Large Subalgebras and the Structure of Crossed Products, Lecture 2: Large Subalgebras and their Basic Properties

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2 June 2015

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1–5 June 2015

- Lecture 1 (1 June 2015): Introduction, Motivation, and the Cuntz Semigroup.
- Lecture 2 (2 June 2015): Large Subalgebras and their Basic Properties.
- Lecture 3 (4 June 2015): Large Subalgebras and the Radius of Comparison.
- Lecture 4 (5 June 2015 [morning]): Large Subalgebras in Crossed Products by $\mathbb{Z}$.
- Lecture 5 (5 June 2015 [afternoon]): Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms.

A rough outline of all five lectures

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if B is a large subalgebra of A, then A and B have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.

Definition

Let A be a C*-algebra, and let $a, b \in (K \otimes A)_+$. We say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Dense subsets

$B \subset A$ is *large* in A if for $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Lemma

In the definition, it suffices to let $S \subset A$ be a subset whose linear span is dense in A, and verify the hypotheses only when $a_1, a_2, \ldots, a_m \in S$.

Unlike other approximation properties (such as tracial rank), it seems not to be possible to take S to be a generating subset, or even a selfadjoint generating subset. (We can do this for the definition of a centrally large subalgebra.)

When A is finite

$B \subset A$ is *large* in A if for $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Proposition

Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra satisfying the definition of a large subalgebra except for condition (5). Then B is large in A.

When A is finite (continued)

From the previous slide:

Proposition

Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra satisfying the definition of a large subalgebra except for the condition $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$. Then B is large in A.

It suffices to prove:

Lemma

Let A be a finite simple infinite dimensional unital C*-algebra. Let $x \in A_+$ satisfy $\|x\| = 1$. Then for every $\varepsilon > 0$ there is $x_0 \in \left(\overline{xAx}\right)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

If we also require $x_0 \precsim_A x$, then we can use x_0 in place of x in the definition.

When A is finite (continued)

To show: $x \in A_+$ with $\|x\| = 1$, $\varepsilon > 0$. Then there is $y \in \left(\overline{xAx}\right)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A y$, then $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Choose a sufficiently small number $\varepsilon_0 > 0$. Choose $f \colon [0, 1] \to [0, 1]$ such that $f = 0$ on $[0,\, 1 - \varepsilon_0]$ and $f(1) = 1$. Construct $a, b_j, c_j, d_j \in \overline{f(x)Af(x)}$ for $j = 1, 2$ such that

$$0 \leq d_j \leq c_j \leq b_j \leq a \leq 1, \quad ab_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \quad \text{and} \quad d_j \neq 0,$$

and $b_1 b_2 = 0$. Take $x_0 = d_1$.

If ε_0 is small enough, $g \precsim_A d_1$, and $\|(1 - g)x(1 - g)\| \leq 1 - \varepsilon$, use

$$\big\|(1 - g)(b_1 + b_2)(1 - g)\big\| = \big\|(b_1 + b_2)^{1/2}(1 - g)^2(b_1 + b_2)^{1/2}\big\|,$$

$$\|(1 - g)x(1 - g)\| = \big\|x^{1/2}(1 - g)^2 x^{1/2}\big\|, \quad \text{and} \quad (b_1 + b_2)^{1/2}x^{1/2} \approx x^{1/2}$$

to get (details omitted)

$$\big\|(1 - g)(b_1 + b_2)(1 - g)\big\| > 1 - \frac{\varepsilon}{3}.$$

When A is finite (continued)

We assumed $g \precsim_A d_1$ and $\|(1-g)x(1-g)\| \leq 1-\varepsilon$, and we want a contradiction. We have

$$0 \leq d_j \leq c_j \leq b_j \leq a \leq 1, \quad ab_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \quad \text{and} \quad d_j \neq 0$$

