathbb{R}, \nu + k)] &= \mathrm{Vect}_k(M, \mathbb{R}) \quad \text{for } \nu > m \\ [M, \mathrm{Gr}_k(\mathbb{C}, \nu + k)] &= \mathrm{Vect}_k(M, \mathbb{C}) \quad \text{for } 2\nu > m \end{aligned} \qquad [4]$$

The natural inclusion of $\mathbb{F}^n$ in $\mathbb{F}^{n+1}$ induces natural inclusions

$$\begin{aligned} \mathrm{Gr}_k(\mathbb{F}, n) &\subset \mathrm{Gr}_k(\mathbb{F}, n+1) \\ V_k(\mathbb{F}, n) &\subset V_k(\mathbb{F}, n+1) \end{aligned} \qquad [5]$$

Let $\mathrm{Gr}_k(\mathbb{F}, \infty)$ and $V_k(\mathbb{F}, \infty)$ be the direct limit spaces under these inclusions; these are the infinite-dimensional Grassmannians and classifying bundles, respectively. The topology on these spaces is the weak or inductive topology. The Grassmannians are called classifying spaces. The isomorphisms of eqn [4] are compatible with the inclusions of eqn [5] and we have

$$[M, \mathrm{Gr}_k(\mathbb{F}, \infty)] = \mathrm{Vect}_k(M, \mathbb{F}) \qquad [6]$$

Spaces with Finite Covering Dimension

A metric space X is said to have a covering dimension at most m if, given any open cover $\{\mathcal{U}_\alpha\}$ of X, there exists a refinement $\{\mathcal{O}_\beta\}$ of the cover so that any intersection of more than $m + 1$ of the $\{\mathcal{O}_\beta\}$ is empty. For example, any manifold of dimension m has covering dimension at most m. More generally, any m-dimensional cell complex has covering dimension at most m.

The isomorphisms of [2]–[4], and [6] continue to hold under the weaker assumption that M is a metric space with covering dimension at most m.

Characteristic Classes of Vector Bundles

The Cohomology of $\mathrm{Gr}_k(\mathbb{F}, \infty)$

The cohomology algebras of the Grassmannians are polynomial algebras on suitably chosen generators:

$$\begin{aligned} H^*(\mathrm{Gr}_k(\mathbb{R}, \infty); \mathbb{Z}_2) &= \mathbb{Z}_2[\mathrm{sw}_1, \ldots, \mathrm{sw}_k] \\ H^*(\mathrm{Gr}_k(\mathbb{C}, \infty); \mathbb{Z}) &= \mathbb{Z}[c_1, \ldots, c_k] \end{aligned} \qquad [7]$$

The Stiefel–Whitney Classes

Let $V \in \mathrm{Vect}_k(M, \mathbb{R})$. We use eqn [6] to find $\Psi : M \to \mathrm{Gr}_k(\mathbb{R}, \infty)$ which classifies V; the map Ψ is uniquely determined up to homotopy and, using eqn [7], one sets

$$\mathrm{sw}_i(V) := \Psi^* \mathrm{sw}_i \in H^i(M; \mathbb{Z}_2)$$

The total Stiefel–Whitney class is then defined by

$$\mathrm{sw}(V) = 1 + \mathrm{sw}_1(V) + \cdots + \mathrm{sw}_k(V)$$

The Stiefel–Whitney class has the properties:

1. If $f : X_1 \to X_2$, then $f^*(\mathrm{sw}(V)) = \mathrm{sw}(f^* V)$.
2. $\mathrm{sw}(V \oplus W) = \mathrm{sw}(V)\mathrm{sw}(W)$.
3. If L is the Möbius bundle over S^1, then $sw_1(L)$ generates $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$.

