MATH 618 (SPRING 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 4

This assignment is due on Canvas on Wednesday 1 May 2024 at 9:00 pm.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

The following problem should be considered to be an example for Rudin, Chapter 10, Problem 25, which was in a previous homework set. However, feel free to use any correct method to solve it (with proof).

Problem 1 (Problem 21 in Chapter 10 of Rudin's book)**.** We want to expand the function

$$f(z) = \frac{1}{1 - z^2} + \frac{1}{3 - z}$$

as a series of the form $\sum_{n=-\infty}^{\infty} c_n z^n$.

How many such expansions are there? In which region is each of them valid? Find the coefficients c_n explicitly for each of these expansions.

Solution. The function f is holomorphic on $\mathbb{C} \setminus \{1, -1, 3\}$. According to Problem 25, there are therefore expansions of the required form on the sets

$$A = \big\{ z \in \mathbb{C} \colon |z| < 1 \big\}, \qquad B = \big\{ z \in \mathbb{C} \colon 1 < |z| < 3 \big\}, \qquad \text{and} \qquad C = \big\{ z \in \mathbb{C} \colon 3 < |z| \big\}.$$

A series of the form $\sum_{n=-\infty}^{\infty} c_n z^n$ converges only if $\sum_{n=0}^{\infty} c_n z^n$ and $\sum_{n=1}^{\infty} c_{-n} \left(\frac{1}{z}\right)^n$ both converge. Therefore the largest open set on which $\sum_{n=-\infty}^{\infty} c_n z^n$ converges is an annulus or disk centered at 0, with outer radius equal to the radius of convergence of $\sum_{n=0}^{\infty} c_n z^n$ and inner radius equal to the reciprocal of the radius of convergence of $\sum_{n=1}^{\infty} c_{-n} w^n$. (One gets a disk if $c_n = 0$ for all $n < 0$.) In particular, since the limits

$$\lim_{z \to 1} f(z), \qquad \lim_{z \to -1} f(z), \qquad \text{and} \qquad \lim_{z \to 3} f(z)$$

are all infinite, no series of the required form can converge on any open set containing any z with $|z| = 1$ or $|z| = 3$. Therefore there are three distinct series, one valid on each of the regions A, B, and C.

In principle, one can find these series by contour integration as in Problem 25 in Chapter 10 of Rudin's book. But the following procedure is easier. All four expansions are based on the geometric series

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n,$$

valid for $|z| < 1$.

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For $|z| < 1$, we have $|z^2| < 1$, so

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}. \tag{1}$$

For $|z| > 1$, we have $|z^{-2}| < 1$, so

$$\frac{1}{1-z^2} = -\left(\frac{1}{z^2}\right)\left(\frac{1}{1-z^{-2}}\right) = -\sum_{n=-1}^{-\infty} z^{2n}. \tag{2}$$

For $|z| < 3$, we have $|z/3| < 1$, so

$$\frac{1}{3-z} = \left(\frac{1}{3}\right)\left(\frac{1}{1-\frac{z}{3}}\right) = \sum_{n=0}^{\infty} 3^{-n-1} z^n. \tag{3}$$

For $|z| > 3$, we have $|3/z| < 1$, so

$$\frac{1}{3-z} = -\left(\frac{1}{z}\right)\left(\frac{1}{1-\frac{3}{z}}\right) = -\sum_{n=0}^{\infty} 3^n z^{-(n+1)} = -\sum_{n=-1}^{-\infty} 3^{-n-1} z^n. \tag{4}$$

For $z \in A$, we combine the series (1) and (3) to get $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ with

$$c_n = \begin{cases} 3^{-n-1} + 1 & n \geq 0 \text{ and even} \\ 3^{-n-1} & n \geq 0 \text{ and odd} \\ 0 & n < 0. \end{cases}$$

For $z \in B$, we combine the series (2) and (3) to get $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ with

$$c_n = \begin{cases} 3^{-n-1} & n \geq 0 \\ -1 & n < 0 \text{ and even} \\ 0 & n < 0 \text{ and odd.} \end{cases}$$

