Lisboa Summer School Course on Crossed Product C*-Algebras: Lecture 4

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Note: Some of the discussion here is not in the notes, and some of the results are in a slightly different order.

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- $\dim\left([0,1]^{\mathbb{Z}}\right) = \infty$.

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One can get this result for the Cantor set by combining the main theorem with known classification results, so Putnam's argument doesn't give any more in the end.

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There is work in progress which will probably give positive results about $C^*(\mathbb{R}, X)$ when $\dim(X) < \infty$ and the action is free and minimal.

Other known results: $\mathbb{Z}^d$ acting on the Cantor set, groups which are not discrete, and actions on simple C*-algebras

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There is also a collection of related results on crossed products of simple C*-algebras by actions of $\mathbb{Z}$ and of finite groups which have the tracial Rokhlin property, and generalizations.

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The proof is an exercise.

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Let X be a compact metric space, and let $h \colon X \to X$ be a homeomorphism. In the transformation group C*-algebra $C^*(\mathbb{Z}, X, h)$, we write u for the standard unitary representing the generator of $\mathbb{Z}$.

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$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X), \, uC_0(X \setminus Y)) \subset C^*(\mathbb{Z}, X, h).$$

$C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a groupoid

Although we will not use formally groupoids in these notes, it should be pointed out that $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a subgroupoid of the transformation group groupoid $\mathbb{Z} \ltimes X$ made from the action of $\mathbb{Z}$ on X generated by h.

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For actions of $\mathbb{Z}^d$, it appears to be necessary to use subalgebras of the crossed product for which the only nice description is in terms of subgroupoids of the transformation group groupoid.

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We first claim that there is $N \in \mathbb{Z}_{>0}$ such that $\bigcup_{n=1}^{N} h^{-n}(Y) = X$. Set $U = \bigcup_{n=1}^{\infty} h^{-n}(Y)$, which is a nonempty open subset of X such that $U \subset h(U)$.

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$$r(y) = \min\{n \geq 1 \colon h^n(y) \in Y\} \leq N.$$

Proof of Lemma 3 (continued)

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It follows that p_k commutes with all elements of $C^*(\mathbb{Z}, X, h)_Y$.

Proof of Lemma 3 (continued)

Define $p_k \in C(X) \subset C^*(\mathbb{Z}, X, h)_Y$ by $p_k = \chi_{X_k}$. Then p_k trivially commutes with every element of $C(X)$. Moreover, suppose $f \in C(X)$ vanishes on Y. Since Y_k, $h^{n(k)}(Y_k) \subset Y$, we have

$$\chi_{Y_k} f = 0 \quad \text{and} \quad \chi_{h^{n(k)}(Y_k)} f = 0$$

Use the action of h on the levels of the tower at the first step, and the equations above at the second step, to get

$$p_k u f = u\big(p_k - \chi_{h^{n(k)}(Y_k)} + \chi_{Y_k}\big) f = u p_k f = u f p_k.$$

It follows that p_k commutes with all elements of $C^*(\mathbb{Z}, X, h)_Y$. So it suffices to prove that $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is AF for each k.

Proof of Lemma 3 (continued)

Now $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is the C*-algebra generated by $C(X_k)$ and

$$u(\chi_{X \setminus Y}) p_k = u(\chi_{X_k \setminus h^{n(k)}(Y_k)}) = \sum_{j=1}^{n(k)-1} u(\chi_{h^j(Y_k)})$$

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One can now check, although it is a bit tedious to write out the details, that there is an isomorphism $\psi_k \colon p_k C^*(\mathbb{Z}, X, h)_Y p_k \to M_{n(k)} \otimes C(Y_k)$

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$$\psi_k(f) = \mathrm{diag}\big(f \circ h|_{Y_k}, \, f \circ h^2|_{Y_k}, \, \ldots, \, f \circ h^{n(k)}|_{Y_k}\big)$$

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Notation

Let A be a C*-algebra, and let $p, q \in A$ be projections. We write $p \sim q$ to mean that p and q are Murray-von Neumann equivalent in A,

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This shows that we can control the size of the leftover in the definition of tracial rank zero using projections in $C(X)$ instead of in the crossed product. This is a big advantage.

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(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

The point is that $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, and (3) says, in view of Lemma 4, that $1 - p$ is "small".

