31. In the discussion of the Riccati equation $dy/dt = r(t) + a(t)y + b(t)y^2$ given in this appendix, we did one change of variables to convert to a Bernoulli equation and then another change of variables to obtain a linear equation. It is possible to combine the two into a single change of variables. If $y_1(t)$ is a particular solution of the Riccati equation above, show that changing to the new dependent variable

$$u(t) = \frac{1}{y(t) - y_1(t)}$$

yields a linear equation for du/dt.

32. Construct your own Bernoulli equation exercise of the form $dy/dt = a(t)y + b(t)y^5$ from the linear equation $dz/dt = z + e^{2t}$.

33. Construct your own Riccati equation exercise from the function $y_1(t) = t^2 + 2$ and the Bernoulli equation $dw/dt = w + t^2 w^2$.

B POWER SERIES: THE ULTIMATE GUESS

One of our most commonly used methods for finding formulas for solutions of differential equations has been picking a reasonable guess for a solution which is then substituted into the equation to see if it is indeed a solution. We improve our chances of success by including unknown constants whose values are determined as part of the process.

The only difficulty with this method is deciding what to guess. It would be sweet if there was a guess that always, or almost always, works. In this appendix we present such a guess.

A First-Order Example

Consider the equation

$$\frac{dy}{dt} = ty + 1.$$

This is a linear equation, and we could try to solve it using an integrating factor (see Section 1.9). However, the integrals involved are problematic, so we would rather guess. But what should we guess? No function $y(t)$ jumps to mind.

Since we do not have any ideas regarding what guess to make, we make a guess that does not require any ideas. We know that almost all nice functions encountered in calculus have Taylor series, so we guess

$$y(t) = a_0 + a_1 t + a_2 t^2 + \dots = \sum_{n=0}^{\infty} a_n t^n.$$

That is, we guess the Taylor series for $y(t)$ centered at $t = 0$ (a Maclaurin series).

As you remember from calculus, an infinite sum is really a limit and the first question with any limit is "Does it exist?". In this appendix we answer all convergence questions with "No comment." Ignoring questions of convergence is often done in this kind of analysis. In fact, there is even a name for it. It is called *formal analysis*. We essentially treat power series as polynomials with a really large degree. This kind of analysis is like skipping dinner and eating double dessert. It tastes good, but it isn't very healthy. There are theorems that justify this formal analysis for most reasonable differential equations including the examples that follow. However, there is also some danger, so if your life (or someone else's) depends on a power series calculation, you should consult an advanced calculus or real analysis textbook for the appropriate convergence result.

Given that disclaimer, we proceed with this power series as our guess. First, we compute the derivative of the power series by differentiating term by term. We get

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + \ldots = \sum_{n=1}^{\infty} n a_n t^n.$$

So the differential equation $dy/dt = ty + 1$ becomes

$$\begin{aligned} &a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots \\ &\qquad = t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots) + 1, \end{aligned}$$

and simplifying the right-hand side, we obtain

$$\begin{aligned} &a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots \\ &\qquad = 1 + a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \ldots. \end{aligned}$$

These two series are equal if and only if the corresponding coefficients are equal (see Exercise 16). Hence, we can think of the equality of these two power series as the infinite list of equations

$$\begin{aligned} a_1 &= 1 \\ 2a_2 &= a_0 \\ 3a_3 &= a_1 \\ 4a_4 &= a_2 \\ 5a_5 &= a_3 \\ &\vdots \end{aligned}$$

with infinitely many unknowns.

Has guessing a power series helped? Basically we have replaced the differential equation for the function $y(t)$ with infinitely many algebraic equations for the infinitely many unknown coefficients of the power series. At first glance, we seem to have replaced one problem with another. However, looking closely at the list of equations for the coefficients, we notice something nice. We do not have to solve the entire infinite system of equations simultaneously. Rather, we can start at the top and work down, solving each successive equation in terms of the previous solutions. That is,

$$a_1 = 1;$$

$$2a_2 = a_0, \quad \text{which implies that} \quad a_2 = \frac{a_0}{2};$$

$$3a_3 = a_1, \quad \text{which implies that} \quad a_3 = \frac{1}{3};$$

$$4a_4 = a_2 = \frac{a_0}{2}, \quad \text{which implies that} \quad a_4 = \frac{a_0}{2 \cdot 4};$$