The cohomology algebra of real projective space is a truncated polynomial algebra:

$$H^*(\mathbb{RP}^k; \mathbb{Z}_2) = \mathbb{Z}_2[x]/x^{k+1} = 0$$

Since $T\mathbb{RP}^k \oplus 1 = (k+1)L$, one has

$$\begin{aligned}\mathrm{sw}(T\mathbb{RP}^k) &= (1+x)^{k+1} \\ &= 1 + kx + \frac{(k+1)k}{2}x^2 + \cdots \end{aligned} \qquad [8]$$

Orientability and Spin Structures

The Stiefel–Whitney classes have real geometric meaning. For example, $\mathrm{sw}_1(V) = 0$ if and only if V is orientable; if $\mathrm{sw}_1(V) = 0$, then $\mathrm{sw}_2(V) = 0$ if and only if V admits a spin structure. With reference to the discussion on the tangent bundle or projective space, eqn [8] yields

$$\mathrm{sw}_1(T\mathbb{RP}^k) = \begin{cases} 0 & \text{if } k \equiv 0 \bmod 2 \\ x & \text{if } k \equiv 1 \bmod 2 \end{cases}$$

Thus, $\mathbb{RP}^k$ is orientable if and only if k is odd. Furthermore,

$$\mathrm{sw}_2(T\mathbb{RP}^k) = \begin{cases} 0 & \text{if } k \equiv 3 \bmod 4 \\ x & \text{if } k \equiv 1 \bmod 4 \end{cases}$$

Thus, $T\mathbb{RP}^k$ is spin if and only if $k \equiv 3 \bmod 4$.

Chern Classes

Let $V \in \mathrm{Vect}_k(M, \mathbb{C})$. We use eqn [6] to find $\Psi : M \to \mathrm{Gr}_k(\mathbb{C}, \infty)$ which classifies V; the map Ψ is uniquely determined up to homotopy and, using eqn [7], one sets

$$c_i(V) := \Psi^* c_i \in H^{2i}(M; \mathbb{Z})$$

The total Chern class is then defined by

$$c(V) := 1 + c_1(V) + \cdots + c_k(V)$$

The Chern class has the properties:

1. If $f : X_1 \to X_2$, then $f^*(c(V)) = c(f^*V)$.
2. $c(V \oplus W) = c(V)c(W)$.
3. Let L be the classifying line bundle over $S^2 = \mathbb{CP}^1$. Then $\int_{S^2} c_1(L) = -1$.

The cohomology algebra of complex projective space also is a truncated polynomial algebra

$$H^*(\mathbb{CP}^k; \mathbb{Z}) = \mathbb{Z}[x]/x^{k+1}$$

where $x = c_1(L)$ and L is the complex classifying line bundle over $\mathbb{CP}^k = \mathrm{Gr}_1(\mathbb{C}, k+1)$. If $T_c\mathbb{CP}^k$ is the complex tangent bundle, then

$$c(T_c\mathbb{CP}^k) = (1+x)^{k+1}$$

The Pontrjagin Classes

Let V be a real vector bundle over a topological space X of rank $r = 2k$ or $r = 2k+1$. The Pontrjagin classes $p_i(V) \in H^{4i}(X; \mathbb{Z})$ are characterized by the properties:

1. $p(V) = 1 + p_1(V) + \cdots + p_k(V)$.
2. If $f : X_1 \to X_2$, then $f^*(p(V)) = p(f^*V)$.
3. $p(V \oplus W) = p(V)p(W)$ mod elements of order 2.
4. $\int_{\mathbb{CP}^2} p_1(T\mathbb{CP}^2) = 3$.

We can complexify a real vector bundle V to construct an associated complex vector bundle $V_{\mathbb{C}}$. We have

$$p_i(V) := (-1)^i c_{2i}(V_{\mathbb{C}})$$

Conversely, if V is a complex vector bundle, we can construct an underlying real vector bundle $V_{\mathbb{R}}$ by forgetting the underlying complex structure. Modulo elements of order 2, we have

$$p(V_{\mathbb{R}}) = c(V)c(V^*)$$

Let $T\mathbb{CP}^k$ be the real tangent bundle of complex projective space. Then

$$p(T\mathbb{CP}^k) = (1 - x^2)^{k+1}$$

Line Bundles

Tensor product makes $\mathrm{Vect}_1(M, \mathbb{F})$ into an abelian group. One has natural equivalences of functors which are group homomorphisms:

$$\begin{aligned}\mathrm{sw}_1 &: \mathrm{Vect}_1(M, \mathbb{R}) \to H^1(M; \mathbb{Z}_2) \\ c_1 &: \mathrm{Vect}_1(M, \mathbb{C}) \to H^2(M; \mathbb{Z})\end{aligned}$$

