PACIFIC JOURNAL OF MATHEMATICS
Vol. 48, No. 1, 1973

ON 2-TRANSITIVE COLLINEATION GROUPS OF FINITE PROJECTIVE SPACES

WILLIAM M. KANTOR

In 1961, A. Wagner proposed the problem of determining all the subgroups of $P\Gamma L(n, q)$ which are 2-transitive on the points of the projective space $PG(n-1, q)$, where $n \geqq 3$. The only known groups with this property are: those containing $PSL(n, q)$, and subgroups of $PSL(4, 2)$ isomorphic to A_7. It seems unlikely that there are others. Wagner proved that this is the case when $n \leqq 5$. In unpublished work, D. G. Higman handled the cases $n = 6, 7$. We will inch up to $n \leqq 9$. Our result is that nothing surprising happens. The same is true if $n = r^\alpha + 1$ for a prime divisor r of $q - 1$.

One of Wagner's results is that it suffices to only consider subgroups of $PGL(n, q)$. Once this is done, it becomes simpler to view the problem as one concerning linear groups: find all those subgroups G of $GL(n, q)$ which are 2-transitive on the 1-spaces of the underlying vector space V. Our approach is based primarily on three facts. (1) Wagner showed that the global stabilizer in G of any 3-space of V induces at least $SL(3, q)$ on that 3-space. (2) Unless $G \geqq SL(n, q)$ or $n = 4$, $q = 2$, and $G \approx A_7$, no nontrivial element of G can fix every 1-space of some n-2-space of V. (3) $G \leqq SL(n, q)$ if $|G|$ is divisible by a prime which is a primitive divisor of $q^m - 1$ for a suitable $m \leqq n - 2$.

Wagner's results are in [10]. Higman's result, and the case $n = 2^\alpha + 1$ and q odd, are mentioned by Dembowski [1], p. 39. The result mentioned above in (2) is an easy consequence of results of Wagner. The idea used in (3) is due to Perin [8] and, independently, to G. Hare and E. Shult.

I am indebted to G. Seitz for several helpful remarks.

2. Notation and preliminaries.

As already mentioned, we will be dealing with linear groups. Let V be an n-dimensional vector space over $GF(q)$. We write $GL(V) = GL(n, q)$ and $SL(V) = SL(n, q)$. It will be convenient to regard everything as taking place in the relative holomorphic $V \cdot GL(V)$. For any subgroups K, L of this semidirect product we can then consider the normalizer $N_L(K)$ and centralizer $C_L(K)$. If $L \leqq GL(V)$ and W is an L-invariant subspace of V, we write $L^W = L/C_L(W)$ for the subgroup of $GL(W)$ induced by L. $C_L(V/W)$ and $L^{V/W}$ are defined similarly. For any group G, as usual G' is its commutator subgroup, $Z(G)$ its center, and $\Phi(G)$ its Frattini subgroup.

A group A is said to be *involved* in a group B if $A \approx C/D$ with $B \geqq C \trianglerighteq D$.

(2.1) If $R \leqq GL(V)$ has prime power order and $(|R|, q) = 1$, then $V = C_V(R) \oplus [V, R]$, where $[V, R] = \langle v - vr \mid v \in V, r \in R\rangle$ is $N_{GL(V)}(R)$-invariant.

Proof. [3], p. 177.

(2.2) Let $R \leqq GL(V)$ have prime power order with $(|R|, q) = 1$. Let W be an R-invariant subspace. Then $\dim C_V(R) = \dim C_W(R) + \dim C_{V/W}(R)$.

Proof. [3], p. 187, or (2.1).

Both (2.1) and (2.2) will be used frequently, generally without reference.

A *primitive divisor* of $q^k - 1$ is a prime r satisfying $r \mid q^k - 1$ but $r \nmid q^i - 1$ for $1 \leqq i < k$; clearly $k \mid r - 1$.

(2.3) (i) If q is a prime power and $k \geqq 2$, then $q^k - 1$ has a primitive divisor unless $k = 6, q = 2$, or $k = 2$ and q is a Mersenne prime.

(ii) Let r be a primitive divisor of $q^k - 1$, and let R be an r-subgroup of $GL(V)$ for a $GF(q)$-space V. If $C_V(R) = 0$, then k divides $\dim V$.

Proof. (i) [12].

(ii) This is clear if $|R| \leqq r$. Let $|R| > r$, and let $R_1 \leqq Z(R)$ have order r. Then $V = W \oplus [V, R_1]$, where $W = C_V(R_1)$ is R-invariant and $C_W(R) = 0$. By induction, k divides $\dim W$ and $\dim [V, R_1]$.

(2.4) Suppose $\dim V = \alpha m$, r is a primitive divisor of $q^m - 1$, and $R \leqq GL(V)$ is an r-group such that $C_V(R) = 0$. Then:

(i) Each noncyclic composition factor of $N = N_{GL(V)}(R)$ is involved in $PSL(\alpha, q^m)$; and

(ii) If R is abelian, each noncyclic composition factor of $N/C_N(R)$ is involved in the symmetric group S_α.