For $z \in C$, we combine the series (2) and (4) to get $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ with

$$c_n = \begin{cases} -3^{-n-1} & n < 0 \text{ and odd} \\ -3^{-n-1} - 1 & n < 0 \text{ and even} \\ 0 & n \geq 0. \end{cases}$$

In the last case, note that $c_0 = 0$, $c_{-1} = -1$, $c_{-2} = -4$, $c_{-3} = -9$, $c_{-4} = -28$, etc. $\square$

Problem 2 (Problem 19 in Chapter 10 of Rudin's book)**.** Let f and g be holomorphic functions on $B_1(0)$, suppose that $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in B_1(0)$, and suppose that

$$\frac{f'\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)} = \frac{g'\left(\frac{1}{n}\right)}{g\left(\frac{1}{n}\right)}$$

for all $n \in \mathbb{Z}_{>0}$ with $n > 1$. Find and prove another simple relation between f and g.

Motivation for the relation: the statement appears to say that the functions $\log \circ f$ and $\log \circ g$ have the same derivative on a set with a cluster point in $B_1(0)$, so they have the same derivative everywhere on $B_1(0)$, so they differ by a constant. To solve the problem this way requires proving that there are holomorphic branches of $\log \circ f$ and $\log \circ g$ on $B_1(0)$. This follows easily from Theorem 13.11 of Rudin (which isn't available to us at this stage), and there are proofs using convexity which are accessible now, but there is an easier way to proceed.

Solution. The relation is that there is a nonzero constant c such that $cf = g$.

Nothing more can be said. Indeed, for any holomorphic function f on $B_1(0)$ with no zeroes in $B_1(0)$, and any $c \in \mathbb{C} \setminus \{0\}$, taking $g = cf$ gives a pair of functions satisfying the condition in the problem.

Now let f and g satisfy the condition in the problem. Set $h(z) = g(z)/f(z)$ for $z \in B_1(0)$. Then for $z \in B_1(0)$ we have

$$h'(z) = \frac{g'(z)f(z) - g(z)f'(z)}{f(z)^2} = \frac{g(z)}{f(z)} \left(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right).$$

This function vanishes on $\left\{ \frac{1}{2}, \frac{1}{3}, \dots \right\}$, which has a cluster point in $B_1(0)$. Since $B_1(0)$ is connected, $h(z) = 0$ for all $z \in B_1(0)$.

It follows (for example, by considering the power series for h), that h is constant, that is, there is $c \in \mathbb{C}$ such that

$$\frac{g(z)}{f(z)} = h(z) = c$$

for all $z \in B_1(0)$. So $g(z) = cf(z)$ for all $z \in B_1(0)$. Since $g(0) \neq 0$, we must have $c \neq 0$. $\square$

The next problem counts as 1.5 ordinary problems.

Problem 3. Let $\Omega \subset \mathbb{C}$ be open. Recall the following.

(1) We define $C_1^{(1)}(\Omega)$ to be the free abelian group on the set of piecewise C^1 curves in Ω, with the element of $C_1^{(1)}(\Omega)$ corresponding to $\gamma \colon [\alpha, \beta] \to \Omega$ being written $[\gamma]$.
(2) If $\Gamma = \sum_{k=1}^{n} m_k [\gamma_k] \in C_1^{(1)}(\Omega)$, with $\gamma_1, \gamma_2, \dots, \gamma_n$ distinct and $m_1, m_2, \dots, m_n \in \mathbb{Z} \setminus \{0\}$, then $\operatorname{Ran}(\Gamma) = \bigcup_{k=1}^{n} \operatorname{Ran}(\gamma_k)$.
(3) We define $C_0(\Omega)$ to be the free abelian group on Ω (not to be confused with the algebra of continuous functions on Ω which vanish at infinity). For $z \in \Omega$, we write $[z]$ for the corresponding element of $C_0(\Omega)$.
(4) The homomorphism $\partial \colon C_1^{(1)}(\Omega) \to C_0(\Omega)$ is the abelian group extension of the map sending $[\gamma]$, for $\gamma \colon [\alpha, \beta] \to \Omega$, to $[\gamma(\beta)] - [\gamma(\alpha)]$.
(5) An element $\Gamma \in C_1^{(1)}(\Omega)$ is a *cycle* if $\partial(\Gamma) = 0$.
(6) An element $\Gamma \in C_1^{(1)}(\Omega)$ is an *elementary cycle* if there are $n \in \mathbb{Z}_{\geq 0}$ and piecewise C^1 curves $\gamma_k \colon [\alpha_k, \beta_k] \to \Omega$, for $k = 1, 2, \dots, n$ such that $\Gamma = \sum_{j=1}^{n} [\gamma_j]$ and:

$$\gamma_2(\alpha_2) = \gamma_1(\beta_1), \quad \gamma_3(\alpha_3) = \gamma_2(\beta_2), \quad \dots,$$

$$\gamma_n(\alpha_n) = \gamma_{n-1}(\beta_{n-1}), \qquad \text{and} \qquad \gamma_1(\alpha_1) = \gamma_n(\beta_n).$$

Prove (this counts as one ordinary problem) that for every cycle $\Gamma \in C_1^{(1)}(\Omega)$ there are elementary cycles $\Gamma_1, \Gamma_2, \dots, \Gamma_n \in C_1^{(1)}(\Omega)$ such that $\operatorname{Ran}(\Gamma_j) \subset \operatorname{Ran}(\Gamma)$ for $j = 1, 2, \dots, n$ and $m_1, m_2, \dots, m_n \in \mathbb{Z}$ such that $\Gamma = \sum_{j=1}^{n} m_k \Gamma_j$.

Further prove (this counts as half an ordinary problem) that if $\Gamma \in C_1^{(1)}(\Omega)$ and $\int_\Gamma f(z)\,dz = 0$ for every continuous function $f \colon \operatorname{Ran}(\Gamma) \to \mathbb{C}$, then Γ is a cycle.

Hint for the first part. Write Γ as a suitable formal integer combination of piecewise C^1 curves. By replacing some of these with orientation reversing reparametrizations, one can reduce to the case in which all the coefficients are strictly positive. This case can be done by induction on the sum of the coefficients. Choose any curve occurring in the sum. If it isn't already closed, there is another curve in the sum which starts at its endpoint. Continue. Eventually the endpoint of the newly chosen curve must be the starting point of one of the curves you already have (but *not* necessarily the first one).

The second part can in fact be done using holomorphic functions defined on the whole complex plane.

Solution to the first part. For the purpose of this solution, call an element $\Gamma \in C_1^{(1)}(\Omega)$ *positive* if one can write $\Gamma = \sum_{j=1}^{n} r_j [\gamma_j]$ for $n \in \mathbb{Z}_{\geq 0}$, strictly positive integers $r_1, r_2, \ldots, r_n$, and distinct piecewise C^1 curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ in Ω. (This representation is obviously unique.) Further define the *weight* of such an element Γ to be $w(\Gamma) = \sum_{j=1}^{n} r_j$. According to this definition, 0 is positive and has weight 0; moreover, if Γ is positive and $w(\Gamma) = 0$, then $\Gamma = 0$. Also, if $\Gamma = \sum_{j=1}^{n} r_j [\gamma_j]$ with $r_j \geq 0$ for $j = 1, 2, \ldots, n$, regardless of whether $\gamma_1, \gamma_2, \ldots, \gamma_n$ are distinct, then Γ is positive.

We first prove the statement for positive cycles Γ, by complete induction on $w(\Gamma)$. The statement is trivial if $w(\Gamma) = 0$. So suppose the result is known when $w(\Gamma) < N$, and the Γ is a positive cycle and $w(\Gamma) = N$. Write $\Gamma = \sum_{j=1}^{n} r_j [\gamma_j]$ as above, with $n \geq 1$ and $\gamma_j \colon [\alpha_j, \beta_j] \to \Omega$ for $j = 1, 2, \ldots, n$.