A real line bundle L is trivial if and only if it is orientable or, equivalently, if $\mathrm{sw}_1(L)$ vanishes. A complex line bundle L is trivial if and only if $c_1(L) = 0$. There are nontrivial vector bundles with vanishing Stiefel–Whitney classes of rank $k > 1$. For example, $\mathrm{sw}_i(TS^k) = 0$ for $i > 0$ despite the fact that TS^k is trivial if and only if $k = 1, 3, 7$.

Curvature and Characteristic Classes

de Rham Cohomology

We can replace the coefficient group $\mathbb{Z}$ by $\mathbb{C}$ at the cost of losing information concerning torsion. Thus, we may regard $p_i(V) \in H^{4i}(M; \mathbb{C})$ if V is real or $c_i(V) \in H^{2i}(M; \mathbb{C})$ if V is complex. Let M be a smooth manifold. Let $C^\infty \Lambda^p M$ be the space of smooth p-forms and let

$$d : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M$$

be the exterior derivative. The de Rham cohomology groups are then defined by

$$H^p_{\mathrm{deR}}(M) := \frac{\ker(d : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M)}{\mathrm{im}(d : C^\infty \Lambda^{p-1} M \to C^\infty \Lambda^p M)}$$

The de Rham theorem identifies the topological cohomology groups $H^p(M;\mathbb{C})$ with the de Rham cohomology groups $H^p_{\mathrm{deR}}(M)$ which are given differential geometrically.

Given a connection on V, the Chern–Weyl theory enables us to compute Pontrjagin and Chern classes in de Rham cohomology in terms of curvature.

Connections

Let V be a vector bundle over M. A connection

$$\nabla : C^\infty(V) \to C^\infty(T^*M \otimes V)$$

on V is a first-order partial differential operator which satisfies the Leibnitz rule, that is, if s is a smooth section to V and if f is a smooth function on M,

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

If X is a tangent vector field, we define

$$\nabla_X s = \langle X, \nabla s\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between the tangent and cotangent spaces. This generalizes to the bundle setting the notion of a directional derivative and has the properties:

1. $\nabla_{fX} s = f\nabla_X s$.
2. $\nabla_X(fs) = X(f)s + f\nabla_X s$.
3. $\nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$.
4. $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$.

The Curvature 2-Form

Let ω_p be a smooth p-form. Then

$$\nabla : C^\infty(\Lambda^p M \otimes V) \to C^\infty(\Lambda^{p+1} M \otimes V)$$

can be extended by defining

$$\nabla(\omega_p \otimes s) = \mathrm{d}\omega_p \otimes s + (-1)^p \omega_p \wedge \nabla s$$

In contrast to ordinary exterior differentiation, ∇^2 need not vanish. We set

$$\Omega(s) := \nabla^2 s$$

This is not a second-order partial differential operator; it is a zeroth-order operator, that is,

$$\begin{aligned}\Omega(fs) &= \mathrm{dd}f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f\nabla^2 s \\ &= f\Omega(s)\end{aligned}$$

The curvature operator Ω can also be computed locally. Let (s_i) be a local frame. Expand

$$\nabla s_i = \sum_j \omega_i^j \otimes s_j$$

to define the connection 1-form ω. One then has

$$\nabla^2 s_i = \left(\mathrm{d}\omega_i^j - \omega_i^k \wedge \omega_k^j\right) \otimes s_k$$

and so

$$\Omega_i^j = \mathrm{d}\omega_i^j - \omega_i^k \wedge \omega_k^j$$

If $\tilde{s} = \psi_i^{js} j$ is another local frame, we compute

$$\tilde{\omega} = \mathrm{d}gg^{-1} + g\omega g^{-1} \quad \text{and} \quad \tilde{\Omega} = g\Omega g^{-1}$$

Although the connection 1-form ω is not tensorial, the curvature is an invariantly defined 2-form-valued endomorphism of V.

