MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 2

This assignment is due on Canvas on Wednesday 17 April 2024 at 9:00 pm.
Little proofreading has been done.
Some parts of problems have several different solutions.

Problem 1 (Problem 2 in Chapter 10 of Rudin's book)**.** Let f be an entire function. Suppose that for every $a \in \mathbb{C}$, in the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_{n,a} (z - a)^n, \tag{1}$$

there is $n \in \mathbb{Z}_{\geq 0}$ such that $c_{n,a} = 0$. Prove that f is a polynomial.

Hint: $n! c_{n,a} = f^{(n)}(a)$.

Rudin wrote (1) as "$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$". Suppressing the dependence on a in the notation for the coefficients makes proper writing of both the problem and its solution awkward.

Solution. For $n \in \mathbb{Z}_{\geq 0}$, set

$$Z_n = \big\{ a \in \mathbb{C} \colon c_{n,a} = 0 \big\}.$$

By hypothesis, we have $\bigcup_{n=0}^{\infty} Z_n = \mathbb{C}$. Therefore there exists $n \in \mathbb{Z}_{\geq 0}$ such that Z_n is uncountable. Since $f^{(n)}(z) = 0$ for all $z \in Z_n$ and $\mathbb{C}$ is a region, Theorem 10.18 of Rudin implies that $Z_n = \mathbb{C}$. Thus $f^{(n)} = 0$. So $f^{(m)} = 0$ for all $m > n$. In particular, for all $m \geq n$, we have $f^{(m)}(0) = 0$. Therefore $c_{m,0} = m! f^{(m)}(0) = 0$. So $f(z) = \sum_{m=0}^{n-1} c_{n,0} z^n$ is a polynomial (of degree at most $n - 1$). $\square$

Problem 2 (Problem 4 in Chapter 10 of Rudin's book)**.** Let f be an entire function. Suppose that there are constants $A, B > 0$ and $k \in \mathbb{Z}_{>0}$ such that $|f(z)| \leq A + B|z|^k$ for all $z \in \mathbb{C}$. Prove that f is a polynomial.

We give three solutions. The first is probably the intended solution, but both the others have been used.

Solution. Since f is entire, there are $c_0, c_1, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$.

Let $n \in \mathbb{Z}_{>0}$ satisfy $n > k$. For all $r > 0$ we have

$$\sup_{z \in B_r(0)} |f(z)| \leq A + B r^k.$$

Combining this estimate with Cauchy's Estimates in the second step, we get

$$|n! c_n| = |f^{(n)}(0)| \leq \frac{n!(A + B r^k)}{r^n}.$$

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Since

$$\lim_{r\to\infty} \frac{n!(A+Br^k)}{r^n} = 0,$$

it follows that $c_n = 0$.

Therefore $f(z) = \sum_{n=0}^{k} c_n z^n$ for all $z \in \mathbb{C}$. That is, f is a polynomial (of degree at most k). $\square$

Second solution. Let $a \in \mathbb{C}$. Then for $r > |a|$ and $z \in B_r(a)$, we have

$$|f(z)| \le A + B|z|^k \le A + B(2r)^k = A + 2^k B r^k.$$

Combining this estimate with Cauchy's Estimates, we get

$$|f^{(k+1)}(a)| \le \frac{(k+1)!(A + 2^k B r^k)}{r^{k+1}}.$$

Since

$$\lim_{r\to\infty} \frac{(k+1)!(A + 2^k B r^k)}{r^{k+1}} = 0,$$

it follows that $f^{(k+1)}(a) = 0$.

Since $a \in \mathbb{C}$ is arbitrary, we have shown that $f^{(k+1)} = 0$. Therefore $f^{(n)} = 0$ for all $n > k$. Since f is entire, there are $c_0, c_1, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$. We have $c_n = f^{(n)}(0)/n!$ for all $n > k$, so $f(z) = \sum_{n=0}^{k} c_n z^n$ for all $z \in \mathbb{C}$. That is, f is a polynomial (of degree at most k). $\square$

Third solution. We prove the result by induction on $k \in \mathbb{Z}_{\ge 0}$. If $k = 0$, then f is bounded, so f is constant by Liouville's Theorem.

