Large Subalgebras and the Structure of Crossed Products, Lecture 5: Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms

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- Lecture 1 (1 June 2015): Introduction, Motivation, and the Cuntz Semigroup.
- Lecture 2 (2 June 2015): Large Subalgebras and their Basic Properties.
- Lecture 3 (4 June 2015): Large Subalgebras and the Radius of Comparison.
- Lecture 4 (5 June 2015 [morning]): Large Subalgebras in Crossed Products by $\mathbb{Z}$.
- Lecture 5 (5 June 2015 [afternoon]): Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms.

A rough outline of all five lectures

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if B is a large subalgebra of A, then A and B have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.

Reminder: Covering dimension

Definition

Let X be a compact Hausdorff space.

1. Let $\mathcal{U}$ be a finite open cover of X. The *order* of $\mathcal{U}$ is the least number n such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty. (The formula $\mathrm{ord}(\mathcal{U}) = -1 + \sup_{x \in X} \sum_{U \in \mathcal{U}} \chi_U(x)$ is often used.)
2. Let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X. Then $\mathcal{V}$ *refines* $\mathcal{U}$ (written $\mathcal{V} \prec \mathcal{U}$) if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.
3. Let $\mathcal{U}$ be a finite open cover of X. We define the *dimension* $\mathcal{D}(\mathcal{U})$ to be the least possible order of a finite open cover which refines $\mathcal{U}$. That is, $\mathcal{D}(\mathcal{U}) = \inf_{\mathcal{V} \prec \mathcal{U}} \mathrm{ord}(\mathcal{V})$.
4. The *covering dimension* $\dim(X)$ is the supremum of $\mathcal{D}(\mathcal{U})$ over all finite open covers $\mathcal{U}$ of X.

Covering dimension (continued)

Recall the definitions involving covers from the previous slide:

1. $\mathrm{ord}(\mathcal{U})$ is the least number n such that the intersection of any $n+1$ distinct elements of $\mathcal{U}$ is empty.
2. $\mathcal{V} \prec \mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.
3. $\mathcal{D}(\mathcal{U}) = \inf_{\mathcal{V} \prec \mathcal{U}} \mathrm{ord}(\mathcal{V})$.
4. $\dim(X) = \sup_{\mathcal{U}} \mathcal{D}(\mathcal{U})$.

We can observe that if X is totally disconnected, then $\dim(X) = 0$. One sees $\dim([0,1]) = 1$ by using open covers consisting of short intervals each of which only intersects its immediate neighbors. One sees $\dim([0,1]^2) = 2$ by using open covers consisting of small neighborhoods of the tiles in a fine hexagonal tiling. It is harder to see what is happening in higher dimensions.

It is a fact that $\dim(X \times Y) \leq \dim(X) + \dim(Y)$, with equality if X and Y are sufficiently nice (for example, finite complexes). However, equality need not hold in general.

Mean dimension

Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism. (For best behavior, h should not have "too many" periodic points.) Lindenstrauss and Weiss defined the *mean dimension* $\mathrm{mdim}(h)$. It is designed so that if K is a sufficiently nice compact metric space (in particular, $\dim(K^n)$ should equal $n \cdot \dim(K)$ for all n), then the shift on $X = K^{\mathbb{Z}}$ should have mean dimension equal to $\dim(K)$.

Given this heuristic, it should not be surprising that if $\dim(X) < \infty$ then $\mathrm{mdim}(h) = 0$.

Definition

Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be two finite open covers of X. Then the *join* $\mathcal{U} \vee \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \vee \mathcal{V} = \{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \}.$$

Definition of mean dimension

Recall: $\mathcal{U} \vee \mathcal{V} = \{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \}$.

Definition

Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism. Then the *mean dimension* of h is

$$\mathrm{mdim}(h) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

The supremum is over all finite open covers of X (just like in the definition of $\dim(X)$).

The definition uses the join of n covers. One needs to prove that the limit exists; we omit this.

If $\dim(X) < \infty$, then the numerator is always at most $\dim(X)$, so the limit is zero. More generally, if h is minimal and has at most countably many ergodic measures, then $\mathrm{mdim}(h) = 0$.

Mean dimension and radius of comparison

Theorem from Lecture 1:

Theorem (Joint work with Hines and Toms)

Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

It is hoped that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for any minimal homeomorphism of an infinite compact metric space X. We can show this for some special systems covered by this theorem, slightly generalizing the construction of Giol and Kerr.

Factor systems

The hypothesis on existence of a surjective map to the Cantor set has other equivalent formulations, one of which is the existence of an equivariant surjective map to the Cantor set. We need a definition.