Unitary Connections

Let $(\cdot\,,\cdot)$ be a nondegenerate Hermitian inner product on V. We say that ∇ is a unitary connection if

$$(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2)$$

Such connections always exist and, relative to a local orthonormal frame, the curvature is skew-symmetric, that is,

$$\Omega + \Omega^* = 0$$

Thus, Ω can be regarded as a 2-form-valued element of the Lie algebra of the structure group, $\mathrm{O}(V)$ in the real setting or $\mathrm{U}(V)$ in the complex setting.

Projections

We can always embed V in a trivial bundle 1^ν of dimension ν; let π_V be the orthogonal projection on V. We project the flat connection to V to define a natural connection on V. For example, if M is embedded isometrically in the Euclidean space $\mathbb{R}^\nu$, this construction gives the Levi-Civita connection on the tangent bundle TM. The curvature of this connection is then given by

$$\Omega = \pi_V \,\mathrm{d}\pi_V \,\mathrm{d}\pi_V$$

Let V_P be the fiber of V over a point $P \in M$. The inclusion $i : V \subset \mathbb{R}^n$ defines the classifying map $f : P \to Gr_k(\mathbb{R}, n)$ where we set

$$f(P) = i(V_P)$$

Chern–Weyl Theory

Let ∇ be a Riemannian connection on a real vector bundle V of rank k. We set

$$p(\Omega) := \det\left(I + \frac{1}{2\pi}\Omega\right)$$

Let Ω^{T} denote the transpose matrix of differential form. Since $\Omega + \Omega^{\mathrm{T}} = 0$, the polynomials of odd degree in Ω vanish and we may expand

$$p(\Omega) = 1 + p_1(\Omega) + \cdots + p_r(\Omega)$$

where $k = 2r$ or $k = 2r + 1$ and the differential forms $p_i(\Omega) \in C^\infty \Lambda^{4i}(M)$ are forms of degree $4i$.

Changing the gauge (i.e., the local frame) replaces Ω by $g\Omega g^{-1}$ and hence $p(\Omega)$ is independent of the local frame chosen. One can show that $\mathrm{d}p_i(\Omega) = 0$; let $[p_i(\Omega)]$ denote the corresponding element of de Rham cohomology. This is independent of the particular connection chosen and $[p_i(\Omega)]$ represents $p_i(V)$ in $H^{4i}(M; \mathbb{C})$.

Similarly, let V be a complex vector bundle of rank k with a Hermitian connection ∇. Set

$$\begin{aligned} c(\Omega) &:= \det\left(I + \frac{\sqrt{-1}}{2\pi}\Omega\right) \\ &= 1 + c_1(\Omega) + \cdots + c_k(\Omega) \end{aligned}$$

Again $c_i(\Omega)$ is independent of the local gauge and $\mathrm{d}c_i(\Omega) = 0$. The de Rham cohomology class $[c_i(\Omega)]$ represents $c_i(V)$ in $H^{2i}(M; \mathbb{C})$.

The Chern Character

The total Chern character is defined by the formal sum

$$\begin{aligned} \mathrm{ch}(\Omega) &:= \mathrm{tr}(\mathrm{e}^{\sqrt{-1}\Omega/2\pi}) \\ &= \sum_\nu \frac{(\sqrt{-1})^\nu}{(2\pi)^\nu \nu!} \mathrm{tr}(\Omega^\nu) \\ &= \mathrm{ch}_0(\Omega) + \mathrm{ch}_1(\Omega) + \cdots \end{aligned}$$

Let $\mathrm{ch}(V) = [\mathrm{ch}(\Omega)]$ denote the associated de Rham cohomology class; it is independent of the particular connection chosen. We then have the relations

$$\begin{aligned} \mathrm{ch}(V \oplus W) &= \mathrm{ch}(V) + \mathrm{ch}(W) \\ \mathrm{ch}(V \otimes W) &= \mathrm{ch}(V)\mathrm{ch}(W) \end{aligned}$$

The Chern character extends to a ring isomorphism from $KU(M) \otimes \mathbb{Q}$ to $H^e(M; \mathbb{Q})$, which is a natural equivalence of functors; modulo torsion, K theory and cohomology are the same functors.

