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Veroneseans, power subspaces and independence

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Abstract. Results are proved indicating that the Veronese map v_d often increases independence of both sets of points and sets of subspaces. For example, any $d+1$ Veronesean points of degree d are independent. Similarly, the dth power map on the space of linear forms of a polynomial algebra also often increases independence of both sets of points and sets of subspaces. These ideas produce $d+1$-independent families of subspaces in a natural manner.

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1 Introduction

In this paper we will study independence questions involving points or subspaces obtained from standard geometric or algebraic objects: Veronese maps and polynomial rings. The proofs are elementary, but some of the results seem unexpected.

We will always be considering integers $d, n > 1$. For any field K, the (vector) *Veronese map*[1] $v_d\colon K^n \to K^N$, $N = \binom{d+n-1}{d}$, is defined in (2.1); the 1-subspaces in $v_d(K^n)$ are the *Veronesean points of degree* d. We will be concerned with the behavior of v_d on sets of subspaces of K^n: in general it increases independence. For example:

Theorem 1.1. *Any $d+1$ Veronesean points of degree d in K^N are independent* (*that is, they span a $d+1$-space*).

The dimension n of the initial space K^n does not play any role in this result or others in this paper. Section 2.2 contains a surprisingly elementary proof. These types of results are in the geometric framework appearing in [12, 9, 3, 2, 10, 11] rather than the more standard Algebraic Geometry occurrences of the Veronese map [6, p. 23], [13, pp. 40–41].

More generally, we will consider independence of sets of subspaces of K^N. We call a set $\mathcal{U}$ of at least $d+1$ such subspaces $d+1$-*independent* if the subspace spanned by

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[1]Traditionally this map is defined sending the projective geometry $\mathbf{P}(K^n)$ on K^n to $\mathbf{P}(K^N)$. For our proofs it is preferable to deal with vectors, which still allows us to act on projective geometries.

any $d+1$ members of $\mathcal{U}$ is their direct sum. For example, 2-independence means that any two members have intersection 0, which is a very familiar geometric situation. With this terminology, Section 2.3 contains an elementary proof of the following generalization of the preceding theorem concerning sets $v_d(\mathcal{U}) := \{\langle v_d(U)\rangle \mid U \in \mathcal{U}\}$ of subspaces K^N:

Theorem 1.2. *If $\mathcal{U}$ is any $e+1$-independent set of at least $de+1$ non-zero subspaces of K^n, then $v_d(\mathcal{U})$ is a $de+1$-independent set of subspaces of K^N.*

Even when $\mathcal{U}$ is a spread we suspect that the resulting $d+1$-independent family $v_d(\mathcal{U})$ is not maximal. It is the natural way of obtaining $v_d(\mathcal{U})$ that seems more interesting than the possible maximality. Note that the dimensions of the subspaces in the preceding theorem are allowed to vary arbitrarily.

For any finite set $\{x_1, \dots, x_n\}$ of indeterminates and any integer $m \geq 1$, we will consider the space A_m consisting of all homogeneous polynomials of degree m in the polynomial algebra $A = K[x_1, \dots, x_n]$. A *powerpoint* is a 1-space $\langle f^d\rangle$ in A_d, where $0 \neq f \in A_1$. Powerpoints in A_d are closely related to Veronesean points in suitable characteristics (cf. Theorem 3.1). As in the case of the Veronese map, we are interested in the behavior of d-fold powers on sets of subspaces of A_1. The case of powerpoints follows easily from Theorem 1.1 (cf. Section 3.1):

Theorem 1.3. *For any field whose characteristic is 0 or greater than d, any $d+1$ powerpoints of A_d are independent.*

More specialized results are possible for small positive characteristics (cf. Theorems 3.2 and 3.3).

Let $\langle T^d\rangle$ denote the subspace spanned by all products of d members of a subset T of A. Section 4.2 again concerns increasing independence of subspaces, once again assuming a restriction on the characteristic:

Theorem 1.4. *Assume that $1 \leq r \leq d$ and $d!/(d-r)! \neq 0$ in K. If $\mathcal{T}$ is any $e+1$-independent set of at least $re+1$ non-zero subspaces of A_1, then $\{\langle T^d\rangle \mid T \in \mathcal{T}\}$ is an $re+1$-independent set in A_d.*

Theorem 1.2 can be used to prove this when $r = d$, while Theorem 1.3 is a special case, although the proofs use very different tools. In Section 5 we prove a somewhat weaker-looking variation on the preceding theorem.

Generalized dual arcs and other configurations are constructed in Section 6 using very elementary properties of the polynomial algebra A. One of these configurations is another infinite family of 3-independent subspaces.

Note: We will always use vector space dimension.

2 The Veronese map

In this section we will prove Theorems 1.1 and 1.2. In passing we use the polynomial ring to reprove a standard result on Veronesean action.

2.1 Background concerning the Veronese map. Consider two integers $d, n > 1$, together with $N = \binom{d+n-1}{d}$. For a field K of arbitrary characteristic and size, we will use the K-space $V = K^n$ of all n-tuples $t = (t_1, \ldots, t_n) = (t_i)$, $t_i \in K$, and the K-space $W = K^N$ of all N-tuples (y_α) for a fixed but arbitrary ordering of all sequences $\alpha = (a_1, \ldots, a_n)$ of integers $a_k \geq 0$ satisfying $\sum_k a_k = d$. Corresponding to α there is a monomial function $t \mapsto t^\alpha := t_1^{a_1} \cdots t_n^{a_n}$ of degree d in the coordinates t_i. (See Section 2.4 for more discussion of this setting.)

The (vector) *Veronese map* $v_d \colon V \to W$ is defined by

$$v_d((t_i)) := (t^\alpha). \tag{2.1}$$

This induces the classical Veronese map $\mathbf{P}(V) \to \mathbf{P}(W)$ on projective spaces [6, p. 23], [13, pp. 40–41]. Some of its geometric aspects have been studied outside Algebraic Geometry in [12, 9, 3, 2, 10, 11].

There is a natural map from homogeneous polynomial functions g in $t_1, \ldots, t_n$ of degree d to linear functionals $W \to K$. Namely, if $g(t_1, \ldots, t_n) = \sum_\alpha a_\alpha t^\alpha$, where $a_\alpha \in K$ using all α as before, then the corresponding linear functional $\tilde{g} \colon W \to K$ is given by $\tilde{g}((y_\alpha)) := \sum_\alpha a_\alpha y_\alpha$. If the field is tiny then it is possible that two monomial functions t^α coincide, so that this correspondence is not bijective, in fact "sending" g to $\tilde{g}$ is not actually a function! However, what matters here is that this recipe produces a linear functional, and that every linear functional on W arises this way.

Clearly, $\tilde{g}$ is a linear functional on W such that

$$\tilde{g}(v_d(t)) = g(t) \quad \text{for all } t = (t_i) \in V. \tag{2.2}$$

2.2 Veronesean points. The following elementary observation implies Theorem 1.1 (see Lemma 2.7 for a much stronger version):

Lemma 2.3. *If z is a point of V not in each of d subspaces $U_1, \ldots, U_d$ of V, then $v_d(z)$ is not in $\langle v_d(U_1), \ldots, v_d(U_d)\rangle$.*

Proof. For $1 \leq j \leq d$ let f_j be a linear function $V \to K$ that vanishes on U_j but not on z. Then $g := \prod_j f_j$ is a homogeneous polynomial function of degree d that vanishes on all U_j but not on z. By (2.2), the corresponding linear functional $\tilde{g}$ on W vanishes on $\langle v_d(U_1), \ldots, v_d(U_d)\rangle$ but not on $v_d(z)$, as required. □

See [2], [3, Theorem 2.10], [1, 4, 14] and [5] for results similar to Theorem 1.1.

Remark 2.4. If $q \geq d$ then the rational normal curve $v_d(\mathbf{P}(K^2))$ spans $K^N = K^{d+1}$ [8, p. 229]. It follows that $v_d(K^2)$ does not contain $d+2$ independent points: Theorem 1.1 is best possible.

Theorem 1.1 can be viewed as a statement about the code C having a check matrix whose columns consist of one non-zero vector in each Veronesean point: C has minimum weight $> d+1$. By the preceding paragraph, the minimum weight is $d+2$ if q is not too small, with codewords of weight $d+2$ arising from $d+2$ points in a 2-space in K^n; and similarly, the next smallest weight is $2d+2$, occurring from $d+1$ points on each of two 2-spaces in a 3-space.

We have not been able to find any reference to this code in the literature. It is probably worth studying, at least from a geometric perspective.

Remark 2.5. The notation v_d is ambiguous, since it omits the original dimension n. With this in mind, these maps can be composed. It is easy to use monomials to check that $v_e(v_d(\mathbf{P}(K^n)))$ is just $v_{ed}(\mathbf{P}(K^n))$ on a subspace of the underlying $\binom{e+N-1}{e}$-dimensional space (where N is as before). For example, for any set X of points of a projective space, $v_e(\mathbf{P}(X))$ is $e+1$-independent by Theorem 1.1; but if $X = v_d(\mathbf{P}(Y))$ for $Y \subseteq A_1$ then $v_e(\mathbf{P}(X))$ is $de+1$-independent.