We construct inductively, starting with $l(1) = 1$, a maximal finite sequence $l(1), l(2), \ldots, l(t)$ in $\{1, 2, \ldots, n\}$ such that, for $k = 1, 2, \ldots, t-1$, we have $\gamma_{l(k+1)}(\alpha_{l(k+1)}) = \gamma_{l(k)}(\beta_{l(k)})$, and such that the numbers

$$\gamma_{l(1)}(\alpha_{l(1)}),\ \gamma_{l(2)}(\alpha_{l(2)}),\ \ldots,\ \gamma_{l(t)}(\alpha_{l(t)})$$

are distinct. Since $\partial(\Gamma) = 0$, there is some $j \in \{1, 2, \ldots, n\}$ such that $\gamma_j(\alpha_j) = \gamma_{l(t)}(\beta_{l(t)})$. By maximality of the sequence, there is $s \in \{1, 2, \ldots, t\}$ such that $\gamma_j(\alpha_j) = \gamma_{l(s)}(\alpha_{l(s)})$. Thus $\gamma_{l(t)}(\beta_{l(t)}) = \gamma_{l(s)}(\alpha_{l(s)})$. Therefore

$$\Delta = [\gamma_{l(s)}] + [\gamma_{l(s+1)}] + \cdots + [\gamma_{l(t)}]$$

is an elementary cycle. Since $r_{l(k)} > 0$ for $k = s, s+1, \ldots, t$, the expression $\Gamma_0 = \Gamma - \Delta$ is again positive, and is a cycle because $\partial \colon C_1^{(1)}(\Omega) \to C_0(\Omega)$ is a homomorphism. Clearly $w(\Gamma_0) < w(\Gamma)$, so Γ_0 is an integer combination of elementary cycles by the induction hypothesis. Therefore $\Gamma = \Delta + \Gamma_0$ is an integer combination of elementary cycles. This completes the induction step, and proves the result for positive cycles.

Now let $\Gamma = \sum_{j=1}^{n} r_j [\gamma_j] \in C_1^{(1)}(\Omega)$ be an arbitrary cycle, with $n \in \mathbb{Z}_{\geq 0}$, $r_1, r_2, \ldots, r_n \in \mathbb{Z}$, and for piecewise C^1 curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ in Ω. Recall that if $\gamma \colon [\alpha, \beta] \to \Omega$ is piecewise C^1, then $-\gamma \colon [-\beta, -\alpha] \to \Omega$ is defined by $(-\gamma)(t) = \gamma(-t)$. Set

$$S = \big\{ j \in \{1, 2, \ldots, n\} \colon r_j < 0 \big\}.$$

Then set

$$\Gamma_0 = \sum_{j \in S \setminus \{1,2,\ldots,n\}} r_j [\gamma_j] - \sum_{j \in S} r_j [-\gamma_j] \qquad \text{and} \qquad \Delta = \sum_{j \in S} r_j ([-\gamma_j] + [\gamma_j]).$$

Obviously $[\gamma]+[-\gamma]$ is a elementary cycle for any γ, so Δ is an integer combination of elementary cycles. Also, Γ_0 is positive, and is a cycle because $\partial\colon C_1^{(1)}(\Omega) \to C_0(\Omega)$ is a homomorphism. Therefore the case already done shows that Γ_0 is an integer combination of elementary cycles. Since $\Gamma_0 + \Delta = \Gamma$, this shows that Γ is an integer combination of elementary cycles. $\square$

Solution to the second part. We prove the contrapositive. Recall that for any entire function f and any piecewise C^1 curve $\gamma\colon [\alpha, \beta] \to \mathbb{C}$, we have (just by the real variable fundamental Theorem of Calculus) $\int_\gamma f'(z)\, dz = f(\gamma(\beta)) - f(\gamma(\alpha))$. Therefore, if $\Gamma \in C_1^{(1)}(\Omega)$ and

$$\partial(\Gamma) = \sum_{j=1}^{n} m_j [z_j], \tag{5}$$

with $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ and $z_1, z_2, \ldots, z_n \in \mathbb{C}$, then