Other Characteristic Classes

The Chern character is defined by the exponential function. There are other characteristic classes which appear in the index theorem that are defined using other generating functions that appear in index theory. Let $\boldsymbol{x} := (x_1, \dots)$ be a collection of indeterminates. Let $s_\nu(\boldsymbol{x})$ be the νth elementary symmetric function;

$$\prod_\nu (1 + x_\nu) = 1 + s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + \cdots$$

For a diagonal matrix $A := \mathrm{diag}(\lambda_1, \dots)$, denote the normalized eigenvalues by $x_j := \sqrt{-1}\lambda_j/2\pi$. Then

$$c(A) = \det\left(1 + \frac{\sqrt{-1}}{2\pi}A\right) = 1 + s_1(\boldsymbol{x}) + \cdots$$

Thus, the Chern class corresponds in a certain sense to the elementary symmetric functions.

Let $f(\boldsymbol{x})$ be a symmetric polynomial or more generally a formal power series which is symmetric. We can express $f(\boldsymbol{x}) = F(s_1(\boldsymbol{x}), \dots)$ in terms of the elementary symmetric functions and define $f(\Omega) = F(c_1(\Omega), \dots)$ by substitution. For example, the Chern character is defined by the generating function

$$f(\boldsymbol{x}) := \sum_{\nu=1}^{k} \mathrm{e}^{x_\nu}$$

The Todd class is defined using a different generating function:

$$\begin{aligned} \mathrm{td}(\boldsymbol{x}) &:= \prod_\nu x_\nu (1 - \mathrm{e}^{-x_\nu})^{-1} \\ &= 1 + \mathrm{td}_1(\boldsymbol{x}) + \cdots \end{aligned}$$

If V is a real vector bundle, we can define some additional characteristic classes similarly. Let $\{\pm\sqrt{-1}\lambda_1, \dots\}$ be the nonzero eigenvalues of a skew-symmetric matrix A. We set $x_j = -\lambda_j/2\pi$ and define the Hirzebruch polynomial L and the $\hat{A}$ genus by

$$\begin{aligned} L(\boldsymbol{x}) &:= \prod_\nu \frac{x_\nu}{\tanh(x_\nu)} \\ &= 1 + L_1(\boldsymbol{x}) + L_2(\boldsymbol{x}) + \cdots \\ \hat{A}(\boldsymbol{x}) &:= \prod_\nu \frac{x_\nu}{2\sinh((1/2)x_\nu)} \\ &= 1 + \hat{A}_1(\boldsymbol{x}) + \hat{A}_2(\boldsymbol{x}) + \cdots \end{aligned}$$

The generating functions

$$\frac{x}{\tanh(x)} \quad \text{and} \quad \frac{x}{2\sinh((1/2)x)}$$

are even functions of x, so the ambiguity in the choice of sign in the eigenvalues plays no role. This defines characteristic classes

$$L_i(V) \in H^{4i}(M;\mathbb{C}) \quad \text{and} \quad \hat{A}_i(V) \in H^{4i}(M;\mathbb{C})$$

Summary of Formulas

We summarize below some of the formulas in terms of characteristic classes:

