Dirac's Belt Trick, Gyroscopes, and the iPad

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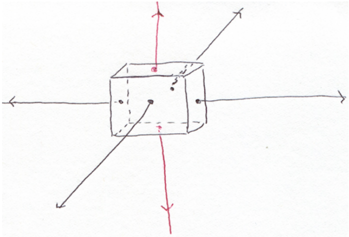
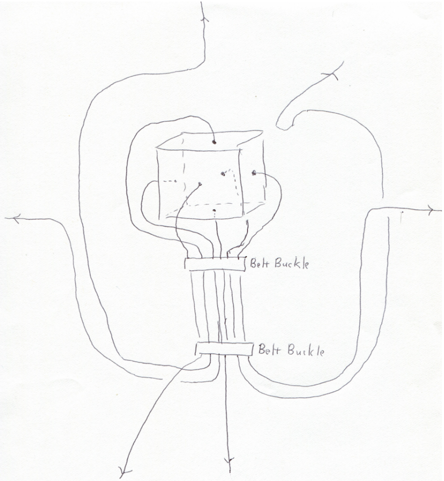
Dirac's Belt Trick

P. A. M. Dirac, 1902 - 1984
Nobel Prize (with Erwin Schrodinger) in 1933

Formulated Dirac equation, a relativistically correct quantum mechanical description of the electron, which predicted the existence of antiparticles.

Dirac's Belt Trick with Strings





TeXShop for Macintosh; *TeX* by Donald Knuth
Free at: `http://pages.uoregon.edu/koch`

```
\documentclass[11pt]{amsart}
\usepackage[paper width = 6in, paperheight = 7in]{geometry}
\usepackage[parfill]{parskip}
\usepackage{graphicx}

\begin{document}

Using \TeX, we can typeset $\sqrt{ {{1 + x + x^2}
\over {e^{2x + \sqrt{5}}}}}$
and the matrix $\left( \begin{array}{cc} 2 & 5 \\
 \sqrt{10} & -7 \end{array} \right)$.
According to calculus
$$\int_0^1 {2x + 3x^2}\ dx =  2 \hspace{.2in} \mbox{and}
 \hspace{.2in} \int_0^\infty e^{- x^2} \ dx
  = {{\sqrt{\pi}} \over 2}$$
The path of a particle in a gravitational field is given by
$\gamma_i(t)$ where $${{d^2 \gamma_i} \over {d t^2}} +
 \sum_{jk} \Gamma^i_{jk} {{d \gamma_i} \over {dt}}
  {{d \gamma_j} \over {dt}} = 0$$

\begin{figure}[htbp]
   \centering
   \includegraphics[width=2in]{MoveTest-math.jpg}
\end{figure}
\end{document}
```

Using TEX, we can typeset $\sqrt{\frac{1+x+x^2}{e^{2x+\sqrt{5}}}}$ and the matrix $\left(\begin{array}{cc} 2 & 5 \\ \sqrt{10} & -7 \end{array} \right)$.

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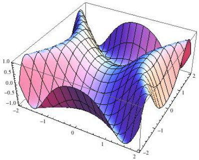
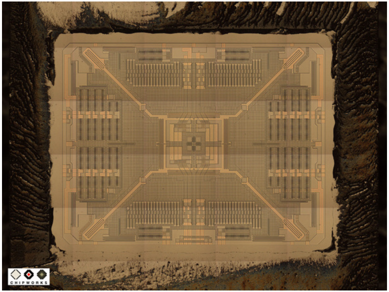


WWDC, Apple's Worldwide Developer's Conference

Gyroscopes in the iPhone and iPad

1. Announced at WWDC 2010
2. Now in iPhones and iPads
3. Actually a small chip



Secret Slide # 1

```
// Turn on gyroscope
motionManager = [[CMMotionManager alloc] init];
motionManager.deviceMotionUpdateInterval = 1.0 / 60.0;
[motionManager startDeviceMotionUpdates];

// Repeat as often as desired
newestDeviceMotion = motionManager.deviceMotion;
...

// Turn off gyroscope
[motionManager stopDeviceMotionUpdates];
[motionManager release];
```

Secret Slide # 2

NewestDeviceMotion contains three descriptions of the attitude of the device. Use whichever is most convenient.