Definition

Let $h \colon X \to X$ and $k \colon Y \to Y$ be homeomorphisms. We say that the system (Y, k) is a *factor* of (X, h) if there is a surjective continuous map $g \colon X \to Y$ (the *factor map*) such that $g \circ h = k \circ g$.

The requirement in the definition is that the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\ k\ } & Y. \end{array}$$

Cantor set factors

Proposition

Let X be a compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Then the following are equivalent:

1. There exists a decreasing sequence $Y_0 \supset Y_1 \supset Y_2 \supset \cdots$ of nonempty compact open subsets of X such that the subset $Y = \bigcap_{n=0}^{\infty} Y_n$ satisfies $h^r(Y) \cap Y = \varnothing$ for all $r \in \mathbb{Z} \setminus \{0\}$.
2. There is a minimal homeomorphism of the Cantor set which is a factor of (X, h).
3. There is a continuous surjective map from X to the Cantor set.
4. For every $n \in \mathbb{Z}_{>0}$ there is a partition $\mathcal{P}$ of X into at least n nonempty compact open subsets.

We omit the proof.

To keep things simple, in this lecture we will assume that h has a particular minimal homeomorphism of the Cantor set as a factor, namely an odometer system.

Odometer as a factor system

Assume h is minimal and $h^n(Y) \cap Y = \varnothing$ for $n \in \mathbb{Z} \setminus \{0\}$. Write $Y = \bigcap_{n=0}^{\infty} Y_n$ with $Y_0 \supset Y_1 \supset \cdots$ and $\operatorname{int}(Y_n) \neq \varnothing$ for all $n \in \mathbb{Z}_{\geq 0}$. Then $C^*(\mathbb{Z}, X, h)_Y = \varinjlim_n C^*(\mathbb{Z}, X, h)_{Y_n}$, and $C^*(\mathbb{Z}, X, h)_{Y_n}$ is a recursive subhomogeneous C*-algebra whose base spaces are closed subsets of X.

The effect of requiring a Cantor system factor is that one can choose Y and $(Y_n)_{n \in \mathbb{Z}_{\geq 0}}$ so that Y_n is both closed and open for all $n \in \mathbb{Z}_{\geq 0}$. This ensures that $C^*(\mathbb{Z}, X, h)_{Y_n}$ is a homogeneous C*-algebra whose base spaces are closed subsets of X. Thus $C^*(\mathbb{Z}, X, h)_Y$ is a simple AH algebra. This is done by taking Y to be the inverse image of a point in the Cantor set.

The further simplification of assuming an odometer factor (definition on next slide) is that one can arrange $C^*(\mathbb{Z}, X, h)_{Y_n} \cong M_{p_n}(C(Y_n))$, that is, there is only one summand. This simplifies the notation but otherwise makes little difference.

Odometers

Definition

Let $d = (d_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{Z}_{>0}$ with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. The *d-odometer* is the minimal system (X_d, h_d) defined as follows. Set

$$X_d = \prod_{n=1}^{\infty} \{0, 1, 2, \ldots, d_n - 1\},$$

which is homeomorphic to the Cantor set. For $x = (x_n)_{n \in \mathbb{Z}_{>0}} \in X_d$, let

$$n_0 = \inf\big(\big\{ n \in \mathbb{Z}_{>0} \colon x_n \neq d_n - 1 \big\}\big).$$

If $n_0 = \infty$ set $h_d(x) = (0, 0, \ldots)$. Otherwise, $h_d(x) = (h_d(x)_n)_{n \in \mathbb{Z}_{>0}}$ is

$$h_d(x)_n = \begin{cases} 0 & n < n_0 \\ x_n + 1 & n = n_0 \\ x_n & n > n_0. \end{cases}$$

It is "addition of $(1, 0, 0, \ldots)$ with carry to the right". When $n_0 \neq \infty$, we have

$$h(x) = \big(0, 0, \ldots, 0, x_{n_0} + 1, x_{n_0+1}, x_{n_0+2}, \ldots\big).$$

Consequences of having an odometer factor

We assume from now on that (X, h) has as a factor system the odometer on $X_d = \prod_{n=1}^{\infty}\{0, 1, 2, \ldots, d_n - 1\}$ for a sequence $d = (d_n)_{n \in \mathbb{Z}_{>0}}$ of integers with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. We will take Y to be the inverse image of $(0, 0, \ldots)$, and take Y_n to be the inverse image of

$$\{0\}^n \times \prod_{k=n+1}^{\infty} \{0, 1, 2, \ldots, d_k - 1\}.$$

Set $p_n = \prod_{k=1}^{n} d_k$. With some work (most of which consists of keeping notation straight), one can show that

$$C^*(\mathbb{Z}, X, h)_Y = \varinjlim_n C(Y_n, M_{p_n}),$$

with the maps $\psi_{n,m} \colon C(Y_m, M_{p_m}) \to C(Y_n, M_{p_n})$ of the system being

$$\psi_{n,m}(f) = \operatorname{diag}\big(f|_{Y_n},\, f \circ h^{p_m}|_{Y_n},\, f \circ h^{2p_m}|_{Y_n},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}|_{Y_n}\big).$$