$$5a_5 = a_3 = \frac{1}{3}, \quad \text{which implies that} \quad a_5 = \frac{1}{3 \cdot 5};$$

$$6a_6 = a_4 = \frac{a_0}{2 \cdot 4}, \quad \text{which implies that} \quad a_6 = \frac{a_0}{2 \cdot 4 \cdot 6};$$

$$\vdots$$

Therefore, the solution $y(t)$ has the form

$$\begin{aligned} y(t) &= a_0 + t + \frac{a_0}{2}t^2 + \frac{1}{3}t^3 + \frac{a_0}{2 \cdot 4}t^4 + \frac{1}{3 \cdot 5}t^5 + \frac{a_0}{2 \cdot 4 \cdot 6}t^6 + \dots \\ &= a_0\left(1 + \frac{1}{2}t^2 + \frac{1}{2 \cdot 4}t^4 + \frac{1}{2 \cdot 4 \cdot 6}t^6 + \dots\right) + \left(t + \frac{1}{3}t^3 + \frac{1}{3 \cdot 5}t^5 + \dots\right). \end{aligned}$$

Since $y(0) = a_0$, we see that this power series is the general solution of the differential equation. The "..." represent higher degree terms that we could compute if we had the patience. We can think of this power series representation of the solution as giving a sequence of polynomials

$$y(t) \approx y_0(t) = a_0$$

$$y(t) \approx y_1(t) = a_0 + t$$

$$y(t) \approx y_2(t) = a_0 + t + \frac{a_0}{2}t^2$$

$$y(t) \approx y_3(t) = a_0 + t + \frac{a_0}{2}t^2 + \frac{1}{3}t^3$$

$$y(t) \approx y_4(t) = a_0 + t + \frac{a_0}{2}t^2 + \frac{1}{3}t^3 + \frac{a_0}{2 \cdot 4}t^4$$

$$\vdots$$

which form better and better approximations of the actual solution (see Figure B.1). This convergence is just like saying $\sqrt{2} \approx 1.4$, $\sqrt{2} \approx 1.41$, $\sqrt{2} \approx 1.414$, ... Each additional term makes the approximation a bit more accurate near $t = 0$. We graph these

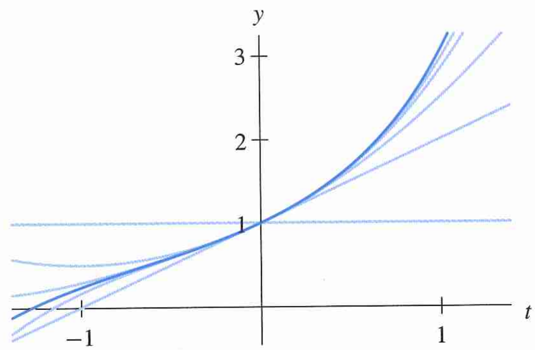

Figure B.1 The graph of the solution to the initial-value problem

$$\frac{dy}{dt} = ty + 1, \quad y(0) = 1,$$

is shown in dark blue. The other graphs are the graphs of the Taylor polynomial approximations to this solution of degrees zero through four.

Taylor approximations for $y(0) = a_0 = 1$ along with the actual solution in Figure B.1. Note that, like most Taylor polynomials, the approximations are accurate near $t = 0$ but not even close far from this point.

A Second-Order Example: Hermite's Equation

The power series technique is particularly useful on certain second-order equations that arise in physics. These equations are linear but do not have constant coefficients.

For example, consider the second-order equation

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2py = 0,$$

where p is a parameter. This equation is called **Hermite's equation**. The coefficient $-2t$ of dy/dt is not a constant, so our usual guess-and-test technique for constant-coefficient equations does not work (try it and see).

Since no better guesses come to mind, we try the guess of last resort, that is, we guess the power series

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots = \sum_{n=0}^{\infty} a_n t^n.$$

To try this quess, we need

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \ldots = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

and

$$\frac{d^2y}{dt^2} = 2a_2 + (3 \cdot 2)a_3 t + (4 \cdot 3)a_4 t^2 + (5 \cdot 4)a_5 t^3 + \ldots = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}.$$

Note that $y(0) = a_0$ and $y'(0) = a_1$, so we can think of the first two coefficients as the initial condition for an initial-value problem.