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$$pbp = pb_0 p = pE(b)p.$$

Proof of Lemma 4 (continued)

Using this equation at the first step, we get

$$\|pcp - pE(c)p\| \leq \|pcp - pbp\| + \|pE(b)p - pE(c)p\| \leq 2\|c - b\| < 2\delta. \quad (1)$$

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This completes the proof.

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Let X be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Let $N \in \mathbb{Z}_{>0}$, and suppose that $Y, h(Y), \ldots, h^N(Y)$ are disjoint. Then the projections χ_Y and $\chi_{h^N(Y)}$ are Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)_Y$.

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An induction argument now shows that $\chi_{h(Y)} \sim \chi_{h^N(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$. Also, $\chi_{X \setminus Y} \sim \chi_{X \setminus h(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$.

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Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $S = C(X) \cup \{u\}$ Then for every finite subset $F \subset S$, every $\varepsilon > 0$, and every nonempty open set $U \subset X$, there exists compact open set $Y \subset X$ and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

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Choose $\delta_0 > 0$ with $\delta_0 < \frac{1}{2}\varepsilon$ and so small that $d(x_1, x_2) < 4\delta_0$ implies $|f(x_1) - f(x_2)| < \frac{1}{4}\varepsilon$ for all $f \in F$.

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Since h is minimal, there is $N > N_0 + 1$ such that $d(h^N(y), \, y) < \delta$.

Proof of Lemma 6 (continued)

Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$.

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Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$. Using minimality again, choose $r_{-N_0}, r_{-N_0+1}, \ldots, r_N \in \mathbb{Z}$

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Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$. Using minimality again, choose $r_{-N_0}, r_{-N_0+1}, \ldots, r_N \in \mathbb{Z}$ such that $h^{r_l}(y) \in U_l$ for $-N_0 \leq l \leq N$.

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$$h^{-N_0}(Y),\ h^{-N_0+1}(Y),\ \ldots,\ Y,\ h(Y),\ \ldots,\ h^N(Y)$$

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Set $q_0 = \chi_Y$. For $-N_0 \leq n \leq N$ set

$$T_n = h^n(Y) \quad \text{and} \quad q_n = u^n q_0 u^{-n} = \chi_{h^n(Y)}.$$

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Then the q_n are mutually orthogonal projections in $C(X)$.

Proof of Lemma 6 (continued)

We now have a sequence of projections, in principle going to infinity in both directions:

$$\ldots, \, q_{-N_0}, \, \ldots, \, q_{-1}, \, q_0, \, q_1, \, \ldots, \, q_{N-N_0}, \, \ldots, \, q_{N-1}, \, q_N, \, \ldots.$$

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The ones shown are orthogonal, and conjugation by u is the shift.

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The ones shown are orthogonal, and conjugation by u is the shift. The projections q_0 and q_N are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs q_{-1} and q_{N-1} down to q_{-N_0} and q_{N-N_0}.

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We now have a sequence of projections, in principle going to infinity in both directions:

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The ones shown are orthogonal, and conjugation by u is the shift. The projections q_0 and q_N are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs q_{-1} and q_{N-1} down to q_{-N_0} and q_{N-N_0}. We are now going to use Berg's technique to splice this sequence along the pairs of indices $(-N_0, \, N - N_0)$ through $(0, N)$, obtaining a loop of length N on which conjugation by u is approximately the cyclic shift.

Proof of Lemma 6 (continued)

Lemma 5 provides a partial isometry $w \in C^*(\mathbb{Z}, X, h)_Y$ such that $w^* w = q_0$ and $w w^* = q_N$.

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$$v(t) = \cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*).$$

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Lemma 5 provides a partial isometry $w \in C^*(\mathbb{Z}, X, h)_Y$ such that $w^* w = q_0$ and $ww^* = q_N$. For $t \in [0, 1]$ define

$$v(t) = \cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*).$$

Then $v(t)$ is a unitary in the corner

$$(q_0 + q_N) C^*(\mathbb{Z}, X, h)_Y (q_0 + q_N)$$

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For $0 \leq k \leq N_0$ define $z_k = u^{-k} v(k/N_0) u^k$.

Proof of Lemma 6 (continued)

We claim that $z_k \in C^*(\mathbb{Z}, X, h)_Y$ for $0 \leq k \leq N_0$.