1. $c_1(\Omega) = \dfrac{\sqrt{-1}\operatorname{tr}(\Omega)}{2\pi}$,
2. $c_2(\Omega) = \dfrac{1}{8\pi^2}\{\operatorname{tr}(\Omega^2) - \operatorname{tr}(\Omega)^2\}$,
3. $p_1(\Omega) = -\dfrac{1}{8\pi^2}\operatorname{tr}(\Omega^2)$,
4. $\operatorname{ch}(V) = k + \left\{c_1 + \dfrac{c_1^2 - 2c_2}{2} + \cdots\right\}(V)$,
5. $\operatorname{td}(V) = \left\{1 + \dfrac{c_1}{2} + \dfrac{(c_1^2 + c_2)}{12} + \dfrac{c_1c_2}{24} + \cdots\right\}(V)$,
6. $\hat{A}(V) = \left\{1 - \dfrac{p_1}{24} + \dfrac{7p_1^2 - 4p_2}{5760} + \cdots\right\}(V)$,
7. $L(V) = \left\{1 + \dfrac{p_1}{3} + \dfrac{7p_2 - p_1^2}{45} + \cdots\right\}(V)$,
8. $\operatorname{td}(V \oplus W) = \operatorname{td}(V)\operatorname{td}(W)$,
9. $\hat{A}(V \oplus W) = \hat{A}(V)\hat{A}(W)$,
10. $L(V \oplus W) = L(V)L(W)$.

The Euler Form

So far, this article has dealt with the structure groups $O(k)$ in the real setting and $U(k)$ in the complex setting. There is one final characteristic class which arises from the structure group $SO(k)$. Suppose $k = 2n$ is even. While a real antisymmetric matrix A of shape $2n \times 2n$ cannot be diagonalized, it can be put in block off 2-diagonal form with blocks,

$$\begin{pmatrix} 0 & \lambda_\nu \\ -\lambda_\nu & 0 \end{pmatrix}$$

The top Pontrjagin class $p_n(A) = x_1^2 \cdots x_n^2$ is a perfect square. The Euler class

$$e_{2n}(A) := x_1 \cdots x_n$$

is the square root of p_n. If V is an oriented vector bundle of dimension $2n$, then

$$e_{2n}(V) \in H^{2n}(M;\mathbb{C})$$

is a well-defined characteristic class satisfying $e_{2n}(V)^2 = p_n(V)$.

If V is the underlying real oriented vector bundle of a complex vector bundle W,

$$e_{2n}(V) = c_n(W)$$

If M is an even-dimensional manifold, let $e_m(M) := e_m(TM)$. If we reverse the local orientation of M, then $e_m(M)$ changes sign. Consequently, $e_m(M)$ is a measure rather than an m-form; we can use the Riemannian measure on M to regard $e_m(M)$ as a scalar. Let R_{ijkl} be the components of the curvature of the Levi-Civita connection with respect to some local orthonormal frame field; we adopt the convention that $R_{1221} = 1$ on the standard sphere S^2 in $\mathbb{R}^3$. If $\varepsilon^{I,J} := (e^I, e^J)$ is the totally antisymmetric tensor, then

$$e_{2n} := \sum_{I,J} \frac{\varepsilon^{I,J} R_{i_1 i_2 j_2 j_1} \cdots R_{i_{m-1} i_m j_m j_{m-1}}}{(8\pi)^n n!}$$

Let $\mathcal{R} := R_{ijji}$ and $\rho_{ij} := R_{ikkj}$ be the scalar curvature and the Ricci tensor, respectively. Then

$$e_2 = \frac{1}{4\pi}\mathcal{R}$$
$$e_4 = \frac{1}{32\pi^2}(\mathcal{R}^2 - 4|\rho|^2 + |R|^2)$$