Suppose now that the result is known for some $k \in \mathbb{Z}_{\ge 0}$, and that f is an entire function and there are constants $A, B > 0$ such that $|f(z)| \le A + B|z|^{k+1}$ for all $z \in \mathbb{C}$. For $z \in \mathbb{C} \setminus \{0\}$, define

$$g_0(z) = \frac{f(z) - f(0)}{z}.$$

Then g is holomorphic on $\mathbb{C} \setminus \{0\}$. Also $\lim_{z\to 0} g(z) = f'(0)$. In particular, the limit exists, so g is bounded on $B_1(0) \setminus \{0\}$, and must therefore have a removable singularity at 0. Thus there is an entire function g such that $g|_{\mathbb{C}\setminus\{0\}} = g_0$.

Define

$$M = \sup_{z \in \overline{B_1(0)}} |g(z)|,$$

which is finite because $\overline{B_1(0)}$ is compact. Set $A_0 = \max(M, |f(0)| + A)$. Let $z \in \mathbb{C}$. If $|z| < 1$, then

$$|g(z)| \le M \le A_0 + B|z|^k.$$

If $|z| \ge 1$, then

$$|g(z)| \le \frac{|f(z)|}{|z|} + \frac{|f(0)|}{|z|} \le \frac{A}{|z|} + B|z^k| + \frac{|f(0)|}{|z|} \le A + B|z^k| + |f(0)| \le A_0 + B|z|^k.$$

Therefore g satisfies the induction hypothesis. So g is a polynomial. Hence $f(z) = zg(z) + f(0)$ is also a polynomial. $\square$

Problem 3 (Problem 6 in Chapter 10 of Rudin's book)**.** Prove that there is a region Ω such that $\exp(\Omega) = B_1(1)$. Prove that there are are many such choices of Ω. Prove that, for any such choice of Ω, the restriction $\exp|_\Omega$ is injective. Fix one such choice of Ω, and define $\log\colon B_1(1) \to \Omega$ to be the inverse function of $\exp|_\Omega$. Prove that $\log'(z) = z^{-1}$. Find the coefficients a_n in the expansion

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

and hence find the coefficients c_n in the expansion

$$\log(z) = \sum_{n=0}^{\infty} c_n (z-1)^n.$$

In which other disks can this be done?

You may use the standard facts about exp, log, sin, and cos in calculus of a real variable, the formula $\exp(x+iy) = e^x(\cos(y) + i\sin(y))$ for $x, y \in \mathbb{R}$, and the polar form $z = r\exp(i\theta)$ of a complex number z, with unique $r \geq 0$ and, if $r > 0$, uniquely determined θ mod $2\pi\mathbb{Z}$. You may also use the following facts about exp:

(1) $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for every $z_1, z_2 \in \mathbb{C}$. (This can be gotten from the power series.)
(2) exp is periodic with period $2\pi i$. (This follows from the formula $\exp(x+iy) = e^x(\cos(y) + i\sin(y))$.)
(3) For every $z \in \mathbb{C} \setminus \{0\}$, there is $b \in \mathbb{C}$ such that $\exp(b) = z$. (Write $z = r\exp(i\theta)$ with $r \in (0,\infty)$ and $\theta \in \mathbb{R}$. Then take $b = \log(r) + i\theta$, using the usual definition of $\log\colon (0,\infty) \to \mathbb{R}$.)

To keep the amount of writing down, I suggest writing a unified proof for all possible choices of the diak.

Solution. The following arguments are valid for any open ball $B_r(a)$ which does not contain 0.

