LARGE SUBALGEBRAS AND THE STRUCTURE OF CROSSED PRODUCTS (DRAFT)

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Abstract. We give a survey of large subalgebras of crossed product C*-algebras, including some recent applications (by several people), mostly to the transformation group C*-algebra $C^*(\mathbb{Z}, X, h)$ of a minimal homeomorphism h of a compact metric space X:

- If there is a continuous surjective map from X to the Cantor set, then $C^*(\mathbb{Z}, X, h)$ has stable rank one (regardless of the mean dimension of h).
- If there is a continuous surjective map from X to the Cantor set, then the radius of comparison of $C^*(\mathbb{Z}, X, h)$ is at most half the mean dimension of h.
- If h has mean dimension zero, then $C^*(\mathbb{Z}, X, h)$ is Z-stable.
- The "extended" irrational rotation algebras, obtained by "cutting" each of the standard unitary generators at one or more points in its spectrum, are AF algebras.

We include some background material, particularly on the Cuntz semigroup. We give or sketch proofs of some of the basic results on large subalgebras, including a much more direct proof than in the paper on large subalgebras of the fact that a large subalgebra and its containing algebra have the same radius of comparison. We describe a more direct proof than in the papers that if $h\colon X \to X$ is a minimal homeomorphism of a compact metric space, $Y \subset X$ is compact, and $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then the orbit breaking subalgebra associated to Y is centrally large in $C^*(\mathbb{Z}, X, h)$. We sketch the proof, using large subalgebras, that if there is a continuous surjective map from X to the Cantor set, then the radius of comparison of $C^*(\mathbb{Z}, X, h)$ is at most half the mean dimension of h. We state a number of open problems.

This is a draft. It has not been properly proofread, and Section 5 is missing entirely.

Large and centrally large subalgebras are a technical tool which has played a key role in several recent results on the structure of the C*-algebras of minimal dynamical systems and some related algebras. These results include:

- The extended irrational rotation algebras are AF.
- Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is Z-stable.
- Let X be a compact metric space such that there is a continuous surjection from X to the Cantor set. Then $\mathrm{rc}\big(C^*(\mathbb{Z}, X, h)\big) \leq \frac{1}{2}\mathrm{mdim}(h)$.
- Let X be a compact metric space such that there is a continuous surjection from X to the Cantor set. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one. (There

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are examples in which this holds but $C^*(\mathbb{Z}, X, h)$ does not have strict comparison of positive elements and is not Z-stable.)

Large subalgebras were also used to give the first proof that if X is a finite dimensional compact metric space with a free minimal action of $\mathbb{Z}^d$, then $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements.

Large subalgebras are a generalization and abstraction of a construction introduced by Putnam in [44], where it was used to prove that if h is a minimal homeomorphism of the Cantor set X, then $K_0(C^*(\mathbb{Z}, X, h))$ is order isomorphic to the K_0-group of a simple AF algebra (Theorem 4.1 and Corollary 5.6 of [44]). Putnam's construction and some generalizations (all of which are centrally large subalgebras in our sense) also played key roles in proofs of other results:

- Let $h\colon X \to X$ be a minimal homeomorphism of the Cantor set. Then $C^*(\mathbb{Z}, X, h)$ is an AT algebra. (Local approximation by circle algebras was proved in Section 2 of [45]. Direct limit decomposition follows from semiprojectivity of circle algebras.)
- Let $h\colon X \to X$ be a minimal homeomorphism of a finite dimensional compact metric space. Then the order on projections over $C^*(\mathbb{Z}, X, h)$ is determined by traces ([29] and Section 4 of [39]).
- Let X be a finite dimensional infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism such that the map $K_0\big(C^*(\mathbb{Z}, X, h)\big) \to \mathrm{Aff}\big(\mathrm{T}\big(C^*(\mathbb{Z}, X, h)\big)\big)$ has dense range. Then $C^*(\mathbb{Z}, X, h)$ has tracial rank zero ([28]).
- Let X be the Cantor set and let $h\colon X \times S^1 \to X \times S^1$ be a minimal homeomorphism. For any $x \in X$, the set $Y = \{x\} \times S^1$ intersects each orbit at most once. The algebra $C^*(\mathbb{Z}, X \times S^1, h)_Y$ (see Definition 1.7 for the notation) is introduced before Proposition 3.3 of [27], where it is called A_x. It is a centrally large subalgebra which plays a key role in the proofs of many of the results there.
- A similar construction, with $X \times S^1 \times S^1$ in place of $X \times S^1$ and with $Y = \{x\} \times S^1 \times S^1$, appears in Section 1 of [48]. It plays a role in that paper similar to that in the previous item.
- Let $h\colon X \to X$ be a minimal homeomorphism of an infinite compact metric space. The large subalgebras $C^*(\mathbb{Z}, X, h)_Y$ of $C^*(\mathbb{Z}, X, h)$ (as in Definition 1.7) with several choices of Y (several one point sets as well as $\{x_1, x_2\}$ with x_1 and x_2 on different orbits) have been used by Toms and Winter [52] to prove that $C^*(\mathbb{Z}, X, h)$ has finite decomposition rank.
- Julian Buck has used large subalgebras in his study of crossed products $C^*(\mathbb{Z}, C(X, D), \alpha)$ in which D is simple and α "lies over" a minimal homeomorphism of X.

There is a competing approach, the method of Rokhlin dimension of group actions [24], which can be used for some of the same problems large subalgebras are good for. When it applies, it often gives stronger results. For example, Szabo has used this method successfully for free minimal actions of $\mathbb{Z}^d$ on finite dimensional compact metric spaces [49]. For many problems involving crossed products for which large subalgebras are a plausible approach, Rokhlin dimension methods should also be considered. Rokhlin dimension has also been successfully applied to problems involving actions on simple C*-algebras, a context in which no useful large subalgebras are known. On the other hand, finite Rokhlin dimension requires some form of topological finite dimensionality. It seems plausible that there might be a generalization of finite Rokhlin dimension which captures actions on infinite dimensional spaces which have mean dimension zero. Such a generalization might be similar to the progression from the study of simple AH algebras with no dimension growth to those with slow dimension growth.

It looks much less likely that Rokhlin dimension methods can be usefully applied to minimal homeomorphisms which do not have mean dimension zero. Large subalgebras have been used to estimate the radius of comparison of $C^*(\mathbb{Z}, X, h)$ when h does not have mean dimension zero (and the radius of comparison is nonzero); see [23]. They have also been used to prove regularity properties of crossed products $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)$ when D is simple, the automorphism $\alpha \in \mathrm{Aut}(C(X, D))$ "lies over" a minimal homeomorphism of X with large mean dimension, and the regularity properties of the crossed product come from D rather than from the action of $\mathbb{Z}$ on X. See [10].

Unfortunately, we are not able to discuss Rokhlin dimension here.

In these lectures, we give an introduction to large subalgebras, and we illustrate their use in the study of crossed products by minimal homeomorphisms.

1. Introduction, Motivation, and the Cuntz Semigroup

1.1. **Definitions and the basic statements.** We get to the definitions as quickly as possible.

Definition 1.1. Let A be a C*-algebra, and let $a, b \in (K \otimes A)_+$. We say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

By convention, if we say that B is a unital subalgebra of a C*-algebra A, we mean that B contains the identity of A.

Definition 1.2 (Definition 4.1 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

We emphasize that the Cuntz subequivalence involving y in (4) is relative to B, not A.

Condition (5) is needed to avoid triviality when A is purely infinite and simple. With $B = \mathbb{C} \cdot 1$, we could then satisfy all the other conditions by taking $g = 1$. In the stably finite case, we can dispense with (5) (see Proposition 2.3 below), but we still need $g \precsim_A x$ in (4). Otherwise, even if we require that B be simple and that the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$ on traces and quasitraces be bijective, we can take A to be any UHF algebra and take $B = \mathbb{C} \cdot 1$. The choice $g = 1$ would always work.

It is crucial to the usefulness of large subalgebras that g in Definition 1.2 need not be a projection. Also, one can do a lot without any kind of approximate commutation condition. Such a condition does seem to be needed for some things. Here is the relevant definition, although we will not make full use of it in these notes.

Definition 1.3 (Definition 3.2 of [5])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *centrally large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.
(6) For $j = 1, 2, \ldots, m$ we have $\|g a_j - a_j g\| < \varepsilon$.

The difference between Definition 1.3 and Definition 1.2 is the approximate commutation condition in Definition 1.3(6).

The following strengthening of Definition 1.3 will be more important in these notes.

Definition 1.4 (Definition 5.1 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *stably large* in A if $M_n(B)$ is large in $M_n(A)$ for all $n \in \mathbb{Z}_{>0}$.

Proposition 1.5 (Proposition 5.6 of [43])**.** Let A_1 and A_2 be infinite dimensional simple unital C*-algebras, and let $B_1 \subset A_1$ and $B_2 \subset A_2$ be large subalgebras. Assume that $A_1 \otimes_{\min} A_2$ is finite. Then $B_1 \otimes_{\min} B_2$ is a large subalgebra of $A_1 \otimes_{\min} A_2$.

In particular, if A is stably finite and $B \subset A$ is large, then B is stably large. We will give a direct proof (Proposition 2.10 below). We don't know whether stable finiteness of A is needed (Question 1.35 below).

We prepare to define the main example used in these notes.

Notation 1.6. For a locally compact Hausdorff space X and an open subset $U \subset X$, we use the abbreviation

$$C_0(U) = \big\{ f \in C_0(X) \colon f(x) = 0 \text{ for all } x \in X \setminus U \big\} \subset C_0(X).$$

This subalgebra is of course canonically isomorphic to the usual algebra $C_0(U)$ when U is considered as a locally compact Hausdorff space in its own right. In particular, if $Y \subset X$ is closed, then

$$C_0(X \setminus Y) = \big\{ f \in C_0(X) \colon f(x) = 0 \text{ for all } x \in Y \big\}. \tag{1.1}$$

Definition 1.7. Let X be a locally compact Hausdorff space and let $h \colon X \to X$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary. (We say more about crossed products at the beginning of Section 4.) Let $Y \subset X$ be a nonempty closed subset, and, following (1.1), define

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

We call it the Y*-orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

The idea of using subalgebras of this type is due to Putnam [44]. We have used a different convention from that used most other places, where one usually takes

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, uC_0(X \setminus Y)\big).$$

The choice of convention in Definition 1.7 has the advantage that, when used in connection with Rokhlin towers, the bases of the towers are subsets of Y rather than of $h(Y)$.

Theorem 1.8. Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 1.3.

We give a proof in Section 4, along with proofs or sketches of proofs of the lemmas which go into the proof.

The key fact about $C^*(\mathbb{Z}, X, h)_Y$ which makes this theorem useful is that it is a direct limit of recursive subhomogeneous C*-algebras (as in Definition 1.1 of [38]) whose base spaces are closed subsets of X. The structure of $C^*(\mathbb{Z}, X, h)_Y$ is therefore much more accessible than the structure of crossed products.

1.2. **Theorems and applications.** We state the main known results about large subalgebras and some recent applications.

Proposition 1.9 (Proposition 5.2 and Proposition 5.5 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple and infinite dimensional.

Theorem 1.10 (Theorem 6.2 and Proposition 6.9 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$, on traces and quasitraces, are bijective.

The proofs of the two parts are quite different. We prove that $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective below (Theorem 2.11).

Let A be a C*-algebra. The Cuntz semigroup $\mathrm{Cu}(A)$ is defined below (Definition 1.20(3)). Let $\mathrm{Cu}_+(A)$ denote the set of elements $\eta \in \mathrm{Cu}(A)$ which are not the classes of projections. (Such elements are sometimes called *purely positive*.)

Theorem 1.11 (Theorem 6.8 of [43])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\iota\colon B \to A$ be the inclusion map. Then $\mathrm{Cu}(\iota)$ defines an order and semigroup isomorphism from $\mathrm{Cu}_+(B) \cup \{0\}$ to $\mathrm{Cu}_+(A) \cup \{0\}$.

It is not true that $\mathrm{Cu}(\iota)$ defines an isomorphism from $\mathrm{Cu}(B)$ to $\mathrm{Cu}(A)$. Example 7.13 of [43] shows that $\mathrm{Cu}(\iota)\colon \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ need not be injective. We suppose this map can also fail to be surjective, but we don't know an example.

Theorem 1.12 (Theorem 6.14 of [43])**.** Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be a large subalgebra. Let $\mathrm{rc}(-)$ be the radius of comparison (Definition 3.1 below). Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

We will prove this result in Section 3 when A is exact. See Theorem 3.2 below.

Proposition 1.13 (Proposition 6.15, Corollary 6.16, and Proposition 6.17 of [43]). Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then:

(1) A is finite if and only if B is finite.
(2) If B is stably large in A, then A is stably finite if and only if B is stably finite.
(3) A is purely infinite if and only if B is purely infinite.

Theorem 1.14 (Theorem 6.3 of [5]). Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Then:

(1) If $\mathrm{tsr}(B) = 1$ then $\mathrm{tsr}(A) = 1$.
(2) If $\mathrm{tsr}(B) = 1$ and $\mathrm{RR}(B) = 0$, then $\mathrm{RR}(A) = 0$.

The following two key technical results are behind many of the theorems stated above. In particular, they are the basis for proving Theorem 1.11, which is used to prove many of the other results.

Lemma 1.15 (Lemmas 6.3 and 6.5 of [43]). Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a stably large subalgebra.

(1) Let $a, b, x \in (K \otimes A)_+$ satisfy $x \neq 0$ and $a \oplus x \precsim_A b$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$, $c \in (M_n \otimes B)_+$, and $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A c \precsim_A (b - \delta)_+$.
(2) Let $a, b \in (K \otimes B)_+$ and $c, x \in (K \otimes A)_+$ satisfy $x \neq 0$, $a \precsim_A c$, and $c \oplus x \precsim_A b$. Then $a \precsim_B b$.

We state some of the applications. In the following theorem, $\mathrm{rc}(A)$ is the radius of comparison of A (see Definition 3.1 below), and $\mathrm{mdim}(h)$ is the mean dimension of h (see Section 5 below).

Theorem 1.16 ([23]). Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

Theorem 1.17 (Theorem 7.1 of [5]). Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one.

There is no finite dimensionality assumption on X. We don't even assume that h has mean dimension zero. In particular, this theorem holds for the examples of Giol and Kerr [21], for which the crossed products are known not to be Z-stable and not to have strict comparison of positive elements. (It is shown in [23] that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for such systems, and in [21] that $\mathrm{mdim}(h) \neq 0$. See the discussion in Section 7 of [5] for details.)

The proof uses Theorem 1.8, Theorem 1.14(1), the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ be the direct limit of an AH system with diagonal maps, and Theorem 4.1 of [16] to show that simple direct limits of AH systems with diagonal maps always have stable rank one, without any dimension growth hypotheses.

It is conjectured that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for all minimal homeomorphisms. In [23], we also prove that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \geq \frac{1}{2}\mathrm{mdim}(h)$ for a reasonably large class of homomorphisms constructed using the methods of Giol and Kerr [21], including the ones in that paper. For all minimal homeomorphisms of this type, there is a continuous surjective map from the space to the Cantor set.

The proof uses Theorem 1.8, Theorem 1.12, the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ be the direct limit of an AH system with diagonal maps, and methods of [34] (see especially Theorem 6.2 there) to estimate radius of comparison of simple direct limits of AH systems with diagonal maps. We would like to use Theorem 6.2 of [34] directly. Unfortunately, the definition of mean dimension of an AH direct system in [34] requires that the base spaces be connected. See Definition 3.6 of [34], which refers to the setup described after Lemma 3.4 of [34].

Theorem 1.18 (Elliott and Niu [18])**.** The "extended" irrational rotation algebras, obtained by "cutting" each of the standard unitary generators at one or more points in its spectrum, are AF algebras.

We omit the precise descriptions of these algebras.

If one cuts just one of the generators, the resulting algebra is a crossed product by a minimal homeomorphism of the Cantor set, with the other unitary playing the role of the image of a generator of the group. If both are cut, the algebra is no longer an obvious crossed product.

In the next theorem, Z is the Jiang-Su algebra. Being Z-stable is one of the regularity conditions in the Toms-Winter conjecture, and for simple separable nuclear C*-algebras it is hoped, and known in many cases, that Z-stability implies classifiability in the sense of the Elliott program.

Theorem 1.19 (Elliott and Niu [18])**.** Let X be a compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is Z-stable.

1.3. **Cuntz comparison.** We give a summary of Cuntz comparison and a few facts about the Cuntz semigroup of a C*-algebra. We refer to [2] for an extensive introduction. The material we need is also either summarized or proved in the first two sections of [43].

Let $M_\infty(A)$ denote the algebraic direct limit of the system $(M_n(A))_{n=1}^\infty$ using the usual embeddings $M_n(A) \to M_{n+1}(A)$, given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a \in M_m(A)$ and $b \in M_n(A)$, we write $a \oplus b$ for the diagonal direct sum

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

By abuse of notation, we will also write $a \oplus b$ when $a, b \in M_\infty(A)$ and we do not care about the precise choice of m and n with $a \in M_m(A)$ and $b \in M_n(A)$.

The main object of study in these notes is how comparison in the Cuntz semigroup of a C*-algebra A relates to comparison in the Cuntz semigroup of a large subalgebra B. We therefore include the algebra in the notation for Cuntz comparison and classes in the Cuntz semigroup.

Parts (1) and (2) of the following definition are originally from [13]. (Part (1) is a restatement of Definition 1.1.)

Definition 1.20. Let A be a C*-algebra.

(1) For $a, b \in (K \otimes A)_+$, we say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^\infty$ in $K \otimes A$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

(2) We say that a and b are *Cuntz equivalent over* A, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$. This relation is an equivalence relation, and we write $\langle a \rangle_A$ for the equivalence class of a.

