LARGE SUBALGEBRAS AND THE STRUCTURE OF CROSSED PRODUCTS

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Abstract. We give a survey of large subalgebras of crossed product C*-algebras, including some recent applications (by several people), mostly to the transformation group C*-algebra $C^*(\mathbb{Z}, X, h)$ of a minimal homeomorphism h of a compact metric space X:

- If there is a continuous surjective map from X to the Cantor set, then $C^*(\mathbb{Z}, X, h)$ has stable rank one (regardless of the mean dimension of h).
- If there is a continuous surjective map from X to the Cantor set, then the radius of comparison of $C^*(\mathbb{Z}, X, h)$ is at most half the mean dimension of h.
- If h has mean dimension zero, then $C^*(\mathbb{Z}, X, h)$ is Z-stable.
- The "extended" irrational rotation algebras, obtained by "cutting" each of the standard unitary generators at one or more points in its spectrum, are AF algebras.
- Let X be a compact metric space and let $h\colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}\big(C^*(\mathbb{Z}, X, h)\big) \leq 1 + 2 \cdot \mathrm{mdim}(h)$.
- Certain algebras of the form $C^*(\mathbb{Z}, C(X, D), \alpha)$, in which D is simple and α "lies over" a minimal homeomorphism of X, have stable rank one.

We include some background material, particularly on the Cuntz semigroup. We give or sketch proofs of some of the basic results on large subalgebras, including a much more direct proof than in the paper on large subalgebras of the fact that a large subalgebra and its containing algebra have the same radius of comparison. We describe a more direct proof than in the papers that if $h\colon X \to X$ is a minimal homeomorphism of a compact metric space, $Y \subset X$ is compact, and $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then the orbit breaking subalgebra associated to Y is centrally large in $C^*(\mathbb{Z}, X, h)$. We sketch the proof, using large subalgebras, that if there is a continuous surjective map from X to the Cantor set, then the radius of comparison of $C^*(\mathbb{Z}, X, h)$ is at most half the mean dimension of h. We state a number of open problems.

Large and centrally large subalgebras are a technical tool which has played a key role in several recent results on the structure of the C*-algebras of minimal dynamical systems and some related algebras. These results include:

- The extended irrational rotation algebras are AF.
- Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is Z-stable.
- Let X be a compact metric space such that there is a continuous surjection from X to the Cantor set. Then $\mathrm{rc}\big(C^*(\mathbb{Z}, X, h)\big) \leq \frac{1}{2}\mathrm{mdim}(h)$.
- Let X be a compact metric space such that there is a continuous surjection from X to the Cantor set. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one. (There

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are examples in which this holds but $C^*(\mathbb{Z}, X, h)$ does not have strict comparison of positive elements and is not Z-stable.)

- Let X be a compact metric space and let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}\big(C^*(\mathbb{Z}, X, h)\big) \leq 1 + 2 \cdot \mathrm{mdim}(h)$.
- We give an example involving a crossed product $C^*(\mathbb{Z}, C(X, D), \alpha)$ in which D is simple and α "lies over" a minimal homeomorphism of X. Let F_∞ be the free group on generators indexed by $\mathbb{Z}$, and for $n \in \mathbb{Z}$ let $u_n \in C^*_{\mathrm{r}}(F_\infty)$ be the unitary which is the image of the corresponding generator of F_∞. Let $h \colon X \to X$ be the restriction of a Denjoy homeomorphism (a nonminimal homeomorphism of the circle whose rotation number is irrational) to its unique minimal set. (See [64].) Thus X is homeomorphic to the Cantor set, and there are $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and a surjective map $\zeta \colon X \to S^1$ such that $\zeta(h(x)) = e^{2\pi i \theta} \zeta(x)$ for all $x \in X$. For $x \in X$ let $\alpha_x \in \mathrm{Aut}(C^*_{\mathrm{r}}(F_\infty))$ be determined by $\alpha_x(u_n) = \zeta(x) u_n$ for $n \in \mathbb{Z}$. Define a kind of noncommutative Furstenberg transformation $\alpha \in \mathrm{Aut}\big(C(X, C^*_{\mathrm{r}}(F_\infty))\big)$ by $\alpha(a)(x) = \alpha_x(a(x))$ for $a \in C(X, C^*_{\mathrm{r}}(F_\infty))$ and $x \in X$. Then $C^*\big(\mathbb{Z}, C(X, C^*_{\mathrm{r}}(F_\infty)), \alpha\big)$ has stable rank one.

Large subalgebras were also used to give the first proof that if X is a finite dimensional compact metric space with a free minimal action of $\mathbb{Z}^d$, then $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements.

Large subalgebras are a generalization and abstraction of a construction introduced by Putnam in [60], where it was used to prove that if h is a minimal homeomorphism of the Cantor set X, then $K_0(C^*(\mathbb{Z}, X, h))$ is order isomorphic to the K_0-group of a simple AF algebra (Theorem 4.1 and Corollary 5.6 of [60]). Putnam's construction and some generalizations (all of which are centrally large subalgebras in our sense) also played key roles in proofs of other results, less recent that those listed above:

- Let $h \colon X \to X$ be a minimal homeomorphism of the Cantor set. Then $C^*(\mathbb{Z}, X, h)$ is an AT algebra. (Local approximation by circle algebras was proved in Section 2 of [61]. Direct limit decomposition follows from semiprojectivity of circle algebras.)
- Let $h \colon X \to X$ be a minimal homeomorphism of a finite dimensional compact metric space. Then $C^*(\mathbb{Z}, X, h)$ satisfies the following K-theoretic version of Blackadar's Second Fundamental Comparability Question: if $\eta \in K_0(A)$ satisfies $\tau_*(\eta) > 0$ for all tracial states τ on A, then there is a projection $p \in M_\infty(A)$ such that $\eta = [p]$. (See [39] and Theorem 4.5(1) of [54]).
- Let X be a finite dimensional infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism such that the map $K_0\big(C^*(\mathbb{Z}, X, h)\big) \to \mathrm{Aff}\big(\mathrm{T}\big(C^*(\mathbb{Z}, X, h)\big)\big)$ has dense range. Then $C^*(\mathbb{Z}, X, h)$ has tracial rank zero ([38]).
- Let X be the Cantor set and let $h \colon X \times S^1 \to X \times S^1$ be a minimal homeomorphism. For any $x \in X$, the set $Y = \{x\} \times S^1$ intersects each orbit at most once. The algebra $C^*(\mathbb{Z}, X \times S^1, h)_Y$ (see Definition 1.7 for the notation) is introduced before Proposition 3.3 of [37], where it is called A_x. It is a centrally large subalgebra which plays a key role in the proofs of some of the results there. For example, the proofs that the crossed products considered there have stable rank one (Theorem 3.12 of [37]) and order on

projections determined by traces (Theorem 3.13 of [37]) rely directly on the use of this subalgebra.

- A similar construction, with $X \times S^1 \times S^1$ in place of $X \times S^1$ and with $Y = \{x\} \times S^1 \times S^1$, appears in Section 1 of [66]. It plays a role in that paper similar to the role of the algebra $C^*(\mathbb{Z},\, X \times S^1,\, h)_Y$ in the previous item.
- Let $h\colon X \to X$ be a minimal homeomorphism of an infinite compact metric space. The large subalgebras $C^*(\mathbb{Z}, X, h)_Y$ of $C^*(\mathbb{Z}, X, h)$ (as in Definition 1.7), with several choices of Y (several one point sets as well as $\{x_1, x_2\}$ with x_1 and x_2 on different orbits), have been used by Toms and Winter [71] to prove that $C^*(\mathbb{Z}, X, h)$ has finite decomposition rank.

There is a competing approach, the method of Rokhlin dimension of group actions [30], which can be used for some of the same problems large subalgebras are good for. When it applies, it often gives stronger results. For example, Szabó has used this method successfully for free minimal actions of $\mathbb{Z}^d$ on finite dimensional compact metric spaces [67]. For many problems involving crossed products for which large subalgebras are a plausible approach, Rokhlin dimension methods should also be considered. Rokhlin dimension has also been successfully applied to problems involving actions on simple C*-algebras, a context in which no useful large subalgebras are known. (But see [49] and [47], where what might be called large systems of subalgebras are used effectively.) On the other hand, finite Rokhlin dimension requires freeness of the action (in a suitable heuristic sense when the algebra is simple), while some form of essential freeness seems likely to be good enough for large subalgebra methods. (This is suggested by the examples in [47].) Finite Rokhlin dimension also requires some form of topological finite dimensionality.

It seems plausible that there might be a generalization of finite Rokhlin dimension which captures actions on infinite dimensional spaces which have mean dimension zero. Such a generalization might be similar to the progression from the study of simple AH algebras with no dimension growth to those with slow dimension growth. It looks much less likely that Rokhlin dimension methods can be usefully applied to minimal homeomorphisms which do not have mean dimension zero. Large subalgebras have been used to estimate the radius of comparison of $C^*(\mathbb{Z}, X, h)$ when h does not have mean dimension zero (and the radius of comparison is nonzero); see [28] and [59]. They can also potentially be used to prove regularity properties of crossed products $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)$ when D is simple, the automorphism $\alpha \in \operatorname{Aut}(C(X, D))$ "lies over" a minimal homeomorphism of X with large mean dimension, and the regularity properties of the crossed product come from D rather than from the action of $\mathbb{Z}$ on X. See [10].

Unfortunately, we are not able to discuss Rokhlin dimension here.

In these lectures, we give an introduction to large subalgebras, and we illustrate their use in the study of crossed products by minimal homeomorphisms.

1. Introduction, Motivation, and the Cuntz Semigroup

1.1. **Definitions and the basic statements.** We get to the definitions as quickly as possible. We need Cuntz subequivalence; more about it will be described below.

Definition 1.1. Let A be a C*-algebra, and let $a, b \in (K \otimes A)_+$. We say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

By convention, if we say that B is a unital subalgebra of a C*-algebra A, we mean that B contains the identity of A.

Definition 1.2 (Definition 4.1 of [58]). Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

We emphasize that the Cuntz subequivalence involving y in (4) is relative to B, not A.

Condition (5) is needed to avoid triviality when A is purely infinite and simple. With $B = \mathbb{C} \cdot 1$, we could then satisfy all the other conditions by taking $g = 1$. In the stably finite case, we can dispense with (5) (see Proposition 2.3 below), but we still need $g \precsim_A x$ in (4). Otherwise, even if we require that B be simple and that the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$ on traces and quasitraces be bijective, we can take A to be any UHF algebra and take $B = \mathbb{C} \cdot 1$. The choice $g = 1$ would always work.

It is crucial to the usefulness of large subalgebras that g in Definition 1.2 need not be a projection. Also, one can do a lot without any kind of approximate commutation condition. Such a condition does seem to be needed for some results. Here is the relevant definition, although we will not make full use of it in these notes.

Definition 1.3 (Definition 3.2 of [5]). Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *centrally large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.
(6) For $j = 1, 2, \ldots, m$ we have $\|g a_j - a_j g\| < \varepsilon$.

The difference between Definition 1.3 and Definition 1.2 is the approximate commutation condition in Definition 1.3(6).

The following strengthening of Definition 1.3 will be more important in these notes.

Definition 1.4 (Definition 5.1 of [58]). Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *stably large* in A if $M_n(B)$ is large in $M_n(A)$ for all $n \in \mathbb{Z}_{>0}$.

Proposition 1.5 (Proposition 5.6 of [58]). Let A_1 and A_2 be infinite dimensional simple unital C*-algebras, and let $B_1 \subset A_1$ and $B_2 \subset A_2$ be large subalgebras. Assume that $A_1 \otimes_{\min} A_2$ is finite. Then $B_1 \otimes_{\min} B_2$ is a large subalgebra of $A_1 \otimes_{\min} A_2$.

In particular, if A is stably finite and $B \subset A$ is large, then B is stably large. We will give a direct proof (Proposition 2.11 below). We don't know whether stable finiteness of A is needed (Question 1.37 below).

We prepare to define the main example used in these notes.

Notation 1.6. For a locally compact Hausdorff space X and an open subset $U \subset X$, we use the abbreviation

$$C_0(U) = \big\{ f \in C_0(X) \colon f(x) = 0 \text{ for all } x \in X \setminus U \big\} \subset C_0(X).$$

This subalgebra is of course canonically isomorphic to the usual algebra $C_0(U)$ when U is considered as a locally compact Hausdorff space in its own right. If $Y \subset X$ is closed, then

$$C_0(X \setminus Y) = \big\{ f \in C_0(X) \colon f(x) = 0 \text{ for all } x \in Y \big\}. \tag{1.1}$$

Definition 1.7. Let X be a locally compact Hausdorff space and let $h \colon X \to X$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary. (We say more about crossed products at the beginning of Section 4.) Let $Y \subset X$ be a nonempty closed subset, and, following (1.1), define

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, C_0(X \setminus Y)u \big) \subset C^*(\mathbb{Z}, X, h).$$

We call it the *Y-orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

The idea of using subalgebras of this type is due to Putnam [60]. We have used a different convention from that used most other places, where one usually takes

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, uC_0(X \setminus Y) \big).$$

The choice of convention in Definition 1.7 has the advantage that, when used in connection with Rokhlin towers, the bases of the towers are subsets of Y rather than of $h(Y)$.

Theorem 1.8. Let X be an infinite compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 1.3.

We give a proof in Section 4, along with proofs or sketches of proofs of the lemmas which go into the proof.

The key fact about $C^*(\mathbb{Z}, X, h)_Y$ which makes this theorem useful is that it is a direct limit of recursive subhomogeneous C*-algebras (as in Definition 1.1 of [53]) whose base spaces are closed subsets of X. The structure of $C^*(\mathbb{Z}, X, h)_Y$ is therefore much more accessible than the structure of crossed products. (We give a slightly expanded discussion in Section 4.)

1.2. **Theorems and applications.** We state the main known results about large subalgebras and some recent applications.

Proposition 1.9 (Proposition 5.2 and Proposition 5.5 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple and infinite dimensional.

In the next section, we prove the simplicity statement (see Proposition 2.7 below) and the stably finite case of the infinite dimensionality statement (see Proposition 2.10 below).

Theorem 1.10 (Theorem 6.2 and Proposition 6.9 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$, on traces and quasitraces, are bijective.

The proofs of the two parts are quite different. We prove that $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective below (Theorem 2.12).

Let A be a C*-algebra. The Cuntz semigroup $\mathrm{Cu}(A)$ is defined below (Definition 1.23(3)). Let $\mathrm{Cu}_{+}(A)$ denote the set of elements $\eta \in \mathrm{Cu}(A)$ which are not the classes of projections. (Such elements are sometimes called *purely positive*.)

Theorem 1.11 (Theorem 6.8 of [58])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\iota \colon B \to A$ be the inclusion map. Then $\mathrm{Cu}(\iota)$ defines an order and semigroup isomorphism from $\mathrm{Cu}_{+}(B) \cup \{0\}$ to $\mathrm{Cu}_{+}(A) \cup \{0\}$.

It is not true that $\mathrm{Cu}(\iota)$ defines an isomorphism from $\mathrm{Cu}(B)$ to $\mathrm{Cu}(A)$. Example 7.13 of [58] shows that $\mathrm{Cu}(\iota) \colon \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ need not be injective. We suppose this map can also fail to be surjective, but we don't know an example.

Theorem 1.12 (Theorem 6.14 of [58])**.** Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be a large subalgebra. Let $\mathrm{rc}(-)$ be the radius of comparison (Definition 3.1 below). Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

We will prove this result in Section 3 when A is exact. See Theorem 3.2 below.

Proposition 1.13 (Proposition 6.15, Corollary 6.16, and Proposition 6.17 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then:

(1) A is finite if and only if B is finite.
(2) If B is stably large in A, then A is stably finite if and only if B is stably finite.
(3) A is purely infinite if and only if B is purely infinite.

Proposition 1.14 (Theorem 6.18 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Suppose that B has property (SP). Then A has property (SP).

Theorem 1.15 (Theorem 6.3 and Theorem 6.4 of [5])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Then:

(1) If $\mathrm{tsr}(B) = 1$ then $\mathrm{tsr}(A) = 1$.
(2) If $\mathrm{tsr}(B) = 1$ and $\mathrm{RR}(B) = 0$, then $\mathrm{RR}(A) = 0$.

Theorem 1.16 (Theorem 2.3 of [3])**.** Let A be an infinite dimensional simple nuclear unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. If B tensorially absorbs the Jiang-Su algebra Z, then so does A.

If A isn't nuclear, the best we can say so far is that A is tracially Z-absorbing in the sense of Definition 2.1 of [29].

The following two key technical results are behind many of the theorems stated above. In particular, they are the basis for proving Theorem 1.11, which is used to prove many of the other results.

Lemma 1.17 (Lemmas 6.3 and 6.5 of [58]). Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a stably large subalgebra.

(1) Let $a, b, x \in (K \otimes A)_{+}$ satisfy $x \neq 0$ and $a \oplus x \precsim_A b$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$, $c \in (M_n \otimes B)_{+}$, and $\delta > 0$ such that $(a - \varepsilon)_{+} \precsim_A c \precsim_A (b - \delta)_{+}$.
(2) Let $a, b \in (K \otimes B)_{+}$ and $c, x \in (K \otimes A)_{+}$ satisfy $x \neq 0$, $a \precsim_A c$, and $c \oplus x \precsim_A b$. Then $a \precsim_B b$.

We state some of the applications. In the following theorem, $\mathrm{rc}(A)$ is the radius of comparison of A (see Definition 3.1 below), and $\mathrm{mdim}(h)$ is the mean dimension of h (see Definition 5.8 below).

Theorem 1.18 ([28]). Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

It is conjectured that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for all minimal homeomorphisms. In [28], we also prove that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \geq \frac{1}{2}\mathrm{mdim}(h)$ for a reasonably large class of homomorphisms constructed using the methods of Giol and Kerr [24], including the ones in that paper. For all minimal homeomorphisms of this type, there is a continuous surjective map from the space to the Cantor set.

The proof of Theorem 1.18 uses Theorem 1.8, Theorem 1.12, the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ be the direct limit of an AH system with diagonal maps, and methods of [46] (see especially Theorem 6.2 there) to estimate radius of comparison of simple direct limits of AH systems with diagonal maps. We would like to use Theorem 6.2 of [46] directly. Unfortunately, the definition of mean dimension of an AH direct system in [46] requires that the base spaces be connected. See Definition 3.6 of [46], which refers to the setup described after Lemma 3.4 of [46].

Theorem 1.19. Let X be a compact metric space. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq 1 + 2 \cdot \mathrm{mdim}(h)$.

Corollary 4.8 of [59] states that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq 1 + 36 \cdot \mathrm{mdim}(h)$. A key ingredient is Theorem 5.1 of [40], an embedding result for minimal homeomorphisms in shifts on cubes, the dimension of the cube depending on the mean dimension of the homeomorphism. The improvement, to appear in a revised version of [59], is based on the use of a stronger embedding result for minimal dynamical systems, Theorem 1.4 of [26]. We really want $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$, as in Theorem 1.18.

Theorem 1.20 (Theorem 7.1 of [5]). Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one.

There is no finite dimensionality assumption on X. We don't even assume that h has mean dimension zero. In particular, this theorem holds for the examples of Giol and Kerr [24], for which the crossed products are known not to be Z-stable and not to have strict comparison of positive elements. (For such systems, it is shown in [28] that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$, and in [24] that $\mathrm{mdim}(h) \neq 0$. See the discussion in Section 7 of [5] for details.)

The proof uses Theorem 1.8, Theorem 1.15(1), the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ be the direct limit of an AH system with diagonal maps, and Theorem 4.1 of [18], according to which simple direct limits of AH systems with diagonal maps always have stable rank one, without any dimension growth hypotheses.

Theorem 1.21 (Elliott and Niu [20])**.** The "extended" irrational rotation algebras, obtained by "cutting" each of the standard unitary generators at one or more points in its spectrum, are AF algebras.

We omit the precise descriptions of these algebras.

If one cuts just one of the generators, the resulting algebra is a crossed product by a minimal homeomorphism of the Cantor set, with the other unitary playing the role of the image of a generator of the group $\mathbb{Z}$. If both are cut, the algebra is no longer an obvious crossed product.

In the next theorem, Z is the Jiang-Su algebra. Being Z-stable is one of the regularity conditions in the Toms-Winter conjecture, and for simple separable nuclear C*-algebras it is hoped, and known in many cases, that Z-stability implies classifiability in the sense of the Elliott program.

Theorem 1.22 (Elliott and Niu [21])**.** Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is Z-stable.

1.3. **Cuntz comparison.** We give a summary of Cuntz comparison and a few facts about the Cuntz semigroup of a C*-algebra. We refer to [1] for an extensive introduction (which does not include all the results that we need). The material we need is either summarized or proved in the first two sections of [58].

Let $M_\infty(A)$ denote the algebraic direct limit of the system $(M_n(A))_{n=1}^{\infty}$ using the usual embeddings $M_n(A) \to M_{n+1}(A)$, given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a \in M_m(A)$ and $b \in M_n(A)$, we write $a \oplus b$ for the diagonal direct sum

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

By abuse of notation, we will also write $a \oplus b$ when $a, b \in M_\infty(A)$ and we do not care about the precise choice of m and n with $a \in M_m(A)$ and $b \in M_n(A)$.

The main object of study in these notes is how comparison in the Cuntz semigroup of a C*-algebra A relates to comparison in the Cuntz semigroup of a large subalgebra B. We therefore include the algebra in the notation for Cuntz comparison and classes in the Cuntz semigroup.

Parts (1) and (2) of the following definition are originally from [13]. (Part (1) is a restatement of Definition 1.1.)

Definition 1.23. Let A be a C*-algebra.

(1) For $a, b \in (K \otimes A)_+$, we say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n\to\infty} v_n b v_n^* = a$. This relation is transitive: $a \precsim_A b$ and $b \precsim_A c$ imply $a \precsim_A c$.

(2) We say that a and b are *Cuntz equivalent over A*, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$. This relation is an equivalence relation, and we write $\langle a \rangle_A$ for the equivalence class of a.

(3) The *Cuntz semigroup* of A is

$$\mathrm{Cu}(A) = (K \otimes A)_+ / \sim_A,$$

together with the commutative semigroup operation, gotten from an isomorphism $M_2(K) \to K$,

$$\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A$$

(the class does not depend on the choice of the isomorphism) and the partial order

$$\langle a \rangle_A \leq \langle b \rangle_A \Longleftrightarrow a \precsim_A b.$$

It is taken to be an object of the category **Cu** given in Definition 4.1 of [1]. We write 0 for $\langle 0 \rangle_A$.

(4) We also define the subsemigroup

$$W(A) = M_\infty(A)_+ / \sim_A,$$

with the same operations and order. (It will follow from Remark 1.24 that the obvious map $W(A) \to \mathrm{Cu}(A)$ is injective.)

(5) Let A and B be C*-algebras, and let $\varphi\colon A \to B$ be a homomorphism. We use the same letter for the induced maps $M_n(A) \to M_n(B)$ for $n \in \mathbb{Z}_{>0}$, $M_\infty(A) \to M_\infty(B)$, and $K \otimes A \to K \otimes B$. We define $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ and $W(\varphi)\colon W(A) \to W(B)$ by $\langle a \rangle_A \mapsto \langle \varphi(a) \rangle_B$ for $a \in (K \otimes A)_+$ or $M_\infty(A)_+$ as appropriate.

It is easy to check that the maps $\mathrm{Cu}(\varphi)$ and $W(\varphi)$ are well defined homomorphisms of ordered semigroups which send 0 to 0. Also, it follows from Lemma 1.26(14) below that if $\eta_1, \eta_2, \mu_1, \mu_2 \in \mathrm{Cu}(A)$ satisfy $\eta_1 \leq \mu_1$ and $\eta_2 \leq \mu_2$, then $\eta_1 + \eta_2 \leq \mu_1 + \mu_2$.

The semigroup $\mathrm{Cu}(A)$ generally has better properties than $W(A)$. For example, certain supremums exist (Theorem 4.19 of [1]), and, when understood as an object of the category **Cu**, it behaves properly with respect to direct limits (Theorem 4.35 of [1]). In this exposition, we mainly use $W(A)$ because, when A is unital, the dimension function d_τ associated to a normalized quasitrace τ (Definition 1.29 below) is finite on $W(A)$ but usually not on $\mathrm{Cu}(A)$. In particular, the radius of comparison (Definition 3.1 below) is easier to deal with in terms of $W(A)$.

We will not need the definition of the category **Cu**.