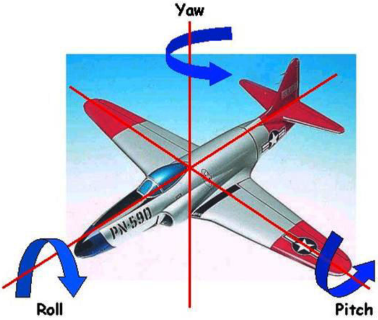
- Euler angles: roll, pitch, and yaw
- Rotation matrix
- Quaternion

Secret Slide # 3

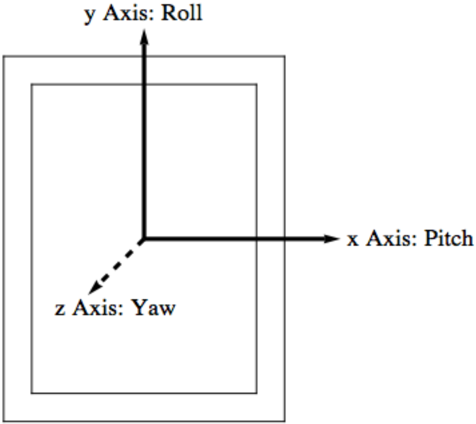
```
// Example code using roll, pitch, yaw
double r = newestDeviceMotion.attitude.roll;
double p = newestDeviceMotion.attitude.pitch;
double y = newestDeviceMotion.attitude.yaw;

// Example code using quaterions
double q0 = newestDeviceMotion.attitude.quaternion.w;
double q1 = newestDeviceMotion.attitude.quaternion.x;
double q2 = newestDeviceMotion.attitude.quaternion.y;
double q3 = newestDeviceMotion.attitude.quaternion.z;
```

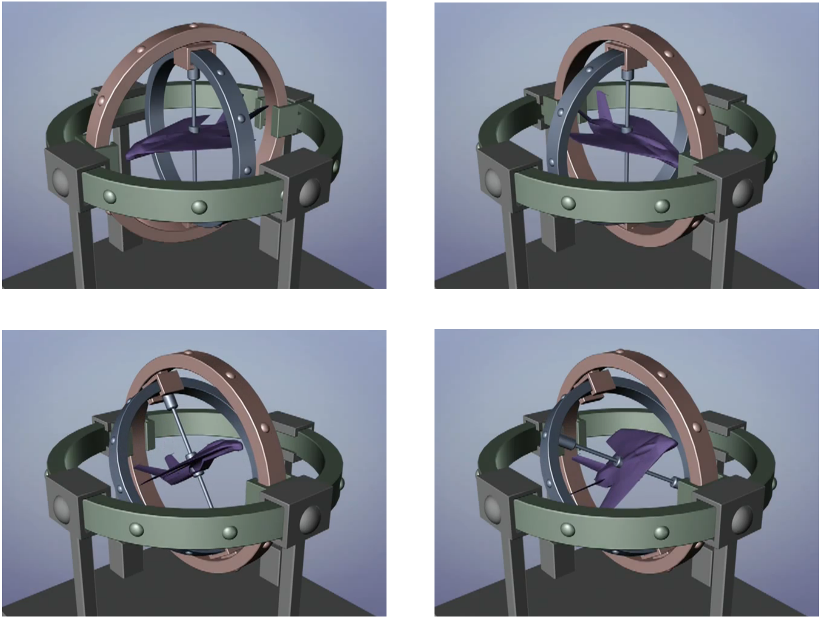
Roll, Pitch, and Yaw



iPad Conventions



Gimbals; Illustrations by Andrew Silke



Gimbal Lock

Rotation on an iPad

Complex Numbers

Quaternions are like complex numbers. Recall the complex rules:

- A point in the plane like $(2,3)$ can be written $2+3i$. This i is a way to keep track of the second coordinate.
- Multiply these numbers using the rule $i^2=-1$. For instance

  $$(2+3i)(4+5i) = 8+10i+12i+15i^2 = 8+22i-15 = -7+22i$$

- If $c=a+bi$, define $\overline{c}=a-bi$. Then $c\overline{c}=a^2+b^2$.
- This trick allows us to divide:

  $$\frac{2+3i}{1+2i} = \frac{(2+3i)(1-2i)}{(1+2i)(1-2i)} = \frac{8+i}{5} = 1.6+0.2i$$

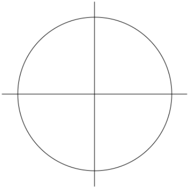
- The distance to the origin is given by the Pythagorean theorem:

  $$|a+bi| = \sqrt{a^2+b^2} = \sqrt{c\overline{c}}$$

- $|c_1c_2| = |c_1|\,|c_2|$

Rotations and Complex Multiplication

Rotations about the origin are given by complex numbers of absolute value one. Indeed, fix c_1 with $|c_1| = 1$. Then $z \to c_1 z$ preserves length because $|c_1 z| = |c_1||z| = |z|$.