(3) The *Cuntz semigroup* of A is

$$\mathrm{Cu}(A) = (K \otimes A)_+ / \sim_A,$$

together with the commutative semigroup operation

$$\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A$$

(the class does not depend on the choice of the isomorphism $M_2(K) \to K$) and the partial order

$$\langle a \rangle_A \leq \langle b \rangle_A \Longleftrightarrow a \precsim_A b.$$

It is taken to be an object of the category **Cu** given in Definition 4.1 of [2]. We write 0 for $\langle 0 \rangle_A$.

(4) We also define the subsemigroup

$$W(A) = M_\infty(A)_+ / \sim_A,$$

with the same operations and order. (It will follow from Remark 1.21 that the obvious map $W(A) \to \mathrm{Cu}(A)$ is injective.)

(5) Let A and B be C*-algebras, and let $\varphi \colon A \to B$ be a homomorphism. We use the same letter for the induced maps $M_n(A) \to M_n(B)$ for $n \in \mathbb{Z}_{>0}$, $M_\infty(A) \to M_\infty(B)$, and $K \otimes A \to K \otimes B$. We define $\mathrm{Cu}(\varphi) \colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ and $W(\varphi) \colon W(A) \to W(B)$ by $\langle a \rangle_A \mapsto \langle \varphi(a) \rangle_B$ for $a \in (K \otimes A)_+$ or $M_\infty(A)_+$ as appropriate.

It is easy to check that the maps $\mathrm{Cu}(\varphi)$ and $W(\varphi)$ are well defined homomorphisms of ordered semigroups which send 0 to 0. Also, it follows from Lemma 1.23(14) below that if $\eta_1, \eta_2, \mu_1, \mu_2 \in \mathrm{Cu}(A)$ satisfy $\eta_1 \leq \mu_1$ and $\eta_2 \leq \mu_2$, then $\eta_1 + \eta_2 \leq \mu_1 + \mu_2$.

The semigroup $\mathrm{Cu}(A)$ generally has better properties than $W(A)$. For example, certain supremums exist (Theorem 4.19 of [2]), and, when understood as an object of the category **Cu**, it behaves properly with respect to direct limits (Theorem 4.35 of [2]). In this exposition, we mainly use $W(A)$ because, when A is unital, the dimension function d_τ associated to a normalized quasitrace τ (Definition 1.26 below) is finite on $W(A)$ but usually not on $\mathrm{Cu}(A)$. In particular, the radius of comparison (Definition 3.1 below) is easier to deal with in terms of $W(A)$.

We will not need the definition of the category **Cu**.

Remark 1.21. We make the usual identifications

$$A \subset M_n(A) \subset M_\infty(A) \subset K \otimes A. \tag{1.2}$$

It is easy to check, by cutting down to corners, that if $a, b \in (K \otimes A)_+$ satisfy $a \precsim_A b$, then the sequence $(v_n)_{n=1}^\infty$ such that $\lim_{n \to \infty} v_n b v_n^* = a$ can be taken to be in the smallest of the algebras in (1.2) which contains both a and b. See Remark 1.2 of [43] for details.

The Cuntz semigroup of a separable C*-algebra can be very roughly thought of as K-theory using open projections in matrices over A'', that is, open supports of positive elements in matrices over A, instead of projections in matrices over A. As justification for this heuristic, we note that if X is a compact Hausdorff space and $f, g \in C(X)_+$, then $f \precsim_{C(X)} g$ if and only if

$$\big\{x \in X \colon f(x) > 0\big\} \subset \big\{x \in X \colon g(x) > 0\big\}.$$

There is a description of $\mathrm{Cu}(A)$ using Hilbert modules over A in place of finitely generated projective modules. See [11].

Unlike K-theory, the Cuntz semigroup is not discrete. If $p, q \in A$ are projections such that $\|p-q\| < 1$, then p and q are Murray-von Neumann equivalent. However, for $a, b \in A_+$, the relation $\|a - b\| < \varepsilon$ says nothing about the classes of a and b in $\mathrm{Cu}(A)$ or $W(A)$, however small $\varepsilon > 0$ is. We can see this in $\mathrm{Cu}(C(X))$. Even if $\big\{x \in X \colon g(x) > 0\big\}$ is a very small subset of X, for every $\varepsilon > 0$ the function $f = g + \frac{\varepsilon}{2}$ has $\langle f \rangle_{C(X)} = \langle 1 \rangle_{C(X)}$. What is true when $\|f - g\| < \varepsilon$ is that

$$\big\{x \in X \colon f(x) > \varepsilon\big\} \subset \big\{x \in X \colon g(x) > 0\big\},$$

so that the function $\max(f - \varepsilon, \, 0)$ satisfies $\max(f - \varepsilon, \, 0) \precsim_{C(X)} g$. This motivates the systematic use of the elements $(a - \varepsilon)_+$, defined as follows.

Definition 1.22. Let A be a C*-algebra, let $a \in A_+$, and let $\varepsilon \geq 0$. Let $f \colon [0, \infty) \to [0, \infty)$ be the function

$$f(\lambda) = (\lambda - \varepsilon)_+ = \begin{cases} 0 & 0 \leq \lambda \leq \varepsilon \\ \lambda - \varepsilon & \varepsilon < \lambda. \end{cases}$$

Then define $(a - \varepsilon)_+ = f(a)$ (using continuous functional calculus).

One must still be much more careful than with K-theory. First, $a \leq b$ does not imply $(a - \varepsilon)_+ \leq (b - \varepsilon)_+$ (although one does get $(a - \varepsilon)_+ \precsim_A (b - \varepsilon)_+$; see Lemma 1.23(17) below). Second, $a \precsim_A b$ does not imply any relation between $(a - \varepsilon)_+$ and $(b - \varepsilon)_+$. For example, if $A = C([0, 1])$ and $a \in C([0, 1])$ is $a(t) = t$ for $t \in [0, 1]$, then for any $\varepsilon \in (0, 1)$ the element $b = \varepsilon a$ satisfies $a \precsim_A b$. But $(a - \varepsilon)_+ \not\precsim_A (b - \varepsilon)_+$, since $(a - \varepsilon)_+$ has open support $(\varepsilon, 1]$ while $(b - \varepsilon)_+ = 0$. The best one can do is in Lemma 1.23(11) below.

We now list a collection of basic results about Cuntz comparison and the Cuntz semigroup. There are very few such results about projections and the K_0-group, the main ones being that if $\|p - q\| < 1$, then p and q are Murray-von Neumann equivalent; that $p \leq q$ if and only if $pq = p$; the relations between homotopy, unitary equivalence, and Murray-von Neumann equivalence; and the fact that addition of equivalence classes respects orthogonal sums. There are many more for Cuntz comparison. We will not use all the facts listed below in these notes (although they are all used in [43]); we include them all so as to give a fuller picture of Cuntz comparison.

Parts (1) through (14) of Lemma 1.23 are mostly taken from [25], with some from [12], [19], [36], and [47], and are summarized in Lemma 1.4 of [43]; we refer to [43] for more on the attributions (although not all the attributions there are to the original sources). Part (15) is Lemma 1.5 of [43]; part (16) is Corollary 1.6 of [43]; part (17) is Lemma 1.7 of [43]; and part (18) is Lemma 1.9 of [43].

We denote by A^+ the unitization of a C*-algebra A. (We add a new unit even if A is already unital.)

Lemma 1.23. Let A be a C*-algebra.

(1) Let $a, b \in A_+$. Suppose $a \in \overline{bAb}$. Then $a \precsim_A b$.

(2) Let $a \in A_+$ and let $f\colon [0, \|a\|] \to [0, \infty)$ be a continuous function such that $f(0) = 0$. Then $f(a) \precsim_A a$.
(3) Let $a \in A_+$ and let $f\colon [0, \|a\|] \to [0, \infty)$ be a continuous function such that $f(0) = 0$ and $f(\lambda) > 0$ for $\lambda > 0$. Then $f(a) \sim_A a$.
(4) Let $c \in A$. Then $c^* c \sim_A c c^*$.
(5) Let $a \in A_+$, and let $u \in A^+$ be unitary. Then $u a u^* \sim_A a$.
(6) Let $c \in A$ and let $\alpha > 0$. Then $(c^* c - \alpha)_+ \sim_A (c c^* - \alpha)_+$.
(7) Let $v \in A$. Then there is an isomorphism $\varphi\colon \overline{v^* v A v^* v} \to \overline{v v^* A v v^*}$ such that, for every positive element $z \in \overline{v^* v A v^* v}$, we have $z \sim_A \varphi(z)$.
(8) Let $a \in A_+$ and let $\varepsilon_1, \varepsilon_2 > 0$. Then

$$\big((a - \varepsilon_1)_+ - \varepsilon_2 \big)_+ = \big(a - (\varepsilon_1 + \varepsilon_2) \big)_+ .$$

(9) Let $a, b \in A_+$ satisfy $a \precsim_A b$ and let $\delta > 0$. Then there is $v \in A$ such that $v^* v = (a - \delta)_+$ and $v v^* \in \overline{b A b}$.
(10) Let $a, b \in A_+$. Then $\|a - b\| < \varepsilon$ implies $(a - \varepsilon)_+ \precsim_A b$.
(11) Let $a, b \in A_+$. Then the following are equivalent:
  (a) $a \precsim_A b$.
  (b) $(a - \varepsilon)_+ \precsim_A b$ for all $\varepsilon > 0$.
  (c) For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A (b - \delta)_+$.
(12) Let $a, b \in A_+$. Then $a + b \precsim_A a \oplus b$.
(13) Let $a, b \in A_+$ be orthogonal (that is, $ab = 0$). Then $a + b \sim_A a \oplus b$.
(14) Let $a_1, a_2, b_1, b_2 \in A_+$, and suppose that $a_1 \precsim_A a_2$ and $b_1 \precsim_A b_2$. Then $a_1 \oplus b_1 \precsim_A a_2 \oplus b_2$.
(15) Let $a, b \in A$ be positive, and let $\alpha, \beta \geq 0$. Then

$$\big(a + b - (\alpha + \beta) \big)_+ \precsim_A (a - \alpha)_+ + (b - \beta)_+ \precsim_A (a - \alpha)_+ \oplus (b - \beta)_+ .$$

(16) Let $\varepsilon > 0$ and $\lambda \geq 0$. Let $a, b \in A$ satisfy $\|a - b\| < \varepsilon$. Then $(a - \lambda - \varepsilon)_+ \precsim_A (b - \lambda)_+$.
(17) Let $a, b \in A$ satisfy $0 \leq a \leq b$. Let $\varepsilon > 0$. Then $(a - \varepsilon)_+ \precsim_A (b - \varepsilon)_+$.
(18) Let $a \in (K \otimes A)_+$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$ and $b \in (M_n \otimes A)_+$ such that $(a - \varepsilon)_+ \sim_A b$.

The following result is sufficiently closely tied to the ideas behind large subalgebras that we include the proof.

Lemma 1.24 (Lemma 1.8 of [43])**.** Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then

$$(a - \varepsilon)_+ \precsim_A \big[(1 - g) a (1 - g) - \varepsilon \big]_+ \oplus g .$$

Proof. Set $h = 2g - g^2$, so that $(1 - g)^2 = 1 - h$. We claim that $h \sim_A g$. Since $0 \leq g \leq 1$, this follows from Lemma 1.23(3), using the continuous function $\lambda \mapsto 2\lambda - \lambda^2$ on $[0, 1]$.

Set $b = \big[(1 - g) a (1 - g) - \varepsilon \big]_+$. Using Lemma 1.23(15) at the second step, Lemma 1.23(6) and Lemma 1.23(4) at the third step, and Lemma 1.23(14) at the last step, we get

$$\begin{aligned}(a-\varepsilon)_+ &= \left[a^{1/2}(1-h)a^{1/2} + a^{1/2}ha^{1/2} - \varepsilon\right]_+ \\ &\precsim_A \left[a^{1/2}(1-h)a^{1/2} - \varepsilon\right]_+ \oplus a^{1/2}ha^{1/2} \\ &\sim_A \left[(1-g)a(1-g) - \varepsilon\right]_+ \oplus h^{1/2}ah^{1/2} \\ &= b \oplus h^{1/2}ah^{1/2} \leq b \oplus \|a\|h \precsim_A b \oplus g.\end{aligned}$$

This completes the proof. □

Notation 1.25. For a unital C*-algebra A, we denote by $\mathrm{T}(A)$ the set of tracial states on A. We denote by $\mathrm{QT}(A)$ the set of normalized 2-quasitraces on A (Definition II.1.1 of [8]; Definition 2.31 of [2]).

Definition 1.26. Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$ for $a \in M_\infty(A)_+$. Further (the use of the same notation should cause no confusion) define $d_\tau \colon (K \otimes A)_+ \to [0, \infty]$ by the same formula, but now for $a \in (K \otimes A)_+$. We also use the same notation for the corresponding functions on $\mathrm{Cu}(A)$ and $W(A)$, as in Proposition 1.27 below.

Proposition 1.27. Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Then d_τ as in Definition 1.26 is well defined on $\mathrm{Cu}(A)$ and $W(A)$. That is, if $a, b \in (K \otimes A)_+$ satisfy $a \sim_A b$, then $d_\tau(a) = d_\tau(b)$.

Proof. This is part of Proposition 4.2 of [19]. □

Also see the beginning of Section 2.6 of [2], especially the proof of Theorem 2.32 there. It follows that d_τ defines a state on $W(A)$. Thus (see Theorem II.2.2 of [8], which gives the corresponding bijection between 2-quasitraces and dimension functions which are not necessarily normalized but are finite everywhere), the map $\tau \mapsto d_\tau$ is a bijection from $\mathrm{QT}(A)$ to the lower semicontinuous dimension functions on A.

1.4. **Cuntz comparison in simple C*-algebras.** We present some results related to Cuntz comparison specifically for simple C*-algebras.

Lemma 1.28 (Lemma 2.1 of [43])**.** Let A be a simple C*-algebra which is not of type I. Then there exists $a \in A_+$ such that $\mathrm{sp}(a) = [0, 1]$.

Proof. The discussion before (1) on page 61 of [1] shows that A is not scattered in the sense of [1]. The conclusion therefore follows from the argument in (4) on page 61 of [1]. □

Lemma 1.29 (Lemma 2.2 and Lemma 2.3 of [43])**.** Let A be a C*-algebra which is not of type I. Let $n \in \mathbb{Z}_{>0}$. Then there exist a unitary $u \in A$ (in A^+ if A is not unital) which is homotopic to 1, and a nonzero positive element $a \in A$, such that the elements

$$a,\, uau^{-1},\, u^2au^{-2},\, \ldots,\, u^nau^{-n}$$

are pairwise orthogonal.

The proof in [43] uses heavy machinery, and there ought to be a simpler proof, particularly when A is simple.

Sketch of the proof of Lemma 1.29. We describe the ideas. (For details, see [43].) We do the unital case; the reduction from the nonunital case to the unital case is easy.

First prove the result for the unitized cone $(CM_{n+1})^+$ in place of A. This is elementary. Then construct an injective unital homomorphism φ_0 from $(CM_{n+1})^+$ to $D = \bigotimes_{m=1}^{\infty} M_{n+1}$. (Use Lemma 1.28 to get started.) A result of Glimm (see Corollary 6.7.4 of [37]) provides a subalgebra $B \subset A$ and a surjective homomorphism $\pi \colon B \to D$. We can assume that B contains the identity of A. Use projectivity of CM_{n+1} (see Theorem 10.2.1 of [31]) to lift φ_0 to an injective unital homomorphism from $(CM_{n+1})^+$ to B, giving an injective unital homomorphism $\varphi \colon (CM_{n+1})^+ \to A$. Now use the result for $(CM_{n+1})^+$ to get the result for A. □

It is now easy to prove the next lemma (although there are shorter proofs).

Lemma 1.30 (Lemma 2.4 of [43]). Let A be a simple C*-algebra which is not of type I. Let $a \in A_+ \setminus \{0\}$, and let $l \in \mathbb{Z}_{>0}$. Then there exist $b_1, b_2, \ldots, b_l \in A_+ \setminus \{0\}$ such that $b_1 \sim_A b_2 \sim_A \cdots \sim_A b_l$, such that $b_j b_k = 0$ for $j \neq k$, and such that $b_1 + b_2 + \cdots + b_l \in \overline{aAa}$.

This lemma has the following corollary.

Corollary 1.31 (Corollary 2.5 of [43]). Let A be a simple unital infinite dimensional C*-algebra. Then for every $\varepsilon > 0$ there is $a \in A_+ \setminus \{0\}$ such that for all $\tau \in \mathrm{QT}(A)$ we have $d_\tau(a) < \varepsilon$.

Lemma 1.32 (Lemma 2.6 of [43]). Let A be a simple C*-algebra, and let $B \subset A$ be a nonzero hereditary subalgebra. Let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A_+ \setminus \{0\}$. Then there exists $b \in B_+ \setminus \{0\}$ such that $b \precsim_A a_j$ for $j = 1, 2, \ldots, n$.