Remark 1.24. We make the usual identifications

$$A \subset M_n(A) \subset M_\infty(A) \subset K \otimes A. \tag{1.2}$$

It is easy to check, by cutting down to corners, that if $a, b \in (K \otimes A)_+$ satisfy $a \precsim_A b$, then the sequence $(v_n)_{n=1}^\infty$ such that $\lim_{n \to \infty} v_n b v_n^* = a$ (as in Definition 1.23(1)) can be taken to be in the smallest of the algebras in (1.2) which contains both a and b. See Remark 1.2 of [58] for details.

The Cuntz semigroup of a separable C*-algebra can be very roughly thought of as K-theory using open projections in matrices over A'', that is, open supports of positive elements in matrices over A, instead of projections in matrices over A. As justification for this heuristic, we note that if X is a compact Hausdorff space and $f, g \in C(X)_+$, then $f \precsim_{C(X)} g$ if and only if

$$\bigl\{ x \in X \colon f(x) > 0 \bigr\} \subset \bigl\{ x \in X \colon g(x) > 0 \bigr\}.$$

A version of this can be made rigorous, at least in the separable case. See [48].

There is a description of $\mathrm{Cu}(A)$ using Hilbert modules over A in place of finitely generated projective modules. See [11].

Unlike K-theory, the Cuntz semigroup is not discrete. If $p, q \in A$ are projections such that $\|p - q\| < 1$, then p and q are Murray-von Neumann equivalent. However, for $a, b \in A_{+}$, the relation $\|a - b\| < \varepsilon$ says nothing about the classes of a and b in $\mathrm{Cu}(A)$ or $W(A)$, however small $\varepsilon > 0$ is. We can see this in $\mathrm{Cu}(C(X))$. Even if $\big\{ x \in X \colon g(x) > 0 \big\}$ is a very small subset of X, for every $\varepsilon > 0$ the function $f = g + \frac{\varepsilon}{2}$ has $\langle f \rangle_{C(X)} = \langle 1 \rangle_{C(X)}$. What is true when $\|f - g\| < \varepsilon$ is that

$$\big\{ x \in X \colon f(x) > \varepsilon \big\} \subset \big\{ x \in X \colon g(x) > 0 \big\},$$

so that the function $\max(f - \varepsilon, \, 0)$ satisfies $\max(f - \varepsilon, \, 0) \precsim_{C(X)} g$. This motivates the systematic use of the elements $(a - \varepsilon)_{+}$, defined as follows.

Definition 1.25. Let A be a C*-algebra, let $a \in A_{+}$, and let $\varepsilon \geq 0$. Let $f \colon [0, \infty) \to [0, \infty)$ be the function

$$f(\lambda) = (\lambda - \varepsilon)_{+} = \begin{cases} 0 & 0 \leq \lambda \leq \varepsilon \\ \lambda - \varepsilon & \varepsilon < \lambda. \end{cases}$$

Then define $(a - \varepsilon)_{+} = f(a)$ (using continuous functional calculus).

One must still be much more careful than with K-theory. First, $a \leq b$ does not imply $(a - \varepsilon)_{+} \leq (b - \varepsilon)_{+}$ (although one does get $(a - \varepsilon)_{+} \precsim_{A} (b - \varepsilon)_{+}$; see Lemma 1.26(17) below). Second, $a \precsim_{A} b$ does not imply any relation between $(a - \varepsilon)_{+}$ and $(b - \varepsilon)_{+}$. For example, if $A = C([0, 1])$ and $a \in C([0, 1])$ is $a(t) = t$ for $t \in [0, 1]$, then for any $\varepsilon \in (0, 1)$ the element $b = \varepsilon a$ satisfies $a \precsim_{A} b$. But $(a - \varepsilon)_{+} \not\precsim_{A} (b - \varepsilon)_{+}$, since $(a - \varepsilon)_{+}$ has open support $(\varepsilon, 1]$ while $(b - \varepsilon)_{+} = 0$. The best one can do is in Lemma 1.26(11) below.

We now list a collection of basic results about Cuntz comparison and the Cuntz semigroup. There are very few such results about projections and the K_0-group, the main ones being that if $\|p - q\| < 1$, then p and q are Murray-von Neumann equivalent; that $p \leq q$ if and only if $pq = p$; the relations between homotopy, unitary equivalence, and Murray-von Neumann equivalence; and the fact that addition of equivalence classes respects orthogonal sums. There are many more for Cuntz comparison. We will not use all the facts listed below in these notes (although they are all used in [58]); we include them all so as to give a fuller picture of Cuntz comparison.

Parts (1) through (14) of Lemma 1.26 are mostly taken from [33], with some from [12], [22], [50], and [65], and are summarized in Lemma 1.4 of [58]; we refer to [58] for more on the attributions (although not all the attributions there are to the original sources). Part (15) is Lemma 1.5 of [58]; part (16) is Corollary 1.6 of [58]; part (17) is Lemma 1.7 of [58]; and part (18) is Lemma 1.9 of [58].

We denote by A^{+} the unitization of a C*-algebra A. (We add a new unit even if A is already unital.)

Lemma 1.26. Let A be a C*-algebra.

(1) Let $a, b \in A_{+}$. Suppose $a \in \overline{bAb}$. Then $a \precsim_{A} b$.
(2) Let $a \in A_{+}$ and let $f \colon [0, \|a\|] \to [0, \infty)$ be a continuous function such that $f(0) = 0$. Then $f(a) \precsim_{A} a$.
(3) Let $a \in A_{+}$ and let $f \colon [0, \|a\|] \to [0, \infty)$ be a continuous function such that $f(0) = 0$ and $f(\lambda) > 0$ for $\lambda > 0$. Then $f(a) \sim_{A} a$.
(4) Let $c \in A$. Then $c^{*} c \sim_{A} c c^{*}$.
(5) Let $a \in A_{+}$, and let $u \in A^{+}$ be unitary. Then $u a u^{*} \sim_{A} a$.

(6) Let $c \in A$ and let $\alpha > 0$. Then $(c^*c - \alpha)_+ \sim_A (cc^* - \alpha)_+$.
(7) Let $v \in A$. Then there is an isomorphism $\varphi\colon \overline{v^*vAv^*v} \to \overline{vv^*Avv^*}$ such that, for every positive element $z \in \overline{v^*vAv^*v}$, we have $z \sim_A \varphi(z)$.
(8) Let $a \in A_+$ and let $\varepsilon_1, \varepsilon_2 > 0$. Then

$$\big((a - \varepsilon_1)_+ - \varepsilon_2\big)_+ = \big(a - (\varepsilon_1 + \varepsilon_2)\big)_+.$$

(9) Let $a, b \in A_+$ satisfy $a \precsim_A b$ and let $\delta > 0$. Then there is $v \in A$ such that $v^*v = (a - \delta)_+$ and $vv^* \in \overline{bAb}$.
(10) Let $a, b \in A_+$. Then $\|a - b\| < \varepsilon$ implies $(a - \varepsilon)_+ \precsim_A b$.
(11) Let $a, b \in A_+$. Then the following are equivalent:
 (a) $a \precsim_A b$.
 (b) $(a - \varepsilon)_+ \precsim_A b$ for all $\varepsilon > 0$.
 (c) For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A (b - \delta)_+$.
 (d) For every $\varepsilon > 0$ there are $\delta > 0$ and $v \in A$ such that

$$(a - \varepsilon)_+ = v[(b - \delta)_+]v^*.$$

(12) Let $a, b \in A_+$. Then $a + b \precsim_A a \oplus b$.
(13) Let $a, b \in A_+$ be orthogonal (that is, $ab = 0$). Then $a + b \sim_A a \oplus b$.
(14) Let $a_1, a_2, b_1, b_2 \in A_+$, and suppose that $a_1 \precsim_A a_2$ and $b_1 \precsim_A b_2$. Then $a_1 \oplus b_1 \precsim_A a_2 \oplus b_2$.
(15) Let $a, b \in A$ be positive, and let $\alpha, \beta \geq 0$. Then

$$\big((a + b - (\alpha + \beta)\big)_+ \precsim_A (a - \alpha)_+ + (b - \beta)_+ \precsim_A (a - \alpha)_+ \oplus (b - \beta)_+.$$

(16) Let $\varepsilon > 0$ and $\lambda \geq 0$. Let $a, b \in A$ satisfy $\|a - b\| < \varepsilon$. Then $(a - \lambda - \varepsilon)_+ \precsim_A (b - \lambda)_+$.
(17) Let $a, b \in A$ satisfy $0 \leq a \leq b$. Let $\varepsilon > 0$. Then $(a - \varepsilon)_+ \precsim_A (b - \varepsilon)_+$.
(18) Let $a \in (K \otimes A)_+$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$ and $b \in (M_n \otimes A)_+$ such that $(a - \varepsilon)_+ \sim_A b$.

The following result is sufficiently closely tied to the ideas behind large subalgebras that we include the proof.

Lemma 1.27 (Lemma 1.8 of [58])**.** Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then

$$(a - \varepsilon)_+ \precsim_A \big[(1 - g)a(1 - g) - \varepsilon\big]_+ \oplus g.$$

Proof. Set $h = 2g - g^2$, so that $(1 - g)^2 = 1 - h$. We claim that $h \sim_A g$. Since $0 \leq g \leq 1$, this follows from Lemma 1.26(3), using the continuous function $\lambda \mapsto 2\lambda - \lambda^2$ on $[0, 1]$.

Set $b = \big[(1 - g)a(1 - g) - \varepsilon\big]_+$. Using Lemma 1.26(15) at the second step, Lemma 1.26(6) and Lemma 1.26(4) at the third step, and Lemma 1.26(14) at the last step, we get

$$\begin{aligned}
(a - \varepsilon)_+ &= \big[a^{1/2}(1 - h)a^{1/2} + a^{1/2}ha^{1/2} - \varepsilon\big]_+ \\
&\precsim_A \big[a^{1/2}(1 - h)a^{1/2} - \varepsilon\big]_+ \oplus a^{1/2}ha^{1/2} \\
&\sim_A \big[(1 - g)a(1 - g) - \varepsilon\big]_+ \oplus h^{1/2}ah^{1/2} \\
&= b \oplus h^{1/2}ah^{1/2} \leq b \oplus \|a\|h \precsim_A b \oplus g.
\end{aligned}$$

This completes the proof. $\square$

Notation 1.28. For a unital C*-algebra A, we denote by $\mathrm{T}(A)$ the set of tracial states on A. We denote by $\mathrm{QT}(A)$ the set of normalized 2-quasitraces on A (Definition II.1.1 of [7]; Definition 2.31 of [1]).

Definition 1.29. Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$ for $a \in M_\infty(A)_+$. Further (the use of the same notation should cause no confusion) define $d_\tau \colon (K \otimes A)_+ \to [0, \infty]$ by the same formula, but now for $a \in (K \otimes A)_+$. We also use the same notation for the corresponding functions on $\mathrm{Cu}(A)$ and $W(A)$, as in Proposition 1.30 below.

Proposition 1.30. Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Then d_τ as in Definition 1.29 is well defined on $\mathrm{Cu}(A)$ and $W(A)$. That is, if $a, b \in (K \otimes A)_+$ satisfy $a \sim_A b$, then $d_\tau(a) = d_\tau(b)$.

Proof. This is part of Proposition 4.2 of [22]. □

Also see the beginning of Section 2.6 of [1], especially the proof of Theorem 2.32 there. It follows that d_τ defines a state on $W(A)$. Thus (see Theorem II.2.2 of [7], which gives the corresponding bijection between 2-quasitraces and dimension functions which are not necessarily normalized but are finite everywhere), the map $\tau \mapsto d_\tau$ is a bijection from $\mathrm{QT}(A)$ to the set of lower semicontinuous dimension functions on A.

1.4. **Cuntz comparison in simple C*-algebras.** We present some results related to Cuntz comparison specifically for simple C*-algebras.

Lemma 1.31 (Proposition 4.10 of [33])**.** Let A be a C*-algebra which is not of type I and let $n \in \mathbb{Z}_{>0}$. Then there exists an injective homomorphism from the cone CM_n over M_n to A.

The proof uses heavy machinery, namely Glimm's result that there is a subalgebra $B \subset A$ and an ideal $I \subset B$ such that the 2^∞ UHF algebra embeds in B/I. Some of what we use this result for can be proved by more elementary methods, but for Lemma 1.35 we don't know such a proof.

Lemma 1.32 (Lemma 2.1 of [58])**.** Let A be a simple C*-algebra which is not of type I. Let $a \in A_+ \setminus \{0\}$, and let $l \in \mathbb{Z}_{>0}$. Then there exist $b_1, b_2, \ldots, b_l \in A_+ \setminus \{0\}$ such that $b_1 \sim_A b_2 \sim_A \cdots \sim_A b_l$, such that $b_j b_k = 0$ for $j \neq k$, and such that $b_1 + b_2 + \cdots + b_l \in \overline{aAa}$.

Proof. Replacing A by $\overline{aAa}$, we can ignore the requirement $b_1 + b_2 + \cdots + b_l \in \overline{aAa}$ of the conclusion. Now fix $n \in \mathbb{Z}_{>0}$. For $j, k = 1, 2, \ldots, n$, we let $e_{j,k} \in M_n$ be the standard matrix unit. In

$$CM_n = \big\{ f \in C([0,1], \, M_n) \colon f(0) = 0 \big\},$$

take $b_j(\lambda) = \lambda e_{j,j}$ for $\lambda \in [0,1]$ and $j = 1, 2, \ldots, n$. Use Lemma 1.31 to embed CM_n in A. □

This lemma has the following corollary.

Corollary 1.33 (Corollary 2.2 of [58])**.** Let A be a simple unital infinite dimensional C*-algebra. Then for every $\varepsilon > 0$ there is $a \in A_+ \setminus \{0\}$ such that for all $\tau \in \mathrm{QT}(A)$ we have $d_\tau(a) < \varepsilon$.

Lemma 1.34 (Lemma 2.4 of [58])**.** Let A be a simple C*-algebra, and let $B \subset A$ be a nonzero hereditary subalgebra. Let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A_+ \setminus \{0\}$. Then there exists $b \in B_+ \setminus \{0\}$ such that $b \precsim_A a_j$ for $j = 1, 2, \ldots, n$.

Sketch of proof. The proof is by induction. The case $n = 0$ is trivial. The induction step requires that for $a, b_0 \in A_+ \setminus \{0\}$ one find $b \in A_+ \setminus \{0\}$ such that $b \in \overline{b_0 A b_0}$ (so that $b \precsim_A b_0$ by Lemma 1.26(1)) and $b \precsim_A a$. Use simplicity to find $x \in A$ such that the element $y = b_0 x a$ is nonzero, and take $b = y y^* \in \overline{b_0 A b_0}$. Using Lemma 1.26(5) and Lemma 1.26(1), we get $b \sim_A y^* y \precsim_A a$. $\square$

The following lemma says, roughly, that a nonzero element of $W(A)$ can be approximated arbitrarily well by elements of $W(A)$ which are strictly smaller.

Lemma 1.35 (Lemma 2.3 of [58])**.** Let A be a simple infinite dimensional C*-algebra which is not of type I. Let $b \in A_+ \setminus \{0\}$, let $\varepsilon > 0$, and let $n \in \mathbb{Z}_{>0}$. Then there are $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that, in $W(A)$, we have

$$n \langle (b - \varepsilon)_+ \rangle_A \leq (n + 1) \langle c \rangle_A \quad \text{and} \quad \langle c \rangle_A + \langle y \rangle_A \leq \langle b \rangle_A.$$

Sketch of proof. We divide the proof into two cases. First assume that $\mathrm{sp}(b) \cap (0, \varepsilon) \neq \varnothing$. Then there is a continuous function $f \colon [0, \infty) \to [0, \infty)$ which is zero on $\{0\} \cup [\varepsilon, \infty)$ and such that $f(b) \neq 0$. We take $c = (b - \varepsilon)_+$ and $y = f(b)$.

Now suppose that $\mathrm{sp}(b) \cap (0, \varepsilon) = \varnothing$. In this case, we might as well assume that b is a projection, and that $\langle (b - \varepsilon)_+ \rangle_A$, which is always dominated by $\langle b \rangle_A$, is equal to $\langle b \rangle_A$. Cutting down by b, we can assume that $b = 1$ (in particular, A is unital), and it is enough to find $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that $n \langle 1 \rangle_A \leq (n + 1) \langle c \rangle_A$ and $\langle c \rangle_A + \langle y \rangle_A \leq \langle 1 \rangle_A$.

Take the unitized cone over M_{n+1} to be $C = (C M_{n+1})^+ = [C_0((0, 1]) \otimes M_{n+1}]^+$, and use the usual notation for matrix units. By Lemma 1.31, we can assume that $C \subset A$. Let $t \in C_0((0, 1])$ be the function $t(\lambda) = \lambda$ for $\lambda \in (0, 1]$. Choose continuous functions $g_1, g_2, g_3 \in C([0, 1])$ such that

$$0 \leq g_3 \leq g_2 \leq g_1 \leq 1, \quad g_1(0) = 0, \quad g_3(1) = 1, \quad g_1 g_2 = g_2, \quad \text{and} \quad g_2 g_3 = g_3.$$

Define

$$x = g_2 \otimes e_{1,1}, \quad c = 1 - x, \quad \text{and} \quad y = g_3 \otimes e_{1,1}.$$

Then $x y = y$ so $c y = 0$. It follows from Lemma 1.26(13) that $\langle c \rangle_A + \langle y \rangle_A \leq \langle 1 \rangle_A$.

It remains to prove that $n \langle 1 \rangle_A \leq (n + 1) \langle c \rangle_A$, and it is enough to prove that in $W(C)$ we have $n \langle 1 \rangle_C \leq (n + 1) \langle 1 - g_2 \otimes e_{1,1} \rangle_C$, that is, in $M_{n+1}(C)$,

$$\mathrm{diag}(1, 1, \ldots, 1, 0) \precsim_C \mathrm{diag}\big(1 - g_2 \otimes e_{1,1}, \, 1 - g_2 \otimes e_{1,1}, \, \ldots, \, 1 - g_2 \otimes e_{1,1}\big). \tag{1.3}$$

To see why this should be true, view $M_{n+1}(C)$ as a set of functions from $[0, 1]$ to $M_{(n+1)^2}$ with restrictions on the value at zero. Since $g_1 g_2 = g_2$, the function $1 - g_2 \otimes e_{1,1}$ is constant equal to 1 on a neighborhood U of 0, and at $\lambda \in U$ the right hand side of (1.3) therefore dominates the left hand side. Elsewhere, both sides of (1.3) are diagonal, with the right hand side being a constant projection of rank $n(n + 1)$ and the left hand side dominating

$$\mathrm{diag}\big(1 - e_{1,1}, \, 1 - e_{1,1}, \, \ldots, \, 1 - e_{1,1}\big),$$

which is a (different) constant projection of rank $n(n+1)$. It is not hard to construct an explicit formula for a unitary $v \in M_{n+1}(C)$ such that

$$\mathrm{diag}(1, 1, \ldots, 1, 0) \leq v \cdot \mathrm{diag}\big(1 - g_2 \otimes e_{1,1}, \, 1 - g_2 \otimes e_{1,1}, \, \ldots, \, 1 - g_2 \otimes e_{1,1}\big) \cdot v^*.$$

See [58] for the details (arranged a little differently). □

1.5. **Open problems.** We discuss some open problems. We start with some which are motivated by particular applications, and then give some which are suggested by results already proved but for which we don't have immediate applications.

The first question is motivated by the hope that large subalgebras can be used to get more information about crossed products than we now know how to get. In most parts, we expect that positive answers would require special hypotheses, if they can be gotten at all.

Question 1.36. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra.

(1) Suppose that B has tracial rank zero. Does it follow that A has tracial rank zero?
(2) Suppose that B is quasidiagonal. Does it follow that A is quasidiagonal?
(3) Suppose that B has finite decomposition rank. Does it follow that A has finite decomposition rank?
(4) Suppose that B has finite nuclear dimension. Does it follow that A has finite nuclear dimension?

It seems likely that "tracial" versions of these properties pass from a large subalgebra to the containing algebra, at least if the tracial versions are defined using cutdowns by positive elements rather than by projections. But we don't know how useful such properties are. As far as we know, they have not been studied.

Next, we ask whether being stably large is automatic.

Question 1.37. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra. Does it follow that $M_n(B)$ is large (or centrally large) in $M_n(A)$ for $n \in \mathbb{Z}_{>0}$?

We know that this is true if A is stably finite. (See Proposition 2.11 below.) Not having the general statement is a technical annoyance. This result would be helpful when dealing with large subalgebras of $C^*(\mathbb{Z}, C(X, D), \alpha)$ when D is simple unital, X is compact metric, and the homeomorphism of $\mathrm{Prim}(C(X, D)) \cong X$ induced by α is minimal. Some results on large subalgebras of such crossed products can be found in [4]; also see Theorem 4.5.

More generally, does Proposition 1.5 still hold without the finiteness assumption?

Question 1.38. Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Is there a useful large or centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$?

We want a centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$ which "locally looks like matrices over corners of A". The paper [49] proves that crossed products by automorphisms with the tracial Rokhlin property preserve the combination of real rank zero, stable rank one, and order on projections determined by traces. The methods were inspired by those of [55], which used large subalgebras (without the name). The proof in [49] does not, however, construct a single large subalgebra. Instead, it constructs a suitable subalgebra (analogous to $C^*(\mathbb{Z}, X, h)_Y$ for a small closed subset $Y \subset X$ with $\mathrm{int}(Y) \neq \varnothing$) for every choice of finite set $F \subset C^*(\mathbb{Z}, A, \alpha)$ and every choice of $\varepsilon > 0$. It is far from clear how to choose these subalgebras to form an increasing sequence so that a direct limit can be built. Similar ideas, under weaker hypotheses (without projections), are used in [47], and there also is far from clear how to choose the subalgebras to form an increasing sequence.

The first intended application is simplification of [49].

Problem 1.39. Let X be a compact metric space, and let G be a countable amenable group which acts minimally and essentially freely on X. Construct a (centrally) large subalgebra $B \subset C^*(G, X)$ which is a direct limit of recursive subhomogeneous C*-algebras as in [53] whose base spaces are closed subsets of X, and which is the (reduced) C*-algebra of an open subgroupoid of the transformation group groupoid obtained from the action of G on X.

In a precursor to the theory of large subalgebras, this is in effect done in [55] when $G = \mathbb{Z}^d$ and X is the Cantor set, following ideas of [23]. The resulting centrally large subalgebra is used in [55] to prove that $C^*(\mathbb{Z}^d, X)$ has stable rank one, real rank zero, and order on projections determined by traces. (More is now known.) We also know how to construct a centrally large subalgebra of this kind when $G = \mathbb{Z}^d$ and X is finite dimensional (unpublished). This gave the first proof that, in this case, $C^*(\mathbb{Z}^d, X)$ has stable rank one and strict comparison of positive elements. (Again, more is now known.)

Unlike for actions of $\mathbb{Z}$, there are no known explicit formulas like that in Theorem 1.8; instead, centrally large subalgebras must be proved to exist via constructions involving many choices. They are direct limits of C*-algebras of open subgroupoids of the transformation group groupoid as in Problem 1.39. In each open subgroupoid, there is a finite upper bound on the size of the orbits; this is why they are recursive subhomogeneous C*-algebras (homogeneous when X is the Cantor set, as in [54]). In fact, the original motivation for the definition of a large subalgebras was to describe the essential properties of these subalgebras, as a substitute for an explicit description.

We presume, as suggested in Problem 1.39, that the construction can be done in much greater generality.

Problem 1.40. Develop the theory of large subalgebras of not necessarily simple C*-algebras.

One can't just copy Definition 1.2. Suppose B is a nontrivial large subalgebra of A. We surely want $B \oplus B$ to be a large subalgebra of $A \oplus A$. Take $x_0 \in A_+ \setminus \{0\}$, and take the element $x \in A \oplus A$ in Definition 1.2 to be $x = (x_0, 0)$. Writing $g = (g_1, g_2)$, we have forced $g_2 = 0$. Thus, not only would $B \oplus B$ not be large in $A \oplus A$, but even $A \oplus B$ would not be large in $A \oplus A$.

In this particular case, the solution is to require that x and y be full elements in A and B. What to do is much less clear if, for example, A is a unital extension of the form

$$0 \longrightarrow K \otimes D \longrightarrow A \longrightarrow E \longrightarrow 0,$$

even if D and E are simple, to say nothing of the general case.

The following problem goes just a small step away from the simple case, and just asking that x and y be full might possibly work for it, although stronger hypotheses may be necessary.

