Lisboa Summer School Course on Crossed Product C*-Algebras

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Let Δ be the modular function of G. We define the adjoint by

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$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1}g,\, h^{-1}x)\, dh$$

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$$f^*(g, x) = \Delta(g)^{-1} \overline{f(g^{-1}, \, g^{-1} x)}.$$

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for $a, b \in A$ and $g, h \in G$,

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We also often write $l^1(G, A, \alpha)$ instead of $L^1(G, A, \alpha)$.

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In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\sum_{g \in G} \|a_g\| < \infty$.

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The integrated form of a covariant representation

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One needs to be more careful with the integral here, because v is generally only strong operator continuous, not norm continuous. Nevertheless, one gets $\|\sigma(a)\| \leq \|a\|_1$, so σ extends to a representation of $L^1(G, A, \alpha)$. We use the same notation σ for this extension.

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$$\begin{aligned}\sigma(au_g)\sigma(bu_h) &= \pi(a)v(g)\pi(b)v(g)^*v(g)v(h) = \pi(a)\pi(\alpha_g(b))v(g)v(h) \\ &= \pi(a\alpha_g(b))v(gh) = \sigma\big([a\alpha_g(b)]u_{gh}\big) = \sigma\big((au_g)(bu_h)\big).\end{aligned}$$

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Exercise

Starting from this computation, fill in the details of the proof that the integrated form representation σ really is a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$.

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In particular, since integrated form representations of $L^1(G, A, \alpha)$ are necessarily contractive, *all* continuous representations of $L^1(G, A, \alpha)$ are necessarily contractive.

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If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$. The point is that $M(L^1(G))$ is the measure algebra of G, and therefore contains the group elements as point masses.

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Exercise

Prove the theorem on the previous slide when G is discrete and A is unital.

For a small taste of the general case, use approximate identities in A to generalize to the case in which A is not necessarily unital.

The universal representation and the crossed product

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Exercise

Give a set theoretically correct definition of the crossed product.

The important point is to preserve the universal property below.

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$.

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It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in the definition above.) The crossed product is, essentially by construction, the universal C*-algebra for covariant representations of (G, A, α),

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There are many notations in use for crossed products, including:

- $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$.
- $C^*(A, G, \alpha)$ and $C^*_{\mathrm{r}}(A, G, \alpha)$.
- $A \rtimes_{\alpha} G$ and $A \rtimes_{\alpha,\mathrm{r}} G$ (used in Williams' book).
- $A \times_{\alpha} G$ and $A \times_{\alpha,\mathrm{r}} G$ (used in Davidson's book).
- $G \times_{\alpha} A$ and $G \times_{\alpha,\mathrm{r}} A$ (used in Pedersen's book).

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Corollary

Let A be a unital C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$. Then the crossed product $C^*(\mathbb{Z}, A, \alpha)$ is the universal C*-algebra generated by a copy of A and a unitary u, subject to the relations $uau^* = \alpha(a)$ for $a \in A$.

The universal representation and the crossed product when G is discrete (continued)

Exercise

Based on the discussion above, write down a careful proof of the theorem.

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For $a \in A$ and $g \in G$, set

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The integrated form of σ, will be called a regular representation of any of $C_{\mathrm{c}}(G, A, \alpha)$, $L^1(G, A, \alpha)$, $C^*(G, A, \alpha)$, and (when defined) $C^*_{\mathrm{r}}(G, A, \alpha)$.

The Hilbert space of the regular covariant representation

The easy way to construct $L^2(G, H)$ is to take it to be the completion of $C_{\mathrm{c}}(G, H)$ in the norm coming from the scalar product

$$\langle \xi, \eta \rangle = \int_G \langle \xi(g), \eta(g) \rangle \, dg.$$

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Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

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If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

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As with crossed products, in these notes we ignore the set theoretic difficulty.

The relationship between reduced and full crossed products

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Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

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We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of the book of Williams.

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We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of the book of Williams. It is, I believe, true that $L^1(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective, and this can probably be proved by working a little harder in the proof of Lemma 2.26 of the book of Williams, but I have not carried out the details and I do not know a reference.

When G is discrete: integrated form of a regular representation

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$$(\sigma(a)\xi)(h) = \sum_{g \in G} \pi_0(\alpha_h^{-1}(a_g))\big(\xi(g^{-1}h)\big).$$

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Lemma

For every $a \in C_{\mathrm{c}}(G, A, \alpha)$, we have $\|a\|_{\infty} \leq \|a\|_{\mathrm{r}} \leq \|a\| \leq \|a\|_{1}$.

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$$\|a_g\| = \|\pi_0(a_g)\| = \|s_g^* \sigma(a) s_1\| \leq \|\sigma(a)\| \leq \|a\|_{\mathrm{r}}.$$

This completes the proof.

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Of course, we can do the same with the full crossed product $C^*(G, A, \alpha)$.

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The last statement follows by continuity from "picking off coordinates" in the regular representation. □

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Injective representations of A always give injective regular representations of the reduced crossed product

It is true for general locally compact groups, not just discrete groups, that the regular representation of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to an injective representation of A is injective. See Theorem 7.7.5 of Pedersen's book.

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3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$.

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The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$.

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Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$. Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group C*-algebra. Not only is the Fourier series of a continuous function always in $l^2(\mathbb{Z})$, but the Fourier series of a function in $L^\infty(S^1)$, which is the group von Neumann algebra of $\mathbb{Z}$, is also always in $l^2(\mathbb{Z})$.

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Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.

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The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!

Preliminaries for computing crossed products

We will shortly do some explicit computations of examples. First, though, we give some useful preliminaries:

- Equivariant maps and functoriality.
- Crossed products of exact sequences.
- Crossed products and direct limits.
- Notation for matrix units.

Equivariant homomorphisms

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If (G, A, α) and (G, B, β) are G-algebras, then a homomorphism $\varphi \colon A \to B$ is said to be *equivariant* (or G-equivariant if the group must be specified) if for every $g \in G$, we have $\varphi \circ \alpha_g = \beta_g \circ \varphi$.

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For a fixed locally compact group G, the G-algebras and equivariant homomorphisms form a category.

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This is straightforward. See the notes for details.

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The analog for reduced crossed products is in general false.

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The proof is done by combining the universal properties of direct limits and crossed products. See the notes.

Notation for matrix units

For any index set S, let $\delta_s \in l^2(S)$ be the standard basis vector, determined by

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$

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$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Example: The trivial action (continued)

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Note how full and reduced crossed products parallel maximal and minimal tensor products.

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So φ_0 is an isometric isomorphism of *-algebras, and therefore extends to an isomorphism of the universal C*-algebras.

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If G acts on $G \times X$ by translation in the first factor and trivially in the second factor, we get the crossed product $C(X, K(l^2(G)))$. (In fact, this is true for an arbitrary action of G on X. See the notes.)