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2-Transitive Groups in Which the Stabilizer of Two Points is Cyclic[1]

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1. INTRODUCTION

The purpose of this paper is to prove the following result:

THEOREM 1.1. *Let G be a finite group 2-transitive on a set Ω. Suppose that the stabilizer $G_{\alpha\beta}$ of two distinct points $\alpha, \beta \in \Omega$ is cyclic, and that G has no regular normal subgroup. Then G is one of the following groups in its usual 2-transitive representation*: $PSL(2, q)$, $PGL(2, q)$, $Sz(q)$, $PSU(3, q)$, $PGU(3, q)$ *or a group of Ree type.*

We note also that the converse of Theorem 1.1 is valid, i.e., in the usual 2-transitive representations of $PSL(2, q)$, $PGL(2, q)$, $Sz(q)$, $PSU(3, q)$, $PGU(3, q)$, and the groups of Ree type, the stabilizer of two points is cyclic.

Here $Sz(q)$ is a Suzuki group [25]. Groups of Ree type will be defined in Section 2.

As with the classification of Zassenhaus groups by Zassenhaus [30], Feit [9], Ito [17], and Suzuki [25], this theorem characterizes several families of 2-transitive groups, without regard for the parity of the characteristic.

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Instead, however, of placing restrictions on the action of G on Ω, we have specified the structure of the relatively small subgroup $G_{\alpha\beta}$ of G. This is analogous to the classification of 2-transitive groups G in which $G_{\alpha\beta}$ has odd order (Suzuki [26], Bender [2, 3]). These latter results are required in our proof.

Two special cases of Theorem 1.1, needed in its proof, are due to Suzuki [27] and O'Nan [20]. Their results characterize $PGU(3, q)$ and $PSU(3, q)$, q odd, by requiring that the structure of G_α be very much like that of the corresponding subgroups of these 2-transitive groups.

Another very special case of Theorem 1.1 is due to Ito [18], who considered 2-transitive groups G in which $|G_{\alpha\beta}| = 2$. Although we have no comparable hold on $|G_{\alpha\beta}|$, the arguments in cases A and C of Section 5 were motivated by Ito's methods.

Our arguments are surprisingly elementary, as a result of which the main part of the proof of Theorem 1.1 (in Section 5) is reasonably self-contained. The proof proceeds as follows. As already noted, by using results of Suzuki [26] and Bender [2, 3], we may assume $G_{\alpha\beta}$ has an involution t. Let Δ be the set of fixed points of t. In Section 4 we observe that the permutation group $C(t)^\Delta$ induced by $C(t)$ on Δ is a 2-transitive group in which the stabilizer of two points is cyclic. Moreover, using a simple counting argument we obtain an important and useful relationship between $n = |\Omega|$, $|\Delta|$, and $G_{\alpha\beta}$.

By induction, $C(t)^\Delta$ either has a regular normal subgroup or is one of the groups we are characterizing. Also, n is either odd or even. Taking each of these possibilities into account, we are led to four cases, which are dealt with in Section 5. In the course of the proof we show that, since G has no regular normal subgroup and $|G_{\alpha\beta}|$ is even, n cannot be odd.

In case A, we assume that n is odd and that $C(t)^\Delta$ has a regular normal elementary abelian p-subgroup, with p an odd prime number. By studying p-subgroups and 2-subgroups of G, we obtain a contradiction.

In case B, n is odd and $C(t)^\Delta$ has no regular normal subgroup, so that $C(t)^\Delta$ is $PSL(2, 2^f)$, $Sz(2^f)$, $PSU(3, 2^f)$, or $PGU(3, 2^f)$, where $2^f \geqslant 4$. In case C, n is even and $C(t)^\Delta$ has a regular normal elementary abelian 2-subgroup. In these cases, as $C(t)^\Delta$ has a relatively large elementary abelian 2-subgroup, we study the preimage in $C(t)$ of this 2-group.

In case D we consider the situation where n is even and $C(t)^\Delta$ has no regular normal subgroup. We proceed in two steps. It is first shown that either G is of Ree type or that G very closely resembles $PSU(3, q)$ or $PGU(3, q)$, q odd: $C(t)^\Delta$ is $PGL(2, q)$, $n = q^3 + 1$, G_α has a normal subgroup regular on $\Omega - \alpha$, and the Sylow 2-subgroups of G are as they should be. We then use an argument of Brauer [1, Chap. 6] to show that $|G_{\alpha\beta}| = q^2 - 1$ or $(q^2 - 1)/(q + 1, 3)$. The aforementioned results of Suzuki [27] and O'Nan [20] now imply that G is $PGU(3, q)$ or $PSU(3, q)$.

Section 2 contains properties of $PSL(2, q)$, $Sz(q)$, $PSU(3, q)$, and groups of Ree type. The definition of groups of Ree type is based on a paper of Ree [23]. There is an error in that paper which has been corrected by Harada [15]. In Theorem 2.3 we observe that Theorem 1.1 also yields a correction to Ree's paper and a result even stronger than that of Ree and that of Harada.

Notation. All groups will be finite. If G is a group and $X \subseteq G$, then $N_G(X)$ and $C_G(X)$, or simply $N(X)$ and $C(X)$, are the normalizer and centralizer of X, respectively. Aut(G) is the automorphism group of G, $G^{(1)}$ is the derived group of G, $O(G)$ is the largest normal subgroup of G of odd order, and $G^\# = G - \{1\}$. If G is a p-group $\Omega_1(G) = \langle x \mid x \in G \text{ and } x^p = 1\rangle$ and $\mho_1(G) = \langle x^p \mid x \in G\rangle$.

We use Wielandt's notation for permutation groups [28]. If G is a permutation group on Ω and $\alpha \in \Omega$, then G_α is the stabilizer of α. If $X \subseteq G$, $\Delta \subseteq \Omega$ and $\Delta^X = \Delta$, then X^Δ denotes the set of permutations induced by X on Δ. G is said to be semiregular on Ω if only $1 \in G$ fixes a point of Ω. G is regular on Ω if it is transitive and semiregular on Ω. An involution $t \in G$ will be called regular if $\langle t\rangle$ is semiregular on Ω.

2. Properties of the Groups

In this section we state those properties of the groups listed in Theorem 1.1 which will be needed in the proof.

LEMMA 2.1. *Let G be $PSL(2, 2^f)$, $Sz(2^f)$, $PSU(3, 2^f)$ or $PGU(3, 2^f)$, $f > 1$, in its usual 2-transitive permutation representation on a set Ω. Let $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.*

(i) *G has a simple normal subgroup of index 1 or 3.*

(ii) *If S is a Sylow 2-subgroup of G, then S fixes some point $\alpha \in \Omega$, $S \lhd G_\alpha$, and S is regular on $\Omega - \alpha$.*

(iii) *$Z(S) = \Omega_1(S)$ has order 2^f, and S has exponent 2 or 4.*

(iv) *$G_{\alpha\beta}$ is cyclic and has a subgroup A of order $2^f - 1$ which is regular on $Z(S)^\#$.*

(v) *A is the set of inverted elements of each involution $(\alpha\beta)\ldots$ interchanging α and β.*

(vi) *If G is $PGU(3, 2^f)$ or $PSU(3, 2^f)$, $C(Z(S))_{\alpha\beta}$ has order $2^f + 1$ or $(2^f + 1)/3$, and fixes $2^f + 1$ points.*

(vii) *If $t \in Z(S)^\#$, then $C(t) = C(Z(S))$.*

(viii) *If G is $Sz(2^f)$, f is odd.*

For q odd, all of the properties of $PGL(2, q)$ which will be needed are well-known. The following lemma will motivate much of Section 5, case D.

LEMMA 2.2. *Let G be $PGU(3, q)$, q odd, in its usual 2-transitive representation on a set Ω. Let $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.*

(i) *G has a simple normal subgroup $M = PSU(3, q)$ of index $(q+1, 3)$.*

(ii) *All involutions of G are conjugate and fix $q + 1$ points.*

(iii) *$G_{\alpha\beta}$ is cyclic of order $q^2 - 1$, and $|M_{\alpha\beta}| = (q^2 - 1)/(q + 1, 3)$.*

(iv) *G_α has a normal subgroup Q of order q^3, regular on $\Omega - \alpha$.*

(v) *$Z(Q) = \Phi(Q)$ has order q, and $G_{\alpha\beta}$ is fixed-point-free on $Q/\Phi(Q)$.*

(vi) *If t is an involution in $G_{\alpha\beta}$, then $C_M(t)$ has a normal subgroup $C_0(t)$ isomorphic to $SL(2, q)$.*

(vii) *$Z(Q) = C_Q(t)$.*

(viii) *The Sylow 2-subgroups of G are quasi-dihedral if $q \equiv 1 \pmod 4$ or wreathed $Z_{2^a} \wr Z_2$ if $q \equiv 3 \pmod 4$.*

In [23], Ree considered groups G satisfying the following conditions:

(R1) G is a group 2-transitive on a finite set Ω, $|\Omega| = n$.

(R2) If $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, $G_{\alpha\beta}$ has a unique element $t \neq 1$ fixing more than 2 points.

(R3) All involutions in G are conjugate.

A family of groups staisfying (R1)-(R3) were discovered earlier by Ree [22]. In its usual representation of degree 126, $PSU(3, 5)$ also satisfies these conditions. A group G will be said to be of Ree type if it satisfies (R1)-(R3) together with

(R4) $C(t) = PSL(2, q) \times \langle t \rangle$ for some odd prime power q.

Ree [23, Proposition 2.3] incorrectly showed that (R1)-(R3) imply (R4). Harada [15] has proved that, when n is even, (R1)-(R3) imply that (R4) holds or G is $PSU(3, 5)$.

Ree also proved the following facts about groups of Ree type:

(R5) $n = q^3 + 1$, $q = 3^{2a+1}$, a a nonnegative integer;

(R6) t fixes $q + 1$ points;

(R7) $G_{\alpha\beta}$ is cyclic of order $q - 1$;

(R8) The Sylow 2-subgroups of G are elementary abelian of order 8.

Groups of Ree type satisfy the conditions of Theorem 1.1. Conversely, assuming Theorem 1.1, we can prove the following:

THEOREM 2.3. *Let G be a group 2-transitive on a finite set Ω. If $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, suppose that $G_{\alpha\beta}$ has a unique nontrivial element t fixing more than two points. Then either*

(i) *G is of Ree type;*

(ii) *G is $PSU(3, 5)$; or*

(iii) *G has a regular normal subgroup, $|G_{\alpha\beta}| = 2$ and G has two classes of involutions.*

Proof. (R1) and (R2) clearly hold and $|t| = 2$. If Δ is the set of fixed points of t, then $C(t)^{\Delta}$ is 2-transitive (Witt [29]) and only $1 \in C(t)^{\Delta}$ fixes three points. Also, $n = |\Omega| \equiv |\Delta| \pmod 2$.

If n is odd, $C(t)^{\Delta}_{\alpha\beta}$ is cyclic of odd order (Suzuki [25]), so that $G_{\alpha\beta}$ is cyclic. By Theorem 1.1, neither (i) nor (ii) holds, and G has a regular normal elementary abelian p-subgroup for some odd prime p. Then $G_{\alpha\beta}$ fixes at least p points of Ω, so that $|G_{\alpha\beta}| = 2$ and G has two classes of involutions (Ito [18, p. 410]).

If n is even, then as in [23, p. 799], G_{α} has a normal subgroup Q regular on $\Omega - \alpha$. Suppose that $\langle t, u\rangle$ is a Klein group in $G_{\alpha\beta}$. Then $Q = C_Q(t) \cdot C_Q(u) \cdot C_Q(tu)$, and since neither u nor tu fixes points other than α and β, $C_Q(u) = C_Q(tu) = 1$. Thus $Q = C_Q(t)$ and t fixes all points of Ω, which is not the case. Consequently, $G_{\alpha\beta}$ has just one involution and, as in [23, Proposition 1.25], $G_{\alpha\beta}$ has a cyclic Sylow 2-subgroup. Then $C(t)^{\Delta}_{\alpha\beta}$ is cyclic (Feit [9], Ito [17]), so that $G_{\alpha\beta}$ is cyclic. By Theorem 1.1, if neither (i) nor (ii) holds then G has a regular normal elementary abelian 2-subgroup and (iii) holds.

3. PRELIMINARY RESULTS

It is well-known that the automorphism group of a cyclic group is abelian. For future reference, we isolate one special situation.

LEMMA 3.1. *Let $S = \langle x, y\rangle$ be a 2-group of order 2^{m+1} such that $|x| = 2^m$ and $|y| = 2$. Then S is defined by one of the following additional relations:*

(i) *$x^y = x$ and S is abelian;*

(ii) *$x^y = x^{-1}$ and S is dihedral;*

(iii) *$x^y = x^{-1+2^{m-1}}$, $m > 2$, and S is quasidihedral;*

(vi) *$x^y = x^{1+2^{m-1}}$, $m > 2$, and S is modular.*

LEMMA 3.2. *A group whose Sylow 2-subgroups are cyclic is solvable.*

Proof. This follows from Burnside's transfer theorem [14, p. 203] and the Feit–Thompson theorem [10].

LEMMA 3.3. *Let X be a 2-group and $Y \trianglelefteq X$, where $|X/Y| = k \geqslant 4$. Let A be a subgroup of* Aut(X) *of odd order centralizing Y and transitive on $(X/Y)^{\#}$. Then either*

(i) *There is a unique A-invariant subgroup X_1 of X such that $X = X_1 \times Y$;*

or

(ii) $k = 4$ *and there is a unique A-invariant subgroup X_1 of X such that X_1 is a quaternion group of order* 8, $X = X_1 Y$, $|X_1 \cap Y| = 2$ *and* $[X_1, Y] = 1$.

Proof. The proof is by induction on $|X|$. Suppose that $k > 4$ and $Y \neq 1$. Let Y_0 be a maximal subgroup of Y normal in X.

If $Y_0 = 1$, then $|X| = 2k$. As $k > 4$, X has an involution not in Y. By the transitivity of A, there is an involution in each coset of Y in X. As $Y \subseteq Z(X)$, X is elementary abelian. Maschke's theorem now implies the result.

Let $Y_0 \neq 1$, and set $\bar{X} = X/Y_0$, $\bar{Y} = Y/Y_0$. By induction $\bar{X} = \bar{X}_0 \times \bar{Y}$, where $Y_0 \subseteq X_0 \subseteq X$ and $\bar{X}_0$ is invariant of order k. As A is transitive on $\bar{X}_0^{\#}$, by induction $X_0 = X_1 \times Y_0$ with X_1 an A-invariant group such that A is transitive on $X_1^{\#}$. Then $X = X_0 Y = X_1 Y$ and $X_1 \cap Y = 1$. Moreover, $X_1 = [A, X_0]$, so that Y normalizes X_1. We thus have $X = X_1 \times Y$, and X_1 is unique because of the action of A.

If $k = 4$ and (i) does not hold, the same argument shows that (ii) holds, although here we must use the simple fact that the Schur multiplier of $SL(2, 3)$ has odd order.

LEMMA 3.4. *Let R be an elementary abelian group of order 2^n, and let B be a solvable subgroup of* Aut(R), *primitive on $R^{\#}$. If $t \in R^{\#}$, then $|C_B(t)| \mid n$.*

Proof. A minimal normal subgroup A of B is regular on $R^{\#}$. Then A is cyclic and $C_B(t)$ acts faithfully as an automorphism group of $GF(2^n)$. Consequently, $|C_B(t)| \mid n$.