Proof. Write $V = W_1 \oplus \cdots \oplus W_\beta$, with each W_i a sum of R-isomorphic irreducible R-spaces and no two W_i having isomorphic irreducible R-subspaces. Set $R_i = C_R(W_i)$. Then $Z(R/R_i)$ is cyclic and nontrivial; let Z_i be its subgroup of order r. By (2.3 ii), $\dim W_i = me_i$ for some e_i. Consequently, $\beta \leqq \alpha$ and $e_i \leqq \alpha$.

N permutes the W_i. Let K be the kernel of this permutation representation. Then N/K is involved in $S_\beta \leqq S_\alpha$, and hence in $GL(\alpha, q^m)$.

Set $K_i = N_{GL(W_i)}(Z_i)$. Then K is contained in $K_1 \times \cdots \times K_\beta$. Moreover, K_i is contained in $\Gamma L(e_i, q^m)$. This proves (i).

Now assume that R is abelian. Then R/R_i is a cyclic group normalized by K. Since $\cap R_i = 1$, it follows that $K/C_K(R)$ is abelian. Since N/K is involved in S_{α}, this proves (ii).

(2.5) Let q be odd, and let $H \leqq GL(V)$. Suppose that $H \trianglerighteq A \neq 1$, where A is an elementary abelian 2-group. Set

$$m = \min \{|H: N_H(B)| \mid B < A, |A: B| = 2\} .$$

Then $m \leqq \dim V$.

Proof. (G. Seitz.) Let $\bar{V}$ be an H-irreducible section of V on which A acts nontrivially. Let $\bar{H}$ and $\bar{A}$ be the groups induced by H and A. Then $\bar{A} \neq 1$, and the corresponding $\bar{m} \geqq m$. We may thus assume that $V = \bar{V}$ is H-irreducible. By Clifford's Theorem ([3], p. 70), $V = V_1 \oplus \cdots \oplus V_t$ with the V_i direct sums of A-isomorphic irreducible A-spaces, no two V_i having a common irreducible constituent. Here A induces a group of order 2 on each V_i, while H is transitive on $\{V_1, \cdots, V_t\}$. Thus, $\{C_A(V_i) \mid i = 1, \cdots, t\}$ is an orbit of H of subgroups of A of index 2. Consequently, $t \geqq m$, so $\dim V \geqq m$.

(2.6) Let L be a finite group and $K \lhd L$ with L/K simple. Suppose L has no proper subgroup L_0 for which $L_0/L_0 \cap K \approx L/K$. Then:

(i) K is nilpotent; and

(ii) Each proper normal subgroup of L is contained in K.

Proof. (i) Let S be a Sylow subgroup of K. By the Frattini argument, $L = KN_L(S)$, so our conditions on L imply that $L = N_L(S)$.

(ii) Let $M \trianglelefteq L$ and $M \nleqq K$. Since $1 \neq MK/K \trianglelefteq L/K$, $MK = L$ and hence $M = L$.

(2.7) Let $d > e \geqq 2$ and $t \geqq 1$. Then $PSL(d, q)$ is not involved in $PSL(e, q^t)$.

Proof. If p is the prime dividing q, then p-Sylow subgroups of $PSL(d, q)$ and $PSL(e, q^t)$ have nilpotence class $d - 1$ and $e - 1$, respectively.

We now come to our main technical lemma.

(2.8) Let $q = p^e$, where p is a prime, and $m = \dim V$. Suppose either $m = 3, 4$, or 5, or $m = 6$ and $p = 2$. Let $L \leqq GL(V)$ and $H, K \lhd L$, where $H \leqq K$, $L/K \approx PSL(3, q)$, and $L/H \approx PSL(3, q)$ or $SL(3, q)$. Assume that L has no proper subgroup L_0 for which $L_0/L_0 \cap K \approx PSL(3, q)$. Finally, assume: (♯) If $1 \neq h \in H$ and $p \nmid |h|$, then $\dim C_V(h) \leqq m - 3$.

Then there are L-invariant subspaces X, Y with $X > Y$ such that the following hold.

(a) $K = P \times C$ with P a p-group, $|C| = (3, q-1)$, and $H = P$ or K.

(b) $L/P \approx SL(3, q)$.

(c) $P^{V/X}, P^{X/Y}$ and P^Y are all 1.

(d) $\dim X/Y = 3$ and $L^{X/Y} = SL(X/Y)$.

(e) If $m \leqq 5$ and $q \neq 2$, then $L^{V/X}$ and L^Y are 1. Moreover, some element g of order p in the center of a p-Sylow subgroup of L satisfies $\dim C_V(g) \geqq m-2$, and even $\dim C_V(g) = m-1$ if $P = 1$.

Proof. Everything is obvious if $m = 3$, so assume $m > 3$. We will proceed by a series of steps.

(i) Clearly $L = L'$. We can apply (2.6) to L. In particular, K is nilpotent.

(ii) Suppose that there are L-invariant subspaces V_1, V_2 with $V_1 \geqq V_2$ and $\dim V_1/V_2 \leqq 2$. We claim that L centralizes V_1/V_2. For, $C_L(V_1/V_2) \trianglelefteq L$, and since L^{V_1/V_2} does not have $PSL(3, q)$ as a homomorphic image, (2.6) implies that $C_L(V_1/V_2) = L$.

(iii) Next, suppose that there are L-invariant subspaces X, Y with $X > Y$, $\dim X/Y = 3$ and $L^{X/Y} \neq 1$. We claim that (a)—(e) hold.