Sketch of proof. The proof is by induction. The case $n = 0$ is trivial. The induction step requires that for $a, b_0 \in A_+ \setminus \{0\}$ one find $b \in A_+ \setminus \{0\}$ such that $b \in \overline{b_0 A b_0}$ (so that $b \precsim_A b_0$ by Lemma 1.23(1)) and $b \precsim_A a$. Use simplicity to find $x \in A$ such that the element $y = b_0 x a$ is nonzero, and take $b = y y^* \in \overline{b_0 A b_0}$. Using Lemma 1.23(5) and Lemma 1.23(1), we get $b \sim_A y^* y \precsim_A a$. □

Lemma 1.33 (Lemma 2.7 of [43]). Let A be a simple infinite dimensional C*-algebra which is not of type I. Let $b \in A_+ \setminus \{0\}$, let $\varepsilon > 0$, and let $n \in \mathbb{Z}_{>0}$. Then there are $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that, in $W(A)$, we have

$$n \langle (b - \varepsilon)_+ \rangle_A \leq (n+1) \langle c \rangle_A \quad \text{and} \quad \langle c \rangle_A + \langle y \rangle_A \leq \langle b \rangle_A.$$

Sketch of proof. We divide the proof into two cases. First assume that $\mathrm{sp}(b) \cap (0, \varepsilon) \neq \varnothing$. Then there is a continuous function $f \colon [0, \infty) \to [0, \infty)$ which is zero on $\{0\} \cup [\varepsilon, \infty)$ and such that $f(b) \neq 0$. We take $c = (b - \varepsilon)_+$ and $y = f(b)$.

Now suppose that $\mathrm{sp}(b) \cap (0, \varepsilon) = \varnothing$. In this case, we might as well assume that b is a projection, and that $\langle (b - \varepsilon)_+ \rangle_A$, which is always dominated by $\langle b \rangle_A$, is equal to $\langle b \rangle_A$. Cutting down by b, we can assume that $b = 1$ (in particular, A is unital), and it is enough to find $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that $n \langle 1 \rangle_A \leq (n+1) \langle c \rangle_A$ and $\langle c \rangle_A + \langle y \rangle_A \leq \langle 1 \rangle_A$.

Take the unitized cone over M_{n+1} to be $C = [CM_{n+1}]^+ = (C_0((0,1]) \otimes M_{n+1})^+$, and use the usual notation for matrix units. Lemma 1.29 provides $a \in A_+ \setminus \{0\}$ and a unitary $u \in A$ such that the elements

$$a, \, uau^{-1}, \, u^2 a u^{-2}, \, \ldots, \, u^n a u^{-n}$$

are pairwise orthogonal. Without loss of generality $\|a\| = 1$. Let $t \in C_0((0,1])$ be the function $t(\lambda) = \lambda$ for $\lambda \in (0,1]$. There is a unital homomorphism $\psi \colon C \to A$ such that $\psi(t \otimes e_{k,k}) = u^{k-1} a u^{-(k-1)}$ for $k = 1, 2, \ldots, n+1$. Choose continuous functions $g_1, g_2, g_3 \in C_0((0,1]$ such that $0 \leq g_3 \leq g_2 \leq g_1 \leq 1$, $g_3(1) = 1$, $g_1 g_2 = g_2$, and $g_2 g_3 = g_3$. Define

$$x = \psi(g_2 \otimes e_{1,1}), \quad c = 1 - x, \quad \text{and} \quad y = \psi(g_3 \otimes e_{1,1}).$$

Then $xy = y$ so $cy = 0$. It follows from Lemma 1.23(13) that $\langle c \rangle_A + \langle y \rangle_A \leq \langle 1 \rangle_A$.

It remains to prove that $n \langle 1 \rangle_A \leq (n+1) \langle c \rangle_A$, and it is enough to prove that in $W(C)$ we have $n \langle 1 \rangle_C \leq (n+1) \langle 1 - g_2 \otimes e_{1,1} \rangle_C$, that is, in $M_{n+1}(C)$,

$$\operatorname{diag}(1, 1, \ldots, 1, 0) \precsim_C \operatorname{diag}\big(1 - g_2 \otimes e_{1,1}, \, 1 - g_2 \otimes e_{1,1}, \, \ldots, \, 1 - g_2 \otimes e_{1,1}\big). \tag{1.3}$$

To see why this should be true, view $M_{n+1}(C)$ as a set of functions from $[0,1]$ to $M_{(n+1)^2}$ with restrictions on the value at zero. Since $g_1 g_2 = g_2$, the function $1 - g_2 \otimes e_{1,1}$ is constant equal to 1 on a neighborhood U of 0, and at $\lambda \in U$ the right hand side of (1.3) therefore dominates the left hand side. Elsewhere, both sides of (1.3) are diagonal, with the right hand side being a constant projection of rank $n(n+1)$ and the left hand side dominating

$$\operatorname{diag}\big(1 - e_{1,1}, \, 1 - e_{1,1}, \, \ldots, \, 1 - e_{1,1}\big),$$

which is a (different) constant projection of rank $n(n+1)$. It is in fact not hard to construct an explicit formula for a unitary $v \in M_{n+1}(C)$ such that

$$\operatorname{diag}(1, 1, \ldots, 1, 0) \leq v \cdot \operatorname{diag}\big(1 - g_2 \otimes e_{1,1}, \, 1 - g_2 \otimes e_{1,1}, \, \ldots, \, 1 - g_2 \otimes e_{1,1}\big) \cdot v^*.$$

See [43] for the details (arranged a little differently). □

1.5. **Open problems.** We discuss some open problems. We start with some which are motivated by particular applications, and then give some which are suggested by results already proved but for which we don't have immediate applications.

The first question is motivated by the hope that large subalgebras can be used to get more information about crossed products than we now know how to get. In most parts, we expect that positive answers would require special hypotheses, if they can be gotten at all.

Question 1.34. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra.

(1) Suppose that B has tracial rank zero. Does it follow that A has tracial rank zero?
(2) Suppose that B is quasidiagonal. Does it follow that A is quasidiagonal?
(3) Suppose that B has finite decomposition rank. Does it follow that A has finite decomposition rank?
(4) Suppose that B has finite nuclear dimension. Does it follow that A has finite nuclear dimension?

It seems likely that "tracial" versions of these properties pass from a large subalgebra to the containing algebra, at least if the tracial versions are defined using cutdowns by positive elements rather than by projections. But we don't know how useful such properties are. As far as we know, they have not been studied.

Next, we ask whether being stably large is automatic.

Question 1.35. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra. Does it follow that $M_n(B)$ is large (or centrally large) in $M_n(A)$ for $n \in \mathbb{Z}_{>0}$?

We know that this is true if A is stably finite. (See Proposition 2.10 below.) Not having the general statement is a technical annoyance. This result would be helpful when dealing with large subalgebras of $C^*(\mathbb{Z}, C(X, D), \alpha)$ when D is simple unital, X is compact metric, and the homeomorphism of $\mathrm{Prim}(C(X, D)) \cong X$ induced by α is minimal. Some results on large subalgebras of such crossed products can be found in [4]; also see Theorem 4.4.

More generally, does Proposition 1.5 still hold without the finiteness assumption?

Question 1.36. Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon \mathbb{Z} \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Is there a useful large or centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$?

We want a centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$ which "locally looks like matrices over corners of A". The paper [35] proves that crossed products by automorphisms with the tracial Rokhlin property preserve the combination of real rank zero, stable rank one, and order on projections determined by traces. The methods were inspired by those of [40], which used large subalgebras (without the name). The proof in [35] does not, however, construct a single large subalgebra. Instead, it constructs a suitable subalgebra (analogous to $C^*(\mathbb{Z}, X, h)_Y$ for a small closed subset $Y \subset X$ with $\mathrm{int}(Y) \neq \varnothing$) for every choice of finite set $F \subset C^*(\mathbb{Z}, A, \alpha)$ and every choice of $\varepsilon > 0$. It is far from clear how to choose these subalgebras to form an increasing sequence so that a direct limit can be built.

The first intended application is simplification of [35].

Problem 1.37. Let X be a compact metric space, and let G be a countable amenable group which acts minimally and essentially freely on X. Construct a (centrally) large subalgebra $B \subset C^*(G, X)$ which is a direct limit of recursive subhomogeneous C*-algebras as in [38] whose base spaces are closed subsets of X, and which is the (reduced) C*-algebra of an open subgroupoid of the transformation group groupoid obtained from the action of G on X.

In a precursor to the theory of large subalgebras, this is in effect done in [40] when $G = \mathbb{Z}^d$ and X is the Cantor set, following ideas of [20]. The resulting centrally large subalgebra is used in [40] to prove that $C^*(\mathbb{Z}^d, X)$ has stable rank one, real rank zero, and order on projections determined by traces. (More is now known.)

We also know how to construct a centrally large subalgebra of this kind when $G = \mathbb{Z}^d$ and X is finite dimensional. This gave the first proof that, in this case, $C^*(\mathbb{Z}^d, X)$ has stable rank one and strict comparison of positive elements. (Again, more is now known.) Unlike for actions of $\mathbb{Z}$, there are no known explicit formulas like that in Theorem 1.8; instead, centrally large subalgebras must be proved to exist via constructions involving many choices. They are direct limits of C*-algebras of open subgroupoids of the transformation group groupoid as in Problem 1.37. In each open subgroupoid, there is a finite upper bound on the size of the orbits; this is why they are recursive subhomogeneous C*-algebras. (In fact, the original motivation for the definition of a large subalgebras was to describe the essential properties of these subalgebras, as a substitute for an explicit description.)

We presume, as suggested in Problem 1.37, that the construction can be done in much greater generality.

Problem 1.38. Develop the theory of large subalgebras of not necessarily simple C*-algebras.

One can't just copy Definition 1.2. Suppose B is a nontrivial large subalgebra of A. We surely want $B \oplus B$ to be a large subalgebra of $A \oplus A$. Take $x_0 \in A_+ \setminus \{0\}$, and take the element $x \in A \oplus A$ in Definition 1.2 to be $x = (x_0, 0)$. Writing $g = (g_1, g_2)$, we have forced $g_2 = 0$. Thus, not only would $B \oplus B$ not be large in $A \oplus A$, but even $A \oplus B$ would not be large in $A \oplus A$.

In this particular case, the solution is to require that x and y be full elements in A and B. What to do is much less clear if, for example, A is a unital extension of the form

$$0 \longrightarrow K \otimes D \longrightarrow A \longrightarrow E \longrightarrow 0,$$

even if D and E are simple, to say nothing of the general case.

The following problem goes just a small step away from the simple case, and just asking that x and y be full might possibly work for it.

Question 1.39. Let X be an infinite compact metric space and let $h \colon X \to X$ be a homeomorphism which has a factor system which is a minimal homeomorphism of an infinite compact metric space (or, stronger, a minimal homeomorphism of the Cantor set). Can one use large subalgebra methods to relate the mean dimension of h to the radius of comparison of $C^*(\mathbb{Z}, X, h)$?

We point out that Lindenstrauss's embedding result for systems of finite mean dimension in shifts built from finite dimensional spaces (Theorem 5.1 of [30]) is proved for homeomorphisms having a factor system which is a minimal homeomorphism of an infinite compact metric space.

Problem 1.40. Develop the theory of large subalgebras of simple but not necessarily unital C*-algebras.

One intended application is to crossed products $C^*\big(\mathbb{Z}, \, C(X, D), \, \alpha\big)$ when X is an infinite compact metric space, D is simple but not unital. and the induced action on X is given by a minimal homeomorphism. (Compare with Theorem 4.4.) Another possible application is to the structure of crossed products $C^*(\mathbb{Z}, X, h)$ when h is a minimal homeomorphism of a noncompact version of the Cantor set. Minimal homeomorphisms of noncompact Cantor sets have been studied in [32] and [33], but, as far as we know, almost nothing is known about their transformation group C*-algebras.

Can the technique of large subalgebras be adapted to L^p operator crossed products, as in [42]? For example, consider the following question.

Question 1.41. Let $p \in [1, \infty) \setminus \{2\}$. Let X be an infinite compact metric space and let $h \colon X \to X$ be a minimal homeomorphism. Suppose that $\dim(X)$ is finite, or that X has a surjective continuous map to the Cantor set. Does it follow that the L^p operator crossed product $F^p(\mathbb{Z}, X, h)$ has stable rank one?

It is at least known (Theorem 5.6 of [42]) that $F^p(\mathbb{Z}, X, h)$ is simple.

The answer to Question 1.41 is unknown even when X is the Cantor set. Putnam's original argument (Section 2 of [45]) depends on continuous deformation of unitaries in $M_n(\mathbb{C})$. The analogous construction for isometries in the L^p version of $M_n(\mathbb{C})$ is not possible. (The only isometries are products of permutation matrices and diagonal isometries.) It seems likely that the machinery will fail if one must replace isometries with more general invertible elements. It also seems likely that a theory of L^p AH algebras will only work if the maps in the direct system are assumed to be diagonal (in a sense related to that at the beginning of Section 2.2 of [16]), and that $F^p(\mathbb{Z}, X, h)$ is unlikely to be such a direct limit even when X is the Cantor set. On the other hand, in many cases $C^*(\mathbb{Z}, X, h)$ has a large subalgebra $C^*(\mathbb{Z}, X, h)_Y$ which is AH with diagonal maps. See Lemma 5.12 of [23]. The idea for Question 1.41 is to adapt C* proofs to show that an L^p AH algebra with no dimension growth and diagonal maps has stable rank one (possibly, following [16], even without assumptions on dimension growth), and to prove that if $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$ then the algebra one might call $F^p(\mathbb{Z}, X, h)_Y$ is large in $F^p(\mathbb{Z}, X, h)$ in a suitable sense. It isn't clear what the abstract definition of a centrally large subalgebra of an L^p operator algebra should be, since we don't know anything about an L^p analog of Cuntz comparison, but possibly one can work with the explicitly given inclusion $F^p(\mathbb{Z}, X, h)_Y \subset F^p(\mathbb{Z}, X, h)$.

For a large subalgebra $B \subset A$, the proofs of most of the relations between A and B do not need B to be centrally large. The exceptions so far are for stable rank one. Do we really need centrally large for the results on stable rank?

Question 1.42. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra (not necessarily centrally large). If B has stable rank one, does it follow that A has stable rank one?

That is, can Theorem 1.14 be generalized from centrally large subalgebras to large subalgebras?

It is not clear how important this question is. In all applications so far, with the single exception of [17], the large subalgebras used are known to be centrally large. In particular, all known useful large subagebras of crossed products are already known to be centrally large.

Question 1.43. Does there exist a large subagebra which is not centrally large? Are there natural examples?

The results of [17] depend on large subagebras which are not proved there to be centrally large, but it isn't known that they are not centrally large.

Question 1.44. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra. If $\mathrm{RR}(B) = 0$, does it follow that $\mathrm{RR}(A) = 0$? What about the converse? Does it help to assume that B is centrally large in the sense of Definition 1.3?

If B has both stable rank one and real rank zero, and is centrally large in A, then A has real rank zero (as well as stable rank one) by Theorem 1.14. The main point of Question 1.44 is to ask what happens if B is not assumed to have stable rank one. The proof in [40] of real rank zero for the crossed product $C^*(\mathbb{Z}^d, X)$ of a free minimal action of $\mathbb{Z}^d$ on the Cantor set X (see Theorem 6.11(2) of [40]; the main part is Theorem 4.6 of [40]) gives reason to hope that if B is large in A and $\mathrm{RR}(B) = 0$, then one does indeed get $\mathrm{RR}(A) = 0$.

Applications to crossed products may be unlikely. It seems possible that $C^*(G, X)$ has stable rank one for every minimal essentially free action of a countable amenable group G on a compact metric space X.

Question 1.45. Let A be an infinite dimensional simple separable unital C*-algebra. Let $B \subset A$ be centrally large in the sense of Definition 1.3. Does it follow that $K_0(B) \to K_0(A)$ is an isomorphism mod infinitesimals?

In other places where this issue occurs (in connection with tracial approximate innerness; see Proposition 6.2 and Theorem 6.4 of [41]), it seems that everything in K_1 should be considered to be infinitesimal.

A six term exact sequence for the K-theory of some orbit breaking subalgebras is given in Example 2.6 of [46].

A positive answer to Question 1.45 would shed some light on both directions in Question 1.44.

Question 1.46. Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be centrally large in the sense of Definition 1.3. If A has stable rank one, does it follow that B has stable rank one?

That is, does Theorem 1.14 have a converse? In many other results in Section 1.2, B has an interesting property if and only if A does.

Question 1.47. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Let $n \in \mathbb{Z}_{>0}$. If $\mathrm{tsr}(B) \leq n$, does it follow that $\mathrm{tsr}(A) \leq n$? If $\mathrm{tsr}(B)$ is finite, does it follow that $\mathrm{tsr}(A)$ is finite?

That is, can Theorem 1.14 be generalized to other values of the stable rank? The proof of Theorem 1.14 uses $\mathrm{tsr}(B) = 1$ in two different places, one of which is not directly related to $\mathrm{tsr}(A)$, so an obvious approach seems unlikely to succeed.

As with Question 1.44, applications to crossed products seem unlikely.

2. Large Subalgebras and their Basic Properties

Recall that, by convention, if we say that B is a unital subalgebra of a C*-algebra A, we mean that B contains the identity of A.

We repeat Definition 1.2.

Definition 2.1 (Definition 4.1 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

We emphasize that the Cuntz subequivalence involving y in (4) is relative to B, not A.

Lemma 2.2. In Definition 2.1, it suffices to let $S \subset A$ be a subset whose linear span is dense in A, and verify the hypotheses only when $a_1, a_2, \ldots, a_m \in S$.

Unlike other approximation properties (such as tracial rank), it seems not to be possible to take S in Lemma 2.2 to be a generating subset, or even a selfadjoint generating subset. (We can do this for the definition of a centrally large subalgebra, Definition 1.3. See Proposition 3.10 of [5].)

By Proposition 4.4 of [43], in Definition 2.1 we can omit mention of $c_1, c_2, \ldots, c_m$, and replace (2) and (3) by the requirement that $\operatorname{dist}\big((1-g)c_j, \, B\big) < \varepsilon$ for $j = 1, 2, \ldots, m$. So far, however, most verifications of Definition 2.1 proceed by constructing elements $c_1, c_2, \ldots, c_m$ as in Definition 2.1.

When A is finite, we do not need condition (5) of Definition 2.1.

