SOFT-PLANES

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Abstract. This note studies projective planes having a collineation group fixing a flag (∞, L_∞) and transitive on the flags (w, W) with $w \notin L_\infty$ and $\infty \notin W$.

1. Introduction

We present a construction of finite projective planes using groups. This does not even require knowing what a finite projective plane is. What is needed is the construction of groups with specific properties (Theorem 2.1).

The planes constructed are *soft-planes*: "sort-of-flag-transitive planes", meaning that there is a collineation group (*a soft-group*) fixing a flag (∞, L_∞) and transitive on the flags (w, W) with $w \notin L_\infty$ and $\infty \notin W$. There are known families of examples of such planes. Our initial goal was to obtain examples not of prime power order (cf. [Pi, p. 294], [Ry, p. 25]), but none has yet been found. A secondary goal was to find examples of prime power order that are neither translation planes nor dual translation planes, but that was also not successful.

Section 2 proves the relationship between soft-planes and groups using undergraduate algebra, while Section 3 contains elementary observations concerning the planes and the groups. Section 3.1 summarizes the purely group-theoretic restrictions presently known, such as those involving normal structure. This paper has an old-fashioned point of view: its methods are in the 55-year-old book [De].

Historical background. Higman and McLaughlin [HM] constructed a flag-transitive geometry using the cosets of two subgroups A and B of a group G, with a "point" Ax on a "line" By iff $Ax \cap By \neq \emptyset$. One of their goals was to study finite projective planes having a flag-transitive collineation group. The present paper arose from the realization that the ideas in [HM] could be used assuming less transitivity. The "new" idea is in Theorems 2.1 and 2.9, which are based on [HM, Lemma 4]. In the 1960s this idea might have seemed novel, but now it seems straightforward.

2. Construction of planes

This note concerns the following result and its converse (Theorem 2.9). The proofs involve little more than the definition of a projective plane and elementary properties of cosets.

Theorem 2.1. *There is a projective plane π of order $n > 1$ if there is a group G having subgroups A, B and M such that*

(1) $|A| = |B| = |M| = nk$ *and* $|G| = n^3k$ *for* $k = |A \cap B|$,
(2) AM *and* BM *are subgroups of order* n^2k,
(3) $G = AMB$, *and*
(4) $AB \cap BA = A \cup B$.

Moreover, π *is a soft-plane with soft-group* G.

Here $AM := \{am \mid a \in A, m \in M\}$.

Remark 2.2. The intriguing aspect of this result is that there is no obvious arithmetic reason to expect n to be a prime power. It seems surprising that no version of Theorem 2.1 is presently in print.

Even though a question about some projective planes is equivalent to one about some groups, the latter setting ought to provide many more tools, involving (normal) subgroups, quotient groups, group extensions, etc. However, as indicated earlier, these additional facilities have not yet led to any new planes.

It seems likely that G is solvable (see Proposition 3.9(iii)).

Examples 2.3. (Heisenberg groups) Using entries from $\mathbb{F}_n$ when n is a prime power,

$$G := \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},\ A := \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M := \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

behave as in the theorem; they produce the desarguesian plane of order n. More general Heisenberg groups arise by replacing $\mathbb{F}_n$ by any finite *semifield* (nonassociative division algebra) [Cr, Hi].

Examples 2.4. Up to duality (i. e., interchanging A and B) the *known* planes arising via (1-4) are the following translation planes:

(i) semifield planes [Cr, Hi] (Examples 2.3),
(ii) likeable planes with $n = q^2$ when $q > 2$ is a prime power with $q \equiv 2 \pmod 3$ or when $q > 5$ is a power of 5 [Ka] (cf. [Be, Wa]),
(iii) a plane due to Sherk [Sh, NRS] with $n = 3^3$,
(iv) Lüneburg-Tits planes [Lü] with $n = q^2$ for $q = 2^{2e+1} > 4$,
(v) a plane due to Jha-Johnson [JJ] with $n = 8^2$ (cf. [BJJM, p. 13]), and
(vi) a plane due to Biliotti-Menichetti [BM] with $n = 8^2$.

Each plane (ii-vi) was found at least 40 years ago without the use of a computer.

Remark 2.5. We briefly describe further properties of some of the examples in (i-vi) *for which* $M \lhd G$ *and* G *is a soft-p-group of order* n^3.

In (i) $M = Z(G)$ and there are many possible nonisomorphic groups G [Hi]. For each of them G/M is elementary abelian.

In (ii) A is abelian: elementary abelian when n is odd but homocyclic of exponent 4 when n is even.

In (iii) $A \cong \mathbb{Z}_3 \times \mathbb{Z}_9$. In (v,vi), A is nonabelian [BJJM, pp. 12-13].

In (iv) A is a Sylow 2-subgroup of a Suzuki group $\mathrm{Sz}(q)$ and BM is the natural 4-dimensional module for $\mathrm{Sz}(q)$.

The planes (ii-vi) were obtained by requiring that BM is an elementary abelian normal subgroup of G (but $AM \ntrianglelefteq G$). Moreover, $G \lesssim \mathrm{A\Gamma L}(4, q)$ in (ii,iv,v,vi) or $\mathrm{AGL}(6, 3)$ in (iii), G is a p-group of class 4 or 5, and $Z(G) = C_M(A)$ has order $\sqrt{n}$ in (ii,iv,v,vi) or 3 in (iii).

While G uniquely determines π, Section 4 contains additional examples of groups G that determine some of the above planes but have different normal structures: examples having $M \ntrianglelefteq G$, $AM \ntrianglelefteq G$ and $BM \ntrianglelefteq G$ and, what seems more interesting, examples of order n^3 having $M \lhd G$, $AM \ntrianglelefteq G$ and $BM \ntrianglelefteq G$.