Arguing as in (ii) we find that $L^{X/Y} = SL(X/Y)$, while $L^{V/X}$ and L^Y are both 1 or $SL(3, q)$. Write $K = P \times C$ with P a p-group and C a p'-group. C induces a group of order 1 or $(3, q-1)$ on V/X, X/Y, and Y. By (2.2), (a) holds unless $|C| = 9$ and $m = 6$. However, in this case $C \leqq Z(L)$, so $L/P = (L/P)'$ is a central extension of $SL(3, q)$ by a group of order 9, and this is impossible [2].

Thus, (a), (b), (c), and (d) hold.

Now let $m \leqq 5$. Then $\dim V/X$ and $\dim Y$ are $\leqq 2$, so $L^{V/X}$ and L^Y are 1 by (ii). If $P \neq 1$ then, by (c), each $g \neq 1$ in P satisfies $\dim C_V(g) \geqq m-2$.

Suppose $P = 1$, so $L \approx SL(3, q)$. By results of Higman [4], §5, if $q \neq 2$ then there is an L-invariant 3-space T, and each element of L inducing a transvection on T is a transvection of V. This proves (e).

(iv) From now on we assume that m and L are chosen with m minimal such that (2.8) is false. Then $m > 3$.

L is irreducible on V. For otherwise, there is an L-invariant subspace W with $V > W > 0$.

Then $L^W \neq 1$ and $L^{V/W} \neq 1$. For suppose, say, that $L^{V/W} = 1$. Consider L^W, K^W, and H^W. By (2.2), (♯) is inherited by L^W. Also, if $L_0 \leqq L$ and $L_0^W/L_0^W \cap K^W \approx PSL(3, q)$ then $L_0K/K \approx L_0/L_0 \cap K$ has $PSL(3, q)$ as a homomorphic image, so that $L_0K = L$ and hence $L_0 = L$.

Consequently, L^W satisfies the hypotheses of (2.8). Then we can find subspaces X and Y of W such that (iii) applies, whereas (2.8) is assumed false. Thus, $L^W \neq 1$ and $L^{V/W} \neq 1$.

By (ii) we must have $m = 6$ and $\dim W = 3$. Then (iii) again applies, and this is again impossible.

(v) By (iv) and the nilpotence of K, $(|K|, q) = 1$.

K is not central in L. For suppose $K \leqq Z(L)$. Since $L = L'$, L is a homomorphic image of the covering group of $PSL(3, q)$. Then L is $PSL(3, q)$ or $SL(3, q)$ (see, e.g., [2]).

On the other hand, L has an irreducible $GF(q)$-representation of degree m, where $4 \leqq m \leqq 6$ and q is even if $m = 6$. No such representation exists by [7] and [9].

(vi) Let r be a prime and R_1 an r-Sylow subgroup of K such that $R_1 \nleqq Z(L)$. Set $R = R_1 \cap H$. Then $R \nleqq Z(L)$ and $R \lhd L$.

Let A be a characteristic elementary abelian subgroup of R. By (#), $|A| \leqq r^{m-3}$.

We claim that $A \leqq Z(L)$. For otherwise, L has a nontrivial $GF(r)$-representation of degree $\leqq m - 3 \leqq 3$. By (2.6 ii), $PSL(3, q)$ is involved in $GL(3, r)$. Thus, $q = 2$ and $r \neq 3$. Since A is a noncyclic elementary abelian subgroup of $GL(6, 2)$, $|A| = 7^2$. Then L acts transitively on $A - \{1\}$. However, not all elements of $A - \{1\}$ are conjugate in $GL(6.2)$.

Thus, $A \leqq Z(L)$. In (iv), $|A| = r$. In particular, $Z(R)$ is cyclic.

(vii) Suppose $r \nmid q - 1$. By (vi), $R \leqq GL(6, q)$ is nonabelian, so $r = 3 | q + 1$ and $m = 6$. Moreover, $R \rhd B$ with $|R{:}\, B| = 3$ and B abelian. By (vi) we can find $B_1 \neq B$ with $R \rhd B_1$, $|R{:}\, B_1| = 3$, and B_1 abelian. Then $B \cap B_1 \leqq Z(R)$ and $|R/Z(R)| \leqq 9$. Consequently, L centralizes $Z(R)$, $R/Z(R)$, and hence also R, which is not the case.

Thus, $r|q-1$. In (iv), $A \leqq L \cap Z(GL(V)) \leqq Z(SL(V))$, so $r|(q-1, m)$.

There are now just three possibilities: $m = 4, r = 2$; $m = 5, r = 5$; and $m = 6, r = 3$.

(viii) Let $m = 4, r = 2$. By (vii), $-1 \in R$. There is an involution $t \neq -1$ in R. Either $\dim C_V(t) \geqq 2$ or $\dim C_V(-t) \geqq 2$. This contradicts (#).

(ix) Let $m = 5, r = 5$. A 5-Sylow subgroup of $GL(5, q)$ has a normal abelian subgroup of index 5 (the "diagonal subgroup"). Thus, we can find $B \leqq R$ with B abelian and $|R{:}\, B| = 1$ or 5. By (vi), $|R{:}\, B|$ is 5 and B is not characteristic in R. Let $B_1 < R$, $B_1 \neq B$, satisfy the same conditions as B. Then $B_1 \cap B \leqq Z(R)$ and $|R{:}\, Z(R)| \leqq 5^2$. By (vi), $Z(R)$ is cyclic, so L centralizes $Z(R)$, $R/Z(R)$, and hence also R, which is not the case.