Proposition 2.3 (Proposition 4.5 of [43])**.** Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra. Suppose that for $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1-g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.

Then B is large in A.

The proof of Proposition 2.3 needs Lemma 2.5 below, which is a version for Cuntz comparison of Lemma 1.15 of [41].

We describe the idea of the proof. (Most of the details are given below.) Given $x \in A_+$ with $\|x\| = 1$, we want $x_0 \in A_+ \setminus \{0\}$ such that $g \precsim_A x_0$ and otherwise as above implies $\|(1-g)x(1-g)\| > 1 - \varepsilon$. (We then use x_0 in place of x in the definition of a large subalgebra.) Choose a sufficiently small number $\varepsilon_0 > 0$. (It will be much smaller than ε.) Choose $f \colon [0,1] \to [0,1]$ such that $f = 0$ on $[0, \, 1-\varepsilon_0]$ and $f(1) = 1$. Construct $b_j, c_j, d_j \in \overline{f(x)Af(x)}$ for $j = 1, 2$ such that

$$0 \leq d_j \leq c_j \leq b_j \leq 1, \quad a b_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \quad \text{and} \quad d_j \neq 0,$$

and $b_1 b_2 = 0$. Take $x_0 = d_1$. If ε_0 is small enough, $g \precsim_A d_1$, and $\|(1-g)x(1-g)\| \leq 1 - \varepsilon$, this gives

$$\|(1-g)(b_1+b_2)(1-g)\| < 1 - \frac{\varepsilon}{3}.$$

One then gets $c_1 + c_2 \precsim_A d_1$. (This is the calculation (2.1) in the proof below.) Now $r = (1 - c_1 - c_2) + d_1$ satisfies $r \precsim_A 1$, so there is $v \in A$ such that $\|v r v^* - 1\| < \frac{1}{2}$. Then $v r^{1/2}$ is right invertible, but $v r^{1/2} d_2 = 0$, so $v r^{1/2}$ is not left invertible. This contradicts finiteness of A.

We now give a more detailed argument.

Lemma 2.4 (Lemma 2.8 of [43])**.** Let A be a C*-algebra, let $x \in A_+$ satisfy $\|x\| = 1$, and let $\varepsilon > 0$. Then there are positive elements $a, b \in \overline{xAx}$ with $\|a\| = \|b\| = 1$, such that $ab = b$, and such that whenever $c \in \overline{bAb}$ satisfies $\|c\| \leq 1$, then $\|xc - c\| < \varepsilon$.

Sketch of proof. Choose continuous functions $f_0, f_1 \colon [0,1] \to [0,1]$ such that $f_1(1) = 1$, f_1 is supported near 1, $|f_0(\lambda) - \lambda| < \varepsilon$ for all $\lambda \in [0,1]$, and $f_0 = 1$ near 1 (so that $f_0 f_1 = f_1$). Take $a = f_0(x)$ and $b = f_1(x)$. Then $\|x - a\| < \varepsilon$ and $ab = b$. ☐

Lemma 2.5 (Lemma 2.9 of [43])**.** Let A be a finite simple infinite dimensional unital C*-algebra. Let $x \in A_+$ satisfy $\|x\| = 1$. Then for every $\varepsilon > 0$ there is $x_0 \in \big(\overline{xAx}\big)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1-g)x(1-g)\| > 1 - \varepsilon$.

Proof. Choose positive elements $a, b \in \overline{x^{1/2} A x^{1/2}}$ as in Lemma 2.4, with $x^{1/2}$ in place of x and $\frac{\varepsilon}{3}$ in place of ε. Then $a, b \in \overline{xAx}$ since $\overline{x^{1/2} A x^{1/2}} = \overline{xAx}$. Since $b \neq 0$, Lemma 1.30 provides nonzero positive orthogonal elements $z_1, z_2 \in \overline{bAb}$ (with $z_1 \sim_A z_2$). We may require $\|z_1\| = \|z_2\| = 1$.

Choose continuous functions $f_0, f_1, f_2 \colon [0, \infty) \to [0, 1]$ such that

$$f_0(0) = 0, \quad f_0 f_1 = f_1, \quad f_1 f_2 = f_2, \quad \text{and} \quad f_2(1) = 1.$$

For $j = 1, 2$ define

$$b_j = f_0(z_j), \quad c_j = f_1(z_j), \quad \text{and} \quad d_j = f_2(z_j).$$

Then

$$0 \leq d_j \leq c_j \leq b_j \leq 1, \quad a b_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \quad \text{and} \quad d_j \neq 0.$$

Also $b_1 b_2 = 0$. Define $x_0 = d_1$. Then $x_0 \in \big(\overline{xAx}\big)_+$.

Let $g \in A_+$ satisfy $0 \leq g \leq 1$ and $g \precsim_A x_0$. We want to show that

$$\|(1 - g) x (1 - g)\| > 1 - \varepsilon,$$

so suppose that $\|(1 - g) x (1 - g)\| \leq 1 - \varepsilon$. The choice of a and b, and the relations $(b_1 + b_2)^{1/2} \in \overline{bAb}$ and $\big\|(b_1 + b_2)^{1/2}\big\| = 1$, imply that

$$\big\| x^{1/2} (b_1 + b_2)^{1/2} - (b_1 + b_2)^{1/2} \big\| < \frac{\varepsilon}{3}.$$

Using this relation and its adjoint at the second step, we get

$$\begin{aligned}
\big\|(1 - g)(b_1 + b_2)(1 - g)\big\| &= \big\|(b_1 + b_2)^{1/2} (1 - g)^2 (b_1 + b_2)^{1/2}\big\| \\
&< \big\|(b_1 + b_2)^{1/2} x^{1/2} (1 - g)^2 x^{1/2} (b_1 + b_2)^{1/2}\big\| + \frac{2\varepsilon}{3} \\
&\leq \big\|x^{1/2} (1 - g)^2 x^{1/2}\big\| + \frac{2\varepsilon}{3} \\
&= \|(1 - g) x (1 - g)\| + \frac{2\varepsilon}{3} \leq 1 - \frac{\varepsilon}{3}.
\end{aligned}$$

Using the equation $(b_1 + b_2)(c_1 + c_2) = c_1 + c_2$ and taking C to be the commutative C*-algebra generated by $b_1 + b_2$ and $c_1 + c_2$, one easily sees that for every $\beta \in [0, 1)$ we have $c_1 + c_2 \precsim_C [(b_1 + b_2) - \beta]_+$. Take $\beta = 1 - \frac{\varepsilon}{3}$, use this fact and Lemma 1.24 at the first step, use the estimate above at the second step, and use $g \precsim_A x_0 = d_1$ at the third step, to get

$$c_1 + c_2 \precsim_A \big[(1 - g)(b_1 + b_2)(1 - g) - \beta\big]_+ \oplus g = 0 \oplus g \precsim_A d_1. \tag{2.1}$$

Set $r = (1 - c_1 - c_2) + d_1$. Use Lemma 1.23(12) at the first step, (2.1) at the second step, and Lemma 1.23(13) and $d_1 (1 - c_1 - c_2) = 0$ at the third step, to get

$$1 \precsim_A (1 - c_1 - c_2) \oplus (c_1 + c_2) \precsim_A (1 - c_1 - c_2) \oplus d_1 \sim_A (1 - c_1 - c_2) + d_1 = r.$$

Thus there is $v \in A$ such that $\|v r v^* - 1\| < \frac{1}{2}$. It follows that $v r^{1/2}$ has a right inverse. But $v r^{1/2} d_2 = 0$, so $v r^{1/2}$ is not invertible. We have contradicted finiteness of A, and thus proved the lemma. $\square$

Proof of Proposition 2.3. Let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, let $x \in A_+ \setminus \{0\}$, and let $y \in B_+ \setminus \{0\}$. Without loss of generality $\|x\| = 1$.

Apply Lemma 2.5, obtaining $x_0 \in \big(\overline{xAx}\big)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1 - g) x (1 - g)\| > 1 - \varepsilon$. Apply the hypothesis with x_0 in place of x and everything else as given, getting $c_1, c_2, \ldots, c_m \in A$ and

$g \in B$. We need only prove that $\|(1-g)x(1-g)\| > 1 - \varepsilon$. But this is immediate from the choice of x_0. ☐

The following strengthening of the definition is often convenient. First, we can always require $\|c_j\| \leq \|a_j\|$. Second, if we cut down on both sides instead of on one side, and the elements a_j are positive, then we may take the elements c_j to be positive.

Lemma 2.6 (Lemma 4.8 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $m, n \in \mathbb{Z}_{\geq 0}$, let $a_1, a_2, \ldots, a_m \in A$, let $b_1, b_2, \ldots, b_n \in A_+$, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in B_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$, $d_1, d_2, \ldots, d_n \in A_+$, and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$, and for $j = 1, 2, \ldots, n$ we have $\|d_j - b_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $\|c_j\| \leq \|a_j\|$, and for $j = 1, 2, \ldots, n$ we have $\|d_j\| \leq \|b_j\|$.
(4) For $j = 1, 2, \ldots, m$ we have $(1-g)c_j \in B$, and for $j = 1, 2, \ldots, n$ we have $(1-g)d_j(1-g) \in B$.
(5) $g \precsim_B y$ and $g \precsim_A x$.
(6) $\|(1-g)x(1-g)\| > 1 - \varepsilon$.

Sketch of proof. To get $\|c_j\| \leq \|a_j\|$ for $j = 1, 2, \ldots, m$, one takes $\varepsilon > 0$ to be a bit smaller in the definition, and scales down c_j for any j for which $\|c_j\|$ is too big. Given that one can do this, following the definition, approximate

$$a_1,\, a_2,\, \ldots,\, a_m,\, b_1^{1/2},\, b_2^{1/2},\, \ldots,\, b_n^{1/2}$$

sufficiently well by

$$c_1, c_2, \ldots, c_m, r_1, r_2, \ldots, r_n,$$

and take $d_j = r_j r_j^*$ for $j = 1, 2, \ldots, n$. ☐

In Definition 4.9 of [43] we defined a "large subalgebra of crossed product type", a strengthening of the definition of a large subalgebra, and in Proposition 4.11 of [43] we gave a convenient way to verify that a subalgebra is a large subalgebra of crossed product type. The large subalgebras we have constructed in crossed products are of crossed product type. Theorem 4.6 of [5] shows that a large subalgebra of crossed product type is in fact centrally large. We will show directly (proof of Theorem 1.8, in Section 4 below) that if X is an infinite compact Hausdorff space, $h \colon X \to X$ is a minimal homeomorphism, and $Y \subset X$ is a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $C^*(\mathbb{Z}, X, h)_Y$ is centrally large in $C^*(\mathbb{Z}, X, h)$. This procedure is easier than using large subalgebras of crossed product type. The abstract version is more useful for subalgebras of crossed products by more complicated groups, but we don't consider these in these notes.

We prove the simplicity statement in Proposition 1.9. The infinite dimensionality statement is easier to prove (provided it is done afterwards), and we refer to the proof of Proposition 5.5 in [43].

Proposition 2.7 (Proposition 5.2 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple.

We need some preliminary work.

Lemma 2.8 (Lemma 1.12 of [43]). Let A be a C*-algebra, let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^{n} a_k$ and $x = \sum_{k=1}^{n} a_k^* a_k$. Then $a^* a \in \overline{xAx}$.

Sketch of proof. Assume $\|a_k\| \leq 1$ for $k = 1, 2, \ldots, n$. Choose $c \in \overline{xAx}$ such that $\|c\| \leq 1$ and $\|c a_k^* a_k - a_k^* a_k\|$ is small for $k = 1, 2, \ldots, n$. Check that $\|c a_k^* - a_k^*\|^2 \leq 2\|c a_k^* a_k - a_k^* a_k\|$, so $\|c a_k^* - a_k^*\|$ is small. Thus $\|c a^* - a^*\|$ is small, whence $\|c a^* a c - a^* a\|$ is small. Therefore $a^* a$ is arbitrarily close to $\overline{xAx}$. ∎

Lemma 2.9 (Lemma 1.13 of [43]). Let A be a C*-algebra and let $a \in A_{+}$. Let $b \in \overline{AaA}$ be positive. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\left\| b - \sum_{k=1}^{n} x_k^* a x_k \right\| < \varepsilon$.

This result is used without proof in the proof of Proposition 2.7(v) of [25]. We prove it when A is unital and $b = 1$, which is the case needed here. In this case, we can get $\sum_{k=1}^{n} x_k^* a x_k = 1$. In particular, we get Corollary 1.14 of [43] this way.

Proof of Lemma 2.9 when $b = 1$. Choose

$$n \in \mathbb{Z}_{>0} \quad \text{and} \quad y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in A$$

such that the element $c = \sum_{k=1}^{n} y_k a z_k$ satisfies $\|c - 1\| < 1$. Set

$$r = \sum_{k=1}^{n} z_k^* a y_k^* y_k a z_k, \quad M = \max\bigl(\|y_1\|, \|y_2\|, \ldots, \|y_n\|\bigr), \quad \text{and} \quad s = M^2 \sum_{k=1}^{n} z_k^* a^2 z_k.$$

Lemma 2.8 implies that $c^* c \in \overline{rAr}$. The relation $\|c - 1\| < 1$ implies that c is invertible, so r is invertible. Since $r \leq s$, it follows that s is invertible. Set $x_k = M a^{1/2} z_k s^{-1/2}$ for $k = 1, 2, \ldots, n$. Then $\sum_{k=1}^{n} x_k^* a x_k = s^{-1/2} s s^{-1/2} = 1$. ∎

Sketch of proof of Proposition 2.7. Let $b \in B_{+} \setminus \{0\}$. We show that there are $n \in \mathbb{Z}_{>0}$ and $r_1, r_2, \ldots, r_n \in B$ such that $\sum_{k=1}^{n} r_k b r_k^*$ is invertible.

Since A is simple, Lemma 2.9 provides $m \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_m \in A$ such that $\sum_{k=1}^{m} x_k b x_k^* = 1$. Set

$$M = \max\bigl(1, \|x_1\|, \|x_2\|, \ldots, \|x_m\|, \|b\|\bigr) \quad \text{and} \quad \delta = \min\left(1, \frac{1}{3mM(2M+1)}\right).$$

By definition, there are $y_1, y_2, \ldots, y_m \in A$ and $g \in B_{+}$ such that $0 \leq g \leq 1$, such that $\|y_j - x_j\| < \delta$ and $(1 - g) y_j \in B$ for $j = 1, 2, \ldots, m$, and such that $g \precsim_B b$. Set $z = \sum_{k=1}^{m} y_j b y_j^*$. The number δ has been chosen to ensure that $\|z - 1\| < \frac{1}{3}$; the estimate is carried out in [43]. It follows that $\left\|(1-g) z (1-g) - (1-g)^2\right\| < \frac{1}{3}$.

Set $h = 2g - g^2$. Lemma 1.23(3), applied to the function $\lambda \mapsto 2\lambda - \lambda^2$, implies that $h \sim_B g$. Therefore $h \precsim_B b$. So there is $v \in B$ such that $\|v b v^* - h\| < \frac{1}{3}$. Now take $n = m + 1$, take $r_j = (1 - g) y_j$ for $j = 1, 2, \ldots, m$, and take $r_{m+1} = v$. Then $r_1, r_2, \ldots, r_n \in B$. One can now check, using $(1 - g)^2 + h = 1$, that $\|1 - \sum_{k=1}^{n} r_k b r_k^*\| < \frac{2}{3}$. Therefore $\sum_{k=1}^{n} r_k b r_k^*$ is invertible, as desired. ∎

Proposition 2.10 (Corollary 5.8 of [43]). Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $n \in \mathbb{Z}_{>0}$. Then $M_n(B)$ is large in $M_n(A)$.

In [43], this result is obtained as a corollary of a more general result (Proposition 1.5 here). A direct proof is easier, and we give it here.

Proof of Proposition 2.10. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in M_n(A)$, let $\varepsilon > 0$, let $x \in M_n(A)_+ \setminus \{0\}$, and let $y \in M_n(B)_+ \setminus \{0\}$. There are $b_{k,l} \in A$ for $k, l = 1, 2, \ldots, n$ such that

$$x^{1/2} = \sum_{k,l=1}^{n} e_{k,l} \otimes b_{k,l} \in M_n \otimes A.$$

Choose $k, l \in \{1, 2, \ldots, n\}$ such that $b_{k,l} \neq 0$. Set $x_0 = b_{k,l}^* b_{k,l} \in A_+ \setminus \{0\}$. Using selfadjointness of $x^{1/2}$, we find that

$$e_{1,1} \otimes x_0 = (e_{l,1} \otimes 1)^* x^{1/2} (e_{k,k} \otimes 1) x^{1/2} (e_{l,1} \otimes 1) \leq (e_{l,1} \otimes 1)^* x (e_{l,1} \otimes 1) \precsim_A x.$$

Similarly, there is $y_0 \in B_+ \setminus \{0\}$ such that $e_{1,1} \otimes y_0 \precsim_B y$.