Question 1.41. Let X be an infinite compact metric space and let $h\colon X \to X$ be a homeomorphism which has a factor system which is a minimal homeomorphism of an infinite compact metric space (or, stronger, a minimal homeomorphism of the

Cantor set). Can one use large subalgebra methods to relate the mean dimension of h to the radius of comparison of $C^*(\mathbb{Z}, X, h)$?

We point out that Lindenstrauss's embedding result for systems of finite mean dimension in shifts built from finite dimensional spaces (Theorem 5.1 of [40]) is proved for homeomorphisms having a factor system which is a minimal homeomorphism of an infinite compact metric space.

Problem 1.42. Develop the theory of large subalgebras of simple but not necessarily unital C*-algebras.

One intended application is to crossed products $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)$ when X is an infinite compact metric space, D is simple but not unital, and the induced action on X is given by a minimal homeomorphism. (Compare with Theorem 4.5.) Another possible application is to the structure of crossed products $C^*(\mathbb{Z}, X, h)$ when h is a minimal homeomorphism of a noncompact version of the Cantor set. Minimal homeomorphisms of noncompact Cantor sets have been studied in [43] and [44], but, as far as we know, almost nothing is known about their transformation group C*-algebras.

Can the technique of large subalgebras be adapted to L^p operator crossed products, as in [57]? For example, consider the following question.

Question 1.43. Let $p \in [1, \infty) \setminus \{2\}$. Let X be an infinite compact metric space and let $h \colon X \to X$ be a minimal homeomorphism. Suppose that $\dim(X)$ is finite, or that X has a surjective continuous map to the Cantor set. Does it follow that the L^p operator crossed product $F^p(\mathbb{Z}, X, h)$ has stable rank one?

It is at least known (Theorem 5.6 of [57]) that $F^p(\mathbb{Z}, X, h)$ is simple.

The answer to Question 1.43 is unknown even when X is the Cantor set. Putnam's original argument (Section 2 of [61]) depends on continuous deformation of unitaries in $M_n(\mathbb{C})$. The analogous construction for isometries in the L^p version of $M_n(\mathbb{C})$ is not possible. (The only isometries are products of permutation matrices and diagonal isometries.) It seems likely that the machinery will fail if one must replace isometries with more general invertible elements. It also seems likely that a theory of L^p AH algebras will only work if the maps in the direct system are assumed to be diagonal (in a sense related to that at the beginning of Section 2.2 of [18]), and that $F^p(\mathbb{Z}, X, h)$ is unlikely to be such a direct limit even when X is the Cantor set. On the other hand, in many cases $C^*(\mathbb{Z}, X, h)$ has a large subalgebra $C^*(\mathbb{Z}, X, h)_Y$ which is AH with diagonal maps. See [28]; a special case is stated (without proof) as Lemma 5.23 below. The idea for Question 1.43 is to adapt C* proofs to show that an L^p AH algebra with no dimension growth and diagonal maps has stable rank one (possibly, following [18], even without assumptions on dimension growth), and to prove that if $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$ then the algebra one might call $F^p(\mathbb{Z}, X, h)_Y$ is large in $F^p(\mathbb{Z}, X, h)$ in a suitable sense. It isn't clear what the abstract definition of a centrally large subalgebra of an L^p operator algebra should be, since we don't know anything about an L^p analog of Cuntz comparison, but possibly one can work with the explicitly given inclusion $F^p(\mathbb{Z}, X, h)_Y \subset F^p(\mathbb{Z}, X, h)$.

For a large subalgebra $B \subset A$, the proofs of most of the relations between A and B do not need B to be centrally large. The exceptions so far are for stable rank one and Z-stability. Do we really need centrally large for these results?

Question 1.44. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra (not necessarily centrally large). If B has stable rank one, does it follow that A has stable rank one?

That is, can Theorem 1.15 be generalized from centrally large subalgebras to large subalgebras?

Question 1.45. Let A be an infinite dimensional simple separable nuclear unital C*-algebra, and let $B \subset A$ be a large subalgebra (not necessarily centrally large). If B is Z-stable, does it follow that A is Z-stable?

That is, can Theorem 1.16 be generalized from centrally large subalgebras to large subalgebras?

It is not clear how important these questions are. In all applications so far, with the single exception of [20] (on the extended irrational rotation algebras), the large subalgebras used are known to be centrally large. In particular, all known useful large subalgebras of crossed products are already known to be centrally large.

Question 1.46. Does there exist a large subalgebra which is not centrally large? Are there natural examples?

The results of [20] depend on large subalgebras which are not proved there to be centrally large, but it isn't known that they are not centrally large.

Question 1.47. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra. If $\mathrm{RR}(B) = 0$, does it follow that $\mathrm{RR}(A) = 0$? What about the converse? Does it help to assume that B is centrally large in the sense of Definition 1.3?

If B has both stable rank one and real rank zero, and is centrally large in A, then A has real rank zero (as well as stable rank one) by Theorem 1.15. The main point of Question 1.47 is to ask what happens if B is not assumed to have stable rank one. The proof in [55] of real rank zero for the crossed product $C^*(\mathbb{Z}^d, X)$ of a free minimal action of $\mathbb{Z}^d$ on the Cantor set X (see Theorem 6.11(2) of [55]; the main part is Theorem 4.6 of [55]) gives reason to hope that if B is large in A and $\mathrm{RR}(B) = 0$, then one does indeed get $\mathrm{RR}(A) = 0$. Proposition 1.14 could also be considered evidence in favor. Nothing at all is known about conditions under which $\mathrm{RR}(A) = 0$ implies $\mathrm{RR}(B) = 0$.

Applications to crossed products may be unlikely. It seems possible that $C^*(G, X)$ has stable rank one for every minimal essentially free action of a countable amenable group G on a compact metric space X.

Question 1.48. Let A be an infinite dimensional simple separable unital C*-algebra. Let $B \subset A$ be centrally large in the sense of Definition 1.3. Does it follow that $K_0(B) \to K_0(A)$ is an isomorphism mod infinitesimals?

In other places where this issue occurs (in connection with tracial approximate innerness; see Proposition 6.2 and Theorem 6.4 of [56]), it seems that everything in K_1 should be considered to be infinitesimal.

A six term exact sequence for the K-theory of some orbit breaking subalgebras is given in Example 2.6 of [62]. Related computations for some special more complicated orbit breaking subalgebras can be found in [63].

A positive answer to Question 1.48 would shed some light on both directions in Question 1.47.

Question 1.49. Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be centrally large in the sense of Definition 1.3. If A has stable rank one, does it follow that B has stable rank one?

That is, does Theorem 1.15 have a converse? In many other results in Section 1.2, B has an interesting property if and only if A does.

Question 1.50. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Let $n \in \mathbb{Z}_{>0}$. If $\mathrm{tsr}(B) \leq n$, does it follow that $\mathrm{tsr}(A) \leq n$? If $\mathrm{tsr}(B)$ is finite, does it follow that $\mathrm{tsr}(A)$ is finite?

That is, can Theorem 1.15 be generalized to other values of the stable rank? The proof of Theorem 1.15 uses $\mathrm{tsr}(B) = 1$ in two different places, one of which is not directly related to $\mathrm{tsr}(A)$, so an obvious approach seems unlikely to succeed.

As with Question 1.47, applications to crossed products seem unlikely.

2. Large Subalgebras and their Basic Properties

2.1. **The definition of a large subalgebra.** In this subsection, we restate the definition of a large subalgebra and give some equivalent versions. Recall that, by convention, if we say that B is a unital subalgebra of a C*-algebra A, we mean that B contains the identity of A.

We repeat Definition 1.2.

Definition 2.1 (Definition 4.1 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

We emphasize that the Cuntz subequivalence involving y in (4) is relative to B, not A.

Lemma 2.2. In Definition 2.1, it suffices to let $S \subset A$ be a subset whose linear span is dense in A, and verify the hypotheses only when $a_1, a_2, \ldots, a_m \in S$.

Unlike other approximation properties (such as tracial rank), it seems not to be possible to take S in Lemma 2.2 to be a generating subset, or even a selfadjoint generating subset. (We can do this for the definition of a centrally large subalgebra, Definition 1.3. See Proposition 3.10 of [5].)

By Proposition 4.4 of [58], in Definition 2.1 we can omit mention of $c_1, c_2, \ldots, c_m$, and replace (2) and (3) by the requirement that $\mathrm{dist}\big((1 - g)a_j, \, B\big) < \varepsilon$ for $j = 1, 2, \ldots, m$. So far, however, most verifications of Definition 2.1 proceed by constructing elements $c_1, c_2, \ldots, c_m$ as in Definition 2.1.

When A is finite, we do not need condition (5) of Definition 2.1.

Proposition 2.3 (Proposition 4.5 of [58])**.** Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra. Suppose that for $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\| c_j - a_j \| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g) c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.

Then B is large in A.

The proof of Proposition 2.3 needs Lemma 2.5 below, which is a version for Cuntz comparison of Lemma 1.15 of [56].

We describe the idea of the proof. (Most of the details are given below.) Given $x \in A_+$ with $\|x\| = 1$, we want $x_0 \in A_+ \setminus \{0\}$ such that $g \precsim_A x_0$ and otherwise as above implies $\|(1-g)x(1-g)\| > 1 - \varepsilon$. (We then use x_0 in place of x in the definition of a large subalgebra.) Choose a sufficiently small number $\varepsilon_0 > 0$. (It will be much smaller than ε.) Choose $f \colon [0, 1] \to [0, 1]$ such that $f = 0$ on $[0, \, 1 - \varepsilon_0]$ and $f(1) = 1$. Construct $a, b_1, b_2, c_1, c_2, d_1, d_2 \in \overline{f(x) A f(x)}$ such that for $j = 1, 2$ we have

$$0 \leq d_j \leq c_j \leq b_j \leq a \leq 1, \quad a b_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \quad \text{and} \quad d_j \neq 0,$$

and $b_1 b_2 = 0$. Take $x_0 = d_1$. If ε_0 is small enough, $g \precsim_A d_1$, and $\|(1-g)x(1-g)\| \leq 1 - \varepsilon$, this gives

$$\|(1-g)(b_1 + b_2)(1-g)\| < 1 - \frac{\varepsilon}{3}.$$

One then gets $c_1 + c_2 \precsim_A d_1$. (This is the calculation (2.1) in the proof below.) Now $r = (1 - c_1 - c_2) + d_1$ satisfies $r \precsim_A 1$, so there is $v \in A$ such that $\|v r v^* - 1\| < \frac{1}{2}$. Then $v r^{1/2}$ is right invertible, but $v r^{1/2} d_2 = 0$, so $v r^{1/2}$ is not left invertible. This contradicts finiteness of A.

We now give a more detailed argument.

Lemma 2.4 (Lemma 2.5 of [58])**.** Let A be a C*-algebra, let $x \in A_+$ satisfy $\|x\| = 1$, and let $\varepsilon > 0$. Then there are positive elements $a, b \in \overline{xAx}$ with $\|a\| = \|b\| = 1$, such that $ab = b$, and such that whenever $c \in \overline{bAb}$ satisfies $\|c\| \leq 1$, then $\|xc - c\| < \varepsilon$.

Sketch of proof. Choose continuous functions $f_0, f_1 \colon [0, 1] \to [0, 1]$ such that $f_1(1) = 1$, f_1 is supported near 1, $|f_0(\lambda) - \lambda| < \varepsilon$ for all $\lambda \in [0, 1]$, and $f_0 = 1$ near 1 (so that $f_0 f_1 = f_1$). Take $a = f_0(x)$ and $b = f_1(x)$. Then $\|x - a\| < \varepsilon$ and $ab = b$. $\square$

Lemma 2.5 (Lemma 2.6 of [58])**.** Let A be a finite simple infinite dimensional unital C*-algebra. Let $x \in A_+$ satisfy $\|x\| = 1$. Then for every $\varepsilon > 0$ there is $x_0 \in \left(\overline{xAx}\right)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1-g)x(1-g)\| > 1 - \varepsilon$.

Proof. Choose positive elements $a, b \in \overline{x^{1/2} A x^{1/2}}$ as in Lemma 2.4, with $x^{1/2}$ in place of x and $\frac{\varepsilon}{3}$ in place of ε. Then $a, b \in \overline{xAx}$ since $\overline{x^{1/2} A x^{1/2}} = \overline{xAx}$. Since $b \neq 0$, Lemma 1.32 provides nonzero positive orthogonal elements $z_1, z_2 \in \overline{bAb}$ (with $z_1 \sim_A z_2$). We may require $\|z_1\| = \|z_2\| = 1$.

Choose continuous functions $f_0, f_1, f_2 \colon [0, \infty) \to [0, 1]$ such that

$$f_0(0) = 0, \quad f_0 f_1 = f_1, \quad f_1 f_2 = f_2, \quad \text{and} \quad f_2(1) = 1.$$

For $j = 1, 2$ define

$$b_j = f_0(z_j), \quad c_j = f_1(z_j), \quad \text{and} \quad d_j = f_2(z_j).$$

Then

$$0 \leq d_j \leq c_j \leq b_j \leq 1, \quad ab_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \quad \text{and} \quad d_j \neq 0.$$

Also $b_1 b_2 = 0$. Define $x_0 = d_1$. Then $x_0 \in \big(\overline{xAx}\big)_+$.

Let $g \in A_+$ satisfy $0 \leq g \leq 1$ and $g \precsim_A x_0$. We want to show that

$$\|(1-g)x(1-g)\| > 1 - \varepsilon,$$

so suppose that $\|(1-g)x(1-g)\| \leq 1 - \varepsilon$. The choice of a and b, and the relations $(b_1 + b_2)^{1/2} \in \overline{bAb}$ and $\big\|(b_1 + b_2)^{1/2}\big\| = 1$, imply that

$$\big\|x^{1/2}(b_1 + b_2)^{1/2} - (b_1 + b_2)^{1/2}\big\| < \frac{\varepsilon}{3}.$$

Using this relation and its adjoint at the second step, we get

$$\begin{aligned} \big\|(1-g)(b_1+b_2)(1-g)\big\| &= \big\|(b_1+b_2)^{1/2}(1-g)^2(b_1+b_2)^{1/2}\big\| \\ &< \big\|(b_1+b_2)^{1/2}x^{1/2}(1-g)^2x^{1/2}(b_1+b_2)^{1/2}\big\| + \frac{2\varepsilon}{3} \\ &\leq \big\|x^{1/2}(1-g)^2x^{1/2}\big\| + \frac{2\varepsilon}{3} \\ &= \|(1-g)x(1-g)\| + \frac{2\varepsilon}{3} \leq 1 - \frac{\varepsilon}{3}. \end{aligned}$$

Using the equation $(b_1 + b_2)(c_1 + c_2) = c_1 + c_2$ and taking C to be the commutative C*-algebra generated by $b_1 + b_2$ and $c_1 + c_2$, one easily sees that for every $\beta \in [0, 1)$ we have $c_1 + c_2 \precsim_C [(b_1 + b_2) - \beta]_+$. Take $\beta = 1 - \frac{\varepsilon}{3}$, use this fact and Lemma 1.27 at the first step, use the estimate above at the second step, and use $g \precsim_A x_0 = d_1$ at the third step, to get

$$c_1 + c_2 \precsim_A \big[(1-g)(b_1+b_2)(1-g) - \beta\big]_+ \oplus g = 0 \oplus g \precsim_A d_1. \tag{2.1}$$

Set $r = (1 - c_1 - c_2) + d_1$. Use Lemma 1.26(12) at the first step, (2.1) at the second step, and Lemma 1.26(13) and $d_1(1 - c_1 - c_2) = 0$ at the third step, to get

$$1 \precsim_A (1 - c_1 - c_2) \oplus (c_1 + c_2) \precsim_A (1 - c_1 - c_2) \oplus d_1 \sim_A (1 - c_1 - c_2) + d_1 = r.$$

Thus there is $v \in A$ such that $\|vrv^* - 1\| < \frac{1}{2}$. It follows that $vr^{1/2}$ has a right inverse. But $vr^{1/2}d_2 = 0$, so $vr^{1/2}$ is not invertible. We have contradicted finiteness of A, and thus proved the lemma. ☐

Proof of Proposition 2.3. Let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, let $x \in A_+ \setminus \{0\}$, and let $y \in B_+ \setminus \{0\}$. Without loss of generality $\|x\| = 1$.

Apply Lemma 2.5, obtaining $x_0 \in \big(\overline{xAx}\big)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1-g)x(1-g)\| > 1-\varepsilon$. Apply the hypothesis with x_0 in place of x and everything else as given, getting $c_1, c_2, \ldots, c_m \in A$ and $g \in B$. We need only prove that $\|(1-g)x(1-g)\| > 1 - \varepsilon$. But this is immediate from the choice of x_0. ☐

The following strengthening of the definition is often convenient. First, we can always require $\|c_j\| \leq \|a_j\|$. Second, if we cut down on both sides instead of on one side, and the elements a_j are positive, then we may take the elements c_j to be positive.

Lemma 2.6 (Lemma 4.8 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $m, n \in \mathbb{Z}_{\geq 0}$, let $a_1, a_2, \ldots, a_m \in A$, let $b_1, b_2, \ldots, b_n \in A_+$, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in B_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$, $d_1, d_2, \ldots, d_n \in A_+$, and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$, and for $j = 1, 2, \ldots, n$ we have $\|d_j - b_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $\|c_j\| \leq \|a_j\|$, and for $j = 1, 2, \ldots, n$ we have $\|d_j\| \leq \|b_j\|$.
(4) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$, and for $j = 1, 2, \ldots, n$ we have $(1 - g)d_j(1 - g) \in B$.
(5) $g \precsim_B y$ and $g \precsim_A x$.
(6) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Sketch of proof. To get $\|c_j\| \leq \|a_j\|$ for $j = 1, 2, \ldots, m$, one takes $\varepsilon > 0$ to be a bit smaller in the definition, and scales down c_j for any j for which $\|c_j\|$ is too big. Given that one can do this, following the definition, approximate

$$a_1,\, a_2,\, \ldots,\, a_m,\, b_1^{1/2},\, b_2^{1/2},\, \ldots,\, b_n^{1/2}$$

sufficiently well by

$$c_1, c_2, \ldots, c_m, r_1, r_2, \ldots, r_n,$$

and take $d_j = r_j r_j^*$ for $j = 1, 2, \ldots, n$. $\square$

In Definition 4.9 of [58] we defined a "large subalgebra of crossed product type", a strengthening of the definition of a large subalgebra, and in Proposition 4.11 of [58] we gave a convenient way to verify that a subalgebra is a large subalgebra of crossed product type. The large subalgebras we have constructed in crossed products are of crossed product type. Theorem 4.6 of [5] shows that a large subalgebra of crossed product type is in fact centrally large. We will show directly (proof of Theorem 1.8, in Section 4 below) that if X is an infinite compact Hausdorff space, $h \colon X \to X$ is a minimal homeomorphism, and $Y \subset X$ is a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $C^*(\mathbb{Z}, X, h)_Y$ is centrally large in $C^*(\mathbb{Z}, X, h)$. This procedure is easier than using large subalgebras of crossed product type. The abstract version is more useful for subalgebras of crossed products by more complicated groups, but we don't consider these in these notes.

2.2. **Proofs of some basic properties of large subalgebras.** In this subsection, we prove two of the basic properties of large subalgebras from Subsection 1.2: if B is large in A, then B is simple (part of Proposition 1.9) and has the "same" tracial states as A (part of Theorem 1.10).

We start with the simplicity statement in Proposition 1.9.

Proposition 2.7 (Proposition 5.2 of [58])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple.

We need some preliminary work.

Lemma 2.8 (Lemma 1.12 of [58])**.** Let A be a C*-algebra, let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^{n} a_k$. Then $a^* a \leq 2^{n-1} \sum_{k=1}^{n} a_k^* a_k$.

Proof. We prove this by induction on n. For $n = 1$, the statement is immediate. Suppose it is known for n; we prove it for $n+1$. Set $x = \sum_{k=1}^{n} a_k$. Then, expanding and cancelling at the third step, using the induction hypothesis at the fourth step, and using $n \geq 1$ at the fifth step, we get

$$\begin{aligned} a^* a &= (x + a_{n+1})^* (x + a_{n+1}) \leq (x + a_{n+1})^* (x + a_{n+1}) + (x - a_{n+1})^* (x - a_{n+1}) \\ &= 2x^* x + 2a_{n+1}^* a_{n+1} \leq 2^n \sum_{k=1}^{n} a_k^* a_k + 2a_{n+1}^* a_{n+1} \leq 2^n \sum_{k=1}^{n+1} a_k^* a_k. \end{aligned}$$

This completes the induction step and the proof. $\square$

Lemma 2.9 (Lemma 1.13 of [58])**.** Let A be a C*-algebra and let $a \in A_+$. Let $b \in \overline{AaA}$ be positive. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\left\| b - \sum_{k=1}^{n} x_k^* a x_k \right\| < \varepsilon$.

This result is used without proof in the proof of Proposition 2.7(v) of [33]. We prove it when A is unital and $b = 1$, which is the case needed here. In this case, we can get $\sum_{k=1}^{n} x_k^* a x_k = 1$. In particular, we get Corollary 1.14 of [58] this way. (This result can also be obtained from Proposition 1.10 of [12], as pointed out after the proof of that proposition.)

Proof of Lemma 2.9 when $b = 1$. Choose

$$n \in \mathbb{Z}_{>0} \quad \text{and} \quad y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in A$$

such that the element $c = \sum_{k=1}^{n} y_k a z_k$ satisfies $\|c - 1\| < 1$. Set

$$r = \sum_{k=1}^{n} z_k^* a y_k^* y_k a z_k, \quad M = \max\big(\|y_1\|, \|y_2\|, \ldots, \|y_n\|\big), \quad \text{and} \quad s = M^2 \sum_{k=1}^{n} z_k^* a^2 z_k.$$

Lemma 2.8 implies that $c^* c \in \overline{rAr}$. The relation $\|c - 1\| < 1$ implies that c is invertible, so r is invertible. Since $r \leq s$, it follows that s is invertible. Set $x_k = M a^{1/2} z_k s^{-1/2}$ for $k = 1, 2, \ldots, n$. Then $\sum_{k=1}^{n} x_k^* a x_k = s^{-1/2} s s^{-1/2} = 1$. $\square$

Sketch of proof of Proposition 2.7. Let $b \in B_+ \setminus \{0\}$. We show that there are $n \in \mathbb{Z}_{>0}$ and $r_1, r_2, \ldots, r_n \in B$ such that $\sum_{k=1}^{n} r_k b r_k^*$ is invertible.

Since A is simple, Lemma 2.9 provides $m \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_m \in A$ such that $\sum_{k=1}^{m} x_k b x_k^* = 1$. Set

$$M = \max\big(1, \|x_1\|, \|x_2\|, \ldots, \|x_m\|, \|b\|\big) \quad \text{and} \quad \delta = \min\left(1, \frac{1}{3mM(2M+1)}\right).$$

By definition, there are $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, such that $\|y_j - x_j\| < \delta$ and $(1 - g) y_j \in B$ for $j = 1, 2, \ldots, m$, and such that $g \precsim_B b$. Set $z = \sum_{k=1}^{m} y_j b y_j^*$. The number δ has been chosen to ensure that $\|z - 1\| < \frac{1}{3}$; the estimate is carried out in [58]. It follows that $\left\| (1-g) z (1-g) - (1-g)^2 \right\| < \frac{1}{3}$.

Set $h = 2g - g^2$. Lemma 1.26(3), applied to the function $\lambda \mapsto 2\lambda - \lambda^2$, implies that $h \sim_B g$. Therefore $h \precsim_B b$. So there is $v \in B$ such that $\|v b v^* - h\| < \frac{1}{3}$. Now take $n = m + 1$, take $r_j = (1 - g) y_j$ for $j = 1, 2, \ldots, m$, and take $r_{m+1} = v$. Then $r_1, r_2, \ldots, r_n \in B$. One can now check, using $(1 - g)^2 + h = 1$, that $\|1 - \sum_{k=1}^{n} r_k b r_k^*\| < \frac{2}{3}$. Therefore $\sum_{k=1}^{n} r_k b r_k^*$ is invertible, as desired. $\square$

The following is a special case of the infinite dimensionality statement in Proposition 1.9 (Proposition 5.5 of [58]), which is easier to prove.

Proposition 2.10 (Stably finite case of Proposition 5.5 of [58])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra and let $B \subset A$ be a large subalgebra. Then B is infinite dimensional.