We first claim that if Ω is a connected set such that $\exp(\Omega) \subset B_r(a)$, then $\exp|_\Omega$ is injective. Let $T = \{-ta\colon t \in [0,\infty)\}$. Write $a = r\exp(i\theta)$ with $r \in (0,\infty)$ and $\theta \in \mathbb{R}$. Then $\exp(z) \in T$ if and only if $\mathrm{Im}(z) \in \theta + 2\pi\mathbb{Z}$. (This is easily checked using the facts above.) The connected components of

$$\{z \in \mathbb{C}\colon \mathrm{Im}(z) \in \theta + 2\pi\mathbb{Z}\}$$

are the strips

$$S_n = \bigl\{z \in \mathbb{C}\colon \theta + 2\pi n < \mathrm{Im}(z) < \theta + 2\pi(n+1)\bigr\}$$

for $n \in \mathbb{Z}$. Since $T \cap B_r(a) = \varnothing$, there is $n \in \mathbb{Z}$ such that $\Omega \subset S_n$. One easily checks (again using the facts above) that $\exp|_{S_n}$ is injective. In particular, $\exp|_\Omega$ is injective.

It also follows that if $\Omega \subset \mathbb{C}$ is one region such that $\exp(\Omega) = B_r(a)$, then the collection of all such regions is

$$\{2\pi i n + \Omega\colon n \in \mathbb{Z}\}.$$

Now we prove the existence of Ω and the formulas involving the function log. Choose any $b \in \mathbb{C}$ such that $\exp(b) = a$. Theorem 10.14 of Rudin provides a holomorphic function $g_0\colon B_r(a) \to \mathbb{C}$ such that $g_0'(z) = z^{-1}$ for all $z \in B_r(a)$. Set

$g(z) = g_0(z) - g_0(a) + b$. Then also $g'(z) = z^{-1}$ for all $z \in B_r(a)$. Furthermore $g(a) = b$.

Define a holomorphic function $h\colon B_r(a) \to \mathbb{C}$ by

$$h(z) = \frac{\exp(g(z))}{z}.$$

(Recall that $0 \notin B_r(a)$.) Using $g'(z) = z^{-1}$, we get

$$h'(z) = \frac{\exp(g(z))g'(z)z - \exp(g(z))}{z} = 0$$

for all $z \in B_r(a)$. It follows that h is constant. (Consider the form the power series for h must take.) Check that $h(a) = 1$. It follows that $\exp(g(z)) = z$ for all $z \in B_r(a)$. We can now take $\Omega = g(B_r(a))$, which is open by the Open Mapping Theorem and connected because g is continuous and $B_r(a)$ is connected. This proves the existence of Ω. Moreover, $g\colon B_r(a) \to \Omega$ is surjective by definition and injective because $\exp(g(z)) = z$ for all $z \in B_r(a)$. Therefore g is the inverse of $\exp|_\Omega$, which shows that the derivative of the inverse of $\exp|_\Omega$ is z^{-1}.

It remains only to compute the series in the case $a = 1$. Whenever $w \in \mathbb{C}$ with $|w| < 1$, we have

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}.$$

Putting $w = 1 - z$, we get, for $|z - 1| < 1$,

$$\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n.$$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

has radius of convergence 1 and its term by term derivative is the series for z^{-1}. Therefore it differs from $g(z)$ be a constant. Taking the number b with $\exp(b) = 1$ to be $b = 0$, we find that the constant is zero by comparing the values at 0. Therefore

$$\log(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

when $|z - 1| < 1$. $\square$

The proof given above is not what most students have done. Instead, most people have proceeded as follows. One uses the formula $\exp(x + iy) = e^x(\cos(y) + i\sin(y))$ to find a region Ω such that exp defines a bijection from Ω to $B_1(1)$. In terms of the standard real version of the logarithm function (also denoted log), the explicit description of Ω (not really needed to solve the problem) is

$$\Omega = \left\{ x + iy\colon x, y \in \mathbb{R},\ -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}, \text{ and } x < \log(2\cos(y)) \right\}.$$

One then uses the Inverse Function Theorem to show that $\log = (\exp|_\Omega)^{-1}$ is holomorphic, and one differentiates the equation $\exp(\log(z)) = z$ to determine its derivative.

The following problem counts as two ordinary problems.