In particular,

$$\psi_{n,0}(f) = \operatorname{diag}\big(f|_{Y_n},\, f \circ h|_{Y_n},\, f \circ h^2|_{Y_n},\, \ldots,\, f \circ h^{p_n - 1}|_{Y_n}\big).$$

Further simplification

$C^*(\mathbb{Z}, X, h)_Y = \varinjlim_n C(Y_n, M_{p_n})$, with maps

$$\psi_{n,m}(f) = \operatorname{diag}\big(f|_{Y_n},\, f \circ h^{p_m}|_{Y_n},\, f \circ h^{2p_m}|_{Y_n},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}|_{Y_n}\big).$$

We have to show that $\operatorname{rc}(C^*(\mathbb{Z}, X, h)_Y) \leq \frac{1}{2}\operatorname{mdim}(h)$.

We will make a further simplification, whose main effect is to simplify the notation, and consider instead the direct limit

$$B = \varinjlim_n C(X, M_{p_n})$$

with maps

$$\psi_{n,m}(f) = \operatorname{diag}\big(f,\, f \circ h^{p_m},\, f \circ h^{2p_m},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}\big).$$

There is one annoyance: we must now assume that h^{p_n} is minimal for all n. (Otherwise, it turns out that the direct limit isn't simple. This actually excludes systems with odometer factors, but never mind.)

(The resulting system is a kind of AH model for the crossed product.)

The main lemma

We would like to use Theorem 6.2 of Niu's paper *Mean dimension and AH-algebras with diagonal maps*. Unfortunately, the definition there of mean dimension of an AH direct system requires that the base spaces be connected, or at least have only finitely many connected components. Our base spaces have surjective maps to the Cantor set. So we proceed more directly, although the arguments are closely related.

Lemma

Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism with no periodic points. Then for every $\varepsilon > 0$ and every finite subset $F \subset C(X)$ there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ there is a compact metric space K and a surjective map $i \colon X \to K$ such that:

1. $\dim(K) < n[\operatorname{mdim}(h) + \varepsilon]$.
2. For $m = 0, 1, \ldots, n - 1$ and $f \in F$ there is $g \in C(K)$ such that $\|f \circ h^m - g \circ i\| < \varepsilon$.

Proof

All sets occurring in open covers will be assumed to be nonempty.

Recall that for any finite open cover $\mathcal{V}$ of a space X, the nerve $K(\mathcal{V})$ is the finite simplicial complex with vertices $[V]$ for $V \in \mathcal{V}$, and in which there is a simplex in $K(\mathcal{V})$ with vertices $[V_0], [V_1], \ldots, [V_n]$ if and only if $V_0 \cap V_1 \cap \cdots \cap V_n \neq \varnothing$. The points $z \in K(\mathcal{V})$ are thus exactly the formal convex combinations

$$z = \sum_{V \in \mathcal{V}} \alpha_V [V] \tag{1}$$

in which $\alpha_V \geq 0$ for all $V \in \mathcal{V}$, $\sum_{V \in \mathcal{V}} \alpha_V = 1$, and $\big\{[V] \colon \alpha_V \neq 0\big\}$ is a simplex in $K(\mathcal{V})$, that is,

$$\bigcap \big\{V \in \mathcal{V} \colon \alpha_V \neq 0\big\} \neq \varnothing.$$

We have $\dim(K(\mathcal{V})) = \operatorname{ord}(\mathcal{V})$. For any partition of unity $(g_V)_{V \in \mathcal{V}}$ with $\operatorname{supp}(g_V) \subset V$ for all $V \in \mathcal{V}$, there is a continuous map $i \colon X \to K(\mathcal{V})$ determined, using (1), by

$$i(x) = \sum g_V(x)[V]$$

for $x \in X$.