(x) Finally, let $m = 6, r = 3$, and $q = 2^i$. Here $3 | q - 1$. On the one hand, $L/C_L(R/\Phi(R))$ can be regarded as a subgroup of $GL(e, 3)$ for some e; on the other hand, using (2.6) and $(|K|, q) = 1$, we find that this group has an elementary abelian 2-subgroup of order q^2 whose normalizer is transitive on the nontrivial elements. By (2.5), $e \geqq q^2 - 1$. However, a 3-Sylow subgroup of $SL(6, q)$ has order $\leqq 3(q-1)^6$. Thus, $3^{q^2-1} \leqq 3^e \leqq |R| < 3q^6$, and since $q \geqq 4$ this is ridiculous.

This contradiction completes the proof of (2.8).

3. Wagner's results and some corollaries. Let V be n-dimensional over $GF(q)$, $n \geqq 3$, and let $G \leqq GL(V)$ be 2-transitive on 1-spaces.

(3.1) For each 3-space T, $N_G(T)^T \geqq SL(T)$.

Proof. Wagner [10], p. 417.

(3.2) If $n \leqq 5$ then $G \geqq SL(V)$, unless $n = 4$, $q = 2$, and $G \approx A_7$.

Poof. Wagner [10], p.422.

(3.3) For each n-1-space W, $N_G(W)$ is 2-transitive on the 1-spaces of V not in W.

Proof. [6], p. 6.

(3.4) If G has an element $g \neq 1$ such that $\dim C_V(g) \geqq n-2$, then $G \geqq SL(V)$ or $n = 4$, $q = 2$, and $G \approx A_7$.

Proof. We may assume that $|g|$ is prime and $n > 5$. Since $\dim [V, g] \leqq 2$ and g centralizes $V/[V, g]$, there is a 3-space $T > [V, g]$ such that $g^T \neq 1$. Then $1 \neq C_G(V/T)^T \trianglelefteq N_G(T)^T$. By (3.1), $C_G(V/T)^T \geqq SL(T)$. Choose $g' \in C_G(V/T)$ with $|g'| \mid q+1$ and $\dim C_T(g') = 1$. Then $\dim C_V(g') = n-2$.

We may thus assume that $(|g|, q) = 1$. Since $g^{[V,g]} \neq 1$, as before $C_G(V/T)^T \geqq SL(T)$ for each 3-space $T > [V, g]$. By the 2-transitivity of G, this holds for every 3-space of V.

Choose $m \leqq n$ maximal with repect to $C_G(V/U)^U \geqq SL(U)$ for all m-spaces U. Suppose $m < n$. By Wagner [10], p. 420, $m \leqq n-2$. Take any subspace W of dimension $m+1$ or $m+2$. For each m-space $U < W$, $C_G(V/U)$ fixes W and centralizes V/W, while $C_G(V/U)^U \geqq SL(U)$. By Wagner [10], p. 420, and (3.2), $C_G(V/W)^W \geqq SL(W)$ for each $m+1$-space W. This contradicts the maximality of m.

(3.5) Let s be a prime and S an s-group maximal with respect to $\dim C_V(S) \geqq 3$. Then $N_G(S)$ is 2-transitive on the 1-spaces of $C_V(S)$.

Proof. Take any 3-space $T \leqq C_V(S)$. Then S is Sylow in $C_G(T)$. By the Frattini argument and (3.1), $(N_G(S) \cap N_G(T))^T = N_G(T)^T \geqq SL(T)$. Our assertion follows immediately.

4. The case $n = r^{\alpha} + 1$.

There is one very easy case of our problem.

(4.1) THEOREM. *Let r be a prime divisor of $q - 1$, and let $\alpha \geqq 1$. Then every collineation group of $PG(r^{\alpha}, q)$ which is 2-transitive on points contains $PSL(r^{\alpha} + 1, q)$.*

We first prove:

(4.2) Let r be a prime divisor of $q - 1$, and let $\alpha \geqq 1$. Let V be an r^{α}-dimensional vector space over $GF(q)$. If $G \leqq \Gamma L(V)$ is transitive on $V - \{0\}$, then $r \mid |G \cap Z(GL(V))|$.

Proof. Let r^{β} be the largest power of r dividing $q^{d} - 1$, where $d = r^{\alpha}$. Then q is not an r^{β}th power, so $r \mid |G \cap GL(V)|$.

Let R be an r-Sylow subgroup of G. By [11], p. 6, each orbit of R on $V - \{0\}$ has length divisible by r^{β}.

R fixes no nontrivial proper subspace of V. For, if it did we would have $r^{\beta} \mid q^{m} - 1$ with $1 \leqq m < d$. Set $e = (d, m)$. Then $r^{\beta} \mid q^{e} - 1$. However, as d/e is a power of $r, (q^{d} - 1)/(q^{e} - 1)$ is divisible by r, and this contradicts the definition of r^{β}.