Characteristic Classes of Principal Bundles

Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G. Let $\pi : \mathcal{P} \to M$ be a principal G bundle over M. For $\xi \in \mathcal{P}$, let

$$\mathcal{V}_\xi := \ker \pi_* : T_\xi\mathcal{P} \to T_{\pi\xi}M \quad \text{and} \quad \mathcal{H}_\xi := \mathcal{V}_\xi^\perp$$

be the vertical and horizontal distributions of the projection π, respectively. We assume that the metric on $\mathcal{P}$ is chosen to be G-invariant and such that $\pi_* : \mathcal{H}_\xi \to T_{\pi\xi}M$ is an isometry; thus, π is a Riemannian submersion. If F is a tangent vector field on M, let $\mathcal{H}F$ be the corresponding vertical lift. Let $\rho_\mathcal{V}$ be orthogonal projection on the distribution $\mathcal{V}$. The curvature is defined by

$$\Omega(F_1, F_2) = \rho_\mathcal{V}[\mathcal{H}(F_1), \mathcal{H}(F_2)]$$

the horizontal distribution $\mathcal{H}$ is integrable if and only if the curvature vanishes. Since the metric is G-invariant, $\Omega(F_1, F_2)$ is invariant under the group action. We may use a local section s to P over a contractible coordinate chart $\mathcal{O}$ to split $\pi^{-1}\mathcal{O} = \mathcal{O} \times G$. This permits us to identify $\mathcal{V}$ with TG and to regard Ω as a $\mathfrak{g}$-valued 2-form. If we replace the section s by a section $\tilde{s}$, then $\tilde{\Omega} = g\Omega g^{-1}$ changes by the adjoint action of G on $\mathfrak{g}$.

If V is a real or complex vector bundle over M, we can put a fiber metric on V to reduce the structure group to the orthogonal group $O(r)$ in the real setting or the unitary group $U(r)$ in the complex setting. Let $\mathcal{P}_V$ be the associated frame bundle. A Riemannian connection ∇ on V induces an invariant splitting of $T\mathcal{P}_V = \mathcal{V} \oplus \mathcal{H}$ and defines a natural metric on $\mathcal{P}_V$; the curvature Ω of the connection ∇ defined here agrees with the definition previously.

Let $\mathcal{Q}(G)$ be the algebra of all polynomials on $\mathfrak{g}$ which are invariant under the adjoint action. If $Q \in \mathcal{Q}(G)$, then $Q(\Omega)$ is well defined. One has $\mathrm{d}Q(\Omega) = 0$. Furthermore, the de Rham cohomology class $Q(P) := [Q(\Omega)]$ is independent of the particular connection chosen. We have

$$\begin{aligned}
\mathcal{Q}(\mathrm{U}(k)) &= \mathbb{C}[c_1, \ldots, c_k] \\
\mathcal{Q}(\mathrm{SU}(k)) &= \mathbb{C}[c_2, \ldots, c_k] \\
\mathcal{Q}(\mathrm{O}(2k)) &= \mathbb{C}[p_1, \ldots, p_k] \\
\mathcal{Q}(\mathrm{O}(2k+1)) &= \mathbb{C}[p_1, \ldots, p_k] \\
\mathcal{Q}(\mathrm{SO}(2k)) &= \mathbb{C}[p_1, \ldots, p_k, e_k]/e_k^2 = p_k \\
\mathcal{Q}(\mathrm{SO}(2k+1)) &= \mathbb{C}[p_1, \ldots, p_k]
\end{aligned}$$

Thus, for this category of groups, no new characteristic classes ensue. Since the invariants are Lie-algebra theoretic in nature,

$$\mathcal{Q}(\mathrm{Spin}(k)) = \mathcal{Q}(\mathrm{SO}(k))$$

Other groups, of course, give rise to different characteristic rings of invariants.

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See also: Cohomology Theories; Gerbes in Quantum Field theory; Instantons: Topological Aspects; *K*-Theory; Mathai-Quillen Formalism; Riemann Surfaces.

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Chern–Simons Models: Rigorous Results

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Introduction

The relationship between topological invariants and functional integrals from quantum Chern–Simons theory discovered by Witten (1989) raised several challenges for mathematicians. Most of the tremendous amount of mathematical activity generated by Witten's discovery has been concerned primarily with issues that arise after one has accepted the functional integral as a formal object. This has left, as an important challenge, the task of giving rigorous meaning to the functional integrals themselves and to rigorously derive their relation to topological invariants. The present article will discuss efforts to put the functional integral itself on a rigorous basis.