Use Lemma 1.30 and simplicity of B (Proposition 2.7) to find systems of nonzero mutually orthogonal and mutually Cuntz equivalent positive elements

$$x_1, x_2, \ldots, x_n \in \overline{x_0 A x_0} \quad \text{and} \quad y_1, y_2, \ldots, y_n \in \overline{y_0 B y_0}.$$

For $j = 1, 2, \ldots, m$, choose elements $a_{j,k,l} \in A$ for $k, l = 1, 2, \ldots, n$ such that

$$a_j = \sum_{k,l=1}^{n} e_{k,l} \otimes a_{j,k,l} \in M_n \otimes A.$$

Apply Proposition 2.3 with mn^2 in place of m, with the elements $a_{j,k,l}$ in place of $a_1, a_2, \ldots, a_m$, with ε/n^2 in place of ε, with x_1 in place of x, and with y_1 in place of y, getting $g_0 \in A_+$ and $c_{j,k,l} \in A$ for $j = 1, 2, \ldots, m$ and $k, l = 1, 2, \ldots, n$. Define $c_j = \sum_{k,l=1}^{n} e_{k,l} \otimes c_{j,k,l}$ for $j = 1, 2, \ldots, m$ and define $g = 1 \otimes g_0$. It is clear that $0 \leq g \leq 1$, that $\|c_j - a_j\| < \varepsilon$ and $(1 - g)c_j \in M_n(B)$ for $j = 1, 2, \ldots, m$. We have $g \precsim_A 1 \otimes x_1$ and $g \precsim_B 1 \otimes y_1$, so Lemma 1.23(1) and Lemma 1.23(13) imply that $g \precsim_A x_0$ and $g \precsim_B y_0$. So $g \precsim_A x$ and $g \precsim_B y$. $\square$

We prove the statement about traces in Theorem 1.10.

Theorem 2.11 (Theorem 6.2 of [43])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective.

Again, we need a lemma.

Lemma 2.12. Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in \mathrm{T}(B)$. Then there exists a unique state ω on A such that $\omega|_B = \tau$.

Proof. Existence of ω follows from the Hahn-Banach Theorem.

For uniqueness, let ω_1 and ω_2 be states on A such that $\omega_1|_B = \omega_2|_B = \tau$, let $a \in A_+$, and let $\varepsilon > 0$. We prove that $|\omega_1(a) - \omega_2(a)| < \varepsilon$. Without loss of generality $\|a\| \leq 1$.

It follows from Proposition 2.7 that B is simple and infinite dimensional. So Corollary 1.31 provides $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$ (for the particular choice of τ we are using). Use Lemma 2.6 to find $c \in A_+$ and $g \in B_+$ such that

$$\|c\| \leq 1, \quad \|g\| \leq 1, \quad \|c - a\| < \frac{\varepsilon}{4}, \quad (1 - g)c(1 - g) \in B, \quad \text{and} \quad g \precsim_B y.$$

For $j = 1, 2$, the Cauchy-Schwarz inequality gives

$$|\omega_j(rs)| \leq \omega_j(rr^*)^{1/2} \omega_j(s^* s)^{1/2} \tag{2.2}$$

for all $r, s \in A$. Also, by Lemma 1.23(3) we have $g^2 \sim_B g \precsim_B y$. Since $\|g^2\| \leq 1$ and $\omega_j|_B = \tau$ is a tracial state, it follows that $\omega_j(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$. Using $\|c\| \leq 1$, we then get

$$|\omega_j(gc)| \leq \omega_j(g^2)^{1/2}\omega_j(c^2)^{1/2} < \frac{\varepsilon}{8}$$

and

$$|\omega_j((1-g)cg)| \leq \omega_j\big((1-g)c^2(1-g)\big)^{1/2}\omega_j(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

So

$$\begin{aligned}\big|\omega_j(c) - \tau((1-g)c(1-g))\big| &= \big|\omega_j(c) - \omega_j((1-g)c(1-g))\big| \\ &\leq |\omega_j(gc)| + |\omega_j((1-g)cg)| < \frac{\varepsilon}{4}.\end{aligned}$$

Also $|\omega_j(c) - \omega_j(a)| < \frac{\varepsilon}{4}$. So

$$\big|\omega_j(a) - \tau((1-g)c(1-g))\big| < \frac{\varepsilon}{2}.$$

Thus $|\omega_1(a) - \omega_2(a)| < \varepsilon$. □

Proof of Theorem 2.11. Let $\tau \in \mathrm{T}(B)$. We show that there is a unique $\omega \in \mathrm{T}(A)$ such that $\omega|_B = \tau$. Lemma 2.12 shows that there is a unique state ω on A such that $\omega|_B = \tau$, and it suffices to show that ω is a trace. Thus let $a_1, a_2 \in A$ satisfy $\|a_1\| \leq 1$ and $\|a_2\| \leq 1$, and let $\varepsilon > 0$. We show that $|\omega(a_1a_2) - \omega(a_2a_1)| < \varepsilon$.

It follows from Proposition 2.7 that B is simple and infinite dimensional. So Corollary 1.31 provides $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$. Use Lemma 2.6 to find $c_1, c_2 \in A$ and $g \in B_+$ such that

$$\|c_j\| \leq 1, \quad \|c_j - a_j\| < \frac{\varepsilon}{8}, \quad \text{and} \quad (1-g)c_j \in B$$

for $j = 1, 2$, and such that $\|g\| \leq 1$ and $g \precsim_B y$. By Lemma 1.23(3) we have $g^2 \sim g \precsim_B y$. Since $\|g^2\| \leq 1$ and $\omega|_B = \tau$ is a tracial state, it follows that $\omega(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$.

We claim that

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1c_2)\big| < \frac{\varepsilon}{4}.$$

Using the Cauchy-Schwarz inequality ((2.2) in the previous proof), we get

$$|\omega(gc_1c_2)| \leq \omega(g^2)^{1/2}\omega(c_2^*c_1^*c_1c_2)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

Similarly, and also at the second step using $\|c_2\| \leq 1$, $(1-g)c_1g \in B$, and the fact that $\omega|_B$ is a tracial state,

$$\begin{aligned}\big|\omega((1-g)c_1gc_2)\big| &\leq \omega\big((1-g)c_1g^2c_1^*(1-g)\big)^{1/2}\omega(c_2^*c_2)^{1/2} \\ &\leq \omega\big(gc_1^*(1-g)^2c_1g\big)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.\end{aligned}$$

The claim now follows from the estimate

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1c_2)\big| \leq \big|\omega((1-g)c_1gc_2)\big| + |\omega(gc_1c_2)|.$$

Similarly

$$\big|\omega((1-g)c_2(1-g)c_1) - \omega(c_2c_1)\big| < \frac{\varepsilon}{4}.$$

Since $(1-g)c_1$, $(1-g)c_2 \in B$ and $\omega|_B$ is a tracial state, we get

$$\omega((1-g)c_1(1-g)c_2) = \omega((1-g)c_2(1-g)c_1).$$

Therefore $|\omega(c_1 c_2) - \omega(c_2 c_1)| < \frac{\varepsilon}{2}$.

One checks that $\|c_1 c_2 - a_1 a_2\| < \frac{\varepsilon}{4}$ and $\|c_2 c_1 - a_2 a_1\| < \frac{\varepsilon}{4}$. It now follows that $|\omega(a_1 a_2) - \omega(a_2 a_1)| < \varepsilon$. □

3. Large Subalgebras and the Radius of Comparison

Let A be a simple unital C*-algebra. We say that *the order on projections over A is determined by traces* if, as happens for type II_1 factors, whenever $p, q \in M_{\infty}(A)$ are projections such that for all $\tau \in \mathrm{T}(A)$ we have $\tau(p) < \tau(q)$, then p is Murray-von Neumann equivalent to a subprojection of q. The question of whether this holds is also known as Blackadar's Second Fundamental Comparability Question (FCQ2; see 1.3.1 of [7]). Without knowing whether every quasitrace is a trace, it is more appropriate to use a condition involving quasitraces. Of course. for exact C*-algebras, it is known that every quasitrace is a trace (Corollary 9.18 of [22]), so it makes no difference.

Simple C*-algebras need not have very many projections, so a more definitive version of this condition is to ask for *strict comparison of positive elements*, that is, whenever $a, b \in M_{\infty}(A)$ (or $K \otimes A$) are positive elements such that for all $\tau \in \mathrm{QT}(A)$ we have $d_{\tau}(a) < d_{\tau}(b)$, then $a \precsim_A b$. (By Proposition 6.12 of [43], it does not matter whether one uses $M_{\infty}(A)$ or $K \otimes A$, but this is not as easy to see as with projections.)

Simple AH algebras with slow dimension growth have strict comparison, but other simple AH algebras need not. Strict comparison is necessary for any reasonable hope of classification in the sense of the Elliott program. According to the Toms-Winter Conjecture, when A is simple, separable, nuclear, unital, and stably finite, it should imply Z-stability, and this is known to hold in a number of cases.

The radius of comparison $\mathrm{rc}(A)$ of A (for a C*-algebra which is unital and stably finite but not necessarily simple) measures the failure of strict comparison. (See [9] for what to do in more general C*-algebras.) For additional context, we point out the following special case of Theorem 5.1 of [51] (which will be needed in Section 5): if X is a compact metric space and $n \in \mathbb{Z}_{>0}$, then

$$\mathrm{rc}(M_n \otimes C(X)) \leq \frac{\dim(X) - 1}{2n}.$$

When X is a finite complex, this inequality is at least approximately an equality.

The following definition of the radius of comparison is adapted from Definition 6.1 of [50].

Definition 3.1. Let A be a stably finite unital C*-algebra.

(1) Let $r \in [0, \infty)$. We say that A has *r-comparison* if whenever $a, b \in M_{\infty}(A)_{+}$ satisfy $d_{\tau}(a) + r < d_{\tau}(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.
(2) The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\left(\left\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\right\}\right).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

Definition 6.1 of [50] actually uses lower semicontinuous dimension functions on A instead of d_{τ} for $\tau \in \mathrm{QT}(A)$, but these are the same functions by Theorem II.2.2 of [8]. It is also stated in terms of the order on the Cuntz semigroup $W(A)$ rather than in terms of Cuntz subequivalence; this is clearly equivalent.

We also note (Proposition 6.3 of [50]) that if every element of $\mathrm{QT}(A)$ is faithful, then the infimum in Definition 3.1(2) is attained, that is, A has $\mathrm{rc}(A)$-comparison. In particular, this is true when A is simple. (See Lemma 1.23 of [43].)

We warn that r-comparison and $\mathrm{rc}(A)$ are sometimes defined using tracial states rather than quasitraces.

It is equivalent to use $K \otimes A$ in place of $M_{\infty}(A)$. See Proposition 6.12 of [43].

We prove here the following special case of Theorem 1.12.

Theorem 3.2. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

The extra assumption is that A is exact, so that every quasitrace is a trace by Corollary 9.18 of [22].

We will give a proof directly from the definition of a large subalgebra. We describe the heuristic argument, using the following simplifications:

(1) The algebra A, and therefore also B, has a unique tracial state τ.
(2) We consider elements of A_{+} and B_{+} instead of elements of $M_{\infty}(A)_{+}$ and $M_{\infty}(B)_{+}$.
(3) For $a \in A_{+}$, when needed, instead of getting $(1-g)c(1-g) \in B$ for some $c \in A_{+}$ which is close to a, we can actually get $(1-g)a(1-g) \in B$. Similarly, for $a \in A$ we can get $(1-g)a \in B$.
(4) For $a, b \in A_{+}$ with $a \precsim_{A} b$, we can find $v \in A$ such that $v^{*}bv = a$ (not just such that $\|v^{*}bv - a\|$ is small).
(5) None of the elements we encounter are Cuntz equivalent to projections, that is, we never encounter anything for which 0 is an isolated point of, or not in, the spectrum.

The most drastic simplification is (3). In the actual proof, to compensate for the fact that we only get approximation, we will need to make systematic use of elements $(a-\varepsilon)_{+}$ for carefully chosen, and varying, values of $\varepsilon > 0$. Avoiding this gives a much better view of the idea behind the argument, and the usefulness of large subalgebras in general.

We first consider the inequality $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. So let $a, b \in A_{+}$ satisfy $d_{\tau}(a) + \mathrm{rc}(B) < d_{\tau}(b)$. The essential idea is to replace b by something slightly smaller which is in B_{+}, say y, and replace a by something slightly larger which is in B_{+}, say x, in such a way that we still have $d_{\tau}(x) + \mathrm{rc}(B) < d_{\tau}(y)$. Then use the definition of $\mathrm{rc}(B)$. With g sufficiently small in the sense of Cuntz comparison, we will take $y = (1-g)b(1-g)$ and (following Lemma 1.24) $x = (1-g)a(1-g) \oplus g$.

Choose $\delta > 0$ such that

$$d_{\tau}(a) + \mathrm{rc}(B) + \delta \leq d_{\tau}(b). \tag{3.1}$$

Applying (3) of our simplification, we can find $g \in B$ with $0 \leq g \leq 1$, such that

$$(1-g)a(1-g) \in B \quad \text{and} \quad (1-g)b(1-g) \in B,$$

and so small in $W(A)$ that

$$d_{\tau}(g) < \frac{\delta}{3}. \tag{3.2}$$

Using Lemma 1.23(4) at the first step, we get

$$(1-g)b(1-g) \sim_{A} b^{1/2}(1-g)^{2}b^{1/2} \leq b. \tag{3.3}$$

Similarly, $(1-g)b(1-g) \precsim_A b$, and this relation implies

$$d_\tau\big((1-g)a(1-g)\big) \leq d_\tau(a). \tag{3.4}$$

Also, $b \precsim_A (1-g)b(1-g) \oplus g$ by Lemma 1.24, so

$$d_\tau\big((1-g)b(1-g)\big) + d_\tau(g) \geq d_\tau(b). \tag{3.5}$$

Using (3.4) at the first step, using (3.1) at the second step, using (3.5) at the third step, and using (3.2) at the fourth step, we get

$$\begin{aligned} d_\tau\big((1-g)a(1-g) \oplus g\big) + \mathrm{rc}(B) + \frac{\delta}{3} &\leq d_\tau(a) + d_\tau(g) + \mathrm{rc}(B) + \frac{\delta}{3} \\ &\leq d_\tau(b) + d_\tau(g) - \frac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big) + 2d_\tau(g) - \frac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big). \end{aligned}$$

So, by the definition of $\mathrm{rc}(B)$,

$$(1-g)a(1-g) \oplus g \precsim_B (1-g)b(1-g).$$

Therefore, using Lemma 1.24 at the first step and (3.3) at the third step, we get

$$a \precsim_A (1-g)a(1-g) \oplus g \precsim_B (1-g)b(1-g) \precsim_A b,$$

that is, $a \precsim_A b$, as desired.

Now we consider the inequality $\mathrm{rc}(A) \geq \mathrm{rc}(B)$. Let $a, b \in B_+$ satisfy $d_\tau(a) + \mathrm{rc}(A) < d_\tau(b)$. Choose $\delta > 0$ such that $d_\tau(a) + \mathrm{rc}(B) + \delta \leq d_\tau(b)$. By lower semicontinuity of d_τ, we always have

$$d_\tau(b) = \sup_{\varepsilon > 0} d_\tau\big((b-\varepsilon)_+\big).$$

So there is $\varepsilon > 0$ such that

$$d_\tau\big((b-\varepsilon)_+\big) > d_\tau(a) + \mathrm{rc}(A). \tag{3.6}$$

Define a continuous function $f\colon [0,\infty) \to [0,\infty)$ by $f(\lambda) = \max(0, \varepsilon^{-1}\lambda(\varepsilon - \lambda))$ for $\lambda \in [0,\infty)$. Then $f(b)$ and $(b-\varepsilon)_+$ are orthogonal positive elements such that $f(b) \neq 0$ (by (5)) and $f(b) + (b-\varepsilon)_+ \leq b$. We have $a \precsim_A (b-\varepsilon)_+$ by (3.6) and the definition of $\mathrm{rc}(A)$. Applying (4) of our simplification, we can find $v \in A$ such that $v^*(b-\varepsilon)_+ v = a$. Applying (3) of our simplification, we can find $g \in B$ with $0 \leq g \leq 1$ such that $(1-g)v^* \in B$ and $g \precsim_B f(b)$. Since

$$v(1-g) \in B \quad \text{and} \quad [v(1-g)]^*(b-\varepsilon)_+[v(1-g)] = (1-g)a(1-g),$$

we get $(1-g)a(1-g) \precsim_B (b-\varepsilon)_+$. Therefore, using Lemma 1.24 at the first step,

$$a \precsim_B (1-g)a(1-g) \oplus g \precsim_B (b-\varepsilon)_+ \oplus g \precsim_B (b-\varepsilon)_+ \oplus f(b) \precsim_B b,$$

as desired.

The original proof of Theorem 3.2 followed the heuristic arguments above, and this is the proof we give below. The proof in [43] uses the same basic ideas, but gives much more. The heuristic arguments above are the basis for the technical results in Lemma 1.15. In [43], these are used to prove Theorem 1.11, which states that, after deleting the classes of the nonzero projections from the Cuntz semigroups $\mathrm{Cu}(B)$ and $\mathrm{Cu}(A)$, the inclusion of B in A is an order isomorphism on what remains. (The inclusion need not be an isomorphism if the classes of the nonzero projections are included. See Example 7.13 of [43].) In Section 3 of [43], it is shown that, in our situation, the part of the Cuntz semigroup without the classes of the nonzero projections is enough to determine the quasitraces, so that the restriction map $\mathrm{QT}(A) \to \mathrm{QT}(B)$ is bijective. It follows that the radius of comparison in this part of the Cuntz semigroup is the same for both A and B, and it turns out that the radius of comparison in this part of the Cuntz semigroup is the same as in the entire Cuntz semigroup.

We will use the characterizations of $\mathrm{rc}(A)$ in the following theorem, which is a special case of results in [9]. The difference between (1) and (3) is that (3) has $n+1$ in one of the places where (1) has n. This result substitutes for the observation that if $a, b \in A_{+}$ satisfy $\tau(a) < \tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then, by compactness of $\mathrm{QT}(A)$ and continuity, we have $\inf_{\tau \in \mathrm{QT}(A)} [\tau(b) - \tau(a)] > 0$. The difficulty is that we need an analog using d_{τ} instead of τ, and $\tau \mapsto d_{\tau}(a)$ is in general only lower semicontinuous, so that $\tau \mapsto d_{\tau}(b) - d_{\tau}(a)$ may be neither upper nor lower semicontinuous.

