MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 2

This assignment is due on Canvas on Wednesday 17 April 2024 at 9:00 pm.

Problem 1 (Problem 2 in Chapter 10 of Rudin's book)**.** Let f be an entire function. Suppose that for every $a \in \mathbb{C}$, in the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_{n,a}(z-a)^n, \tag{1}$$

there is $n \in \mathbb{Z}_{\geq 0}$ such that $c_{n,a} = 0$. Prove that f is a polynomial.

Hint: $n! c_{n,a} = f^{(n)}(a)$.

Rudin wrote (1) as "$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$". Suppressing the dependence on a in the notation for the coefficients makes proper writing of both the problem and its solution awkward.

Problem 2 (Problem 4 in Chapter 10 of Rudin's book)**.** Let f be an entire function. Suppose that there are constants $A, B > 0$ and $k \in \mathbb{Z}_{>0}$ such that $|f(z)| \leq A + B|z|^k$ for all $z \in \mathbb{C}$. Prove that f is a polynomial.

Problem 3 (Problem 6 in Chapter 10 of Rudin's book)**.** Prove that there is a region Ω such that $\exp(\Omega) = B_1(1)$. Prove that there are are many such choices of Ω. Prove that, for any such choice of Ω, the restriction $\exp|_{\Omega}$ is injective. Fix one such choice of Ω, and define $\log\colon B_1(1) \to \Omega$ to be the inverse function of $\exp|_{\Omega}$. Prove that $\log'(z) = z^{-1}$. Find the coefficients a_n in the expansion

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

and hence find the coefficients c_n in the expansion

$$\log(z) = \sum_{n=0}^{\infty} c_n (z-1)^n.$$

In which other disks can this be done?

You may use the standard facts about exp, log, sin, and cos in calculus of a real variable, the formula $\exp(x+iy) = e^x(\cos(y) + i\sin(y))$ for $x, y \in \mathbb{R}$, and the polar form $z = r\exp(i\theta)$ of a complex number z, with unique $r \geq 0$ and, if $r > 0$, uniquely determined θ mod $2\pi\mathbb{Z}$. You may also use the following facts about exp:

(1) $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for every $z_1, z_2 \in \mathbb{C}$. (This can be gotten from the power series.)
(2) exp is periodic with period $2\pi i$. (This follows from the formula $\exp(x+iy) = e^x(\cos(y) + i\sin(y))$.)
(3) For every $z \in \mathbb{C} \setminus \{0\}$, there is $b \in \mathbb{C}$ such that $\exp(b) = z$. (Write $z = r\exp(i\theta)$ with $r \in (0,\infty)$ and $\theta \in \mathbb{R}$. Then take $b = \log(r) + i\theta$, using the usual definition of $\log\colon (0,\infty) \to \mathbb{R}$.)

Date: 10 April 2024.

To keep the amount of writing down, I suggest writing a unified proof for all possible choices of the diak.

The following problem counts as two ordinary problems.

Problem 4 (Problem 17 in Chapter 10 of Rudin's book)**.** Determine the largest regions in which the following functions are defined and holomorphic:

$$f(z) = \int_0^1 \frac{1}{1+tz}\,dt. \tag{2}$$

$$g(z) = \int_0^\infty \frac{e^{tz}}{1+t^2}\,dt. \tag{3}$$

$$h(z) = \int_{-1}^1 \frac{e^{tz}}{1+t^2}\,dt. \tag{4}$$

Hint: Use Problem 16 in Chapter 10 of Rudin's book (in the previous homework assignment), or combine Morera's Theorem and Fubini's Theorem. In either case, be sure to verify the hypotheses of the theorems you use.