In particular, if C is a conic in K^3 then $v_2(C) = v_2(v_2(\mathbf{P}(K^2)))$ is 5-independent: it is a rational normal curve in a 5-space. Similarly, it is natural to ask for the independence properties of v_d-images of geometrically natural sets of points. For example, by Theorem 1.2, $v_2(\text{hyperoval})$ and $v_2(\text{ovoid})$ are 5-independent (and 5 is best possible).

2.3 Families of subspaces. The following special case of Theorem 1.2 contains Theorem 1.1:

Theorem 2.6. *If $\mathcal{U}$ is any set of at least $d+1$ non-zero subspaces of K^n pairwise intersecting in 0, then $\langle v_d(\mathcal{U})\rangle$ is a $d+1$-independent set of subspaces of K^N.*

This is an immediate consequence of the following

Lemma 2.7. *If U_0 is a subspace of V intersecting each of d subspaces $U_1,\dots,U_d$ of V only in 0, then $\langle v_d(U_0)\rangle \cap \langle v_d(U_1),\dots,v_d(U_d)\rangle = 0$.*

Proof. We will construct a linear map L on W whose kernel contains $v_d(U_j)$, $1 \le j \le d$, and meets $\langle v_d(U_0)\rangle$ only in 0. By Corollary 2.19, one may arbitrarily change a basis of V while leaving the set $v_d(V) \subseteq W$ invariant. Thus we may assume that $U_0 = \{(t_1,\dots,t_m,0,\dots,0) \mid t_i \in K\}$; we will view U_0 as K^m. Let $W_0 \cong K^{N_0}$, $N_0 = \binom{d+m-1}{d}$, be the span of the set of vectors in W having a non-zero coordinate for some member of $v_d(U_0)$ and zero outside of $v_d(U_0)$ (i.e., ith coordinate 0 for all $i > m$). Then $v_d\colon U_0 \to W_0$ *can be viewed as the Veronese map on* U_0. (If K is small then W_0 might not be the span of $v_d(U_0)$, which adds a minor complication to our argument.)

Throughout this proof, β will range over all sequences $(b_1,\dots,b_m,0,\dots,0)$ of n integers $b_k \ge 0$ satisfying $\sum_k b_k = d$.

Let $\mathbb{H}$ denote the K-space of all homogeneous polynomial functions g on V of degree d such that $g(U_j) = 0$ for $1 \le j \le d$. We will construct many elements of $\mathbb{H}$. First note that *every monomial function t^β on U_0 of degree d is the restriction of some member of* $\mathbb{H}$. For, write $t^\beta = t_{\sigma(1)}\cdots t_{\sigma(d)}$ for a function $\sigma\colon \{1,\dots,d\} \to \{1,\dots,m\}$. If $1 \le i \le d$ let λ_i denote any linear functional on V such that $\lambda_i((t_1,\dots,t_m,0,\dots,0)) = t_{\sigma(i)}$ and $\lambda_i(U_i) = 0$ (recall that $U_0 \cap U_i = 0$). Then $\prod_{i=1}^d \lambda_i((t_1,\dots,t_m,0,\dots,0)) = \prod_{i=1}^d t_{\sigma(i)} = t^\beta$ and $\prod_{i=1}^d \lambda_i(U_j) = 0$ for $1 \le j \le d$. Thus, $\prod_{i=1}^d \lambda_i$ behaves as required.

It follows that every homogeneous polynomial function on U_0 of degree d is the restriction of some member of $\mathbb{H}$.

Let $\mathbb{W}$ denote the set of all linear functionals on W that vanish on $v_d(U_j)$ for $1 \le j \le d$. Set $W_\bullet := \langle v_d(U_0)\rangle$. The crucial step of the proof of the lemma is that

$$\text{Every linear functional } \mu_\bullet \text{ on } W_\bullet \text{ is the restriction of some } \mu \in \mathbb{W}. \tag{2.8}$$

For, arbitrarily extend $\mu_\bullet$ to a linear functional μ_0 on W_0 (this is irrelevant if $W_0 = W_\bullet$). As noted in Section 2.1, there is a homogeneous polynomial function g_0 on U_0 of degree d such that $\mu_0 = \tilde{g}_0$. We have seen that g_0 is the restriction to U_0 of some $g \in \mathbb{H}$. Consequently, for (2.8) it suffices to show that $\mu := \tilde{g}$ coincides with $\mu_\bullet$ on $W_\bullet$. If $t \in U_0$ then we can apply (2.2) using both V and U_0:

$$\tilde{g}(v_d(t)) = g(t) = g_0(t) = \tilde{g}_0(v_d(t)) = \mu_0(v_d(t)) = \mu_\bullet(v_d(t)).$$

Since $\tilde{g}$ and $\mu_\bullet$ are linear on $W_\bullet = \langle v_d(U_0)\rangle$, it follows that $\tilde{g} = \mu_\bullet$ on $W_\bullet$, which proves (2.8).

Set $N_\bullet := \dim W_\bullet$. By (2.8), $\mathbb{W}$ has a subset $\{\mu_i \mid 1 \le i \le N_\bullet\}$ whose restrictions to $W_\bullet$ form a basis of the dual space $W_\bullet^*$. Then $\mu_i(v_d(U_j)) = 0$ for $1 \le i \le N_\bullet$, $1 \le j \le d$, by the definition of $\mathbb{W}$.

Define $L\colon W \to K^{N_\bullet}$ by $L((y_\alpha)) = (\mu_i((y_a)))$. Then L is linear, and $L(v_d(U_j)) = (\mu_i(v_d(U_j))) = 0$ for $1 \le j \le d$. Since $\{\mu_i \mid 1 \le i \le N_\bullet\}$ restricts to a basis of $W_\bullet^*$,

$$\langle v_d(U_0)\rangle \cap \langle v_d(U_1), \dots, v_d(U_d)\rangle \le W_\bullet \cap \ker L = W_\bullet \cap \bigcap_i \ker \mu_i = 0. \qquad \square$$

Remark 2.9. The most familiar examples of 2-independent families are spreads. It would be interesting to know for which r the set in Theorem 2.6 is r-independent when Σ is a Desarguesian spread of k-spaces of a $2k$-space.

The preceding lemma also yields Theorem 1.2:

Proof of Theorem 1.2. Consider distinct $U_0, \dots, U_{de} \in \mathcal{U}$, and suppose that $\sum_{i=0}^{de} y_i = 0$ for some $y_i \in \langle v_d(U_i)\rangle$. By symmetry, it suffices to show that $y_0 = 0$.

Let Π be any partition of $\{1, \dots, de\}$ into d subsets π of size e. For $\pi \in \Pi$ let $U_\pi := \langle U_i \mid i \in \pi\rangle$. Then $U_0 \cap U_\pi = 0$ since $\{U_0, U_i \mid i \in \pi\}$ is $e+1$-independent. By the preceding lemma,

$$\begin{aligned} -y_0 = \sum_1^{de} y_i &\in \langle v_d(U_0)\rangle \cap \sum_{\pi\in\Pi}\sum_{i\in\pi} \langle v_d(U_i)\rangle \\ &\le \langle v_d(U_0)\rangle \cap \langle v_d(U_\pi) \mid \pi \in \Pi\rangle = 0. \end{aligned} \qquad \square$$

Remarks 2.10. 1. We used the rather weak inclusion $\langle v_d(A), v_d(B)\rangle \le \langle v_d(\langle A, B\rangle)\rangle$ for subspaces A, B of K^n: in general the right side is far larger than the left.

2. The proof shows that we did not need independence for all $e+1$-subsets of $\mathcal{U}$. For each $de+1$-subset $\mathcal{U}'$ of $\mathcal{U}$ we only needed a family $\mathbb{W}$ of independent $e+1$-subsets of $\mathcal{U}'$

such that the complement of each member of $\mathcal{U}'$ is partitioned by some of the members of $\mathbb{W}$.

The minimal version of this is as follows: each $de+1$-subset $\mathcal{U}'$ of $\mathcal{U}$ is equipped with the structure of a 2-design with $v = de + 1, k = e + 1, \lambda = 1$, such that each block is $e + 1$-independent. In this situation, "almost all" triples from $\mathcal{U}$ need *not* be independent and yet the proof shows that $v_d(\mathcal{U})$ nevertheless must be $de + 1$-independent.

2.4 Veronesean action. This section develops two algebraic results that play a small role in the proofs in this paper. One is that linear transformations of the space of homogeneous polynomials of degree one induce endomorphisms of degree zero of the polynomial algebra $K[X]$ (see Remark 2.12). The other is the often-quoted result that there is an action of $\mathrm{GL}(K^n)$ on K^N that stabilizes the set of Veronesean vectors, inducing an action permutation-equivalent to its action on K^n (used in Lemma 2.7), which we prove using polynomial rings and their morphisms.

