Tuesday, March 17, 2009

Calculus I (Math 251, CRN 23180), Final

Name:____________________________

1	2	3	4	5	6	7	8	9	10	TOTAL

INSTRUCTIONS: Look through the exam before you start, and answer the questions that seem easiest to you first. Work on the exam sheets – you may use the backs of the pages if you need more space. Show all your work!!

There are 100 points distributed among 10 questions. You may find the exam too long to finish. If that is so, don't panic. Just make sure you start with the problems that you know how to do.

(1) Find the equation of the tangent line to the curve

$x^2 + 4xy + y^2 = 13$

at the point $(2, 1)$.

We have a point on the line, so we need the slope. We use implicit differentiation to find $\frac{dy}{dx}$. *Differentiating the equation above, we get*

$$2x + 4y + 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0.$$

Evaluating at $(2, 1)$ *gives us*

$$4 + 4 + 8\frac{dy}{dx} + 2\frac{dy}{dx} = 0 \text{ so } \frac{dy}{dx} = \frac{8}{10} = .8$$

So the equation for the line is then

$(y - 1) = .8(x - 2)$ or equivalently $y = .8x - .6$.

(2) How many solutions does

$e^x = 1000x$

have? Explain why your answer is correct.

We can consider the function $f(x) = e^x - 1000x$. *A solution* x *to the given equation is the same as a number* x *with* $f(x) = 0$. *To understand when* $f(x)$ *can be zero, we graph* $f(x)$. $f(0) = 1$. $f'(x) = e^x - 1000$. *So* f *is increasing when* $x > \ln(1000)$ *and* f *is decreasing when* $x < \ln(1000)$. f *is continuous and differentiable, so it has a global minimum at* $a = \ln(1000)$.

$$f(a) = e^{\ln(1000)} - 1000\ln(1000) = 1000 - 1000\ln(1000) = 1000(1 - \ln(1000)).$$

So $f(a)$ *is negative. The limit of* $f(x)$ *as* x *approaches infinity or negative infinity is infinite. So the graph of* $f(x)$ *crosses the* x*-axis twice, once to the left of* a *and once* *to the right of a. So there are two zeroes of $f(x)$ and thus two solutions to the given equation.*

(3) Find the points on the ellipse

$$4x^2 + y^2 = 4$$

farthest from $(1, 0)$.

If (x, y) is a point on the ellipse, we wish to maximize the distance to $(1, 0)$. That is the same as maximizing the square of the distance to $(1, 0)$. The square of the distance is $(x-1)^2 + y^2$. Using our equation, $y^2 = 4 - 4x^2$. So we need to maximize

$$f(x) = (x-1)^2 + (4 - 4x^2)$$

for values of x that can be on the ellipse. Those values are $x \in [-1, 1]$. So we'll need to check the end points and critical points.

$$f'(x) = 2(x-1) - 8x = -1 - 6x.$$

This is 0 when $x = -1/6$. We'll check $x = -1, -1/6$ and 1.

$$f(-1) = 4, f(-1/6) = \frac{49}{36} + (4 - \frac{4}{36}) = 4 + \frac{45}{36}, f(1) = 0.$$

The largest value is the middle one, so the point maximizing the distance from $(-1, 0)$ is $(-1/6, \sqrt{4 - 4/36})$. (There are actually two such points since we would get the same distance to the point with the corresponding negative y-coordinate.)

(4) Find the absolute maximum and minimum of $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$.

Notice that the input to $\ln$ is always positive in that range, so the function is well-defined, continuous and differentiable. The absolute maximum and minimum will occur at either endpoints or critical points, so we need to start by finding critical points.

$$f'(x) = \frac{2x+1}{x^2 + x + 1}.$$

This is 0 when $x = -1/2$. So we need to check $x = -1, -1/2, 1$.

$$f(-1) = \ln(1) = 0. f(-1/2) = \ln(3/4) < 0. f(1) = \ln(3) > 0.$$

So the minimum is $\ln(3/4)$ and the maximum is $\ln(3)$.

(5) Calculate the following limits

(a)

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$$

Multiply the fraction by $1 = \frac{\sqrt{x}+1}{\sqrt{x}+1}$ and we get

$$\lim_{x \to 1} \frac{x - 1}{(x-1)(\sqrt{x} + 1)} = \frac{1}{2}.$$

(b)

$$\lim_{x \to -\infty} \frac{1 - 10x + 56x^2}{5x^3 - 1}.$$

Dividing top and bottom by x^3 and taking limits gives 0.

(c)

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$$

Using L'Hôpital:

$$\lim_{x\to 0}\frac{e^x + e^{-x}}{1} = 2.$$

(d)

$$\lim_{x\to 1^+}\frac{x^2+1}{x^2-1}$$

The numerator approaches 2, the denominator approaches 0 *from above, so the limit is* ∞.