Proof of Theorem 2.1. The plane π can be described as follows:

distinguished line L_∞ and distinguished point ∞ on it
other points: cosets Ax and cosets BMx for $x \in G$
other lines: cosets By and cosets AMy for $y \in G$

$$\begin{array}{lll} Ax \textit{ is on } By & \iff & Ax \cap By \neq \emptyset \\ Ax \textit{ is on } AMy & \iff & Ax \subset AMy \\ By \textit{ is on } BMx & \iff & By \subset BMx \\ BMx \textit{ is on } L_\infty & \textit{and} & AMy \textit{ is on } \infty. \end{array} \tag{2.6}$$

Thus, *a point-coset is on a line-coset if and only if the cosets meet.*

That Ax is on $By \iff Ax \cap By \neq \emptyset$ is taken from [HM, Lemma 4][1].

Here and later A, B, AM and BM have different lives as subgroups of G and as points or lines of π. The structure of G is not involved in the proof, which only uses the most basic properties of cosets: the proof is elementary.

Right multiplications by elements of G induce collineations of π.

It is easy to use (1) and (2) in the theorem to check that there are n^2+n+1 points and lines, with $n+1$ points per line and $n+1$ lines per point. We check that any two points are on one and only one line in a few simple steps involving (2.6).

(i) First note that $AM \cap B = A \cap B$ (since $AM \cap B \geq A \cap B$ and, by (1,2,3), $|B|/|A \cap B| = n = |G|/|AM| = |B|/|AM \cap B|$). Similarly, $BM \cap A = A \cap B$.

(ii) *Distinct points A and Am, $m \in M$, are on AM and no other line.* (If A and Am are on Ba with $a \in A$ then $a'm = b'a$ with $a' \in A, b' \in B$. Then $b' \in B \cap AM = A \cap B$ by (i), so $m \in A$ and $Am = A$.)

(iii) By (4), *two points not on L_∞ are on at most one line* [HM, Lemma 4]. (We may assume that we have two points A and Ab of B with $b \in B\backslash A$. If they are on a line Bz then $z = b_1a_1 = b_2a_2b$ for some $b_1, b_2 \in B$ and $a_1, a_2 \in A$. Now $(b_2^{-1}b_1)a_1 = a_2b \in BA \cap AB = A \cup B$ by (4). Since $b \notin A$ we have $a_1 \in B$ and $Bz = Ba_1 = B$.)

(iv) *The points A and ∞ are on a unique line* ($A \subset AMy \iff AMy = AM$).

The points A and BMx are on a unique line. (By (3), $BMx = BMa$ with $a \in A$, so A and BMx are on Ba. By (2.6), a line on A and BMx must look like $Ba' \subset BMx = BMa$ with $a' \in A$, so $a'a^{-1} \in BM \cap A = A \cap B$ by (i), and $Ba' = Ba$.)

(v) We know that A is on a unique line with each of the $(n-1)+1$ points $\neq A$ on AM and with each of the $n((n-1)+1)$ points $\neq A$ on the n lines Ba, $a \in A$, and hence with each of n^2+n points $\neq A$. Thus, *π is a projective plane of order n.*

(vi) Finally, *G acts as a soft-group for π*: it is transitive on the flags (Ax, By) with $x, y \in G$. (For (Ax, By) to be a flag we must have $ax = by$ for some $a \in A, b \in B$, so $(A, B)ax = (Ax, By)$.) $\square$

Remark 2.7. Let $\mathfrak{I}$ be the set of points on the line of π on both A and ∞ (an *ideal line*), and let $\mathfrak{i}$ be the set of lines on the point on both B and L_∞ (an *ideal point*). Then

(i) $\mathfrak{I} = \{Am \mid m \in M\} \dot\cup \{\infty\}$ and $\mathfrak{i} = \{Bm \mid m \in M\} \dot\cup \{L_\infty\}$,
(ii) $AM = G_\mathfrak{I}$ and $BM = G_\mathfrak{i}$, and
(iii) $M = G_{\mathfrak{I}\mathfrak{i}}$.

Proof. We will use (2.6).

(i) $\mathfrak{I}$ is the set of points on the line AM.

(ii) $G_\mathfrak{I} = G_{AM} = AM$.

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[1]In a much more general context the geometric use of cosets and their intersections seems to have originated in [Ti, p. 272].

(iii) By (2.6) and (ii), $G_{\mathfrak{I}}$ is transitive on the points $\neq \infty$ on L_∞, so $|G_{\mathfrak{I}\mathfrak{i}}| = |AM|/n = |M|$, where $G_{\mathfrak{I}\mathfrak{i}} \geq M$ by (i). $\square$

Corollary 2.8. *The subgroups A, B and M in* Theorem 2.1 *also satisfy*

(i) $A \cap M = A \cap B = B \cap M$,
(ii) $AM \cap B = A \cap B = BM \cap A$,
(iii) $AM \cap BM = M$,
(iv) $AM = A\dot{\cup}(G\backslash ABA)$ *and* $BM = B\dot{\cup}(G\backslash BAB)$, *and*
(v) *A and B uniquely determine M in $G = \langle A, B\rangle$.*

Proof. (i) By Remark 2.7(i,iii), $A \cap B$ fixes $\mathfrak{i}$ and $\mathfrak{I}$, so lies in $A \cap M$; and $A \cap M$ fixes the line on A and $\mathfrak{i}$, so lies in $A \cap B$. Similarly, $A \cap B = B \cap M$.

(ii) See (i) in the proof of Theorem 2.1.

(iii) See Remark 2.7.

(iv) The points Ag $(g \in G)$ are the points Am $(m \in M)$ of $\mathfrak{I}$, together with the points $Aba \neq A$ $(a \in A, b \in B\backslash A)$ on the lines Ba on A.