2.4.1 Symmetric algebras and polynomial rings. Let V be an arbitrary vector space over K of dimension n. The symmetric algebra $S(V)$ is the K-algebra of symmetric tensors — that is, the free commutative K-algebra generated by the vector space V. It is a graded algebra

$$S(V) = K \oplus V \oplus S_2(V) \oplus \cdots$$

where $S_d(V)$ is the vector space spanned by the d-fold symmetric tensors. If $X = \{x_1, x_2, \ldots, x_n\}$ is any basis of V then $S(V)$ is isomorphic to the polynomial ring

$$A = K[X] = K \oplus A_1 \oplus \cdots \oplus A_d \oplus \cdots \tag{2.11}$$

where A_d is the vector space of homogeneous polynomials of degree d. Thus selecting the basis X of A_1 produces a basis $\{x^\alpha\}$ of A_d consisting of the monomials $x^\alpha := x_1^{a_1} \cdots x_n^{a_n}$, where $\alpha = (a_1, \ldots, a_n)$ is a sequence of non-negative integers for which $d = \sum a_i$.

2.4.2 The substitution-transformation $\boldsymbol{\rho_d}$. Let $f : A_1 \to W$ be any linear transformation, where W is a K-vector space. Then f extends to a K-algebra homomorphism $\bar{f} : A[X] \to S(W)$ of graded algebras, by mapping any polynomial $p(x_1, \ldots, x_n)$ to $p(f(x_1), f(x_2), \ldots, f(x_n))$, a "polynomial" in the algebra $S(W)$. By restriction of $\bar{f}$, we set

$$\rho_d(f) := \bar{f}|_{A_d} : A_d \to S(W)_d.$$

Its value at any monomial $x^\alpha = \prod x_i^{a_i}$ is $\prod f(x_i)^{a_i}$ in $S(W)_d$. Thus $\rho_d(f)$ is simply the linear morphism on A_d which results from *substituting* $f(x_i)$ for x_i.

Remark 2.12. Note that when $W \leq A_1$, f has been extended to an *endomorphism* of the algebra $K[X]$.

Now what happens when we apply ρ_d to a functional $\lambda : A_1 \to K$? Since $\bar{\rho}(\lambda)$ is defined by substitution of each x_i by the scalar $\lambda(x_i)$ in each polynomial of $K[X]$, it induces a functional $\rho_d(\lambda) : A_d \to K$ of A_d.

Lemma 2.13. *Some properties of ρ_d:*

(1) ρ_d *transforms any linear transformation* $A_1 \to A_1$ *to a linear transformation of* A_d *into itself. If* T *is the identity transformation of* A_1*, then* $\rho_d(T)$ *is the identity transformation of* A_d*.*

(2) *If* T *is in the group* $\mathrm{GL}(A_1)$*, then* $\rho_d(T)$ *is an invertible transformation of* A_d*.*

(3) *If* $\lambda : A_1 \to K$ *is a functional of* A_1*, then* $\rho_d(\lambda)$ *is a functional of* A_d*.*

(4) *If* $S : A_1 \to A_1$ *is a linear transformation and if* $T : A_1 \to W$*, where* W *is either the* K*-vector space* A_1 *or the* K*-algebra* K *itself, then*

$$\rho_d(T \circ S) = \rho_d(T) \circ \rho_d(S), \tag{2.14}$$

and is also K*-linear.*

(5) *Suppose* R*,* S *are linear transformations* $A_1 \to A_1$ *while* $T : A_1 \to W$ *is also* K*-linear, where* W *is as in* (4)*. Then*

$$\rho_d(T \circ S \circ R) = \rho_d(T) \circ \rho_d(S) \circ \rho_d(R). \tag{2.15}$$

Proof. The first part of (1) follows from the fact that $\rho_d(T)$ is defined by substituting $T(x_i)$ for x_i in any homogeneous polynomial of degree d. If T is the identity map on A_1, then substitution of x_i for x_i, does not change anything — that is, $\rho_d(T)$ is the identity transformation of A_d.

Statement (3) was explained in the paragraph preceding the lemma.

Statement (4) is also a consequence of $\rho_d(T)$ being defined by "substitution". Since S is a linear transformation of A_1 into itself, we may utilize the basis $X = \{x_1, \ldots, x_n\}$ to write

$$S(x_i) = \sum_{j=1}^{n} c_{ij} x_j, \text{ where } c_{ij} \in K, i \in [1, n].$$

Then $\rho_d(S)$ takes a monomial $x^\alpha = \prod x_i^{a_i}$ to $\prod(\sum_j c_{ij} x_j)^{a_i}$. Since $\rho_d(T)$ takes any monomial $\prod x_j^{b_j}$ of degree d to $\prod T(x_j)^{b_j}$, we see that

$$\rho_d(T) \circ \rho_d(S) : x^\alpha = \prod x_i^{a_i} \mapsto \prod_i (\sum c_{ij} T(x_j))^{a_i}. \tag{2.16}$$

But since $T \circ S$ takes x_i to $\sum_j c_{ij} T(x_j)$ we see that it also takes the monomial x^α to the right side of Equation (2.16). Thus we have

$$\rho_d(T \circ S) = \rho_d(T) \circ \rho_d(S),$$

establishing statement (4).

Remembering that W is permitted to be A_1 in statement (4), statement (5) follows from applying Equation (2.14) several times.

For statement (2) suppose T is invertible, so there exists a $T^{-1} : A_1 \to A_1$ such that $T \circ T^{-1} = \mathrm{id}_1$, the identity transformation of A_1. Now by (1) and (2.14), the identity transformation id_d of A_d can be written as

$$\mathrm{id}_d = \rho_d(\mathrm{id}_1) = \rho_d(T \circ T^{-1}) = \rho_d(T) \circ \rho_d(T^{-1}),$$

proving that $\rho_d(T)$ is invertible. □

2.4.3 Veronesean functionals. Suppose $\lambda \in A_1^*$ is the functional $A_1 \to K$ that takes the basis element x_i to the scalar t_i. Then $\rho_d(\lambda)$ is the functional of A_d that takes the basis element x^α to $t^\alpha \in K$. We call a functional of this type (that is, one that maps x^α to t^α where $t = (t_1, \ldots, t_n)$) a *Veronesean functional* of A_d. These are very special elements of A_d^*.

Theorem 2.17. *The group* $\rho_d(\mathrm{GL}(A_1))$ *induces an action on* A_d^* *that stabilizes the set of non-zero Veronesean functionals in* A_d^* *and induces on this set an action that is permutation-equivalent to the action of* $\mathrm{GL}(A_1)$ *on the non-zero vectors of* A_1^**. Explicitly, if* $T \in \mathrm{GL}(A_1)$ *acts on* A_1^* *by sending the functional* λ *to* $\lambda \circ T$*, then* $\rho_d(T)$ *acts on* $\rho_d(\lambda)$*, the corresponding Veronesean functional, by sending it to* $\rho_d(\lambda) \circ \rho_d(T) = \rho_d(\lambda \circ T)$*, another Veronesean functional.*

Proof. If $\lambda \in A_1^*$ and S and T are elements of $\mathrm{GL}(A_1)$, then by Lemma 2.13

$$\rho_d(\lambda \circ S \circ T) = \rho_d(\lambda) \circ \rho_d(S \circ T) = \rho_d(\lambda) \circ \rho_d(S) \circ \rho_d(T), \tag{2.18}$$

for any $\lambda \in A_1^*$.

By (2.18), we have a right action of $\rho_d(\mathrm{GL}(A_1))$ on the set of Veronesean functionals. Since these functionals are in one-to-one correspondence with the elements of A_1^*, the equation

$$\rho_d(\lambda \circ T) = \rho_d(\lambda) \circ \rho_d(T)$$

exhibits the permutation-equivalence of the action of $\mathrm{GL}(A_1)$ on A_1^* and the action of its isomorphic copy $\rho_d(\mathrm{GL}(A_1))$ on the Veronesean functionals of A_d^*. □

2.4.4 The Veronesean action.

Corollary 2.19. *There is an action of* $\mathrm{GL}(A_1)$ *on the non-zero Veronesean vectors of* K^N *that is permutation equivalent to its action on the non-zero vectors of* A_1^**, or, equivalently, its action as* $\mathrm{GL}(K^n)$ *on* K^n*.*

Proof. As before, $\alpha = (a_1, \ldots, a_n)$ is a sequence of non-negative integers summing to d, so that $x^\alpha = \prod x_i^{a_i}$ is a monomial of degree d. We define the scalar $t^\alpha = \prod t_i^{a_i}$ whenever $t = (t_1, \ldots, t_n) \in K^n$. If N is the number of monomials of degree d in n indeterminates, then the classical *Veronesean vectors* are the N-tuples of the form (t^α). Define the action of $\mathrm{GL}(A_1)$ on A_d^* by $f^T := f \circ \rho_d(T)$, for every functional $f \in A_d^*$ and $T \in \mathrm{GL}(A_1)$. Equation (2.14) shows that this meets the definition of a group action. By Theorem 2.17, this action stabilizes the set of Veronesean functionals.