Rotation:

$$z \to c_1 \; z$$

Hamilton and the Discovery of Quaternions

Sir William Ronan Hamilton was a great Irish physicist and mathematician. In 1843, Hamilton tried to define a multiplication on three dimensional vectors. Hamilton later wrote in a letter to one of his sons “Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edward and yourself used to ask me: ‘Well, Papa, can you multiply triples?’ Whereto I was always obliged to reply, with a sad shake of the head, ‘No, I can only add and subtract them.’ ”

The Discovery of Quaternions

Eventually Hamilton discovered that multiplication works if we work in *four* dimensions. In that case, we can add, subtract, multiply, and divide, and all the usual grade school properties remain true *except* that multiplication is not commutative. Elements of the resulting object are called *quaternions*.

Multiplying Quaternions

- A *quaternion* is formed by four numbers q_0, q_1, q_2, q_3. We always write
$$q = q_0 + q_1 i + q_2 j + q_3 k$$
- Multiply these numbers using Hamilton's multiplication rules
$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1$$
$$ij = k = -ji \quad jk = i = -kj \quad ki = j = -ik$$
- For example
$$(2+7i+j)(i+k) = 2i - 7 - k + 2k - 7j + i = -7 + 3i - 7j + k$$

Conjugation and Division

- If $q = q_0 + q_1 i + q_2 j + q_3 k$, we define $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$.
- Amazingly, $q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$
- So we can divide using the standard trick:

$$\frac{1+i}{2j-k} = \frac{(1+i)(-2j+k)}{(2j-k)(-2j+k)} = \frac{-2j-2k+k-j}{5} =$$

$$\frac{-3j-k}{5} = -\frac{3}{5}j - \frac{1}{5}k$$

Norm

- The distance to the origin is given by the Pythagorean theorem:

$$||q_0 + q_1 i + q_2 j + q_3 k|| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = \sqrt{q\overline{q}}$$

- And it is still true that

$$||q_1 q_2|| = ||q_1|| ||q_2||$$

Another Way to Write Quaternions

It is possible to think of a quaternion as a real number q_0 and a three dimensional vector $q_1 i + q_2 j + q_3 k$. To emphasize that we have a real number and a vector, write

$$< r, v >$$

Everyone knows how to multiply two reals, or scalar multiply a vector by a real. So it suffices to explain how to multiply two vectors, and the formula is

$$< 0, v >< 0, w >=< -v \cdot w, v \times w >$$

Algebra with Vectors

To show the advantage of the new quaternion notation, consider the product $q\overline{q}$ discussed earlier. This product is calculated below using vector notation. Since $v \times v = 0$, the result is the real number $r^2 + ||v||^2 = q_0^2 + q_2^2 + q_2^2 + q_3^2$:

$$< r, v >< r, -v >=< r^2 + v \cdot v, rv - rv - v \times v >$$

I leave it to you to prove a second result: $\overline{q_1 q_2} = \overline{q_2}\ \overline{q_1}$.

Finally $||q_1 q_2||^2 = q_1 q_2 \overline{q_1 q_1} = q_1\ q_2\ \overline{q_2}\ \overline{q_1}$. The middle two terms give $||q_2||^2$, which is real and so commutes with everything, so the final product is $||q_1||^2 ||q_2||^2$.

Boughton Bridge

William Rowan Hamilton discovered the quaternions in Dublin on October 16, 1843, during a walk with his wife. He immediately carved the equations on Boughton Bridge (now called Broom Bridge). They vanished, but the bridge remains.

Rotations in R^n

Definition: A *rotation* about the origin in R^n is a linear transformation $R : R^n \to R^n$ which preserves distance to the origin, so $||Rv|| = ||v||$ for all v.

Note: Rotations actually preserve all distances and angles in R^n.

Let $SO(n)$ be the group of all rotations of R^n.