Let $x \in Z(R) \cap GL(V)$ have order r. Since $r \mid q - 1$, x can be diagonalized. By the preceding paragraph, x is a scalar transformation, that is, $x \in Z(GL(V))$.

(4.3) Let r be a prime divisor of $q - 1$, and let $\alpha \geqq 1$. Then a collineation group of the affine space $AG(r^{\alpha}, q)$ which is 2-transitive on points contains the translation group.

Proof. (4.2).

Now (4.1) follows immediately from (3.3) and (4.3).

5. Primes dividing $|G|$.

We will consider the following situation in the remainder of this paper.

Let V be an n-dimensional $GF(q)$-space, $n \geqq 6$, and G be a subgroup of $GL(V)$, 2-transitive on 1-spaces, such that $G \ngeqq SL(V)$. We may clearly assume that $G > Z = Z(GL(V))$.

In this section let s be a prime dividing $(|G|, q^{m} - 1)$, $1 < m \leqq n - 2$, such that s is a primitive divisor of $q^{m} - 1$. (5.1) is essentially due to Perin [8] and, independently, to E. Shult and G. Hare.

(5.1) If $m = n - 2$ then $q = 2$ and n is even.

(5.2) Suppose that $n = \alpha m + \beta$, $\alpha < \beta \leqq m + 2$, and an element of order s centralizes some 3-space X. Then, for some n' satisfying $5 < n' < n$ and $n' \equiv n \pmod{m}$, there is a subgroup of $GL(n', q)$, not containing $SL(n', q)$, which is 2-transitive on the points of $PG(n' - 1, q)$.

Clearly (5.2) has an inductive flavor. Since the proofs are similar, we will only prove the second of the above results.

Proof of (5.2). Choose $S \leqq C_G(X)$ as in (3.5). Set $W = C_V(S)$, $W^* = [V, S]$, and $N = N_G(S)$. Then $V = W \oplus W^*$, $C_{W^*}(S) = 0$, and N^W is 2-transitive on 1-spaces.

Set $n' = \dim W$, so $n' \geqq 3$. By (2.3 ii), since $\beta \leqq m + 2$ we have $\dim W^* = \gamma m$ with $\gamma \leqq \alpha$. Then $n' = n - \gamma m \geqq n - \alpha m = \beta > \alpha \geqq \gamma$.

We must show that $n' > 5$ and $N^W \ngeqq SL(W)$. Deny this. Then either $N^W \geqq SL(W)$ or $n' = 4, q = 2$, and $N^W \approx A_7$. In particular, the commutator subgroup N'^W contains a nontrivial element centralizing an n'-2-space.

In this situation, $C_{N'}(W^*)^W \leqq Z(GL(W))$. For otherwise, $C_{N'}(W^*)^W \trianglelefteq N'^W$ implies that $C_{N'}(W^*)^W = N'^W$. Then $C_{N'}(W^*)$ has a nontrivial element g centralizing an n'-2-space of W. Hence, $\dim C_V(g) \geqq n - 2$, which contradicts (3.4).

It follows that N'^{W^*} has $PSL(n', q)$ as a homomorphic image, unless $n' = 4$ and $q = 2$, in which case A_7 may be a homomorphic image.

Since $C_{W^*}(S) = 0$, we can apply (2.4): each noncyclic composition factor of N^{W^*} is involved in $PSL(\gamma, q^m)$. Since $n' > \gamma$, by (2.7) $PSL(n', q)$ cannot be such a composition factor. Thus, $n' = 4, q = 2, \gamma \leqq 3$, and A_7 is a composition factor of N'^{W^*}. However, A_7 is not involved in $PSL(3, 2^m)$. This is a contradiction.

REMARK. It is useful to note that the above proof holds under slightly weaker hypotheses: s is a primitive divisor of $q^m - 1$, $S \neq 1$ is an s-subgroup of G with $W = C_V(S)$ of dimension $n' \geqq 3$, $(n - n')/m < n'$, and $N_G(S)^W$ is 2-transitive on 1-spaces.

We conclude this section with two miscellaneous results.

(5.3) Assume that G has a cyclic subgroup H of order $q^n - 1$ containing an r-Sylow subgroup of G for some prime r dividing $q^2 + q + 1$. Then $q = 2$ and n is even.

Proof. Suppose $q \neq 2$ or $q = 2$ and n is odd. By (2.3), H is transitive on $V - \{0\}$. Thus, H is transitive on the 3-spaces fixed by its subgroup R of order r.

On the other hand, by (3.1) each 3-space is fixed by a conjugate of R. Thus, G is transitive on 3-spaces, and this contradicts Perin [8] or (5.1) since $n \geqq 6$.

(5.4) Assume that G has a cyclic subgroup of order $q^{n-1} - 1$ fixing some $n - 1$-space W and transitive on $W - \{0\}$. Then $N_G(W)$ is 2-transitive on the 1-spaces of $W, q = 2$, and n is even.

Proof. We may assume that $G - Z$ has no element fixing all 1- spaces in W. By [6], Lemma 7.3, $N_G(W)$ is 2-transitive on the 1-spaces of W. The result now follows from (2.3) and (5.1).

6. The case $n \leqq 9$. Let n, V, G, and Z be as in § 5, so $G \ngeqq SL(V)$. Let p be the prime dividing q.

Assume that $6 \leqq n \leqq 9$.