Unfortunately, the results in [9] are stated in terms of $\mathrm{Cu}(A)$ rather than $W(A)$.

Theorem 3.3. Let A be a stably finite simple unital C*-algebra. Then:

(1) The radius of comparison $\mathrm{rc}(A)$ is the infimum of all m/n with $m, n \in \mathbb{Z}_{>0}$ such that whenever $a, b \in M_{\infty}(A)_{+}$ and $n\langle a \rangle_A + m\langle 1 \rangle_A \leq n\langle b \rangle_A$ in $W(A)$, then $a \precsim_A b$.
(2) The radius of comparison $\mathrm{rc}(A)$ is the least number $s \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > s$, and $a, b \in M_{\infty}(A)_{+}$ satisfy $n\langle a \rangle_A + m\langle 1 \rangle_A \leq n\langle b \rangle_A$ in $W(A)$, then $a \precsim_A b$.
(3) The radius of comparison $\mathrm{rc}(A)$ is the least number $t \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > t$, and $a, b \in M_{\infty}(A)_{+}$ satisfy $(n+1)\langle a \rangle_A + m\langle 1 \rangle_A \leq n\langle b \rangle_A$ in $W(A)$, then $a \precsim_A b$.

Proof. It is easy to check that there is in fact a least $s \in [0, \infty]$ satisfying the condition in (2), and similarly that there is a least $t \in [0, \infty]$ as in (3).

We will first prove this for $K \otimes A$ and $\mathrm{Cu}(A)$ in place of $M_{\infty}(A)$ and $W(A)$. So let r be the infimum in (1), let s and t be the numbers given in (2) and (3), and let r_0, s_0, and t_0 be the numbers defined the same way but with $K \otimes A$ and $\mathrm{Cu}(A)$ in place of $M_{\infty}(A)$ and $W(A)$. Clearly $s_0 \geq r_0 \geq t_0$. Since A is simple and stably finite and $\langle 1 \rangle_A$ is a full element of $\mathrm{Cu}(A)$, Proposition 3.2.3 of [9], the preceding discussion in [9], and Definition 3.2.2 of [9] give $t_0 = \mathrm{rc}(A)$. So we need to show that $s_0 \leq t_0$.

We thus assume $m, n \in \mathbb{Z}_{>0}$ and $m/n > t_0$, and that $a, b \in (K \otimes A)_{+}$ satisfy $n\langle a \rangle_A + m\langle 1 \rangle_A \leq n\langle b \rangle_A$ in $\mathrm{Cu}(A)$. We must prove that $a \precsim b$. For any functional ω on $\mathrm{Cu}(A)$ (as at the beginning of Section 2.4 of [9]), we have $n\omega(\langle a \rangle_A) + m\omega(\langle 1 \rangle_A) \leq n\omega(\langle b \rangle_A)$, so $\omega(\langle a \rangle_A) + (m/n)\omega(\langle 1 \rangle_A) \leq \omega(\langle b \rangle_A)$. Since $m/n > t_0$, Proposition 3.2.1 of [9] implies that $a \precsim_A b$.

It remain to prove that $r_0 = r$, $s_0 = s$, and $t_0 = t$. The proofs of all three are the same as the proof of Proposition 6.12 of [43]. $\square$

Lemma 3.4. Let $M \in (0, \infty)$, let $f \colon [0, \infty) \to \mathbb{C}$ be a continuous function such that $f(0) = 0$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever A is a C*-algebra and $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$, then $\|f(a) - f(b)\| < \varepsilon$.

This is a standard polynomial approximation argument. We have not found it written down (although there are similar arguments in [43] and many other places). We therefore give it for completeness.

Proof of Lemma 3.4. Choose $n \in \mathbb{Z}_{>0}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that the polynomial function $g(\lambda) = \sum_{k=1}^{n} \alpha_k \lambda^k$ satisfies $|g(\lambda) - f(\lambda)| < \frac{\varepsilon}{3}$ for $\lambda \in [-M-1, \, M+1]$. Define

$$\delta = \min \left(1, \, \frac{\varepsilon}{1 + 3 \sum_{k=1}^{n} |\alpha_k| k (M+1)^{k-1}} \right).$$

Now let A be a C*-algebra and let $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$. Then $\|b\| \leq M + 1$. So for $m \in \mathbb{Z}_{>0}$ we have

$$\|a^m - b^m\| \leq \sum_{k=1}^{m} \|a^{k-1}\| \cdot \|a - b\| \cdot \|b^{m-k}\| < m(M+1)^{m-1} \delta.$$

Therefore

$$\|g(a) - g(b)\| \leq \sum_{k=1}^{n} |\alpha_k| k (M+1)^{k-1} \delta < \frac{\varepsilon}{3}.$$

So

$$\|f(a) - f(b)\| \leq \|f(a) - g(a)\| + \|g(a) - g(b)\| + \|g(b) - f(b)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof. $\square$

Proposition 3.5. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) \leq \mathrm{rc}(B)$.

Proof. We use the criterion of Theorem 3.3(2). Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(B)$, and let $a, b \in M_{\infty}(A)_{+}$ satisfy $n \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$ in $W(A)$. We want to prove that $a \precsim_A b$. Without loss of generality $\|a\|, \|b\| \leq 1$. It suffices to prove that $(a - \varepsilon)_{+} \precsim_A b$ for every $\varepsilon > 0$.

So let $\varepsilon > 0$. We may assume $\varepsilon < 1$. Let $x \in M_{\infty}(A)_{+}$ be the direct sum of n copies of a, let $y \in M_{\infty}(A)_{+}$ be the direct sum of n copies of b, and let $q \in M_{\infty}(A)_{+}$ be the direct sum of m copies of the identity of A. The relation $n \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$ means that $x \oplus q \precsim_A y$. By Lemma 1.23(11b), there exists $\delta > 0$ such that

$$\big((x \oplus q) - \tfrac{1}{3} \varepsilon \big)_{+} \precsim_A (y - \delta)_{+}.$$

Since $\varepsilon < 3$, this is equivalent to

$$\big(x - \tfrac{1}{3} \varepsilon \big)_{+} \oplus q \precsim_A (y - \delta)_{+}. \tag{3.7}$$

Choose $l \in \mathbb{Z}_{>0}$ so large that $a, b \in M_l \otimes A$. Since $m/n > \mathrm{rc}(B)$, there is $k \in \mathbb{Z}_{>0}$ such that

$$\mathrm{rc}(B) < \frac{m}{n} - \frac{2}{k}.$$

Set

$$\varepsilon_0 = \min \big(\tfrac{1}{3} \varepsilon, \tfrac{1}{2} \delta \big).$$

Using Lemma 3.4, choose $\varepsilon_1 > 0$ with $\varepsilon_1 \leq \varepsilon_0$ and so small whenever D is a C*-algebra and $z \in D_{+}$ satsifies $\|z\| \leq 1$, then $\|z_0 - z\| < \varepsilon_1$ implies

$$\|(z_0 - \varepsilon_0)_{+} - (z - \varepsilon_0)_{+}\| < \varepsilon_0, \quad \big\| \big(z_0 - \tfrac{1}{3} \varepsilon \big)_{+} - \big(z - \tfrac{1}{3} \varepsilon \big)_{+} \big\| < \varepsilon_0,$$

and

$$\big\| \big(z_0 - \big(\varepsilon_0 + \tfrac{1}{3} \varepsilon \big) \big)_{+} - \big(z - \big(\varepsilon_0 + \tfrac{1}{3} \varepsilon \big) \big)_{+} \big\| < \varepsilon_0.$$

Since A is infinite dimensional and simple, Lemma 1.30 provides $z \in A_+ \setminus \{0\}$ such that $(k+1)\langle z \rangle_A \leq \langle 1 \rangle_A$. Using Proposition 2.10, choose $g \in M_l(B)_+$ and $a_0, b_0 \in M_l(A)_+$ satisfying

$$0 \leq g, a_0, b_0 \leq 1, \quad \|a_0 - a\| < \varepsilon_1, \quad \|b_0 - b\| < \varepsilon_1, \quad g \precsim_A z,$$

and such that

$$(1-g)a_0(1-g),\, (1-g)b_0(1-g) \in M_l \otimes B.$$

From $g \precsim_A z$ and $(k+1)\langle z \rangle_A \leq \langle 1 \rangle_A$ we get

$$\sup_{\tau \in \mathrm{T}(A)} d_\tau(g) < \frac{1}{k}. \tag{3.8}$$

Set

$$a_1 = \big[(1-g)a_0(1-g) - \big(\varepsilon_0 + \tfrac{1}{3}\varepsilon\big)\big]_+ \quad \text{and} \quad b_1 = \big[(1-g)b_0(1-g) - \varepsilon_0\big]_+,$$

which are in $M_l \otimes B$. We claim that a_0, a_1, b_0, and b_1 have the following properties:

(1) $(a-\varepsilon)_+ \precsim_A \big[a_0 - \big(\varepsilon_0 + \tfrac{1}{3}\varepsilon\big)\big]_+$.
(2) $\big[a_0 - \big(\varepsilon_0 + \tfrac{1}{3}\varepsilon\big)\big]_+ \precsim_B a_1 \oplus g$.
(3) $a_1 \precsim_A \big(a - \tfrac{1}{3}\varepsilon\big)_+$.
(4) $(b-\delta)_+ \precsim_A (b_0 - \varepsilon_0)_+$.
(5) $(b_0 - \varepsilon_0)_+ \precsim_B b_1 \oplus g$.
(6) $b_1 \precsim_A b$.

We give full details of the proofs for (1), (2), and (3) (involving a_0 and a_1). The proofs for (4), (5), and (6) (involving b_0 and b_1) are a bit more sketchy.

We prove (1). The choice of ε_1 implies

$$\big\| \big[a_0 - \big(\tfrac{1}{3}\varepsilon + \varepsilon_0\big)\big]_+ - \big[a - \big(\tfrac{1}{3}\varepsilon + \varepsilon_0\big)\big]_+ \big\| < \varepsilon_0 \leq \tfrac{1}{3}\varepsilon.$$

At the last step in the following computation use this and Lemma 1.23(10), at the first step use $\varepsilon_0 \leq \frac{1}{3}\varepsilon$, and at the second step use Lemma 1.23(8), to get

$$\begin{aligned} (a-\varepsilon)_+ &\leq \big[a - \big(\tfrac{2}{3}\varepsilon + \varepsilon_0\big)\big]_+ \\ &= \big[\big(a - \big(\tfrac{1}{3}\varepsilon + \varepsilon_0\big)\big)_+ - \tfrac{1}{3}\varepsilon\big]_+ \precsim_A \big[a_0 - \big(\tfrac{1}{3}\varepsilon + \varepsilon_0\big)\big]_+. \end{aligned}$$

For (4) (the corresponding argument for b_0), we use $\varepsilon_0 \leq \frac{1}{2}\delta$ at the first step; since

$$\|(b-\varepsilon_0)_+ - (b_0 - \varepsilon_0)_+\| < \varepsilon_0,$$

we get

$$(b-\delta)_+ \leq (b - 2\varepsilon_0)_+ = \big[(b-\varepsilon_0)_+ - \varepsilon_0\big]_+ \precsim_A (b_0 - \varepsilon_0)_+.$$

For (2), we use Lemma 1.24 with a_0 in place of a and with $\frac{1}{3}\varepsilon + \varepsilon_0$ in place of ε. For (5), we use Lemma 1.24 with b_0 in place of a and with ε_0 in place of ε.

For (3), begin by recalling that $\|a_0 - a\| < \varepsilon_1$, whence

$$\|(1-g)a_0(1-g) - (1-g)a(1-g)\| < \varepsilon_1.$$

Therefore

$$\big\| \big[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\big]_+ - \big[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\big]_+ \big\| < \varepsilon_0.$$

Using Lemma 1.23(8) at the first step, this fact and Lemma 1.23(10) at the second step, Lemma 1.23(6) at the third step, and Lemma 1.23(17) and $a^{1/2}(1-g)^2a^{1/2} \leq a$ at the last step, we get

$$\begin{aligned} a_1 &= \left[\left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \varepsilon_0\right]_+ \\ &\precsim_A \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \sim_A \left[a^{1/2}(1-g)^2a^{1/2} - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+, \end{aligned}$$

as desired.

For (6) (the corresponding part involving b_1), just use

$$\|(1-g)b_0(1-g) - (1-g)b(1-g)\| < \varepsilon_1 \leq \varepsilon_0$$

to get, using Lemma 1.23(4) at the second step,

$$b_1 \precsim_A (1-g)b(1-g) \sim_A b^{1/2}(1-g)^2b^{1/2} \leq b.$$

The claims (1)–(6) are now proved.

Now let $\tau \in \mathrm{T}(A)$. Recall that x and y are the direct sums of n copies of a and b. Therefore $\left(x - \frac{1}{3}\varepsilon\right)_+$ is the direct sum of n copies of $\left(a - \frac{1}{3}\varepsilon\right)_+$ and $(y-\delta)_+$ is the direct sum of n copies of $(b-\delta)_+$. So the relation (3.7) implies

$$n \cdot d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right) + m \leq n \cdot d_\tau\big((b-\delta)_+\big). \tag{3.9}$$

Using (4) and (5) at the first step and (3.8) at the third step, we get the estimate

$$d_\tau\big((b-\delta)_+\big) \leq d_\tau(b_1) + d_\tau(g) < d_\tau(b_1) + k^{-1}. \tag{3.10}$$

The relation (3) implies

$$d_\tau(a_1) \leq d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right). \tag{3.11}$$

Using (3.8) at the second step, (3.11) at the third step, (3.9) at the fourth step, and (3.10) at the fifth step, we get

$$\begin{aligned} n \cdot d_\tau(a_1 \oplus g) + m &= n \cdot d_\tau(a_1) + m + n \cdot d_\tau(g) \\ &\leq n \cdot d_\tau(a_1) + m + nk^{-1} \\ &\leq n \cdot d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right) + m + nk^{-1} \\ &\leq n \cdot d_\tau\big((b-\delta)_+\big) + nk^{-1} \\ &\leq n \cdot d_\tau(b_1) + 2nk^{-1}. \end{aligned}$$

It follows that

$$d_\tau(a_1 \oplus g) + \frac{m}{n} - \frac{2}{k} \leq d_\tau(b_1).$$

This holds for all $\tau \in \mathrm{T}(A)$, and therefore, by Theorem 2.11, for all $\tau \in \mathrm{T}(B)$.

Subalgebras of exact C*-algebras are exact so Corollary 9.18 of [22] implies that $\mathrm{QT}(B) = \mathrm{T}(B)$. Since

$$\frac{m}{n} - \frac{2}{k} > \mathrm{rc}(B),$$

and since $a_1, b_1, g \in M_l \otimes B$, it follows that $a_1 \oplus g \precsim_B b_1$. Using this relation at the third step, (1) at the first step, (2) at the second step, and (6) at the last step, we then get

$$(a-\varepsilon)_+ \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \precsim_A a_1 \oplus g \precsim_B b_1 \precsim_A b.$$

This completes the proof that $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. □

Proposition 3.6. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) \geq \mathrm{rc}(B)$.

Proof. We use Theorem 3.3(3). Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(A)$. Let $l \in \mathbb{Z}_{>0}$, and let $a, b \in (M_l \otimes B)_+$ satisfy

$$(n+1)\langle a\rangle_B + m\langle 1\rangle_B \leq n\langle b\rangle_B$$

in $W(B)$. We must prove that $a \precsim_B b$. Without loss of generality $\|a\| \leq 1$. Moreover, by Lemma 1.23(11), it is enough to show that for every $\varepsilon > 0$ we have $(a-\varepsilon)_+ \precsim_B b$. So let $\varepsilon > 0$. Without loss of generality $\varepsilon < 1$.