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$$a_k = q_0 u^k = (uq_{-1})(uq_{-2}) \cdots (uq_{-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

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(because $N_0 < N$), and using $T_{-k} \cap T_{N-k} = T_0 \cap T_N = \varnothing$, we can write

$$z_k = (a_k + b_k)^* v(k/N_0)(a_k + b_k) \in C^*(\mathbb{Z}, X, h)_Y.$$

Proof of Lemma 6 (continued)

We claim that $z_k \in C^*(\mathbb{Z}, X, h)_Y$ for $0 \leq k \leq N_0$. For $k = 0$ this is true by construction. For $1 \leq k \leq N_0$, with

$$a_k = q_0 u^k = (uq_{-1})(uq_{-2}) \cdots (uq_{-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

and

$$b_k = q_N u^k = (uq_{N-1})(uq_{N-2}) \cdots (uq_{N-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

(because $N_0 < N$), and using $T_{-k} \cap T_{N-k} = T_0 \cap T_N = \varnothing$, we can write

$$z_k = (a_k + b_k)^* v(k/N_0)(a_k + b_k) \in C^*(\mathbb{Z}, X, h)_Y.$$

Therefore z_k is a unitary in the corner

$$(q_{-k} + q_{N-k}) C^*(\mathbb{Z}, X, h)_Y (q_{-k} + q_{N-k}).$$

Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

Proof of Lemma 6 (continued)

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$$\| u z_{k+1} u^* - z_k \| = \| v(k/N_0) - v((k-1)/N_0) \| \leq 2\pi/N_0 < \tfrac{1}{2}\varepsilon.$$

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Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$,

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Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$, and for $N - N_0 \leq n \leq N$ write $k = N - n$ and set $e_n = z_k q_{-k} z_k^*$.

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Moreover, adding estimates on the differences of the matrix entries at the second step,

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Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

$$\|uz_{k+1}u^* - z_k\| = \|v(k/N_0) - v((k-1)/N_0)\| \leq 2\pi/N_0 < \tfrac{1}{2}\varepsilon.$$

Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$, and for $N - N_0 \leq n \leq N$ write $k = N - n$ and set $e_n = z_k q_{-k} z_k^*$. The two definitions for $n = N - N_0$ agree because $z_{N_0} q_{-N_0} z_{N_0}^* = q_{N-N_0}$, and moreover $e_N = e_0$.

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Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

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$$\|ue_{n-1}u^* - e_n\| \leq 2\|uz_{N-n+1}u^* - z_{N-n}\| < \varepsilon.$$

Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

$$\|uz_{k+1}u^* - z_k\| = \|v(k/N_0) - v((k-1)/N_0)\| \leq 2\pi/N_0 < \tfrac{1}{2}\varepsilon.$$

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$$\|ue_{n-1}u^* - e_n\| \leq 2\|uz_{N-n+1}u^* - z_{N-n}\| < \varepsilon.$$

Also, clearly $e_n \in C^*(\mathbb{Z}, X, h)_Y$ for all n.

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$.

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Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\| pa - ap \| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

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First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

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$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\|upu^* - p\| < \varepsilon$.

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\| pa - ap \| < \varepsilon$ for all $a \in F \cup \{ u \}$.
2. $pap \in p C^* (\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{ u \}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^* (\mathbb{Z}, X, h)$.

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (u e_{n-1} u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\| upu^* - p \| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$,

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\|upu^* - p\| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$, we get $pup \in C^*(\mathbb{Z}, X, h)_Y$.

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\|upu^* - p\| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$, we get $pup \in C^*(\mathbb{Z}, X, h)_Y$. This is (1) and (2) for the element $u \in F \cup \{u\}$.

Proof of Lemma 6 (continued)

Next, let $f \in F$.

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Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$.

Proof of Lemma 6 (continued)

Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than δ.

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Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than δ. Therefore $T_{n-N} \cup T_n$ has diameter less than $2\delta + \delta_0 \leq 3\delta_0$.

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$$S_1 = T_1, \;\; S_2 = T_2, \;\; \ldots, \;\; S_{N-N_0-1} = T_{N-N_0-1},$$

$$S_{N-N_0} = T_{N-N_0} \cup T_{-N_0}, \;\; S_{N-N_0+1} = T_{N-N_0+1} \cup T_{-N_0+1}, \;\; \ldots,$$
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are disjoint,

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Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than δ. Therefore $T_{n-N} \cup T_n$ has diameter less than $2\delta + \delta_0 \leq 3\delta_0$. Since f varies by at most $\frac{1}{4}\varepsilon$ on any set with diameter less than $4\delta_0$, and since the sets

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$$S_1 = T_1, \;\; S_2 = T_2, \;\; \ldots, \;\; S_{N-N_0-1} = T_{N-N_0-1},$$

$$S_{N-N_0} = T_{N-N_0} \cup T_{-N_0}, \;\; S_{N-N_0+1} = T_{N-N_0+1} \cup T_{-N_0+1}, \;\; \ldots,$$

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are disjoint, there is $g \in C(X)$ which is constant on each of these sets and satisfies $\|f - g\| < \frac{1}{2}\varepsilon$.