for $j = 1, 2$, and $b_1 b_2 = 0$. We also have

$$\big\|(1-g)(b_1+b_2)(1-g)\big\| > 1 - \frac{\varepsilon}{3}. \tag{1}$$

From $(b_1+b_2)(c_1+c_2) = c_1+c_2$ one gets, for any $\beta \in [0,1)$,

$$c_1 + c_2 \precsim_A [(b_1+b_2) - \beta]_+. \tag{2}$$

(If we are in $C(X)$, whenever $(c_1+c_2)(x) \neq 0$, we have $(b_1+b_2)(x) = 1 > \beta$.) Take $\beta = 1 - \frac{\varepsilon}{3}$. Combine (2) with the second lemma on the list of basic results on Cuntz equivalence at the first step, (1) at the second step, and $g \precsim_A d_1$ at the last step, to get

$$c_1 + c_2 \precsim_A \big[(1-g)(b_1+b_2)(1-g) - \beta\big]_+ \oplus g = 0 \oplus g \precsim_A d_1.$$

When A is finite (continued)

In search of a contradiction, we have gotten

$$c_1 + c_2 \precsim_A d_1$$

with

$$c_1 d_1 = d_1, \quad c_1 c_2 = 0, \quad \text{and} \quad c_2 \neq 0.$$

This looks rather suspicious.

Set $r = (1 - c_1 - c_2) + d_1$. Use basic result (12) at the first step, $c_1 + c_2 \precsim_A d_1$ at the second step, and basic result (13) and $d_1(1 - c_1 - c_2) = 0$ at the third step, to get

$$1 \precsim_A (1-c_1-c_2) \oplus (c_1+c_2) \precsim_A (1-c_1-c_2) \oplus d_1 \sim_A (1-c_1-c_2) + d_1 = r.$$

Thus, there is $v \in A$ such that $\|vrv^* - 1\| < \frac{1}{2}$. It follows that $vr^{1/2}$ has a right inverse. Recall that $c_2 d_2 = d_2$ and $d_2 \neq 0$. So $rd_2 = 0$, whence $vr^{1/2} d_2 = 0$. Thus $vr^{1/2}$ is not invertible. We have contradicted finiteness of A, and thus proved the lemma.

Lemma

Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $m, n \in \mathbb{Z}_{\geq 0}$, let $a_1, a_2, \ldots, a_m \in A$, let $b_1, b_2, \ldots, b_n \in A_+$, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in B_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$, $d_1, d_2, \ldots, d_n \in A_+$, and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. $\|c_j - a_j\| < \varepsilon$ and $\|d_j - b_j\| < \varepsilon$.
3. $\|c_j\| \leq \|a_j\|$ and $\|d_j\| \leq \|b_j\|$.
4. $(1-g)c_j \in B$ and $(1-g)d_j(1-g) \in B$.
5. $g \precsim_B y$ and $g \precsim_A x$.

Sketch of proof.

To get $\|c_j\| \leq \|a_j\|$ one takes $\varepsilon > 0$ to be a bit smaller, and scales down c_j for any j for which $\|c_j\|$ is too big. To get d_j, approximate $b_j^{1/2}$ sufficiently well by r_j (without increasing the norm), and take $d_j = r_j r_j^*$. □

Simplicity of a large subalgebra

Recall from Lecture 1:

Proposition

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple.

(The result stated in Lecture 1 also included infinite dimensionality. Once one has simplicity, infinite dimensionality is easy to prove, and we omit it.) The proof of this proposition uses two preliminary lemmas.

Lemma

Let A be a C*-algebra, let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^n a_k$ and $x = \sum_{k=1}^n a_k^* a_k$. Then $a^* a \in \overline{xAx}$.

Lemma

Let A be a unital C*-algebra and let $a \in A_+$. Suppose $\overline{AaA} = A$. Then there exist $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\sum_{k=1}^n x_k^* a x_k = 1$.

The first lemma

From the previous slide:

Lemma

Let A be a C*-algebra, let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^{n} a_k$ and $x = \sum_{k=1}^{n} a_k^* a_k$. Then $a^* a \in \overline{xAx}$.

Sketch of proof.