LEMMA 3.5. *If G is a group 2-transitive on Ω, and if $|Z(G_\alpha)|$ is even for some $\alpha \in \Omega$, then G has a regular normal subgroup.*

Proof. Let x be an involution in $Z(G_\alpha)$. Since G_α is transitive on $\Omega - \alpha$, x fixes just the point α. Consequently, $|\Omega|$ is odd and we can choose a Sylow 2-subgroup S of G with $x \in S \subseteq G_\alpha$. If $x^g \in S$, then x^g fixes only α, so $g \in G_\alpha$ and $x^g = x$. Now Glauberman's Z^*-theorem [13] implies that $0(G) \neq 1$. The result now follows from the Feit–Thompson theorem [10].

LEMMA 3.6. *Let G be a 2-transitive group on a set Ω. If p is a prime dividing $|\Omega|$, and if θ is the permutation character of G, then $\theta \in B_0(p, G)$ (the principal p-block of G).*

Proof. It suffices to show that, for each x in G,

$$\frac{|G|}{|C(x)|}\frac{\theta(x)}{n-1} \equiv \frac{|G|}{|C(x)|} \pmod p,$$

where $n = |\Omega|$. Since $p \mid n$, $n = kp$ and $(n-1, p) = 1$.

First, suppose that $C(x)$ contains a Sylow p-subgroup P of G. Then P fixes no points of Ω, so that $\theta(x) = lp - 1$ for some nonnegative integer l. Also $|G : C(x)|\,((\theta(x)/n-1) - 1)$ is an algebraic integer and $n - 1$ is prime to p. Thus

$$|G : C(x)|\left(\frac{\theta(x)}{n-1} - 1\right) = |G : C(x)|\, p\left(\frac{l-k}{n-1}\right) \equiv 0 \pmod p.$$

Now suppose that $C(x)$ does not contain a Sylow p-subgroup of G. Then $p \mid |G : C(x)|$, and again the result follows from the fact that $(p, n-1) = 1$.

4. Beginning of the Proof

Let G be a counterexample to Theorem 1.1 of least order. Set $n = |\Omega|$.

LEMMA 4.1. *Let $1 \neq U \subseteq G_{\alpha\beta}$, and let Δ be the set of fixed points of U. Call a subset of Ω a* line *if it has the form Δ^g, $g \in G$.*

(i) *$N(U)^\Delta$ is 2-transitive, and $N(U)^\Delta_{\alpha\beta}$ is cyclic.*

(ii) *Two distinct points of Ω are on precisely one line.*

(iii) *Each point is on $(n-1)/(k-1)$ lines, where $k = |\Delta|$.*

(iv) *There are $n(n-1)/k(k-1)$ lines.*

(v) *$n \geq k^2 - k + 1$.*

Proof. As U is weakly closed in $G_{\alpha\beta}$, $N(U)^\Delta$ is 2-transitive and (ii) holds (Witt [29]). $N(U)^\Delta_{\alpha\beta}$ is clearly cyclic. There are $n - 1$ points $\neq \alpha$, each on a unique line on α, proving (iii). Counting in two ways the ordered pairs (γ, Δ^g) with $\gamma \in \Delta^g$, $g \in G$, we obtain (iv). If $\gamma \notin \Delta$ then each point of Δ is on a line through γ. By (iii), $(n-1)/(k-1) \geq k$.

In particular, either $N(U)^\Delta$ satisfies the hypotheses of Theorem 1.1, and hence is known, or $N(U)^\Delta$ has a regular normal subgroup.

LEMMA 4.2. *$G_{\alpha\beta}$ contains an involution t.*

Proof. Suppose that $|G_{\alpha\beta}|$ is odd. By results of Bender [2, 3], G has a normal subgroup M which acts on Ω as $PSL(2, q)$. $Sz(q)$, or $PSU(3, q)$ in its usual permutation representation. As $G \subseteq \mathrm{Aut}(M)$, G satisfies the conclusions of Theorem 1.1, contrary to the fact that G is a counterexample of least order.

The following notation will be used throughout the proof of Theorem 1.1. t is an involution in $G_{\alpha\beta}$, Δ its set of fixed points, and W the pointwise stabilizer of Δ in G. Set $k = |\Delta|$. Clearly $n \equiv k \pmod 2$. Let $t' = (\alpha\beta)\ldots$ be a conjugate of t, and Δ', W' the corresponding set of fixed points and pointwise stabilizer. Note that all involutions fixing at least two points are conjugate. Lines are defined as in Lemma 4.1 with $U = \langle t \rangle$.

Let c be the number of involutions $(\alpha\beta)\ldots$, and d the number of involutions $(\alpha\beta)\ldots$ fixing at most one point of Ω.

LEMMA 4.3.

(i) $n = k((c - d)(k - 1) + 1)$.

(ii) *c is the number of elements of $G_{\alpha\beta}$ inverted by any involution $u = (\alpha\beta)\ldots$.*

(iii) *c is even.*

(iv) *If n is odd, there are precisely $c - d$ conjugates of t in $C(t)_\alpha - \{t\}$, and each of these is regular on $\Delta - \alpha$.*

(v) *If n is even, there are precisely $(c - d)(k - 1)$ conjugates of t in $C(t) - \{t\}$ and each of these is regular on Δ.*

(vi) $c - d = c$ *or* $\frac{1}{2}c$.

(vii) $c - d \neq 1$ *and* $d \neq 1$.

(viii) *If n is odd, d is the number of involutions fixing just the point α.*

Proof. We shall use the terminology of Lemma 4.1.

(i) There are $c - d$ conjugates of t of the form $(\alpha\beta)\ldots$, hence $(c - d)(n - 1)$ conjugates of t moving α. That is, there are $(c - d)(n - 1)$ lines not on α. By Lemma 4.1 (iii) and (iv),

$$(c - d)(n - 1) = n(n - 1)/k(k - 1) - (n - 1)/(k - 1),$$

or

$$c - d = (n - k)/k(k - 1).$$

(ii) $x = (\alpha\beta)\ldots$ is an involution if and only if u inverts ux, where $ux \in G_{\alpha\beta}$.

(iii) This is clear from (ii), since $G_{\alpha\beta}$ is cyclic of even order.

(iv) t centralizes t^g if and only if t fixes Δ^g. As k is odd, if t fixes Δ^g, then t fixes a point of Δ^g. Also, if $\gamma^t \neq \gamma$ then t fixes the line through γ and γ^t. Thus, t fixes $(n - k)/(k - 1)$ lines $\neq \Delta$, each of which meets Δ. The number meeting Δ at α is, by Lemma 4.1 (i), $(n-k)/(k-1)k = c-d$. By Lemma 4.1 (ii), each such line meets Δ in a single point.

(v) In this case, t fixes $(n-k)/k = (c-d)(k-1)$ lines $\neq \Delta$, each of which does not meet Δ.

(vi) By Lemma 3.1, $\langle t' \rangle G_{\alpha\beta} - G_{\alpha\beta}$ has 1 or 2 classes of involutions under $G_{\alpha\beta}$. If $x = (\alpha\beta)...$ is an involution then $t'x \in G_{\alpha\beta}$, so that $C(t')_{\alpha\beta} = C(x)_{\alpha\beta}$. Thus, if $d \neq 0$, then $c - d = |G_{\alpha\beta} : C(t')_{\alpha\beta}| = d$.

(vii) If $c - d = 1$, then $n = k^2$ by (i). The points of Ω, together with the lines $\Delta^g, g \in G$, thus form a finite affine plane, and G is 2-transitive on the points of the plane. By a result of Ostrom–Wagner [21, Theorem 1], G has a regular normal subgroup, which is not the case. If $d = 1$, then $c - d = 1$ by (vi).

(viii) d is the number of involutions interchanging α and β and fixing a single point. Let r be the number of involutions fixing a single point. We count in two ways the pairs $(j, (\lambda, \mu))$, where j is an involution fixing a single point and interchanging λ and μ.

Since the number of such involutions j is nr and each interchanges $(n-1)/2$ points, this number is $(nr)(n-1)/2$. Since there are $n(n-1)/2$ two element subsets of Ω and there are d such involutions j interchanging the elements of a fixed two element subset, this number is also $(n(n-1)/2)(d)$. Thus $d = r$.

LEMMA 4.4. *Let n be even.*

(i) *If $U \leqslant W$ fixes more then two points, then $t' \in C(U)$.*

(ii) *W is semiregular on $\Omega - \Delta$.*

Proof.

(i) Let $U \leqslant W$ and Γ be the set of fixed points of U. Let $l = |\Gamma| > 2$. By Lemma 4.1 $N(U)^\Gamma$ is 2-transitive. As t^Γ fixes k points of Γ, l is even. As $N(U)/C(U)$ is abelian and $l > 2$, $C(U)^\Gamma$ is transitive. Since $G_{\alpha\beta} \leqslant C(U)$, $|N(U)/C(U)| \mid (l-1)$. The result follows since $l - 1$ is odd.

(ii) Let $\Gamma \supset \Delta$ and suppose that $U \neq 1$ is the pointwise stabilizer of Γ. Then $|U|$ is odd. Set $l = |\Gamma|$. If c and d are as in Lemma 4.3, and c' and d' are the corresponding numbers for $N(U)^\Gamma$, then $c' - d' \geqslant \frac{1}{2}c'$. Each involution $(\alpha\beta)...$ centralizes U, so that $c = c'$. As in Lemma 4.3 (i), we have

$$\begin{aligned} l - k &= (c' - d')\,k(k-1) \geqslant \tfrac{1}{2}ck(k-1) \geqslant \tfrac{1}{2}(c-d)\,k(k-1) \\ &= \tfrac{1}{2}(n-k) > \tfrac{1}{2}n - k. \end{aligned}$$

However, $n \geqslant l^2 - l + 1$ by Lemma 4.1 (v), a contradiction.

LEMMA 4.5. $k \geqslant 3$.

Proof. Otherwise $k = 2$ and n is even. By Lemma 4.4 (iii), $W = G_{\alpha\beta}$ is semiregular on $\Omega - \{\alpha, \beta\}$. Thus, G is a Zassenhaus group of even degree, and hence is $PSL(2, q)$ or $PGL(2, q)$ [30], which is not the case.

5. PROOF OF THEOREM 1.1

In view of Lemma 4.1 (i), there are four cases:

(A) n is odd and $C(t)^{\Delta}$ has a regular normal subgroup;
(B) n is odd and $C(t)^{\Delta}$ has no regular normal subgroup;
(C) n is even and $C(t)^{\Delta}$ has a regular normal subgroup;
(D) n is even and $C(t)^{\Delta}$ has no regular normal subgroup.

We recall also that G is a counterexample to Theorem 1.1 of least order.

Case A. Here n is odd and $C(t)^{\Delta}$ has a regular normal elementary abelian p-subgroup L^{Δ} of order k, where $L \supseteq W$ and p is an odd prime. Since $N(W)/C(W)$ is abelian, and L^{Δ} is the unique minimal normal subgroup of $C(t)^{\Delta}$, $L \subseteq C(W)$, and hence L has a normal Sylow p-subgroup P. Thus, $C(t) \subseteq N(P)$. Let P^* be a Sylow p-subgroup of $C(t)$ containing P.

Let t^* be an involution in $C(t)_{\alpha} - \{t\}$ (Lemma 4.3 (iv)), Δ^* its set of fixed points and W^* the pointwise stabilizer of Δ^*. Then $t^{*\Delta} \in Z(C(t)_{\alpha}^{\Delta})$ by Lemma 4.3 (iv).

LEMMA A.1. $n = k(c(k - 1) + 1)$ *and all involutions in G are conjugate.*

Proof. By Lemma 4.3 (i), we must show that $d = 0$. Let x be an involution fixing only α. If $\gamma \neq \alpha$, then x fixes the line through γ and γ^x, hence fixes a point of this line, since k is odd. Thus x fixes each line Δ^g on α. If x' is also an involution fixing only α, then x' also fixes each such line Δ^g, $\alpha \in \Delta^g$, and x and x' agree on Δ^g. (They both invert the regular normal subgroup of $C(t^g)^{\Delta^g}$.) Thus $x = x'$. It follows that $d = 1$, contradicting Lemma 4.3 (vii). (Alternatively, Lemma 3.5 can be used to obtain a contradiction.)

LEMMA A.2. *If the Sylow* 2-*subgroup* R *of* $G_{\alpha\beta}$ *is not contained in* W, *then* $c = 2$ *and* $t^* \in C(W)$.

Proof. Let $x \in G_{\alpha\beta}$ be a 2-element such that x^{Δ} is an involution. Then $\langle t^*, x\rangle^{\Delta}$ is a Klein group acting on P^{Δ}, and $t^{*\Delta}$ inverts P^{Δ}. Thus $(t^*x)^{\Delta}$ centralizes an element $\neq 1$ of P^{Δ} not centralized by x^{Δ}. In particular, $(t^*x)^{\Delta}$ fixes distinct points $\gamma, \delta \in \Delta - \{\alpha, \beta\}$, where we may assume $\gamma^{t^*} = \delta$. Then $t^*x \in G_{\gamma\delta}W = G_{\gamma\delta} \subseteq C(W)$ and $x \in G_{\alpha\beta} \subseteq C(W)$ imply that $t^* \in C(W)$.

If y is another involution fixing Δ and only α on Δ, then $t^{*\Delta} = y^{\Delta}$, so $y \in \langle W, t^* \rangle$. Since $t^* \in C(W)$, there are only two possibilities for y. By Lemma 4.3 (iv), $c - d = 2$. Since $d = 0$, $c = 2$.

LEMMA A.3. $N(P^*) = C(P^*)(C(t) \cap N(P^*))$.

Proof. By definition $t \in C(P^*)$. Let S be a Sylow 2-subgroup of $P^*C(P^*)$ containing t. As n is odd, S fixes a point γ. Since $t \in S$, $\gamma \in \Delta$. Then $S \subseteq C(P)$ fixes at least 2 points of Δ, and hence is cyclic. Since $P^*C(P^*) \lhd N(P^*)$, the Frattini argument implies that

$$N(P^*) = P^*C(P^*)(C(t) \cap N(P^*)) = C(P^*)(C(t) \cap N(P^*)).$$

LEMMA A.4. $c \neq 2$.

Proof. Let $c = 2$, so that t^* centralizes $0(W)$. By Lemma A.1, $n = k(2k-1)$ and k is the highest power of p dividing n. It follows that P^* is a Sylow p-subgroup of G.

If $N(P^*) \subseteq C(t)$, then $|G : C(t)| = |G : N(P^*)|/|C(t) : N(P^*)| \equiv 1 \pmod{p}$. However, $|G : C(t)| = n(n-1)/k(k-1) = (2k+1)(2k-1)$, a contradiction.

Thus, $N(P^*) \not\subseteq C(t)$. By Lemma A.3, $C(P^*) \not\subseteq C(t)$, so that $C(P) \not\subseteq C(t)$. Also, as in Lemma A.3, $C(P)$ has a cyclic Sylow 2-subgroup S containing t, so that $PC(P) = 0(PC(P))S$.