(6.1) $n \neq 6$.

Proof. Suppose $n = 6$. If $q = 2$ then $q^5 - 1$ is a prime. By (5.4), the stabilizer of a 5-space W is 2-transitive on $W - \{0\}$. By (3.2) and (3.4), $G \geqq SL(V)$, which is not the case.

Thus, $q > 2$. Let r be a prime dividing $q - 1$.

Suppose that there is 3-space T for which $N_G(T) - Z$ contains an element inducing a scalar transformation of order r on T. Using Z, we find that $r \mid |C_G(T)|$. Let R be an r-Sylow subgroup of $C_G(T)$. By (3.4), $T = C_V(R)$. By (3.5), $N_G(R)^T \geqq SL(T)$. Also, $N_G(R)$ normalizes the 3-space $[V, R]$. An element of order p in the center of a p-Sylow subgroup of $N_G(R)$ centralizes 2-spaces of both $C_V(R)$ and $[V, R]$, and hence centralizes a 4-space of $V = C_V(R) \oplus [V, R]$. This contradicts (3.4). Thus, no element of $G - Z$ of order r has an eigenspace of dimension > 2.

Now take any 3-space T, and write $T = X \oplus Y$ with $\dim X = 2$ and $\dim Y = 1$. Set $F = N_G(X) \cap N_G(Y)$, so $F^X = GL(X)$. Take $R \leqq F$ of order r with $R \nleqq Z$ and $R^T \leqq Z(F^T)$. By the Frattini argument, $N_F(R)^X = GL(X)$. Let $E \leqq N_F(R)$ be minimal with respect to $E^X = SL(X)$.

Since R is diagonalizable and each of its eigenspaces has dimension 1 or 2, we can write $V = X \oplus W_1 \oplus W_2$ with $W_1 > Y$, $\dim W_i = 2$, and W_i invariant under $N_G(R)$. If $q \neq 3$, $E = E'$ centralizes W_1, so an element of E of order p centralizes a 4-space, which contradicts (3.4). If $q = 3$, R cannot have more than two eigenspaces as $|R| = 2$, which is again a contradiction.

(6.2) q is even.

Proof. Assume that q is odd. There is an involution $t \in G - Z$. Since $n \geqq 6$, $\dim C_V(t)$ or $\dim C_V(-t)$ is $\geqq 3$. Let S be a 2-group in G maximal with respect to $\dim C_V(S) \geqq 3$. Set $W = C_V(S)$ and $W^* = [V, S]$, so $V = W \oplus W^*$. Set $M = N_G(S)$. By (3.5), M^W is 2-transitive on 1-spaces. Since $M > Z$ and all involutions in M^W centralize at most a 2-space (by the maximality of S), $\dim W \leqq 4$. Consequently, by (3.2), $M^W \geqq SL(W)$.

By (4.1) and (6.1), $n = 7$ or 8, so $\dim W^* \leqq 5$.

We claim that $C_M(W^*)^W \leqq Z(GL(W))$. For otherwise, $C_M(W^*)^W \trianglelefteq M^W$ yields $C_M(W^*)^W \geqq SL(W)$. Then $C_M(W^*)$ contains a nontrivial transvection of V, which contradicts (3.4).

Thus, $C_M(W^*)$ is cyclic and M'^{W^*} has $PSL(W)$ as a homomorphic image.

Suppose that $\dim W = 4$. Then $\dim W^* = 3$ or 4. Use of M'^{W^*} yields $\dim W^* = 4$ and $M'^{W^*} \geqq SL(W^*)$. If $g \neq 1$ is in the center of a p-Sylow subgroup of M' then g^W and g^{W^*} are transvections, and this contradicts (3.4).

Thus, $\dim W = 3$. Let $L \leqq M$ be minimal with respect to having $PSL(3, q)$ as a homomorphic image. Let $H = C_L(W) \leqq K \triangleleft L$ with $L/K \approx PSL(3, q)$. Then (2.8) applies to W^*, L^{W^*}, K^{W^*}, and H^{W^*}.

Choose $g \in L$ so that g^{W^*} is as in (2.8 e). If $g \in H = C_L(W)$, then $\dim C_V(g) \geqq n - 2$. If $H^{W^*} = 1$ then $H = 1$, and both g^W and g^{W^*} are transvections, so once again $\dim C_V(g) \geqq n - 2$. In either case we have contradicted (3.4).

(6.3) $n \neq 7, 8$.

Proof. Let $n = 7$ or 8. Fix a prime $r \mid q + 1$.

Take any 3-space T. By (3.1), $N_G(T)^T \geqq SL(T)$. Also, $N_G(T)$ acts on V/T. By (3.4), $C_G(V/T)^T \leqq Z(GL(T))$ (since otherwise, $C_G(V/T)$ would have an element of order r), so $C_G(V/T)$ is solvable. Thus, $N_G(T)^{V/T}$ has $PSL(3, q)$ as a composition factor. By (2.8), there is an r-group $R \neq 1$ in $N_G(T)$ such that $\dim C_{V/T}(R) \geqq 2$, and then $\dim C_V(R) \geqq 3$.

This contradicts (5.2) with $n = 2 \cdot 2 + 3$ or $2 \cdot 2 + 4$.

(6.4) If $n = 9$ then $q = 2$ or 4.