$$\int_\Gamma f'(z)\, dz = \sum_{j=1}^{n} m_j f(z_j).$$

Now let $\Gamma \in C_1^{(1)}(\Omega)$ and suppose $\partial(\Gamma) \neq 0$. In (5), we can assume that $z_1, z_2, \ldots, z_n$ are distinct and $m_1, m_2, \ldots, m_n$ are nonzero. Also $n \geq 1$. Define $p(z) = \prod_{j=2}^{n} (z - z_j)$ for $z \in \mathbb{C}$. Then

$$\int_\Gamma p'(z)\, dz = \sum_{j=1}^{n} m_j p(z_j) = m_1 p(z_1) \neq 0.$$

This completes the solution. $\square$

The following is a rewording (to be more careful) of Rudin, Chapter 10, Problem 28. Do this problem, but possibly with the modifications suggested afterwards. It counts as 1.5 ordinary problems.

Problem 4 (Problem 28 in Chapter 10 of Rudin's book)**.** Let Γ be a closed curve in the plane (continuous but not necessarily piecewise C^1), with parameter interval $[0,\, 2\pi]$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Choose a sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$ of closed curves given by trigonometric polynomials which converges uniformly to Γ. Show that for all sufficiently large m and n, we have $\mathrm{Ind}_{\Gamma_m}(\alpha) = \mathrm{Ind}_{\Gamma_n}(\alpha)$. Define $\mathrm{Ind}_\Gamma(\alpha)$ to be this common value. Prove that it does not depend on the choice of the sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$. Prove that Lemma 10.39 now holds for closed curves which are merely continuous. Use this result to prove that Theorem 10.40 holds for closed curves which are merely continuous.

The problem says to use trigonometric polynomials for the approximation, but feel free to use piecewise linear functions instead, or some other convenient approximation. Furthermore, it is probably better not to use sequences, despite the statement of the problem. (Of course, don't use Theorem 10.40 of Rudin, but you will want Lemma 10.39.)

For reference, here are the statements of Lemma 10.39 and Theorem 10.40.

Lemma 1. Let $\Gamma_0, \Gamma_1\colon [0,\, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Suppose that

$$|\Gamma_1(t) - \Gamma_0(t)| < |\alpha - \Gamma_0(t)|$$

for all $t \in [0,\, 2\pi]$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Theorem 2. Let $\Omega \subset \mathbb{C}$ be open, and let $\Gamma_0, \Gamma_1 \colon [0,\, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in Ω which are homotopic in Ω. Let $\alpha \in \mathbb{C} \setminus \Omega$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

We state the steps in the solution as several lemmas. The proofs are all short.

Lemma 3. Let $\Gamma \colon [0,\, 2\pi] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$. Let $\varepsilon > 0$. Then there is a piecewise C^1 closed curve γ in $\mathbb{C}$ such that $|\gamma(t) - \Gamma(t)| < \varepsilon$ for all $t \in [0,\, 2\pi]$.

We omit the details of the proof. It is easy to do using approximation by trigonometric polynomials (as suggested by Rudin), piecewise linear functions (with care taken to ensure that $\gamma(2\pi) = \gamma(0)$), or by using the Stone-Weierstrass Theorem to show that the C^∞ functions from the circle to $\mathbb{C}$ are uniformly dense in the continuous functions from the circle to $\mathbb{C}$.

Lemma 4. Let $\Gamma \colon [0,\, 2\pi] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Let $\gamma_1, \gamma_2 \colon [0,\, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in $\mathbb{C}$ such that for all $t \in [0,\, 2\pi]$, we have

$$|\gamma_1(t) - \Gamma(t)| < \tfrac{1}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma)) \qquad \text{and} \qquad |\gamma_2(t) - \Gamma(t)| < \tfrac{1}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma)).$$

Then $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_2}(\alpha)$.

It will later become clear that one can use $\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma))$ in place of $\frac{1}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma))$, but this result is easier and sufficient.