Proof. Suppose B is finite dimensional. Proposition 2.7 tells us that B is simple, so there is $n \in \mathbb{Z}_{>0}$ such that $B \cong M_n$. It follows from the discussion after Theorem 3.3 of [9] that there is a quasitrace τ on A. Apply Corollary 1.33 to get $x \in A_+ \setminus \{0\}$ such that $d_\tau(x) < (n+1)^{-1}$. We may assume that $\|x\| = 1$. Clearly $B \neq A$, so there is $a \in A$ such that $\operatorname{dist}(a, B) > 1$. Apply Definition 2.1, getting $g \in B$ and $c \in A$ such that

$$0 \leq g \leq 1, \qquad \|a - c\| < \frac{1}{2}, \qquad (1-g)c \in B, \quad \text{and} \quad g \precsim_A x.$$

Then $c \notin B$, so $g \neq 0$. Also, $d_\tau(g) \leq d_\tau(x) < (n+1)^{-1}$. Now $\sigma = \tau|_B$ is a quasitrace on B, so must be the normalized trace on B, and $0 < d_\sigma(g) = d_\tau(g) < (n+1)^{-1}$. There are no elements $g \in (M_n)_+$ with $0 < d_\sigma(g) < (n+1)^{-1}$, so we have a contradiction. □

Proposition 2.11 (Corollary 5.8 of [58])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $n \in \mathbb{Z}_{>0}$. Then $M_n(B)$ is large in $M_n(A)$.

In [58], this result is obtained as a corollary of a more general result (Proposition 1.5 here). A direct proof is easier, and we give it here.

Proof of Proposition 2.11. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in M_n(A)$, let $\varepsilon > 0$, let $x \in M_n(A)_+ \setminus \{0\}$, and let $y \in M_n(B)_+ \setminus \{0\}$. There are $b_{k,l} \in A$ for $k, l = 1, 2, \ldots, n$ such that

$$x^{1/2} = \sum_{k,l=1}^{n} e_{k,l} \otimes b_{k,l} \in M_n \otimes A.$$

Choose $k, l \in \{1, 2, \ldots, n\}$ such that $b_{k,l} \neq 0$. Set $x_0 = b_{k,l}^* b_{k,l} \in A_+ \setminus \{0\}$. Using selfadjointness of $x^{1/2}$, we find that

$$e_{1,1} \otimes x_0 = (e_{l,1} \otimes 1)^* x^{1/2} (e_{k,k} \otimes 1) x^{1/2} (e_{l,1} \otimes 1) \leq (e_{l,1} \otimes 1)^* x (e_{l,1} \otimes 1) \precsim_A x.$$

Similarly, there is $y_0 \in B_+ \setminus \{0\}$ such that $e_{1,1} \otimes y_0 \precsim_B y$.

Use Lemma 1.32 and simplicity (Proposition 2.7) and infinite dimensionality (Proposition 2.10) of B to find systems of nonzero mutually orthogonal and mutually Cuntz equivalent positive elements

$$x_1, x_2, \ldots, x_n \in \overline{x_0 A x_0} \quad \text{and} \quad y_1, y_2, \ldots, y_n \in \overline{y_0 B y_0}.$$

For $j = 1, 2, \ldots, m$, choose elements $a_{j,k,l} \in A$ for $k, l = 1, 2, \ldots, n$ such that

$$a_j = \sum_{k,l=1}^{n} e_{k,l} \otimes a_{j,k,l} \in M_n \otimes A.$$

Apply Proposition 2.3 with mn^2 in place of m, with the elements $a_{j,k,l}$ in place of $a_1, a_2, \ldots, a_m$, with ε / n^2 in place of ε, with x_1 in place of x, and with y_1 in place of y, getting $g_0 \in A_+$ and $c_{j,k,l} \in A$ for $j = 1, 2, \ldots, m$ and $k, l = 1, 2, \ldots, n$. Define $c_j = \sum_{k,l=1}^{n} e_{k,l} \otimes c_{j,k,l}$ for $j = 1, 2, \ldots, m$ and define $g = 1 \otimes g_0$. It is clear that $0 \leq g \leq 1$, that $\|c_j - a_j\| < \varepsilon$ and $(1-g)c_j \in M_n(B)$ for $j = 1, 2, \ldots, m$. We have $g \precsim_A 1 \otimes x_1$ and $g \precsim_B 1 \otimes y_1$, so Lemma 1.26(1) and Lemma 1.26(13) imply that $g \precsim_A x_0$ and $g \precsim_B y_0$. Therefore $g \precsim_A x$ and $g \precsim_B y$. □

We prove the statement about traces in Theorem 1.10, assuming that the algebras are stably finite (the interesting case).

Theorem 2.12 (Stably finite case of Theorem 6.2 of [58])**.** Let A be an infinite dimensional stably finite simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective.

Again, we need a lemma.

Lemma 2.13. Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in \mathrm{T}(B)$. Then there exists a unique state ω on A such that $\omega|_B = \tau$.

Proof. Existence of ω follows from the Hahn-Banach Theorem.

For uniqueness, let ω_1 and ω_2 be states on A such that $\omega_1|_B = \omega_2|_B = \tau$, let $a \in A_+$, and let $\varepsilon > 0$. We prove that $|\omega_1(a)-\omega_2(a)| < \varepsilon$. Without loss of generality $\|a\| \leq 1$.

It follows from Proposition 2.7 and Proposition 2.10 that B is simple and infinite dimensional. So Corollary 1.33 provides $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$ (for the particular choice of τ we are using). Use Lemma 2.6 to find $c \in A_+$ and $g \in B_+$ such that

$$\|c\| \leq 1, \quad \|g\| \leq 1, \quad \|c - a\| < \frac{\varepsilon}{4}, \quad (1-g)c(1-g) \in B, \quad \text{and} \quad g \precsim_B y.$$

For $j = 1, 2$, the Cauchy-Schwarz inequality gives

$$|\omega_j(rs)| \leq \omega_j(rr^*)^{1/2}\omega_j(s^*s)^{1/2} \tag{2.2}$$

for all $r, s \in A$. Also, by Lemma 1.26(3) we have $g^2 \sim_B g \precsim_B y$. Since $\|g^2\| \leq 1$ and $\omega_j|_B = \tau$ is a tracial state, it follows that $\omega_j(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$. Using $\|c\| \leq 1$ and the Cauchy-Schwarz inequality, we then get

$$|\omega_j(gc)| \leq \omega_j(g^2)^{1/2}\omega_j(c^2)^{1/2} < \frac{\varepsilon}{8}$$

and

$$|\omega_j((1-g)cg)| \leq \omega_j\big((1-g)c^2(1-g)\big)^{1/2}\omega_j(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

So

$$\begin{aligned} \big|\omega_j(c) - \tau((1-g)c(1-g))\big| &= \big|\omega_j(c) - \omega_j((1-g)c(1-g))\big| \\ &\leq |\omega_j(gc)| + |\omega_j((1-g)cg)| < \frac{\varepsilon}{4}. \end{aligned}$$

Also $|\omega_j(c) - \omega_j(a)| < \frac{\varepsilon}{4}$. So

$$\big|\omega_j(a) - \tau((1-g)c(1-g))\big| < \frac{\varepsilon}{2}.$$

Thus $|\omega_1(a) - \omega_2(a)| < \varepsilon$. ∎

The uniqueness statement in Lemma 2.13 is used to prove that the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is injective.

One might hope that Lemma 2.13 would enable the following idea for the proof that $\mathrm{T}(A) \to \mathrm{T}(B)$ is surjective.

We first observe that a state ω is tracial whenever $\omega(uau^*) = \omega(u)$ for all $a \in A$ and all unitaries $u \in A$. Indeed, putting au for a gives $\omega(ua) = \omega(au)$ for all $a \in A$ and all unitaries $u \in A$. Since A is the linear span of its unitaries, it follows that $\omega(ba) = \omega(ab)$ for all $a, b \in A$.

Now let A and B be as in Theorem 2.12, let $\tau \in \mathrm{T}(B)$, and $u \in A$. Let ω be the unique state on A which extends τ (Lemma 2.13). We would like to argue that the state $\rho(a) = \omega(uau^*)$ for $a \in A$ is equal to ω because it also extends τ. The first thing which goes wrong is that if $b \in B$ and $u \in A$ is unitary, then ubu^* need not even be in B. So the is no immediate reason to think that ρ extends τ.

If the unitary u is actually in B, then ρ does indeed extend ω. Thus, the uniqueness statement in Lemma 2.13 implies that $\omega(uau^*) = \omega(a)$ for all $a \in A$ and all unitaries $u \in B$. We can still replace a by au as above, and deduce that $\omega(ba) = \omega(ab)$ for all $a \in A$ and $b \in B$. In particular, $\omega(vb) = \omega(bv)$ for all $b \in B$ and unitaries $v \in A$. But to get $\omega(vbv^*) = \omega(b)$ from this requires putting bv^* in place of b, and bv^* isn't in B.

Proof of Theorem 2.12. Let $\tau \in \mathrm{T}(B)$. We show that there is a unique $\omega \in \mathrm{T}(A)$ such that $\omega|_B = \tau$. Lemma 2.13 shows that there is a unique state ω on A such that $\omega|_B = \tau$, and it suffices to show that ω is a trace. Thus let $a_1, a_2 \in A$ satisfy $\|a_1\| \leq 1$ and $\|a_2\| \leq 1$, and let $\varepsilon > 0$. We show that $|\omega(a_1 a_2) - \omega(a_2 a_1)| < \varepsilon$.

It follows from Proposition 2.7 and Proposition 2.10 (without stable finiteness, we must appeal to Proposition 5.5 of [58]) that B is simple and infinite dimensional. So Corollary 1.33 provides $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$. Use Lemma 2.6 to find $c_1, c_2 \in A$ and $g \in B_+$ such that

$$\|c_j\| \leq 1, \quad \|c_j - a_j\| < \frac{\varepsilon}{8}, \quad \text{and} \quad (1-g)c_j \in B$$

for $j = 1, 2$, and such that $\|g\| \leq 1$ and $g \precsim_B y$. By Lemma 1.26(3), we have $g^2 \sim g \precsim_B y$. Since $\|g^2\| \leq 1$ and $\omega|_B = \tau$ is a tracial state, it follows that $\omega(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$.

We claim that

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1 c_2)\big| < \frac{\varepsilon}{4}.$$

Using the Cauchy-Schwarz inequality ((2.2) in the previous proof), we get

$$|\omega(g c_1 c_2)| \leq \omega(g^2)^{1/2} \omega(c_2^* c_1^* c_1 c_2)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

Similarly, and also at the second step using $\|c_2\| \leq 1$, $(1-g)c_1 g \in B$, and the fact that $\omega|_B$ is a tracial state,

$$\begin{aligned} \big|\omega((1-g)c_1 g c_2)\big| &\leq \omega\big((1-g)c_1 g^2 c_1^*(1-g)\big)^{1/2} \omega(c_2^* c_2)^{1/2} \\ &\leq \omega\big(g c_1^*(1-g)^2 c_1 g\big)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}. \end{aligned}$$

The claim now follows from the estimate

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1 c_2)\big| \leq \big|\omega((1-g)c_1 g c_2)\big| + |\omega(g c_1 c_2)|.$$

Similarly

$$\big|\omega((1-g)c_2(1-g)c_1) - \omega(c_2 c_1)\big| < \frac{\varepsilon}{4}.$$

Since $(1-g)c_1$, $(1-g)c_2 \in B$ and $\omega|_B$ is a tracial state, we get

$$\omega((1-g)c_1(1-g)c_2) = \omega((1-g)c_2(1-g)c_1).$$

Therefore $|\omega(c_1 c_2) - \omega(c_2 c_1)| < \frac{\varepsilon}{2}$.

One checks that $\|c_1 c_2 - a_1 a_2\| < \frac{\varepsilon}{4}$ and $\|c_2 c_1 - a_2 a_1\| < \frac{\varepsilon}{4}$. It now follows that $|\omega(a_1 a_2) - \omega(a_2 a_1)| < \varepsilon$. $\square$

3. Large Subalgebras and the Radius of Comparison

Let A be a simple unital C*-algebra. We say that *the order on projections over A is determined by traces* if, as happens for type II_1 factors, whenever $p, q \in M_\infty(A)$ are projections such that for all $\tau \in \mathrm{T}(A)$ we have $\tau(p) < \tau(q)$, then p is Murray-von Neumann equivalent to a subprojection of q. The question of whether this holds is also known as Blackadar's Second Fundamental Comparability Question (FCQ2; see 1.3.1 of [6]). Without knowing whether every quasitrace is a trace, it is more appropriate to use a condition involving quasitraces. For exact C*-algebras, every quasitrace is a trace (Theorem 5.11 of [27]), so it makes no difference.

Simple C*-algebras need not have very many projections, so a more definitive version of this condition is to ask for *strict comparison of positive elements*, that is, whenever $a, b \in M_\infty(A)$ (or $K \otimes A$) are positive elements such that for all $\tau \in \mathrm{QT}(A)$ we have $d_\tau(a) < d_\tau(b)$, then $a \precsim_A b$. (By Proposition 6.12 of [58], it does not matter whether one uses $M_\infty(A)$ or $K \otimes A$, but this is not as easy to see as with projections.)

Simple AH algebras with slow dimension growth have strict comparison, but other simple AH algebras need not. (For example, see [69].) Strict comparison seems to be necessary for any reasonable hope of classification in the sense of the Elliott program. According to the Toms-Winter Conjecture, when A is simple, separable, nuclear, unital, and stably finite, strict comparison should imply Z-stability, and this is known to hold in a number of cases.

The radius of comparison $\mathrm{rc}(A)$ of A (for a C*-algebra which is unital and stably finite but not necessarily simple) measures the failure of strict comparison. (See [8] for what to do in more general C*-algebras.) For additional context, we point out the following special case of Theorem 5.1 of [70] (which will be needed in Section 5, where it is restated as Theorem 5.32): if X is a compact metric space and $n \in \mathbb{Z}_{>0}$, then

$$\mathrm{rc}(M_n \otimes C(X)) \leq \frac{\dim(X) - 1}{2n}.$$

Under some conditions on X (being a finite complex is enough), this inequality is at least approximately an equality. See [19].

The following definition of the radius of comparison is adapted from Definition 6.1 of [68].

Definition 3.1. Let A be a stably finite unital C*-algebra.

(1) Let $r \in [0, \infty)$. We say that A has *r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.

(2) The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\big(\big\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\big\}\big).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

Definition 6.1 of [68] actually uses lower semicontinuous dimension functions on A instead of d_τ for $\tau \in \mathrm{QT}(A)$, but these are the same functions by Theorem II.2.2 of [7]. It is also stated in terms of the order on the Cuntz semigroup $W(A)$ rather than in terms of Cuntz subequivalence; this is clearly equivalent.

We also note (Proposition 6.3 of [68]) that if every element of $\mathrm{QT}(A)$ is faithful, then the infimum in Definition 3.1(2) is attained, that is, A has $\mathrm{rc}(A)$-comparison. In particular, this is true when A is simple. (See Lemma 1.23 of [58].)

We warn that r-comparison and $\mathrm{rc}(A)$ are sometimes defined using tracial states rather than quasitraces.

It is equivalent to use $K \otimes A$ in place of $M_{\infty}(A)$. See Proposition 6.12 of [58].

We prove here the following special case of Theorem 1.12.

Theorem 3.2. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

The extra assumption is that A is exact, so that every quasitrace is a trace by Theorem 5.11 of [27].

We will give a proof directly from the definition of a large subalgebra. We describe the heuristic argument, using the following simplifications:

(1) The algebra A, and therefore also B, has a unique tracial state τ.
(2) We consider elements of A_{+} and B_{+} instead of elements of $M_{\infty}(A)_{+}$ and $M_{\infty}(B)_{+}$.
(3) For $a \in A_{+}$, when applying the definition of a large subalgebra (Definition 2.1), instead of getting $(1-g)c(1-g) \in B$ for some $c \in A_{+}$ which is close to a, we can actually get $(1-g)a(1-g) \in B$. Similarly, for $a \in A$ we can get $(1-g)a \in B$.
(4) For $a, b \in A_{+}$ with $a \precsim_{A} b$, we can find $v \in A$ such that $v^{*}bv = a$ (not just such that $\|v^{*}bv - a\|$ is small).
(5) None of the elements we encounter are Cuntz equivalent to projections, that is, we never encounter anything for which 0 is an isolated point of, or not in, the spectrum.

The most drastic simplification is (3). In the actual proof, to compensate for the fact that we only get approximation, we will need to make systematic use of elements $(a-\varepsilon)_{+}$ for carefully chosen, and varying, values of $\varepsilon > 0$. Avoiding this complication gives a much better view of the idea behind the argument, and the usefulness of large subalgebras in general.

We first consider the inequality $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. So let $a, b \in A_{+}$ satisfy $d_{\tau}(a) + \mathrm{rc}(B) < d_{\tau}(b)$. The essential idea is to replace b by something slightly smaller which is in B_{+}, say y, and replace a by something slightly larger which is in B_{+}, say x, in such a way that we still have $d_{\tau}(x) + \mathrm{rc}(B) < d_{\tau}(y)$. Then use the definition of $\mathrm{rc}(B)$. With g sufficiently small in the sense of Cuntz comparison, we will take $y = (1-g)b(1-g)$ and (following Lemma 1.27) $x = (1-g)a(1-g) \oplus g$.

Choose $\delta > 0$ such that

$$d_{\tau}(a) + \mathrm{rc}(B) + \delta \leq d_{\tau}(b). \tag{3.1}$$

Applying (3) of our simplification, we can find $g \in B$ with $0 \leq g \leq 1$, such that

$$(1-g)a(1-g) \in B \quad \text{and} \quad (1-g)b(1-g) \in B,$$

and so small in $W(A)$ that

$$d_{\tau}(g) < \frac{\delta}{3}. \tag{3.2}$$

Using Lemma 1.26(4) at the first step, we get

$$(1-g)b(1-g) \sim_{A} b^{1/2}(1-g)^{2}b^{1/2} \leq b,$$

so

$$(1-g)b(1-g) \precsim_{A} b. \tag{3.3}$$

Similarly, $(1-g)a(1-g) \precsim_A a$, and this relation implies

$$d_\tau\big((1-g)a(1-g)\big) \leq d_\tau(a). \tag{3.4}$$

Also, $b \precsim_A (1-g)b(1-g) \oplus g$ by Lemma 1.27, so

$$d_\tau\big((1-g)b(1-g)\big) + d_\tau(g) \geq d_\tau(b). \tag{3.5}$$

Using (3.4) at the first step, using (3.1) at the second step, using (3.5) at the third step, and using (3.2) at the fourth step, we get

$$\begin{aligned} d_\tau\big((1-g)a(1-g) \oplus g\big) + \mathrm{rc}(B) + \frac{\delta}{3} &\leq d_\tau(a) + d_\tau(g) + \mathrm{rc}(B) + \frac{\delta}{3} \\ &\leq d_\tau(b) + d_\tau(g) - \frac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big) + 2d_\tau(g) - \frac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big). \end{aligned}$$

So, by the definition of $\mathrm{rc}(B)$,

$$(1-g)a(1-g) \oplus g \precsim_B (1-g)b(1-g).$$

Therefore, using Lemma 1.27 at the first step and (3.3) at the third step, we get

$$a \precsim_A (1-g)a(1-g) \oplus g \precsim_B (1-g)b(1-g) \precsim_A b,$$

that is, $a \precsim_A b$, as desired.

Now we consider the inequality $\mathrm{rc}(A) \geq \mathrm{rc}(B)$. Let $a, b \in B_+$ satisfy $d_\tau(a) + \mathrm{rc}(A) < d_\tau(b)$. Choose $\delta > 0$ such that $d_\tau(a) + \mathrm{rc}(A) + \delta \leq d_\tau(b)$. By lower semicontinuity of d_τ, we always have

$$d_\tau(b) = \sup_{\varepsilon > 0} d_\tau\big((b-\varepsilon)_+\big).$$

So there is $\varepsilon > 0$ such that

$$d_\tau\big((b-\varepsilon)_+\big) > d_\tau(a) + \mathrm{rc}(A). \tag{3.6}$$

Define a continuous function $f \colon [0,\infty) \to [0,\infty)$ by $f(\lambda) = \max(0, \varepsilon^{-1}\lambda(\varepsilon - \lambda))$ for $\lambda \in [0,\infty)$. Then $f(b)$ and $(b-\varepsilon)_+$ are orthogonal positive elements such that $f(b) \neq 0$ (by (5)) and $f(b) + (b-\varepsilon)_+ \leq b$. We have $a \precsim_A (b-\varepsilon)_+$ by (3.6) and the definition of $\mathrm{rc}(A)$. Applying (4) of our simplification, we can find $v \in A$ such that $v^*(b-\varepsilon)_+ v = a$. Applying (3) of our simplification, we can find $g \in B$ with $0 \leq g \leq 1$ such that $(1-g)v^* \in B$ and $g \precsim_B f(b)$. Since

$$v(1-g) \in B \quad \text{and} \quad [v(1-g)]^*(b-\varepsilon)_+[v(1-g)] = (1-g)a(1-g),$$

we get $(1-g)a(1-g) \precsim_B (b-\varepsilon)_+$. Therefore, using Lemma 1.27 at the first step,

$$a \precsim_B (1-g)a(1-g) \oplus g \precsim_B (b-\varepsilon)_+ \oplus g \precsim_B (b-\varepsilon)_+ \oplus f(b) \precsim_B b,$$

as desired.

The original proof of Theorem 3.2 followed the heuristic arguments above, and this is the proof we give below. The proof in [58] uses the same basic ideas, but gives much more. The heuristic arguments above are the basis for the technical results in Lemma 1.17. In [58], these are used to prove Theorem 1.11, which states that, after deleting the classes of the nonzero projections from the Cuntz semigroups $\mathrm{Cu}(B)$ and $\mathrm{Cu}(A)$, the inclusion of B in A is an order isomorphism on what remains. (The inclusion need not be an isomorphism if the classes of the nonzero projections are included. See Example 7.13 of [58].) In Section 3 of [58], it is shown that, in our situation, the part of the Cuntz semigroup without the classes of the nonzero projections is enough to determine the quasitraces, so that the restriction map $\mathrm{QT}(A) \to \mathrm{QT}(B)$ is bijective. It follows that the radius of comparison in this part of the Cuntz semigroup is the same for both A and B, and it turns out that the radius of comparison in this part of the Cuntz semigroup is the same as in the entire Cuntz semigroup.

We will use the characterizations of $\mathrm{rc}(A)$ in the following theorem, which is a special case of results in [8]. The difference between (1) and (2) is that (2) has $n+1$ in one of the places where (1) has n. This result substitutes for the observation that if $a, b \in A_+$ satisfy $\tau(a) < \tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then, by compactness of $\mathrm{QT}(A)$ and continuity, we have $\inf_{\tau \in \mathrm{QT}(A)} [\tau(b) - \tau(a)] > 0$. The difficulty is that we need an analog using d_τ instead of τ, and $\tau \mapsto d_\tau(a)$ is in general only lower semicontinuous, so that $\tau \mapsto d_\tau(b) - d_\tau(a)$ may be neither upper nor lower semicontinuous.

Unfortunately, the results in [8] are stated in terms of $\mathrm{Cu}(A)$ rather than $W(A)$.

Theorem 3.3. Let A be a stably finite simple unital C*-algebra. Then:

(1) The radius of comparison $\mathrm{rc}(A)$ is the least number $s \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > s$, and $a, b \in M_\infty(A)_+$ satisfy
$$n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$$
in $W(A)$, then $a \precsim_A b$.

(2) The radius of comparison $\mathrm{rc}(A)$ is the least number $t \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > t$, and $a, b \in M_\infty(A)_+$ satisfy
$$(n+1)\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$$
in $W(A)$, then $a \precsim_A b$.

Proof. It is easy to check that there is in fact a least $s \in [0, \infty]$ satisfying the condition in (1), and similarly that there is a least $t \in [0, \infty]$ as in (2).

We will first prove this for $K \otimes A$ and $\mathrm{Cu}(A)$ in place of $M_\infty(A)$ and $W(A)$. So let s_0 and t_0 be the numbers defined as in (1) and (2), except with $K \otimes A$ and $\mathrm{Cu}(A)$ in place of $M_\infty(A)$ and $W(A)$. Again, it is clear that there are least such numbers s_0 and t_0. Clearly $s_0 \geq t_0$. Since A is simple and stably finite and $\langle 1\rangle_A$ is a full element of $\mathrm{Cu}(A)$, Proposition 3.2.3 of [8], the preceding discussion in [8], and Definition 3.2.2 of [8] give $t_0 = \mathrm{rc}(A)$. So we need to show that $s_0 \leq t_0$.

We thus assume $m, n \in \mathbb{Z}_{>0}$ and $m/n > t_0$, and that $a, b \in (K \otimes A)_+$ satisfy $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ in $\mathrm{Cu}(A)$. We must prove that $a \precsim_A b$. For any functional ω on $\mathrm{Cu}(A)$ (as at the beginning of Section 2.4 of [8]), we have $n\omega(\langle a\rangle_A) + m\omega(\langle 1\rangle_A) \leq n\omega(\langle b\rangle_A)$, so $\omega(\langle a\rangle_A) + (m/n)\omega(\langle 1\rangle_A) \leq \omega(\langle b\rangle_A)$. Since $m/n > t_0$, Proposition 3.2.1 of [8] implies that $a \precsim_A b$.

It remain to prove that $s_0 = s$ and $t_0 = t$. We prove that $s_0 = s$; the proof that $t_0 = t$ is the same. Let $m, n \in \mathbb{Z}_{>0}$. We have to prove the following. Suppose that m and n have the property that whenever $a, b \in M_\infty(A)_+$ satisfy
$$n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$$
in $W(A)$, then $a \precsim_A b$. Then whenever $a, b \in (K \otimes A)_+$ satisfy
$$n\langle a\rangle_{K\otimes A} + m\langle 1\rangle_{K\otimes A} \leq n\langle b\rangle_{K\otimes A}$$
in $\mathrm{Cu}(A)$, then $a \precsim_{K\otimes A} b$. We also need to prove the reverse implication.

The reverse implication is easy, so we prove the forwards implication. Let $a, b \in (K \otimes A)_+$ satisfy

$$n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$$

in $\mathrm{Cu}(A)$. Let $\varepsilon > 0$; by Lemma 1.26(11), it suffices to prove that $(a-\varepsilon)_+ \precsim_{K\otimes A} b$. We may clearly assume that $\varepsilon < 1$. Using an isomorphism $K \otimes K \to K$, let $x \in (K \otimes A)_+$ be the direct sum of n copies of a, let $y \in (K \otimes A)_+$ be the direct sum of n copies of b, and let $q \in (K \otimes A)_+$ be the direct sum of m copies of the identity of A. The relation $n\langle a\rangle_{K\otimes A} + m\langle 1\rangle_{K\otimes A} \leq n\langle b\rangle_{K\otimes A}$ means that $x \oplus q \precsim_{K\otimes A} y$. By Lemma 1.26(11c), there exists $\delta > 0$ such that

$$\big((x \oplus q) - \varepsilon\big)_+ \precsim_{K\otimes A} (y - \delta)_+.$$

Since $\varepsilon < 1$ and q is a projection, this relation is equivalent to

$$(x - \varepsilon)_+ \oplus q \precsim_{K\otimes A} (y - \delta)_+.$$

Since $(x - \varepsilon)_+$ is the direct sum of n copies of $(a - \varepsilon)_+$ and $(y - \delta)_+$ is the direct sum of n copies of $(b - \delta)_+$, we therefore have

$$n\langle (a-\varepsilon)_+\rangle_{K\otimes A} + m\langle 1\rangle_{K\otimes A} \leq n\langle (b-\delta)_+\rangle_{K\otimes A}.$$

It follows from Lemma 1.9 of [58] that $\langle (a-\varepsilon)_+\rangle_{K\otimes A}$ and $\langle (b-\delta)_+\rangle_{K\otimes A}$ are actually classes of elements $c, d \in M_\infty(A)_+$, and it is easy to check that inequalities among classes in $W(A)$ which hold in $\mathrm{Cu}(A)$ must also hold in $W(A)$. The assumption therefore implies that $c \precsim_A d$. Thus

$$(a - \varepsilon)_+ \sim_{K\otimes A} c \precsim_A d \sim_{K\otimes A} (b - \delta)_+ \leq b,$$

whence $(a - \varepsilon)_+ \precsim_{K\otimes A} b$, as desired. $\square$

Lemma 3.4. Let $M \in (0, \infty)$, let $f\colon [0, \infty) \to \mathbb{C}$ be a continuous function such that $f(0) = 0$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever A is a C*-algebra and $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a-b\| < \delta$, then $\|f(a)-f(b)\| < \varepsilon$.

This is a standard polynomial approximation argument. We have not found it written in the literature. There are similar arguments in [58] and many other places. It is also stated (in a slightly different form) as Lemma 2.5.11(2) of [36]; the proof there is left to the reader (although a related proof is given). We therefore give it for completeness.

Proof of Lemma 3.4. Choose $n \in \mathbb{Z}_{>0}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that the polynomial function $g(\lambda) = \sum_{k=1}^{n} \alpha_k \lambda^k$ satisfies $|g(\lambda)-f(\lambda)| < \frac{\varepsilon}{3}$ for $\lambda \in [-M-1,\, M+1]$. Define

$$\delta = \min\left(1,\ \frac{\varepsilon}{1 + 3\sum_{k=1}^{n} |\alpha_k| k(M+1)^{k-1}}\right).$$

Now let A be a C*-algebra and let $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$. Then $\|b\| \leq M + 1$. So for $m \in \mathbb{Z}_{>0}$ we have

$$\|a^m - b^m\| \leq \sum_{k=1}^{m} \|a^{k-1}\| \cdot \|a - b\| \cdot \|b^{m-k}\| < m(M+1)^{m-1}\delta.$$

Therefore

$$\|g(a) - g(b)\| \leq \sum_{k=1}^{n} |\alpha_k| k(M+1)^{k-1}\delta < \frac{\varepsilon}{3}.$$

So

$$\|f(a) - f(b)\| \leq \|f(a) - g(a)\| + \|g(a) - g(b)\| + \|g(b) - f(b)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof. $\square$

Proposition 3.5. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) \leq \mathrm{rc}(B)$.

Proof. We use the criterion of Theorem 3.3(1). Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(B)$, and let $a, b \in M_\infty(A)_+$ satisfy $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ in $W(A)$. We want to prove that $a \precsim_A b$. Without loss of generality $\|a\|, \|b\| \leq 1$. It suffices to prove that $(a - \varepsilon)_+ \precsim_A b$ for every $\varepsilon > 0$.

So let $\varepsilon > 0$. We may assume $\varepsilon < 1$. Let $x \in M_\infty(A)_+$ be the direct sum of n copies of a, let $y \in M_\infty(A)_+$ be the direct sum of n copies of b, and let $q \in M_\infty(A)_+$ be the direct sum of m copies of the identity of A. The relation $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ means that $x \oplus q \precsim_A y$. By Lemma 1.26(11b), there exists $\delta > 0$ such that

$$\left((x \oplus q) - \tfrac{1}{3}\varepsilon\right)_+ \precsim_A (y - \delta)_+.$$

Since $\varepsilon < 3$, this is equivalent to

$$\left(x - \tfrac{1}{3}\varepsilon\right)_+ \oplus q \precsim_A (y - \delta)_+. \tag{3.7}$$

Choose $l \in \mathbb{Z}_{>0}$ so large that $a, b \in M_l \otimes A$. Since $m/n > \mathrm{rc}(B)$, there is $k \in \mathbb{Z}_{>0}$ such that

$$\mathrm{rc}(B) < \frac{m}{n} - \frac{2}{k}.$$

Set

$$\varepsilon_0 = \min\left(\tfrac{1}{3}\varepsilon, \tfrac{1}{2}\delta\right).$$

Using Lemma 3.4, choose $\varepsilon_1 > 0$ with $\varepsilon_1 \leq \varepsilon_0$ and so small that whenever D is a C*-algebra and $z \in D_+$ satisfies $\|z\| \leq 1$, then $\|z_0 - z\| < \varepsilon_1$ implies

$$\|(z_0 - \varepsilon_0)_+ - (z - \varepsilon_0)_+\| < \varepsilon_0, \quad \left\|\left(z_0 - \tfrac{1}{3}\varepsilon\right)_+ - \left(z - \tfrac{1}{3}\varepsilon\right)_+\right\| < \varepsilon_0,$$

and

$$\left\|\left(z_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right)_+ - \left(z - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right)_+\right\| < \varepsilon_0.$$

Since A is infinite dimensional and simple, Lemma 1.32 provides $z \in A_+ \setminus \{0\}$ such that $(k+1)\langle z\rangle_A \leq \langle 1\rangle_A$. Using Proposition 2.11 and Lemma 2.6, choose $g \in M_l(B)_+$ and $a_0, b_0 \in M_l(A)_+$ satisfying

$$0 \leq g, a_0, b_0 \leq 1, \quad \|a_0 - a\| < \varepsilon_1, \quad \|b_0 - b\| < \varepsilon_1, \quad g \precsim_A z,$$

and such that

$$(1-g)a_0(1-g),\ (1-g)b_0(1-g) \in M_l \otimes B.$$

From $g \precsim_A z$ and $(k+1)\langle z\rangle_A \leq \langle 1\rangle_A$ we get

$$\sup_{\tau \in \mathrm{T}(A)} d_\tau(g) < \frac{1}{k}. \tag{3.8}$$

Set

$$a_1 = \left[(1-g)a_0(1-g) - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \quad \text{and} \quad b_1 = \left[(1-g)b_0(1-g) - \varepsilon_0\right]_+,$$

which are in $M_l \otimes B$. We claim that a_0, a_1, b_0, and b_1 have the following properties:

(1) $(a - \varepsilon)_+ \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+$.
(2) $\left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \precsim_B a_1 \oplus g$.

(3) $a_1 \precsim_A \left(a - \frac{1}{3}\varepsilon\right)_+$.
(4) $(b-\delta)_+ \precsim_A (b_0 - \varepsilon_0)_+$.
(5) $(b_0 - \varepsilon_0)_+ \precsim_B b_1 \oplus g$.
(6) $b_1 \precsim_A b$.

We give full details of the proofs for (1), (2), and (3) (involving a_0 and a_1). The proofs for (4), (5), and (6) (involving b_0 and b_1) are a bit more sketchy.

We prove (1). Since $\|a_0 - a\| < \varepsilon_1$, the choice of ε_1 implies

$$\left\| \left[a_0 - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+ - \left[a - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+ \right\| < \varepsilon_0 \leq \tfrac{1}{3}\varepsilon.$$

At the last step in the following computation use this inequality and Lemma 1.26(10), at the first step use $\varepsilon_0 \leq \frac{1}{3}\varepsilon$, and at the second step use Lemma 1.26(8), to get

$$\begin{aligned}(a-\varepsilon)_+ &\leq \left[a - \left(\tfrac{2}{3}\varepsilon + \varepsilon_0\right)\right]_+ \\ &= \left[\left(a - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right)_+ - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left[a_0 - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+.\end{aligned}$$

For (4) (the corresponding argument for b_0), we use $\varepsilon_0 \leq \frac{1}{2}\delta$ at the first step; since

$$\|(b - \varepsilon_0)_+ - (b_0 - \varepsilon_0)_+\| < \varepsilon_0,$$

we get

$$(b-\delta)_+ \leq (b - 2\varepsilon_0)_+ = \left[(b-\varepsilon_0)_+ - \varepsilon_0\right]_+ \precsim_A (b_0 - \varepsilon_0)_+.$$

For (2), we use Lemma 1.27 with a_0 in place of a and with $\frac{1}{3}\varepsilon + \varepsilon_0$ in place of ε. For (5), we use Lemma 1.27 with b_0 in place of a and with ε_0 in place of ε.

For (3), begin by recalling that $\|a_0 - a\| < \varepsilon_1$, whence

$$\|(1-g)a_0(1-g) - (1-g)a(1-g)\| < \varepsilon_1.$$

Therefore

$$\left\| \left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \right\| < \varepsilon_0.$$

Using Lemma 1.26(8) at the first step, this fact and Lemma 1.26(10) at the second step, Lemma 1.26(6) at the third step, and Lemma 1.26(17) and $a^{1/2}(1-g)^2 a^{1/2} \leq a$ at the last step, we get

$$\begin{aligned}a_1 &= \left[\left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \varepsilon_0\right]_+ \\ &\precsim_A \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \sim_A \left[a^{1/2}(1-g)^2 a^{1/2} - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+,\end{aligned}$$

as desired.

For (6) (the corresponding part involving b_1), just use

$$\|(1-g)b_0(1-g) - (1-g)b(1-g)\| < \varepsilon_1 \leq \varepsilon_0$$

to get, using Lemma 1.26(4) at the second step,

$$b_1 \precsim_A (1-g)b(1-g) \sim_A b^{1/2}(1-g)^2 b^{1/2} \leq b.$$

The claims (1)–(6) are now proved.

Now let $\tau \in \mathrm{T}(A)$. Recall that x and y are the direct sums of n copies of a and b. Therefore $\left(x - \frac{1}{3}\varepsilon\right)_+$ is the direct sum of n copies of $\left(a - \frac{1}{3}\varepsilon\right)_+$ and $(y-\delta)_+$ is the direct sum of n copies of $(b-\delta)_+$. So the relation (3.7) implies

$$n \cdot d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right) + m \leq n \cdot d_\tau\left((b-\delta)_+\right). \tag{3.9}$$

Using (4) and (5) at the first step and (3.8) at the third step, we get the estimate

$$d_\tau\left((b-\delta)_+\right) \leq d_\tau(b_1) + d_\tau(g) < d_\tau(b_1) + k^{-1}. \tag{3.10}$$

The relation (3) implies

$$d_\tau(a_1) \leq d_\tau\big(\big(a - \tfrac{1}{3}\varepsilon\big)_+\big). \tag{3.11}$$

Using (3.11) and (3.8) at the second step, (3.9) at the third step, and (3.10) at the fourth step, we get

$$\begin{aligned} n \cdot d_\tau(a_1 \oplus g) + m &= n \cdot d_\tau(a_1) + m + n \cdot d_\tau(g) \\ &\leq n \cdot d_\tau\big(\big(a - \tfrac{1}{3}\varepsilon\big)_+\big) + m + nk^{-1} \\ &\leq n \cdot d_\tau\big((b-\delta)_+\big) + nk^{-1} \\ &\leq n \cdot d_\tau(b_1) + 2nk^{-1}. \end{aligned}$$

It follows that

$$d_\tau(a_1 \oplus g) + \frac{m}{n} - \frac{2}{k} \leq d_\tau(b_1).$$

This holds for all $\tau \in \mathrm{T}(A)$, and therefore, by Theorem 2.12, for all $\tau \in \mathrm{T}(B)$.

Subalgebras of exact C*-algebras are exact (by Proposition 7.1(1) of [32]), so Theorem 5.11 of [27] implies that $\mathrm{QT}(B) = \mathrm{T}(B)$. Since

$$\frac{m}{n} - \frac{2}{k} > \mathrm{rc}(B),$$

and since $a_1, b_1, g \in M_l \otimes B$, it follows that $a_1 \oplus g \precsim_B b_1$. Using this relation at the third step, (1) at the first step, (2) at the second step, and (6) at the last step, we then get

$$(a-\varepsilon)_+ \precsim_A \big[a_0 - \big(\varepsilon_0 + \tfrac{1}{3}\varepsilon\big)\big]_+ \precsim_A a_1 \oplus g \precsim_B b_1 \precsim_A b.$$

This completes the proof that $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. ∎

Proposition 3.6. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) \geq \mathrm{rc}(B)$.

Proof. We use Theorem 3.3(2). Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(A)$. Let $l \in \mathbb{Z}_{>0}$, and let $a, b \in (M_l \otimes B)_+$ satisfy

$$(n+1)\langle a\rangle_B + m\langle 1\rangle_B \leq n\langle b\rangle_B$$

in $W(B)$. We must prove that $a \precsim_B b$. Without loss of generality $\|a\| \leq 1$. Moreover, by Lemma 1.26(11), it is enough to show that for every $\varepsilon > 0$ we have $(a-\varepsilon)_+ \precsim_B b$. So let $\varepsilon > 0$. Without loss of generality $\varepsilon < 1$.

Choose $k \in \mathbb{Z}_{>0}$ such that

$$\frac{km}{kn+1} > \mathrm{rc}(A).$$

Then in $W(B)$ we have

$$(kn+1)\langle a\rangle_B + km\langle 1\rangle_B \leq k(n+1)\langle a\rangle_B + km\langle 1\rangle_B \leq kn\langle b\rangle_B.$$

Let $x \in M_\infty(B)_+$ be the direct sum of $kn+1$ copies of a, let $z \in M_\infty(B)_+$ be the direct sum of kn copies of b, and let $q \in M_\infty(B)_+$ be the direct sum of km copies of 1. Then, by definition, $x \oplus q \precsim_B z$. Therefore Lemma 1.26(11) provides $\delta > 0$ such that $\big(x \oplus q - \frac{1}{4}\varepsilon\big)_+ \precsim_B (z-\delta)_+$. Since $\varepsilon < 4$, we have

$$\big(x \oplus q - \tfrac{1}{4}\varepsilon\big)_+ = \big(x - \tfrac{1}{4}\varepsilon\big)_+ \oplus \big(q - \tfrac{1}{4}\varepsilon\big)_+ \sim_B \big(x - \tfrac{1}{4}\varepsilon\big)_+ \oplus q,$$

so

$$(kn+1)\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_+ \big\rangle_B + km\langle 1\rangle_B \leq kn\langle (b-\delta)_+\rangle_B.$$

Lemma 1.35 provides $c \in (M_l \otimes B)_+$ and $y \in (M_l \otimes B)_+ \setminus \{0\}$ such that

$$kn\langle (b-\delta)_+\rangle_B \leq (kn+1)\langle c\rangle_B \quad \text{and} \quad \langle c\rangle_B + \langle y\rangle_B \leq \langle b\rangle_B \tag{3.12}$$

in $W(B)$. Then

$$(kn+1)\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_+\big\rangle_B + km\langle 1\rangle_B \leq (kn+1)\langle c\rangle_B.$$

Applying the map $W(A) \to W(B)$, we get

$$(kn+1)\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_+\big\rangle_A + km\langle 1\rangle_A \leq (kn+1)\langle c\rangle_A.$$

For $\tau \in \mathrm{T}(A)$, we apply d_τ and divide by $kn+1$ to get

$$d_\tau\big(\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_+\big\rangle\big) + \frac{km}{kn+1} \leq d_\tau(c).$$

Since $\mathrm{QT}(A) = \mathrm{T}(A)$ (by Theorem 5.11 of [27]) and

$$\frac{km}{kn+1} > \mathrm{rc}(A),$$

it follows that $\big(a - \frac{1}{4}\varepsilon\big)_+ \precsim_A c$. In particular, there is $v \in M_l \otimes A$ such that

$$\big\| vcv^* - \big(a - \tfrac{1}{4}\varepsilon\big)_+\big\| < \tfrac{1}{4}\varepsilon.$$

Since B is large in A, we can apply Proposition 2.11 and Lemma 2.6 to find $v_0 \in M_l \otimes A$ and $g \in M_l \otimes B$ with $0 \leq g \leq 1$ and such that

$$g \precsim_B y, \quad \|v_0\| \leq \|v\|, \quad \|v_0 - v\| < \frac{\varepsilon}{4\|v\|\|c\| + 1}, \quad \text{and} \quad (1-g)v_0 \in M_l \otimes B.$$

It follows that $\|v_0^* c v_0 - v^* c v\| < \frac{\varepsilon}{2}$, so

$$\big\|(1-g)v_0 c[(1-g)v_0]^* - (1-g)\big(a - \tfrac{1}{4}\varepsilon\big)_+(1-g)\big\| < \tfrac{3}{4}\varepsilon.$$

Therefore, using Lemma 1.26(10) at the first step,

$$\big[(1-g)\big(a - \tfrac{1}{4}\varepsilon\big)_+(1-g) - \tfrac{3}{4}\varepsilon\big]_+ \precsim_B (1-g)v_0 c[(1-g)v_0]^* \precsim_B c. \tag{3.13}$$

Using Lemma 1.27 at the first step, with $\big(a - \frac{1}{4}\varepsilon\big)_+$ in place of a and $\frac{3}{4}\varepsilon$ in place of ε, as well as Lemma 1.26(8), using (3.13) at the second step, using the choice of g at the third step, and using the second part of (3.12) at the fourth step, we get

$$(a-\varepsilon)_+ \precsim_B \big[(1-g)\big(a - \tfrac{1}{4}\varepsilon\big)_+(1-g) - \tfrac{3}{4}\varepsilon\big]_+ \oplus g \precsim_B c \oplus g \precsim_B c \oplus y \precsim_B b.$$

This is the relation we need, and the proof is complete. ☐

Proof of Theorem 3.2. Combine Proposition 3.5 and Proposition 3.6. ☐

4. Large Subalgebras in Crossed Products by $\mathbb{Z}$

4.1. **Crossed products and orbit breaking subalgebras.** We begin with a review of crossed products by discrete groups. General references include [72] and Chapter 7 of [52].

Definition 4.1. Let G be a (discrete) group, let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. The *skew group ring* $A[G]$ is the set of all formal sums

$$\sum_{g\in G} a_g u_g$$

with $a_g \in A$ for all $g \in G$ and $a_g = 0$ for all but finitely many $g \in G$. The product and adjoint are determined by requiring that:

(1) u_g is unitary for $g \in G$.
(2) $u_g u_h = u_{gh}$ for $g, h \in G$.
(3) $u_g a u_g^* = \alpha_g(a)$ for $g \in G$ and $a \in A$.

Thus,

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_{g^{-1}}]) \cdot u_{gh} = (a\alpha_g(b)) \cdot u_{gh}$$

and

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, and these operations are extended linearly. We omit the proof that this really makes $A[G]$ a *-algebra.

Letting $1 \in G$ be the group identity, it is clear that $a \mapsto a \cdot u_1$ is a unital homomorphism of *-algebras. We often identify A with $\{a \cdot u_1 : a \in A\} \subset A[G]$.

For a compact Hausdorff space X and an action $(g, x) \mapsto gx$ of G on X, we take $A = C(X)$ and define $\alpha \colon G \to \operatorname{Aut}(C(X))$ by $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C(X)$, and $x \in X$.

The crossed product $C^*(G, A, \alpha)$ (or $C^*(G, X)$ if the action comes from an action of G on X as above) is the completion of $A[G]$ in a suitable C* norm. It is chosen so as to give a one to one correspondence between unital representations $\sigma \colon C^*(G, A, \alpha) \to L(H)$ on a Hilbert space H and unital covariant representations of (G, A, α) on H, that is, pairs (v, π) consisting of a unitary representation $g \mapsto v_g$ of G on H and a unital representation π of A on H such that $v_g \pi(a) v_g^* = \pi(\alpha_g(a))$ for $g \in G$ and $a \in A$. The relation between σ and (v, π) is given as follows. If $a = \sum_{g \in G} a_g u_g \in A[G]$ as above, then $\sigma(a) = \sum_{g \in G} \pi(a_g) v_g$.

For completeness, we describe how to construct a family of unital covariant representations, the regular covariant representations. Let $\pi_0 \colon A \to L(H_0)$ be a unital representation of A on a Hilbert space H_0. (Here, we actually care mostly about faithful representations.) Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g \in G}$ in $\prod_{g \in G} H_0$ such that $\sum_{g \in G} \|\xi_g\|^2 < \infty$, with the scalar product

$$\big\langle (\xi_g)_{g \in G},\, (\eta_g)_{g \in G} \big\rangle = \sum_{g \in G} \langle \xi_g, \eta_g \rangle.$$

Then define $(v_g \xi)_h = \xi_{g^{-1}h}$ for $g, h \in G$ and $\xi \in H$ and $(\pi(a)\xi)_h = \pi_0(\alpha_h^{-1}(a))(\xi_h)$ for $a \in A$, $h \in G$, and $\xi \in H$. For $a = \sum_{g \in G} a_g u_g \in A[G]$, this gives

$$(\sigma(a)\xi)_h = \sum_{g \in G} \pi_0(\alpha_h^{-1}(a_g))(\xi_{g^{-1}h})$$

for all $h \in G$.

In [52], the *reduced crossed product* $C^*_{\mathrm{r}}(G, A, \alpha)$ (or $C^*_{\mathrm{r}}(G, X)$ if the action comes from an action of G on X as above) is defined to be be the completion of $A[G]$ in the norm $\|a\| = \|\sigma(a)\|$ when π_0 is taken to be the universal representation of A. It follows from Theorem 7.7.5 of [52] that $C^*_{\mathrm{r}}(G, A, \alpha)$ is independent of the choice of π_0 as long as π_0 is injective. For amenable groups, including $\mathbb{Z}$, the natural map $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism. (See Theorem 7.7.7 of [52].) In the cases we are interested in, $C^*(G, A, \alpha)$ is simple. When this happens, one can obtain $C^*(G, A, \alpha)$ by starting with any unital representation π_0 of A on a nonzero Hilbert space and completing $A[G]$ in the norm coming from the representation σ above.

The map $a \mapsto a u_1$, interpreted in the setup above as a map $A \to A[G]$, is an injective unital homomorphism from A to $C^*(G, A, \alpha)$, and also an injective unital homomorphism from A to $C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, we identify A with a subalgebra of $C^*(G, A, \alpha)$ and also with a subalgebra of $C^*_{\mathrm{r}}(G, A, \alpha)$. There is a faithful conditional expectation $E_\alpha\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ (written just E when α is understood) such that for $a = \sum_{g \in G} a_g u_g \in A[G]$ (so that $a_g = 0$ for all but finitely many $g \in G$) one has $E_\alpha(a) = a_1$. One must work to prove existence, since continuity on the dense subalgebra $A[G]$ is not obvious. Further work is required to prove faithfulness. (Composing with the natural map $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$, one gets a conditional expectation $C^*(G, A, \alpha) \to A$, which is not faithful unless the natural map is an isomorphism.)

If G is amenable (for general G, if one uses $C^*_{\mathrm{r}}(G, A, \alpha)$ instead of $C^*(G, A, \alpha)$), every element $a \in C^*(G, A, \alpha)$ has a formal series expansion $\sum_{g \in G} a_g u_g$ with coefficients $a_g \in A$ for $g \in G$. The coefficients a_g are obtained as $a_g = E(a u_g^*)$. In general, this series does *not* converge in norm, so care must be taken whenever it is used.

This section deals with the special case $G = \mathbb{Z}$, and mainly with the case $A = C(X)$. We prove that if X is infinite, $\alpha \in \mathrm{Aut}(C(X))$ comes from a minimal homeomorphism h, and Y intersects each orbit of h at most once, then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$. For easy reference, we summarize the notation from the discussion above in this special case.

Notation 4.2. Let X be a compact metric space, and let $h\colon X \to X$ be a homeomorphism. We take the corresponding automorphism $\alpha \in \mathrm{Aut}(C(X))$ to be given by $\alpha(f)(x) = f(h^{-1}(x))$ for $f \in C(X)$ and $x \in X$. The crossed product is $C^*(\mathbb{Z}, X, h)$. (Since $\mathbb{Z}$ is amenable, the full and reduced crossed products are the same.) We let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary corresponding to the generator $1 \in \mathbb{Z}$. Thus, $u f u^* = f \circ h^{-1}$ for all $f \in C(X)$, and for $n \in \mathbb{Z}$ the standard unitary u_n (following the notation above) is $u_n = u^n$. The dense subalgebra $C(X)[\mathbb{Z}]$ is the set of all finite sums

$$a = \sum_{k=-n}^{n} f_k u^k \tag{4.1}$$

with $n \in \mathbb{Z}_{\geq 0}$ and $f_{-n}, f_{-n+1}, \ldots, f_n \in C(X)$. We identify $C(X)$ with a subalgebra of $C^*(\mathbb{Z}, X, h)$ in the standard way: it is all a as in (4.1) such that $f_k = 0$ for $k \neq 0$. The standard conditional expectation $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ is given on $C(X)[\mathbb{Z}]$ by $E(a) = f_0$ when a is as in (4.1).

In order to state more general results, we generalize the construction of Definition 1.7 in Section 1.1. Notation 4.3 and Definition 4.4 below differ from Notation 1.6 and Definition 1.7 in that they consider $C_0(X, D)$ for a C*-algebra D instead of just $C_0(X)$.

Notation 4.3. For a locally compact Hausdorff space X, a C*-algebra D, and an open subset $U \subset X$, we use the abbreviation

$$C_0(U, D) = \big\{ f \in C_0(X, D) \colon f(x) = 0 \text{ for all } x \in X \setminus U \big\} \subset C_0(X, D).$$

This subalgebra is of course canonically isomorphic to the usual algebra $C_0(U, D)$ when U is considered as a locally compact Hausdorff space in its own right. As in Notation 1.6, if $D = \mathbb{C}$ we omit it from the notation.

In particular, if $Y \subset X$ is closed, then

$$C_0(X \setminus Y,\, D) = \big\{ f \in C_0(X, D) \colon f(x) = 0 \text{ for all } x \in Y \big\}. \tag{4.2}$$

Definition 4.4. Let X be a locally compact Hausdorff space, let D be a unital C*-algebra, and let $h\colon X \to X$ be a homeomorphism. Let $\alpha \in \mathrm{Aut}(C(X, D))$ be an automorphism which "lies over h", in the sense that there exists a function $x \mapsto \alpha_x$ from X to $\mathrm{Aut}(D)$ such that $\alpha(a)(x) = \alpha_x\big(a(h^{-1}(x))\big)$ for all $x \in X$ and $a \in C_0(X, D)$. Let $Y \subset X$ be a nonempty closed subset, and, following (4.2), define

$$C^*\big(\mathbb{Z}, C_0(X, D), \alpha\big)_Y = C^*\big(C_0(X, D), C_0(X \setminus Y, D)u\big) \subset C^*\big(\mathbb{Z}, C_0(X, D), \alpha\big).$$

We call it the *Y-orbit breaking subalgebra* of $C^*\big(\mathbb{Z}, C_0(X, D), \alpha\big)$.

We give a sketch of the proof of Theorem 1.8, namely that if $h\colon X \to X$ is a minimal homeomorphism and $Y \subset X$ is a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 1.3. Our presentation differs from that of [58] and [5] in that we prove the result directly rather than via large subalgebras of crossed product type.

Under some technical conditions on α and D, similar methods can be used to prove the analogous result for $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)_Y$. The following theorem is a consequence of results in [4].

Theorem 4.5. Let X be an infinite compact metric space, let $h\colon X \to X$ be a minimal homeomorphism, let D be a simple unital C*-algebra which has a tracial state, and let $\alpha \in \mathrm{Aut}(C(X, D))$ lie over h. Assume that D has strict comparison of positive elements, or that the automorphisms α_x in Definition 4.4 are all approximately inner. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)_Y$ is a centrally large subalgebra of $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)$ in the sense of Definition 1.3.

The ideas of the proof of Theorem 1.8 are all used in the proof of the general theorem behind Theorem 4.5, but additional work is needed to deal with the presence of D.

4.2. **The proof of Theorem 1.8.** We describe the proof of Theorem 1.8, omitting a few details.

Proposition 4.6 (Proposition 7.5 of [58])**.** Let X be a compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, X, h)$ and $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ be as in Notation 4.2. Let $Y \subset X$ be a nonempty closed subset. For $n \in \mathbb{Z}$, set

$$Y_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0. \end{cases}$$

Then

$$C^*(\mathbb{Z}, X, h)_Y = \big\{ a \in C^*(\mathbb{Z}, X, h)\colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z} \big\} \tag{4.3}$$

and

$$\overline{C^*(\mathbb{Z}, X, h)_Y \cap C(X)[\mathbb{Z}]} = C^*(\mathbb{Z}, X, h)_Y. \tag{4.4}$$

Sketch of proof. Define B to be the right hand side of (4.3), that is,

$$B = \big\{ a \in C^*(\mathbb{Z}, X, h)\colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z} \big\},$$

and define

$$B_0 = B \cap C(X)[\mathbb{Z}].$$

We claim that B_0 is dense in B. To see this, let $b \in B$ and for $k \in \mathbb{Z}$ define $b_k = E(bu^{-k}) \in C_0(X \setminus Y_k)$. Then for $n \in \mathbb{Z}_{>0}$, the element

$$a_n = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) b_k u^k$$

is clearly in B_0, and Theorem VIII.2.2 of [15] implies that $\lim_{n\to\infty} a_n = b$. The claim follows. In particular, (4.4) will now follow from (4.3), so we need only prove (4.3).

Next, one proves that B_0 is a *-algebra. It is enough to prove that if $f \in C_0(X \setminus Y_m)$ and $g \in C_0(X \setminus Y_n)$, then $(fu^m)(gu^n) \in B_0$ and $(fu^m)^* \in B_0$. The proof involves manipulations with h and the sets Y_n, and the proof that $(fu^m)(gu^n) \in B_0$ must be broken into six cases: all combinations of signs of m, n, and $m+n$ which can actually occur. We refer to [58] for the details.

Since $C(X) \subset B_0$ and $C_0(X \setminus Y)u \subset B_0$, it follows that $C^*(\mathbb{Z}, X, h)_Y \subset \overline{B_0} = B$.

The next step is to show that for all $n \in \mathbb{Z}$ and $f \in C_0(X \setminus Y_n)$, we have $fu^n \in C^*(\mathbb{Z}, X, h)_Y$. For $n = 0$ this is trivial. Let $n > 0$, and let $f \in C_0(X \setminus Y_n)$. Define $f_0 = (\mathrm{sgn} \circ f)|f|^{1/n}$ and for $j = 1, 2, \ldots, n-1$ define $f_j = |f \circ h^j|^{1/n}$. The definition of Y_n implies that $f_0, f_1, \ldots, f_{n-1} \in C_0(X \setminus Y)$. Therefore the element

$$a = (f_0 u)(f_1 u) \cdots (f_{n-1} u)$$

is in $C^*(\mathbb{Z}, X, h)_Y$. A computation (carried out in [58]) shows that $a = fu^n$. The case $n < 0$ is reduced to the case $n > 0$ by taking adjoints; see [58] for details.

It now follows that $B_0 \subset C^*(\mathbb{Z}, X, h)_Y$. Combining this result with $\overline{B_0} = B$ and $C^*(\mathbb{Z}, X, h)_Y \subset B$, we get $C^*(\mathbb{Z}, X, h)_Y = B$. □

Corollary 4.7 (Corollary 7.6 of [58])**.** Let X be a compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $Y \subset X$ be a nonempty closed subset. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary, as in Notation 4.2, and let $v \in C^*(\mathbb{Z}, X, h^{-1})$ be the analogous standard unitary in $C^*(\mathbb{Z}, X, h^{-1})$. Then there exists a unique homomorphism $\varphi\colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ such that $\varphi(f) = f$ for $f \in C(X)$ and $\varphi(v) = u^*$, the map φ is an isomorphism, and

$$\varphi\big(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z}, X, h)_Y.$$

See [58] for the straightforward proof.

Lemma 4.8 (Lemma 7.4 of [58])**.** Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $K \subset X$ be a compact set such that $h^n(K) \cap K = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let $U \subset X$ be a nonempty open subset. Then there exist $l \in \mathbb{Z}_{\geq 0}$, compact sets $K_1, K_2, \ldots, K_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$, such that $K \subset \bigcup_{j=1}^{l} K_j$ and such that $h^{n_1}(K_1)$, $h^{n_2}(K_2)$, $\ldots$, $h^{n_l}(K_l)$ are disjoint subsets of U.

Sketch of proof. Choose a nonempty open subset $V \subset X$ such that $\overline{V}$ is compact and contained in U. Use minimality of h to cover K with the images of V under finitely many negative powers of h, say $h^{-n_1}(V)$, $h^{-n_2}(V)$, $\ldots$, $h^{-n_l}(V)$. Set $K_j = h^{-n_j}\big(\overline{V}\big) \cap K$ for $j = 1, 2, \ldots, l$. □

The next lemma is straightforward if one only asks that $f \precsim_{C^*(\mathbb{Z}, X, h)} g$ (Cuntz subequivalence in the crossed product), and then doing it for one value of n is equivalent to doing it for any other. Getting $f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g$ for both positive n and negative n is a key step in showing $C^*(\mathbb{Z}, X, h)_Y$ a large subalgebra of $C^*(\mathbb{Z}, X, h)$.

Lemma 4.9 (Lemma 7.7 of [58])**.** Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let $U \subset X$ be a nonempty open subset and let $n \in \mathbb{Z}$. Then there exist $f, g \in C(X)_+$ such that

$$f|_{h^n(Y)} = 1, \quad 0 \leq f \leq 1, \quad \operatorname{supp}(g) \subset U, \quad \text{and} \quad f \precsim_{C^*(\mathbb{Z},X,h)_Y} g.$$

Proof. We first prove this when $n = 0$.

Apply Lemma 4.8 with Y in place of K, obtaining $l \in \mathbb{Z}_{\geq 0}$, compact sets $Y_1, Y_2, \ldots, Y_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$. Set $N = \max(n_1, n_2, \ldots, n_l)$. Choose disjoint open sets $V_1, V_2, \ldots, V_l \subset U$ such that $h^{n_j}(Y_j) \subset V_j$ for $j = 1, 2, \ldots, l$. Then $Y_j \subset h^{-n_j}(V_j)$, so the sets $h^{-n_1}(V_1)$, $h^{-n_2}(V_2)$, $\ldots$, $h^{-n_l}(V_l)$ cover Y. For $j = 1, 2, \ldots, l$, define

$$W_j = h^{-n_j}(V_j) \cap \left(X \setminus \bigcup_{n=1}^{N} h^{-n}(Y) \right).$$

Then $W_1, W_2, \ldots, W_l$ form an open cover of Y. Therefore there are $f_1, f_2, \ldots, f_l \in C(X)_+$ such that for $j = 1, 2, \ldots, l$ we have $\operatorname{supp}(f_j) \subset W_j$ and $0 \leq f_j \leq 1$, and such that the function $f = \sum_{j=1}^{l} f_j$ satisfies $f(x) = 1$ for all $x \in Y$ and $0 \leq f \leq 1$. Further define $g = \sum_{j=1}^{l} f_j \circ h^{-n_j}$. Then $\operatorname{supp}(g) \subset U$.

Let $u \in C^*(\mathbb{Z}, X, h)$ be as in Notation 4.2. For $j = 1, 2, \ldots, l$, set $a_j = f_j^{1/2} u^{-n_j}$. Since f_j vanishes on $\bigcup_{n=1}^{n_j} h^{-n}(Y)$, Proposition 4.6 implies that $a_j \in C^*(\mathbb{Z}, X, h)_Y$. Therefore, in $C^*(\mathbb{Z}, X, h)_Y$ we have

$$f_j \circ h^{-n_j} = a_j^* a_j \sim_{C^*(\mathbb{Z},X,h)_Y} a_j a_j^* = f_j.$$

Consequently, using Lemma 1.26(12) at the second step and Lemma 1.26(13) and disjointness of the supports of the functions $f_j \circ h^{-n_j}$ at the last step, we have

$$f = \sum_{j=1}^{l} f_j \precsim_{C^*(\mathbb{Z},X,h)_Y} \bigoplus_{j=1}^{l} f_j \sim_{C^*(\mathbb{Z},X,h)_Y} \bigoplus_{j=1}^{l} f_j \circ h^{-n_j} \sim_{C^*(\mathbb{Z},X,h)_Y} g.$$

This completes the proof for $n = 0$.

Now suppose that $n > 0$. Choose functions f and g for the case $n = 0$, and call them f_0 and g. Since $f_0(x) = 1$ for all $x \in Y$, and since $Y \cap \bigcup_{l=1}^{n} h^{-l}(Y) = \varnothing$, there is $f_1 \in C(X)$ with $0 \leq f_1 \leq f_0$, $f_1(x) = 1$ for all $x \in Y$, and $f_1(x) = 0$ for $x \in \bigcup_{l=1}^{n} h^{-l}(Y)$. Set $v = f_1^{1/2} u^{-n}$ and $f = f_1 \circ h^{-n}$. Then $f(x) = 1$ for all $x \in h^n(Y)$ and $0 \leq f \leq 1$. Proposition 4.6 implies that $v \in C^*(\mathbb{Z}, X, h)_Y$. We have

$$v^* v = u^n f_1 u^{-n} = f_1 \circ h^{-n} = f \quad \text{and} \quad v v^* = f_1.$$

Using Lemma 1.26(4), we thus get

$$f \sim_{C^*(\mathbb{Z},X,h)_Y} f_1 \leq f_0 \precsim_{C^*(\mathbb{Z},X,h)_Y} g.$$

This completes the proof for the case $n > 0$.

Finally, we consider the case $n < 0$. In this case, we have $-n-1 \geq 0$. Apply the cases already done with h^{-1} in place of h. We get $f, g \in C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}$ such that $f(x) = 1$ for all $x \in (h^{-1})^{-n-1}(h^{-1}(Y)) = h^n(Y)$, such that $0 \leq f \leq 1$, such that $\operatorname{supp}(g) \subset U$, and such that $f \precsim_{C^*(\mathbb{Z},X,h^{-1})_{h^{-1}(Y)}} g$. Let $\varphi\colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ be the isomorphism of Corollary 4.7. Then

$$\varphi(f) = f, \quad \varphi(g) = g, \quad \text{and} \quad \varphi\big(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z}, X, h)_Y.$$

Therefore $f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g$. □

The following result is a special case of Lemma 7.9 of [58]. The basic idea has been used frequently; related arguments can be found, for example, in the proofs of Theorem 3.2 of [17], Lemma 2 and Theorem 1 in [2], Lemma 10 of [34], and Lemma 3.2 of [50]. (The papers listed are not claimed to be representative or to be the original sources; they are ones I happen to know of.)

Lemma 4.10. Let X be an infinite compact space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in B_+ \setminus \{0\}$. Then there exists $b \in C(X)_+ \setminus \{0\}$ such that $b \precsim_B a$.

Sketch of proof. Without loss of generality $\|a\| \leq 1$. The conditional expectation $E_\alpha \colon C^*_{\mathrm{r}}(G, X) \to C(X)$ is faithful. Therefore $E_\alpha(a) \in C(X)$ is a nonzero positive element. Set $\varepsilon = \frac{1}{6}\|E_\alpha(a)\|$. Choose $c \in B \cap C(X)[\mathbb{Z}]$ such that $\|c - a^{1/2}\| < \varepsilon$ and $\|c\| \leq 1$. One can check that $\|E_\alpha(c^*c)\| > 4\varepsilon$. There are $n \in \mathbb{Z}_{\geq 0}$ and $g_{-n}, g_{-n+1}, \ldots, g_n \in C(X)$ such that $c^*c = \sum_{k=-n}^{n} g_k u^k$. We have $g_0 = E_\alpha(c^*c) \in C(X)_+$ and $\|g_0\| > 4\varepsilon$. Therefore there is $x \in X$ such that $g_0(x) > 4\varepsilon$. Choose $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$, and the sets $h^k(\mathrm{supp}(f))$ are disjoint for $k = -n, -n+1, \ldots, n$. One can then check that $fc^*cf = fg_0 f$, so that $\|fc^*cf\| > 4\varepsilon$. Therefore $(fc^*cf - 2\varepsilon)_+$ is a nonzero element of $C(X)$. Using Lemma 1.26(6) at the first step, Lemma 1.26(17) and $cf^2c^* \leq cc^*$ at the second step, and Lemma 1.26(10) and $\|cc^* - a\| < 2\varepsilon$ at the last step, we then have

$$(fc^*cf - 2\varepsilon)_+ \sim_B (cf^2c^* - 2\varepsilon)_+ \precsim_B (cc^* - 2\varepsilon)_+ \precsim_B a.$$

This completes the proof. □

Corollary 4.11. Let X be an infinite compact Hausdorff space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in A_+ \setminus \{0\}$ and let $b \in B_+ \setminus \{0\}$. Then there exists $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_{C^*(\mathbb{Z}, X, h)} a$ and $f \precsim_B b$.

Proof. Applying Lemma 4.10 to both a (with $C^*(\mathbb{Z}, X, h)$ in place of B) and b (with B as given), we see that it is enough to prove the corollary for $a, b \in C(X)_+ \setminus \{0\}$. Also, without loss of generality $\|a\| \leq 1$.

Choose $x_0 \in X$ such that $b(x_0) \neq 0$. Since the orbit of x_0 is dense, there is $n \in \mathbb{Z}$ such that $a(h^n(x_0)) \neq 0$. Define $f \in C(X)$ by $f(x) = b(x)a(h^n(x))$ for $x \in X$. Then $f \neq 0$ since $f(x_0) \neq 0$. We have $f \precsim_B b$ because $\|a\| \leq 1$ implies $f \leq b$. Also, $f = (b^{1/2}u^{-n})a(b^{1/2}u^{-n})^*$ so $f \precsim_{C^*(\mathbb{Z}, X, h)} a$. □

The next result is a standard type of approximation lemma.

Lemma 4.12. Let A be a C*-algebra, and let $S \subset A$ be a subset which generates A as a C*-algebra and such that $a \in S$ implies $a^* \in S$. Then for every finite subset $F \subset A$ and every $\varepsilon > 0$ there are a finite subset $T \subset S$ and $\delta > 0$ such that whenever $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta$ for all $b \in T$, then $\|ca - ac\| < \varepsilon$ for all $a \in F$.

Proof. Let $B \subset A$ be the set of all $a \in A$ such that for every $\varepsilon > 0$ there are $T(a, \varepsilon) \subset S$ and $\delta(a, \varepsilon) > 0$ as in the statement of the lemma, that is, $T(a, \varepsilon)$ is finite and whenever $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta(a, \varepsilon)$ for all $b \in T(a, \varepsilon)$, then $\|ca - ac\| < \varepsilon$.

We have $S \subset B$, as is seen by taking $T(a, \varepsilon) = \{a\}$ and $\delta(a, \varepsilon) = \varepsilon$. If $a \in B$, then also $a^* \in B$, as is seen by taking

$$T(a^*, \varepsilon) = \big\{ b^* \colon b \in T(a, \varepsilon) \big\} \quad \text{and} \quad \delta(a^*, \varepsilon) = \delta(a, \varepsilon).$$

We show that B is closed under addition. So let $a_1, a_2 \in B$ and let $\varepsilon > 0$. Define

$$T = T\big(a_1, \tfrac{\varepsilon}{2}\big) \cup T\big(a_2, \tfrac{\varepsilon}{2}\big) \quad \text{and} \quad \delta = \min\big(\delta\big(a_1, \tfrac{\varepsilon}{2}\big), \, \delta\big(a_2, \tfrac{\varepsilon}{2}\big)\big).$$

Suppose $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta$ for all $b \in T$. Then

$$\|ca_1 - a_1 c\| < \tfrac{\varepsilon}{2} \quad \text{and} \quad \|ca_2 - a_2 c\| < \tfrac{\varepsilon}{2},$$

so

$$\|c(a_1 + a_2) - (a_1 + a_2)c\| \leq \|ca_1 - a_1 c\| + \|ca_2 - a_2 c\| < \varepsilon.$$

This shows that $a_1 + a_2 \in B$. To show that if $a_1, a_2 \in B$ then $a_1 a_2 \in B$, we use a similar argument, taking

$$\varepsilon_0 = \frac{\varepsilon}{1 + \|a_1\| + \|a_2\|}$$

and using the choices

$$T = T(a_1, \varepsilon_0) \cup T(a_2, \varepsilon_0) \quad \text{and} \quad \delta = \min\big(\delta(a_1, \varepsilon_0), \, \delta(a_2, \varepsilon_0)\big),$$

and the estimate

$$\|ca_1 a_2 - a_1 a_2 c\| \leq \|ca_1 - a_1 c\| \|a_2\| + \|a_1\| \|ca_2 - a_2 c\|.$$

Finally, we claim that B is closed. So let $a \in \overline{B}$ and let $\varepsilon > 0$. Choose $a_0 \in B$ such that $\|a - a_0\| < \frac{\varepsilon}{3}$. Define

$$T = T\big(a_0, \tfrac{\varepsilon}{3}\big) \quad \text{and} \quad \delta = \delta\big(a_0, \tfrac{\varepsilon}{3}\big).$$

Suppose $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta$ for all $b \in T$. Then $\|ca_0 - a_0 c\| < \frac{\varepsilon}{3}$, so

$$\|ca - ac\| \leq 2\|c\| \|a - a_0\| + \|ca_0 - a_0 c\| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The claim is proved.

Since S generates A as a C*-algebra, we have $B = A$. Now let $F \subset A$ be finite and let $\varepsilon > 0$. The conclusion of the lemma follows by taking

$$T = \bigcup_{a \in F} T(a, \varepsilon) \quad \text{and} \quad \delta = \min_{a \in F} \delta(a, \varepsilon).$$

This completes the proof. $\square$

Proof of Theorem 1.8. Set $A = C^*(\mathbb{Z}, X, h)$ and $B = C^*(\mathbb{Z}, X, h)_Y$. Since h is minimal, it is well known that A is simple and finite. Also clearly A is infinite dimensional.

We claim that the following holds. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, and let $f \in C(X)_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$ and $g \in C(X)$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B f$.
(5) $\|gu - ug\| < \varepsilon$.

Suppose the claim has been proved; we show that the theorem follows. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, let $r \in A_+ \setminus \{0\}$, and let $s \in B_+ \setminus \{0\}$. (The elements r and s play the roles of x and y in Definition 1.3. Here, we use x and y for elements of X.) Apply Lemma 2.5 with r in place of x, getting $r_0 \in A_+ \setminus \{0\}$. In Lemma 4.12, take $S = C(X) \cup \{u, u^*\}$, take $F = \{a_1, a_2, \ldots, a_m\}$, and let $\varepsilon > 0$ be as given. Let the finite set $T \subset S$ and $\delta > 0$ be as in the conclusion. We may assume that $u, u^* \in T$. Apply Corollary 4.11 with r_0 in place of a and s in place of b, getting $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_A r_0$ and $f \precsim_B s$. Apply the claim with $a_1, a_2, \ldots, a_m$ as given, and with $\min(\varepsilon, \delta)$ in place of ε.

We can now verify the conditions of Definition 1.3. Conditions (1), (2), and (3) there are conditions (1), (2), and (3) here. Condition (4) there follows from condition (4) here and the relations $f \precsim_A r_0 \precsim_A r$ and $f \precsim_B s$. Condition (5) there follows from $g \precsim_A r_0$ and the choice of r_0. It remains only to verify condition (6) there, namely $\|g a_j - a_j g\| < \varepsilon$ for $j = 1, 2, \ldots, m$. It suffices to check that $\|gb - bg\| < \delta$ for all $b \in T$. We have $\|gu - ug\| < \delta$ by construction. Also, $gu^* - u^* g = -u^*(gu - ug)u^*$, so $\|gu^* - u^* g\| < \delta$. Finally, if $b \in T$ is any element other than u or u^*, then $b \in C(X)$, so $gb = bg$. This completes the proof that the claim implies the conclusion of the theorem.

We now prove the claim. Choose $c_1, c_2, \ldots, c_m \in C(X)[\mathbb{Z}]$ such that $\|c_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, m$. (This estimate is condition (2).) Choose $N \in \mathbb{Z}_{>0}$ such that there are $c_{j,l} \in C(X)$ for $j = 1, 2, \ldots, m$ and $l = -N, \, -N+1, \, \ldots, \, N-1, \, N$ with

$$c_j = \sum_{l=-N}^{N} c_{j,l} u^l.$$

Choose $N_0 \in \mathbb{Z}_{>0}$ such that $\frac{1}{N_0} < \varepsilon$. Define

$$I = \big\{ -N - N_0, \, -N - N_0 + 1, \, \ldots, \, N + N_0 - 1, \, N + N_0 \big\}.$$

Set $U = \{x \in X : f(x) \neq 0\}$, and choose nonempty disjoint open sets $U_l \subset U$ for $l \in I$. For each such l, use Lemma 4.9 to choose $f_l, r_l \in C(X)_+$ such that $r_l(x) = 1$ for all $x \in h^l(Y)$, such that $0 \leq r_l \leq 1$, such that $\operatorname{supp}(f_l) \subset U_l$, and such that $r_l \precsim_B f_l$.

Choose an open set W containing Y such that the sets $h^l(W)$ are disjoint for $l \in I$, and choose $r \in C(X)$ such that

$$0 \leq r \leq 1, \quad r|_Y = 1, \quad \text{and} \quad \operatorname{supp}(r) \subset W.$$

Set

$$g_0 = r \cdot \prod_{l \in I} r_l \circ h^l.$$

Set $g_l = g_0 \circ h^{-l}$ for $l \in I$. Then $0 \leq g_l \leq r_l \leq 1$. Define λ_l for $l \in I$ by

$$\lambda_{-N-N_0} = 0, \quad \lambda_{-N-N_0+1} = \frac{1}{N_0}, \quad \lambda_{-N-N_0+2} = \frac{2}{N_0}, \quad \ldots, \quad \lambda_{-N-1} = 1 - \frac{1}{N_0},$$

$$\lambda_{-N} = \lambda_{-N+1} = \cdots = \lambda_{N-1} = \lambda_N = 1,$$

$$\lambda_{N+1} = 1 - \frac{1}{N_0}, \quad \lambda_{N+2} = 1 - \frac{2}{N_0}, \quad \ldots, \quad \lambda_{N+N_0-1} = \frac{1}{N_0}, \quad \lambda_{N+N_0} = 0.$$

Set $g = \sum_{l \in I} \lambda_l g_l$. The supports of the functions g_l are disjoint, so $0 \leq g \leq 1$. This is condition (1). Using Lemma 1.26(13) at the first and fourth steps and

Lemma 1.26(14) at the third step, we get

$$g \precsim_B \bigoplus_{l \in I} g_l \leq \bigoplus_{l \in I} r_l \precsim_B \bigoplus_{l \in I} f_l \sim_{C(X)} \sum_{l \in I} f_l \precsim_{C(X)} f.$$

This is condition (4).

We check condition (5). We have

$$\|gu - ug\| = \|g - ugu^*\| = \|g - g \circ h^{-1}\| = \left\| \sum_{l \in I} \lambda_l g_0 \circ h^{-l} - \sum_{l \in I} \lambda_l g_0 \circ h^{-l-1} \right\|.$$

In the second sum in the last term, we change variables to get $\sum_{l+1 \in I} \lambda_{l-1} g_0 \circ h^{-l}$. Use $\lambda_{-N-N_0} = \lambda_{N+N_0} = 0$ and combine terms to get

$$\|gu - ug\| = \left\| \sum_{l=-N-N_0+1}^{N+N_0} (\lambda_l - \lambda_{l-1}) g_0 \circ h^{-l} \right\|.$$

The expressions $g_0 \circ h^{-l}$ are orthogonal and have norm 1, so

$$\|gu - ug\| = \max_{-N-N_0+1 \leq l \leq N+N_0} |\lambda_l - \lambda_{l-1}| = \frac{1}{N_0} < \varepsilon.$$

It remains to verify condition (3). Since $1 - g$ vanishes on the sets

$$h^{-N}(Y),\, h^{-N+1}(Y),\, \ldots,\, h^{N-2}(Y),\, h^{N-1}(Y),$$

Proposition 4.6 implies that $(1-g)u^l \in B$ for $l = -N, -N+1, \ldots, N-1, N$. For $j = 1, 2, \ldots, m$, since $c_{j,l} \in C(X) \subset B$ for $l = -N, -N+1, \ldots, N-1, N$, we get

$$(1-g)c_j = \sum_{l=-N}^{N} c_{j,l} \cdot (1-g)u^l \in B.$$

This completes the verification of condition (3), and the proof of the theorem. □

5. Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms

The purpose of this section is to describe some of the ideas involved in Theorem 1.18 and its proof. We describe the mean dimension of a homeomorphism, and we give proofs of simple special cases or related statements for some of the steps in the proof.

An explanation of mean dimension starts with dimension theory. Dimension theory attempts to assign a dimension to each topological space (usually in some restricted class) in such a way as to generalize the dimension of a manifold, in particular, the relation $\dim(\mathbb{R}^n) = n$, and to preserve expected properties of the dimension. There are a number of books on dimension theory; the one I have so far found most useful is [51]. (A warning on terminology there: "bicompact" is used for "compact Hausdorff". See Definition 1.5.4 of [51].) The mean dimension of a homeomorphism h of a space X should perhaps be thought of as saying how much more of the space X one sees with every iteration of h, with "how much one sees" being measured in some sense by dimension.

One warning about dimension theories: they find the dimension of the highest dimensional part of the space. A space like $\mathbb{R}^n$ is homogeneous (in a very strong sense: the diffeomorphism group acts transitively), as is a connected compact manifold without boundary. Even for a connected compact manifold with boundary, it seems intuitively clear that the dimension as seen at any point should be the same. However, a finite complex or a disconnected compact manifold may well have parts which should be considered to have different dimensions. All dimension theories I know of assign to a finite simplicial complex the dimension given by the largest standard (combinatorial) dimension of any of its simplices, even if there are other simplices of much lower dimension which are not contained in any higher dimensional simplex. (See Section 2.6 of [51] for a presentation of the basics of simplicial complexes. Following a common abuse of terminology, we say here that a topological space is a simplicial complex when, formally, we mean that it is homeomorphic to the geometric realization of a simplicial complex.) There are at least some notions of "local dimension" at a point, which attempt to account for this kind of behavior, but the theory seems to be much less well developed.

We will primarily be interested in spaces X which admit minimal homeomorphisms, or minimal actions of other countable groups. Such spaces clearly have at least a weak form of homogeneity, since each orbit is dense and the action is transitive on orbits. We know little about what one can really get from this, the examples of Gjerde and Johansen [25] show that it does not imply that the local dimension is the same at every point.

There are at least two quite different general approaches to the problem of assigning dimensions to spaces. One assumes the existence of a metric, and attempts to quantify how the "size" of a ball in the space grows with its radius. This approach leads to the Hausdorff dimension and its relatives. The result depends on the metric, need not be an integer, and can be strictly positive for the Cantor set (depending on the metric one uses). Such dimensions have so far played no role in the structure theory of C*-algebras, which is not surprising since $C(X)$ does not depend on the metric on X.

The approach more useful here relies entirely on topological properties of X, takes integer values, and is zero on the Cantor set, regardless of the metric. The three most well known dimension theories of this kind are the covering dimension $\dim(X)$ (Section 3.1 of [51]), the small inductive dimension $\mathrm{ind}(X)$ (Section 4.1 of [51]), and the large inductive dimension $\mathrm{Ind}(X)$ (Section 4.2 of [51]). There are three others that should be mentioned: for compact X, the topological stable rank $\mathrm{tsr}(C(X, \mathbb{R}))$ of the algebra $C(X, \mathbb{R})$ of continuous *real* valued functions on X; for metrizable X the infimum, over all metrics ρ defining the topology, of the Hausdorff dimension of (X, ρ); and for compact metrizable X the cohomological dimension as described in [16] (with integer coefficients). For nonempty compact metrizable X, these all agree (except that one must use $\mathrm{tsr}(C(X, \mathbb{R})) - 1$, and in the case of cohomological dimension with integer coefficients require that $\dim(X) < \infty$), and for specific pairs of dimension theories, it is often known that they agree under much weaker conditions. For $\dim(X)$, $\mathrm{ind}(X)$, and $\mathrm{Ind}(X)$ see Corollary 4.5.10 of [51]. Agreement with $\mathrm{tsr}(C(X, \mathbb{R})) - 1$ is essentially Proposition 3.3.2 of [51] (not stated in that language). Agreement with the infimum of the Hausdorff dimensions of (X, ρ) is in Section 7.4 of [31]. When $\dim(X) < \infty$, agreement with cohomological dimension with integer coefficients is Theorem 1.4 of [16]; without the condition $\dim(X) < \infty$, Theorem 7.1 [16] shows that agreement can fail.

We give a second warning about dimension theories: they do not necessarily have the properties one expects, or the properties they have are weaker than what one expects. Some such examples are presented or at least mentioned [51]. We mention just a few. The notes to Chapter 8 of [51] mention an example due to Filippov: if $1 \leq m \leq n \leq 2m - 1$, there is a compact Hausdorff space X such that

$$\dim(X) = 1, \qquad \mathrm{ind}(X) = m, \quad \text{and} \quad \mathrm{Ind}(X) = n.$$

The conventions usually take $\dim(\varnothing) = -1$, so that the standard inequality

$$\dim(X \times Y) \leq \dim(X) + \dim(Y)$$

fails if $X = \varnothing$ or $Y = \varnothing$. However, this inequality can fail even if $X \neq \varnothing$ and $Y \neq \varnothing$. See Example 9.3.7 of [51]. If X and Y are compact Hausdorff spaces, then one does get $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ (Proposition 3.2.6 of [51]; see Section 9.3 of [51] for an assortment of weaker conditions under which this inequality holds). There are nonempty compact metric spaces X and Y such that $\dim(X \times Y) < \dim(X) + \dim(Y)$; in [16] combine Example 1.3(1), Example 1.9, and the example after Corollary 3.8. There is even a nonempty compact metric space X such that $\dim(X \times X) < 2\dim(X)$; for example, combine [45] and [35].

The dimension theory most useful so far for minimal homeomorphisms is the covering dimension, which is defined using open covers. We thus start by stating the basic concepts used to define the covering dimension. We make all our definitions for finite open covers of compact Hausdorff spaces, although the earlier ones make sense in much greater generality (for more general spaces, not requiring that the covers be open, and sometimes not even requiring that the covers be finite).

By a finite open cover $\mathcal{U}$ of a compact Hausdorff space X, we mean a finite collection $\mathcal{U}$ of open subsets of X such that $X = \bigcup_{U \in \mathcal{U}} U$. (This convention follows [41].) Possibly (following Section 3.1 of [51]) one should instead use indexed families $(U_i)_{i \in I}$ of open subsets, for a finite index set I; this formulation allows repetitions among the sets. We will not need this refinement. (It is easy to check that it makes no difference in the definition of covering dimension, since one can simply delete repeated sets.)

Notation 5.1. Let X be a compact Hausdorff space. We write $\mathrm{Cov}(X)$ for the set of all finite open covers of X.

Definition 5.2. Let X be a compact Hausdorff space, and let $\mathcal{U}$ be a finite open cover of X. The *order* $\mathrm{ord}(\mathcal{U})$ of $\mathcal{U}$ is the least number $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n + 2$ distinct elements of $\mathcal{U}$ is empty.

That is, $\mathrm{ord}(\mathcal{U})$ is the largest $n \in \mathbb{Z}_{>0}$ such that there are $n+1$ distinct sets in $\mathcal{U}$ whose intersection is not empty. An alternative formulation is

$$\mathrm{ord}(\mathcal{U}) = -1 + \sup_{x \in X} \sum_{U \in \mathcal{U}} \chi_U(x).$$

The normalization is chosen so that if $\mathcal{U}$ is cover of X by disjoint open sets, and $X \neq \varnothing$, then $\mathrm{ord}(\mathcal{U}) = 0$: the intersection of any two distinct sets in $\mathcal{U}$ is empty, but the sets themselves need not be empty.

Definition 5.3. Let X be a compact Hausdorff space, and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X. Then $\mathcal{V}$ *refines* $\mathcal{U}$ (written $\mathcal{V} \prec \mathcal{U}$) if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.

That is, every set in $\mathcal{V}$ is contained in some set in $\mathcal{V}$.

Definition 5.4. Let X be a compact Hausdorff space, and let $\mathcal{U}$ be a finite open cover of X. We define the *dimension* $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$ by

$$\mathcal{D}(\mathcal{U}) = \inf \big(\big\{ \operatorname{ord}(\mathcal{V}) \colon \mathcal{V} \in \operatorname{Cov}(X) \text{ and } \mathcal{V} \prec \mathcal{U} \big\} \big).$$

That is, $\mathcal{D}(\mathcal{U})$ is the least possible order of a finite open cover which refines $\mathcal{U}$.

Definition 5.5 (Definition 3.1.1 of [51])**.** Let X be a nonempty compact Hausdorff space. The *covering dimension* $\dim(X)$ is

$$\dim(X) = \sup \big(\big\{ \mathcal{D}(\mathcal{U}) \colon \mathcal{U} \in \operatorname{Cov}(X) \big\} \big).$$

By convention, $\dim(\varnothing) = -1$.

That is, $\dim(X)$ is the supremum of $\mathcal{D}(\mathcal{U})$ over all finite open covers $\mathcal{U}$ of X.

We will say that a compact Hausdorff space X is *totally disconnected* if there is a base for the topology of X consisting of compact open sets. (This seems to be the standard definition for this class of spaces. In [51], a different definition is used, but for compact Hausdorff spaces it is equivalent. See Proposition 3.1.3 of [51].)

We have $\dim([0,1]) \neq 0$ by Exercise **??**. To show $\dim([0,1]) \leq 1$, consider open covers of $[0,1]$ consisting of intervals

$$[0, \beta_0), \, (\alpha_1, \beta_1), \, (\alpha_2, \beta_2), \, \ldots, \, (\alpha_{n-1}, \beta_{n-1}), \, (\alpha_n, 1]$$

such that $\alpha_j \leq \beta_{j-1}$ but $\beta_{j-1} < \alpha_{j+1}$, and $\beta_j - \alpha_j$ is small, for all j. The intervals this cover $[0,1]$, but $[0, \beta_0)$ is disjoint from (α_2, β_2), etc.

One sees that $\dim([0,1]^2) = 2$ by using open covers consisting of small neighborhoods of the tiles in a fine hexagonal tiling. In general, one has $\dim(\mathbb{R}^n) = n$ (Theorem 3.2.7 of [51]), but proving this is nontrivial. Most proofs rely on some version of the Brouwer Fixed Point Theorem, and thus, in effect, on algebraic topology.

We now consider the mean dimension, introduced in [41], of a homeomorphism $h \colon X \to X$. (For best behavior, h should not have "too many" periodic points.) It is designed so that if K is a sufficiently nice compact metric space (in particular, $\dim(K^n)$ should equal $n \cdot \dim(K)$ for all $n \in \mathbb{Z}_{>0}$), then the shift on $X = K^{\mathbb{Z}}$ should have mean dimension equal to $\dim(K)$. Given this heuristic, it should not be surprising that if $\dim(X) < \infty$ then $\operatorname{mdim}(h) = 0$.

Definition 5.6. Let X be a compact Hausdorff space, and let $\mathcal{U}$ and $\mathcal{V}$ be two finite open covers of X. Then the *join* $\mathcal{U} \vee \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \vee \mathcal{V} = \big\{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \big\}.$$

Definition 5.7. Let X be a compact Hausdorff space, let $\mathcal{U}$ be a finite open cover of X, and let $h \colon X \to X$ be a homeomorphism. We define

$$h(\mathcal{U}) = \big\{ h(U) \colon U \in \mathcal{U} \big\}.$$

Definition 5.8 (Definition 2.6 of [41])**.** Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism. Then the *mean dimension* of h is (see Corollary 5.11 below for existence of the limit)

$$\operatorname{mdim}(h) = \sup_{\mathcal{U} \in \operatorname{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U}) \big)}{n}.$$

The expression in the definition uses the join of n covers.

Existence of the limit depends on the following result.

Proposition 5.9 (Corollary 2.5 of [41])**.** Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be two finite open covers of X. Then $\mathcal{D}(\mathcal{U} \vee \mathcal{V}) \leq \mathcal{D}(\mathcal{U}) + \mathcal{D}(\mathcal{V})$.

We omit the proof, but the idea is similar to that of the proof of Proposition 3.2.6 of [51] ($\dim(X \times Y) \leq \dim(X) + \dim(Y)$ for nonempty compact Hausdorff spaces X and Y). The point is that an open cover $\mathcal{U}$ has $\mathrm{ord}(\mathcal{U}) \leq m$ if and only if there is a finite simplicial complex K of dimension at most m which approximates X "as seen by $\mathcal{U}$", and if K and L are finite simplicial complexes which approximate X as seen by $\mathcal{U}$ and by $\mathcal{V}$, then $K \times L$ is a finite simplicial complex with dimension $\dim(K) + \dim(L)$ which approximates X as seen by $\mathcal{U} \vee \mathcal{V}$.

Lemma 5.10. Let $(\alpha_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $[0, \infty)$ which is subadditive, that is, $\alpha_{m+n} \leq \alpha_m + \alpha_n$ for all $m, n \in \mathbb{Z}_{>0}$. Then $\lim_{n \to \infty} n^{-1} \alpha_n$ exists and is equal to $\inf_{n \in \mathbb{Z}_{>0}} n^{-1} \alpha_n$.

Proof. We follow part of the proof of Theorem 6.1 of [41]. Define $\beta = \inf_{n \in \mathbb{Z}_{>0}} n^{-1} \alpha_n$. Let $\varepsilon > 0$. Choose $N_0 \in \mathbb{Z}_{>0}$ such that $N_0^{-1} \alpha_{N_0} < \beta + \frac{\varepsilon}{2}$. Choose $N \in \mathbb{Z}_{>0}$ so large that

$$N \geq N_0 \quad \text{and} \quad \frac{N_0 \alpha_1}{N} < \frac{\varepsilon}{2}.$$

Let $n \geq N$. Since $N \geq N_0$, there are $r \in \mathbb{Z}_{>0}$ and $s \in \{0, 1, \ldots, N_0 - 1\}$ such that $n = r N_0 + s$. Then, using subadditivity at the first step,

$$\frac{\alpha_n}{n} \leq \frac{r \alpha_{N_0} + s \alpha_1}{n} = \frac{r \alpha_{N_0}}{r N_0 + s} + \frac{s \alpha_1}{n} < \frac{\alpha_{N_0}}{N_0} + \frac{N_0 \alpha_1}{n} < \beta + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \beta + \varepsilon.$$

This completes the proof. ∎

Corollary 5.11. Let X be a compact metric space let $\mathcal{U}$ be a finite open cover of X, and let $h \colon X \to X$ be a homeomorphism. Then the limit

$$\lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}$$

in Definition 5.8 exists.

Proof. Combine Lemma 5.10 and Proposition 5.9. ∎

The following result is immediate.

Proposition 5.12. Let X be a compact metric space with finite covering dimension, and let $h \colon X \to X$ be a homeomorphism. Then $\mathrm{mdim}(h) = 0$.

Proof. Let $\mathcal{U}$ be a finite open cover of X. Then, by definition,

$$\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big) \leq \dim(X),$$

so

$$\mathrm{mdim}(h) \leq \lim_{n \to \infty} \frac{\dim(X)}{n} = 0.$$

This completes the proof. ∎

The following result is less obvious, but not difficult (although we refer to [41] for the proof). In particular, it shows that every uniquely ergodic minimal homeomorphism has mean dimension zero.

Proposition 5.13. Let X be a compact metric space, let $h \colon X \to X$ be a homeomorphism, and assume that h has at most countably many ergodic invariant Borel probability measures. Then $\mathrm{mdim}(h) = 0$.

Proof. In [41], see Theorem 5.4 and the discussion after Definition 5.2. □

Proposition 5.12 covers most of the common examples of minimal homeomorphisms. However, not all minimal homeomorphisms have mean dimension zero. We start with the standard nonminimal example.

Definition 5.14. Let K be a set. The *shift* $h_K \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is the bijection given by $h_K(x)_k = x_{k+1}$ for $x = (x_k)_{k \in \mathbb{Z}} \in K^{\mathbb{Z}}$ and $k \in \mathbb{Z}$.

Theorem 5.15 (Proposition 3.1 of [41])**.** Let K be a compact metric space, and let h_K be as in Definition 5.14. Then $\mathrm{mdim}(h_K) \leq \dim(K)$.

Theorem 5.16 (Proposition 3.3 of [41])**.** Let $d \in \mathbb{Z}_{>0}$, set $K = [0,1]^d$, and and let h_K be as in Definition 5.14. Then $\mathrm{mdim}(h_K) = d$.

We omit the proofs. To understand the result heuristically, in Definition 5.8 consider a finite open cover $\mathcal{U}_0$ of K, for $n \in \mathbb{Z}$ let $p_n \colon K^{\mathbb{Z}} \to K$ be the projection on the nth coordinate, and consider the finite open cover

$$\mathcal{U} = \big\{ p_0^{-1}(U) \colon U \in \mathcal{U}_0 \big\}.$$

Then the cover $\mathcal{U} \vee h_K^{-1}(\mathcal{U}) \vee \cdots \vee h_K^{-n+1}(\mathcal{U})$ sees only n of the coordinates in $K^{\mathbb{Z}}$, so that

$$\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big) \leq \dim(K^n) \leq n \dim(K).$$

The proof of Theorem 5.15 requires only one modification of this idea, namely that the original cover $\mathcal{U}$ must be allowed to depend on an arbitrary finite number of coordinates rather than just one. The proof of Theorem 5.16 requires more work.

One does not expect $\mathrm{mdim}(h_K) = \dim(K)$ in general, because of the possibility of having $\dim(K^n) < n \dim(K)$. When $\dim(K) < \infty$, by combining Theorem 1.4 of [16] and Theorem 3.16(b) of [16] and the discussion afterwards, one sees that $\dim(K^n)$ is either always $n \dim(K)$ or always $n \dim(K) - n + 1$. In the first case,

$$\lim_{n \to \infty} \frac{\dim(K^n)}{n} = \dim(K),$$

while in the second case,

$$\lim_{n \to \infty} \frac{\dim(K^n)}{n} = \dim(K) - 1.$$

Moreover, the second case actually occurs. (For example, combine [45] and [35].) A modification of the proof of Theorem 5.15 should easily give the upper bound $\mathrm{mdim}(h_K) \leq \dim(K) - 1$ in the second case. This suggests the following question, which, as far as we know, has not been addressed.