Problem 4 (Problem 17 in Chapter 10 of Rudin's book)**.** Determine the largest regions in which the following functions are defined and holomorphic:

$$f(z) = \int_0^1 \frac{1}{1+tz}\,dt. \tag{2}$$

$$g(z) = \int_0^\infty \frac{e^{tz}}{1+t^2}\,dt. \tag{3}$$

$$h(z) = \int_{-1}^1 \frac{e^{tz}}{1+t^2}\,dt. \tag{4}$$

Hint: Use Problem 16 in Chapter 10 of Rudin's book (in the previous homework assignment), or combine Morera's Theorem and Fubini's Theorem. In either case, be sure to verify the hypotheses of the theorems you use.

We do (2) using Problem 16 in Chapter 10 of Rudin's book. We give both versions of the solution for (3). We do (4) using Morera's Theorem and Fubini's Theorem.

Solution for (2). Set $\Omega = \mathbb{C} \setminus (-\infty, -1]$. Then Ω is a connected open subset of $\mathbb{C}$. We claim that f is holomorphic on Ω and that for $z \notin \Omega$, the integrand in the definition of f is not an L^1 function (so that the definition does not make sense).

So suppose $z \in \mathbb{R}$ and $z \leq -1$. Then, changing variables at the second step, we have

$$\int_0^1 \left|\frac{1}{1+tz}\right| dt = \left|\frac{1}{z}\right| \int_0^1 \left|\frac{1}{z^{-1}+t}\right| dt = \left|\frac{1}{z}\right| \int_{z^{-1}}^{1+z^{-1}} \left|\frac{1}{t}\right| dt \geq \left|\frac{1}{z}\right| \int_0^{1+z^{-1}} \frac{1}{t}\,dt = \infty.$$

This shows that the integrand is not an L^1 function.

We use Problem 16 in Chapter 10 of Rudin's book to show that f is defined and holomorphic on Ω. Let $z_0 \in \Omega$. Then $\operatorname{dist}(z_0, (-\infty, -1]) > 0$. Set $r = \frac{1}{2}\operatorname{dist}(z_0, (-\infty, -1])$. Then $\overline{B_r(z_0)}$ is a compact subset of Ω. The formula

$$\varphi(z,t) = \frac{1}{1+tz}$$

defines a continuous function $c\colon \overline{B_r(z)} \times [0,1] \to \mathbb{C}$, which is therefore bounded, and for every $t \in [0,1]$, the function $z \mapsto \varphi(z,t)$ is clearly holomorphic on $B_r(z_0)$. We can therefore apply the result of Problem 16 in Chapter 10 of Rudin's book with $B_r(z_0)$ in place of Ω, with $X = [0,1]$, with Lebesgue measure in place of μ, and with φ as given, to conclude that f is holomorphic on $B_r(z_0)$. We have shown that for every $z_0 \in \Omega$, there is an open set containing z_0 on which f is defined and holomorphic. The conclusion follows. $\square$

Solution for (3). Set $\Omega = \{z \in \mathbb{C} \colon \operatorname{Re}(z) < 0\}$. Then Ω is a connected open subset of $\mathbb{C}$. We show that Ω is the largest open set in $\mathbb{C}$ on which g is defined, and that g is holomorphic on Ω. The set Ω is connected, and thus is in fact a region.

We first claim that g is not defined on any larger open set. It suffices to prove that if $\operatorname{Re}(z) > 0$, then the function

$$t \mapsto \left|\frac{e^{tz}}{1+t^2}\right|$$

is not integrable on $[0, \infty)$. We have

$$\lim_{t \to \infty} \left| \frac{e^{tz}}{1+t^2} \right| = \lim_{t \to \infty} \frac{e^{t\operatorname{Re}(z)}}{1+t^2} = \infty.$$

So there exists $M \in [0, \infty)$ such that

$$\left| \frac{e^{tz}}{1+t^2} \right| > 1$$

for all $t \in [M, \infty)$, and it is immediate that

$$\int_0^\infty \left| \frac{e^{tz}}{1+t^2} \right| dt = \infty.$$

This proves the claim.