(e)

$$\lim_{x\to 0^+} x\ln(1/x)$$

Using L'Hôpital:

$$\lim_{x\to 0^+}\frac{\ln(1/x)}{1/x} = \lim_{x\to 0^+}\frac{(-1/x^2)/(1/x)}{-1/x^2} = \lim_{x\to 0^+}\frac{1}{1/x} = 0.$$

(6) Calculate the following derivatives

(a)

$$\frac{d}{dx}(\sqrt{x+1} + \frac{1}{\sqrt[3]{x}})$$

$\frac{1}{2}(x+1)^{-1/2} - \frac{1}{3}x^{-4/3}$.

(b)

$$\frac{d}{dt}(\cos(t^2) - [\cos(t)]^2$$

$$-\sin(t^2)\cdot 2t - 2[\cos(t)]\cdot[-\sin(t)].$$

(c)

$$\frac{d}{du}\log_2(u+u^{-1})$$

Using $\log_a(x) = \ln(x)\frac{1}{\ln(a)}$ *we get*

$$\frac{d}{du}\log_2(u+u^{-1}) = \frac{d}{du}\frac{1}{\ln(2)}\ln(u+u^{-1}) = \frac{1}{\ln(2)(u+u^{-1})}(1-u^{-2})$$

(d)

$$\frac{d}{dt}\sqrt{\frac{t}{t^2+1}}$$

$$\frac{1}{2\sqrt{\frac{t}{t^2+1}}}\frac{(t^2+1)-2t^2}{(t^2+1)^2} = \frac{1}{2\sqrt{\frac{t}{t^2+1}}}\frac{1-t^2}{(t^2+1)^2}$$

(e)

$$\frac{d}{dx}[1 + x\sin(x)]^{10}$$

$10[1 + x\sin x]^9(\sin x + x\cos x)$.

(7) A cylindrical can is to be made using 290 square centimeters of aluminum. Assuming no waste, and no overlap, find the maximum volume of the can.

Let the can have radius r and height h. The area is

$290 = 2\pi r^2 + 2\pi rh$ *so* $h = \dfrac{290 - 2\pi r^2}{2\pi r}$.

The volume is

$V(r) = \pi r^2 h = \dfrac{r}{2}(290 - 2\pi r^2) = 145r - \pi r^3$.

Given that $r \in [0, \sqrt{145/\pi}]$ and that the volume is 0 at the endpoints and positive in between, the maximum has to be at a critical point.

$V'(r) = 145 - 3\pi r^2$

so this is a critical point when $r = \sqrt{145/3\pi}$. The volume is then

$$145\sqrt{145/3\pi} - \frac{145}{3}\sqrt{145/3\pi} = \frac{290}{3}\sqrt{145/3\pi}$$

(8) A population of amoebae triples in size every two hours. If there are initially 50 amoebae, find the instantaneous rate of growth at time 5 hours.

(9) Sketch the graph of $f(x) = e^{2-x-x^2}$. Indicate critical points, local and global minima and maxima, regions where the function is increasing/decreasing, inflection points, and regions where it is concave up/concave down.

$f'(x) = f(x)(-1 - 2x)$. $f(x)$ is always positive, so we have a single critical point when $x = -1/2$. The function is decreasing for $x > -1/2$ and increasing when $x < -1/2$.

$f''(x) = f(x)(-1 - 2x)^2 - 2f(x) = f(x)(4x^2 + 4x - 1)$.

Since $f(x)$ is positive, we have inflection points when

$4x^2 + 4x - 1 = 0$ *so when* $x = \dfrac{-4 \pm \sqrt{32}}{8} = \dfrac{-1}{2} \pm \dfrac{\sqrt{2}}{2}$.

So the function is concave up on $(-\infty, -1/2 - \sqrt{2}/2)$, concave down on $(-1/2 - \sqrt{2}/2, -1/2 + \sqrt{2}/2)$ and concave up on $(-1/2 + \sqrt{2}/2, \infty)$.

(10) A hockey team plays in an arena with a seating capacity of 15,000 spectators. With ticket prices set at \$12, attendance is 11,000. If the ticket price is reduced by a dollar, attendance increases by 1000 people. How should the price be set to maximize revenue?

The revenue is given by

$R(p) = (11000 + 1000(12 - p))p = 11000p + 12000p - 1000p^2 = 1000(23p - p^2)$

where p is the price in dollars. This is a quadratic open down, so has a maximum at its critical point. $R'(p) = 1000(23 - 2p)$, so the critical point is $23/2 = 11.5$. So the ticket price should be \$11.50.