Choose $k \in \mathbb{Z}_{>0}$ such that

$$\frac{km}{kn+1} > \mathrm{rc}(A).$$

Then in $W(B)$ we have

$$(kn+1)\langle a\rangle_B + km\langle 1\rangle_B \leq k(n+1)\langle a\rangle_B + km\langle 1\rangle_B \leq kn\langle b\rangle_B.$$

Let $x \in M_\infty(B)_+$ be the direct sum of $kn+1$ copies of a, let $z \in M_\infty(B)_+$ be the direct sum of kn copies of b, and let $q \in M_\infty(B)_+$ be the direct sum of km copies of 1. Then, by definition, $x \oplus q \precsim_B z$. Therefore Lemma 1.23(11) provides $\delta > 0$ such that $\left(x \oplus q - \frac{1}{4}\varepsilon\right)_+ \precsim_B (z-\delta)_+$. Since $\varepsilon < 4$, we have

$$\left(x \oplus q - \tfrac{1}{4}\varepsilon\right)_+ = \left(x - \tfrac{1}{4}\varepsilon\right)_+ \oplus \left(q - \tfrac{1}{4}\varepsilon\right)_+ \sim_B \left(x - \tfrac{1}{4}\varepsilon\right)_+ \oplus q,$$

so

$$(kn+1)\left\langle \left(a - \tfrac{1}{4}\varepsilon\right)_+\right\rangle + km\langle 1\rangle \leq kn\langle (b-\delta)_+\rangle.$$

Lemma 1.33 provides $c \in (M_l \otimes B)_+$ and $y \in (M_l \otimes B)_+ \setminus \{0\}$ such that

$$kn\langle (b-\delta)_+\rangle_B \leq (kn+1)\langle c\rangle_B \quad \text{and} \quad \langle c\rangle_B + \langle y\rangle_B \leq \langle b\rangle_B \tag{3.12}$$

in $W(B)$. Then

$$(kn+1)\left\langle \left(a - \tfrac{1}{4}\varepsilon\right)_+\right\rangle_B + km\langle 1\rangle_B \leq (kn+1)\langle c\rangle_B.$$

Applying the map $W(A) \to W(B)$, we get

$$(kn+1)\left\langle \left(a - \tfrac{1}{4}\varepsilon\right)_+\right\rangle_A + km\langle 1\rangle_A \leq (kn+1)\langle c\rangle_A.$$

For $\tau \in \mathrm{T}(A)$, we apply d_τ and divide by $kn+1$ to get

$$d_\tau\left(\left\langle \left(a - \tfrac{1}{4}\varepsilon\right)_+\right\rangle\right) + \frac{km}{kn+1} \leq d_\tau(c).$$

Since $\mathrm{QT}(A) = \mathrm{T}(A)$ (by Corollary 9.18 of [22]) and

$$\frac{km}{kn+1} > \mathrm{rc}(A),$$

it follows that $\left(a - \frac{1}{4}\varepsilon\right)_+ \precsim_A c$. In particular, there is $v \in M_l \otimes A$ such that

$$\left\| vcv^* - \left(a - \tfrac{1}{4}\varepsilon\right)_+\right\| < \tfrac{1}{4}\varepsilon.$$

Since B is large in A, we can apply Proposition 2.10 and Lemma 2.6 to find $v_0 \in M_l \otimes A$ and $g \in M_l \otimes B$ with $0 \leq g \leq 1$ and such that

$$g \precsim_B y, \quad \|v_0\| \leq \|v\|, \quad \|v_0 - v\| < \frac{\varepsilon}{4\|v\| \cdot \|c\| + 1}, \quad \text{and} \quad (1-g)v_0 \in M_l \otimes B.$$

It follows that $\|v_0^* c v_0 - v^* c v\| < \frac{\varepsilon}{2}$, so

$$\left\| (1-g)v_0 c[(1-g)v_0]^* - (1-g)\left(a - \tfrac{1}{4}\varepsilon\right)_+(1-g)\right\| < \tfrac{3}{4}\varepsilon. \tag{3.13}$$

Therefore, using Lemma 1.23(10) at the first step,

$$\big[(1-g)\big(a-\tfrac{1}{4}\varepsilon\big)_{+}(1-g)-\tfrac{3}{4}\varepsilon\big]_{+}\precsim_B (1-g)v_0c[(1-g)v_0]^{*}\precsim_B c. \tag{3.14}$$

Using Lemma 1.24 at the first step, with $\big(a-\frac{1}{4}\varepsilon\big)_{+}$ in place of a and $\frac{3}{4}\varepsilon$ in place of ε, as well as Lemma 1.23(8), using (3.14) at the second step, using (3.13) at the third step, and using the second part of (3.12) at the fourth step, we get

$$(a-\varepsilon)_{+}\precsim_B \big[(1-g)\big(a-\tfrac{1}{4}\varepsilon\big)_{+}(1-g)-\tfrac{3}{4}\varepsilon\big]_{+}\oplus g\precsim_B c\oplus g\precsim_B c\oplus y\precsim_B b.$$

This is the relation we need, and the proof is complete. □

Proof of Theorem 3.2. Combine Proposition 3.5 and Proposition 3.6. □

4. Large Subalgebras in Crossed Products by $\mathbb{Z}$

In this section, we let $h\colon X\to X$ be a homeomorphism of a compact Hausdorff space X. Following Putnam [44], for $Y\subset X$ closed we define the Y-orbit breaking subalgebra $C^*(\mathbb{Z},X,h)_Y\subset C^*(\mathbb{Z},X,h)$. We prove that if X is infinite, h is minimal, and Y intersects each orbit at most once, then $C^*(\mathbb{Z},X,h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z},X,h)$. In order to state more general results, we generalize the construction of Definition 1.7 in Section 1.1.

Notation 4.1. Let A be a C*-algebra, and let $\alpha\in\operatorname{Aut}(A)$. We identify A with a subalgebra of $C^*(\mathbb{Z},A,\alpha)$ in the standard way. We let $u\in M(C^*(\mathbb{Z},A,\alpha))$ be the standard unitary corresponding to the generator $1\in\mathbb{Z}$. Following standard notation for discrete groups, we let $A[\mathbb{Z}]$ denote the dense *-subalgebra of $C^*(\mathbb{Z},A,\alpha)$ consisting of sums $\sum_{k=-n}^{n}a_ku^k$ with $n\in\mathbb{Z}_{\geq 0}$ and $a_{-n},a_{-n+1},\ldots,a_n\in A$. We let $E_\alpha\colon C^*(\mathbb{Z},A,\alpha)\to A$ denote the standard conditional expectation, defined on $A[\mathbb{Z}]$ by $E_\alpha\big(\sum_{k=-n}^{n}a_ku^k\big)=a_0$. When α is understood, we just write E.

For a locally compact Hausdorff space X and a homeomorphism $h\colon X\to X$, we use obvious analogs of this notation for $C^*(\mathbb{Z},X,h)$, with the automorphism of $C(X)$ being given by $\alpha(f)(x)=f(h^{-1}(x))$ for $f\in C(X)$ and $x\in X$. In particular, we have $ufu^*=f\circ h^{-1}$.

Notation 4.2 and Definition 4.3 below differ from Notation 1.6 and Definition 1.7 in that they consider $C_0(X,D)$ for a C*-algebra D instead of just $C_0(X)$.

Notation 4.2. For a locally compact Hausdorff space X, a C*-algebra D, and an open subset $U\subset X$, we use the abbreviation

$$C_0(U,D)=\big\{f\in C_0(X,D)\colon f(x)=0\text{ for all }x\in X\setminus U\big\}\subset C_0(X,D).$$

This subalgebra is of course canonically isomorphic to the usual algebra $C_0(U,D)$ when U is considered as a locally compact Hausdorff space in its own right. As in Notation 1.6, if $D=\mathbb{C}$ we omit it from the notation.

In particular, if $Y\subset X$ is closed, then

$$C_0(X\setminus Y,\,D)=\big\{f\in C_0(X,D)\colon f(x)=0\text{ for all }x\in Y\big\}. \tag{4.1}$$

Definition 4.3. Let X be a locally compact Hausdorff space, let D be a unital C*-algebra, and let $h\colon X\to X$ be a homeomorphism. Let $\alpha\in\operatorname{Aut}(C(X,D))$ be an automorphism which "lies over h", in the sense that there exists a function $x\mapsto\alpha_x$ from X to $\operatorname{Aut}(D)$ such that $\alpha(a)(x)=\alpha_x(a(h^{-1}(x)))$ for all $x\in X$ and $a\in C_0(X,D)$. Let $Y\subset X$ be a nonempty closed subset, and, following (4.1), define

$$C^*\big(\mathbb{Z},\,C_0(X,D),\,\alpha\big)_Y=C^*\big(C_0(X,D),\,C_0(X\setminus Y,\,D)u\big)\subset C^*\big(\mathbb{Z},\,C_0(X,D),\,\alpha\big).$$

We call it the *Y-orbit breaking subalgebra* of $C^*\big(\mathbb{Z},\, C_0(X, D),\, \alpha\big)$.

We give a sketch of the proof of Theorem 1.8, namely that if $h\colon X \to X$ is a minimal homeomorphism and $Y \subset X$ is a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 1.3.

Under some technical conditions on α and D, similar methods can be used to prove the analogous result for $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)_Y$. The following theorem is a consequence of results in [4].

Theorem 4.4. Let X be an infinite compact metric space, let $h\colon X \to X$ be a minimal homeomorphism, let D be a simple unital C*-algebra which has a tracial state, and let $\alpha \in \mathrm{Aut}(C(X, D))$ lie over h. Assume that D has strict comparison of positive elements, or that the automorphisms α_x in Definition 4.3 are all approximately inner. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)_Y$ is a centrally large subalgebra of $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)$ in the sense of Definition 1.3.

The ideas of the proof of Theorem 1.8 are all used in the proof of the general theorem behind Theorem 4.4, but additional work is needed to deal with the presence of D.

Proposition 4.5 (Proposition 7.5 of [43])**.** Let X be a compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, X, h)$ and $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ be as in Notation 4.1. Let $Y \subset X$ be a nonempty closed subset. For $n \in \mathbb{Z}$, set

$$Y_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0. \end{cases}$$

Then

$$C^*(\mathbb{Z}, X, h)_Y = \big\{a \in C^*(\mathbb{Z}, X, h)\colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z}\big\} \tag{4.2}$$

and

$$\overline{C^*(\mathbb{Z}, X, h)_Y \cap C(X)[\mathbb{Z}]} = C^*(\mathbb{Z}, X, h)_Y. \tag{4.3}$$

Sketch of proof. Define

$$B = \big\{a \in C^*(\mathbb{Z}, X, h)\colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z}\big\}$$

and

$$B_0 = B \cap C(X)[\mathbb{Z}].$$

We claim that B_0 is dense in B. To see this, let $b \in B$ and for $k \in \mathbb{Z}$ define $b_k = E(bu^{-k}) \in C_0(X \setminus Y_k)$. Then for $n \in \mathbb{Z}_{>0}$, the element

$$a_n = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) b_k u^k.$$

is clearly in B_0, and Theorem VIII.2.2 of [14] implies that $\lim_{n\to\infty} a_n = b$. The claim follows. In particular, (4.3) will now follow from (4.2), so we need only prove (4.2).

Next, one proves that B_0 is a *-algebra. It is enough to prove that if $f \in C_0(X \setminus Y_m)$ and $g \in C_0(X \setminus Y_n)$, then $(fu^m)(gu^n) \in B_0$ and $(fu^m)^* \in B_0$. The proof involves manipulations with h and the sets Y_n, and the proof that $(f u^m)(g u^n) \in B_0$ must be broken into six cases: all combinations of signs of m, n, and $m+n$ which can actually occur. We refer to [43] for the details.

Since $C(X) \subset B_0$ and $C_0(X \setminus Y)u \subset B_0$, it follows that $C^*(\mathbb{Z}, X, h)_Y \subset \overline{B_0} = B$.

The next step is to show that for all $n \in \mathbb{Z}$ and $f \in C_0(X \setminus Y_n)$, we have $f u^n \in C^*(\mathbb{Z}, X, h)_Y$. For $n = 0$ this is trivial. Let $n > 0$, and let $f \in C_0(X \setminus Y_n)$. Define $f_0 = (\mathrm{sgn} \circ f)|f|^{1/n}$ and for $j = 1, 2, \ldots, n-1$ define $f_j = |f \circ h^j|^{1/n}$. The definition of Y_n implies that $f_0, f_1, \ldots, f_{n-1} \in C_0(X \setminus Y)$. Therefore the element

$$a = (f_0 u)(f_1 u) \cdots (f_{n-1} u)$$

is in $C^*(\mathbb{Z}, X, h)_Y$. A computation (carried out in [43]) shows that $a = f u^n$. The case $n < 0$ is reduced to the case $n > 0$ by taking adjoints; see [43] for details.

It now follows that $B_0 \subset C^*(\mathbb{Z}, X, h)_Y$. Combining this result with $\overline{B_0} = B$ and $C^*(\mathbb{Z}, X, h)_Y \subset B$, we get $C^*(\mathbb{Z}, X, h)_Y = B$. □

Corollary 4.6 (Corollary 7.6 of [43])**.** Let X be a compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $Y \subset X$ be a nonempty closed subset. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary, as in Notation 4.1, and let $v \in C^*(\mathbb{Z}, X, h^{-1})$ be the analogous standard unitary in $C^*(\mathbb{Z}, X, h^{-1})$. Then there exists a unique homomorphism $\varphi\colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ such that $\varphi(f) = f$ for $f \in C(X)$ and $\varphi(v) = u^*$, the map φ is an isomorphism, and

$$\varphi\big(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z}, X, h)_Y.$$

See [43] for the straightforward proof.

Lemma 4.7 (Lemma 7.4 of [43])**.** Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $K \subset X$ be a compact set such that $h^n(K) \cap K = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let $U \subset X$ be a nonempty open subset. Then there exist $l \in \mathbb{Z}_{\geq 0}$, compact sets $K_1, K_2, \ldots, K_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$, such that $K \subset \bigcup_{j=1}^{l} K_j$ and such that $h^{n_1}(K_1)$, $h^{n_2}(K_2)$, $\ldots$, $h^{n_l}(K_l)$ are disjoint subsets of U.

Sketch of proof. Choose a nonempty open subset $V \subset X$ such that $\overline{V}$ is compact and contained in U. Use minimality of h to cover K with the images of V under finitely many negative powers of h, say $h^{-n_1}(V)$, $h^{-n_2}(V)$, $\ldots$, $h^{-n_l}(V)$. Set $K_j = h^{-n_j}\big(\overline{V}\big) \cap K$ for $j = 1, 2, \ldots, l$. □

The next lemma is one of the key steps. It is straightforward if one only asks that $f \precsim_{C^*(\mathbb{Z}, X, h)} g$. Getting $f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g$ for both positive n and negative n is a key step in showing $C^*(\mathbb{Z}, X, h)_Y$ a large subalgebra of $C^*(\mathbb{Z}, X, h)$.

Lemma 4.8 (Lemma 7.7 of [43])**.** Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let $U \subset X$ be a nonempty open subset and let $n \in \mathbb{Z}$. Then there exist $f, g \in C(X)_+$ such that

$$f|_{h^n(Y)} = 1, \quad 0 \leq f \leq 1, \quad \mathrm{supp}(g) \subset U, \quad \text{and} \quad f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g.$$

Proof. We first prove this when $n = 0$.

Apply Lemma 4.7 with Y in place of K, obtaining $l \in \mathbb{Z}_{\geq 0}$, compact sets $Y_1, Y_2, \ldots, Y_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$. Set $N = \max(n_1, n_2, \ldots, n_l)$. Choose disjoint open sets $V_1, V_2, \ldots, V_l \subset U$ such that $h^{n_j}(Y_j) \subset V_j$ for $j = 1, 2, \ldots, l$.

Then $Y_j \subset h^{-n_j}(V_j)$, so the sets $h^{-n_1}(V_1)$, $h^{-n_2}(V_2)$, $\ldots$, $h^{-n_l}(V_l)$ cover Y. For $j = 1, 2, \ldots, l$, define

$$W_j = h^{-n_j}(V_j) \cap \left(X \setminus \bigcup_{n=1}^{N} h^{-n}(Y) \right).$$

Then $W_1, W_2, \ldots, W_l$ form an open cover of Y. Therefore there are $f_1, f_2, \ldots, f_l \in C(X)_+$ such that for $j = 1, 2, \ldots, l$ we have $\mathrm{supp}(f_j) \subset W_j$ and $0 \leq f_j \leq 1$, and such that the function $f = \sum_{j=1}^{l} f_j$ satisfies $f(x) = 1$ for all $x \in Y$ and $0 \leq f \leq 1$. Further define $g = \sum_{j=1}^{l} f_j \circ h^{-n_j}$. Then $\mathrm{supp}(g) \subset U$.

Let $u \in C^*(\mathbb{Z}, X, h)$ be as in Notation 4.1. For $j = 1, 2, \ldots, l$, set $a_j = f_j^{1/2} u^{-n_j}$. Since f_j vanishes on $\bigcup_{n=1}^{n_j} h^{-n}(Y)$, Proposition 4.5 implies that $a_j \in C^*(\mathbb{Z}, X, h)_Y$. Therefore, in $C^*(\mathbb{Z}, X, h)_Y$ we have

$$f_j \circ h^{-n_j} = a_j^* a_j \sim_{C^*(\mathbb{Z}, X, h)_Y} a_j a_j^* = f_j.$$

Consequently, using Lemma 1.23(12) at the second step and Lemma 1.23(13) and disjointness of the supports of the functions $f_j \circ h^{-n_j}$ at the last step, we have

$$f = \sum_{j=1}^{l} f_j \precsim_{C^*(\mathbb{Z}, X, h)_Y} \bigoplus_{j=1}^{l} f_j \sim_{C^*(\mathbb{Z}, X, h)_Y} \bigoplus_{j=1}^{l} f_j \circ h^{-n_j} \sim_{C^*(\mathbb{Z}, X, h)_Y} g.$$

This completes the proof for $n = 0$.

Now suppose that $n > 0$. Choose functions f and g for the case $n = 0$, and call them f_0 and g. Since $f_0(x) = 1$ for all $x \in Y$, and since $Y \cap \bigcup_{l=1}^{n} h^{-l}(Y) = \varnothing$, there is $f_1 \in C(X)$ with $0 \leq f_1 \leq f_0$, $f_1(x) = 1$ for all $x \in Y$, and $f_1(x) = 0$ for $x \in \bigcup_{l=1}^{n} h^{-l}(Y)$. Set $v = f_1^{1/2} u^{-n}$ and $f = f_1 \circ h^{-n}$. Then $f(x) = 1$ for all $x \in h^n(Y)$ and $0 \leq f \leq 1$. Proposition 4.5 implies that $v \in C^*(\mathbb{Z}, X, h)_Y$. We have

$$v^* v = u^n f_1 u^{-n} = f_1 \circ h^{-n} = f \quad \text{and} \quad v v^* = f_1.$$

Using Lemma 1.23(4), we thus get

$$f \sim_{C^*(\mathbb{Z}, X, h)_Y} f_1 \leq f_0 \precsim_{C^*(\mathbb{Z}, X, h)_Y} g.$$

This completes the proof for the case $n > 0$.