Define $\varphi \colon \Omega \times [0, \infty) \to \mathbb{C}$ by $\varphi(z, t) = e^{tz}$ for $(t, z) \in \Omega \times [0, \infty)$. Then for all $(t, z) \in \Omega \times [0, \infty)$ we have $t\operatorname{Re}(z) \leq 0$, so

$$|\varphi(t, z)| = e^{t\operatorname{Re}(z)} \leq 1.$$

Therefore φ is bounded. It is obvious that φ is continuous and that for every $t \in [0, \infty)$, the function $z \mapsto \varphi(z, t)$ is holomorphic on Ω. We can therefore apply the result of Problem 16 in Chapter 10 of Rudin's book with $X = [0, \infty)$, with Ω and φ as given, and with μ being the complex measure on $[0, \infty)$ given by

$$\mu(E) = \int_E \frac{1}{1+t^2} \, dt$$

for every Lebesgue measurable set $E \subset [0, \infty)$. We conclude that g is holomorphic on Ω. $\square$

We can't take μ in the solution above to be Lebesgue measure: this measure is not finite and hence not a complex measure. One can get around this with a bit more work, as follows.

Second solution for (3) (sketch). We define Ω as in the first solution, and we show as there that g can't be defined on any larger open set. Define $\varphi \colon \Omega \times [0, \infty) \to \mathbb{C}$ by

$$\varphi(z, t) = \frac{e^{tz}}{1+t^2}$$

for $(t, z) \in \Omega \times [0, \infty)$. Then for all $(t, z) \in \Omega \times [0, \infty)$ we have $t\operatorname{Re}(z) \leq 0$, so

$$|\varphi(t, z)| = \frac{e^{t\operatorname{Re}(z)}}{1+t^2} \leq \frac{1}{1+t^2} \leq 1.$$

Therefore φ is bounded on $\Omega \times [0, \infty)$.

For $n \in \mathbb{Z}_{>0}$ and $z \in \mathbb{C}$, set

$$g_n(z) = \int_0^n \frac{e^{tz}}{1+t^2} \, dt.$$

The integral clearly exists for all $z \in \mathbb{C}$. Also, $\varphi|_{\Omega \times [0,n]}$ is continuous and bounded, and for every $t \in [0, 1]$, the function $z \mapsto \varphi(z, t)$ is holomorphic on Ω. We can therefore apply the result of Problem 16 in Chapter 10 of Rudin's book with $X = [0, n]$, with Ω and φ as given, and with μ being Lebesgue measure on $[0, n]$, to conclude that g_n is holomorphic on Ω. (In fact, a slight refinement shows that g_n is holomorphic on $\mathbb{C}$.)

Next, we claim that $g(z)$ is defined for all $z \in \Omega$. Let $z \in \Omega$. Then

$$\left| \frac{e^{tz}}{1+t^2} \right| = \frac{e^{t\mathrm{Re}(z)}}{1+t^2} \leq \frac{1}{1+t^2}. \tag{5}$$

Since $t \mapsto \frac{1}{1+t^2}$ is integrable on $\mathbb{R}$, the claim follows.

We now claim that $g_n \to g$ uniformly. Let $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{>0}$ so large that

$$\int_N^\infty \frac{1}{1+t^2}\, dt < \varepsilon.$$

Let $z \in \Omega$. Then, using (5) at the third step, we have

$$\begin{aligned} |g_n(z) - g(z)| &= \left| \int_n^\infty \frac{e^{tz}}{1+t^2}\, dt - \int_0^\infty \frac{e^{tz}}{1+t^2}\, dt \right| \\ &\leq \int_n^\infty \left| \frac{e^{tz}}{1+t^2} \right| dt \leq \int_n^\infty \frac{1}{1+t^2}\, dt \leq \int_N^\infty \frac{1}{1+t^2}\, dt < \varepsilon. \end{aligned}$$

The claim follows.

It now follows from Theorem 10.8 of Rudin that g is holomorphic. □

Now we give the solution using Morera's Theorem and Fubini's Theorem.

Third solution for (3). Set $\Omega = \{z \in \mathbb{C} \colon \mathrm{Re}(z) < 0\}$. Then Ω is a connected open subset of $\mathbb{C}$. We claim that Ω is the largest open set in $\mathbb{C}$ on which g is defined, and that g is holomorphic on Ω. The set Ω is connected, and thus is in fact a region.

