MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 3

This assignment is due on Canvas on Wednesday 24 April 2024 at 9:00 pm.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Problem 1 (Problem 3 in Chapter 10 of Rudin's book)**.** Suppose that f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. What conclusion can you draw?

Solution. The conclusion is that there is a constant c such that $|c| \leq 1$ and $f = cg$.

No stronger conclusion is possible, since, if g is an entire function and $c \in \mathbb{C}$ satisfies $|c| \leq 1$, then $f = cg$ is an entire function such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$.

We prove the claimed conclusion. Set $Z = \big\{ z \in \mathbb{C} \colon g(z) = 0 \big\}$. There are two cases. First, assume that Z has a limit point in $\mathbb{C}$. Then $g = 0$. Therefore also $f = 0$, and $c = 1$ (or $c = 0$, or $c = \frac{1}{\pi}$) will do.

Suppose now that Z has no limit point in $\mathbb{C}$. Define a holomorphic function h_0 on $\mathbb{C} \setminus Z$ by $h_0(z) = f(z)/g(z)$ for all $z \in \mathbb{C} \setminus Z$. Then $|h_0(z)| \leq 1$ for all $z \in \mathbb{C} \setminus Z$. Since the points of Z are isolated, it follows that they are all removable singularities of h_0. Therefore there exists an entire function h such that $h|_{\mathbb{C} \setminus Z} = h_0$. By continuity, $|h(z)| \leq 1$ for all $z \in \mathbb{C}$. Therefore Liouville's Theorem implies that there is a constant $c \in \mathbb{C}$ such that $h(z) = c$ for all $z \in \mathbb{C}$. Clearly $|c| \leq 1$. It follows that $f(z) = cg(z)$ for all $z \in \mathbb{C} \setminus Z$. By continuity, we must have $f(z) = cg(z)$ for all $z \in \mathbb{C}$. $\square$

The following problem counts as 1.5 ordinary problems. Most of it is in Problem 14 in Chapter 10 of Rudin's book.

Problem 2 (Expansion of Problem 14 in Chapter 10 of Rudin's book)**.** Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be connected open sets. Let $f \colon \Omega_1 \to \mathbb{C}$ and $g \colon \Omega_2 \to \mathbb{C}$ be nonconstant functions. Suppose that $f(\Omega_1) \subset \Omega_2$ and that $g \circ f$ is holomorphic.

(1) If f is holomorphic, can we conclude anything about g?

(2) If g is holomorphic, can we conclude anything about f?

Can you improve the situation by adding mild extra hypotheses?

Outcome. In neither case can we conclude that the other function is holomorphic. However, in both cases, under reasonable additional assumptions, we can conclude that the other function is holomorphic. In the first case, the positive result can be interpreted as something about g in the general case. Specifically, $f(\Omega_1)$ is open (by the Open Mapping Theorem), and we can conclude that $g|_{f(\Omega_1)}$ is holomorphic.

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In the second case, f need not be continuous, but, if f is continuous, then f is holomorphic. □

Counterexample for Part (1). Take $\Omega_1 = \mathbb{C} \setminus \{0\}$, $\Omega_2 = \mathbb{C}$, $f(z) = z$ for $z \in \Omega_1$, and

$$g(z) = \begin{cases} z & z \neq 0 \\ 17\pi^3 & z = 0. \end{cases}$$

Then $g(f(z)) = z$ for all $z \in \Omega_1$, but g is not continuous. □

Requiring that g be continuous doesn't help, as the following example shows.

Second counterexample for Part (1). Take $\Omega_1 = \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$, $\Omega_2 = \mathbb{C}$, $f(z) = z$ for $z \in \Omega_1$, and

$$g(z) = \begin{cases} z & \operatorname{Im}(z) \geq 0 \\ \overline{z} & \operatorname{Im}(z) \leq 0. \end{cases}$$

Then $g(f(z)) = z$ for all $z \in \Omega_1$, but $g'(z)$ does not exist for $z \in \mathbb{R}$. □

However, the following proposition implies, in particular, that g is holomorphic on $f(\Omega_1)$. (The set $f(\Omega_1)$ is a region: it is open by the Open Mapping Theorem, and is connected because it is the image of a connected set under a continuous function.)

Proposition 1. Assume the hypotheses for Part (1) above, and assume in addition that $f(\Omega_1) = \Omega_2$. Then g is holomorphic.

We do not need to assume that g is continuous.

Proof of Proposition 1. Define $Z \subset \Omega_2$ by

$$Z = \big\{z \in \Omega_2 \; : f'(w) = 0 \text{ for every } w \in \Omega_1 \text{ such that } f(w) = z\big\}.$$

We first claim that $g'(a)$ exists for $a \in \Omega_2 \setminus Z$. To see this, let $a \in \Omega_2 \setminus Z$. Choose $b \in \Omega_1$ such that $f(b) = a$. Then $f'(b) \neq 0$. Therefore there are open sets $U \subset \Omega_1$ and $V \subset \Omega_2$ with $b \in U$ such that $f|_U$ is a bijection from U to V with a holomorphic inverse $h\colon V \to U$. For $z \in V$, we then have $g(z) = (g \circ f)(h(z))$. Since h and $g \circ f$ are holomorphic, so is $g|_V$. In particular, $g'(a)$ exists. This proves the claim.

We next claim that every point of Z is isolated in Ω_2. So let $a \in Z$. Choose $b \in \Omega_1$ such that $f(b) = a$. Then $f'(b) = 0$. Since f is not constant and Ω_1 is connected, the zeros of f' are isolated, so there is an open set $U \subset \Omega_1$ such that $b \in U$ and $f'(w) \neq 0$ for $w \in U \setminus \{b\}$. The Open Mapping Theorem implies that $f(U)$ is open. Since $f(U)$ contains no points of Z except for a, the claim follows.

We finish the proof by showing that g is continuous at every point $a \in Z$. This will imply that $g'(a)$ exists, by the theorem on removability of singularities. So let $\varepsilon > 0$. Choose $b \in \Omega_1$ such that $f(b) = a$. Choose $\rho > 0$ such that $|w - b| < \rho$ implies $|g(f(w)) - g(f(b))| < \varepsilon$. The Open Mapping Theorem implies that there is $\delta > 0$ such that $B_\delta(a) \subset f(B_\rho(b))$. If $|z - a| < \delta$, then there is $w \in B_\rho(b)$ such that $f(w) = z$, and then

$$|g(z) - g(a)| = |g(f(w)) - g(f(b))| < \varepsilon.$$

This proves continuity at a. □

The following alternate proof of continuity of g is taken from a student solution. Once one has this, the rest of the problem is fairly easy.

Alternate proof of continuity of g. Let $w \in \Omega_2$. Choose $z \in \Omega_1$ such that $f(z) = w$. Following Theorem 10.32 of Rudin, choose a neighborhood V of z, a number $r > 0$, $m \in \mathbb{Z}_{>0}$, and a bijective holomorphic function $k \colon V \to B_r(0)$, such that for all $y \in V$ we have $f(y) = w + k(y)^m$.

We prove sequential continuity of g at w. Since $f(V)$ is open, it is enough to let $(w_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $f(V)$ such that $w_n \to w$, and prove that $g(w_n) \to g(w)$. For $n \in \mathbb{Z}_{>0}$ choose $z_n \in V$ such that $f(z_n) = w_n$. We claim that $z_n \to z$. To prove this, use $f(z_n) \to w$ and $f(z_n) = w + k(z_n)^m$ to deduce that $k(z_n)^m \to 0$, from which it follows that $k(z_n) \to 0$. Continuity of k^{-1} now implies $z_n \to z$, proving the claim.

Using continuity of $g \circ f$, we now see that $g(w_n) = (g \circ f)(z_n) \to (g \circ f)(z) = g(w)$, as desired. □

Counterexample for Part (2). Take $\Omega_1 = \Omega_2 = \mathbb{C}$, take

$$f(z) = \begin{cases} 1 & z \neq 0 \\ -1 & z = 0, \end{cases}$$

and take $g(z) = z^2$. Then $g \circ f$ is the constant function 1, but f is not continuous. □

The following example shows that it does not help to assume that f is injective.

Second counterexample for Part (2). Take $\Omega_1 = \Omega_2 = \mathbb{C}$, for $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0, 2\pi)$ take $f(z) = r^{1/2} e^{i\theta/2}$, and take $g(z) = z^2$. Then $(g \circ f)(z) = z$ for all $z \in \mathbb{C}$, but f is not continuous. □

We now give an additional condition under which the answer to the second part is yes. (This is not required for a solution to the problem as stated in Rudin's book, but is certainly worth doing. It is required with the extra request, "Can you improve the situation by adding mild extra hypotheses?")

Proposition 2. Assume the hypotheses for Part (2) above, and assume in addition that f is continuous. Then f is holomorphic.

Proof. We first claim that $g \circ f$ is not constant. If the claim is false, then there is $c \in \mathbb{C}$ such that $g(f(z)) = c$ for all $z \in \Omega_1$. Since Ω_2 is connected and g is not constant, the set $S = \{z \in \Omega_2 \colon g(z) = c\}$ is discrete. Since Ω_1 is connected, f is continuous, and $f(\Omega_1) \subset S$, it follows that f is constant, contradicting the hypotheses. This proves the claim.

Now set $Z = \{z \in \Omega_1 \colon g'(f(a)) = 0\}$. We claim that Z is discrete. Suppose not, and let $a \in \Omega_1$ be a limit point of Z. Then there is a sequence $(z_n)_{n \in \mathbb{Z}_{>0}}$ in $Z \setminus \{a\}$ such that $\lim_{n \to \infty} z_n = a$. Since g is holomorphic and not constant, $f(a)$ is an isolated point of $\{w \in \Omega_2 \colon g'(w) = 0\}$. Since f is continuous, it follows that $f(z_n) = f(a)$ for all sufficiently large n. Then $g(f(z_n)) = g(f(a))$ for all sufficiently large n. It follows that a is a limit point of $\{z \in \Omega_1 \colon g(f(z)) = g(f(a))\}$. Since $g \circ f$ is not constant, this contradicts the assumption that $g \circ f$ is holomorphic. The claim is proved.

We next claim that $f|_{\Omega_2 \setminus Z}$ is holomorphic. Let $a \in \Omega_2 \setminus Z$. Since $g'(f(a)) \neq 0$, there are open sets $U \subset \Omega_2$ and $V \subset \mathbb{C}$ with $f(a) \in U$ such that $g|_U$ is a bijection from U to V with a holomorphic inverse $h\colon V \to U$. For $z \in f^{-1}(U)$, we then have $f(z) = h((g \circ f)(z))$. Since h and $g \circ f$ are holomorphic, so is $f|_{f^{-1}(U)}$. In particular, $f'(a)$ exists. This proves the claim.

We have shown that f is holomorphic on the complement of a discrete set. Since f is continuous, the theorem on removability of singularities implies that f is holomorphic. $\square$

The following problem counts as 2.5 ordinary problems. It is all but the last part of Problem 25 in Chapter 10 of Rudin. You will want the Global Cauchy Formula, part of Theorem 10.35 of Rudin, but I don't think anything from later is needed. Even the Global Cauchy Formula can be avoided by a slight trick.

Problem 3 (Most of Problem 25 in Chapter 10 of Rudin's book)**.** Let $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. Let A be the annulus

$$A = \bigl\{ z \in \mathbb{C} \colon r_1 < |z| < r_2 \bigr\}.$$

(1) Let $\varepsilon > 0$ satisfy $r_1 + \varepsilon < r_2 - \varepsilon$. Define closed curves $\gamma_1, \gamma_2 \colon [0, 2\pi] \to A$ by

$$\gamma_1(t) = (r_1 + \varepsilon)e^{-it} \qquad \text{and} \qquad \gamma_2(t) = (r_2 - \varepsilon)e^{it}$$

for $t \in [0, 2\pi]$. For a holomorphic function f on A, prove that if $z \in \mathbb{C}$ satisfies $r_1 + \varepsilon < |z| < r_2 - \varepsilon$, then

$$f(z) = \frac{1}{2\pi i} \left(\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right).$$

(2) Let f be a holomorphic function on A. Use part (1) to prove that there are a holomorphic function f_1 on $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ and a holomorphic function f_2 on $B_{r_2}(0)$ such that $f = f_1|_A + f_2|_A$. Further prove that it is possible to require that $f_1(z) \to 0$ as $|z| \to \infty$, and that then f_1 and f_2 are unique.

(3) Use the decomposition of part (2) to associate to each holomorphic function f on A its *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

with coefficients $c_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. Prove that the coefficients c_n are uniquely determined by f, and that the series converges uniformly to f on compact subsets of A.

(4) Let f be a bounded holomorphic function on A. Prove that the functions f_1 and f_2 of part (2) (satisfying $f_1(z) \to 0$ as $|z| \to \infty$) are bounded.

(5) How much of parts (1), (2), (3), and (4) can be extended to the cases $r_1 = 0$, $r_2 = \infty$, or both?

To keep the amount of writing down, I suggest writing appropriate steps as lemmas which can be used in part (5) as well as in the earlier parts.

The last part of Problem 25 in Chapter 10 of Rudin asks how much of parts (1), (2), (3), and (4) can be extended to regions bounded by finitely many (more than two) circles. I could not figure out what was intended here. If the circles are concentric, you don't get a region. If even two circles are not concentric, the situation is much more complicated than in the rest of the problem.

The following convenient notation will be used throughout the solution.

Notation 3. For $r \in (0, \infty)$, we define a closed curve $\sigma_r \colon [0, 2\pi] \to \mathbb{C}$ by $\sigma_r = re^{it}$ for $t \in [0, 2\pi]$. Also, for $r, s \in [0, \infty]$ with $r < s$, we set

$$A_{r,s} = \big\{ z \in \mathbb{C} \colon r < |z| < s \big\}.$$

Thus, the set A of part (1) is equal to A_{r_1, r_2}, but the notation also covers the cases of part (5).

Remark 4. Using the notation of part (1), we then have $\gamma_2 = \sigma_{r_2 - \varepsilon}$. Moreover, $\mathrm{Ran}(\sigma_{r_1+\varepsilon}) = \mathrm{Ran}(\gamma_1)$ and

$$\int_{\sigma_{r_1+\varepsilon}} g(\zeta)\, d\zeta = -\int_{\gamma_1} g(\zeta)\, d\zeta$$

for every continuous function g on $\mathrm{Ran}(\sigma_{r_1+\varepsilon})$.

In view of Remark 4, the following lemma gives part (1) and also the corresponding part of part (5).

Lemma 5. Let $r_1, r_2 \in [0, \infty]$ satisfy $r_1 < r_2$. Let $s_1, s_2 \in (0, \infty)$ satisfy $r_1 < s_1 < s_2 < r_2$. Let f be a holomorphic function on A_{r_1, r_2}, and let $z \in A_{s_1, s_2}$. Then

$$f(z) = \frac{1}{2\pi i} \left(\int_{\sigma_{s_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\sigma_{s_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right).$$

Proof. By Theorem 10.11 of Rudin, for $r \in (0, \infty)$, we have $\mathrm{Ind}_{\sigma_r}(z) = 0$ when $|z| > r$ and $\mathrm{Ind}_{\sigma_r}(z) = 1$ when $|z| < r$. Therefore $\mathrm{Ind}_{\sigma_{s_2} - \sigma_{s_1}}(z) = 0$ when $z \in \mathbb{C} \setminus A_{r_1, r_2}$ and $\mathrm{Ind}_{\sigma_{s_2} - \sigma_{s_1}}(z) = 1$ when $z \in \mathbb{C} \setminus A_{s_1, s_2}$. The result now follows from the global Cauchy formula, Theorem 10.35 of Rudin. $\square$

Lemma 6. Let $r \in (0, \infty)$, set $S_r = \{z \in \mathbb{C} \colon |z| = r\}$, and let $h \colon S_r \to \mathbb{C}$ be continuous. Then the function $g \colon B_r(0) \to \mathbb{C}$, given by

$$g(z) = \frac{1}{2\pi i} \int_{\sigma_r} \frac{h(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in B_r(0)$, is holomorphic on $B_r(0)$ and has a unique representation as a power series

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

for $z \in B_r(0)$. Moreover, this series converges uniformly on compact subsets of $B_r(0)$.

Proof. It follows from Theorem 10.7 of Rudin that f is representable by power series on $B_r(0)$, which implies that f is holomorphic and implies the existence of the series (1). Uniqueness follows from uniqueness of the power series representation of a function, and uniform convergence on compact sets follow from the fact that power series uniformly on compact subsets of the open disk of convergence. (I didn't find the last statement in Rudin, but it was proved in the lectures.) $\square$

Lemma 7. Let $r \in (0, \infty)$, set $S_r = \{z \in \mathbb{C} \colon |z| = r\}$, and let $h \colon S_r \to \mathbb{C}$ be continuous. Then the function $g \colon \mathbb{C} \setminus \overline{B_r(0)} \to \mathbb{C}$, given by

$$g(z) = \frac{1}{2\pi i} \int_{\sigma_r} \frac{h(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in \mathbb{C} \setminus \overline{B_r(0)}$, is holomorphic on $\mathbb{C} \setminus \overline{B_r(0)}$, has a unique representation as a series

$$g(z) = \sum_{n=1}^{\infty} c_n z^{-n} \tag{2}$$

for $z \in \mathbb{C}\setminus\overline{B_r(0)}$, and this series converges uniformly on compact subsets of $\mathbb{C}\setminus\overline{B_r(0)}$. Moreover, $g(z) \to 0$ as $|z| \to \infty$.

One can prove this lemma by the same method as used for Theorem 10.7 of Rudin. Specifically, for $|\zeta| = r$ and $|z| > r$, write

$$\frac{1}{\zeta - z} = -\left(\frac{1}{\zeta z}\right)\left(\frac{1}{\frac{1}{\zeta} - \frac{1}{z}}\right).$$

Since $\left|\frac{1}{z}\right| < \left|\frac{1}{\zeta}\right|$, the right hand side can be expanded as a power series in $\frac{1}{z}$. However, by some manipulation, one can reduce to Lemma 6. (The resulting argument isn't really shorter.) However, the alternate solution below is shorter.

Proof of Lemma 7. Define a continuous function $k \colon S_{1/r} \to \mathbb{C}$ by $k(\zeta) = -\frac{1}{\zeta} h\left(\frac{1}{\zeta}\right)$ for $\zeta \in S_{1/r}$. Apply Lemma 6 with k in place of h and with $\frac{1}{r}$ in place of r, getting a holomorphic function $f \colon B_{1/r}(0) \to \mathbb{C}$ with a power series representation

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \tag{3}$$

which converges uniformly on compact subsets of $B_{1/r}(0)$.

We claim that for all $z \in \mathbb{C} \setminus \overline{B_r(0)}$, we have

$$\frac{1}{2\pi i} \int_{\sigma_{1/r}} \frac{k(\zeta)}{\zeta - z} \, d\zeta = \left(\frac{1}{z}\right) f\left(\frac{1}{z}\right).$$

Given the claim, it follows that g is holomorphic on $\mathbb{C} \setminus \overline{B_r(0)}$ and that the series expansion (2) is valid if we take $c_n = b_{n-1}$ for $n \in \mathbb{Z}_{>0}$. Moreover, as $|z| \to \infty$, we have $f\left(\frac{1}{z}\right) \to f(0)$ and $\frac{1}{z} \to 0$, so $g(z) \to 0$. Finally, if $K \subset \mathbb{C} \setminus \overline{B_r(0)}$ is compact, then $L = \{z^{-1} \colon z \in K\}$ is a compact subset of $B_{1/r}(0)$. Since the series (3) converges uniformly on L, it follows that the series (2) converges uniformly on K to $g(z) = \frac{1}{z} f\left(\frac{1}{z}\right)$. Uniqueness of the series for g follows from uniqueness of the series for f.

It remains to prove the claim. We have, changing variables from t to $-t$ at the fourth step,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_r}\frac{k(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{k\left(\frac{1}{r}e^{it}\right)}{\frac{1}{r}e^{it} - z}\left(\frac{1}{r}\right) ie^{it}\,dt \\ &= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{re^{-it}h\left(re^{-it}\right)}{\left(\frac{1}{re^{-it}} - z\right)re^{-it}} i\,dt \\ &= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{h\left(re^{-it}\right)}{z\left(\frac{1}{z} - re^{-it}\right)} rie^{-it}\,dt \\ &= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{h\left(re^{it}\right)}{z\left(\frac{1}{z} - re^{it}\right)} rie^{it}\,dt \\ &= \frac{1}{2\pi i}\left(\frac{1}{z}\right)\int_0^{2\pi}\frac{h\left(re^{it}\right)}{re^{it} - \frac{1}{z}} rie^{it}\,dt \\ &= \frac{1}{2\pi i}\left(\frac{1}{z}\right)\int_{\sigma_r}\frac{h(\zeta)}{\zeta - \frac{1}{z}}\,d\zeta = \frac{1}{z} f\left(\frac{1}{z}\right).\end{aligned}$$

The claim is proved. ∎

Alternate proof of Lemma 7. It follows from Theorem 10.7 of Rudin that g is representable by power series on $\mathbb{C} \setminus \overline{B_r(0)}$, which implies that g is holomorphic on $\mathbb{C} \setminus \overline{B_r(0)}$. Using the definition of the path integral, we get

$$g(z) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{h(re^{i\theta})ire^{i\theta}}{re^{i\theta} - z}\,d\theta.$$

As $|z| \to \infty$, the integrand converges uniformly to zero. Therefore $\lim_{|z|\to\infty} g(z) = 0$. By the theorem on removable singularities, the formula

$$k(z) = \begin{cases} 0 & z = 0 \\ g\left(\frac{1}{z}\right) & 0 < |z| < \frac{1}{r} \end{cases}$$

defines a holomorphic function on $B_{1/r}(0)$. Therefore it has a power series representation $k(z) = \sum_{n=0}^{\infty} c_n z^n$, which converges uniformly on compact subsets of $B_{1/r}(0)$. Since $k(0) = 0$, we have $c_0 = 0$. Therefore the series $\sum_{n=1}^{\infty} c_n z^{-n}$ converges uniformly to $g(z)$ on compact subsets of $\mathbb{C} \setminus \overline{B_r(0)}$. Uniqueness of the series for g follows from uniqueness of the series for k. ∎

Lemma 8. Let $r_1, r_2 \in [0, \infty]$ satisfy $r_1 < r_2$. Let f be a holomorphic function on A_{r_1,r_2}. Then there exists a holomorphic function g on $A_{r_1,\infty}$ such that, whenever $r \in (0, \infty)$ and $z \in \mathbb{C}$ satisfy $r_1 < r < \min(r_2, |z|)$, we have

$$g(z) = \frac{1}{2\pi i}\int_{\sigma_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. For $r \in (r_1, r_2)$, Lemma 7 provides a holomorphic function $g_r \colon \mathbb{C}\setminus\overline{B_r(0)} \to \mathbb{C}$ such that

$$g_r(z) = \frac{1}{2\pi i}\int_{\sigma_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for $z \in \mathbb{C} \setminus \overline{B_r(0)}$.

Suppose $r, s \in (r_1, r_2)$ satisfy $r < s$. We claim that $g_r|_{\mathbb{C} \setminus \overline{B_s(0)}} = g_s$. So let $z \in \mathbb{C} \setminus \overline{B_s(0)}$. Choose $s_0 \in \mathbb{R}$ with $s < s_0 < \min(r_2, |z|)$. As in the proof of Lemma 5, we have $\mathrm{Ind}_{\sigma_s - \sigma_r}(w) = 0$ for all $w \notin A_{r_1, s_0}$. Since the function

$$\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$$

is holomorphic on A_{r_1, s_0}, we get

$$g_s(z) - g_r(z) = \frac{1}{2\pi i} \int_{\sigma_s} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\sigma_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\sigma_r - \sigma_s} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0.$$

The claim is proved.

The claim implies that there is a well defined function $g \colon A_{r_1, \infty} \to \mathbb{C}$ such that $g(z) = g_r(z)$ whenever $z \in A_{r_1, \infty}$ and $r_1 < r < \min(r_2, |z|)$. It remains only to show that g is holomorphic. Let $z \in A_{r_1, \infty}$. Choose $r \in \mathbb{R}$ such that $r_1 < r < \min(r_2, |z|)$. Then g agrees with g_r on an open set containing z, and $g_r'(z)$ exists, so $g'(z)$ exists. □

Lemma 9. Let $r_1, r_2 \in [0, \infty]$ satisfy $r_1 < r_2$. Let f be a holomorphic function on A_{r_1, r_2}. Then there exists a holomorphic function h on $B_{r_2}(0)$ such that, whenever $r \in (0, \infty)$ and $z \in \mathbb{C}$ satisfy $\max(r_1, |z|) < r < r_2$, we have

$$h(z) = \frac{1}{2\pi i} \int_{\sigma_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof. The proof is essentially the same as the proof of Lemma 8, but uses Lemma 6 in place of Lemma 7. □

The following lemma gives existence in part (2) and existence of the series representation in part (3), as well as the corresponding parts of part (5).

Lemma 10. Let $r_1, r_2 \in [0, \infty]$ satisfy $r_1 < r_2$. Let f be a holomorphic function on A_{r_1, r_2}. Then there exist a holomorphic function f_1 on $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ (on $\mathbb{C} \setminus \{0\}$ if $r_1 = 0$) and a holomorphic function f_2 on $B_{r_2}(0)$ such that $f = f_1|_A + f_2|_A$, such that $f_1(z) \to 0$ as $|z| \to \infty$, and such that there are series representations

$$f_1(z) = \sum_{n=-\infty}^{-1} c_n z^n \qquad \text{and} \qquad f_2(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{4}$$

the series for f_1 converging uniformly on compact sets in $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ (in $\mathbb{C} \setminus \{0\}$ if $r_1 = 0$) and the series for f_1 converging uniformly on compact sets in $B_{r_2}(0)$.

Proof. Let g be the holomorphic function of Lemma 8, set $f_1 = -g$, and let f_2 be the holomorphic function h of Lemma 9. It follows from Lemma 5 that for $z \in A_{r_1, r_2}$ we have

$$0 = h(z) - g(z) = f_1(z) + f_2(z).$$

It remains to prove the existence of the series in (4). Existence of the series for f_2 follows from the fact that f_2 is holomorphic on $B_{r_2}(0)$. We prove the existence of the series for f_1. For every $r \in (r_1, r_2)$, Lemma 7 provides unique $c_{-1,r}$, $c_{-2,r}$, $\ldots \in \mathbb{C}$ such that the series $\sum_{n=1}^{\infty} c_{-n,r} z^{-n}$ converges to $f_1(z)$ for all $z \in A_{r, r_2}$. Moreover, the convergence is uniform on compact subsets of A_{r, r_2}. Fix $r_0 \in (r_1, r_2)$. Let $K \subset A_{r_1, r_2}$ be compact. We claim that $\sum_{n=1}^{\infty} c_{-n,r_0} z^{-n}$ converges uniformly to $f_1(z)$ on K. Since K is compact and contained in A_{r_1, r_2}, we have $\inf_{z \in K} |z| > r_1$.

Choose r such that $r_1 < r < \inf_{z \in K} |z|$ and $r < r_0$. Then $\sum_{n=1}^{\infty} c_{-n,r} z^{-n}$ converges uniformly to $f_1(z)$ on compact subsets of A_{r,r_2}. In particular, we have uniform convergence on K and on compact subsets of A_{r_0,r_2}. Uniqueness of $c_{-1,r_0}, c_{-2,r_0}, \ldots$ therefore implies that $c_{-n,r} = c_{-n,r_0}$ for all $n \in \mathbb{Z}_{>0}$. So $\sum_{n=1}^{\infty} c_{-n,r_0} z^{-n}$ converges uniformly to $f_1(z)$ on K. $\square$

Without the condition on the limit of f_1 at ∞, the functions f_1 and f_2 are certainly not unique. For any entire function g, one may replace f_1 by $f_1 + g$ and f_2 by $f_2 - g$.

The following lemma gives uniqueness of the series representation in part (3), as well as the corresponding part of part (5).

Lemma 11. Let $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. Let $c_n, d_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. Suppose that the series $\sum_{n=-\infty}^{\infty} c_n z^n$ and $\sum_{n=-\infty}^{\infty} d_n z^n$ both converge on A_{r_1,r_2} to the same function f. Then $c_n = d_n$ for all $n \in \mathbb{Z}$.

Proof. Since the series $\sum_{n=0}^{\infty} c_n z^n$ converges on A_{r_1,r_2}, the general properties of power series imply that this series converges uniformly on compact subsets of $B_{r_2}(0)$.

The series $\sum_{n=-1}^{-\infty} c_n \left(\frac{1}{z}\right)^n$ converges whenever $\frac{1}{z} \in A_{r_1,r_2}$. This series is a power series, so the general properties of power series imply that it converges uniformly on compact subsets of $B_{1/r_1}(0)$. Therefore $\sum_{n=-1}^{-\infty} c_n z^n$ converges uniformly on compact subsets of $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ (of $\mathbb{C} \setminus \{0\}$ if $r_1 = 0$).

We conclude that $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on compact subsets of A_{r_1,r_2}. Similarly $\sum_{n=0}^{\infty} d_n z^n$ converges uniformly on compact subsets of A_{r_1,r_2}.

Choose $r \in (r_1, r_2)$. Define a closed curve γ by $\cancel{\gamma(t) = re^{-t}}$ $\gamma(t) = re^{-it}$. Recall that $\int_\gamma z^m \, dz = 2\pi i$ when $m = -1$ and $\int_\gamma z^m \, dz = 0$ otherwise. In the following computation, the interchanges of summation and integration are justified by uniform convergence of the series on $\mathrm{Ran}(\gamma)$. For $n \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} \int_\gamma c_m z^{m-n-1} \, dz \\ &= \frac{1}{2\pi i} \int_\gamma z^{-n-1} f(z) \, dz = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} \int_\gamma d_m z^{m-n-1} \, dz = d_n. \end{aligned}$$

This completes the proof. $\square$

The following lemma proves part (4) and the corresponding parts of part (5).

Lemma 12. Let $r_1, r_2 \in [0, \infty]$ satisfy $r_1 < r_2$. Let f be a bounded holomorphic function on A_{r_1,r_2}. Let f_1 and f_2 be as in Lemma 10. Then f_1 and f_2 are bounded.

The conclusion of the lemma (and the requested statement in part (4)) is that f_1 is bounded on $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ (on $\mathbb{C} \setminus \{0\}$ if $r_1 = 0$) and that f_2 is bounded on $B_{r_2}(0)$, not merely that they are bounded on A_{r_1,r_2}.

If $r_1 = 0$, it follows that f_1 has a removable singularity at 0. This isn't really new, since it already follows that f has a removable singularity at 0.

Proof of Lemma 12. Fix $s_1, s_2 \in \mathbb{R}$ with $r_1 < s_1 < s_2 < r_2$. We first observe that f_2 is bounded on $\overline{B_{s_2}(0)}$ because $\overline{B_{s_2}(0)}$ is compact. Also, there is M such that $|f_1(z)| < 1$ for all $z \in \mathbb{C} \setminus B_M(0)$, and f_1 is bounded on $\overline{B_M(0)} \setminus B_{s_1}(0)$ ~~be~~ by compactness, so f_1 is bounded on $\mathbb{C} \setminus B_{s_1}(0)$.

Now assume that f is bounded. On $B_{r_2}(0) \setminus \overline{B_{s_2}(0)}$, we have $f_2 = f - f_1$. The function f_1 is bounded there by the previous paragraph. The function f is bounded there by hypothesis. So f_2 is bounded on $B_{r_2}(0) \setminus \overline{B_{s_2}(0)}$, and hence on $\overline{B_{s_2}(0)} \cup \left[B_{r_2}(0) \setminus \overline{B_{s_2}(0)}\right] = B_{r_2}(0)$.

The proof that f_1 is bounded on $\mathbb{C} \setminus \overline{B_{r_1}(0)}$ (on $\mathbb{C} \setminus \{0\}$ if $r_1 = 0$) is essentially the same. $\square$