Substituting these series into Hermite's equation

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2py = 0,$$

we get

$$\begin{aligned}&\left(2a_2 + (3\cdot 2)a_3t + (4\cdot 3)a_4t^2 + (5\cdot 4)a_5t^3 + \dots\right)\\ &\quad -2t\left(a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \dots\right)\\ &\qquad + 2p\left(a_0 + a_1t + a_2t^2 + a_3t^3 + \dots\right) = 0.\end{aligned}$$

Collecting terms on the left-hand side yields the power series

$$\begin{aligned}&(2pa_0 + 2a_2)\\ &+ (2pa_1 - 2a_1 + 6a_3)\,t\\ &+ (2pa_2 - 4a_2 + 12a_4)\,t^2\\ &+ (2pa_3 - 6a_3 + 20a_5)\,t^3\\ &+ \dots = 0\end{aligned}$$

In order for this power series to be identically zero, all of the coefficients must be zero (see Exercise 16). Consequently, solving the differential equation turns into the problem of solving the infinite family of algebraic equations with infinitely many unknowns

$$\begin{aligned}2pa_0 + 2a_2 &= 0\\ 2(p-1)a_1 + 6a_3 &= 0\\ 2(p-2)a_2 + 12a_4 &= 0\\ 2(p-3)a_3 + 20a_5 &= 0\\ &\vdots\end{aligned}$$

Because $a_0 = y(0)$ and $a_1 = y'(0)$ are determined by the initial condition, we can solve this list of equations starting from the top to obtain

$$\begin{aligned}a_2 &= -p\,a_0\\ a_3 &= -\frac{p-1}{3}\,a_1\\ a_4 &= -\frac{p-2}{6}\,a_2 = \frac{(p-2)p}{6}\,a_0\\ a_5 &= -\frac{p-3}{10}\,a_3 = \frac{(p-3)(p-1)}{30}\,a_1\\ &\vdots\end{aligned}$$

Note that the coefficients with even subscripts are multiples of a_0 while those with odd subscripts are multiples of a_1. The resulting general solution is

$$\begin{aligned} y(t) &= a_0 + a_1 t - p a_0 t^2 - \frac{p-1}{3} a_1 t^3 + \frac{(p-2)p}{6} a_0 t^4 + \frac{(p-3)(p-1)}{30} a_1 t^5 - \dots \\ &= a_0 \left(1 - p t^2 + \frac{(p-2)p}{6} t^4 \mp \dots \right) \\ &\quad + a_1 \left(t - \frac{p-1}{3} t^3 + \frac{(p-3)(p-1)}{30} t^5 \mp \dots \right). \end{aligned}$$

With this series representation of the general solution of Hermite's equation, we are tempted to think that our discussion of this example is complete. However there are some important special cases that we should consider.

The parameter p is a nonnegative integer

Suppose the parameter p is a nonnegative integer. For example, consider $p = 0$. Then all of the coefficients having p as a factor are zero, and we have

$$y(t) = a_0 \left(1 - 0 \cdot t^2 + 0 \cdot t^4 - \dots \right) + a_1 \left(t - \frac{0-1}{3} t^3 + \frac{(0-3)(0-1)}{30} t^5 - \dots \right).$$

Moreover, if we specify the initial condition $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$, we get that the constant function

$$y(t) = 1$$

is the solution to Hermite's equation with $p = 0$ and $(y(0), y'(0)) = (1, 0)$.

Next we consider $p = 1$. Then all of the terms having $p - 1$ as a factor are zero, and we get

$$y(t) = a_0 \left(1 - t^2 + \frac{(1-2)p}{6} t^4 - \dots \right) + a_1 \left(t - 0 t^3 + 0 t^5 - \dots \right),$$

Taking $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$, we get that the function

$$y(t) = t$$

is the solution of Hermite's equation with $p = 1$ and $(y(0), y'(0)) = (0, 1)$.

Taking $p = 2$, we see that all of the coefficients having $p - 2$ as a factor equal zero. Again taking $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$, we get that the function

$$y(t) = 1 - 2t^2$$

is the solution of Hermite's equation with $p = 2$ and $(y(0), y'(0)) = (1, 0)$.

This pattern continues. If p is a positive even integer and $(y(0), y'(0)) = (1, 0)$, then the resulting Taylor series has only finitely many nonzero terms. The same holds if p is a positive odd integer and $(y(0), y'(0)) = (0, 1)$. These solutions are called the **Hermite polynomials** $H_p(t)$. We have shown that the first three Hermite polynomials are $H_0(t) = 1$, $H_1(t) = t$, and $H_2(t) = 1 - 2t^2$, and we can easily generate many more.