Let $z \in \mathbb{C}$. Then for all $t \in \mathbb{R}$, we have

$$\left| \frac{e^{tz}}{1+t^2} \right| = \frac{e^{t\mathrm{Re}(z)}}{1+t^2}.$$

If $\mathrm{Re}(z) \leq 0$, this expression is dominated by $t \mapsto \frac{1}{1+t^2}$, which is integrable on $[0, \infty)$. Therefore $g(z)$ is defined. If $\mathrm{Re}(z) > 0$, then

$$\lim_{t \to \infty} \frac{e^{t\mathrm{Re}(z)}}{1+t^2} = \infty,$$

so the integrand in the definition of $g(z)$ is not an L^1 function (for details see the first solution), and $g(z)$ is not defined. Thus Ω is the largest open set in $\mathbb{C}$ on which g is defined.

Next, we show that g is continuous on the slightly larger set

$$\overline{\Omega} = \{z \in \mathbb{C} \colon \mathrm{Re}(z) \leq 0\}.$$

(There is no benefit to using this larger set, but the proof works for it, so we might as well do it.) It is easiest to use the Dominated Convergence Theorem (although this theorem is more powerful than is really needed). Let $(z_n)_{n \in \mathbb{Z}_{>0}}$ be any convergent sequence in $\overline{\Omega}$. Let $z = \lim_{n \to \infty} z_n$. We show that $\lim_{n \to \infty} g(z_n) = g(z)$. Set

$$c(t, z) = \frac{e^{tz}}{1+t^2}$$

for $t \in [-1, 1]$ and $z \in \overline{\Omega}$. Then $\lim_{n \to \infty} c(t, z_n) = c(t, z)$ for all $t \in [0, \infty]$. Moreover,

$$|c(t, z)| = \frac{e^{t\mathrm{Re}(z)}}{1+t^2} \leq \frac{1}{1+t^2}. \tag{6}$$

Using the integrable function $t \mapsto \frac{1}{1+t^2}$ as the dominating function, we apply the Dominated Convergence Theorem to conclude that $\lim_{n\to\infty} g(z_n) = g(z)$. This completes the proof that h is continuous.

We now use Morera's Theorem to show that h is holomorphic on Ω. Let Δ be a triangle in Ω. We take $\partial\Delta$ to be parametrized as a piecewise C^1 curve $\gamma\colon [\alpha,\beta] \to \mathbb{C}$. In the following calculation, we provide the justification for the use of Fubini's Theorem at the second step afterwards, and the fourth step follows from Cauchy's Theorem for a triangle:

$$\begin{aligned}\int_{\partial\Delta} h(z)\,dz &= \int_\alpha^\beta \left(\int_0^\infty \frac{\exp(t\gamma(s))}{1+t^2}\,dt\right)\gamma'(s)\,ds \\ &= \int_0^\infty \left(\int_\alpha^\beta \frac{\exp(t\gamma(s))}{1+t^2}\gamma'(s)\,ds\right)dt \\ &= \int_0^\infty \left(\int_{\partial\Delta} \frac{e^{tz}}{1+t^2}\,dz\right)dt = \int_0^\infty 0\,dt = 0.\end{aligned}$$

Fubini's Theorem is justified as follows. There is a finite set $S \subset [\alpha,\beta]$ such that the integrand as a function of (s,t) is continuous on $([\alpha,\beta]\setminus S)\times[0,\infty]$. Since $S \times [-1,1]$ has measure zero, the integrand is measurable. By the definition of a piecewise C^1 curve, the quantity

$$R = \sup_{s\in[\alpha,\beta]\setminus S} |\gamma'(s)|$$

is finite. The estimate (6) now implies that the absolute value of the integrand is at most

$$\left|\frac{\exp(t\gamma(s))}{1+t^2}\right| \cdot |\gamma'(s)| \le \frac{R}{1+t^2},$$

which is integrable on $[\alpha,\beta]\times[-1,1]$. So the hypotheses of Fubini's Theorem are satisfied.