Assume $\|a_k\| \leq 1$ for $k = 1, 2, \ldots, n$. Choose $c \in \overline{xAx}$ such that $\|c\| \leq 1$ and $\|c a_k^* a_k - a_k^* a_k\|$ is small for $k = 1, 2, \ldots, n$. Check that $\|c a_k^* - a_k^*\|^2 \leq 2 \|c a_k^* a_k - a_k^* a_k\|$, so $\|c a_k^* - a_k^*\|$ is small. Then $\|c a^* - a^*\|$ is small, so that $\|c a^* a c - a^* a\|$ is small. Therefore $a^* a$ is arbitrarily close to $\overline{xAx}$. □

The second lemma

From the slide before the previous slide:

Lemma

Let A be a unital C*-algebra and let $a \in A_+$. Suppose $\overline{AaA} = A$. Then there exist $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\sum_{k=1}^{n} x_k^* a x_k = 1$.

Proof.

Choose $n \in \mathbb{Z}_{>0}$ and $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in A$ such that the element $c = \sum_{k=1}^{n} y_k a z_k$ satisfies $\|c - 1\| < 1$. Set

$$r = \sum_{k=1}^{n} z_k^* a y_k^* y_k a z_k, \quad M = \max_k \|y_k\|, \quad \text{and} \quad s = M^2 \sum_{k=1}^{n} z_k^* a^2 z_k.$$

The previous lemma implies that $c^* c$ is in the hereditary subalgebra generated by r. The relation $\|c - 1\| < 1$ implies that c is invertible, so r is invertible. Since $r \leq s$, it follows that s is invertible. Set $x_k = M a^{1/2} z_k s^{-1/2}$. Then check that $\sum_{k=1}^{n} x_k^* a x_k = s^{-1/2} s s^{-1/2} = 1$. □

Proof of simplicity of B

Recall that we want to prove:

Proposition

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple.

Let $b \in B_+ \setminus \{0\}$. We show that there are $n \in \mathbb{Z}_{>0}$ and $r_1, r_2, \ldots, r_n \in B$ such that $\sum_{k=1}^{n} r_k b r_k^*$ is invertible.

Use the previous lemma to find $x_1, x_2, \ldots, x_m \in A$ such that $\sum_{k=1}^{m} x_k b x_k^* = 1$. Set

$$M = \max\left(1, \|x_1\|, \ldots, \|x_m\|, \|b\|\right) \quad \text{and} \quad \delta = \min\left(1, \frac{1}{3mM(2M+1)}\right).$$

By definition, there are $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, $\|y_j - x_j\| < \delta$, $(1 - g) y_j \in B$, and $g \precsim_B b$. Set $z = \sum_{k=1}^{m} y_j b y_j^*$. The number δ has been chosen to ensure that $\|z - 1\| < \frac{1}{3}$; we omit details. Then $\left\|(1 - g) z (1 - g) - (1 - g)^2\right\| < \frac{1}{3}$.

Proof of simplicity of B (continued)

We are proving:

Proposition

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple.

We took $b \in B_+ \setminus \{0\}$. We got $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, $\|y_j - x_j\| < \delta$, $(1 - g) y_j \in B$, and $g \precsim_B b$. We defined $z = \sum_{k=1}^{m} y_j b y_j^*$, and got $\left\|(1 - g) z (1 - g) - (1 - g)^2\right\| < \frac{1}{3}$.

Set $h = 2g - g^2$. Use basic result (3) on Cuntz comparison on the map $\lambda \mapsto 2\lambda - \lambda^2$ on $[0, 1]$, to get $h \sim_B g$. So $h \precsim_B b$. Choose $v \in B$ such that $\|v b v^* - h\| < \frac{1}{3}$.

Take $n = m + 1$, take $r_j = (1 - g) y_j$ for $j = 1, 2, \ldots, m$, and take $r_{m+1} = v$. Then $r_1, r_2, \ldots, r_n \in B$. One can now check, using $(1 - g)^2 + h = 1$, that $\|1 - \sum_{k=1}^{n} r_k b r_k^*\| < \frac{2}{3}$. Therefore $\sum_{k=1}^{n} r_k b r_k^*$ is invertible. This proves simplicity of B.

Traces

For a unital C*-algebra A, we denote by $\mathrm{T}(A)$ the set of tracial states on A. We denote by $\mathrm{QT}(A)$ the set of normalized 2-quasitraces on A.

If you haven't heard of quasitraces, just pretend they are all tracial states. This is true on exact C*-algebras (in particular, on nuclear ones), and it is possible that it is always true.

Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.