Proof. Suppose $n = 9$ and $q > 4$ is even.

(i) By (5.2) with $n = 2 \cdot 3 + 3$, no nontrivial element of order dividing $(q^2 + q + 1)/(q + 1, 3)$ can centralize a 1-space.

(ii) Let T be any 3-space. Let $L \leqq N_G(T)$ be minimal with respect to having $PSL(3, q)$ as a homomorphic image. By (3.4), $C_G(V/T)^T \leqq Z(GL(T))$, so (2.8) applies to $L^{V/T}$. Consequently, by (i) there is a 6-space $Y > T$ such that $L^{Y/T} = SL(Y/T)$ and $L^{V/Y} = SL(V/Y)$.

(iii) Let s be a prime dividing $q + 1$. By (ii), there is an element of order s centralizing a 3-space.

Let S be an s-group maximal with respect to $\dim C_V(S) \geqq 3$. By (3.5), $N_G(S)$ is 2-transitive on the 1-spaces of $C_V(S)$. In view of (i), it follows from (3.2), (6.1), and (6.3) that $\dim C_V(S) = 3$.

Let $T = C_V(S)$ in (ii), and choose $L \leqq N_G(S)$ there. By (i) and the proof of (2.4), $(LS)^{[V,S]}$ acts as a subgroup of $\Gamma L(3, q^2)$, with S inducing scalar transformations.

(iv) Since $q > 4$, by (2.3 i) there is a prime $r \neq 3$ dividing $q - 1$. Moreover, if $q \neq 16$ we can choose $r \neq 5$.

We claim that some element of order r centralizes a 4-space. For, since $r \neq 3$, in (iii) we can find $g \in L - Z$ of order r such that $g^{[V,S]}$ has an eigenspace of dimension $\geqq 4$. Consequently, some element of $\langle g, Z \rangle$ of order r centralizes a 4-space.

(v) Let R be an r-group maximal with respect to $\dim C_V(R) \geqq 3$; by (iv), $R \neq 1$. Set $T = C_V(R)$ and $T^* = [V, R]$. By (3.5), $N_G(R)^T$ is 2-transitive on 1-spaces, so $\dim T = 3$ by (i). We can thus choose $L \leqq N_G(R)$ in (ii).

We claim that LR centralizes R and that R is diagonalizable. Certainly $(LR)^{T^*} \leqq GL(T^*)$. Suppose $r > 5$. Then an r-Sylow subgroup of $GL(6, q)$ is diagonalizable, and hence abelian. By (2.4 ii) (with $m = 1, \alpha = 6$), each composition factor of $L/C_L(R)$ is involved in S_6. By (2.6 ii), $L = C_L(R)$, so $R \leqq Z(LR)$.

Consider the case $r = 5, q = 16$. Suppose $L > C_L(R)$. Then L acts nontrivially on $R/\Phi(R)$, where $|R/\Phi(R)| \leqq 5^7$. By (2.6 ii), $16 + 1$ divides $|GL(7, 5)|$, which is not the case.

Thus, L centralizes R. There is an s-group $S_0 < L$ such that $\dim C_{T^*}(S_0) = 2$. Since R normalizes $C_{T^*}(S_0)$ and $[T^*, S_0]$, it follows that R is again diagonalizable. Thus, $R \leqq Z(LR)$.

(vi) T^* is the direct sum of R-invariant subspaces, each invariant under LR. By (ii) and (v), there are 3-spaces X and X' such that $T^* = X \oplus X'$, R^X and $R^{X'}$ consist of scalar transformations, $L^X = SL(X')$, and $L^{X'} = SL(X')$.

Consequently, for each $h \in R$, $\dim C_V(h) = 3, 6,$ or 9.

(vii) By (iv), there is an r-group $R_1 \neq 1$ maximal with respect to $\dim C_V(R_1) \geqq 4$. By (vi), $W = C_V(R_1)$ has dimension 6. Set $M = N_G(R_1)$.

Take any 3-space $T < W$. Let $R \geqq R_1$ be an r-Sylow subgroup of $C_G(T)$. If $R = R_1$ then $N_M(T)^T \geqq SL(T)$ by the Frattini argument. If $R > R_1$ then the choice of R_1 implies that $C_V(R) = T$, and hence that R is an r-group maximal with respect to $\dim C_V(R) \geqq 3$; by (v), $C_G(R)^T \geqq SL(T)$, so again $N_M(T)^T \geqq SL(T)$.

Consequently, M^W is 2-transitive on 1-spaces. Then $(q^6-1)/(q-1)$ divides $|G|$, and this contradicts (5.2).

(6.5) If $n = 9$ then $q \neq 4$.

Proof. Suppose $n = 9$ and $q = 4$. We will try to imitate the proof of (6.4) using $r = 3$. Steps (i) and (ii) of that proof still hold.

We begin by showing the existence of $x \in G$ of order 3 such that $x^y = x^{-1}$ for some 2-element y. Take T and L as in (ii). Then we can find $x, y \in L$ with $|x| = 3$, y a 2-element, and $x^y = x^{-1}a$, $a \in C_L(T)$.

By (2.8), $C_L(T) = P \times C$ with P a 2-group and $|C| = 1$ or 3. Then $\langle x \rangle$ is Sylow in $\langle x, y \rangle P$. By the Frattini argument, some element of $\langle y \rangle P$ inverts $\langle x \rangle$, and we may assume this is y.