We have shown that g is continuous and $\int_{\partial\Delta} h(z)\,dz = 0$ for every triangle Δ in $\mathbb{C}$. So g is holomorphic by Morera's Theorem. $\square$

The solution we give for (4) is essentially the same as the third solution for (3). The steps have all been copied so that it can be read independently. There are, however, several small differences. (It is also easy to use the result of Problem 16 in Chapter 10 of Rudin's book.)

Solution for (4). We claim that this formula defines an entire function.

For every $z \in \mathbb{C}$, the integrand is a continuous function of t defined on a compact set, and hence integrable. Therefore $h(z)$ is defined for all $z \in \mathbb{C}$. Next, we check that h is continuous. It is easiest to use the Dominated Convergence Theorem (although this theorem is more powerful than is really needed). Let $(z_n)_{n\in\mathbb{Z}_{>0}}$ be any convergent sequence in $\mathbb{C}$. Let $z = \lim_{n\to\infty} z_n$. We show that $\lim_{n\to\infty} h(z_n) = h(z)$. Let $K \subset \mathbb{C}$ be a compact set which contains $\{z_n : n \in \mathbb{Z}_{>0}\}$. Set

$$c(t,z) = \frac{e^{tz}}{1+t^2}$$

for $t \in [-1,1]$ and $z \in K$. Then c is a continuous function on the compact set $[-1,1]\times K$. So there exists $M \in [0,\infty)$ such that $|c(t,z)| \le M$ for all

$(t, z) \in [-1, 1] \times K$. Moreover, $\lim_{n\to\infty} c(t, z_n) = c(t, z)$ for all $t \in [-1, 1]$. Using the constant function M as the dominating function, we apply the Dominated Convergence Theorem to conclude that $\lim_{n\to\infty} h(z_n) = h(z)$. This completes the proof that h is continuous.

We now use Morera's Theorem to show that h is holomorphic. Let Δ be a triangle in $\mathbb{C}$. We take $\partial\Delta$ to be parametrized as a piecewise C^1 curve $\gamma\colon [\alpha, \beta] \to \mathbb{C}$. In the following calculation, we provide the justification for the use of Fubini's Theorem at the second step afterwards, and the fourth step follows from Cauchy's Theorem for a triangle:

$$\begin{aligned}
\int_{\partial\Delta} h(z)\,dz &= \int_\alpha^\beta \left(\int_{-1}^1 \frac{\exp(t\gamma(s))}{1+t^2}\,dt \right) \gamma'(s)\,ds \\
&= \int_{-1}^1 \left(\int_\alpha^\beta \frac{\exp(t\gamma(s))}{1+t^2} \gamma'(s)\,ds \right) dt \\
&= \int_{-1}^1 \left(\int_{\partial\Delta} \frac{e^{tz}}{1+t^2}\,dz \right)\,dt = \int_{-1}^1 0\,dt = 0.
\end{aligned}$$

Fubini's Theorem is justified as follows. There is a finite set $S \subset [\alpha, \beta]$ such that the integrand as a function of (s, t) is continuous on $([\alpha, \beta] \setminus S) \times [-1, 1]$. Since $S \times [-1, 1]$ has measure zero, the integrand is measurable. By compactness of $[\alpha, \beta] \times [-1, 1]$, the quantity

$$M = \sup_{s\in[\alpha,\beta]} \sup_{t\in[-1,1]} \left| \frac{\exp(t\gamma(s))}{1+t^2} \right|$$

is finite, and by the definition of a piecewise C^1 curve, the quantity

$$R = \sup_{s\in[\alpha,\beta]\setminus S} |\gamma'(s)|$$

is finite. Therefore the integrand is dominated in absolute value by the constant function with value MR. This function is integrable on $[\alpha, \beta] \times [-1, 1]$, so the hypotheses of Fubini's Theorem are satisfied.

We have shown that h is continuous and $\int_{\partial\Delta} h(z)\,dz = 0$ for every triangle Δ in $\mathbb{C}$. So h is holomorphic by Morera's Theorem. $\square$