The vector space isomorphism $\tau : K^N \to A_d^*$ which maps (y_α) to the functional of A_d whose value on x^α is y_α, bijectively maps the set $v_d(K^n)$ of Veronesean vectors of K^N to the set of Veronesean functionals of A_d^*. Conjugation by τ then transports this action of $\mathrm{GL}(A_1)$ on non-zero Veronesean functionals described in the previous paragraph to an equivalent action on the non-zero Veronesean vectors.

Similarly, let $\mu : K^n \to A_1^*$ be the vector space isomorphism which maps an n-tuple (t_i) to the functional on A_1^* whose value at x_i is t_i. Then conjugation by μ^{-1}

transports the action of $\mathrm{GL}(A_1)$ on A_1 to an action as the full linear group on K^n. One can express this in terms of the (vector) Veronesean mapping introduced in (2.1). Thus, setting $t = (t_i) \in K^n$,

$$\tau(v_d(t)) = \rho_d(\mu(t)).$$

Then for any $S \in \mathrm{GL}(A_1)$, we have

$$v_d(t)^S := \tau^{-1}\rho_d(\mu(t)) \circ \rho_d(g) \circ \tau = v_d(\mu^{-1} \circ g \circ \mu) := v_d((t^S)).$$

Equality of the extremal members of this equation justifies the last remark of the corollary. □

See [7, (2.3)] and [3, Theorem 2.10] for other approaches to this corollary.

3 Powerpoints

For the rest of this paper, $x_1, \dots, x_n$ will denote indeterminates over K, and $A := K[x_1, \dots, x_n]$ is the graded algebra (2.11), so that $A_i A_j \subseteq A_{i+j}$ for all non-negative integers i, j. (If P and Q are sets of polynomials then PQ *will denote the set of all products pq,* $p \in P$, $q \in Q$. In general, it is not a subspace even if P and Q are.) If we replace $\{x_1, \dots, x_n\}$ by any other basis of A_1 then we still obtain the same subspaces A_d (cf. Section 2.4).

In this and the next two sections we will be concerned with *powers* U^d *of subspaces* U *of* A_1. For now we will consider the set $P_d(A_1)$ of *powerpoints* U^d: the case in which U has dimension 1, in which case so does U^d.

3.1 Powerpoints and Veronesean points. It is elementary and standard that these two types of points are closely related for suitable characteristics:

Theorem 3.1. *If* $\mathrm{char}K > d$ *or* $\mathrm{char}K = 0$*, then there is a linear isomorphism* $\sigma\colon A_d \to K^N, N = \binom{d+n-1}{d}$*, such that*

(a) σ *sends the set of powerpoints in* A_d *to the set* $v_d(\boldsymbol{P}(K^n))$ *of Veronesean points in* K^N*, and*

(b) $\sigma([\eta((t_i))]^d) = (t^\alpha)$ *if* $\eta\colon K^n \to A_1$ *sends* $(t_i) \mapsto \sum_i t_i x_i$*.*

Here (t^α) *was defined in the preceding section.*

Proof. By the Multinomial Theorem, each powerpoint is spanned by a polynomial of the form

$$(t_1x_1 + \cdots + t_nx_n)^d = \sum_\alpha c(\alpha)t^\alpha x^\alpha$$

with $t_i \in K$ and multinomial coefficients $c(\alpha)$. All $c(\alpha)$ are non-zero in view of the assumed characteristic. Hence, the map σ defined by $\sigma\colon \sum_\alpha c(\alpha)k_\alpha x^\alpha \mapsto (k_\alpha)$, $k_\alpha \in K$, behaves as required. □

Proof of Theorem 1.3. Theorem 3.1 shows that linear independence of powerpoints corresponds to linear independence of Veronesean points. Now use Theorem 1.1. □

3.2 Small characteristic. We now use Remark 2.12 to prove additional independence results in small characteristics, a situation excluded in Theorem 1.3:

Theorem 3.2. *Assume that r is such that $|K| > (r+1)^2/2$ and $\binom{d}{i} \neq 0$ in K for $0 \le i \le r$. Then any $r+1$ powerpoints of A_d are independent.*

Proof. If $n = 2$ then all powerpoints are spanned either by x_1^d or

$$(x_2 + tx_1)^d = \sum_0^r \binom{d}{i} t^i x_2^{d-i} x_1^i + \sum_{r+1}^d \binom{d}{i} t^i x_2^{d-i} x_1^i$$

for some $t \in K$. Since $\binom{d}{i} \neq 0$ for $0 \le i \le r$, it suffices to note that the Vandermonde determinant $\det(t_j^i)_0^r \neq 0$ for any $r+1$ different elements $t_j \in K$.

If $n > 2$, assume that the result holds for $n-1$ indeterminates x_i. Consider $r+1$ distinct powerpoints $\langle f_1^d \rangle, \ldots, \langle f_{r+1}^d \rangle$ and a linear dependence relation $\sum_1^{r+1} k_i f_i^d = 0$, $k_i \in K$. Apply the endomorphism T of A fixing $x_1, \ldots, x_{n-1}$ and sending x_n to an arbitrarily chosen linear combination f of $x_1, \ldots, x_{n-1}$ (cf. Remark 2.12). This produces an identity $\sum_1^{r+1} k_i T(f_i)^d = 0$ in the ring $T(A) = K[x_1, \ldots, x_{n-1}]$. If the powerpoints $\langle T(f_i)^d \rangle$, $1 \le i \le r+1$, are distinct then induction implies that all k_i are 0.

If $\langle T(f_i)^d \rangle = \langle T(f_j)^d \rangle$ for some $i \neq j$ then $T\langle f_i \rangle = T\langle f_j \rangle$, so that $\langle f_i \rangle$ and $\langle f_j \rangle$ are congruent modulo $\ker T = \langle x_n - f \rangle$. Therefore, we only need to choose f so that the point $\langle x_n - f \rangle$ of A_1 does not lie on the line joining any two of our points $\langle f_i \rangle$. Assume that $|K| = q$ is finite. The union of those lines has size at most $\binom{r+1}{2}(q-1) + r + 1$. There are q^{n-1} points $\langle x_n - f \rangle$ as f varies. Since we have assumed that $q > (r+1)^2/2$, it follows that $q^{n-1} > \binom{r+1}{2}(q-1) + r + 1$ and a suitable f exists. When K is infinite the argument is even easier. □

A variant of the previous argument can be used in characteristic 2:

Theorem 3.3. *Let $K = \mathrm{GF}(2^m)$ and $d = 2^i + 1$ with $(i, m) = 1$ and $m \ge 3$. Then any 4 powerpoints of A_d are independent.*

Proof. If $n = 2$ and $d = 2^i + 1 = s + 1$, then each powerpoint is spanned by x_1^d or $(x_2 + tx_1)^d = x_2^d + tx_2^s x_1 + t^s x_2 x_1^s + t^d x_1^d$ for some $t \in K$. By [8, Lemma 21.3.14], the points $\langle (0,0,0,1) \rangle$ and $\langle (1, t, t^s, t^{s+1}) \rangle$, $t \in K$, form a 4-independent set. (NB: By contrast, in odd characteristic p, using $s = p^i$ the analogous set of points always has 4 dependent members, so that the analogue of the theorem does not hold.)

Now suppose that $n > 2$. We are given 4 distinct powerpoints $\langle f_1^d \rangle, \ldots, \langle f_4^d \rangle$, and we will assume a linear dependence relation $\sum_1^4 k_i f_i^d = 0$ for scalars k_i. Apply the endomorphism T of A fixing $x_1, \ldots, x_{n-1}$ and sending x_n to an arbitrarily chosen linear combination f of $x_1, \ldots, x_{n-1}$ (cf. Remark 2.12) in order to obtain an identity $\sum_1^4 k_i T(f_i)^d = 0$ in the ring $K[x_1, \ldots, x_{n-1}]$. If the powerpoints $\langle T(f_i)^d \rangle$ are distinct then we will have reduced the number of indeterminates x_i, as desired: the k_i are all 0.

As in the proof of the preceding theorem, we only need to choose f so that the point $\ker T = \langle x_n - f \rangle$ does not lie on the line joining any two of the points $\langle T(f_i) \rangle$. The union of those lines has size at most $\binom{4}{2}(q-1)+4$, where $q = 2^m$. There are q^{n-1} points $\langle x_n - f\rangle$ as f varies. Then a suitable f exists since $m \geq 3$ implies that $q^{n-1} > 6(q-1)+4$. □

We emphasize that the preceding theorem is a higher-dimensional generalization of a standard result in $\mathrm{PG}(3,q)$ [8, Lemma 21.3.14]. In fact, since this is only a question of four points an approach that is easier than the above simply plays with the space spanned by the f_i. As in Remark 2.4 there is an associated code that may be worth some study. For example, an elementary examination of possible dependence relations among the polynomials f_i^d shows that all minimum weight codewords arise from 2-spaces of K^n.

4 Independence of power subspaces

Let A be as in (2.11). Recall that, if T is a subspace of A_1, then $\langle T^d\rangle$ is the subspace of A_d spanned by all d-fold products of linear polynomials in T. Note that $\dim\langle T^d\rangle = \binom{d+\dim T-1}{d}$ since the monomials of degree d in a basis of T form a basis of $\langle T^d\rangle$.