Finally, we consider the case $n < 0$. In this case, we have $-n-1 \geq 0$. Apply the cases already done with h^{-1} in place of h. We get $f, g \in C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}$ such that $f(x) = 1$ for all $x \in (h^{-1})^{-n-1}(h^{-1}(Y)) = h^n(Y)$, such that $0 \leq f \leq 1$, such that $\mathrm{supp}(g) \subset U$, and such that $f \precsim_{C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}} g$. Let $\varphi \colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ be the isomorphism of Corollary 4.6. Then

$$\varphi(f) = f, \quad \varphi(g) = g, \quad \text{and} \quad \varphi\big(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z}, X, h)_Y.$$

Therefore $f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g$. $\square$

The following result is a special case of Lemma 7.9 of [43]. The basic idea has been used frequently; related arguments can be found, for example, in the proofs of Theorem 3.2 of [15], Lemma 2 and Theorem 1 in [3], Lemma 10 of [26], and Lemma 3.2 of [36]. (The papers listed are not claimed to be representative or to be the original sources; they are ones I happen to know of.)

Lemma 4.9. Let X be an infinite compact space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in B_+ \setminus \{0\}$. Then there exists $b \in C(X)_+ \setminus \{0\}$ such that $b \precsim_B a$.

Sketch of proof. Without loss of generality $\|a\| \leq 1$. The conditional expectation $E_\alpha\colon C^*_{\mathrm{r}}(G, X) \to C(X)$ is faithful. Therefore $E_\alpha(a) \in C(X)$ is a nonzero positive element. Set $\varepsilon = \frac{1}{6}\|E_\alpha(a)\|$. Choose $c \in B \cap C(X)[\mathbb{Z}]$ such that $\|c - a^{1/2}\| < \varepsilon$ and $\|c\| \leq 1$. One can check that $\|E_\alpha(c^*c)\| > 4\varepsilon$. There are $n \in \mathbb{Z}_{\geq 0}$ and $g_{-n}, g_{-n+1}, \ldots, g_n \in C(X)$ such that $c^*c = \sum_{k=-n}^{n} g_k u^k$. We have $g_0 = E_\alpha(c^*c) \in C(X)_+$ and $\|g_0\| > 4\varepsilon$. Therefore there is $x \in X$ such that $g_0(x) > 4\varepsilon$. Choose $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$, and the sets $h^k(\mathrm{supp}(f))$ are disjoint for $k = -n, -n+1, \ldots, n$. One can then check that $fc^*cf = fg_0f$, so that $\|fc^*cf\| > 4\varepsilon$. Therefore $(fc^*cf - 2\varepsilon)_+$ is a nonzero element of $C(X)$. Using Lemma 1.23(6) at the first step, Lemma 1.23(17) and $cf^2c^* \leq cc^*$ at the second step, and Lemma 1.23(10) and $\|cc^* - a\| < 2\varepsilon$ at the last step, we then have

$$(fc^*cf - 2\varepsilon)_+ \sim_B (cf^2c^* - 2\varepsilon)_+ \precsim_B (cc^* - 2\varepsilon)_+ \precsim_B a.$$

This completes the proof. $\square$

Corollary 4.10. Let X be an infinite compact Hausdorff space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in A_+ \setminus \{0\}$ and let $b \in B_+ \setminus \{0\}$. Then there exists $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_{C^*(\mathbb{Z},X,h)} a$ and $f \precsim_B b$.

Lemma 4.11. Let A be a C*-algebra, and let $S \subset A$ be a subset which generates A as a C*-algebra. Then for every finite subset $F \subset A$ and every $\varepsilon > 0$ there are a finite subset $F_0 \subset S$ and $\varepsilon_0 > 0$ such that whenever $b \in A$ satisfies $\|b\| \leq 1$ and $\|ba - ab\| < \varepsilon_0$ for all $a \in F_0$, then $\|ba - ab\| < \varepsilon$ for all $a \in F$.

Proof of Theorem 1.8. Since h is minimal, it is well known that A is simple and finite. Also clearly A is infinite dimensional.

We claim that it suffices to do the following. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, and let $f \in C(X)_+ \setminus \{0\}$. We find $c_1, c_2, \ldots, c_m \in A$ and $g \in C(X)$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in C^*(\mathbb{Z}, X, h)_Y$.
(4) $g \precsim_B f$.
(5) $\|gu - ug\| < \varepsilon$.

To see this, first observe that if in (5) we required $\|ga_j - a_jg\| < \varepsilon$ for $j = 1, 2, \ldots, m$, then the claim would follow from Corollary 4.10 and Lemma 2.5 (as in the proof of Proposition 2.3). There is no need to require that the finite set used in (2) and (3) be the same as the finite set used in (5). Applying Lemma 4.11 and choosing a smaller value of ε, in (5) we can replace $\|ga_j - a_jg\| < \varepsilon$ for $j = 1, 2, \ldots, m$ by $\|ga - ag\| < \varepsilon$ for all a in a finite subset of the generating set $C(X) \cup \{u\}$. Since $g \in C(X)$, it is automatic that g commutes exactly with the elements of $C(X)$. So we need only require that $\|gu - ug\| < \varepsilon$. This proves the claim.

Choose $c_1, c_2, \ldots, c_m \in C(X)[\mathbb{Z}]$ such that $\|c_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, m$. (This estimate is condition (2).) Choose $N \in \mathbb{Z}_{>0}$ such that for $j = 1, 2, \ldots, m$

there are $c_{j,l} \in C(X)$ for $l = -N, \, -N+1, \, \ldots, \, N-1, \, N$ with

$$c_j = \sum_{l=-N}^{N} c_{j,l} u^l.$$

Choose $N_0 \in \mathbb{Z}_{>0}$ such that $\frac{1}{N_0} < \varepsilon$. Define

$$I = \big\{ -N - N_0, \, -N - N_0 + 1, \, \ldots, \, N + N_0 - 1, \, N + N_0 \big\}.$$

Set $U = \{x \in X \colon f(x) \neq 0\}$, and choose nonempty disjoint open sets $U_l \subset U$ for $l \in I$. For each such l, use Lemma 4.8 to choose $f_l, r_l \in C(X)_+$ such that $r_l(x) = 1$ for all $x \in h^l(Y)$, such that $0 \leq r_l \leq 1$, such that $\mathrm{supp}(f_l) \subset U_l$, and such that $r_l \precsim_B f_l$.

Choose an open set W containing Y such that the sets $h^l(W)$ are disjoint, for $l \in I$, and choose $r \in C(X)$ such that

$$0 \leq r \leq 1, \quad r|_Y = 1, \quad \text{and} \quad \mathrm{supp}(r) \subset W.$$

Set

$$g_0 = r \cdot \prod_{l=-N-N_0}^{N+N_0} r_l \circ h^l.$$

Set $g_l = g_0 \circ h^{-l}$ for $l = -N - N_0, \, -N - N_0 + 1, \, \ldots, \, N + N_0 - 1, \, N + N_0$. Then $0 \leq g_l \leq r_l \leq 1$. Define λ_l for $l \in I$ by

$$\lambda_{-N-N_0} = 0, \quad \lambda_{-N-N_0+1} = \frac{1}{N_0}, \quad \lambda_{-N-N_0+2} = \frac{2}{N_0}, \quad \ldots, \quad \lambda_{-N-1} = 1 - \frac{1}{N_0},$$
$$\lambda_{-N} = \lambda_{-N+1} = \cdots = \lambda_{N-1} = \lambda_N = 1,$$
$$\lambda_{N+1} = 1 - \frac{1}{N_0}, \quad \lambda_{N+2} = 1 - \frac{2}{N_0}, \quad \ldots, \quad \lambda_{N+N_0-1} = \frac{1}{N_0}, \quad \lambda_{N+N_0} = 0.$$

Set $g = \sum_{l \in I} \lambda_l g_l$. The supports of the functions g_l are disjoint, so $0 \leq g \leq 1$. This is condition (1). Using Lemma 1.23(13) at the first and fourth steps and Lemma 1.23(14) at the third step, we get

$$g \precsim_B \bigoplus_{l \in I} g_l \leq \bigoplus_{l \in I} r_l \precsim_B \bigoplus_{l \in I} f_l \sim_{C(X)} \sum_{l \in I} f_l \precsim_{C(X)} f.$$

This is condition (4).

We check condition (5). We have

$$\|gu - ug\| = \|g - ugu^*\| = \|g - g \circ h^{-1}\| = \left\| \sum_{l \in I} \lambda_l g_0 \circ h^{-l} - \sum_{l \in I} \lambda_l g_0 \circ h^{-l-1} \right\|.$$

In the second sum in the last term, we change variables to get $\sum_{l+1 \in I} \lambda_{l-1} g_0 \circ h^{-l}$. Use $\lambda_{-N-N_0} = \lambda_{N+N_0} = 0$ and combine terms to get

$$\|gu - ug\| = \left\| \sum_{l=-N-N_0+1}^{N+N_0} (\lambda_l - \lambda_{l-1}) g_0 \circ h^{-l} \right\|.$$

The expressions $g_0 \circ h^{-l}$ are orthogonal and have norm 1, so

$$\|gu - ug\| = \max_{-N-N_0+1 \leq l \leq N+N_0} |\lambda_l - \lambda_{l-1}| = \frac{1}{N_0} < \varepsilon.$$

It remains to verify condition (3). Since $1 - g$ vanishes on the sets

$$h^{-N}(Y), \, h^{-N+1}(Y), \, \ldots, \, h^{N-2}(Y), \, h^{N-1}(Y),$$

Proposition 4.5 implies that $(1-g)u^l \in B$ for $l = -N, -N+1, \ldots, N-1, N$. For $j = 1, 2, \ldots, m$, since $c_{j,l} \in C(X) \subset B$ for $l = -N, -N+1, \ldots, N-1, N$, we get

$$(1-g)c_j = \sum_{l=-N}^{N} c_{j,l} \cdot (1-g)u^l \in B.$$

This completes the verification of condition (3), and the proof of the theorem. □

5. Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms

This section has not yet been written.

References

[1] C. A. Akemann and F. Shultz, *Perfect C*-algebras*, Memoirs Amer. Math. Soc., vol. 55 no. **326**(1985).

[2] P. Ara, F. Perera, and A. S. Toms, *K-Theory for operator algebras. Classification of C*-algebras*, pages 1–71 in: *Aspects of Operator Algebras and Applications*, P. Ara, F Lledó, and F. Perera (eds.), Contemporary Mathematics vol. 534, Amer. Math. Soc., Providence RI, 2011.

[3] R. J. Archbold and J. S. Spielberg, *Topologically free actions and ideals in discrete C*-dynamical systems*, Proc. Edinburgh Math. Soc. (2) **37**(1994), 119–124.

[4] D. Archey, J. Buck, and N. C. Phillips, in preparation.

[5] D. Archey and N. C. Phillips, *Permanence of stable rank one for centrally large subalgebras and crossed products by minimal homeomorphisms*, preprint (arXiv: 1505.00725v1 [math.OA]).

[6] D. Archey and N. C. Phillips, *Centrally large subalgebras and Z-stability*, in preparation.

[7] B. Blackadar, *Comparison theory for simple C*-algebras*, pages 21–54 in: *Operator Algebras and Applications*, D. E. Evans and M. Takesaki (eds.) (London Math. Soc. Lecture Notes Series no. 135), Cambridge University Press, Cambridge, New York, 1988.

[8] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras*, J. Funct. Anal. **45**(1982), 297–340.

[9] B. Blackadar, L. Robert, A. P. Tikuisis, A. S. Toms, and W. Winter, *An algebraic approach to the radius of comparison*, Trans. Amer. Math. Soc. **364**(2002), 3657–3674.

[10] J. Buck, in preparation.

[11] K. T. Coward, G. A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C*-algebras*, J. reine angew. Math. **623**(2008), 161–193.

[12] J. Cuntz, *The structure of multiplication and addition in simple C*-algebras*, Math. Scand. **40**(1977), 215–233.

[13] J. Cuntz, *Dimension functions on simple C*-algebras*, Math. Ann. **233**(1978), 145–153.

[14] K. R. Davidson, *C*-Algebras by Example*, Fields Institute Monographs no. 6, Amer. Math. Soc., Providence RI, 1996.

[15] G. A. Elliott, *Some simple C*-algebras constructed as crossed products with discrete outer automorphism groups*, Publ. RIMS Kyoto Univ. **16**(1980), 299–311.

[16] G. A. Elliott, T. M. Ho, and A. S. Toms, *A class of simple C*-algebras with stable rank one*, J. Funct. Anal. **256**(2009), 307–322.

[17] G. A. Elliott and Z. Niu, *All irrational extended rotation algebras are AF algebras*, preprint.

[18] G. A. Elliott and Z. Niu, *C*-algebra of a minimal homeomorphism of zero mean dimension*, preprint (arXiv:1406.2382v2 [math.OA]).

[19] G. A. Elliott, L. Robert, and L. Santiago, *The cone of lower semicontinuous traces on a C*-algebra*, Amer. J. Math. **133**(2011), 969–1005.

[20] A. Forrest, *A Bratteli diagram for commuting homeomorphisms of the Cantor set*, International J. Math. **11**(2000), 177–200.

[21] J. Giol and D. Kerr, *Subshifts and perforation*, J. reine angew. Math. **639**(2010), 107–119.

[22] U. Haagerup and S. Thorbjørnsen, *Random matrices and K-theory for exact C*-algebras*, Documenta Math. **4**(1999), 341–450 (electronic).

[23] T. Hines, N. C. Phillips, and A. S. Toms, *Mean dimension and radius of comparison for minimal homeomorphisms with Cantor factors*, in preparation.

[24] I. Hirshberg, W. Winter, and J. Zacharias, *Rokhlin dimension and C*-dynamics*, preprint (arXiv: 1209.1618v2 [math.OA]).

[25] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C*-algebras*, Amer. J. Math. **122**(2000), 637–666.

[26] A. Kishimoto and A. Kumjian, *Crossed products of Cuntz algebras by quasi-free automorphisms*, pages 173–192 in: *Operator Algebras and their Applications (Waterloo, ON, 1994/1995)*, Fields Inst. Commun. vol. 13, Amer. Math. Soc., Providence, RI, 1997.

[27] H. Lin and H. Matui, *Minimal dynamical systems on the product of the Cantor set and the circle*, Commun. Math. Phys. **257**(2005), 425–471.

[28] H. Lin and N. C. Phillips, *Crossed products by minimal homeomorphisms*, J. reine ang. Math. **641**(2010), 95–122.

[29] Q. Lin and N. C. Phillips, *Ordered K-theory for C*-algebras of minimal homeomorphisms*, pages 289–314 in: *Operator Algebras and Operator Theory*, L. Ge, etc. (eds.), Contemporary Mathematics vol. 228, Amer. Math. Soc., Providence RI, 1998.

[30] E. Lindenstrauss, *Mean dimension, small entropy factors and an embedding theorem*, Inst. Hautes Études Sci. Publ. Math. **89**(2000), 227–262.

[31] T. A. Loring, *Lifting Solutions to Perturbing Problems in C*-Algebras*, Fields Institute Monographs no. 8, American Mathematical Society, Providence RI, 1997.

[32] H. Matui, *Topological orbit equivalence of locally compact Cantor minimal systems*, Ergod. Th. Dynam. Sys. **22**(2002), 1871–1903.

[33] H. Matui, *Topological spectrum of locally compact Cantor minimal systems*, Proc. Amer. Math. Soc. **132**(2004), 87–95.

[34] Z. Niu, *Mean dimension and AH-algebras with diagonal maps*, J. Funct. Anal. **266**(2014), 4938–4994.

[35] H. Osaka and N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergod. Th. Dynam. Sys. **26**(2006), 1579–1621.

[36] C. Pasnicu and N. C. Phillips, *Crossed products by spectrally free actions*, J. Funct. Anal., to appear.

[37] G. K. Pedersen, *C*-Algebras and their Automorphism Groups*, Academic Press, London, New York, San Francisco, 1979.

[38] N. C. Phillips, *Recursive subhomogeneous algebras*, Trans. Amer. Math. Soc. **359**(2007), 4595–4623.

[39] N. C. Phillips, *Cancellation and stable rank for direct limits of recursive subhomogeneous algebras*, Trans. Amer. Math. Soc. **359**(2007), 4625–4652.

[40] N. C. Phillips, *Crossed products of the Cantor set by free minimal actions of* $\mathbb{Z}^d$, Commun. Math. Phys. **256**(2005), 1–42.

[41] N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C*-algebras*, Amer. J. Math. **133**(2011), 581–636.

[42] N. C. Phillips, *Crossed products of* L^p *operator algebras and the K-theory of Cuntz algebras on* L^p *spaces*, preprint (arXiv: 1309.6406 [math.FA]).

[43] N. C. Phillips, *Large subalgebras*, preprint (arXiv: 1408.5546v1 [math.OA]).

[44] I. F. Putnam, *The C*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math. **136**(1989), 329–353.

[45] I. F. Putnam, *On the topological stable rank of certain transformation group C*-algebras*, Ergod. Th. Dynam. Sys. **10**(1990), 197–207.

[46] I. F. Putnam, *On the K-theory of C*-algebras of principal groupoids*, Rocky Mtn. J. Math. **28**(1998), 1483–1518.

[47] M. Rørdam, *On the structure of simple C*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107**(1992), 255–269.

[48] W. Sun, *Crossed product C*-algebras of minimal dynamical systems on the product of the Cantor set and the torus*, preprint (arXiv:1102.2801v1 [math.OA]).

[49] G. Szabo, *The Rokhlin dimension of topological* $\mathbb{Z}^m$*-actions*, preprint (arXiv:1308.5418v4 [math.OA]).

[50] A. S. Toms, *Flat dimension growth for C*-algebras*, J. Funct. Anal. **238**(2006), 678–708.

[51] A. S. Toms, *Comparison theory and smooth minimal C*-dynamics*, Commun. Math. Phys. **289**(2009), 401–433.

[52] A. S. Toms and W. Winter, *Minimal dynamics and K-theoretic rigidity: Elliott's conjecture*, Geom. Funct. Anal. **23**(2013), 467–481.

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