We next claim that some element of order 3 centralizes a 4-space. For, assume that this is false, and choose x, y as above. Since $q = 4$, x is diagonalizable and has at most 3 eigenspaces. However, no element of $\langle x, Z \rangle - \{1\}$ centralizes a 4-space, so $C_V(x) = T$ is a 3-space and x has two other 3-dimensional eigenspaces T_1, T_2. Moreover, by our assumption, $C_G(T)$ has a cyclic 3-Sylow subgroup. Thus, by the Frattini argument, $N_G(\langle x \rangle)^T \geqq SL(T)$, so $C_G(x)^T \geqq SL(T)$. Since $|GL(T): SL(T)| = 3$, $y^T \in SL(T)$, so we can find $c \in C_G(X)$ such that $c^{-1}y \in C_G(T)$. Clearly $c^{-1}y$ inverts x, so there is an involution $t \in \langle c^{-1}y \rangle$. Here, t centralizes T and centralizes 2-spaces of each T_i, so $\dim C_V(t) \geqq 7$. This contradicts (3.4), and proves our claim.

Now define R, T, T^*, and L as in (v). We will be able to obtain a contradiction precisely as in (vi) and (vii) if we can show that $R \leqq Z(LR)$ and R is diagonalizable.

By (2.6), $L \triangleright K$ with $L/K \approx PSL(3, 4)$ and K nilpotent. By (2.2) and (2.8), $K = P \times C$ with $|C| = 3$ or 9 and P a 2-group; moreover, there is an L-invariant 3-space $X < T^*$ such that $L^X = SL(X)$, $L^{T^*/X} = SL(T^*/X)$, and P centralizes T, X, and T^*/X. By (3.4), no nontrivial element of P centralizes a 4-space of T^*. Consequently, P is elementary abelian of order $\leqq 4^3$. Thus, if $P \not\leqq Z(L)$ then $PSL(3, 4)$ is isomorphic to a subgroup of $GL(6, 2)$, which is not the case ([7], [9]). Thus, $K \leqq Z(L)$.

Now suppose that L acts nontrivially on R, and hence on $R/\Phi(R)$. Since $R \leqq GL(6, 4)$, $|R/\Phi(R)| \leqq 3^6 \cdot 3^2$. Thus, $PSL(3, 4)$ or $SL(3, 4)$ is isomorphic to a subgroup of $GL(8, 3)$. Then $GL(8, 3)$ has an elementary abelian subgroup of order 4^2 whose normalizer is transitive on the nontrivial elements. By (2.5), this is impossible.

Consequently, $L \leqq C_G(R)$. An element of L of order 5 centralizes 1-spaces of X and T^*/X. It follows that T^* is the sum of R-invariant 2-spaces. Thus, R is diagonalizable and $R \leqq Z(LR)$. This completes the proof of (6.5).

Last, and least:

(6.6) If $n = 9$ then $q \neq 2$.

Proof. Suppose $n = 9$ and $q = 2$. Using (5.1) and (5.2) we find that $|G| = 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7 \cdot 17 \cdot 73$ for some α, β.

Let S be a 73-Sylow subgroup of G. By (5.3), $|C_G(S)| = 73$. Thus, $|N_G(S)| = 3^\gamma \cdot 73$ with $\gamma \leqq 2$.

By Sylow's theorem, $2^\alpha \cdot 3^{\beta-\gamma} \cdot 5 \cdot 7 \cdot 17 \equiv 1 \pmod{73}$. A little arithmetic shows that this is impossible.

In view of (3.2) and the results of this section, we can now state:

THEOREM 6.7. *Let H be a subgroup of $P\Gamma L(n, q)$ which is 2-transitive on the points of $PG(n-1, q)$. If $3 \leqq n \leqq 9$, then $H \geqq PSL(n, q)$ or $n = 4, q = 2$, and $H \approx A_7$.*

REFERENCES

1. P. Dembowski, *Finite Geometries*, Springer, Berlin-Heidelberg-New York, 1968.
2. W. Feit, *The current situation in the theory of finite simple groups*, Proc. Int. Cong. Math. Nice, 1970, Vol. 1, 55-93.
3. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
4. G. Hare, *Thesis*, Southern Illinois University, 1970.
5. D. G. Higman, *Flag-transitive collineation groups of finite projective spaces*, Illinois J. Math., **6** (1962), 434-446.
6. W. M. Kantor, *2-Transitive symmetric designs*, Trans. Amer. Math. Soc., **146** (1969), 1-28.
7. C. Mark, *Thesis*, University of Toronto, 1939.
8. D. Perin, *On collineation groups of finite projective spaces*, Math. Z., **126** (1972), 135-142.
9. R. Steinberg, *Representations of algebraic groups*, Nagoya Math. J., **22** (1963), 33-56.
10. A. Wagner, *On collineation groups of finite projective spaces*, I. Math. Z., **76** (1961), 411-426.
11. H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.
12. K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. f. Math. u. Phys., **3** (1892), 265-284.

Received November 6, 1971 and in revised form October 13, 1972. This research was supported in part by NSF Grant GP28420.

UNIVERSITY OF OREGON