(v) Use (iv) and (iii) to obtain M, and Theorem 2.1(3) to obtain $G = \langle A, B\rangle$. $\square$

Theorem 2.9. *Every projective plane of order n having a collineation group with a flag-orbit of size n^3 arises via* Theorem 2.1. (*In particular, every soft-plane arises via* Theorem 2.1.)

Proof. Given a projective plane $\bar\pi$ and a collineation group G with a flag-orbit of size n^3, we need to show that G is a soft-group with respect to a distinguished flag $(\bar\infty, L_{\bar\infty})$, we need subgroups A, B and M behaving as in Theorem 2.1 that produce a plane π, and finally we must prove that $\pi \cong \bar\pi$.

(i) *Soft-group.* For a flag $(\bar A, \bar B)$ in the given flag-orbit let A and B be the stabilizers of $\bar A$ and $\bar B$ in G, respectively. Then $n^3 = |G{:}A\cap B| = |G{:}B||B{:}A\cap B|$ with $|G{:}B| \leq n^2+n+1$ and $|B{:}A\cap B| \leq n+1$. Since $|G{:}B|$ and $|B{:}A\cap B|$ are factors of n^3 it follows that $|G{:}B| = n^2$ and $|B{:}A\cap B| = n$. Similarly, $|A{:}A\cap B| = n$ and $|X| = |G{:}A| = n^2$ where $X := \bar A^G$.

Clearly $|\bar B^g \cap X|$ is the same for all $g \in G$. Let $k := |\bar B \cap X|$, so k is n or $n+1$ since $\bar B \cap X$ contains the n points $\bar A^b$, $b \in B$. Since $\bar A$ is on $n = |A{:}A\cap B|$ lines $\bar B^a$, $a \in A$, we have $n^2 = |X| \geq |\cup\{\bar B^a \cap X \mid a \in A\}| = 1 + n(k-1)$. Thus, $k = n$, and $\bar B$ has a point not in $\bar A^G$. Dually, $\bar A$ is on a line $\bar{\mathfrak{I}} \notin \bar B^G$.

The $n(k-1) = n(n-1)$ points of $X\backslash\{\bar A\}$ on the n lines $\bar B^a$, $a \in A$, leave $(n^2-1) - n(n-1) = n-1$ points of $X\backslash\{\bar A\}$ that lie in lines containing $\bar A$ and hence lie in $\bar{\mathfrak{I}}\backslash\{\bar A\}$, so $|\bar{\mathfrak{I}} \cap X| \geq n$. If $g \in G$ then $\bar A^g$ is on a unique line $\bar{\mathfrak{I}}^g$ of $\bar{\mathfrak{I}}^G$, so $\bar{\mathfrak{I}}^G$ induces a partition of X into $n^2/|\bar{\mathfrak{I}} \cap X|$ sets. Since $|\bar{\mathfrak{I}} \cap X| = n$ or $n+1$ we have $|\bar{\mathfrak{I}} \cap X| = n$, so $|\bar{\mathfrak{I}}^G| = n$.

We now have n^2+n lines in $\bar B^G \cup \bar{\mathfrak{I}}^G$, so there is one further line L of π. Since L contains no point of X and has $n^2+n+1-|X|$ points it is the complement of X. Clearly G fixes L. Dually, G fixes a point p. Then $p \in L$, so G is a soft-group with respect to the flag (p, L).

(ii) *Behavior of A, B and M.* We saw that $|G| = n^3k$ where $k := |A \cap B|$, and that $|A| = |B| = |G|/n^2 = nk$, proving most of Theorem 2.1(1).

Using the distinguished flag $(\bar\infty, L_{\bar\infty})$, G is flag-transitive the geometry $\bar\Pi$ whose points are those of $\bar\pi$ not on $L_{\bar\infty}$ and whose lines are those of $\bar\pi$ not on $\bar\infty$.

Let $T := G_{\bar{\mathfrak{I}}} \geq A$ where (as noted above) $\bar{\mathfrak{I}}$ is the line of $\bar\pi$ on $\bar\infty$ and $\bar A$. Since G is point-transitive on $\bar\Pi$, $|G{:}T| = |\bar{\mathfrak{I}}^G| = n$ (so $|T| = n^2k$) and T is transitive on $\bar{\mathfrak{I}}\backslash\{\bar\infty\}$ (since any element of G sending a point of $\bar{\mathfrak{I}}$ to a point of $\bar{\mathfrak{I}}$ lies in T).

Every line of $\bar{\Pi}$ meets $\bar{\mathfrak{I}}\backslash\{\bar{\infty}\}$, and can be moved by an element of T to a line of $\bar{\Pi}$ on the point $\bar{A}$, while A is transitive on the lines of $\bar{\Pi}$ on $\bar{A}$. Then T is transitive on the lines of $\bar{\Pi}$, so $G = TG_{\bar{B}} = TB$.

Let $\bar{\mathfrak{i}}$ be the point $\bar{B} \cap L_{\bar{\infty}}$ of $\bar{\pi}$ (compare Remark 2.7) and $M := G_{\bar{\mathfrak{I}}\bar{\mathfrak{i}}} = T_{\bar{\mathfrak{i}}}$. Since T and A are transitive on $L_{\bar{\infty}}\backslash\{\bar{\infty}\}$ we have $|M| = |T|/|\bar{\mathfrak{i}}^T| = n^2k/n$, $T = AT_{\bar{\mathfrak{i}}} = AM$ and $G = (AM)B$, proving (1), (3), and half of (2). Since B fixes $\bar{\infty}$ and is transitive on $\bar{B}\backslash\{\bar{\mathfrak{i}}\}$ it is transitive on the n lines $\neq L_{\bar{\infty}}$ on $\bar{\infty}$; so is $G_{\bar{\mathfrak{i}}} \geq B$, so $G_{\bar{\mathfrak{i}}} = B(G_{\bar{\mathfrak{i}}})_{\bar{\mathfrak{I}}} = BM$ has order $|G|/n = n^2k$, completing the proof of (2).

Let Π denote the geometry with points and lines the cosets of A and B and incidence determined by nonempty intersection. The map $\varphi\colon \bar{A}^x \mapsto Ax, \bar{B}^x \mapsto Bx$ for $x \in G$ induces an isomorphism $\bar{\Pi} \to \Pi$. (This is well-defined: if $\bar{A}^x = \bar{A}^{x'}$ then $x'x^{-1} \in G_{\bar{A}} = A$. It is an isomorphism: if $\bar{A}^x$ is on $\bar{B}^y$ then some $g \in G$ sends $\bar{A} \mapsto \bar{A}^x$ and $\bar{B} \mapsto \bar{B}^y$, so $g \in G_{\bar{A}}x \cap G_{\bar{B}}y = (\bar{A}^x)^\varphi \cap (\bar{B}^h)^\varphi$.) Using [HM, Lemma 4] (or the proof of Theorem 2.1(iii)), (4) holds since any two points of Π are on at most one line.

(iii) *Isomorphism.* First note $(*)$: By Remark 2.7(ii), $G_{\bar{\mathfrak{I}}} = T = AM = G_{\mathfrak{I}}$ and $G_{\bar{\mathfrak{i}}} = BM = G_{\mathfrak{i}}$; and $\bar{\mathfrak{i}}$ is on the lines $\bar{B}^x, x \in G_{\bar{\mathfrak{i}}}$, while $\mathfrak{i}$ is on the lines $Bx, x \in G_{\mathfrak{i}}$, with similar statements for $\bar{\mathfrak{I}}$ and $\mathfrak{I}$.

The subgroups A, B and M of G together with Theorem 2.1 produce a projective plane π on which G acts as a soft-group with respect to a flag (∞, L_∞). By $(*)$, extending φ by sending $\bar{\infty} \mapsto \infty$, $L_{\bar{\infty}} \mapsto L_\infty$, $\bar{\mathfrak{I}}^x \mapsto \mathfrak{I}x$ and $\bar{\mathfrak{i}}^x \mapsto \mathfrak{i}x$ is well-defined and yields the desired isomorphism $\bar{\pi} \to \pi$. □

Corollary 2.10. *If n is a power of a prime p then a Sylow p-subgroup of G can be used in* Theorem 2.1 *to produce π.*

Proof. Since G is transitive on the n^3 flags opposite (∞, L_∞) the same is true of a Sylow p-subgroup P of G. Now apply Theorem 2.9 to P. □

Remark 2.11. If E is a group of automorphisms of a field or semifield in Examples 2.3 then E normalizes G, A, B and M, so GE acts on π. If E is not a p-group then GE also is not a p-group: *in* Theorem 2.1 *G can be solvable but not nilpotent.*

Theorem 2.1(4) is equivalent to *super-noncommutativity*:

(4′) If $a \in A\backslash B$ and $b \in B\backslash A$ then $ab \neq b'a'$ whenever $b' \in B, a' \in A$.

One instance of this follows easily from the other conditions in Theorem 2.1:

Lemma 2.12. *Assume that* $(1, 2, 3)$ *hold. If $a \in A$, $b \in B$ and $[a, b] \in AM\backslash A$ then $ab \neq b'a'$ whenever $b' \in B, a' \in A$.*

Proof. If $x := [a, b] \in AM\backslash A$ and $ab = b'a'$ with $b' \in B$, $a' \in A$, then $b'a' = bax$. Now $b^{-1}b' = axa'^{-1} \in AM \cap B = A \cap B$ (proved exactly as in step (i) of the proof of Theorem 2.1), whereas $x \notin A$. □

Lemma 2.13. *If $a \in A\backslash B$ then $B^a \cap B \leq A \cap B$.*

Proof. If $b = b'^a$ for $b, b' \in B$ then $ab = b'a \in AB \cap BA = A \cup B$ by (4). Since $a \notin B$ it follows that $b \in A$. □

3. Some properties of G and π

Let G and π be as in Theorem 2.1.

3.1. **Group-theoretic summary.** We are about to prove group-theoretic or geometric results concerning the situation in Theorem 2.1. For group theorists we summarize the purely group-theoretic results.

1. If $BM \lhd G$ then BM contains a normal subgroup of G that is elementary abelian of order n^2 (Proposition 3.2).

2. If BM is abelian then it is normal in G and elementary abelian of order n^2 (Proposition 3.4).

3. If $[A, B] \subseteq M$, or if AM and BM are normal in G, then G has a normal subgroup behaving as in Examples 2.3 (Corollary 3.3).

4. If $n \equiv 2 \ (mod\ 4)$ then $n = 2$ (Proposition 3.8).

5. If $M \lhd G$ then $A \cap B = 1$, and $|G| = n^3$ is either odd or a power of 2 (Proposition 3.9(ii,iii)).

6. If $M \lhd G$ and $C_A(M) \neq 1$ then M is elementary abelian. Thus, if $M \lhd G$ and n is not a prime power then, by conjugation, both A and B induce groups of $|M|$ automorphisms of M (Proposition 3.10).

Remarks 2.11 and Section 4 contain more information concerning the possible normal structure of G.