Before we can prove Theorem 1.4 we need a few algebraic preliminaries. In this section we will use the uncommon notation $V^{(d)}$ to denote a cartesian power, in order to distinguish it from powers in rings.

4.1 The universal nature of symmetric tensors. For a K-vector space V and a commutative K-algebra B, we will need an almost-basic property of symmetric d-multilinear K-forms $V^{(d)} \to B$; that is, multilinear forms $f(v_1,\dots,v_d)$ assuming values in B and invariant under all permutations of the $v_i \in V$.

As in Section 2.4, we view the algebra $S(V)$ of symmetric tensors as the polynomial algebra $A = K[X]$ for a basis X of V, viewing V as A_1 and the subspace $S(V)_d$ spanned by the d-fold symmetric tensors as A_d. A standard and elementary universal property of symmetric tensors is the case $B = K$ of the following

Theorem 4.1. *Let $f\colon V^{(d)} \to B$ be a symmetric d-multilinear K-form with values in a commutative K-algebra B without zero divisors. Then there is a K-linear mapping $\bar f\colon A_d = S(V)_d \to B$ such that, for every $(v_1,\dots,v_d) \in V^{(d)}$,*

$$f(v_1,\dots,v_d) = \bar f(v_1 \cdot v_2 \cdots v_d).$$

Proof. Let K' be the field of fractions of B. Then $K' \otimes_K A = K'[X]$ and $K' \otimes_K A_d = K'[X]_d$.

Let $V' = K' \otimes_K V$. There is a symmetric multilinear K'-form f' determined by f together with a K-basis $X := \{x_1,\dots,x_n\}$ of A_1: for $v_i' = \sum_j \beta_{ij} x_j \in V'$, $i = 1,\dots,d$, $\beta_{ij} \in K'$, define

$$f'(v_1',\dots,v_d') := \sum_\sigma \Big[\prod_{i=1}^d \beta_{i\sigma(i)}\Big] f(x_{\sigma(1)},\dots,x_{\sigma(d)}),$$

where the above sum is over all sequences $\sigma = (\sigma(1), \dots, \sigma(d))$ with entries in $\{1, \dots, n\}$. This definition is forced by multilinearity and the requirement that $f'(v'_1, \dots, v'_d) = f(v'_1, \dots, v'_d)$ if all $v'_i \in V$.

By the field case of the theorem there is a K'-linear mapping $\bar{f}'\colon K' \otimes_K A_d = K'[X]_d \to K'$ such that, for all $v'_i \in V'$,

$$\bar{f}'(v'_1 v'_2 \cdots v'_d) = f'(v'_1, v'_2, \dots, v'_d).$$

If all $v'_i = v_i \in V = A_1$ then the right side is just $f(v_1, v_2, \dots, v_d) \in B$. Hence, the desired K-linear mapping is $\bar{f} := \bar{f}'|_{A_d}\colon A_d = K[X]_d \to B$, since $f(v_1, \dots, v_d) = f'(v_1, \dots, v_d) = \bar{f}'(v_1 \cdots v_n)$ for every $(v_1, \dots, v_d) \in V^{(d)}$. □

4.2 Proof of Theorem 1.4. We begin with the analogue of Lemma 2.7:

Lemma 4.2. *Suppose that* $1 \le r \le d$ *with* $d!/(d-r)! \neq 0$ *in* K. *If* T_0 *is a subspace of* A_1 *intersecting each of* r *subspaces* $T_1, \dots, T_r$ *of* A_1 *only in* 0, *then* $\langle T_0^d \rangle \cap \langle T_1^d, \dots, T_r^d \rangle = 0$.

Proof. It suffices to prove that there is a subspace N_0 of A_d such that N_0 contains T_j^d for all $j \ge 1$ and $N_0 \cap \langle T_0^d \rangle = 0$. Change coordinates in A_1 so that $x_1, \dots, x_k$ is a basis of T_0. Let $B := K[x_1, \dots, x_k]$, so that $B_d := B \cap A_d$ is $\langle T_0^d \rangle$.

If $1 \le j \le r$ then $T_0 \oplus T_j$ is a direct summand of A_1, so that there is a linear transformation $\lambda_j\colon A_1 \to A_1$ such that $\lambda_j(x_i) = x_i$, $i = 1, \dots, k$, and $\lambda_j(T_j) = 0$. (The behavior of λ_j on a complement to $T_0 \oplus T_j$ in A_1 is irrelevant to the proof.)

If $r < j \le d$ then $\lambda_j\colon A_1 \to A_1$ will be the identity map.

Let Θ be a (left) transversal for the pointwise stabilizer S_{d-r} of $1, \dots, r$ in the symmetric group S_d on $\{1, \dots, d\}$, so that $|\Theta| = d!/(d-r)!$. Define a d-multilinear K-form $L_0\colon A_1^{(d)} \to A_d$ by

$$L_0(v_1, \dots, v_d) := \sum_{\pi \in \Theta} \prod_{j=1}^{d} \lambda_j(v_{\pi(j)}). \tag{4.3}$$

We claim that L_0 *is symmetric*. For, let $\pi \in \Theta$, $\rho \in S_d$, and write $\rho\pi = \pi'\sigma$ with $\pi' \in \Theta$, $\sigma \in S_{d-r}$. Then

$$\begin{aligned}\prod_{j=1}^{r} \lambda_j(v_{\rho\pi(j)}) \prod_{j=r+1}^{d} \lambda_j(v_{\rho\pi(j)}) &= \prod_{j=1}^{r} \lambda_j(v_{\pi'\sigma(j)}) \prod_{j=r+1}^{d} v_{\pi'\sigma(j)} \\ &= \prod_{j=1}^{r} \lambda_j(v_{\pi'(j)}) \prod_{j=r+1}^{d} v_{\pi'(j)},\end{aligned}$$

since $\{\pi'\sigma(j) \mid r+1 \le j \le d\}$ is the complement in $\{1, \dots, d\}$ of $\{\pi'\sigma(j) \mid 1 \le j \le r\} = \{\pi'(j) \mid 1 \le j \le r\}$, and hence is $\{\pi'(j) \mid r+1 \le j \le d\}$. Consequently, ρ permutes the summands that define L_0, which proves the claim.

By Theorem 4.1, there is a linear transformation $\bar{L}_0\colon A_d \to B$ such that

$$\bar{L}_0(v_1 \cdots v_d) = L_0(v_1, \dots, v_d) = \sum_{\pi \in \Theta} \prod_{j=1}^{d} \lambda_j(v_{\pi(j)}) \tag{4.4}$$

for all $v_i \in A_1$. Clearly $\bar{L}_0(A_d) \subseteq A_d$. We will show that $N_0 := \ker \bar{L}_0$ *behaves as required at the start of this proof.*

Consider $j \geq 1$. If all $v_i \in T_j$, then $v_{\pi(j)} \in T_j \subseteq \ker \lambda_j$, and each summand on the right side of (4.4) is 0. Thus, $\bar{L}_0(T_j^d) = 0$.

It remains to determine the action of $\bar{L}_0$ on $\langle T_0^d \rangle$. We first calculate $\bar{L}_0$ on each monomial $x^\alpha = x_1^{a_1} \cdots x_k^{a_k}$, $\sum_i a_i = d$. Since $\lambda_j(x_i) = x_i$ for all $i \leq k$ and all j, (4.4) gives

$$\bar{L}_0(x^\alpha) = |\Theta| x_1^{a_1} \cdots x_k^{a_k}. \tag{4.5}$$

Here $|\Theta| = d!/(d-r)! \neq 0$ by hypothesis.

Thus, as x^α ranges over all monomials of degree d in $x_1, \dots, x_k$, their $\bar{L}_0$-images form a K-basis for B_d. Consequently, $\bar{L}_0$ restricted to $\langle T_0^d \rangle$ is a surjection $\langle T_0^d \rangle \to B_d$, and hence an isomorphism since $\dim\langle T_0^d \rangle = \dim B_d$. Thus, $N_0 \cap \langle T_0^d \rangle = \ker \bar{L}_0 \cap \langle T_0^d \rangle = 0$, as required. □

Proof of Theorem 1.4. The case $e = 1$ of Theorem 1.4 follows immediately from the preceding lemma. The general case is obtained exactly as in the proof of Theorem 1.2 near the end of Section 2.3. □

When $r = d$, an entirely different proof of Theorem 1.4 is obtained by combining Theorems 2.6 and 3.1. Theorem 1.4 clearly contains Theorem 1.3 as a special case, but it does not quite contain Theorem 3.2: the requirements on r are less stringent in the latter result. (For example, if $d = 5$ and the characteristic is $r = 3$, then 3 divides $5!/(5-3)!$ but none of the binomial coefficients $\binom{5}{i}$.)

5 r-independence of power subspaces

In this section we will use subspaces of polynomials to prove a (weak) variation on the results in the preceding section:

Theorem 5.1. *Let $r \geq 1$. If $d > 1$ is not a power of* $\operatorname{char} K$ *and if $\mathcal{T}$ is any r-independent set of subspaces of A_1, then $\{\langle T^d \rangle \mid T \in \mathcal{T}\}$ is an $r+1$-independent set in A_d.*

5.1 Calculating with spaces of polynomials. We will make frequent use of the following elementary observation and its consequences.