Proof of Lemma 6 (continued)

Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N.

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Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$.

Proof of Lemma 6 (continued)

Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$. For $N - N_0 < n \leq N$ we use $e_n \in (q_{n-N} + q_n) C^*(\mathbb{Z}, X, h)_Y (q_{n-N} + q_n)$ to get,

$$ge_n = g(q_{n-N} + q_n)e_n = \lambda_n(q_{n-N} + q_n)e_n = e_n(q_{n-N} + q_n)g = e_n g.$$

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Since $\|f - g\| < \frac{1}{2}\varepsilon$ and $ge = eg$, it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

Proof of Lemma 6 (continued)

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Since $\|f - g\| < \frac{1}{2}\varepsilon$ and $ge = eg$, it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

This is (1) for f. That $pfp \in C^*(\mathbb{Z}, X, h)_Y$ follows from the fact that f and p are in this subalgebra.

Proof of Lemma 6 (continued)

Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$. For $N - N_0 < n \leq N$ we use $e_n \in (q_{n-N} + q_n) C^*(\mathbb{Z}, X, h)_Y (q_{n-N} + q_n)$ to get,

$$ge_n = g(q_{n-N} + q_n)e_n = \lambda_n(q_{n-N} + q_n)e_n = e_n(q_{n-N} + q_n)g = e_n g.$$

Since $\|f - g\| < \frac{1}{2}\varepsilon$ and $ge = eg$, it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

This is (1) for f. That $pfp \in C^*(\mathbb{Z}, X, h)_Y$ follows from the fact that f and p are in this subalgebra. So we also have (2) for f.

Proof of Lemma 6 (continued)

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$$1 - p = e \leq \sum_{l=-N_0}^{N} q_l \sim \sum_{l=-N_0}^{N} \chi_{h^{r_l}(Y)} = \chi_Z.$$

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This completes the proof.

Some comments on the general case

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$$C^*(\mathbb{Z}, X, h)_{Y_0} \subset C^*(\mathbb{Z}, X, h)_{Y_1} \subset C^*(\mathbb{Z}, X, h)_{Y_2} \subset \cdots$$

and

$$\overline{\bigcup_{n=0}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}} = C^*(\mathbb{Z}, X, h)_{\{y_0\}}.$$

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This completes the proof that τ_μ is a tracial state on $C^*(\mathbb{Z}, X, h)$.

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$$\tau(g_j f u^n) = \tau\left(g_j^{1/2} f u^n g_j^{1/2}\right) = \tau\left(g_j^{1/2} f \left(g_j^{1/2} \circ h^{-n}\right) u^n\right) = \tau(0) = 0.$$

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Summing over j gives $\tau(fu^n) = 0$. This completes the proof.

Tracial states on the subalgebra

Lemma

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Proof of the lemma (continued)

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$$f \circ h^{-1} = ufu^* = (uf_1)(uf_2)^* \in C^*(\mathbb{Z}, X, h)_{\{y\}}.$$

We now use the trace property at the second step to get

$$\int_X (f \circ h^{-1}) \, d\mu = \tau\big((uf_1)(uf_2)^*\big) = \tau\big((uf_2)^*(uf_1)\big) = \tau(f) = \int_X f \, d\mu.$$

Thus μ is h-invariant.

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$.

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$),

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$,

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication),

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if $fu^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(fu^m) = 0$.

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if $fu^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(fu^m) = 0$. The argument is the same as in the proof of the previous Proposition;

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if $fu^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(fu^m) = 0$. The argument is the same as in the proof of the previous Proposition; one merely needs to note that the elements g_j used there are in $C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}$.

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form fu^m, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if $fu^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(fu^m) = 0$. The argument is the same as in the proof of the previous Proposition; one merely needs to note that the elements g_j used there are in $C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}$. This completes the proof.