3.2. **Elements fixing a line pointwise** (cf. [De, Secs. 3.1, 4.3])**.** For a line W and a point w of π let $\dot\Gamma(w, W)$ denote the group of all elements of $\mathrm{Aut}\,\pi$ that fix all points on W and all lines on w; these are called (w, W)-*elations* if w is on W and (w, W)-*homologies* otherwise. Here W is an *axis* and w is a *center* of each element of $\dot\Gamma(w, W)$. Let $\dot\Gamma(L_\infty)$ and $\dot\Gamma(\infty)$ denote the groups of elations in $\mathrm{Aut}\,\pi$ with axis L_∞ resp. center ∞. If $\dot\Gamma(L_\infty)$ is transitive on the n^2 points not on L_∞ then π is a *translation plane* whose dual is a *dual translation plane*. Propositions 4.1(ii) and 4.2(ii) show that the plane π in Theorem 2.1 can be a translation plane whose translation group $\dot\Gamma(L_\infty)$ is not contained in G.

Instead of $\dot\Gamma(w, W)$ we will focus on $\Gamma(w, W) := \dot\Gamma(w, W) \cap G$. Similarly, $\Gamma(L_\infty) := \dot\Gamma(L_\infty) \cap G$ and $\Gamma(\infty) := \dot\Gamma(\infty) \cap G$, which are normal subgroups of G.

Lemma 3.1. *If some nontrivial element h of G induces a homology of π with axis L_∞ then π is a translation plane with translation group $\Gamma(L_\infty) < BM$.*

Proof. Since h fixes $\mathfrak{i}$ it lies BM by Remark 2.7(ii). Since BM is transitive on the points not on L_∞, $\Gamma(L_\infty) < \langle h^{BM} \rangle \le BM$ and $|\Gamma(L_\infty)| = n^2$ by [De, 4.3.2]. □

For most instances in Examples 2.4 there is a soft-group containing nontrivial homologies behaving as in the lemma.

Proposition 3.2. *If $BM \lhd G$ then n is a prime power and π is a translation plane with translation group $\Gamma(L_\infty) < BM$.*

Proof. Let $a \in A\backslash B$. Then $B^a \le BM$, and $B \cap B^a \le A \cap B$ by Lemma 2.13. By Theorem 2.1(1,2), $|BM| \ge |BB^a| = |B||B^a|/|B \cap B^a| \ge nk \cdot nk/|A \cap B| = n^2k = |BM|$, so $B \cap B^a = A \cap B$ and $BM = BB^a$.

Then $H := \cap_{a \in A} B^a \ge A \cap B$. Since B^a is the stabilizer of the line Ba, H is the stabilizer of all lines on A, and consists of homologies since it fixes the line L_∞ not on A. By Lemma 3.1, if $H \neq 1$ then n is a prime power and π is a translation plane with translation group inside BM.

Assume that $H = 1$, so $B \cap B^a = A \cap B = 1$. Then $B^a \cap M = 1$ (if $b^a = m \in M$ with $b \in B$ then $b \in AM \cap B = A \cap B = 1$ by Corollary 2.8(ii)). By Theorem 2.1(1), $\{B^a, M \mid a \in A\}$ consists of $n + 1$ subgroups of order n such that the group BM of order n^2 is the product of any two of them. By [De, p. 131] this produces an affine translation plane with point-set and translation group BM; its lines are $B^a x$ and Mx for $a \in A$, $x \in BM$ (and $\{B^a x \mid x \in BM\}$ is the parallel class of B^a). That plane is isomorphic to the affine plane obtained by removing L_∞ and its points from π in (2.6), via $x \mapsto Ax$, $B^a x \mapsto Bax$ and $Mx \mapsto AMx$ for $a \in A$, $x \in BM$. (This is well-defined: for $x, x' \in BM$ and $a, a' \in A$, if $B^a x = B^{a'} x'$ then $a = a'$ (using parallel classes) and $x = b^a x'$ with $b \in B$, so $Bax = Bab^a x' = Ba'x'$. Moreover, incidence is preserved: if $x \in B^a x'$ then $x = b^a x'$ with $b \in B$, so $Ax \cap Bax' = Aa^{-1}bax' \cap Bax'$ contains bax'; while $x \in Mx' \Rightarrow Ax \subset AMx'$.) □

Corollary 3.3. *If either of the following holds then π is a projective plane over a semifield, and G has a normal subgroup behaving as in* Examples 2.3:

(i) AM and BM are normal in G, or
(ii) $[A, B] \subseteq M$.

Proof. (i) Use the proposition together with [De, (3.1.22)(f)] and [Hi], where the stated subgroup is $\langle \Gamma(\infty), \Gamma(L_\infty)\rangle$.

(ii) Let $a \in A \backslash B$. Then $BB^a \subseteq BM$, so $|B||B^a|/|B \cap B^a| \le |B||M|/|B \cap M|$. By Lemma 2.13 and Corollary 2.8(i), $B \cap B^a \le A \cap B = B \cap M$, so $|B||M|/|B \cap M| \le |B||B^a|/|B \cap B^a|$. Then $BB^a = BM$. Similarly, $BB^{a^{-1}} = BM$, so $BM = BB^a = (BB^{a^{-1}})^a = (BM)^a$. Then $BM \lhd G$. Similarly, $AM \lhd G$, so (ii) follows from (i). □

Proposition 3.4. *If BM is abelian then n is a prime power, π is a translation plane and BM is the translation group $\Gamma(L_\infty)$.*

Proof. Since BM is transitive on the n^2 points not on L_∞ it has order n^2, so $k = 1$. Since BM is transitive on the n lines $\ne L_\infty$ on $\mathfrak{i}$ (cf. Remark 2.7(ii)), the kernel of that action has order $|BM|/n = n$ and hence contains only elations: it is $\Gamma(\mathfrak{i}, L_\infty)$ and is transitive on the points $\ne \mathfrak{i}$ on B. The transitivity of G on $L_\infty \backslash \{\infty\}$ implies the transitivity of $\Gamma(L_\infty)$ on the points not on L_∞. Now $\Gamma(L_\infty) \le BM$, where both groups have order n^2. □

If X is a subgroup of G let X_0 denote the set of elations in X. Examples 2.4 with $BM \lhd G$ have $|A_0| = q$ (ii,iv,vi), 3 in (iii) and $2q = 16$ in (v).