To understand this, take $A = C(X)$ and $g \in C(X)$ with $0 \leq g \leq 1$, and take τ to be given by a probability measure μ on X. ($\tau(f) = \int_X f \, d\mu$.) Set $U = \{x \in X \colon g(x) \neq 0\}$. Then $g^{1/n} \nearrow \chi_U$ and $d_\tau(g) = \mu(U)$.

Some facts: d_τ gives a well defined functional $d_\tau \colon W(A) \to [0, \infty)$ (and also $d_\tau \colon \mathrm{Cu}(A) \to [0, \infty]$) such that $d_\tau(\langle a \rangle_A)$ is "the trace of the open support of a". It preserves order and addition, and commutes with countable increasing supremums when they exist. In particular, $d_\tau(a) = \sup_{\varepsilon > 0} d_\tau((a - \varepsilon)_+)$. Also, $0 \leq a \leq 1$ implies $\tau(a) \leq d_\tau(a)$.

Bijection on traces

Recall from Lecture 1:

Theorem

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective.

(The result stated in Lecture 1 also included the same thing for quasitraces. That result requires much more work, since it depends on the fact that the inclusion of A in B induces an isomorphism on the subsemigroups of purely positive elements.)

The proof of this proposition uses a preliminary lemma.

Lemma

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in \mathrm{T}(B)$. Then there exists a unique state ω on A such that $\omega|_B = \tau$.

From the previous slide:

Lemma

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in \mathrm{T}(B)$. Then there exists a unique state ω on A such that $\omega|_B = \tau$.

Existence of ω follows from the Hahn-Banach Theorem.

For uniqueness, let ω_1 and ω_2 be states with $\omega_1|_B = \omega_2|_B = \tau$, let $a \in A_+$, and let $\varepsilon > 0$. We show $|\omega_1(a) - \omega_2(a)| < \varepsilon$. We can assume $\|a\| \leq 1$.

We saw above that B is simple and infinite dimensional. The third lemma on the list of basic results on Cuntz equivalence can be used to find $y \in B_+ \setminus \{0\}$ such that $\sup_{\sigma \in \mathrm{QT}(B)} d_\sigma(y)$ is as small as we want. (For orthogonal elements with $b_1 \sim_B b_2 \sim_B \cdots \sim_B b_n$, we must have $d_\sigma(b_1) = d_\sigma(b_2) = \cdots = d_\sigma(b_n)$, so $n d_\sigma(b_1) \leq 1$.) Choose $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$. Since B is large, there are $c \in A_+$ and $g \in B_+$ such that $\|c\| \leq 1$, $\|g\| \leq 1$, $\|c - a\| < \frac{\varepsilon}{4}$, $(1-g)c(1-g) \in B$, and $g \precsim_B y$. So $\omega_j(g^2) = \tau(g^2) \leq d_\tau(g^2) < \frac{\varepsilon^2}{64}$.

We have $a \in A$ and we want to prove that $|\omega_1(a) - \omega_2(a)| < \varepsilon$. We have

$$\|c\| \leq 1, \quad \|g\| \leq 1, \quad \|c - a\| < \tfrac{\varepsilon}{4}, \quad (1-g)c(1-g) \in B, \quad \omega_j(g^2) < \tfrac{\varepsilon^2}{64}.$$

The Cauchy-Schwarz inequality gives

$$|\omega_j(rs)| \leq \omega_j(rr^*)^{1/2} \omega_j(s^*s)^{1/2}$$

for all $r, s \in A$. Using $\|c\| \leq 1$, we then get

$$|\omega_j(gc)| \leq \omega_j(g^2)^{1/2} \omega_j(c^2)^{1/2} < \tfrac{\varepsilon}{8},$$

$$|\omega_j((1-g)cg)| \leq \omega_j\big((1-g)c^2(1-g)\big)^{1/2} \omega_j(g^2)^{1/2} < \tfrac{\varepsilon}{8}.$$

So (omitting some algebra at the second step)

$$\begin{aligned} \big|\omega_j(c) - \tau((1-g)c(1-g))\big| &= \big|\omega_j(c) - \omega_j((1-g)c(1-g))\big| \\ &\leq |\omega_j(gc)| + |\omega_j((1-g)cg)| < \tfrac{\varepsilon}{4}. \end{aligned}$$

Also $|\omega_j(c) - \omega_j(a)| < \frac{\varepsilon}{4}$. So

$$\big|\omega_j(a) - \tau((1-g)c(1-g))\big| < \tfrac{\varepsilon}{2}.$$

Thus $|\omega_1(a) - \omega_2(a)| < \varepsilon$, as desired. The lemma is proved.