Let P_0 be a Sylow p-subgroup of $0(PC(P))$. As $P^* \subseteq N(P)$, we may assume that $P_0 \subseteq P^*$. However, $C_{P^*}(P) \subseteq P$. For otherwise, $P^* \supset P$, $P^* - P$ has an element g fixing a point of Δ centralizing P, and then $g \in P^* \cap W \subseteq P$. Thus $P_0 = P$ is a Sylow p-subgroup of $0(PC(P))$.

It follows that $0(PC(P)) = P \times U$, where $(|U|, 2p) = 1$. Set $U_0 = U \cap C(t)$. As $C(P) \not\subseteq C(t)$, $U_0 \subset U$. Also, $U_0 \subseteq W$. Since $P \subseteq C(W)$, it follows that $0(W) = (P \cap W) \times U_0$. The Klein group $\langle t^*, t \rangle$ acts on U, so that $U = C_U(t)\, C_U(t^*)\, C_U(tt^*)$. Moreover, $U_0 \subseteq 0(W) \subseteq C(\langle t, t^* \rangle)$, so that for some involution $t_1 \neq t$ of $\langle t, t^* \rangle$ we have $U_1 = C_U(t_1) \supset U_0$.

Let q be a prime dividing $|U_1 : U_0|$ and let Q be a Sylow q-subgroup of U_1. Then Q acts on the set Δ_1 of fixed points of t_1. As $q \neq p$, Q fixes some point $\delta \in \Delta_1$. If δ is the only fixed point of Q, then $P \subseteq C(U) \subseteq C(U_1)$ implies that P fixes δ. Since $p \mid n$, P must fixed at least 2 points, and hence P is cyclic. However, P^{Δ} is elementary abelian, so that $k = p$ and W contains a Sylow p-subgroup of $G_{\alpha\beta}$. Thus, $|P| \mid |W|$, whereas $P \supset P \cap W$. This contradiction shows that Q fixes at least 2 points, so that Q is cyclic.

As t acts on $C_U(t_1) = U_1$, t normalizes a conjugate of Q, which we may assume to be Q. Then t inverts $Q/Q \cap U_0$, and since Q is cyclic, t inverts Q. The group $\langle t, Q \rangle$ acts on Δ_1. Let $t_1 = t^g$, $g \in G$, and let L^g/W^g be the regular normal subgroup of $C(t^g)^{\Delta^g}$. Then tL^g is in $Z(C(t^g)/L^g)$, so that $Q = [t, Q] \subseteq L^g$. As $(|Q|, p) = 1$, $Q \subseteq W^g$. However, since $c = 2$, t centralizes $0(W^g)$, a contradiction.

Let M be a minimal normal subgroup of G.

LEMMA A.5. *M is a simple group whose Sylow 2-subgroups are not dihedral.*

Proof. M is simple and $C(M) = 1$ by results of Burnside [8, pp. 200–202]. If M has a dihedral Sylow 2-subgroup, then $M \approx A_7$ or $PSL(2, q)$ for some odd q (Gorenstein and Walter [12]). Then $G \subseteq \mathrm{Aut}(M)$. As $G_{\alpha\beta}$ is cyclic, this is impossible by a result of Lüneburg [19, p. 422].

LEMMA A.6. *Let R be the Sylow 2-subgroup of $G_{\alpha\beta}$ and S a Sylow 2-subgroup of $C(t)_\alpha$ containing $R \cdot \langle t^*\rangle$.*

(i) $R \subseteq W$.

(ii) *$S_0 = S \cap M$ is a Sylow 2-subgroup of M.*

(iii) *S_0^Δ is cyclic.*

(iv) $|S_0^\Delta| \geqslant 4$.

(v) *S_0 is nonabelian.*

Proof.

(i) Lemmas A.2 and A.4.

(ii) $n - 1 = (k - 1)(ck + 1)$ by Lemma A.1, and c is even by Lemma 4.3 (iii). It follows that S is a Sylow 2-subgroup of G.

(iii) Otherwise, S^Δ is a generalized quaternion group by (i). As $N(W)/C(W)$ is abelian, $t^{*\Delta} \in C(W)^\Delta$, so that $t^* \in C(W)$. Precisely as in the proof of Lemma A.2, $c = 2$, a contradiction.

(iv) By Lemma A.1, M contains all involutions of G. Moreover, $|C_M(t)|$ and $|W \cap M|$ are independent of the involution t.

Suppose that $|S_0^\Delta| < 4$. Then $S_0 = \langle t^*\rangle R_0$, where $R_0 = R \cap M$. If S_0 is abelian then $N(S_0)$ controls fusion in S_0, so that S_0 is a Klein group, contradicting Lemma A.5. If S_0 is modular (Lemma 3.1), then t^* centralizes the subgroup R_1 of index 2 in R_0. Then R_1 acts faithfully on Δ^*, so that $|R_1| = 2$ and S_0 is dihedral of order 8, again contradicting Lemma A.5.
Thus, S_0 is quasi-dihedral. By a theorem of Grün [14, p. 214],

$$S_0 = S_0 \cap M^{(1)} = \langle S_0 \cap N_M(S_0)^{(1)}, S_0 \cap S_0^{(1)g} \mid g \in M\rangle. \tag{1}$$

$N_M(S_0) = S_0C_M(S_0)$ implies that $S_0 \cap N_M(S_0)^{(1)} \subseteq S_0^{(1)}$. If $S_0 \cap S_0^{(1)g}$ has order 2, it lies in the dihedral group $\langle t^* \rangle\, S_0^{(1)}$. If $| S_0 \cap S_0^{(1)g} | \geqslant 4$, then $t \in S_0 \cap S_0^{(1)g}$, so that $t = t^g$. However, $S_0^{(1)} \subseteq R_0 \subseteq W$, so that $S_0^{(1)g} = S_0^{(1)}$. Thus, $S_0 \cap M^{(1)} \subseteq \langle t^* \rangle\, S_0^{(1)}$, a contradiction.

(v) Otherwise, as $S_0{}^{\Delta}$ and $S_0 \cap W$ are cyclic, we have $S_0 = S_1 \times S_2$ with S_1, S_2 cyclic. By Lemma A.1 and the fact that $N_M(S_0)$ controls fusion in S_0 [14, p. 203], $| S_1 | = | S_2 |$. Now a result of Brauer [5, p. 317] shows that S_0 is a Klein group, contradicting (iv).

We can now complete case A. Once again we use (1).

By Lemma A.6, $1 \neq S_0^{(1)} \subseteq R_0$, so that $N_M(S_0) \subseteq C_M(t)$. Also, by Burnside's transfer theorem, $C_M(t)_\alpha^\Delta = 0(C_M(t)_\alpha^\Delta)\, S_0{}^{\Delta}$. Thus, $S_0 \cap N_M(S_0)^{(1)} \subseteq R_0$.

Let $L = S_0 \cap S_0^{(1)g} \neq 1$, where $g \in M$. Then $L \subseteq S_0^{(1)g}$ implies that $t^g \in L$. By (1), we may assume that for some g, $R_0 L = S_0$, since S_0/R_0 is cyclic.

$t^g \neq t$, as otherwise $L \subseteq S_0^{(1)g} = W \cap S_0^{(1)g} \subseteq R_0$. Thus, $R_0 \cap L = 1$. If $| R_0 | = 2^r$ and $| S_0 | = 2^{r+s}$, then $| L | = 2^s$. Since $s > 1$ (Lemma A.6 (iv)), either $t^* \in C_M(R_0)$ or t^* is a square in $N_M(R_0)/C_M(R_0)$. In either case, t^* centralizes the subgroup R_1 of index 2 in R_0. Then R_1 acts faithfully on Δ^*, so that $2^{r-1} = | R_1 | \leqslant | S_0{}^{\Delta} | = 2^s$.

Thus, $| L | = 2^s \geqslant 2^{r-1}$. However, $L \subseteq S_0^{(1)g} \subset R_0{}^g$. It follows that $| L | = 2^{r-1}$. As L is cyclic and $\mathrm{Aut}(R_0)$ is abelian of exponent 2^{r-2}, t^g must centralize R_0. However, R_0 then acts faithfully on Δ^g, and we have $2^r = | R_0 | \leqslant | S_0{}^{\Delta} | = 2^s = 2^{r-1}$, a final contradiction.

Case B. As n is odd and $C(t)^{\Delta}$ has no regular normal subgroup, by Lemma 4.1 $C(t)^{\Delta}$ is $PSL(2, 2^f)$, $Sz(2^f)$, $PSU(3, 2^f)$ or $PGU(3, 2^f)$, where $f \geqslant 2$ and $k - 1$ is some power of 2.

LEMMA B.1. *$| W | = 2$ or 4, and $| W | = 2$ if $C(t)^{\Delta}$ is $PSL(2, 2^f)$.*

Proof. As $N(W)^{\Delta} = C(t)^{\Delta}$, $|N(W) : C(W)| = 1$ or 3 (Lemma 2.1 (i)). In particular, if $t^* \neq t$ is a conjugate of t in $C(t)_\alpha$ (Lemma 4.3 (iv)), then $W \subseteq C(t^*)$. Let Δ^* be the set of fixed points of t^*, so that $\Delta \cap \Delta^* = \{\alpha\}$.

The Sylow 2-subgroup of W acts semiregularly on $\Omega - \Delta$, and, in particular, on $\Delta^* - \alpha$. By Lemma 2.1 (iii), the Sylow 2-subgroup of W has order 2 or 4, 2 if $C(t)^{\Delta}$ is $PSL(2, 2^f)$.

Suppose that U is a subgroup of $0(W)$ of prime order. Then U fixes a point of $\Delta^* - \alpha$. Let Γ be the set of fixed points of U. Then $\Gamma \supset \Delta$ and t acts on Γ, so that $| \Gamma |$ is odd. By Lemma 4.1 (i), $N(U)^{\Gamma}$ is 2-transitive. As $C(t)$ normalizes U, the minimality of G implies that $N(U)^{\Gamma}$ has a regular normal subgroup E. Then $E \cap C(t)^{\Delta}$ is a regular normal subgroup of $C(t)^{\Delta}$, a contradiction.

LEMMA B.2. *$C(t)^{\Delta}$ is neither $PSU(3, 2^f)$ nor $PGU(3, 2^f)$.*

Proof. Otherwise let S_0 be a Sylow 2-subgroup of $C(t)_\alpha$. Let $S/W = \Omega_1(S_0/W) = Z(S_0/W)$, so that S contains each involution in $C(t)_\alpha$ (Lemma 2.1). S is centralized by a group $H \subseteq G_{\alpha\beta}$ of order $2^f + 1$ or $(2^f+1)/3$, so that $H \subseteq C(t^*)_\alpha$ and H fixes points of $\Delta^* - \alpha$ (Lemma 2.1 (vi)). The set Γ of fixed points of H is not contained in Δ and t^Γ fixes $2^f + 1$ points. Then $|\Gamma|$ is odd. By Lemma 4.1 (i) and the minimality of G, $N(H)^\Gamma$ has a regular normal subgroup. However, $N(H) \cap C(t)$ acts on both Γ and $\Delta \cap \Gamma$, and $(N(H) \cap C(t))^{\Delta \cap \Gamma}$ does not have a regular normal subgroup, a contradiction.

LEMMA B.3. *Let S_0 be a Sylow 2-subgroup of $C(t)_\alpha$. Then $\Omega_1(S_0) = R \times \langle t \rangle$ is elementary abelian of order 2^{f+1}, where $C(t)_\alpha$ acts transitively on $R^\#$. Either Rt is the set of all involutions in S_0 conjugate to t, or $2^f = 4$ and all involutions in S_0 are conjugate to t.*

Proof. Set $S/W = \Omega_1(S_0/W) = Z(S_0/W)$. Then all involutions in $C(t)_\alpha$ are in S. There is a group $A \subseteq C(t)_\alpha$ of order $2^f - 1$ transitive on $(S/W)^\#$. If $2^f \geqslant 8$, then, by Lemma 3.3, S is abelian and $\Omega_1(S_0) = R \times \langle t \rangle$ has order 2^{f+1} with R A-invariant. If $2^f = 4$, then $C(t)^\Delta$ is $PSL(2, 4)$ and $|W| = 2$ (Lemma B.1). Since $t^* \in S_0 - W$ and A is irreducible on S/W, we again have $S_0 = R \times \langle t \rangle$ elementary abelian.

Now $\Omega_1(S_0) \subseteq C(t)_\alpha \cap C(t^*)_\alpha$, so that $\Omega_1(S_0)$ contains all the involutions in both $C(t)_\alpha$ and $C(t^*)_\alpha$. Thus, $N = N(\Omega_1(S_0)) \supseteq \langle C(t)_\alpha, C(t^*)_\alpha \rangle$. Also, $C_N(t)$ has orbits on $\Omega_1(S_0) - \langle t \rangle$ of length $2^f - 1$ or $2(2^f - 1)$. Thus, either $\Omega_1(S_0)$ consists entirely of conjugates of t, or 2^f conjugates of t lie in $\Omega_1(S_0)$. In the latter case, the conjugates of t in $\Omega_1(S_0)$ consist of $R^\# \cup \{t\}$ or Rt.

Suppose that $d = 0$ and $2^f \geqslant 8$. Then all involutions in $\Omega_1(S_0)$ are conjugate to t. Moreover, S_0 is a Sylow 2-subgroup of G and $\Omega_1(S_0)$ is weakly closed in S_0, so that N is transitive on $\Omega_1(S_0)^\#$. Also, $\Omega_1(S_0)$ fixes just α, so that $N \leqslant G_\alpha$ and $C_N(t) \leqslant C(t)_\alpha$. Consider $N/C_N(\Omega_1(S_0))$ as a transitive permutation group on $\Omega_1(S_0)^\#$ and apply Burnside's p-complement theorem to first the odd prime divisors of $|C_N(t)/C_N(\Omega_1(S_0))|$ and then to $p = 2$. It follows that $N/C_N(\Omega_1(S_0))$ contains a regular normal subgroup and acts as a primitive solvable group on $\Omega_1(S_0)^\#$. This contradicts Lemma 3.4.

We may then assume that $d \neq 0$ and $\Omega_1(S_0)$ has 2^f conjugates of t, and $t^G \cap \Omega_1(S_0) = R^\# \cup \{t\}$ or Rt. Suppose that $t^g \in R, g \in G$. As $\Omega_1(S_0)$ contains all involutions in $C(t^g)_\alpha$, we have $\Omega_1(S_0)^g = \Omega_1(S_0)$. Then R^g is contained in $R \cup \{t\}$, and it follows that $R^g = R$. Then $g \in N(R)$, whereas $t^g \in R$ and $t \notin R$, a contradiction. Thus, Rt is the set of all conjugates of t in $\Omega_1(S_0)$.

LEMMA B.4. *$C(t)^\Delta$ is not $PSL(2, 2^f)$.*

Proof. Suppose that $C(t)^\Delta = PSL(2, 2^f)$ with $2^f \geqslant 8$. By Lemma 4.3,

$c - d = 2^f - 1$. Then $n - 1 = 2^{3f}$ by Lemma 4.3 (i). Let Q be a Sylow 2-subgroup of G_α containing t.