Question 5.17. Let K be a compact metric space, and let h_K be as in Definition 5.14. Does it follow that $\mathrm{mdim}(h_K) = \dim(K)$ or $\mathrm{mdim}(h_K) = \dim(K) - 1$?

Shifts are not minimal (unless K has at most one point), but one can construct minimal subshifts with large mean dimension. A basic construction of this type is given in [41].

Theorem 5.18 (Proposition 3.5 of [41])**.** There exists a minimal invariant subset $X \subset ([0,1]^2)^{\mathbb{Z}}$ such that $\mathrm{mdim}(h_{[0,1]^2}|_X) > 1$.

A related construction is used in [24] to produce many more examples, including ones with arbitrarily large mean dimension.

We now recall the statement of Theorem 1.18.

Theorem 5.19 ([28])**.** Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

It is hoped that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for any minimal homeomorphism of an infinite compact metric space X. This has been proved in [28] for some special systems covered by Theorem 5.19, slightly generalizing the construction of Giol and Kerr.

The hypothesis on existence of a surjective map to the Cantor set has other equivalent formulations, one of which is the existence of an equivariant surjective map to the Cantor set. We need a definition.

Definition 5.20. Let X and Y be compact metric spaces. Let $h \colon X \to X$ and $k \colon Y \to Y$ be homeomorphisms. We say that the system (Y, k) is a *factor* of (X, h) if there is a surjective continuous map $g \colon X \to Y$ (the *factor map*) such that $g \circ h = k \circ g$.

The requirement in the definition is that the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "g"] \\ Y \arrow[r, "k"] & Y. \end{tikzcd}$$

Proposition 5.21. Let X be a compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Then the following are equivalent:

(1) There exists a decreasing sequence $Y_0 \supset Y_1 \supset Y_2 \supset \cdots$ of nonempty compact open subsets of X such that the subset $Y = \bigcap_{n=0}^{\infty} Y_n$ satisfies $h^r(Y) \cap Y = \varnothing$ for all $r \in \mathbb{Z} \setminus \{0\}$.
(2) There is a minimal homeomorphism of the Cantor set which is a factor of (X, h).
(3) There is a continuous surjective map from X to the Cantor set.
(4) For every $n \in \mathbb{Z}_{>0}$ there is a partition $\mathcal{P}$ of X into at least n nonempty compact open subsets.

We omit the proof.

Assume h is minimal and $h^n(Y) \cap Y = \varnothing$ for $n \in \mathbb{Z} \setminus \{0\}$. Write $Y = \bigcap_{n=0}^{\infty} Y_n$ with $Y_0 \supset Y_1 \supset \cdots$ and $\mathrm{int}(Y_n) \neq \varnothing$ for all $n \in \mathbb{Z}_{\geq 0}$. Then $C^*(\mathbb{Z}, X, h)_Y = \varinjlim_n C^*(\mathbb{Z}, X, h)_{Y_n}$, and $C^*(\mathbb{Z}, X, h)_{Y_n}$ is a recursive subhomogeneous C*-algebra whose base spaces are closed subsets of X. (See Theorem 2.22 of [59].) The effect of requiring a Cantor system factor is that one can choose Y and $(Y_n)_{n \in \mathbb{Z}_{\geq 0}}$ so that Y_n is both closed and open for all $n \in \mathbb{Z}_{\geq 0}$. Doing so ensures that $C^*(\mathbb{Z}, X, h)_{Y_n}$ is a homogeneous C*-algebra whose base spaces are closed subsets of X. Thus $C^*(\mathbb{Z}, X, h)_Y$ is a simple AH algebra. We get such a set Y by taking the inverse image of a point in the Cantor set.

To keep things simple, in these notes we will assume that h has a particular minimal homeomorphism of the Cantor set as a factor, namely an odometer system. The further simplification of assuming an odometer factor is that one can arrange $C^*(\mathbb{Z}, X, h)_{Y_n} \cong M_{p_n}(C(Y_n))$, that is, there is only one summand. This simplifies the notation but otherwise makes little difference.

Definition 5.22. Let $d = (d_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{Z}_{>0}$ with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. The *d-odometer* is the minimal system (X_d, h_d) defined as follows. Set

$$X_d = \prod_{n=1}^{\infty} \{0, 1, 2, \ldots, d_n - 1\},$$

which is homeomorphic to the Cantor set. For $x = (x_n)_{n \in \mathbb{Z}_{>0}} \in X_d$, let

$$n_0 = \inf\big(\{n \in \mathbb{Z}_{>0} \colon x_n \neq d_n - 1\}\big).$$

If $n_0 = \infty$ set $h_d(x) = (0, 0, \ldots)$. Otherwise, $h_d(x) = (h_d(x)_n)_{n \in \mathbb{Z}_{>0}}$ is

$$h_d(x)_n = \begin{cases} 0 & n < n_0 \\ x_n + 1 & n = n_0 \\ x_n & n > n_0. \end{cases}$$

The homeomorphism is "addition of $(1, 0, 0, \ldots)$ with carry to the right". When $n_0 \neq \infty$, we have

$$h(x) = \big(0, 0, \ldots, 0, x_{n_0} + 1, x_{n_0+1}, x_{n_0+2}, \ldots\big).$$

We omit the proof of the following lemma. Some work is required, most of which consists of keeping notation straight. A more general version (assuming an arbitrary minimal homeomorphism of the Cantor set as a factor) is in [28].

Lemma 5.23. Let X be a compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. We assume that (X, h) has as a factor system the odometer on $X_d = \prod_{n=1}^{\infty} \{0, 1, 2, \ldots, d_n - 1\}$ for a sequence $d = (d_n)_{n \in \mathbb{Z}_{>0}}$ of integers with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. Let Y be the inverse image of $(0, 0, \ldots)$ under the factor map, and let Y_n be the inverse image of

$$\{0\}^n \times \prod_{k=n+1}^{\infty} \{0, 1, 2, \ldots, d_k - 1\}.$$

For $n \in \mathbb{Z}_{>0}$ set $p_n = \prod_{k=1}^{n} d_k$. For $m, n \in \mathbb{Z}_{\geq 0}$ with $n \geq m$, define

$$\psi_{n,m} \colon C(Y_m, M_{p_m}) \to C(Y_n, M_{p_n})$$

by

$$\psi_{n,m}(f) = \operatorname{diag}\big(f|_{Y_n},\, f \circ h^{p_m}|_{Y_n},\, f \circ h^{2p_m}|_{Y_n},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}|_{Y_n}\big)$$

for $f \in C(Y_m, M_{p_m})$. Then

$$C^*(\mathbb{Z}, X, h)_Y \cong \varinjlim_n C(Y_n, M_{p_n}).$$

The map $\psi_{n,0}$ in the statement of the lemma has the particularly suggestive formula

$$\psi_{n,0}(f) = \operatorname{diag}\big(f|_{Y_n},\, f \circ h|_{Y_n},\, f \circ h^2|_{Y_n},\, \ldots,\, f \circ h^{p_n - 1}|_{Y_n}\big).$$

The problem is now reduced to showing that if $A = \varinjlim_n C(Y_n, M_{p_n})$, with maps

$$\psi_{n,m}(f) = \operatorname{diag}\big(f|_{Y_n},\, f \circ h^{p_m}|_{Y_n},\, f \circ h^{2p_m}|_{Y_n},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}|_{Y_n}\big),$$

then $\mathrm{rc}(A) \leq \frac{1}{2}\mathrm{mdim}(h)$.

We will make a further simplification, and prove instead the following theorem, also from [28].

Theorem 5.24 ([28])**.** Let X be an infinite compact metric space. Let $d = (d_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of integers with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{>0}$ set $p_n = \prod_{k=1}^{n} d_k$. Let $h \colon X \to X$ be a homeomorphism, and suppose that h^{p_n} is minimal for all $n \in N$. For $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, define $\psi_{n,m} \colon C(X, M_{p_m}) \to C(X, M_{p_n})$ by

$$\psi_{n,m}(f) = \operatorname{diag}\big(f, \, f \circ h^{p_m}, \, f \circ h^{2p_m}, \, \ldots, \, f \circ h^{(p_n/p_m - 1)p_m}\big)$$

for $f \in C(X, M_{p_m})$. Using these maps, define

$$B = \varinjlim_{n} C(X, M_{p_n}).$$

Then $\operatorname{rc}(B) \leq \frac{1}{2}\operatorname{mdim}(h)$.

The following lemma (whose easy proof is left as an exercise) ensures that the direct system in Theorem 5.24 actually makes sense.

Lemma 5.25 ([28])**.** Let X, h, d, and $\psi_{n,m}$ for $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, be as in Theorem 5.24, but without any minimality assumptions on h. Then for $k, m, n \in \mathbb{Z}_{>0}$ with $k \leq m \leq n$, we have $\psi_{n,m} \circ \psi_{m,k} = \psi_{n,k}$.

The algebra B in Theorem 5.24 is a kind of AH model for the crossed product $C^*(\mathbb{Z}, X, h)$. In particular, it is always an AH algebra, while we needed the assumption of a Cantor set factor system to find a large subalgebra of $C^*(\mathbb{Z}, X, h)$ which is an AH algebra. This model has the defect that we must now assume that h^{p_n} is minimal for all $n \in N$. Otherwise, it turns out that the direct limit isn't simple. (This minimality condition on the powers actually excludes systems with odometer factors.) The proof of the following lemma is a fairly direct consequence of the simplicity criterion in Proposition 2.1(iii) of [14].

Lemma 5.26 ([28])**.** Let X, h, d, and $\psi_{n,m}$ for $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, be as in Theorem 5.24, but without any minimality assumptions on h. Set $B = \varinjlim_n C(X, M_{p_n})$. Then B is simple if and only if h^{p_n} is minimal for all $n \in N$.

The main effect of passing to the situation of Theorem 5.24 is to further simplify the notation. For minimal homeomorphisms without Cantor set factor systems, the replacement of a direct limit of recursive subhomogeneous algebras with an AH algebra of the sort appearing in Theorem 5.24 is a much more substantial simplification. There are difficulties (presumably technical) in the more general context which we don't (yet) know how to solve.

We would like to use Theorem 6.2 of [46] to prove Theorem 5.24 (and also Theorem 5.19). Unfortunately, the definition there of mean dimension of an AH direct system requires that the base spaces be connected, or at least have only finitely many connected components. If (X, h) has a Cantor set factor system, the base spaces in the AH model (and also in the direct system in Lemma 5.23) have surjective maps to the Cantor set. So we proceed more directly, although the arguments are closely related.

Lemma 5.27. Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism with no periodic points. Then for every $\varepsilon > 0$ and every finite subset $F \subset C(X)$ there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ there is a compact metric space K and a surjective map $i \colon X \to K$ satisfying:

(1) $\dim(K) < n[\operatorname{mdim}(h) + \varepsilon]$.

(2) For $m = 0, 1, \ldots, n-1$ and $f \in F$ there is $g \in C(K)$ such that $\| f \circ h^m - g \circ i \| < \varepsilon$.

The argument depends on nerves of covers and their geometric realizations. See Section 2.6 of [51], especially Definition 2.6.1, Definition 2.6.2, Definition 2.6.7, and the proof of Proposition 2.6.8, for more details of the theory than are presented here.

Definition 5.28. Let X be a topological space, and let $\mathcal{V}$ be a finite open cover of X, with $\varnothing \notin \mathcal{V}$. The *nerve* $K(\mathcal{V})$ is the finite simplicial complex with vertices $[V]$ for $V \in \mathcal{V}$, and in which there is a simplex in $K(\mathcal{V})$ with vertices $[V_0], [V_1], \ldots, [V_n]$ if and only if $V_0 \cap V_1 \cap \cdots \cap V_n \neq \varnothing$.

The points $z \in K(\mathcal{V})$ (really, points z in its geometric realization) are thus exactly the formal convex combinations

$$z = \sum_{V \in \mathcal{V}} \alpha_V [V] \tag{5.1}$$

in which $\alpha_V \geq 0$ for all $V \in \mathcal{V}$, $\sum_{V \in \mathcal{V}} \alpha_V = 1$, and $\big\{ [V] \colon \alpha_V \neq 0 \big\}$ is a simplex in $K(\mathcal{V})$, that is,

$$\bigcap \big\{ V \in \mathcal{V} \colon \alpha_V \neq 0 \big\} \neq \varnothing.$$

Lemma 5.29. Let X be a topological space, and let $\mathcal{V}$ be a finite open cover of X, with $\varnothing \notin \mathcal{V}$. Then $\dim(K(\mathcal{V})) = \operatorname{ord}(\mathcal{V})$.

Proof. It is immediate that $\operatorname{ord}(\mathcal{V})$ is the largest (combinatorial) dimension of a simplex occurring in $K(\mathcal{V})$. It follows from standard results in dimension theory (in [51], see Proposition 3.1.5, Theorem 3.2.5, and Theorem 3.2.7) that this dimension is equal to $\dim(K(\mathcal{V}))$. □

Lemma 5.30. Let X be a topological space, and let $\mathcal{V}$ be a finite open cover of X, with $\varnothing \notin \mathcal{V}$. Let $(g_V)_{V \in \mathcal{V}}$ be a partition of unity on X such that $\operatorname{supp}(g_V) \subset V$ for all $V \in \mathcal{V}$. Then there is a continuous map $i \colon X \to K(\mathcal{V})$ determined, using (5.1), by

$$i(x) = \sum_{V \in \mathcal{V}} g_V(x)[V]$$

for $x \in X$.

At this point, we leave traditional topology.

Lemma 5.31. Let X be a compact Hausdorff space, and let $\mathcal{V}$, $(g_V)_{V \in \mathcal{V}}$, and $i \colon X \to K(\mathcal{V})$ be as in Lemma 5.30. Let $(x_V)_{V \in \mathcal{V}}$ be a collection of points in X such that $x_V \in V$ for $V \in \mathcal{V}$. Then there is a linear map $P \colon C(X) \to C(K(\mathcal{V}))$ (not a homomorphism) defined, following (5.1), by

$$P(f)\left(\sum_{V \in \mathcal{V}} \alpha_V [V] \right) = \sum_{V \in \mathcal{V}} \alpha_V f(x_V)$$

for $f \in C(X)$. Moreover:

(1) $\|P\| \leq 1$.
(2) For all $f \in C(X)$, we have

$$\|P(f) \circ i - f\| \leq \sup_{V \in \mathcal{V}} \sup_{x, y \in V} |f(x) - f(y)|.$$

The key point is part (2): if $f \in C(X)$ varies by at most $\delta > 0$ over each set $V \in \mathcal{V}$, then $P(f)$ is a function on $K(\mathcal{V})$ whose pullback to X is close to f. That is, if $\mathcal{V}$ is sufficiently fine, then we can approximate a finite set of functions on X by functions on the finite (in particular, finite dimensional) simplicial complex $K(\mathcal{V})$. Moreover, the dimension of $K(\mathcal{V})$ is controlled by the order of $\mathcal{V}$.

Proof of Lemma 5.31. It is easy to check that $P(f)$ is continuous, that P is linear, and that $\|P\| \leq 1$.

For (2), let $r > 0$ and suppose that for all $V \in \mathcal{V}$ and $x, y \in V$ we have $|f(x) - f(y)| \leq r$. Let $x \in X$ and estimate:

$$\begin{aligned}\big|P(f)(i(x)) - f(x)\big| &= \left|P(f)\left(\sum_{V\in\mathcal{V}} g_V(x)[V]\right) - \sum_{V\in\mathcal{V}} g_V(x)f(x)\right| \\ &\leq \sum_{V\in\mathcal{V}} g_V(x)|f(x_V) - f(x)| \leq \sum_{V\in\mathcal{V}} g_V(x)r = r.\end{aligned}$$

This completes the proof. ☐

Proof of Lemma 5.27. Choose a finite open cover $\mathcal{U}$ of X such that for all $U \in \mathcal{U}$, $x, y \in U$, and $f \in F$, we have $|f(x) - f(y)| < \frac{\varepsilon}{2}$. By definition, we have

$$\lim_{n\to\infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n} \leq \mathrm{mdim}(h).$$

Therefore there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have

$$\frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n} < \mathrm{mdim}(h) + \varepsilon.$$

Let $n \geq N$. Then there is a finite open cover $\mathcal{V}$ of X which refines

$$\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})$$

and such that

$$\mathrm{ord}(\mathcal{V}) < n[\mathrm{mdim}(h) + \varepsilon]. \tag{5.2}$$

Since X is a compact metric space, we can choose a partition of unity $(g_V)_{V\in\mathcal{V}}$ on X such that $\mathrm{supp}(g_V) \subset V$ for all $V \in \mathcal{V}$. Apply Lemma 5.30, getting $i\colon X \to K(\mathcal{V})$, and let $P\colon C(X) \to C(K(\mathcal{V}))$ be as in Lemma 5.31.

Let $f \in F$ and let $m \in \{0, 1, \ldots, n-1\}$. Since $\mathcal{V}$ refines $h^{-m}(\mathcal{U})$, it follows that for all $V \in \mathcal{V}$ and $x, y \in V$ we have $\big|(f \circ h^m)(x) - (f \circ h^m)(y)\big| < \frac{\varepsilon}{2}$. So

$$\big\|P(f \circ h^m) \circ i - f \circ h^m\big\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

We are done with the proof except for the fact that i might not be surjective. So define $K = i(X) \subset K(\mathcal{V})$. Since the dimension of a subspace can't be larger than the dimension of the whole space (see Proposition 3.1 5 of [51]),

$$\dim(K) \leq \dim(K(\mathcal{V})) = \mathrm{ord}(\mathcal{V}) < n[\mathrm{mdim}(h) + \varepsilon].$$

In place of $P(f \circ h^m)$ we use $P(f \circ h^m)|_K$. This completes the proof. ☐

The proof of Theorem 5.24 requires two further results. For both proofs, we refer to the original sources. The first is a special case of Theorem 5.1 of [70].

Theorem 5.32 (see Theorem 5.1 of [70])**.** Let X be a compact metric space and let $n \in \mathbb{Z}_{>0}$. Then

$$\mathrm{rc}(M_n \otimes C(X)) \leq \frac{\dim(X) - 1}{2n}.$$

Lemma 5.33 (Lemma 6.1 of [46])**.** Let B be a simple unital exact C*-algebra and let $r \in [0, \infty)$. Suppose:

(1) For every finite subset $S \subset B$ and every $\varepsilon > 0$, there is a unital C*-algebra D such that $\mathrm{rc}(D) < r + \varepsilon$ and an injective unital homomorphism $\rho \colon D \to B$ such that $\mathrm{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.
(2) For every $s \in [0, 1]$ and every $\varepsilon > 0$, there exists a projection $p \in B$ with $|\tau(p) - s| < \varepsilon$ for all $\tau \in \mathrm{T}(B)$.

Then $\mathrm{rc}(B) \leq r$.

Proof of Theorem 5.24. We use Lemma 5.33. Certainly B is simple, unital, and exact. Since $C(X, M_{p_n}) \hookrightarrow B$ and $C(X, M_{p_n})$ has projections of constant rank k for any $k \in \{0, 1, \ldots, p_n\}$, condition (2) in Lemma 5.33 is satisfied.

We need to show that for every finite subset $S \subset B$ and every $\varepsilon > 0$, there is a unital C*-algebra D such that $\mathrm{rc}(D) < \frac{1}{2}\mathrm{mdim}(h) + \varepsilon$ and an injective unital homomorphism $\rho \colon D \to B$ such that $\mathrm{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

For $n \in \mathbb{Z}_{\geq 0}$ let $\psi_n \colon C(X, M_{p_n}) \to B$ be the map obtained from the direct limit description of B. Let $S \subset B$ be finite and let $\varepsilon > 0$. Choose $m \in \mathbb{Z}_{>0}$ and a finite set

$$S_0 \subset C(X, M_{p_m}) = M_{p_m}(C(X))$$

such that for every $a \in S$ there is $b \in S_0$ with $\|\psi_m(b) - a\| < \frac{1}{2}\varepsilon$. Let $F \subset C(X)$ be the set of all matrix entries of elements of S_0. Use Lemma 5.27 to find $N \in \mathbb{Z}_{\geq 0}$ such that for all $l \geq N$ there are a compact metric space K and a surjective map $i \colon X \to K$ such that $\dim(K) < l[\mathrm{mdim}(h) + \varepsilon]$ and for $r = 0, 1, \ldots, l - 1$ and $f \in F$ there is $g \in C(K)$ with

$$\|f \circ h^r - g \circ i\| < \frac{\varepsilon}{2p_m^2}.$$

Choose $n \geq m$ such that $p_n \geq N$. Choose K and i for $l = p_n$, so that

$$\dim(K) < p_n[\mathrm{mdim}(h) + \varepsilon]$$

and for $r = 0, 1, \ldots, p_n - 1$ and $f \in F$ there is $g \in C(K)$ with

$$\|f \circ h^r - g \circ i\| < \frac{\varepsilon}{2p_m^2}.$$

Define an injective homomorphism $\rho_0 \colon C(K) \to C(X)$ by $\rho_0(f) = f \circ i$ for $f \in C(K)$. Set $D = M_{p_n}(C(K))$ and define

$$\rho = \psi_n \circ (\mathrm{id}_{M_{p_n}} \otimes \rho_0) \colon D \to B.$$

Then ρ is also injective.

By Theorem 5.32,

$$\mathrm{rc}(D) \leq \frac{\dim(K) - 1}{2p_n} < \frac{\mathrm{mdim}(h) + \varepsilon}{2} < \frac{\mathrm{mdim}(h)}{2} + \varepsilon.$$

It remains to prove that $\mathrm{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

Let $a \in S$. Choose $b \in S_0$ such that $\|\psi_m(b) - a\| < \frac{\varepsilon}{2}$. For $j, k \in \{0, 1, \ldots, p_m - 1\}$, we let $e_{j,k} \in M_{p_m}$ be the standard matrix unit (except that we start the indexing at 0 rather than 1). Then there are $b_{j,k} \in F$ for $j, k \in \{0, 1, \ldots, p_m - 1\}$ such that

$b = \sum_{j,k=0}^{p_m-1} e_{j,k} \otimes b_{j,k}$. By construction, for $r = 0, 1, \ldots, p_n - 1$ there is $g_{j,k,r} \in C(K)$ such that

$$\| g_{j,k,r} \circ i - b_{j,k} \circ h^r \| < \frac{\varepsilon}{2p_m^2}.$$

For $t = 0, 1, \ldots, p_n/p_m - 1$, define

$$c_t = \sum_{j,k=0}^{p_m-1} e_{j,k} \otimes g_{j,k,tp_m} \in M_{p_m}(C(K)).$$

Then define

$$c = \operatorname{diag}\bigl(c_0, c_1, \ldots, c_{p_n/p_m-1}\bigr) \in M_{p_n}(C(K)).$$

We claim that $\| \rho(c) - a \| < \varepsilon$, which will finish the proof. We have, using the definition of $\psi_{n,m}$ at the third step,

$$\begin{aligned} \| \rho(c) - a \| & \leq \| a - \psi_m(b) \| + \| \psi_m(b) - \rho(c) \| \\ & < \frac{\varepsilon}{2} + \| \psi_{n,m}(b) - c \| \\ & < \frac{\varepsilon}{2} + \bigl\| \operatorname{diag}\bigl(f, \, f \circ h^{p_m}, \, f \circ h^{2p_m}, \, \ldots, \, f \circ h^{(p_n/p_m-1)p_m}\bigr) \\ & \qquad\qquad - \operatorname{diag}\bigl(c_0, c_1, c_2, \ldots, c_{p_n/p_m-1}\bigr) \bigr\| \\ & \leq \frac{\varepsilon}{2} + \max_{0 \leq t \leq p_n/p_m-1} \| f \circ h^{p_m r} - c_t \| \\ & \leq \frac{\varepsilon}{2} + \max_{0 \leq t \leq p_n/p_m-1} \sum_{j,k=1}^{p_m} \| g_{j,k,r} \circ i - b_{j,k} \circ h^r \| \\ & \leq \frac{\varepsilon}{2} + p_m^2 \left(\frac{\varepsilon}{2p_m^2} \right) = \varepsilon. \end{aligned}$$

This completes the proof. ∎

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