Lemma 3.5. (i) $A_0 = \Gamma(\infty, \mathfrak{I}), B_0 = \Gamma(\mathfrak{i}, L_\infty),$ *and* $M_0 = \Gamma(\infty, L_\infty)$,

(ii) $M_0 \lhd G$, *and M_0 contains every subgroup of M normal in G,*
(iii) *M normalizes A_0 and B_0,*
(iv) *$|A_0|, |B_0|$ and $|M_0|$ divide n, and*
(v) *Every elation in G lies in $\Gamma(L_\infty) \cup \Gamma(\infty)$.*

Proof. (i) A_0 consists of elations fixing ∞, L_∞ and A and so having axis $\mathfrak{I}$ and center ∞. Every $(\infty, \mathfrak{I})$-elation fixes A. The cases B_0 and M_0 are similar.

(ii) Clearly $\Gamma(\infty, L_\infty) \lhd G$, where $\Gamma(\infty, L_\infty) \le G_{\mathfrak{I}\mathfrak{i}} = M$ by Remark 2.7(iii). Let $M_1 \le M$ with $M_1 \lhd G$. If $1 \ne x \in M_1$ then x has center ∞: in view of Remark 2.7(i), x sends each line $\mathfrak{I}b = \{Am \mid m \in M\}b \cup \{\infty\}$ (for $b \in B$) to itself (since $\{Am \mid m \in M\}bx = \{Amx^{b^{-1}}b \mid mx^{b^{-1}} \in M\} = \{Am \mid m \in M\}b$ as $x \in M_1 \lhd G$). Dually, x has axis L_∞, so $M_1 \le \Gamma(\infty, L_\infty) = M_0$.

(iii) If $a \in A_0$ it has axis $\mathfrak{I}$. If $m \in M$ then $Am \in \mathfrak{I}$ by Remark 2.7(i), so $Ama = Am$. Then the elation mam^{-1} is in A, so M normalizes A_0.

(iv) [De, p. 187].

(v) Every nontrivial elation fixes ∞ and L_∞. An elation with center $w \ne \infty$ has $w \in L_\infty$; its axis contains ∞ and w and so is L_∞. □

If s is an integer and p is a prime then s_p denotes the largest power of p dividing s.

Proposition 3.6. *If $B_0 \neq 1$ then*

(i) $\Gamma(L_\infty)$ *and* $B_0 = \Gamma(\mathfrak{i}, L_\infty)$ *are elementary abelian p-groups for some prime p,*
(ii) $\{B_0^a \backslash \{1\} \mid a \in A\}$ *is a G-invariant partition of* $\Gamma(L_\infty)\backslash M_0$ *into n subsets,*
(iii) $n/|M_0| = (|\Gamma(L_\infty){:}M_0| - 1)/(|B_0| - 1)$ *and* $|M_0| = n_p \geq |B_0| > 1$,
(iv) *If n is not a power of p then n determines $|B_0|$ and $|\Gamma(L_\infty)|$, and*
(v) *If n is not a power of p and $A_0 \neq 1$ then* $|\Gamma(\infty)| = |\Gamma(L_\infty)| \leq n_2^2$ *and* $\langle \Gamma(\infty), \Gamma(L_\infty)\rangle = \Gamma(\infty)\Gamma(L_\infty)$ *is a special group with center* $M_0 = \Gamma(\infty) \cap \Gamma(L_\infty)$. *Moreover, elements of* $\Gamma(\infty)\backslash M_0$ *and* $\Gamma(L_\infty)\backslash M_0$ *never commute.*

Proof. (i) Since $B_0^a = \Gamma(\mathfrak{i}a, L_\infty)$ for $a \in A$, there are elations in $\Gamma(L_\infty)$ with axis L_∞ and different centers, so this follows from [De, 4.3.4(b)].

(ii) Every element of $\Gamma(L_\infty)\backslash M_0$ is an elation whose center is in $L_\infty \backslash \{\infty\}$ and so is $\mathfrak{i}a$ for some $a \in A$. Then $\{B_0^a \backslash \{1\} \mid a \in A\}$ is a partition of $\Gamma(L_\infty)\backslash M_0$ of size $|\mathfrak{i}A| = n$. It is G-invariant since G acts on $L_\infty \backslash \{\infty\}$.

(iii) By (ii), $|\Gamma(L_\infty)| - |M_0| = n(|B_0| - 1)$, so $n/|M_0| = \big(|\Gamma(L_\infty){:}M_0| - 1\big)/(|B_0| - 1)$. By (i) and Lemma 3.5(iv), $n/|M_0|$ is an integer not divisible by p. Then $|M_0| = n_p$ since $|M_0$ is a p-power, and $n_p \geq |B_0| > 1$ by Lemma 3.5(iv).

(iv) By (iii), $0 < (n/|M_0|) - 1 = |B_0|\big((|\Gamma(L_\infty)|/|M_0||B_0|) - 1\big)/(|B_0| - 1)$. Since $\Gamma(L_\infty) \geq M_0B_0 \cong M_0 \oplus B_0$, $|\Gamma(L_\infty)|/|M_0||B_0|$ is a p-power and is not 1. Then $|B_0|$ is the largest p-power dividing $(n/|M_0|) - 1 = (n/n_p) - 1$, and this also determines $|\Gamma(L_\infty)|$.