Proof (continued)

Continuing with an arbitrary finite open cover $\mathcal{V}$ of X and the notation above, for every $V \in \mathcal{V}$ choose a point $x_V \in V$. Following (1), define $P\colon C(X) \to C(K(\mathcal{V}))$ by

$$P(f)\left(\sum_{V\in\mathcal{V}} \alpha_V [V]\right) = \sum_{V\in\mathcal{V}} \alpha_V f(x_V)$$

for $f \in C(X)$. One easily checks that $P(f)$ is in fact continuous.

Let $f \in C(X)$. We claim that if $r > 0$ and for all $V \in \mathcal{V}$ and $x, y \in V$ we have $|f(x) - f(y)| < r$, then $\|P(f) \circ i - f\| < r$. Let $x \in X$ and estimate:

$$|P(f)(i(x)) - f(x)| = \left| P(f)\left(\sum_{V\in\mathcal{V}} g_V(x)[V]\right) - \sum_{V\in\mathcal{V}} g_V(x) f(x) \right|$$
$$\leq \sum_{V\in\mathcal{V}} g_V(x)|f(x_V) - f(x)| < \sum_{V\in\mathcal{V}} g_V(x) r = r.$$

By continuity and compactness, this implies $\|P(f) \circ i - f\| < r$, proving the claim.

Proof (continued)

Now choose a finite open cover $\mathcal{U}$ of X such that for all $U \in \mathcal{U}$, $x, y \in U$, and $f \in F$, we have $|f(x) - f(y)| < \varepsilon$. By definition, we have

$$\lim_{n\to\infty} \frac{\mathcal{D}(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U}))}{n} \leq \mathrm{mdim}(h).$$

Therefore there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have

$$\frac{\mathcal{D}(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U}))}{n} < \mathrm{mdim}(h) + \varepsilon.$$

Proof (continued)

Let $n \geq N$. Then there is a finite open cover $\mathcal{V}$ of X which refines

$$\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})$$

and such that

$$\mathrm{ord}(\mathcal{V}) < n[\mathrm{mdim}(h) + \varepsilon]. \qquad (2)$$

Apply the first part of the proof with this choice of $\mathcal{V}$, getting $i\colon X \to K(\mathcal{V})$ and $P\colon C(X) \to C(K(\mathcal{V}))$ as there. Let $f \in F$ and let $m \in \{0, 1, \ldots, n-1\}$. Since $\mathcal{V}$ refines $h^{-m}(\mathcal{U})$, it follows that for all $V \in \mathcal{V}$ and $x, y \in V$ we have $|(f \circ h^m)(x) - (f \circ h^m)(y)| < \varepsilon$. So $\|P(f \circ h^m) \circ i - f \circ h^m\| < \varepsilon$.

We are done with the proof except for the fact that i might not be surjective. So define $K = i(X) \subset K(\mathcal{V})$. Since the dimension of a subspace can't be larger than the dimension of the whole space,

$$\dim(K) \leq \dim(K(\mathcal{V})) = \mathrm{ord}(\mathcal{V}) < n[\mathrm{mdim}(h) + \varepsilon].$$

In place of $P(f \circ h^m)$ we use $P(f \circ h^m)|_K$. This completes the proof.

Starting the proof of the main theorem

Recall: $B = \varinjlim_n C(X, M_{p_n})$ with maps

$$\psi_{n,m}(f) = \mathrm{diag}\big(f,\, f \circ h^{p_m},\, f \circ h^{2p_m},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}\big).$$

We are assuming that B is simple, and we want $\mathrm{rc}(B) \leq \frac{1}{2}\mathrm{mdim}(h)$.

Lemma (Niu)

Let B be a simple unital exact C*-algebra and let $r \in [0, \infty)$. Suppose:

1. For every finite subset $S \subset B$ and every $\varepsilon > 0$, there is a unital C*-algebra D such that $\mathrm{rc}(D) < r + \varepsilon$ and an injective unital homomorphism $\rho\colon D \to B$ such that $\mathrm{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.
2. For every $s \in [0, 1]$ and every $\varepsilon > 0$, there exists a projection $p \in B$ with $|\tau(p) - s| < \varepsilon$ for all $\tau \in \mathrm{T}(B)$.

Then $\mathrm{rc}(B) \leq r$.

Certainly our B is simple, unital, and exact. Since $C(X, M_{p_n}) \hookrightarrow B$ and $C(X, M_{p_n})$ has projections of constant rank k for any $k \in \{0, 1, \ldots, p_n\}$, condition (2) is satisfied.