Let P be a p-subgroup of $G_{\alpha\beta}$ for an odd prime p. We claim that $C_{G_\alpha}(P) = G_{\alpha\beta} = N_{G_\alpha}(P)$. We first note that $G_\alpha = G_{\alpha\beta} \cdot Q$, so that $N_{G_\alpha}(P) = G_{\alpha\beta}(N_Q(P))$. Thus, $N_Q(P)$ is transitive on the fixed points of P other than α, so that either P fixes just α and β or P fixes an odd number of points. Also, t fixes just α and β in its action on the fixed points of P. Consequently, P must fix just the points α and β, and since $N_Q(P) \subseteq G_\alpha$, $N_Q(P) \subseteq G_{\alpha\beta}$. This proves the claim.

If P is chosen to be a Sylow p-subgroup of $G_{\alpha\beta}$, then, since $n - 1 = 2^{3f}$, P is a Sylow p-subgroup of G_α and by the above, G_α has a normal p-complement. It follows that G_α has a normal Sylow 2-subgroup, so that $Q \trianglelefteq G_\alpha$. Moreover, if $x \in (G_{\alpha\beta})^\#$ has odd order, then $C_Q(x) = \langle t \rangle$.

In the notation of Lemma B.3, $R^\#$ consists of all $2^f - 1$ involutions in $C(t)_\alpha$ fixing only 1 point of Ω. By Lemma 4.3 (vi) and (vii), R contains all $d = c - d = 2^f - 1$ involutions of G fixing only α. Thus, $R \lhd G_\alpha$. Also, $R \subseteq Q \lhd G_\alpha$.

Let R_1/R be a minimal normal subgroup of G_α/R with $R_1 \subseteq Q$. Then $R_1/R \subseteq Z(Q/R)$, so that either $R_1 = R\langle t \rangle$, or $t \notin R_1$, $G_{\alpha\beta}$ is irreducible on R_1/R and $| R_1/R | = 2^f$.

Suppose that $R_1 = R\langle t \rangle$. Then Q permutes the 2^f involutions Rt. Since $| Q/R_1 | = 2^{2f}$, we have $t_1^g = t_1$ with $t_1 \in Rt$ and $g \in Q - R_1$, so that $C(t_1)_\alpha \supseteq \langle R_1, g \rangle$. However, t and t_1 are conjugate, so that R_1 is a Sylow 2-subgroup of $C(t)$ (Lemma B.1), a contradiction. Thus, $| R_1/R | = 2^f$.

Let R_2/R_1 be a minimal normal subgroup of G_α/R_1 with $R_2 \subseteq Q$. As before, either $R_2 = R_1\langle t \rangle$ or $t \notin R_2$ and $| R_2/R_1 | = 2^f$. Suppose that $R_2 = R_1\langle t \rangle$. Then $| Q/R_2 | = 2^f$ and $0(G_{\alpha\beta})$ is transitive on $(Q/R_2)^\#$. Since we are assuming that $2^f \geq 8$, we can apply Lemma 3.3 to Q/R_1 in order to obtain a group $\tilde{R}_2 \lhd Q$ normalized by $0(G_{\alpha\beta})$ and such that $Q = \tilde{R}_2\langle t \rangle$ and $t \notin \tilde{R}_2$.

We may thus assume that $Q = R_2\langle t \rangle$ with $R_2 \lhd G_\alpha$ and $t \notin R_2$. Then $R_2 G_{\alpha\beta} = G_\alpha$ and $R_2 \cap G_{\alpha\beta} = 1$. Since $R_2 \lhd G_\alpha$, t is conjugate to no element of R_2. By considering the image of t under the transfer map $G \to Q/R_2$, we find that G has a normal subgroup $\tilde{G}$ such that $G = \tilde{G} \cdot \langle t \rangle$, $t \notin \tilde{G}$, and $R_2 \subset \tilde{G}$.

Since R_2 is transitive on $\Omega - \alpha$, $\tilde{G}$ is 2-transitive on Ω. Also, $\tilde{G}_{\alpha\beta} = 0(G_{\alpha\beta})$. We have seen that all elements of $0(G_{\alpha\beta})^\#$ fix just α and β. Moreover, the involutions in $\tilde{G}_\alpha$ are precisely the $2^f - 1$ involutions in R. By the minimality of G, $\tilde{G}$ is $PSL(2, 2^f)$ or $Sz(2^f)$. In either case, the Sylow 2-subgroup of G has order $< 2^{3f}$, a contradiction. Consequently, we must have $2^f = 4$.

Then $C(t)^\Delta = PSL(2, 4)$, $k = 5$, and $c - d = 3$ or 6. By Lemma 4.3 (i), $n = 5 \cdot 13$ or 5^3. Suppose first that $c - d = 6$. In the notation of Lemma B.3, $S_0 = \Omega_1(S_0)$ is elementary abelian of order 8 and all involutions are conjugate. Then $N(S_0)$ is transitive on $S_0^{\#}$. It follows that $7 \mid |G|$ which is not the case.

Thus, $c - d = 3$ and $n = 5 \cdot 13$. Here, $|G_{\alpha\beta}| = |W| \, |C(t)^{\Delta}_{\alpha\beta}| = 2 \cdot 3$, $|G| = (5 \cdot 13)(2^6)(2 \cdot 3)$. The centralizer of a Sylow 13-subgroup P has order 13 or $5 \cdot 13$, and it is easy to check that $|G : N(P)| \not\equiv 1 \pmod{13}$, contradicting Sylow's Theorem.

LEMMA B.5. *$C(t)^{\Delta}$ is not $Sz(2^f)$.*

Proof. Suppose that $C(t)^{\Delta}$ is $Sz(2^f)$, so that $2^f \geqslant 8$ and $k = 2^{2f} + 1$. By Lemma B.3, $C(t)_\alpha - \{t\}$ has precisely $2^f - 1$ conjugates of t, so that $c - d = 2^f - 1$ and $n - 1 = 2^{3f}(2^{2f} - 2^f + 1)$. Since $0(G_{\alpha\beta})$ has order $2^f - 1$, it is a Hall subgroup of G_α.

Let $x \in 0(G_{\alpha\beta})^{\#}$, so that x fixes just 2 points of Δ. If Γ is the set of fixed points of x, then t fixes just 2 points of Γ, so that if x does not fix just α and β, then by Lemma 4.1 (i), $N(\langle x\rangle)^{\Gamma} = PSL(2, q)$ or $PGL(2, q)$ with q odd. As $\langle x\rangle$ is cyclic, $N(\langle x\rangle)/C(\langle x\rangle)$ is abelian, and since $|\Gamma| > 2$ and $|\Gamma \cap \Delta| = 2$, no involution fixes each point of Γ. It follows that there is a Klein group $\langle t, u\rangle$ centralizing $\langle x\rangle$. Then u^{Δ} centralizes x^{Δ}, which is impossible. Thus, x fixes just the points α and β.

Now, as in the proof of Lemma B.4, G_α has a normal subgroup Q with $Q \cdot 0(G_{\alpha\beta}) = G_\alpha$ and $Q \cap 0(G_{\alpha\beta}) = 1$. Moreover, if $x \in 0(G_{\alpha\beta})^{\#}$, then $C_Q(x) = W$.

Let P be a Sylow p-subgroup for p a prime dividing $2^{2f} - 2^f + 1$. As $(|0(G_{\alpha\beta})|, |Q|) = 1$, we may assume that $0(G_{\alpha\beta}) \subseteq N_{G_\alpha}(P)$, so that $0(G_{\alpha\beta})$ is fixed-point-free on P. If P_0 is a minimal normal subgroup of $P \cdot 0(G_{\alpha\beta})$ with $P_0 \subseteq P$, then $(2^f - 1) \mid (|P_0| - 1)$, so that $|P_0| \geqslant 2^f$. If $P_0 \subset P$, then $|P/P_0| \geqslant 2^f$ and $|P| \geqslant 2^{2f} > 2^{2f} - 2^f + 1$, which is impossible. Then $P_0 = P$. Similarly, $|P| = 2^{2f} - 2^f + 1$. For otherwise, let L be a Sylow l-subgroup for a prime $l \neq p$, dividing $2^{2f} - 2^f + 1$. Then $2^{2f} - 2^f + 1 \geqslant |P| \cdot |L| \geqslant 2^f \cdot 2^f$, a contradiction.

Now f is odd, so that $3 \mid (2^{2f} - 2^f + 1)$ and P is a 3-group of order $2^{2f} - 2^f + 1$. Let $3^a = 2^{2f} - 2^f + 1$. Then $3^a \equiv 1 \pmod 4$ implies a is even. Write $a = 2b$. Then $(3^b + 1)(3^b - 1) = 2^f(2^f - 1)$. Since $(3^b + 1, 3^b - 1) = 2$, we have $2^{f-1} \mid (3^b - \epsilon)$, $\epsilon = \pm 1$. Also, $3^{2b} - 1 < 2^{2f} - 1$, so that $3^b < 2^f$. Then $3^b - \epsilon < 2^f$, as otherwise $3^b + 1 = 2^f$ and $3^b - 1 = 2^f - 1$. Thus, $2^{f-1} \mid (3^b - \epsilon)$ and $(3^b - \epsilon) < 2^f$, so that $2^{f-1} = 3^b - \epsilon$. Now $2^{f-1} + 2\epsilon = 3^b + \epsilon = 2(2^f - 1)$. This contradiction completes case B.

Case C. Here n is even and $C(t)^{\Delta}$ has a regular normal elementary abelian 2-subgroup L^{Δ}, where $L \supseteq W$. Since $N(W)/C(W)$ is abelian, $L \subseteq C(W)$. Then $L = S \times 0(W)$ with S a Sylow 2-subgroup of L and $S \cap W \subseteq Z(S)$.

Since k is even, $C(t)_\alpha^\Delta$ has cyclic Sylow 2-subgroups. Then $C(t)^\Delta$ is solvable (Lemma 3.2). By a result of Huppert [16], $C(t)^\Delta$ can be regarded as a subgroup of the group of 1-dimensional affine semilinear mappings on $GF(k)$. In such groups, all regular involutions are in the regular normal subgroup, namely, S^Δ.

By Lemma 4.3 (v), S contains each of the $(c - d)(k - 1)$ involutions in $C(t) - \{t\}$ conjugate in G to t.

LEMMA C.1. *$c = 2$, $d = 0$, and $n = k(2k - 1)$.*

Proof. By Burnside's transfer theorem [14, p. 203], $C(t)_\alpha$ has a normal 2-complement A. Clearly, A is transitive on $(S/S \cap W)^\#$. As $t' \in S - S \cap W$ and $S \cap W \subseteq Z(S)$, each coset of $S \cap W$ in S contains precisely 2 involutions. Thus, $S - \{t\}$ contains just $2(k - 1)$ involutions. It follows that $c - d = 2$. By Lemma 4.3 (iii) and (i), we have $d = 0$ and $n = k(2(k - 1) + 1)$.

LEMMA C.2. *$k = 4$.*

Proof. Suppose that $k \neq 4$. Since k is even and $k > 2$, $k > 4$. Now $S \cap W \subseteq Z(S)$ and A is transitive on $(S/Z(S))^\#$. By Lemma 3.3, S is abelian. Then $R = \Omega_1(S)$ is elementary abelian of order $2k$. All elements of $R^\#$ are conjugate. Let $t^g \in R, g \in G$. Then $R^{g^{-1}} \subseteq C(t)$, whereas R contains all the involutions in $C(t)$. It follows that $g \in N(R)$.

Thus, $N(R)$ is transitive on $R^\#$. Also, $N(R) \cap C(t)$ is transitive on $(R/\langle t\rangle)^\#$ and hence has orbits on $R - \langle t\rangle$ of length $k - 1$ or $2(k - 1)$. Since $|N(R) : N(R) \cap C(t)|$ is odd, the Sylow 2-subgroups of $H = N(R)/C(R)$ are cyclic. By Lemma 3.2, H is solvable. Also, H is primitive on $R^\#$, and $(2k - 1)(k - 1) \mid |H|$. However, if $2k = 2^{f+1}$, then $|H| \mid (2k - 1)(f + 1)$ by Lemma 3.4. Then $(2^f - 1) \mid (f + 1)$, contradicting the fact that $f > 2$.

We now complete case C. By Lemmas C.1 and C.2, $n = 4 \cdot 7 = 28$ and $C(t)^\Delta$ is A_4 or S_4. Since $0(W) \subseteq C(t')$, Lemma 4.4 implies that $0(W) = 1$. Also, $t^{\Delta'} \neq 1$, so that $|W| = 2$ or 4.

Since $|C(t)_{\alpha\beta}^\Delta| \leqslant 2$, $|G_{\alpha\beta}| = 2, 4$, or 8. If $|G_{\alpha\beta}| = 2$, then G is of Ree type [23, 18]. If $|G_{\alpha\beta}| = 8$, then G is $PSU(3, 3)$ [27]. Thus, we must have $|G_{\alpha\beta}| = 4$, $|G| = 28 \cdot 27 \cdot 4$. Since $d = 0$, a Sylow 7-subgroup of G is self-centralizing, and it is easy to check that $|G : N(S)| \not\equiv 1 \pmod 7$, contradicting Sylow's theorem.

Case D. Now n is even and $C(t)^\Delta$ has no regular normal subgroup. Thus, $C(t)^\Delta$ is $PSL(2, q)$, $PGL(2, q)$, for some odd prime power $q > 3$, or $PSU(3, q)$, $PGU(3, q)$, with q an odd prime power, or a group of Ree type. By Lemma 4.3 (v), the involution $(t')^\Delta$ is regular. By Lemma 2.2 (ii) and (R6) of Section 2, the latter possibilities for $C(t)^\Delta$ cannot occur, so $C(t)^\Delta$ is $PSL(2, q)$ or $PGL(2, q)$, q

an odd power, $q > 3$. By Lemma 4.4 (i), $|C(t)^{\Delta} : C(W)^{\Delta}| \leqslant 2$. Since $G_{\alpha\beta} \subseteq C(W)$, it follows that $C(W) = C(t)$.

Clearly, $k = q + 1$.

LEMMA D.1. *$c = q - 1$, and either*

(i) *$n = q^3 + 1$ and $d = 0$, or*

(ii) *$n = 1 + q(q^2 + 1)/2$ and $d = (q - 1)/2$.*

Proof. By definition, c is the number of involutions $x = (\alpha\beta)\ldots$. Such an involution x normalizes $G_{\alpha\beta}$, so centralizes t and fixes Δ. Moreover, x^{Δ} is regular.

In its usual permutation representation, $PGL(2, q)$ has precisely $q(q - 1)/2$ regular involutions, all of which are conjugate under $PSL(2, q)$. Thus, there are $(q - 1)/2$ involutions in $C(t)^{\Delta}$ which interchange α and β and are conjugate to t'^{Δ}. Moreover, with x as in the previous paragraph, t'^{Δ} and x^{Δ} are conjugate. If we have $x^{\Delta} = t'^{\Delta}$, then $xt' \in W \subseteq C(t')$ implies that $x \in t'\langle t\rangle$. Thus, $c = q - 1$.

By Lemma 4.3 (iv) and (ii), $c - d = q - 1$ or $(q - 1)/2$, and $n - 1 = (k - 1)((c - d)(k + 1) = q^3$ or $q(q^2 + 1)/2$.

LEMMA 2.