(v) By (i), $\langle \Gamma(\infty), \Gamma(L_\infty)\rangle = \Gamma(\infty)\Gamma(L_\infty)$ is a p-group with commutator subgroup $[\Gamma(\infty), \Gamma(L_\infty)] \leq \Gamma(\infty) \cap \Gamma(L_\infty) \leq Z\big(\Gamma(\infty)\Gamma(L_\infty)\big)$, so $\Gamma(\infty)\Gamma(L_\infty)$ is a special group. Also by (i), $M_0 \leq Z\big(\Gamma(\infty)\Gamma(L_\infty)\big)$.

If $x \in \Gamma(\infty)\backslash M_0$ and $y \in \Gamma(L_\infty)\backslash M_0$ commute then y fixes the axis of x, whereas L_∞ is the only line on ∞ fixed by y. Then $Z(\Gamma(\infty)\Gamma(L_\infty)) = M_0$. □

Corollary 3.7. *If n is even and not a square then all involutions of G lie in* $\Gamma(L_\infty) \cup \Gamma(\infty)$. *Moreover,* B_0, M_0 *and* A_0 *are nontrivial, and* $\Gamma(L_\infty) \geq B_0 \times M_0$ *and* $\Gamma(\infty) \geq A_0 \times M_0$ *are elementary abelian 2-groups.*

Proof. Since n is not a square, all involutions are elations [De, 4.1.9] and hence lie in $\Gamma(\infty) \cup \Gamma(L_\infty)$. By Lemma 3.5(i) and Proposition 3.6(i), $\Gamma(L_\infty) \geq B_0 \times M_0$ and $\Gamma(\infty) \geq A_0 \times M_0$ are elementary abelian. □

Proposition 3.8. *If $n \equiv 2 \ (mod\ 4)$ then $n = 2$.*

Proof. Since n is not a square, k is odd and $|G|_2 = n_2^3 = 8$ by Theorem 2.1(1). By Corollary 3.7, $D := \Gamma(\infty)\Gamma(L_\infty)$ is a normal dihedral subgroup of G. Any $a \in A$ of odd order centralizes D, where $D \geq \Gamma(L_\infty) \geq B_0$, so a lies in $A \cap B$ by Theorem 2.1(4). Then $|A|/2$ divides $|A \cap B| = |A|/n$, so $n = 2$. □

3.3. **The case $M \lhd G$.** *We now assume that the subgroup M occurring in* Theorem 2.1 *is normal in G.* All of the planes π in Examples 2.4 arise from soft-groups $G \leq \operatorname{Aut}\pi$ for which M behaves in this manner, but in general M need not be normal in G (Propositions 4.1(i) and 4.2(i))).

Proposition 3.9. (i) $M = M_0 = \Gamma(\infty, L_\infty)$,

(ii) $A \cap B = 1$, $|\Gamma(\infty, L_\infty)| = |M| = n$ *and* $|G| = n^3$,
(iii) $|G|$ *is either odd or a power of* 2, *and*

(iv) *Every element of G can be written amb for unique $a \in A, m \in M, b \in B$.*

Proof. (i) Lemma 3.5(i,ii).

(ii) Since M consists of (∞, L_∞)-elations it is regular on $\mathfrak{I}\backslash\{\infty\}$, so Theorem 2.1(1) implies that $n = |M| = n|A \cap B|$ and $|G| = n^3$.

(iii) Assume that n is even. Since $A \cap B = 1$, each involution in B does not fix any flag in Π and so is an elation. By Proposition 3.6(i), $\Gamma(L_\infty) > M$ is elementary abelian. Then $|M| = n$ and $|G| = n^3$ are powers of 2.

(iv) Use Theorem 2.1(1,3) and (ii). □

Proposition 3.10. *If $C_B(M) \neq 1$ then M is elementary abelian of order n.*

If n is not a prime power then, by conjugation, both A and B induce groups of $|M|$ automorphisms of M.

Proof. Let $1 \neq c \in C_B(M)$. Then c fixes each line on $\mathfrak{i}$ since $(Bm)c = Bcm = Bm$ for $m \in M$ (cf. Remark 2.7(i)). Homologies have order dividing $n-1$, but $|G| = n^3$ by Proposition 3.9(ii), so c is an elation. Then $c \in B_0$ since $c \in C_B(M)$. Now Proposition 3.6(i) implies that M is elementary abelian of order n.

If n is not a prime power then $C_B(M) = 1$, so B induces a group of $|B| = |M|$ automorphisms of M. □

4. Additional groups

Examples 2.3 and 2.4 focussed on planes not on the groups that produce them. Here we deal with additional groups that occur as soft-groups of planes in those examples. For G in Examples 2.3 (cf. [De, Sec. 5.3]) or 2.4(ii) (using groups in [Ka] and provided below in the proof of Proposition 4.2), *we assume that the underlying field or semifield has an automorphism α of order p* that acts on matrix entries, inducing a collineation (also called α) of the associated plane π. Note that $k = 1$, and that, in Examples 2.4(ii,iv), $p \neq 3$ and $|\mathrm{Aut}\mathbb{F}_q|$ is odd when $q = 2^{2e+1}$.

Then $\tilde{G} := G\langle\alpha\rangle \leq \mathrm{Aut}\,\pi$ is a soft-p-group. The stabilizer $\tilde{G}_\Phi$ of the flag $\Phi := (A, B)$ of π is $\langle\alpha\rangle$. The nonabelian groups $\tilde{A} := A\langle\alpha\rangle$, $\tilde{B} := B\langle\alpha\rangle$ and $\tilde{M} := M\langle\alpha\rangle$ play the roles of A, B and M for $\tilde{G}$.