Bijection on traces

Recall that we want to prove:

Theorem

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective.

Let $\tau \in \mathrm{T}(B)$. We show that there is a unique $\omega \in \mathrm{T}(A)$ such that $\omega|_B = \tau$. We know that there is a unique state ω on A such that $\omega|_B = \tau$, and it suffices to show that ω is a trace. Thus let $a_1, a_2 \in A$ satisfy $\|a_1\| \leq 1$ and $\|a_2\| \leq 1$, and let $\varepsilon > 0$. We show that $|\omega(a_1 a_2) - \omega(a_2 a_1)| < \varepsilon$.

As in the proof of the lemma, find $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$. Since B is large, there are $c_1, c_2 \in A$ and $g \in B_+$ such that

$$\|c_j\| \leq 1, \quad \|c_j - a_j\| < \frac{\varepsilon}{8}, \quad \text{and} \quad (1-g)c_j \in B$$

for $j = 1, 2$, and such that $\|g\| \leq 1$ and $g \precsim_B y$. As before, $\omega(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$.

Bijection on traces (continued)

We got $0 \leq g \leq 1$, $\|c_j\| \leq 1$, $\|c_j - a_j\| < \frac{\varepsilon}{8}$, $(1-g)c_j \in B$, $\omega(g^2) < \frac{\varepsilon^2}{64}$.

We claim that

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1c_2)\big| < \frac{\varepsilon}{4}.$$

Using the Cauchy-Schwarz inequality as in the proof of the lemma, we get

$$|\omega(gc_1c_2)| \leq \omega(g^2)^{1/2}\omega(c_2^*c_1^*c_1c_2)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

Similarly, and also at the second step using $\|c_2\| \leq 1$, $(1-g)c_1g \in B$, and the fact that $\omega|_B$ is a tracial state,

$$\begin{aligned}\big|\omega((1-g)c_1gc_2)\big| &\leq \omega\big((1-g)c_1g^2c_1^*(1-g)\big)^{1/2}\omega(c_2^*c_2)^{1/2} \\ &\leq \omega\big(gc_1^*(1-g)^2c_1g\big)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.\end{aligned}$$

The claim now follows from the estimate (an algebra step is omitted)

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1c_2)\big| \leq \big|\omega((1-g)c_1gc_2)\big| + |\omega(gc_1c_2)|.$$

Bijection on traces (continued)

We got $\|a_j\| \leq 1$, $\|c_j\| \leq 1$, $\|c_j - a_j\| < \frac{\varepsilon}{8}$, $(1-g)c_j \in B$, and (at the bottom of the previous slide)

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1c_2)\big| < \frac{\varepsilon}{4}.$$

A similar argument gives

$$\big|\omega((1-g)c_2(1-g)c_1) - \omega(c_2c_1)\big| < \frac{\varepsilon}{4}.$$

Since $(1-g)c_1$, $(1-g)c_2 \in B$ and $\omega|_B$ is a tracial state, we get

$$\omega((1-g)c_1(1-g)c_2) = \omega((1-g)c_2(1-g)c_1).$$

Therefore $|\omega(c_1c_2) - \omega(c_2c_1)| < \frac{\varepsilon}{2}$.

One checks that $\|c_1c_2 - a_1a_2\| < \frac{\varepsilon}{4}$ and $\|c_2c_1 - a_2a_1\| < \frac{\varepsilon}{4}$. It now follows that $|\omega(a_1a_2) - \omega(a_2a_1)| < \varepsilon$.

We have $|\omega(a_1a_2) - \omega(a_2a_1)| < \varepsilon$ for all $\varepsilon > 0$, so $\omega(a_1a_2) = \omega(a_2a_1)$.

Bijection on traces

We have thus proved:

Theorem

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective.