MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 3

Problem 1 (Problem 3 in Chapter 10 of Rudin's book)**.** Suppose that f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. What conclusion can you draw?

The following problem counts as 1.5 ordinary problems. Most of it is in Problem 14 in Chapter 10 of Rudin's book.

Problem 2 (Expansion of Problem 14 in Chapter 10 of Rudin's book)**.** Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be connected open sets. Let $f\colon \Omega_1 \to \mathbb{C}$ and $g\colon \Omega_2 \to \mathbb{C}$ be nonconstant functions. Suppose that $f(\Omega_1) \subset \Omega_2$ and that $g \circ f$ is holomorphic.

(1) If f is holomorphic, can we conclude anything about g?
(2) If g is holomorphic, can we conclude anything about f?

Can you improve the situation by adding mild extra hypotheses?

The following problem counts as 2.5 ordinary problems. It is all but the last part of Problem 25 in Chapter 10 of Rudin. You will want the Global Cauchy Formula, part of Theorem 10.35 of Rudin, but I don't think anything from later is needed. Even the Global Cauchy Formula can be avoided by a slight trick.

Problem 3 (Most of Problem 25 in Chapter 10 of Rudin's book)**.** Let $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. Let A be the annulus

$$A = \big\{ z \in \mathbb{C} \colon r_1 < |z| < r_2 \big\}.$$

(1) Let $\varepsilon > 0$ satisfy $r_1 + \varepsilon < r_2 - \varepsilon$. Define closed curves $\gamma_1, \gamma_2\colon [0, 2\pi] \to A$ by

$$\gamma_1(t) = (r_1 + \varepsilon)e^{-it} \qquad \text{and} \qquad \gamma_2(t) = (r_2 - \varepsilon)e^{it}$$

for $t \in [0, 2\pi]$. For a holomorphic function f on A, prove that if $z \in \mathbb{C}$ satisfies $r_1 + \varepsilon < |z| < r_2 - \varepsilon$, then

$$f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right).$$

(2) Let f be a holomorphic function on A. Use part (1) to prove that there are a holomorphic function f_1 on $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ and a holomorphic function f_2 on $B_{r_2}(0)$ such that $f = f_1|_A + f_2|_A$. Further prove that it is possible to require that $f_1(z) \to 0$ as $|z| \to \infty$, and that then f_1 and f_2 are unique.
(3) Use the decomposition of part (2) to associate to each holomorphic function f on A its *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

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with coefficients $c_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. Prove that the coefficients c_n are uniquely determined by f, and that the series converges uniformly to f on compact subsets of A.

(4) Let f be a bounded holomorphic function on A. Prove that the functions f_1 and f_2 of part (2) (satisfying $f_1(z) \to 0$ as $|z| \to \infty$) are bounded.

(5) How much of parts (1), (2), (3), and (4) can be extended to the cases $r_1 = 0$, $r_2 = \infty$, or both?

To keep the amount of writing down, I suggest writing appropriate steps as lemmas which can be used in part (5) as well as in the earlier parts.

The last part of Problem 25 in Chapter 10 of Rudin asks how much of parts (1), (2), (3), and (4) can be extended to regions bounded by finitely many (more than two) circles. I could not figure out what was intended here. If the circles are concentric, you don't get a region. If even two circles are not concentric, the situation is much more complicated than in the rest of the problem.