Our analysis of the Hermite's equation can be compared to our study of constant-coefficient, second-order equations such as

$$\frac{d^2y}{dt^2} + n^2 y = 0.$$

The solutions of this constant-coefficient equation are linear combinations of $\sin nt$ and $\cos nt$. The Hermite polynomials share many properties with the functions $\sin nt$ and $\cos nt$, and both families of functions appear frequently in applications. Legendre's equation, another equation with similar properties, is studied in Exercise 15.

EXERCISES FOR APPENDIX B

In Exercises 1–4, use the guess-and-test method to find the power series expansion centered at $t = 0$ for the general solution up to degree four, that is, up to and including the t^4 term. (You may find the general solution using other methods and then find the Taylor series centered at $t = 0$ to check your computation if you like.)

1. $\dfrac{dy}{dt} = y$

2. $\dfrac{dy}{dt} = -y + 1$

3. $\dfrac{dy}{dt} = -2ty$

4. $\dfrac{dy}{dt} = t^2 y + 1$

In Exercise 5–8, find the power series expansion for the general solution up to degree four, that is, up to and including the t^4 term.

5. $\dfrac{dy}{dt} = -y + e^{2t}$

6. $\dfrac{dy}{dt} = 2y + \sin t$

7. $\dfrac{d^2y}{dt^2} + 2y = \cos t$

8. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + y = \sin 2t$

9. Verify that $y(t) = \tan t$ is a solution of

$$\frac{dy}{dt} = y^2 + 1,$$

and compute a power series solution to find the terms up to degree six (up to and including the t^6 term) of the Taylor series centered at $t = 0$ of $\tan t$.

In Exercises 10–13, find the general solution up to degree six, that is, up to and including the t^6 term.

10. $\dfrac{d^2y}{dt^2} + 2y = 0$

11. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

12. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + t^2 y = \cos t$

13. $\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^{-2t}$

14. Compute the Hermite polynomials $H_3(t)$, $H_4(t)$, and $H_5(t)$. Check that they are solutions of the appropriate Hermite's equation.

15. Legendre's equation is the second-order differential equation

$$(1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + \nu(\nu+1)y = 0,$$

where ν is a constant. Guess the Taylor series centered at $t = 0$ as a solution $y(t)$, that is,

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots.$$

(a) Compute the coefficients a_2, a_3, and a_4 in terms of the initial conditions $a_0 = y(0)$ and $a_1 = y'(0)$.

(b) By choosing ν to be a positive integer and using the initial conditions $(y(0), y'(0)) = (1, 0)$ or $(y(0), y'(0)) = (0, 1)$, show that there are polynomial solutions $P_\nu(t)$ of Legendre's equation.

(c) Show that the first three of these polynomials are $P_0(t) = 1$, $P_1(t) = t$, and $P_2(t) = 1 - 3t^2$.

(d) Compute $P_\nu(t)$ for $\nu = 3, 4, 5$, and 6.

(e) Verify that $kP_\nu(t)$ is a solution of the same Legendre's equation as $P_\nu(t)$ if k is a constant. (Consequently, $P_\nu(t)$ is only determined up to a constant. For example, $P_2(t)$ is sometimes given as $P_2(t) = (3t^2 - 1)/2$).

16. In this appendix we used the fact that two power series in t are equal if and only if all of their coefficients are equal. To justify this statement, let

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

and

$$g(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \dots$$

and suppose that $f(t) = g(t)$ for all t for which both sides are defined.

(a) Verify that $a_0 = b_0$.

(b) Compute $f'(t)$ and $g'(t)$. Since $f(t) = g(t)$, we must have $f'(t) = g'(t)$ for all t. Verify that $a_1 = b_1$.

(c) Explain why $a_n = b_n$ for all n.

17. Consider the linear differential equation

$$\frac{dy}{dt} = -y + e^{-t}.$$

The natural first guess for a particular solution of this equation is $y(t) = ae^{-t}$, but this guess does not yield a solution because it is a solution of the associated homogeneous equation.

Guess a power series solution $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$ and use the initial condition $y(0) = a_0 = 0$ to find a particular solution. Check that this series is the power series for te^{-t}.