Proposition 5.2. *In* (2.11) *let U_1 and U_2 be subspaces of A_1 such that $U_1 \cap U_2 = 0$. If $d > 1$, then*

$$\langle A_{d-1} U_1 \rangle \cap \langle A_{d-1} U_2 \rangle = \langle A_{d-2} U_1 U_2 \rangle \tag{5.3}$$

$$\langle (U_1 + U_2)^d \rangle = \bigoplus_{k=0}^{d} \langle U_1^k U_2^{d-k} \rangle \tag{5.4}$$

$$\langle U_1^d \rangle \cap \langle A_{d-1} U_2 \rangle = 0. \tag{5.5}$$

Proof. Let $X_1 := \{x_1, \ldots, x_\ell\}$ and $X_2 := \{x_{\ell+1}, \ldots, x_m\}$ be respective bases for U_1 and U_2. Let $X \supseteq X_1 \dot\cup X_2$ be a basis of V.

Both sides of each of the above equations are subspaces of A_d. The left side of (5.4) is the subspace of A_d spanned by all monomials of degree d with factors chosen from $X_1 \dot\cup X_2$. Partitioning these monomials according to the number of factors of X_1 they contain proves (5.4).

For (5.5) note that $\langle U_1^d \rangle$ is spanned by monomials in X_1 of degree d, while $\langle A_{d-1} U_2 \rangle$ is spanned by monomials containing at least one factor from X_2.

For (5.3), consider the following pairwise disjoint sets of monomials in X:

- $Y_i \subset A_{d-1} X_i$ is the set of monomials in X with at least one factor from X_i and no factor from X_{3-i}, for $i = 1, 2$, and
- $Y_{12} \subset A_{d-2} X_1 X_2$ is the set of all monomials in X having at least one factor from X_1 and at least one from X_2.

It follows that $\langle Y_1 \rangle \cap \langle Y_2 \rangle = 0$ and $\langle A_{d-1} U_i \rangle = \langle Y_i \rangle \oplus \langle Y_{12} \rangle$ for $i = 1, 2$. Consequently, $\langle A_{d-1} U_1 \rangle \cap \langle A_{d-1} U_2 \rangle = \langle Y_{12} \rangle = \langle A_{d-2} U_1 U_2 \rangle$. □

We can now show that the dth power operator commutes with intersections:

Corollary 5.6. *For any subspaces B and C of A_1,*

$$\langle B^d \rangle \cap \langle C^d \rangle = \langle (B \cap C)^d \rangle. \tag{5.7}$$

Proof. We may assume that $d > 1$. Set $C_1 := B \cap C$, and choose a subspace C_2 such that $C = C_1 \oplus C_2$. Since $d > 1$, (5.4) yields

$$\langle C^d \rangle = \bigoplus_{j=0}^{d} \langle C_1^j C_2^{d-j} \rangle = \langle C_1^d \rangle \oplus \langle C_2 C^{d-1} \rangle. \tag{5.8}$$

Since $B \cap C_2 = B \cap (C \cap C_2) = C_1 \cap C_2 = 0$, (5.5) forces $\langle B^d \rangle \cap \langle C^{d-1} C_2 \rangle = 0$. On the other hand, $\langle B^d \rangle$ contains $\langle C_1^d \rangle$, the first summand at the end of (5.8). Thus,

$$\langle B^d \rangle \cap \langle C^d \rangle = \langle C_1^d \rangle = \langle (B \cap C)^d \rangle.$$ □

5.2 Proof of Theorem 5.1. We begin with a special case:

Proposition 5.9. *Suppose that $A_1 = T_1 \oplus \cdots \oplus T_r$ with $\dim T_i = s$, and that T_{r+1} is an s-space in A_1 such that the set $\{T_1, \ldots, T_{r+1}\}$ is r-independent. Then $\langle T_1^d, \ldots, T_r^d \rangle = \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, and one of the following holds:*

(1) $\langle T_{r+1}^d \rangle \cap (\langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle) = 0$, *or*

(2) $\mathrm{char} K$ *is a prime p, d is a power of p, and* $\dim[\langle T_{r+1}^d \rangle \cap (\langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle)] = s$.

Proof. Let $\{y_1, \ldots, y_s\}$ be a basis of T_{r+1}. If x_{ij} is the projection of y_j into T_i, $1 \le i \le r$, then $y_j = \sum_i x_{ij}$ with each $x_{ij} \neq 0$ due to r-independence, and $X_i := \{x_{ij} \mid 1 \le j \le s\}$ is a basis of T_i for $i \le r$. Then $\langle T_1^d, \ldots, T_r^d \rangle = \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$ since $\langle T_i^d \rangle$ is spanned by monomials in X_i.

If $0 \neq f \in \langle T_{r+1}^d \rangle \cap (\langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle)$, then

$$f = \sum_{\alpha} k_\alpha \prod_{j=1}^{s} y_j^{a_j}, \tag{5.10}$$

where $k_\alpha \in K$ and the sum is indexed by all $\alpha = (a_1, \dots, a_s)$ with all $a_i \geq 0$ and $\sum_i a_i = d$. Since $f \in \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, when expanded as a linear combination of monomials in $\bigcup_i X_i$ of degree d the coefficients of the monomials in (5.10) with "mixed terms" — i.e., monomials containing members of $X_i X_j$ with $i \neq j$ — must be zero.

Let $\alpha = (a_1, \dots, a_s)$ be as above and suppose that (at least) two of the numbers a_ℓ and a_m are positive ($\ell \neq m$). Then the product $\prod_{j=1}^{s} x_{*j}^{a_j}$, where $* = 1$ except that $* = 2$ when $j = m$, contains a term in $X_\ell X_m$, and this product occurs only once in (5.10). Since $f \in \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, it follows that the coefficient k_α in (5.10) is zero for all $\alpha = (a_1, \dots, a_s)$ having at least two non-zero terms.

Now (5.10) reduces to

$$f = \sum_i k_i y_i^d. \tag{5.11}$$

By the Binomial Theorem, f involves a non-zero mixed term containing a member of $X_1 X_2$ unless K has characteristic $p > 0$ and $d = p^e$ for some e. Then $y_i^d = (\sum_i x_{ij})^d = \sum_i x_{ij}^d \in \langle T_1^d \rangle \oplus \cdots \oplus \langle T_r^d \rangle$, and (2) holds. □

Proof of Theorem 5.1. It suffices to prove that, if $T_1, \dots, T_{r+1}$ are distinct members of $\mathcal{T}$, then

$$\langle T_{r+1}^d \rangle \cap \langle T_1^d, \dots, T_r^d \rangle = 0 \tag{5.12}$$

(compare Lemma 4.2). Set $T := \langle T_1, \dots, T_r \rangle = \langle T_1 \rangle \oplus \cdots \oplus \langle T_r \rangle$ (by r-independence) with corresponding projections $\pi_i \colon T \to T_i$.

Let $U_{r+1} := T_{r+1} \cap T$. We may assume that $U_{r+1} \neq 0$, as otherwise $\langle T_1, \dots, T_{r+1} \rangle = T_1 \oplus \cdots \oplus T_{r+1}$ and (5.12) is clear (use Corollary 5.6 with $C = T$).

Once again let $\{y_1, \dots, y_s\}$ be a basis for U_{r+1} and $x_{ij} := \pi_i(y_j)$, $1 \leq j \leq s$, $1 \leq i \leq r$. If, for some i, $\{x_{ij} \mid 1 \leq j \leq s\}$ is linearly dependent, then some $y \neq 0$ in U_{r+1} satisfies $\pi_i(y) = 0$, and r-independence produces the contradiction $y \in U_{r+1} \cap \ker \pi_i \leq T_{r+1} \cap \oplus_{j \neq i} T_j = 0$.

Thus, $U_i := \langle x_{ij} \mid 1 \leq j \leq s \rangle$ is an s-subspace of T_i for each i. Since $y_j \in T$ we have $y_j \in \langle \pi_i(y_j) \mid 1 \leq i \leq r \rangle$ and hence $U_{r+1} \leq U := \langle U_1, \dots, U_r \rangle$. Since $U_i \leq T_i$ and $\{T_1, \dots, T_{r+1}\}$ is an r-independent family, so is the family $\{U_1, \dots, U_{r+1}\}$ of subspaces of T. Apply Proposition 5.9 to this family with U_i and U in the roles of T_i and A_1:

$$\langle U_{r+1}^d \rangle \cap \langle U_1^d, \dots, U_r^d \rangle = 0, \tag{5.13}$$

which resembles our goal (5.12).

We claim that

$$\langle T_{r+1}^d \rangle \cap \langle T_1^d, \dots, T_r^d \rangle \leq \langle U_{r+1}^d \rangle \cap \langle T_1^d, \dots, T_r^d \rangle \tag{5.14}$$

(and later we will show that the right hand side is 0). For, select a complement W_i to U_i in T_i for $i = 1, \dots, r+1$, and let $W := \langle W_1, \dots, W_r\rangle$. Then $T = U \oplus W$ and $\langle T_1, \dots, T_{r+1}\rangle = U \oplus W \oplus W_{r+1}$. By Corollary 5.6, $\langle T_{r+1}^d\rangle \cap \langle T_1^d, \dots, T_r^d\rangle \le \langle T_{r+1}^d\rangle \cap \langle T^d\rangle = \langle (T_{r+1} \cap T)^d\rangle = \langle U_{r+1}^d\rangle$, which proves (5.14).

By (5.5), $\langle U^d\rangle \cap \langle A_{d-1}W\rangle = 0$. Since $\langle U_1^d, \dots, U_r^d\rangle \le \langle U^d\rangle$, by the modular law

$$\begin{aligned}\langle U^d\rangle \cap [\langle U_1^d, \dots, U_r^d\rangle) \oplus \langle A_{d-1}W\rangle] &= \langle U_1^d, \dots, U_r^d\rangle \oplus (\langle U^d\rangle \cap \langle A_{d-1}W\rangle)\\ &= \langle U_1^d, \dots, U_r^d\rangle,\end{aligned}$$

and hence (since $T_i^d = (U_i \oplus W_i)^d \le U_i^d \oplus A_{d-1}W_i$ by (5.4))

$$\begin{aligned}\langle U^d\rangle \cap \langle T_1^d, \dots, T_r^d\rangle &\le \langle U^d\rangle \cap [\langle U_1^d, \dots, U_r^d\rangle + \langle A_{d-1}W_1, \dots, A_{d-1}W_k\rangle]\\ &= \langle U^d\rangle \cap [\langle U_1^d, \dots, U_r^d\rangle + \langle A_{d-1}W\rangle] = \langle U_1^d, \dots, U_r^d\rangle.\end{aligned}$$

Since $\langle U_{r+1}^d\rangle \le \langle U^d\rangle$, (5.13) yields

$$\begin{aligned}\langle U_{r+1}^d\rangle \cap \langle T_1^d, \dots, T_r^d\rangle &= \langle U_{r+1}^d\rangle \cap [\langle U^d\rangle \cap \langle T_1^d, \dots, T_r^d\rangle]\\ &\le \langle U_{r+1}^d\rangle \cap \langle U_1^d, \dots, U_r^d\rangle = 0.\end{aligned}$$

Now (5.14) implies (5.12), as required. □

6 Generalized dual arcs and polynomial rings

6.1 Definitions. A *generalized dual arc of* $V = K^n$ *with vector dimensions* $(n, n_1, \dots, n_d)$ is a set $\mathcal{D}$ of n_1-subspaces of V such that the intersection of any j of them has dimension $n_j > 0$, $j = 2, \dots, d$, and the intersection of any $d+1$ of them is 0. This notion was introduced in [10, 11], but using projective dimension instead of vector space dimension. When $d = 2$ and $n_d = 1$, $\mathcal{D}$ is a *dual arc*.

Suppose $\mathcal{D}$ is a generalized dual arc with vector dimensions $(n, n_1, \dots, n_d)$. If $D \in \mathcal{D}$, then $\mathcal{D}' := \{D \cap D' \mid D' \in \mathcal{D} - \{D\}\}$ is a generalized dual arc with vector dimensions $(n_1, \dots, n_d)$. In general, this procedure can be iterated.

For any subset Δ of a sublattice $\mathbf{L}$ of the lattice $\mathbf{L}(V)$ of all subspaces of V, let $\mathbf{L}(\Delta)$ denote the sublattice generated by Δ (the smallest sublattice containing Δ), and let $\mathbf{I}(\Delta)$ denote the *ideal* generated by Δ (the set of all elements of $\mathbf{L}(\Delta)$ that are bounded above by at least one element of Δ). We call Δ *regular* if $V = \langle\Delta\rangle$, and, for each intersection U of finitely many members of Δ,

$$U = \langle U \cap D \mid D \in \Delta,\ D \not\subseteq U\rangle. \tag{6.1}$$

We call Δ *strongly regular* if it is regular and

$$U \cap \langle D_1, \dots, D_\ell\rangle = \langle U \cap D_i \mid 1 \le i \le \ell\rangle \tag{6.2}$$

for any $D_1, \dots, D_\ell \in \Delta$ and any subspace U that is an intersection of finitely many members of Δ. There are many stronger versions of this concept possible[2], but this definition is geared to conform to the definitions appearing in [10, 11].

[2]For example, U might be restricted to range over all elements of $\mathbf{L}(\Delta) \cap \mathbf{I}(\Delta)$.

6.2 An elementary construction. Consider (2.11), fix $d \geq 2$, and let $\mathcal{D}$ be the following set of K-subspaces of A_d:

$$\mathcal{D} := \{A_{d-1}y \mid 0 \neq y \in A_1\}. \tag{6.3}$$

If $z = \alpha y, \alpha \in K^*$, then $A_{d-1}y = A_{d-1}z$, so $\mathcal{D}$ is parametrized by the 1-spaces of A_1. If $D_j := A_{d-1}y_j$, $1 \leq j \leq d$, are distinct elements of $\mathcal{D}$, *we claim that*

$$D_1 \cap D_2 \cap \cdots \cap D_j = A_{d-j}y_1 \cdots y_j. \tag{6.4}$$

Clearly the right side of (6.4) is contained in the left. The left side is precisely all homogeneous polynomials of degree d that are divisible by all of the polynomials $y_1, \ldots, y_j$. Since the polynomials y_i are primes of the unique factorization domain A no two of which are associates, each polynomial on the left side of (6.4) is a multiple of $y_1 \cdots y_j$ and so lies in the right side, as claimed.

The dimension of the subspace in (6.4) is $\dim A_{d-j}$. This proves

Theorem 6.5. *$\mathcal{D}$ is a generalized dual arc in A_d with vector dimensions*

$$\left(\binom{n+d-1}{d}, \binom{n+d-2}{d-1}, \ldots, n_{d-1} = \binom{n}{1} = n, n_d = 1\right).$$

In [11] there are objects with similar properties constructed without the use of polynomials. Note that $\langle \mathcal{D} \rangle = A_d$.

6.3 Regularity properties of this construction: Examples.

Example 6.6. *Strong regularity can fail for* (6.3). Suppose $n = \dim A_1 \geq 3$ and $d = 2$. We will see in the next Example that $\mathcal{D}$ in (6.3) is regular since d is small. However, *$\mathcal{D}$ is not strongly regular*. For, consider the subspaces $D_i := A_1 x_i$, $i = 1, 2$, and $U := A_1(x_1 + x_2) < \langle D_1, D_2 \rangle$ belonging to $\mathcal{D}$.

Since $U \cap D_i = \langle x_i(x_1 + x_2) \rangle$, $i = 1, 2$, $\dim\langle U \cap D_1, U \cap D_2 \rangle = 2 < n = \dim U = \dim(U \cap \langle D_1, D_2 \rangle)$, so that $U \cap \langle D_1, D_2 \rangle \neq \langle U \cap D_1, D_2 \cap D' \rangle$ and (6.2) fails, as required.

Example 6.7. *Even regularity can fail for* (6.3), *but only for finite K and enormous d.* Since $\langle \mathcal{D} \rangle = A_d$, in order to see this we must examine (6.1).

Each $U \neq 0$ in (6.1) has the form $U := \bigcap_{i=1}^{k} (A_{d-1}y_i) = A_{d-k}\pi$ for distinct $\langle y_1 \rangle, \ldots, \langle y_k \rangle$ in A_1 and $\pi := y_1 \cdots y_k$, where $d \geq k$. Regularity asserts that each such U is spanned by the subspaces $U \cap (A_{d-1}y)$ where y ranges over the set Y' of linear polynomials *not* in $\langle y_1 \rangle \cup \cdots \cup \langle y_k \rangle$.

By (6.4),

$$\langle U \cap (A_{d-1}y) \mid y \in Y' \rangle = \langle A_{d-k-1}y\pi \mid y \in Y' \rangle = \langle A_{d-k-1}Y' \rangle \pi.$$

In particular, if $A_1 = \langle Y' \rangle$ then $\langle U \cap (A_{d-1}y) \mid y \in Y' \rangle = A_{d-k}\pi = U$.

Thus, if $\mathcal{D}$ is *not* regular then, for some U, the corresponding set Y' must span a proper subspace H of A_1. Then the k-set $\{\langle y_1 \rangle, \ldots, \langle y_k \rangle\}$ must contain all 1-spaces of

A_1 not in H. Thus, K is finite and $k \geq |K|^{n-1}$, the number of points not in a hyperplane of $\mathbf{P}(A_1)$. Then $d \geq k \geq |K|^{n-1}$.

Conversely, if $d \geq |K|^{n-1}$ choose $U := \bigcap_{i=1}^{k}(A_{d-1}y_i)$, where $\{\langle y_1\rangle, \ldots, \langle y_k\rangle\}$ consists of all $k = |K|^{n-1}$ points outside the hyperplane $H = \langle x_1, \ldots, x_{n-1}\rangle$ of A_1. Then the previous argument produces a subspace

$$\langle U \cap (A_{d-1}y) \mid y \in Y' \rangle = \langle A_{d-k-1}H \rangle \pi$$

of dimension smaller than that of $U = A_{d-k}\pi$, which proves nonregularity.

Thus, we have proved

Proposition 6.8. *A generalized dual arc $\mathcal{D}$ in* (6.3) *need not be strongly regular.*
Moreover, $\mathcal{D}$ is not regular if and only if K is a finite field $\mathrm{GF}(q)$ *and $d \geq q^{n-1}$.*

6.4 A generalization of the construction. Let I_k denote the set of 1-spaces spanned by the various homogeneous polynomials of degree k that are powers of irreducible polynomials (the degrees of these irreducible polynomials are allowed to vary). If $d \geq k$, then

$$\mathcal{D} := \{A_{d-k}y \mid y \in I_k\}$$

is a set of $|I_k|$ subspaces of A_d, each of dimension $\dim A_{d-k}$. Choose the positive integer c such that $0 \leq d - kc < k$. By unique factorization in A, if $2 \leq m \leq c$ then the intersection of any m members of $\mathcal{D}$ is a subspace of dimension $\dim A_{d-mk}$, and the intersection of any $c + 1$ members of $\mathcal{D}$ is 0. Thus,

Theorem 6.9. *$\mathcal{D}$ is a generalized dual arc with vector dimensions*

$$(\dim A_d, \dim A_{d-k}, \ldots, \dim A_{d-ck}).$$

Theorem 6.5 is the special case $k = 1$. This time, $\mathcal{D}$ need not span A_d (e.g., if $k = \mathrm{char}K$).

6.5 Further variations.

6.5.1 An example leading to a 3-independent family.

Lemma 6.10. *If $\dim_K A_1 = n$ and H is a k-space in A_1, then A_1H is a subspace of A_2 of dimension $kn - \binom{k}{2}$.*

Proof. Clearly

$$\langle A_1H \rangle = A_1x_1 + A_1x_2 + \cdots + A_1x_k, \tag{6.11}$$

assuming that $\{x_1, \ldots, x_k\}$ is a basis of H. The intersection of any two summands on the right side of (6.11) is a 1-space while the intersection of any three is 0. Consequently, by elementary linear algebra and inclusion-exclusion,

$$\dim\Big(\sum_{i=1}^{k} A_1x_i\Big) = \sum_{i=1}^{k} \dim A_1x_i - \sum_{1\leq i<j\leq k} \dim(A_1x_i \cap A_1x_j) = kn - \binom{k}{2}. \qquad \square$$

Lemma 6.12. *Let $\mathcal{S}$ be a partial spread of k-subspaces of A_1, where $\dim A_1 = 2k$.*

(1) *If $H \in \mathcal{S}$ then $\dim\langle A_1 H\rangle = k(3k+1)/2$.*
(2) *For any distinct $H_1, H_2 \in \mathcal{S}$, $\langle A_1 H_1\rangle \cap \langle A_1 H_2\rangle = \langle H_1 H_2\rangle$ has dimension k^2.*
(3) *For any distinct $H_1, H_2, H_3 \in \mathcal{S}$, $\dim\big(\bigcap_{i=1}^{3}\langle A_1 H_i\rangle\big) = \binom{k}{2}$.*

Proof. (1) follows from Lemma 6.10 with $n = 2k$.

(2) follows from (5.3).

(3) We may assume that $X_1 = \{x_1, \dots, x_k\}$ and $X_2 = \{x_{k+1}, \dots, x_{2k}\}$ are bases of H_1 and H_2, respectively, such that $\{x_i + x_{k+i} \mid 1 \le i \le k\}$ is a basis of H_3. Using (2), if $y \in \langle A_1 H_1\rangle \cap \langle A_1 H_2\rangle \cap \langle A_1 H_3\rangle = \langle H_1 H_2\rangle \cap \langle A_1 H_3\rangle$ then we can write

$$y = \sum_{i=1}^{k} y_i(x_i + x_{k+i}) \tag{6.13}$$

where $y_i = \sum_{j=1}^{k}(\alpha_{ij}x_j + \beta_{ij}x_{k+j})$. Since X_1X_2 is a basis of $\langle H_1H_2\rangle$, for $1 \le i, j \le k$ the coefficients of x_i^2, x_ix_j and $x_{k+i}x_{k+j}$ in the polynomial y must be 0:

$$\alpha_{ii} = \beta_{ii} = 0, \quad \alpha_{ij} = -\alpha_{ji}, \quad \text{and} \quad \beta_{ij} = -\beta_{ji}, \quad 1 \le i, j \le k. \tag{6.14}$$

The coefficient of x_ix_{k+j} in (6.13) is $\beta_{ij} + \alpha_{ji} = \beta_{ij} - \alpha_{ij}$, by (6.14). Similarly, the coefficient of x_jx_{k+i} is $\alpha_{ij} - \beta_{ij}$. Then

$$y = \sum_{1 \le i < j \le k} (\beta_{ij} - \alpha_{ij})(x_ix_{k+j} - x_jx_{k+i}). \tag{6.15}$$

Conversely, by reversing the steps, it is clear that any y as in (6.15) lies in $\langle H_1H_2\rangle = \langle A_1H_1\rangle \cap \langle A_1H_2\rangle$ and also in $\langle A_1H_3\rangle$. It follows that the desired dimension is the number of pairs i, j such that $1 \le i < j \le k$. □

Again consider a partial spread $\mathcal{S}$ and $\mathcal{D} = \{A_1H \mid H \in \mathcal{S}\}$ in Lemma 6.12: a set of subspaces of the $\binom{2k+1}{2}$-space A_2 of dimension $k(3k+1)/2$, any two meeting in a space of dimension k^2 and any three meeting in a subspace of dimension $k(k-1)/2$. Subtracting these numbers from $\dim A_2$, we see that these intersections have codimensions $\binom{k+1}{2}$, $2\binom{k+1}{2}$, and $3\binom{k+1}{2}$, respectively. This implies the following:

Proposition 6.16. *Let $\mathcal{S}$ be a partial spread of k-spaces of the $2k$-space A_1, and consider the set $\mathcal{D} = \{A_1H \mid H \in \mathcal{S}\}$ of subspaces of A_2. The dual set $\mathcal{D}^*$ is a 3-independent family of $\binom{k+1}{2}$-spaces in the dual $\binom{2k+1}{2}$-space A_2^*.*

Intersection dimensions of four members of the preceding set $\mathcal{D}$ are not, in general, constant.

6.5.2 Three 4-space structures in dimension 10.

Let $K = \mathrm{GF}(q)$.

Example 6.17. Let $n = \dim A_1 = 3$ and $\mathcal{D}_1 = \{A_2 f \mid 0 \neq f \in A_1\}$. Then $\mathcal{D}_1$ is a set of $1+q+q^2$ 6-subspaces of the 10-space A_3. Any two members of $\mathcal{D}_1$ meet at a 3-space and so generate a 9-space. Since the intersection of any three is a 1-space, any three span a space of dimension $6+6+6-3-3-3+1 = 10$, hence the entire space. Then in the dual space A_3^* we obtain a set $\mathcal{D}_1^*$ of $1+q+q^2$ 4-spaces, any two of which meet at a 1-space, and any three of which meet at 0. Thus, *$\mathcal{D}_1^*$ is a dual arc in $A_3(K^{(3)})^*$ with vector dimensions* $(10,4,1)$.

Example 6.18. If $n = 4$ then $\mathcal{D}_2 := \{A_1 f \mid 0 \neq f \in A_1\}$ consists of $1+q+q^2+q^3$ 4-spaces of the 10-space A_2. Any two members of $\mathcal{D}_2$ intersect at a 1-space, so $\mathcal{D}_2$ is *another dual arc in $A_2(K^{(4)})$ with vector dimensions* $(10,4,1)$*, but it has more members than $\mathcal{D}_1^*$.*

Example 6.19. Let V be any vector space over K of dimension n. The exterior algebra $\bigwedge V = K \oplus V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus \cdots$ is a graded algebra that can replace the polynomial ring in the construction in Section 6.2. While we obtain a set of subspaces of a graded component of this algebra with good pairwise intersections, triple intersections show that it is no longer a generalized dual arc.

Suppose $\dim V = 5$. Then $\mathcal{D}_3 := \{V \wedge \langle v \rangle \mid 0 \neq v \in V\}$ is a set of $1+q+q^2+q^3+q^4$ 4-spaces of the 10-space $V \wedge V$. The intersection of any two members of $\mathcal{D}_3$ is a 1-space. Any 1-space that is the intersection of two members of $\mathcal{D}_3$ in fact lies in $1+q$ members of $\mathcal{D}_3$, so this is *not a dual arc*.

Besides illustrating a use both of duals and exterior algebra, these last three examples possess a numerology that raises a question. In view of the fact that the 10-spaces $A_3(K^3)^*$, $A_2(K^4)$ and $K^5 \wedge K^5$ are isomorphic, is there a relationship among the structures $\mathcal{D}_1^*$, $\mathcal{D}_2$ and $\mathcal{D}_3$?

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