The Dimension of $SO(n)$

The dimension of $SO(n)$ is

$$(n-1)+(n-2)+\ldots+1=\frac{n(n-1)}{2}$$

Examples:

$\dim SO(2) = 1$
$\dim SO(3) = 3$
$\dim SO(4) = 6$

Quaternions as Rotations in Four Dimensions

Just as multiplication by a complex number of absolute value 1 gives a rotation of R^2, so multiplication by a quaternion q_1 of absolute value 1 gives a rotation of R^4, because $||q_1 q|| = ||q_1|| ||q|| = ||q||$. Thus we obtain a large number of rotations of R^4:

$$R : q \to q_1 \; q$$

Rotations in Four Dimensions

Unfortunately, the dimension of the unit sphere in R^4 is three, while the dimension of $SO(4)$ is 6. So we only have half of the rotations of R^4.

Remember that multiplication is not commutative. The missing rotations have the form $R(q) = qq_1$ for a fixed q_1 of norm one.

The most general rotation of R^4 is $q \to q_1 q q_2$ for unit quaternions q_1 and q_2.

Actually $(q_1, q_2) = (1, 1)$ and $(q_1, q_2) = (-1, -1)$ both give the identity, so

$$SO(4) = S^3 \times S^3 / \pm (1, 1)$$

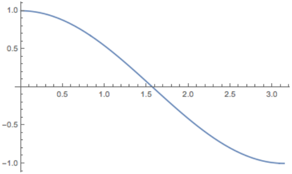
Rotations in Three Dimensions

Once we know how to rotate R^4, it is easy to rotate vectors in R^3. If a rotation of R^4 leaves $< 1, 0 >$ fixed, it rotates the three dimensional subspace of the quaternions perpendicular to $< 1, 0 >$. Since $q_1 1 q_2 = 1$ exactly when $q_2 = q_1^{-1}$, we conclude that $v \to qvq^{-1}$ is a rotation of the three dimensional vector v, considered as a quaternion $< 0, v >$.

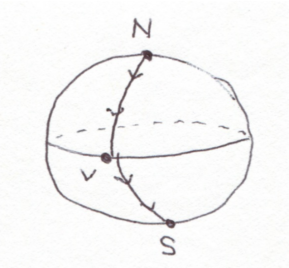
If $||q|| = 1$, then $||q||^2 = q\overline{q} = 1$, so $q^{-1} = \overline{q}$. Note that both $q = 1$ and $q = -1$ give the identity map.

We conclude that $SO(3) = S^3/\{\pm 1\}$ where S^3 is the group of unit quaternions. The most general rotation of R^3 is $v \to qv\overline{q}$ for a unit quaternion q.

What Rotation Corresponds to $< r, v >$?



A unit quaternion has the form $< r, v >$ where $r^2 + ||v||^2 = 1$. Consequently r can be written uniquely as $\cos\theta$ where $0 \leq \theta \leq \pi$.



Theorem

If $q =< \cos\theta, v >$ is a unit quaternion, the rotation $v \to qvq^{-1}$ of R^3 has axis v and angle of rotation 2θ.

What Rotation Corresponds to $< r, v >$?

Suppose q is a unit quaternion of the form $< r, v >$. The corresponding rotation maps $w \in R^3$ to $qwq^{-1} = qw\overline{q}$. This equals $< r, v >< 0, w >< r, -v >$ and a short calculation gives $< 0, r^2 w + 2r(v \times w) + (v \cdot w)v + v \times (v \times w) >$

So the unit quaternion q gives the rotation

$$w \to r^2 w + 2r(v \times w) + (v \cdot w)v + v \times (v \times w)$$

If w points in the same direction as v, then $v \times w = 0$ and we get $r^2 w + (v \cdot w)v$. Writing $w = \alpha v$, we get $r^2 \alpha v + \alpha ||v||^2 v = (r^2 + ||v||^2)\alpha v = \alpha v = w$. So vectors on the line through v are fixed, and we have a rotation with axis v.

Rotation by $q =< r, v >$, Continued

Suppose w is perpendicular to v. Then w is mapped to $r^2 w + 2r(v \times w) + v \times (v \times w)$.