Proving the main theorem

$B = \varinjlim_n C(X, M_{p_n})$ with maps

$$\psi_{n,m}(f) = \operatorname{diag}\big(f,\, f \circ h^{p_m},\, f \circ h^{2p_m},\, \ldots,\, f \circ h^{(p_n/p_m - 1)p_m}\big).$$

We need to show that for every finite subset $S \subset B$ and every $\varepsilon > 0$, there is a unital C*-algebra D such that $\operatorname{rc}(D) < \frac{1}{2}\operatorname{mdim}(h) + \varepsilon$ and an injective unital homomorphism $\rho \colon D \to B$ such that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

We will use:

Theorem (Toms)

Let X be a compact metric space and let $n \in \mathbb{Z}_{>0}$. Then

$$\operatorname{rc}(M_n \otimes C(X)) \leq \frac{\dim(X) - 1}{2n}.$$

Proving the main theorem (continued)

For $n \in \mathbb{Z}_{\geq 0}$ let $\psi_n \colon C(X, M_{p_n}) \to B$ be the map obtained from the direct limit description of B. Let $S \subset B$ be finite and let $\varepsilon > 0$. Choose $m \in \mathbb{Z}_{>0}$ and a finite set

$$S_0 \subset C(X, M_{p_m}) = M_{p_m}(C(X))$$

such that for every $a \in S$ there is $b \in S_0$ with $\|\psi_m(b) - a\| < \frac{1}{2}\varepsilon$. Let $F \subset C(X)$ be the set of all matrix entries of elements of S_0. Use the main lemma to find $N \in \mathbb{Z}_{\geq 0}$ such that for all $l \geq N$ there is a compact metric space K and a surjective map $i \colon X \to K$ such that $\dim(K) < l[\operatorname{mdim}(h) + \varepsilon]$ and for $r = 0, 1, \ldots, l - 1$ and $f \in F$ there is $g \in C(K)$ with

$$\|f \circ h^r - g \circ i\| < \frac{\varepsilon}{2p_m^2}.$$

Choose $n \geq m$ such that $p_n \geq N$. Choose K and i for $l = p_n$, so that

$$\dim(K) < p_n[\operatorname{mdim}(h) + \varepsilon].$$

We have $\dim(K) < p_n[\operatorname{mdim}(h) + \varepsilon]$ and $i \colon X \to K$ such that for $r = 0, 1, \ldots, p_n - 1$ and $f \in F$ there is $g \in C(K)$ with

$$\|f \circ h^r - g \circ i\| < \frac{\varepsilon}{2p_m^2}.$$

We need to find a unital C*-algebra D such that $\operatorname{rc}(D) < \frac{1}{2}\operatorname{mdim}(h) + \varepsilon$ and an injective unital homomorphism $\rho \colon D \to B$ such that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

Define an injective homomorphism $\rho_0 \colon C(K) \to C(X)$ by $\rho_0(f) = f \circ i$ for $f \in C(K)$. Set $D = M_{p_n}(C(K))$ and define

$$\rho = \psi_n \circ (\operatorname{id}_{M_{p_n}} \otimes \rho_0) \colon D \to B.$$

Then ρ is also injective.

By the theorem of Toms above,

$$\operatorname{rc}(D) \leq \frac{\dim(K) - 1}{2p_n} < \frac{\operatorname{mdim}(h) + \varepsilon}{2} < \frac{\operatorname{mdim}(h)}{2} + \varepsilon.$$

It remains to prove that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

Proving the main theorem (continued)

It remains to prove that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

Let $a \in S$. Choose $b \in S_0$ such that $\|\psi_m(b) - a\| < \frac{1}{2}\varepsilon$. For $j, k \in \{0, 1, \ldots, p_m - 1\}$, we let $e_{j,k} \in M_{p_m}$ be the standard matrix unit (except that we start the indexing at 0 rather than 1). Then there are $b_{j,k} \in F$ for $j, k \in \{0, 1, \ldots, p_m - 1\}$ such that $b = \sum_{j,k=0}^{p_m - 1} e_{j,k} \otimes b_{j,k}$. By construction, for $r = 0, 1, \ldots, p_n - 1$ there is $g_{j,k,r} \in C(K)$ such that

$$\big\| g_{j,k,r} \circ i - b_{j,k} \circ h^r \big\| < \frac{\varepsilon}{2p_m^2}.$$

For $t = 0, 1, \ldots, p_n/p_m - 1$, define

$$c_t = \sum_{j,k=0}^{p_m - 1} e_{j,k} \otimes \big(g_{j,k,tp_m}|_K\big) \in M_{p_m}(C(K)).$$

Then define

$$c = \operatorname{diag}\big(c_0, c_1, \ldots, c_{p_n/p_m - 1}\big) \in M_{p_n}(C(K)).$$

We claim that $\|\rho(c) - a\| < \varepsilon$, which will finish the proof. This is a computation, which is omitted.