(i) *$G_{\alpha\beta}$ is semiregular on $\Omega - \Delta$.*

(ii) *$|W| \mid (q + 1)$ and $|G_{\alpha\beta}| \mid (q^2 - 1)$.*

(iii) *If $1 \neq U \subseteq W$, then $N(U) = C(U) = C(t)$.*

(iv) *If $1 \neq U \subseteq G_{\alpha\beta}$ and $U \cap W = 1$, then $N(U)_{\alpha} = C(U)_{\alpha}$.*

Proof. (i) By Lemma 4.4 (ii), W is semiregular on $\Omega - \Delta$. Suppose that $G_{\alpha\beta}$ is not semiregular on $\Omega - \Delta$ and U is $G_{\alpha\beta\gamma}$ for some $\gamma \in \Omega - \Delta$, such that $G_{\alpha\beta\gamma} \neq 1$. Then $U \cap W = 1$, and so $|U|$ is odd. Take x any element interchanging α and β. Now $|U^{\Delta}| = |U|$ and x^{Δ} inverts U^{Δ}, so x inverts U.

Now let Γ be the fixed point set of U. Then $\Delta \cap \Gamma = \{\alpha, \beta\}$, as $C(t)^{\Delta}$ is a Zassenhaus group. Also, t^{Γ} fixes just α and β, so that $|\Gamma|$ is even and $N(U)^{\Gamma}$ is $PSL(2, q')$ or $PGL(2, q')$. Since $\mathrm{Aut}(U)$ is abelian, $C(U)^{\Gamma}$ contains $PSL(2, q')$. Thus, some element interchanging α and β centralizes U, in contradiction to the previous paragraph.

(ii) We know that $W \subseteq C(t')$. Thus, W acts on Δ'. Since $\Delta \cap \Delta' = \varnothing$, $W^{\Delta'}$ is semiregular by (i).

(iii) Since W is semiregular on $\Omega - \Delta$, the fixed point set of U is Δ. Thus $N(U) \subseteq C(t) = C(W) \subseteq C(U)$.

(iv) By (i), U fixes only α and β. Thus, $N(U)_{\alpha}$ fixes β and $N(U)_{\alpha} \subseteq G_{\alpha\beta} \subseteq C(U)$.

LEMMA D.3. *G_α has a normal subgroup Q of order $n - 1$ such that Q is regular on $\Omega - \alpha$. Also,* $(|Q|, |G_{\alpha\beta}|) = 1$.

Proof. By Lemmas D.1 and D.2 (ii), $G_{\alpha\beta}$ is a Hall subgroup of G_α. If $P \neq 1$ is a Sylow p-subgroup of $G_{\alpha\beta}$, then, by Lemma D.2, $N(P)_\alpha = C(P)_\alpha$. By repeated use of Burnside's transfer theorem, we find that G_α has a normal subgroup Q such that $G_\alpha = Q \cdot G_{\alpha\beta}$ and $Q \cap G_{\alpha\beta} = 1$. Then $|Q| = n - 1$ and Q is regular on $\Omega - \alpha$.

LEMMA D.4. *G is simple.*

Proof. If G is not simple, let M be a proper normal subgroup of G. Then M is transitive and $G = M \cdot G_\alpha = M \cdot Q \cdot G_{\alpha\beta}$. Thus, $MQ \leq G$, and $(MQ)\, G_{\alpha\beta} = G$. Therefore, MQ is 2-transitive on Ω.

Suppose first that $MQ = G$. Then $|G : M| \mid |Q|$, and $(|G_{\alpha\beta}|, |Q|) = 1$, by Lemma D.3, so $G_{\alpha\beta} \subseteq M$. Thus, $t \in M$ and $[t, Q] \subseteq Q \cap M$, so that $Q = (M \cap Q)\, C_Q(t)$. Thus, $G = M \cdot C_Q(t)$. Since $t' \in M$, $C_M(t)^\Delta \neq 1$. Since $C(t)^\Delta$ is $PSL(2, q)$ or $PGL(2, q)$, $C_M(t)^\Delta \lhd C(t)^\Delta$, and $G_{\alpha\beta} \subset C_M(t)$, we have $C(t) \subseteq C_M(t)$. Thus, $G = M$, a contradiction.

Consequently, $L = MQ \lhd G$, with L 2-transitive on Ω. As $L \subset G$, either L has a regular normal subgroup, or L is $PSL(2, q')$, $PGL(2, q')$, $PSU(3, q')$, $PGU(3, q')$, or a group of Ree type. L has no regular normal subgroup, as G does not. If L is $PSL(2, q')$ or $PGL(2, q')$, then $G \subseteq P\Gamma L(2, q')$. Since $G_{\alpha\beta}$ is cyclic, $G = PSL(2, q')$ or $PGL(2, q')$, which is not the case. If L is $PSU(3, q')$ or $PGU(3, q')$, then $q' + 1 = |\Delta| = q + 1$ and $|G_{\alpha\beta}| \geqslant |L_{\alpha\beta}| \geqslant (q^2 - 1)/3$. By Lemma D.2 (ii), the results of Suzuki [25] and O'Nan [20] imply that G is $PSU(3, q)$ or $PGU(3, q)$, which is again not the case.

Finally, suppose that L is of Ree type. By Section 2, $C_L(t) = H \times \langle t \rangle$, where $H = PSL(2, q')$. Then $q = q'$, $n = q^3 + 1$ and $d = 0$ (Lemma D.1). Consequently, all involutions of G are in L. Let V be a Sylow 2-subgroup of $C_L(t)$ such that $t' \in V$ and let $\langle x \rangle$ be the Sylow 2-subgroup of $G_{\alpha\beta}$. We claim that $x = t$. If $x \in W$, then $L\langle x \rangle$ has Sylow 2-subgroups of type $(2, 2, |x|)$ and all involutions of $L\langle x \rangle$ are conjugate in G. Thus $|x| = 2$ and $x = t$. Suppose now that $x \notin W$, so that $C_G(t)^\Delta = C_L(t)^\Delta \langle x \rangle^\Delta$ and $x^2 \in W$. As above $x^2 = t$. Now $C_G(V) = V \times O(W)$ and $(N_G(V) \cap C(t))^\Delta$ is isomorphic to S_4. As all involutions of V are conjugate in $N_L(V)$, $7 \mid |N_L(V)|$. It follows that $|N_G(V)/C_G(V)| = 2 \cdot 3 \cdot 7$ and $N_L(V)/C_L(V)$ contains a subgroup isomorphic to S_3. This is impossible, hence $x = t$ as claimed. Thus V is a Sylow 2-subgroup of G.

Now $C(V) = V \times O(W)$. Since V is also a Sylow 2-subgroup of $C_L(t')$ containing $\langle t', t \rangle$, $C(V) = V \times O(W')$. Therefore, $O(W) = O(W')$. Thus, $O(W)$ fixes $\Delta \cup \Delta'$ pointwise, and by Lemma D.2, $O(W) = 1$. It follows that $W = \langle t \rangle$, so that $L = G$, contrary to hypothesis.

LEMMA D.5. *$n = q^3 + 1$ and all involutions of G are conjugate.*

Proof. If $n \neq q^3 + 1$, by Lemma D.1 we have $|Q| = n - 1 = q(q^2 + 1)/2$. The proof will be divided into three parts.

(i) We first show that $Q = Q_0 L$, where $Q_0 = C_Q(t)$ and L are elementary abelian groups of order q and $(q^2 + 1)/2$, respectively. Moreover, $G_{\alpha\beta}$ normalizes both Q_0 and L, and $G_{\alpha\beta}$ is fixed-point-free and irreducible on L.

By Lemma D.3, $(|Q|, |G_{\alpha\beta}|) = 1$. Taking $Q_0 = C_Q(t)$, we see that Q_0 is an elementary abelian group of order q such that $G_{\alpha\beta}$ acts irreducibly on Q_0. Moreover, if l is some prime divisor of $(q^2 + 1)/2$, then $G_{\alpha\beta}$ normalizes a Sylow l-subgroup L of Q.

Identifying Q with $\Omega - \alpha$, where $1 \in Q$ corresponds to $\beta \in \Omega - \alpha$ (as we may by Lemma D.3), we see, using Lemma D.2 (i), that all fixed points of elements of $G_{\alpha\beta}^{\#}$ lie in Q_0, which corresponds to Δ. Thus, $G_{\alpha\beta}$ is fixed-point-free on L.

Since $G_{\alpha\beta}$ is fixed-point free on L, and $|G_{\alpha\beta}| \geqslant 2\,|G_{\alpha\beta}^{\Delta}| \geqslant q - 1$, we have $|L| \geqslant q$. However, $|Q : Q_0| = (q^2 + 1)/2$, so that l is the only prime divisor of $(q^2 + 1)/2$, L is elementary abelian of order $(q^2 + 1)/2$, and $G_{\alpha\beta}$ acts irreducibly on L.

(ii) We next show that $Q = Q_0 \times L$ is abelian.

Now $N_Q(Q_0) = Q_0 L'$, with $L' = N_L(Q_0)$. Since $G_{\alpha\beta}$ normalizes L and Q_0, $G_{\alpha\beta}$ normalizes L'. Since $G_{\alpha\beta}$ acts irreducibly on L, $L' = 1$ or L. If $L' = 1$, $[Q : N_Q(Q_0)] = [Q : Q_0] = (q^2 + 1)/2$. But $(q^2 + 1)/2 \not\equiv 1 (\mathrm{mod}\ p)$, contradicting Sylow's theorem. Therefore, $Q_0 \lhd Q$.

Now $LG_{\alpha\beta}$ is a Frobenius group with Frobenius kernel L and Frobenius complement $G_{\alpha\beta}$ by (i). Since $LG_{\alpha\beta}$ normalizes Q_0 and $W \subseteq G_{\alpha\beta}$ centralizes Q_0, L centralizes Q_0. Thus, (ii) follows.

(iii) We now derive a contradiction using an argument of Suzuki [27, Lemma 12].

There are $q(\frac{1}{2}(q^2 + 1) - 1) = q(q^2 - 1)/2$ linear characters ζ of Q not having L in their kernels. Each induced character ζ^{G_α} is irreducible since $G_{\alpha\beta}$ is fixed-point free on L. In this manner we obtain $s = q(q^2 - 1)/2\,|G_{\alpha\beta}|$ irreducible characters $\phi_1, \phi_2, \ldots, \phi_s$ of G_α, each of degree $|G_{\alpha\beta}|$, and these exhaust the irreducible characters of G_α not having L in their kernels.

Let $\chi_1, \chi_2, \ldots, \chi_s$ be the exceptional characters of G corresponding to $\phi_1, \ldots, \phi_s$ [11, pp. 146, 147]. These are all of the irreducible characters of G whose restrictions to G_α do not contain all the ϕ_i with the same multiplicity.

There is a linear character $\lambda \neq 1$ of G_α, having Q in its kernel, such that $\lambda^{t'}$ and λ agree on $G_{\alpha\beta}$. (For example, the linear character of G_α whose kernel is the product of Q and the subgroup of index 2 in $G_{\alpha\beta}$.) Then $\lambda^G = \sigma + \zeta$ with σ and ζ distinct non-principal irreducible characters of G.

If $1^G = 1 + \theta$, then $\theta \neq \sigma, \zeta$, as otherwise either σ or ζ would be linear, contradicting Lemma D.4.

For each i and j, we have $((1 - \lambda)^G, \chi_i - \chi_j) = \pm((1 - \lambda)^G, (\phi_i - \phi_j)^G) = \pm(1 - \lambda, \phi_i - \phi_j) = 0$, since $1 - \lambda = 0$ on Q. Suppose that $(1 - \lambda)^G$ contains an exceptional character χ_i. It follows then that $((1 - \lambda)^G, \chi_j) = ((1 - \lambda)^G, \chi_i) \neq 0$ for each j, so that $(1 - \lambda)^G$ contains each of the s exceptional characters. Then $3 \geqslant s = q(q^2 - 1)/2 \mid G_{\alpha\beta} \mid$ whereas $q \geqslant 5$ and $\mid G_{\alpha\beta} \mid \mid (q^2 - 1)$ (Lemma D.2 (ii)), a contradiction.

In particular, neither σ nor ζ is an exceptional character. Suppose that the restrictions of both σ and ζ to G_α contain a character ϕ_i. Then these restrictions contain each of the s characters ϕ_i, and we have $1 + q(q^2 + 1)/2 = \sigma(1) + \zeta(1) \geqslant s\phi_1(1) + s\phi_1(1) = 2s \mid G_{\alpha\beta} \mid = q(q^2 - 1)$, a contradiction.

We may thus assume that the restriction of σ to G_α contains no ϕ_i, and hence has L in its kernel. It follows that the kernel of σ contains L, contradicting the simplicity of G (Lemma D.4).

LEMMA D.6. *$C(t)$ has a unique normal subgroup $C_0(t)$, having the properties: $W \subseteq C_0(t)$, $C_0(t)^\Delta = PSL(2, q)$, and $C_0(t)^{(1)} = PSL(2, q)$ or $SL(2, q)$.*

Proof. As $C(t)^\Delta = PSL(2, q)$ or $PGL(2, q)$, there is a unique $C_0(t)$ satisfying the first two conditions. Since $W \subseteq Z(C_0(t))$, a result of Schur [24] implies that either $C_0(t)^{(1)} = PSL(2, q)$ or $SL(2, q)$, or $q = 9$ and $C_0(t)^{(1)} \cap W$ is cyclic of order dividing 6. Suppose then $q = 9$ and that $3 \mid \mid C_0(t)^{(1)} \cap W \mid$ and let L be a Sylow 3-subgroup of $C_0(t)^{(1)}$. Then L is a nonabelian group of order 27 and exponent 3. On the other hand, $C_Q(t)$ is an elementary abelian group of order 9, contained in $C_0(t)^{(1)}$, with $C_Q(t) \cap W = 1$, contradicting the structure of L.

LEMMA D.7. *$C(t)^\Delta$ is not $PSL(2, q)$.*

Proof. If $C(t)^\Delta = PSL(2, q)$, then $C(t) = C_0(t)$ and $C(t)^{(1)} = PSL(2, q)$ or $SL(2, q)$ be Lemma D.6.

(i) Suppose that $C(t)^{(1)} = PSL(2, q)$. Since $C(t)^\Delta = PSL(2, q)$ and $C(t)^\Delta$ contains the regular involution t'^Δ, $q \equiv 3 \pmod 4$. Also, $q + 1 > 4$. Since $C(t) = C(t)^{(1)} \times W$, $C(t) \cap C(t') = (C(t)^{(1)} \cap C(t')) \times W$, where $C(t)^{(1)} \cap C(t')$ is dihedral of order $q + 1$. Thus, $Z(C(t) \cap C(t')) = \langle t' \rangle \times W$. Interchanging t and t', $Z(C(t) \cap C(t')) = \langle t \rangle \times W'$. Since $W \cap W' = 1$, $W' = \langle t' \rangle$. By Section 2, G is of Ree type, in contradiction to the minimality of G.

(ii) Suppose that $C(t)^{(1)} = SL(2, q)$. Let $\langle x \rangle$ be the Sylow 2-subgroup of W, $\langle x' \rangle$ a Sylow 2-subgroup of W'. Since t' centralizes W, $W \subseteq C(t') = C(W')$, by Lemma D.2 (iii). Thus, $\langle x, x' \rangle$ is abelian of exponent $\mid x \mid$.

Since $C(t')^{\Delta'}$ is $PSL(2, q)$, $x^{\Delta'}$ is inverted by an involution $y^{\Delta'}$ of $C(t')^{\Delta'}$. Since all involutions of $C(t')^{\Delta'}$ are conjugate, we may suppose y is an involution. Then $x^y = x^{-1}w$, with $w \in \langle x' \rangle$. Therefore $|w| \leqslant |x|$.

We note first that $x \neq t$. For if $x = t$, $C(t) = C(t)^{(1)} \times 0(W)$, and the Sylow 2-subgroup of $C(t)$ is generalized quaternion, contradicting the fact that $\langle t, t' \rangle$, a Klein group, is contained in $C(t)$.

Suppose next that $|w| < |x|$. Then $t^y = t$, so $y \in C(t) = C(x)$. Then $x^2 = w \in W'$ and $x^2 \in W$, so $x^2 = 1$, and $x = t$, in contradiction to the previous paragraph.

Thus, $|w| = |x|$, and $(xy)^2 = w$, where w generates $\langle x' \rangle$. But any Sylow 2-subgroup of $C(t')$ is the central product of $\langle w \rangle$ and a generalized quaternion group. Therefore, w cannot be a square in $C(t')$, a contradiction.

LEMMA D.8. *If $q \equiv 1 \pmod 4$, then G has quasi-dihedral Sylow 2-sub-groups.*

Proof. By Lemma D.5, $C(t)$ contains a Sylow 2-subgroup S of G. We may assume that $S = \langle t' \rangle S_{\alpha\beta}$. Here, S^{Δ} is dihedral of order $\geqslant 8$, by Lemma D.7. Thus, if S is not quasi-dihedral, then S must be dihedral. However, if S is dihedral, then $C(t)$ has a normal 2-complement [12, p. 260], which is not the case.

LEMMA D.9. *If $q \equiv 3 \pmod 4$, then a Sylow 2-subgroup of G is a wreathed group $Z_{2^j} \wr Z_2$, where 2^j is the largest power of 2 dividing $q + 1$.*

Proof. Let S be a Sylow 2-subgroup of $C(t)$ such that S contains the Sylow 2-subgroup $\langle x' \rangle$ of W'. If $C_0(t)$ is as in Lemma D.6, then $|C(t) : C_0(t)| = 2$, and $C_0(t)^{(1)} = PSL(2, q)$ or $SL(2, q)$.

(i) Suppose that $C_0(t)^{(1)} = PSL(2, q)$, so that $C_0(t) = C_0(t)^{(1)} \times W$. Then we may assume that $S = DR$, where D is a dihedral Sylow 2-subgroup of $C_0(t)^{(1)}$, R is a cyclic Sylow 2-subgroup of $G_{\alpha\beta}$, and $t' \in Z(S) \cap D$.

Then, since $t' \in Z(S)$, S acts on Δ'. Now the kernel of restriction map is W', so $S \cap W' \lhd S$. Since $N(S \cap W') = C(S \cap W')$ by Lemma D.2 (iii), $S \cap W' \subseteq Z(S)$. Also, $Z(S) \subseteq \langle R, t' \rangle$, so $S \cap W' \subseteq \langle R \cap W', t' \rangle$. Since R is a Sylow 2-subgroup of $G_{\alpha\beta}$, $\{\alpha, \beta\} \subseteq \Delta$, and $\Delta' \cap \Delta = \varnothing$, $R \cap W' = 1$. Therefore, $S \cap W' = \langle t' \rangle$. Thus, the Sylow 2-subgroup of W is of order 2.

Since the Sylow 2-subgroup of the two-point stabilizer of $PGL(2, q)$, $q \equiv 3 \pmod 4$, is cyclic of order 2, $|R| = 4$. Now, $D^{\Delta'} \lhd S^{\Delta'}$ and $R^{\Delta'} \cap D^{\Delta'} = 1$. On the other hand, $R^{\Delta'}$ is a cyclic subgroup of order 4 of the dihedral group $S^{\Delta'}$. Therefore, $R^{\Delta'} \lhd S^{\Delta'}$. Thus, $R^{\Delta'} \cap D^{\Delta'} \supseteq Z(S^{\Delta'})$, a contradiction.

(ii) Suppose that $C_0(t)^{(1)} = SL(2, q)$. Define x, y, and w as in the proof of Lemma D.7, part (ii). We may assume that $y \in C_0(t')$. Then, $x^y = x^{-1}w$, with y an involution, $w \in \langle x' \rangle$, and $|w| \leqslant |x|$.

If $x = t$, then $C_0(t) = C_0(t)^{(1)} \times 0(w)$ contains only one involution, t. On the other hand, since $q \equiv 3 \pmod 4$, $t' \in C_0(t)$, a contradiction. As before, we have $|x| = |w|$.

Suppose that $|x| < 2^f$. Then, $x^{\Delta'} \in C_0(t')^{\Delta'}$, so that both x and y are in $C_0(t')$. As $(xy)^2 = w$ and all involutions in $C_0(t')^{\Delta'}$ are conjugate to $t^{\Delta'}$, we have $xyu = w_1 \in W'$ for some element $u \in C(t') = C(W')$ with $u^2 = 1$. Then $w = (xy)^2 = (uw_1)^2 = w_1{}^2$, whereas $\langle w \rangle$ is the Sylow 2-subgroup of W', a contradiction.

Thus, $|x| = 2^f$. The group $\langle x, w, y \rangle$ is a Sylow 2-subgroup of $C(t)$, and therefore, of G. Since x and w commute, x and x^y commute. Moreover, $\langle x \rangle \cap \langle x^y \rangle = \langle x \rangle \cap \langle x^{-1}w \rangle$ and the involution of $\langle x^{-1}w \rangle$ is tt'. Thus, $\langle x \rangle \cap \langle x^y \rangle = 1$, and $\langle x, w, y \rangle$ is wreathed, as asserted.

At this stage, G is a simple group of order $(1 + q^3)\, q^3(q - 1)\, |W|$, where $|W| \mid (q + 1)$ (Lemma D.2 (ii)). If $|W| = q + 1$ or $(q + 1)/(q + 1, 3)$, then the results of Suzuki [27] or O'Nan [20] will complete the proof. It follows that we must build up the order of W. In doing this, we modify an argument of Brauer [1, Chap. 6]. Brauer carried out the process of building up $|W|$ in the case where the Sylow 2-subgroups of G are quasi-dihedral. In the following argument, we will treat the cases of quasi-dihedral and wreathed Sylow 2-subgroups simultaneously.

LEMMA D.10. *Let p be an odd prime divisor of $q + 1$. Then there is a Sylow p-subgroup P of $C(t)$ such that $P \subseteq C(t) \cap C(t')$. There are involutions y_1 in $C(t)$ and y_2 in $C(t')$ such that $(t')^{y_1} = tt'$, $t^{y_2} = tt'$, y_1 and y_2 normalize P, and $\langle y_1, y_2 \rangle$ induces on $\langle t, t' \rangle$ a group of automorphisms isomorphic to S_3. Moreover, $y_1{}^{\Delta}$ inverts P^{Δ} and $y_2^{\Delta'}$ inverts $P^{\Delta'}$. Finally, if $p > 3$, P is a Sylow p-subgroup of G.*

Proof. Since $C(t)^{\Delta} = PGL(2, q)$, there is a dihedral group L^{Δ} of order $2(q + 1)$ in $C(t)^{\Delta}$. Since t' fixes no points of Δ, we may choose this dihedral group so that t'^{Δ} is central in it. Let P be a Sylow p-subgroup of $C(t)$ such that P^{Δ} is the Sylow p-subgroup of L^{Δ}. As $|G| = (q^3 + 1)\, q^3(q - 1)\, |W|$, if $p > 3$, then P is a Sylow p-subgroup of G. Since $W \subseteq Z(C(t))$, $P \subseteq C(t')$.

Assume that $q \equiv 3 \pmod 4$. Then all involutions in $PSL(2, q)$ are regular. It follows that there is an involution y^{Δ} in $C(t)^{\Delta}$, which fixes no point of Δ, with the property that y^{Δ} centralizes t'^{Δ} and y^{Δ} inverts P^{Δ}.

Now suppose that $q \equiv 1 \pmod 4$. Then in the dihedral group L^{Δ}, there is an involution y^{Δ} inverting P^{Δ} and centralizing t'^{Δ}. As t'^{Δ} fixes no point of Δ, either y^{Δ} or $(yt')^{\Delta}$ is an involution fixing no point of Δ.

Thus, there always is an involution $y^{\Delta} \in C(t)^{\Delta}$ such that y^{Δ} centralizes t'^{Δ}, y^{Δ} fixes no point of Δ, and y^{Δ} inverts P^{Δ}. Now t'^{Δ} is regular and all regular involutions of $C(t)^{\Delta}$ are conjugate to t'^{Δ}. Therefore, there is an involution $y_1 \in C(t)$ with $y_1^{\Delta} = y^{\Delta}$. Similarly, in $C(t')$ there is an involution $y_2 \in C(t')$ such that $y_2^{\Delta'}$ centralizes $t^{\Delta'}$ and inverts $P^{\Delta'}$.

Since t'^{Δ} is centralized by y_1^{Δ}, $(t')^{y_1} = t'$ or tt'. In the first case, $\langle y, t, t' \rangle$ is an elementary abelian group of order 8 ($y \notin \langle t, t' \rangle$, as $y \notin C(P)$). However, this contradicts Lemmas D.8 and D.9. Thus, $(t')^{y_1} = tt'$, and similarly, $t^{y_2} = tt'$. This completes the proof.

LEMMA D.11.

(i) *If the Sylow 2-subgroups of G are wreathed, then the characters in* $B_0(2, G)$ *have degrees* $1, q^3, q(q-1), q^2-q+1, q(q^2-q+1), (q-1)(q^2-q+1)$, *and* $(q+1)(q^2-q+1)$.

(ii) *If the Sylow 2-subgroups of G are quasi-dihedral, then the characters in* $B_0(2, G)$ *have degrees* 1, q^3, $q(q-1)$, q^2-q+1, $q(q^2-q+1)$, *and* $(q+1)(q^2-q+1)$.

In either case, the permutation character θ *is the only character in* $B_0(2, G)$ *of degree* q^3.

Proof. By Lemma 3.6, the permutation character θ is in $B_0(2, G)$. In [6], Brauer found the degrees of the characters in $B_0(2, G)$ for a group G having wreathed Sylow 2-subgroups and no normal subgroup of index 2. The degrees are $1, \epsilon f^3, f(f+1), f^2+f+1, \epsilon f(f^2+f+1), \epsilon(f+1)(f^2+f+1)$, and $\epsilon(f+1)(f^2+f+1)$, where f is an integer and $\epsilon = \pm 1$. Brauer also states that there is only one character of degree ϵf^3. Thus, to prove (i) it suffices to show that $f = -q$. From the fact that $\theta(1) = q^3$, for q an odd prime power, it follows that $q^3 = \epsilon f^3$ or $q^3 = f^2 + f + 1$. Brauer also states that $f(f+1) \mid |C(t)|$. Since $(q^3 - 1) \nmid |C(t)|$, we have $q^3 = \epsilon f^3$, and $q = \pm f$. If $q = f$, then $(q^2 + q + 1) \mid |G|$, which is not the case. Thus, $q = -f$, proving (i).

To prove (ii), we first recall that all involutions of G are conjugate and $q \equiv 1 \pmod 4$. Thus, $\langle x, t' \rangle$ is a Sylow 2-subgroup of G, with $\langle x \rangle$ the Sylow 2-subgroup of $G_{\alpha\beta}$. Then, $x, t' \in C(t) - C(t)^{(1)}$, and $xt' \in C(t)^{(1)} = SL(2, q)$. Since all elements of order 4 of $SL(2, q)$ are conjugate, xt' is conjugate to the element of order 4 in $\langle x \rangle$. We may now apply the results of Brauer [7, Section 8]. Brauer has shown that $B_0(2, G)$ consists of characters $\chi_0 = 1_G, \chi_1, \chi_2, \chi_3, \chi_4$, and characters $\chi^{(j)}$, such that the characters $\chi^{(j)}$ have the same degree, x. Moreover, if we set, $x_i = \chi_i^{(1)}$, there are signs, $\delta_1, \delta_2, \delta_3$, and an integer $m \equiv 1 \pmod 4$, such that

$$1 + \delta_1 x_1 = \delta_1 x = -\delta_2 x_2 - \delta_3 x_3, \qquad 1 + \delta_2 x_2 = \delta_2 x_4$$
$$\delta_1\delta_2\delta_3 = 1, \qquad x_1 x_2 = m^2 x_3, \qquad x_2 \equiv -m \equiv -1 \pmod 4),$$
$$x_1 \equiv \delta_1 \pmod 4, \qquad x_4 \equiv 0 \pmod 2,$$
$$\chi_1(t) = \delta_1 m, \qquad \chi_2(t) = -\delta_2 m, \qquad \chi_3(t) = -\delta_3.$$

First we show that $\theta = \chi_1$. Since $1 + \delta_1 x_1 = \delta_1 x$, and $x_1 \equiv \delta_1 \pmod 4$, x is even, and $\theta \neq \chi^{(j)}$ for any j. Also, $\theta(t) = q \equiv 1 \pmod 4$ and $\theta(1) = q^3 \equiv 1 \pmod 4$, so that $\theta \neq \chi_2, \chi_3$, or χ_4. Thus, $\theta = \chi_1$. Since $\chi_1(t) = \delta_1 m$, $q = \delta_1 m$.

Now $1 \equiv q^3 = x_1 \equiv \delta_1 \pmod 4$, so that $\delta_1 = 1$. As $\delta_1\delta_2\delta_3 = 1$, $\delta_2 = \delta_3$. Since $x_1 x_2 = m^2 x_3$, $x_3 = q x_2$. Using the relation $1 + \delta_1 x_1 = -\delta_2 x_2 - \delta_3 x_3$, we obtain $x_2 = q^2 - q + 1$, $x_3 = q(q^2 - q + 1)$, and $\delta_2 = -1$. From $1 + \delta_2 x_2 = \delta_2 x_4$, it follows that $x_4 = q(q-1)$. Finally, $1 + \delta_1 x_1 = \delta_1 x$ implies that $x = q^3 + 1 = (q+1)(q^2 - q + 1)$. This proves (ii), and the proof shows that $\theta = \chi_1$ is the only character of degree q^3 in $B_0(2, G)$.

LEMMA D.12. *Let $p \neq 3$ be an odd prime dividing $q + 1$. Then the p-part of $|W|$ equals the p-part of $q + 1$. The 2-part of $|W|$ equals the 2-part of $q + 1$.*

Proof. The last assertion follows from Lemmas D.8 and D.9. If $p \neq 3$ is an odd prime and $p \mid (q + 1)$, we first show that $p \mid |W|$.

Suppose that $p \nmid |W|$ and let P, y_1, y_2 be as in Lemma D.10. Then P is cyclic and y_1 inverts P, so that $C(P) \subset N(P)$. We claim that $N(P) = \langle C(P), y_1 \rangle$. For, let S be a Sylow 2-subgroup of $\langle C(P), y_1 \rangle$ containing t, t', and y_1. Then $t = t'^{y_1} t' \in S^{(1)}$. By Lemmas D.8 and D.9, the derived group of a Sylow 2-subgroup of G is cyclic. Thus, $\langle t \rangle = \Omega_1(S^{(1)})$. Also, $N(P)/C(P)$ is cyclic, so that $C(P)S \trianglelefteq N(P)$. By the Frattini argument, $N(P) \subseteq C(P)\, SC(t) = C(P)\, C(t)$. However, in $C(t)$ the only nontrivial automorphism induced on P is involutory and induced by y_1. Thus, $N(P) = \langle C(P), y_1 \rangle$.

We now apply results of Brauer [7, Section 9] concerning the principal p-block of G. $B_0(p, G)$ contains $2 + (|P| - 1)/2$ characters 1, $\psi_1, \psi^{(i)}$. These are all real-valued, and there is a sign δ such that $\delta + \psi_1(1) = \psi^{(i)}(1)$ and $\psi_1(u) = \delta$ for each p-singular element u of G.

Suppose that $\psi_1(1)$ is even. Then each $\psi^{(i)}(1)$ is odd, and since $\psi^{(i)}$ is real-valued, $\psi^{(i)}$ is in $B_0(2, G)$ (Brauer [6]). By Lemma 3.6, the permutation character θ of degree q^3 is in $B_0(p, G)$, so that $\theta = \psi_1$ or $\psi^{(i)}$, for some i. Now each $\psi^{(i)}$ has degree q^3 and each $\psi^{(i)}$ is in $B_0(2, G)$. This contradicts Lemma D.11, as $(|P| - 1)/2 > 1$.

Therefore, $\psi_1(1)$ is odd and $\psi^{(i)}(1)$ is even. By Lemma 3.6, $\theta \in B_0(p, G)$, so $\theta = \psi_1$. Since $p \nmid |G_{\alpha\beta}|$, $\delta = \psi_1(u) = \theta(u) = -1$ for all p-singular elements u. Then $\psi^{(i)}(1) = q^3 - 1$, which does not divide $|G|$. This is a contradiction.

We thus have $p \mid |W|$. Again let P, y_1, and y_2 be as in Lemma D.10. The dihedral group $D = \langle y_1, y_2 \rangle$ normalizes P. As $P/P \cap W$ is cyclic and $P \cap W \subseteq Z(C(t))$, P is abelian. Since y_1 inverts $P/P \cap W$, we have $P = P_0 \times (P \cap W)$, where $|P_0|$ is the p-part of $q + 1$ and y_2 inverts P_0.

Suppose $|P_0| > |P \cap W|$. From the structure of P, it follows that D modulo $C_D(P) \cdot 0_p(D)$ is abelian. By Lemma D.10, D contains a 3-element inverted by y_1. Thus, since $p \neq 3$, the Sylow 3-subgroup of D centralizes P. However, $C(P) \subseteq C(P \cap W) = C(t)$ by Lemma D.2 (iii). Since $y_1 y_2$ induces an automorphism of order 3 on $\langle t, t' \rangle$, the Sylow 3-subgroup of D is not contained in $C(P)$. This is a contradiction, so that $|P_0| = |P \cap W|$, as asserted.

LEMMA D.13. *If 3^h is the 3-part of $q + 1$ and 3^k is the 3-part of $|W|$, then $h - k \geqslant 2$ and $k \geqslant 1$.*

Proof. If $3 \nmid (q + 1)$, then, by Lemma D.12, $|W| = q + 1$ and Suzuki's result [27] implies that $G = PGU(3, q)$. Thus, $3 \mid (q + 1)$. If $h - k \leqslant 1$, then the results of Suzuki [27] and O'Nan [20] imply that $G = PSU(3, q)$ or $PGU(3, q)$. Therefore, $h - k \geqslant 2$, so it remains to show that $k \geqslant 1$.

Suppose that $3 \nmid |W|$, and let P, y_1, y_2 be as in Lemma D.10. Since $9 \mid (q + 1), 3 \mid (q^2 - q + 1)$, and $9 \nmid (q^2 - q + 1)$, so that a Sylow 3-subgroup of G has order $3|P| = 3^{h+1}$. Also, P is cyclic. Since $\langle y_1, y_2 \rangle$ induces S_3 on $\langle t, t' \rangle$ and normalizes P, $\langle y_1, y_2 \rangle$ has a 3-element z centralizing P but not t. Thus, $B = \langle P, z \rangle$ is an abelian Sylow 3-subgroup of G. Also, B is cyclic or of type $(3^h, 3)$.

Since B is abelian, $N(B)$ controls the fusion of B. Thus, there is a 2-element x in $N(B)$ with $u^x = u^{-1}$ for $\langle u \rangle = P$, and $C(B) \subset N(B)$.

If B is cyclic, then $|N(B) : C(B)| = 2$.

If B is abelian of type $(3^h, 3)$, then $N(B)$ does not act indecomposably on B as $3^h > 3$. Thus, there are elements v_1, v_2 in B such that $|v_1| = 3^h$, $|v_2| = 3$, $B = \langle v_1 \rangle \times \langle v_2 \rangle$, and $\langle v_1 \rangle, \langle v_2 \rangle$ are both $N(B)$-invariant. In particular, v_2 is not conjugate to any element of $\langle v_1 \rangle$, and $N(B)/C(B)$ is an elementary abelian group of order 2 or 4. If $|N(B) : C(B)| = 4$, then $|C(v_2)|$ is even. However, $\langle v_1^3 \rangle = \mho_1(B) = \langle u^3 \rangle$, and each element of $C(t)$ of order 3 is a cube in $C(t)$, so that v_2 must fuse to an element of $\langle u^3 \rangle = \langle v_1^3 \rangle$, which is not the case.

Thus, we always have $|N(B) : C(B)| = 2$ and $N(B) = \langle C(B), x \rangle$. Since B centralizes no conjugate of t, $|C(B)|$ is odd and $|x| = 2$. Since B is abelian and G is simple, a result of Grün [14, p. 215] implies that x inverts B.

We may now use the same argument as in the proof of Lemma D.12 to obtain a contradiction.

LEMMA D.14. *Let P be a Sylow 3-subgroup of $C(t)$. Then $P = P_0 \times P_1$, where P_1 is the Sylow 3-subgroup of W, $|P_0| = 3^h$, $|P_1| = 3^k$, $P_0 = \langle u_0 \rangle$, $P_1 = \langle u_1 \rangle$. There is a 3-element z such that $B = \langle P, z \rangle$ is a Sylow 3-subgroup of G and $|B : P| = 3$. There is an involution y in $N(B)$ such that y inverts z and u_0 and centralizes u_1.*

Proof. Let P and $y_1 = y$ be as in Lemma D.10. As $P \cap W \subseteq Z(P)$ and $P/P \cap W$ is cyclic, P is abelian. Since y inverts $P/P \cap W$ and centralizes $P \cap W$, we have $P = P_0 \times P_1 = \langle u_0 \rangle \times \langle u_1 \rangle$ with P_0 inverted by y and $P_1 = P \cap W$ centralized by y. Letting y_2 be as in Lemma D.10 and $\langle z \rangle$ a Sylow 3-subgroup of $\langle y_1, y_2 \rangle$, we have $z \notin C(t)$, $z \in N(P)$, with z inverted by y. Since $9 \nmid (q^2 - q + 1)$, $B = \langle P, z \rangle$ is a Sylow 3-subgroup of G and $[B : P] = 3$.

LEMMA D.15. *Set $u_0^z = u_0^a u_1^b$ and $u_1^z = u_0^c u_1^d$, where a, b, c, and d are integers. Then*

(i) $|u_1| = 3^k = 3$;

(ii) $a = 1 + 3^{h-1}a_0$, *with* $a_0 \not\equiv 0 \pmod 3$;

(iii) $c = 3^{h-1}c_0$, *with* $c_0 \not\equiv 0 \pmod 3$;

(iv) $d \equiv 1 \pmod 3$;

(v) $a_0 + bc_0 \equiv 0 \pmod 3$.

Proof. Let $\bar{y}$ and $\bar{z}$ be the automorphisms induced on $P_0 \times P_1$ by y and z, respectively. Then $\bar{y}$ corresponds to $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\bar{z}$ corresponds to $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, where the first columns are taken modulo 3^h and the second columns are taken modulo 3^k.

Now $\bar{z}^3 = 1$, so that

$$\bar{z}^{-1} = \bar{z}^2 = \begin{bmatrix} a^2 + bc & (a + d)b \\ (a + d)c & d^2 + bc \end{bmatrix}.$$

Also, $\bar{z}^{\bar{y}} = \bar{z}^{-1}$, so that

$$\bar{z}\bar{y} = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix} = \bar{y}\bar{z}^{-1} = \begin{bmatrix} -(a^2 + bc) & -(a + d)b \\ (a + d)c & d^2 + bc \end{bmatrix}.$$

Consequently, we must have

$$\begin{aligned} a^2 + bc &\equiv a \pmod{3^h}, \\ (a + d)c &\equiv -c \pmod{3^h}, \\ (a + d)b &\equiv -b \pmod{3^k}, \\ d^2 + bc &\equiv d \pmod{3^k}. \end{aligned} \tag{2}$$

Also, $\bar{z}^{-1} = \bar{y}\bar{z}\bar{y} = \left[\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right]$, so that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 - bc & (a - d)b \\ (d - a)c & d^2 - bc \end{bmatrix}.$$

Thus, we also have

$$
\begin{aligned}
a^2 - bc &\equiv 1 \pmod{3^h},\\
(a-d)c &\equiv 0 \pmod{3^h},\\
(a-d)b &\equiv 0 \pmod{3^k},\\
d^2 - bc &\equiv 1 \pmod{3^k}.
\end{aligned} \tag{3}
$$

We note that, since $| u_1^z | = | u_1 | = 3^k$, we must have $c = 3^{h-k}c_0$ for some integer c_0. Now suppose that $c_0 \equiv 0 \pmod 3$, and let $u^* = u_1^{3^{k-1}}$. Then $(u^*)^z \in \langle u^* \rangle$ and $u^* \in W^\#$. Thus, $z \in N(\langle u^* \rangle) = C(t)$ (Lemma D.2 (iii)) which is not the case. Thus, $c_0 \not\equiv 0 \pmod 3$.

From (2) and (3) we obtain

$$
\begin{aligned}
(2a+1)(a-1) &= 2a^2 - a - 1 \equiv 0 \pmod{3^h},\\
(2d+1)(d-1) &= 2d^2 - d - 1 \equiv 0 \pmod{3^k},\\
a^2 &\equiv 1 + bc \equiv d^2 \pmod{3^k}.
\end{aligned} \tag{4}
$$

Since $| u_0^z | = | u_0 |$ and $h - k \geqslant 2$, $3 \nmid a$. Thus, by (4), we have $a \equiv \pm d \pmod{3^k}$. If $a \equiv 2 \pmod 3$, then $2a + 1 \equiv 2 \pmod 3$, and this contradicts (4). Thus, $a \equiv 1 \pmod 3$. Since $(a - 1, 2a + 1) = 3$, by (4) either $a \equiv 1 \pmod{3^{h-1}}$ or $a \equiv -\frac{1}{2} \pmod{3^{h-1}}$.

Suppose that $a \equiv -\frac{1}{2} \pmod{3^{h-1}}$. Then from (2) it follows that $3^{h-k}bc_0 = bc \equiv a - a^2 \equiv -\frac{3}{4} \pmod{3^{h-1}}$, and consequently $h - k \leqslant 1$, a contradiction. Thus, $a \equiv 1 \pmod{3^{h-1}}$, and we can write $a = 1 + a_0 3^{h-1}$. By (2), $3^{h-k}bc_0 = bc \equiv -a_0 3^{h-1} \pmod{3^h}$. Since $3 \nmid c_0$, it follows that $3^{k-1} \mid b$. Write $b = 3^{k-1}b_0$.

Now $u_0^z = u_0^a u_1^{3^{k-1}b_0}$, so that $\langle u_0^3 \rangle$ is normalized by z. As $\bar{z}$ has order 3, u_0^9 is centralized by z. Set $\tilde{u}_0 = u_0^{3^{h-k}}$, so that $\tilde{u}_0 \in \langle u_0^9 \rangle$ and $\tilde{u}_0$ is centralized by z. Since $u_1^z = \tilde{u}_0^{c_0} u_1^d$ and $u_1 = u_1^{z^3} = u_0^{c_0(1+d+d^2)} u_1^{d^3}$, we have $c_0(1 + d + d^2) \equiv 0 \pmod{3^k}$, and since $c_0 \not\equiv 0 \pmod 3$, $1 + d + d^2 \equiv 0 \pmod{3^k}$. Then $d \equiv 1 \pmod 3$, and $d^2 + d + 1 = (d-1)^2 + 3d \equiv 0 \pmod 9$ if $k \geqslant 2$. Thus, if $k \geqslant 2$, we have $3d \equiv 0 \pmod 9$, whereas $d \equiv 1 \pmod 3$, a contradiction. Thus $k = 1$.

We now have $c = 3^{h-1}c_0$, $c_0 \not\equiv 0 \pmod 3$, $d \equiv 1 \pmod 3$, and $k = 1$. From (2) we also get $a_0 + bc_0 \equiv 0 \pmod 3$.

It only remains to show that $a_0 \not\equiv 0 \pmod 3$. If $a_0 \equiv 0 \pmod 3$, then a may be taken as 0, and $b \equiv 0 \pmod 3$. Thus, $u_0^z = u_0$ and $u_1^z = u_0^{3^{h-1}c_0} u_1$. The group $B/\langle u_0 \rangle$ is abelian, since it is of order 9, and $Z(B) = \langle u_0 \rangle$. Thus, B has class 2. Consider the action of $N(B)$ on $B/Z(B)$. The involution y inverts the coset $zZ(B)$ and centralizes $u_1 Z(B)$. Clearly, the action of $N(B)$ on $B/Z(B)$ must be that of some 2-group in $GL(2, 3)$. If the order of this 2-group is greater then 2, there is some 2-element $g \in N(B)$, such that g inverts

$B/Z(B)$. Therefore, g permutes the four subgroups of P of order 3, while normalizing $\Omega_1(\langle u_0\rangle)$, so that g normalizes some conjugate of $\langle u_1\rangle$ in P. Then, g must centralize this conjugate of $\langle u_1\rangle$ (Lemma D.2 (iii)), and consequently g centralizes an element of $B/Z(B)$. Since this cannot occur, the action of $N(B)$ on $B/Z(B)$ must be of order 2 and $N(B)$ has a normal subgroup of index 3. The Hall–Wielandt theorem [14, p. 212] implies that G has a normal subgroup of index 3, and this contradicts Lemma D.4. Therefore, $a_0 \not\equiv 0 \pmod 3$, as asserted.

LEMMA D.16.

(i) $N(P) = \langle C(P), B, y\rangle$;
(ii) $N(B) = \langle B, y\rangle = \langle P, z, y\rangle$.

Proof.

(i) Let $g \in N(P)$. From the structure of P (Lemmas D.14 and D.15), it follows that g normalizes $\Omega_1(P)$ and $\Omega_1(\langle u_0{}^3\rangle)$. Since B permutes the 3 subgroups of $\Omega_1(P)$ other than $\Omega_1(\langle u_0{}^3\rangle)$ transitively, for some $h \in B$, $gh \in N(\langle u_1\rangle) = C(t)$ (Lemma D.2 (iii)). Since $C(t) \cap N(P) = (C(P) \cap C(t))\langle y\rangle$, $gh \in \langle C(P), y\rangle$, so $g \in \langle C(P), B, y\rangle$.

(ii) Since B is not of class 2 (Lemma D.15), and since P is an abelian subgroup of index 3 in B, P is weakly closed in B. Therefore, $N(B) \subseteq N(P)$. By (i) it suffices to show that $N(B) \cap C(P) = P$.

Let R be a Sylow r-subgroup of $C(P)$, $r \neq 3$. Then, in the notation of Lemma D.10, y_1, y_2 normalize R, and $z \in \langle y_1, y_2\rangle$. Since R is homocyclic on 2 generators, and $\langle y_1, y_2\rangle$ induces a dihedral group of automorphisms of R, if z centralizes some element of $R^{\#}$, it centralizes all of R. Then $z \in C(t)$, a contradiction.

Thus, if $\bar{R}$ is a Sylow r-subgroup of $N(B) \cap C(P)$, $r \neq 3$, we have $[\bar{R}, B] = 1$, so $\bar{R} \subseteq C(z)$. Thus, $\bar{R} = 1$. Therefore, $N(B) \cap C(P) \subseteq P$, and the result follows.

LEMMA D.17. *Set $l = 3^{h-1}$. Then, $B^{(1)} = \langle u_0{}^l, u_1\rangle = \Omega_1(P)$ and $Z(B) = \langle u_0{}^3\rangle = \mho_1(P)$.*

Proof. This follows from the structure of B given in Lemma D.15.

In the following we let S be a Sylow 2-subgroup of $C(t) \cap N(P)$ containing the Klein group $\langle t, t'\rangle$. If S is quasi-dihedral, set $T = \langle t, t'\rangle$. If S is wreathed, take T to be the homocyclic abelian subgroup of index 2 in S. Thus, in either case $\langle t, t'\rangle \subseteq T$.

LEMMA D.18.

(i) $C(z) \cap C(T) \subseteq B$.

(ii) $C(P) = C(T)$.

(iii) $N(P) = N(T) = \langle C(T), y, z\rangle$.

(iv) $C(B) = Z(B)$.

(v) $N(B) = \langle B, y\rangle$.

Proof. To prove (i), recall that $C(T)$ is an abelian group of order $(q+1) \mid W \mid$. If R is a Sylow r-subgroup of $C(T)$, with $r \neq 3$, R is homocyclic on two generators. In the notation of Lemma D.10, z belongs to the dihedral group $\langle y_1, y_2\rangle$. If z centralizes some element of $R^{\#}$, z centralizes all of R and, in particular, $R \cap W \neq 1$. By Lemma D.2, $z \in C(t)$, a contradiction. Thus, $C(z) \cap C(T) \subseteq B$, as claimed.

Since $P \subseteq C(T)$ and $C(T)$ is abelian, $C(T) \subseteq C(P)$. As $P \cap W \neq 1$, $C(P) \subseteq C(t)$. Since $C(t)^{\Delta} = PGL(2, q)$, $C(P)^{\Delta} \subseteq C(T)^{\Delta}$. Since $W \subseteq C(T)$, $C(P) \subseteq C(T)$. So (ii) follows.

Then, (iii), (iv), and (v) follow from Lemmas D.15 and D.16.

LEMMA D.19. *Let $g \in P^{\#}$.*

(i) *If $g \in \langle u_1\rangle^{z^r}$ with $r = 0, 1, -1$ then $C(g) = C(t)^{z^r}$.*

(ii) *If $g \in \mho_1(P)$, then $C(g) = \langle C(T), z\rangle$.*

(iii) *In all other cases, $C(g) = C(T)$.*

Proof. Lemma D.2 (iii) implies (i). Suppose that g is not in $\langle u_1\rangle^{z^r}$ for $r = 0, 1, -1$ and set $K = C(g)$. Since $C(T)$ is an abelian group and contains P, $C(T) \subseteq K$.

We claim that T is a Sylow 2-subgroup of K. For if T is not a Sylow 2-subgroup of K, let $\bar{T} \subset K$, with $\mid \bar{T} : T \mid = 2$. Let t_1 be an involution of $Z(\bar{T})$ and let Δ_1 be the set of fixed points of t_1. Then it follows that g^{Δ_1} is centralized by a Klein group in $C(t_1)^{\Delta_1}$. But since $C(t_1)^{\Delta_1} = PGL(2, q)$ and $g^{\Delta_1} \neq 1$, this is impossible.

We note next that $C(t) \cap K = C(T)$. Clearly, $C(T) \subseteq C(t) \cap K$. Since $g^{\Delta} \neq 1$ and $C(t)^{\Delta} = PGL(2, q)$, $(C(t) \cap K)^{\Delta} \subseteq C(P)^{\Delta}$. Since $C(P) = C(T)$ and $W \subseteq C(T)$, $C(t) \cap K \subseteq C(T)$. Likewise, $C(t') \cap K = C(T)$ and $C(tt') \cap K = C(T)$.

Now let R be a Sylow 3-subgroup of K containing P. Then, $R = P$ or $\mid R : P \mid = 3$. In the latter case, $R \subseteq N(P) = \langle C(P), z, y\rangle$ (Lemma D.16 (i)). Since $C(P) = C(T) \subseteq K$, $\langle P, z\rangle \subseteq K$. Thus, we may take $R = \langle P, z\rangle$. If $R = \langle P, z\rangle$, $g \in \mho_1(P)$. If $R = P$, $g \notin \mho_1(P)$.

Now $y \notin K$, since the Sylow 2-subgroups of K are abelian. Thus, $N_K(P) = C(P) \cdot R$. Also, $Z(J(R)) = P$. By Glauberman's theorem [11, p. 280], K has a normal 3-complement, say L. Then, $N_L(T) = C_L(T)$, so L has a normal 2-complement, M.

Then $M = C_M(t)\,C_M(t')\,C_M(tt')$. Since $C(t) \cap K = C(T)$, $C(t') \cap K = C(T)$, and $C(tt') \cap K = C(T)$, $M \subseteq C(T)$. Therefore, $L \subseteq C(T)$. So $K = R \cdot C(T)$ and the result follows:

LEMMA D.20. *If $v \in B - P$, then $|C(v)|$ is odd and v is conjugate to no element of P.*

Proof. Suppose that $v^g = u \in P, g \in G$. Then $C(v)^g = C(u)$ and $\langle Z(B), v\rangle^g$ is contained in some Sylow 3-subgroup of $C(u)$. Thus, by replacing g by gc for some $c \in C(u)$, we may assume $Z(B)^g \subseteq B$. Thus, $\langle u_0^3\rangle^g \subseteq B$ and $\langle u_0^9\rangle^g \subseteq P$. As $h \geq 3$, $\langle u_0^{3^{h-1}}\rangle^g \subseteq P$. As u_1 is not a cube in $N(\langle u_1\rangle) = C(t)$ (Lemma D.2 (iii)), $\langle u_0^{3^{h-1}}\rangle$ is not conjugate to $\langle u_1\rangle$. Therefore, $\langle u_0^{3^{h-1}}\rangle^g = \langle u_0^{3^{h-1}}\rangle$, so $g \in N(P)$ (Lemma D.19). Then, $v^g \in P$, with $v \in B - P$, a contradiction.

If $|C(v)|$ is even, then v centralizes a conjugate of t. Then, v is contained in some conjugate of P, which is a contradiction.

LEMMA D.21. *If $v \in B - P$, then $\langle v, Z(B)\rangle$ is a Sylow 3-subgroup of $C(v)$.*

Proof. By Lemma D.20, v is conjugate to no element of P. For any $u \in B - P$, $C_B(u) = \langle u, Z(B)\rangle$. Thus, $C_B(v)$ is a Sylow 3-subgroup of $C(v)$.

LEMMA D.22. *If $v_1, v_2 \in B - P$ are conjugate in G, then they are conjugate in $N(B)$.*

Proof. Suppose that $v_1^g = v_2, g \in G$. By Lemma D.21, $C(v_1)$ has Sylow 3-subgroup $\langle v_1, Z(B)\rangle$. Thus, we may assume that $\langle v_1, Z(B)\rangle^g = \langle v_2, Z(B)\rangle$, so that $Z(B)^g \subseteq \langle v_2, Z(B)\rangle$. Let $l = 3^{h-1}$. Then, $(u_0^l)^g \in \langle v_2, Z(B)\rangle$. By Lemma D.20, $(u_0^l)^g \in Z(B)$. Thus, $g \in N(\langle u_0^l\rangle) = \langle C(T), z, y\rangle = N(T)$ (Lemmas D.18 (iii) and D.19 (ii)). Also, $C(T)$ has a normal 3-complement H such that $C(T) = P \times H$. Let $g = g_1 g_2$, where $g_1 \in B\langle y\rangle$ and $g_2 \in H$. Then $v_1^{g_1} \in B$, so that $[v_1^{g_1}, g_2] \in H \cap B = 1$. Therefore, $v_2 = v_1^{g_1 g_2} = v_1^{g_1}$ and $g_1 \in N(B)$.

LEMMA D.23. *If $v \in B - P$, then $C(v)$ has a normal 3-complement.*

Proof. By Lemma D.21 if $v \in B - P$, then $\langle v, Z(B)\rangle$ is a Sylow 3-subgroup of $C(v)$. By Lemma D.20 $|C(v)|$ is odd. Therefore, all elements of $C(v)$ normalizing $\langle v, Z(B)\rangle$ must centralize it. By Burnside's transfer theorem [14, p. 203], $C(v)$ has a normal 3-complement.

LEMMA D.24. *There is an element $\tilde{z} \in B - P$ with $\tilde{z}^3 \in B^{(1)} = \Omega_1(P)$.*

Proof. By Lemma D.17, $B^{(1)} = \Omega_1(P)$. Since B is nonabelian, $B/B^{(1)}$ is abelian of type $(3^{h-1}, 3)$. Thus, we can clearly choose $\tilde{z} \in B - P$ such that $\tilde{z}^3 \in B^{(1)}$.

LEMMA D.25. *If $\tilde{z}$ is chosen as in Lemma* D.24, *then $B - P$ is the disjoint union of the sets $\tilde{z}u_0{}^jB^{(1)}$, $\tilde{z}^{-1}u_0{}^jB^{(1)}$ for $j = 0, 1,..., 3^{h-1} - 1$. If $u \in \tilde{z}u_0{}^jB^{(1)}$, then $u^G \cap B = \tilde{z}u_0{}^jB^{(1)} \cup \tilde{z}^{-1}u_0^{-j}B^{(1)}$.*

Proof. The first statement follows from the structure of B. If $u \in B - P$, then, by Lemma D.21, $|C_B(u)| = |\langle u, Z(B)\rangle| = 3^h$. Thus, if $u \in \tilde{z}u_0{}^jB^{(1)}$, then $u^B = \tilde{z}u_0{}^jB^{(1)}$. Since y inverts $B/B^{(1)}$, the result follows from Lemmas D.22 and D.16.

We can now complete the proof of Theorem 1.1. Let λ be a linear character of B with kernel P. We can choose λ so that, if $u \in \tilde{z}P$, then $\lambda(u) = \rho$, where ρ is a primitive cube root of unity. Thus, if v lies in $\tilde{z}^{-1}u_0^{-j}B^{(1)}$, $\lambda(v) = \rho^{-1}$.

If χ is any irreducible character of G, consider $\chi(u)$ and $(1 - \lambda)(u)$ for $u \in B - P$. If $u \in \tilde{z}u_0{}^jB^{(1)}$, then $\chi(u) = \chi(\tilde{z}u_0{}^j) = \chi(\tilde{z}^{-1}u_0^{-j})$ (by Lemma D.25) and $(1 - \lambda)(u) = 1 - \rho$. If $u \in \tilde{z}^{-1}u_0^{-j}B^{(1)}$, then $(1 - \lambda)(u) = 1 - \rho^{-1}$.

Now consider $B_0(3, G)$, the principal 3 block of G, and let $X(r)$, $r \in G$, denote the column whose entry for $\chi \in B_0(3, G)$ is $\chi(r)$. If $r \in B - P$ it follows from Lemma D.23 that $X(r)$ is the column of generalized decomposition numbers for the block $B_0(3, G)$ and the 3-section of r [5, (2.6), Corollary 5]. Defining inner products of columns as usual we have:

(a) If $v \in B - P$, then $(X(v), X(v))$ is the 3-part, 3^h, of $C(v)$ (see Brauer [5, 2.7]).

(b) If v_1, v_2 are in $B - P$ and v_1 is not conjugate in G to v_2, then $(X(v_1), X(v_2)) = 0$ (Brauer [4, (7C)]).

(c) If $v \in B - P$, then $(X(1), X(v)) = 0$ (Brauer [4, (7C)]).

Set

$$R = \frac{1}{3^{h+2}} \sum_{u \in B} X(u)\, \overline{(1 - \lambda)(u)}.$$

Then R is the column whose entry in position $\chi \in B_0(3, G)$ is just $(\chi, 1 - \lambda)_B$. In order to compute R it suffices to let u range over $B - P$, since $1 - \lambda$ vanishes on P. We thus have

$$R = \frac{1}{3^{h+2}} \sum_{j=0}^{3^{h-1}-1} 9X(\tilde{z}u_0{}^j)(1 - \rho + 1 - \rho^{-1}) = \frac{1}{3^{h+2}} \sum_{j=0}^{3^{h-1}-1} 27X(\tilde{z}u_0{}^j)$$

$$= \frac{1}{3^{h-1}} \sum_{j=0}^{3^{h-1}-1} X(\tilde{z}u_0{}^j).$$

Since the 3^{h-1} elements $\tilde{z}u_0{}^j$, $j = 0,..., 3^{h-1} - 1$, lie in distinct classes of G (Lemma D.25), it follows from (a) and (b) that

$$(R, R) = \frac{1}{3^{2h-2}} \cdot 3^h \cdot 3^{h-1} = 3.$$

However, each entry in R is an integer, so that R must have exactly 3 nonzero entries, each ± 1. From the definition of R it is clear that 1_G contributes a 1. Let θ be the permutation character, which, by Lemma 3.6, is in $B_0(3, G)$. If $v \in B - P$, then v fixes no points of Ω, so that $\theta(v) = -1$. Thus, the entry in the θ position of R is

$$\frac{1}{3^{h+2}} \sum_{u \in B} \theta(u)\overline{(1 - \lambda)(u)} = \frac{1}{3^{h+2}} \sum_{u \in B-P} \theta(u)\overline{(1 - \lambda)(u)}$$
$$= -\frac{1}{3^{h+2}} \sum_{u \in B} \overline{(1 - \lambda)(u)} = -1.$$

Now let χ be the third character in $B_0(3, G)$ that contributes a non-zero entry to R. By (c) we have

$$(X(1), R) = \frac{1}{3^{h+2}} \sum_{u \in B} (X(1), X(u)\overline{(1 - \lambda)(u)}) = 0.$$

However, $(X(1), R) = 1 \cdot 1 + q^3(-1) + \chi(1)\delta$, where $\delta = \pm 1$. Therefore, $\chi(1)\delta = q^3 - 1$, $\delta = 1$, and $\chi(1) = q^3 - 1$. However, $q^3 - 1$ does not divide $|G|$. This final contradiction completes the proof of Theorem 1.1.

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