Let e_3 be a unit vector in the direction of v. Then $v = ||v||e_3$. Since $r^2 + ||v||^2 = 1$, we can write $r = \cos\theta$ and $||v|| = \sin\theta$. A short calculation shows that the above formula maps w to

$$\cos^2\theta \;\, w + 2\cos\theta\sin\theta \;\, (e_3 \times w) + \sin^2\theta \;\, (e_3 \times (e_3 \times w))$$

Let e_1 and e_2 be vectors perpendicular to e_3, so e_1, e_2, e_3 forms a right handed coordinate system. Then a very short calculation from the last formula shows that

$$e_1 \to (\cos^2\theta - \sin^2\theta)e_1 + 2\cos\theta\sin\theta e_2 = \cos 2\theta \;\, e_1 + \sin 2\theta \;\, e_2$$

$$e_2 \to (\cos^2\theta - \sin^2\theta)e_2 - 2\cos\theta\sin\theta e_1 = \cos 2\theta \;\, e_1 - \sin 2\theta \;\, e_2$$

Let's Try It on the iPad

The Main Picture of S^3

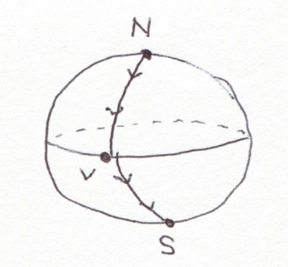
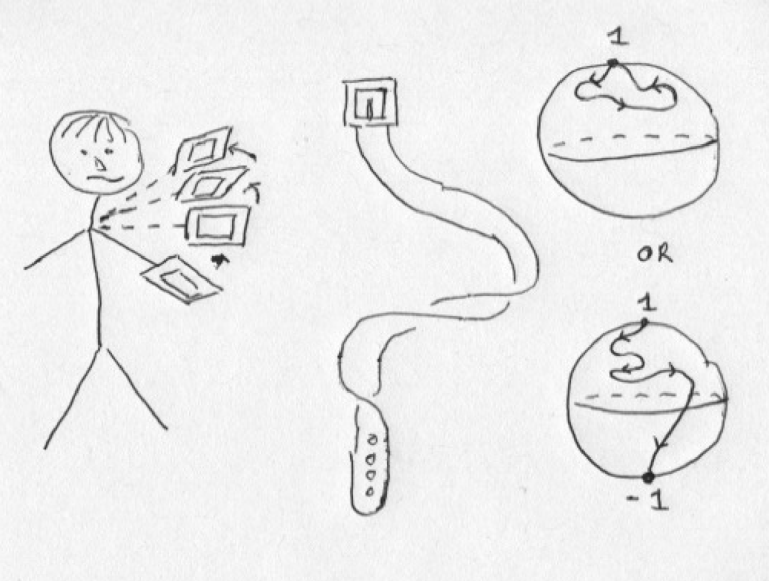


Figure: Unit Quaternions

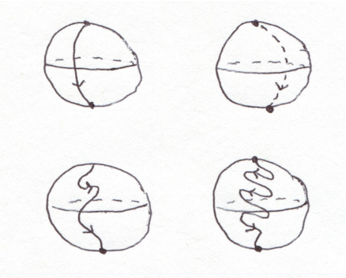
The unit quaternions form a sphere S^3. I like to draw this as the sphere S^2, thinking of the vertical axis as the real component of quaternions, and the plane of the equator as 3-space. If v is a unit vector in the equator, the great circle through this vector from the north pole to the south pole corresponds to rotations about v through angles 2θ, which trace a full 2π rotation as we move from north to south. The north and south poles, corresponding to $< \pm 1, 0 >$, both map to the identity rotation.

The Key Picture



The Belt Trick in Dimension 4

Our argument also shows that you cannot remove an odd number of twists from the belt even if you are allowed to twist it into the fourth dimension. We just replace the previous picture of $SO(3)$ with the picture of $SO(4)$ below. The initial odd twist about an axis in R^3 has the form $h \to qh\overline{q}$ and gives the top paths below. A homotopy to actual four dimensional rotations gives the bottom picture. But during the homotopy, both paths will continue to end at the south pole.



Coincidences in Low Dimensions

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- A cross product on R^n is a nontrivial bilinear product $v \times w$ such that v and w are perpendicular to $v \times w$, and $||v \times w||^2 = ||v||^2||w||^2 - (v \cdot w)^2$. Hurwitz's theorem implies that such products only exist in dimensions 3 and 7.

More Coincidences

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- No other $SO(n)$ contains a non-trivial normal subgroup.
- $SO(4)$ is essentially $S^3 \times S^3 = SO(3) \times SO(3)$.
- No other $SO(n)$ or $SU(n)$ has an almost product structure.