Proposition 4.1. *For any given characteristic p, for infinitely many n there are both desarguesian planes and nondesarguesian semifield planes having collineation groups behaving as follows:*

(i) *$\tilde{G} \leq \mathrm{Aut}\,\pi$ is a soft-p-group of order n^3p such that the corresponding subgroups $\tilde{A}\tilde{M}$, $\tilde{B}\tilde{M}$ and $\tilde{M}$ are not normal in $\tilde{G}$; and*

(ii) *$\ddot{G} \leq \mathrm{Aut}\,\pi$ is a soft-p-group of order n^3 not isomorphic to the group G in* Examples 2.3 *such that the corresponding subgroups $\ddot{A}\ddot{M}$ and $\ddot{B}\ddot{M}$ are not normal in $\ddot{G}$ while $\ddot{M} \lhd \ddot{G}$. (Moreover, the translation group of the associated translation plane is not contained in $\ddot{G}$.)*

Proposition 4.2. *For any given characteristic $p > 3$, for infinitely many n there are planes π in* Examples 2.4(ii) *having collineation groups behaving as follows:*

(i) *$\tilde{G} \leq \mathrm{Aut}\,\pi$ is a soft-p-group of order q^6p such that the corresponding subgroups $\tilde{B}\tilde{M}$ are not normal in $\tilde{G}$; and*

(ii) *$\ddot{G} \leq \mathrm{Aut}\,\pi$ is a soft-p-group of order q^6 not isomorphic to the group G in* Examples 2.4(ii) *such that the corresponding subgroups $\ddot{A}\ddot{M}$ and $\ddot{B}\ddot{M}$ are* *not normal in $\ddot{G}$ while $\ddot{M} \lhd \ddot{G}$. (Moreover, the translation group of the associated translation plane is not contained in $\ddot{G}$.)*

Since the proofs of these propositions are very similar we will only sketch a proof of the second one.

Proof. Let $F := \mathbb{F}_q$ and $G := A \ltimes F^4$, where the elementary abelian group $A := \{A(t,u) \mid t,u \in F\}$ acts on F^4 via

$$[A(t,u),(x,y,z,w)] = -\big(0, xt, xu+yt, xf(t,u)+yu+zt\big) \tag{4.3}$$

with $f(t,u) := tu - \frac{1}{3}t^3 + l(t)$ for a suitable additive map $l\colon F \to F$ (cf. [Ka]). (Here $l = 0$ if the final characteristic 5 instances of Examples 2.4(ii) are ignored.) Examples 2.4(ii) are obtained using G and the subgroups A, $B := (F,F,0,0)$ and $M := (0,0,F,F)$. The action of α on G is $A(t,u)(x,y,z,w) \mapsto A(t^\alpha,u^\alpha)(x^\alpha,y^\alpha,z^\alpha,w^\alpha)$.

(i) We noted that $\tilde{G} = G\langle\alpha\rangle$ is a soft-p-group. If $a^\alpha \neq a \in A$ then $\alpha^a = \alpha[\alpha,a] \notin BM\langle\alpha\rangle$, so $\tilde{M} = M\langle\alpha\rangle$ and $\tilde{B}\tilde{M} = BM\langle\alpha\rangle$ are not normal in $\tilde{G}$.

(ii) By (4.3), $G' = [A,F^4] = (0,F,F,F)$. Then $\tilde{G} = AG'(F,0,0,0)\langle\alpha\rangle$ implies that $\tilde{G}/G' \cong A(F,0,0,0)\langle\alpha\rangle$, where $A(F,0,0,0)$ is elementary abelian by (4.3). Now $\tilde{G} \rhd AG'$ and $\tilde{G}/AG' \cong (F,0,0,0)\langle\alpha\rangle$.

Let H be an α-invariant subgroup of index p in $(F,0,0,0)$. Let $\ddot{G}$ be a subgroup of index p in $\tilde{G}$ containing $AG'H$ such that $\ddot{G} \neq AG'(F,0,0,0)$, $AG'H\langle\alpha\rangle$.

We will show that $\ddot{G}$ behaves as required.

First, $\ddot{G}$ is a soft-group for π: $|\Phi^{\ddot{G}}| = |\ddot{G}|/|\ddot{G}_\Phi|$ with $\alpha \notin \ddot{G}_\Phi \leq \tilde{G}_\Phi = \langle\alpha\rangle$ and $|\alpha| = p$, so $|\Phi^{\ddot{G}}| = |\ddot{G}| = q^6$, as claimed.

There are subgroups $\ddot{A}, \ddot{B}, \ddot{M}$ of $\ddot{G}$ behaving as in Theorem 2.9. By Proposition 3.2, $\ddot{B}\ddot{M} \ntrianglelefteq \ddot{G}$ since the subgroup $(F,0,0,0)$ of the translation group BM of π is not in $\ddot{G}$. Also by Proposition 3.2, $\ddot{A}\ddot{M} \ntrianglelefteq \ddot{G}$ since π is not a dual translation plane.

Clearly $\ddot{M} = \tilde{M} \cap \ddot{G} = M\langle\alpha\rangle \cap \ddot{G} = M \lhd \tilde{G}$.

Finally, we claim that $\ddot{G} \ncong G$. By (4.3), $C_{\tilde{G}}(G') = C_{\tilde{G}}\big((0,F,F,F)\big) = F^4$ and $C_{\tilde{G}}(AG') = (0,0,0,F)$. Since $\ddot{G}$ contains an element of $G\alpha$, it follows that $C_{\tilde{G}}(\ddot{G}) \leq (0,0,0,C_F(\alpha))$, so $Z(\ddot{G})$ is smaller than $Z(G) = (0,0,0,F)$ and $\ddot{G} \ncong G$. □

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