Proof of Lemma 4. The triangle inequality implies that for all $t \in [0,\, 2\pi]$, we have

$$|\alpha - \gamma_1(t)| > \tfrac{2}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma)) \qquad \text{and} \qquad |\gamma_2(t) - \gamma_1(t)| < \tfrac{2}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma)).$$

Since γ_1 and γ_2 are piecewise C^1 closed curves, the result now follows from Lemma 10.39 of Rudin. $\square$

It follows from Lemma 3 that the quantity $\mathrm{Ind}_\Gamma(\alpha)$ in the following definition exists, and from Lemma 4 that it is well defined. Also, it is obvious that it agrees with the original definition when Γ is already piecewise C^1, since we can take $\gamma = \Gamma$.

Definition 5. Let $\Gamma \colon [0,\, 2\pi] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Let $\gamma \colon [0,\, 2\pi] \to \mathbb{C}$ be a piecewise C^1 closed curve in $\mathbb{C}$ such that for all $t \in [0,\, 2\pi]$, we have

$$|\gamma(t) - \Gamma(t)| < \tfrac{1}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma))$$

We define $\mathrm{Ind}_\Gamma(\alpha) = \mathrm{Ind}_\gamma(\alpha)$.

We can now prove the generalization of Lemma 10.39 of Rudin to continuous closed curves.

Lemma 6. Let $\Gamma_0, \Gamma_1 \colon [0,\, 2\pi] \to \mathbb{C}$ be continuous closed curves in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Suppose that

$$|\Gamma_1(t) - \Gamma_0(t)| < |\alpha - \Gamma_0(t)|$$

for all $t \in [0,\, 2\pi]$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Proof. Set

$$\rho = \inf_{t \in [0,\, 2\pi]} \big(|\alpha - \Gamma_0(t)| - |\Gamma_1(t) - \Gamma_0(t)| \big).$$

Then $\rho > 0$ since $[0,\, 2\pi]$ is compact. Set

$$\varepsilon = \min\left(\frac{\rho}{3},\, \frac{1}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma_1)),\, \frac{1}{3}\mathrm{dist}(\alpha,\, \mathrm{Ran}(\Gamma_2))\right).$$

Choose (Lemma 3) piecewise C^1 closed curves $\gamma_0, \gamma_1 \colon [0,\, 2\pi] \to \mathbb{C}$ such that

$$|\gamma_0(t) - \Gamma_0(t)| < \varepsilon \qquad \text{and} \qquad |\gamma_1(t) - \Gamma_1(t)| < \varepsilon$$

for all $t \in [0,\, 2\pi]$. Then $\mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$ and $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$. The triangle inequality implies that for all $t \in [0,\, 2\pi]$, we have

$$|\gamma_1(t) - \gamma_0(t)| < \frac{2\rho}{3} + |\Gamma_1(t) - \Gamma_0(t)| < \frac{2\rho}{3}|\alpha - \Gamma_0(t)| < \frac{2\rho}{3} + \frac{\rho}{3} + |\alpha - \gamma_0(t)|.$$

So Lemma 10.39 of Rudin implies that $\mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha)$. □

Now we can give the generalization of Theorem 10.40 of Rudin.

Theorem 7. Let $\Omega \subset \mathbb{C}$ be open, and let $\Gamma_0, \Gamma_1 \colon [0,\, 2\pi] \to \mathbb{C}$ be continuous closed curves in Ω which are homotopic in Ω. Let $\alpha \in \mathbb{C} \setminus \Omega$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Proof. Let $(s, t) \mapsto \Gamma_s(t)$, for $s \in [0, 1]$ and $t \in [0,\, 2\pi]$, be a homotopy as in the hypotheses, with Γ_0 and Γ_1 as already given. Let

$$K = \big\{\Gamma_s(t) \colon s \in [0, 1] \text{ and } t \in [0,\, 2\pi]\big\}$$

Then $K \subset \Omega$ and K is compact, so $\varepsilon = \mathrm{dist}(K,\, \mathbb{C} \setminus \Omega) > 0$. Since $(s, t) \mapsto \Gamma_s(t)$ is uniformly continuous, there exists $\delta > 0$ such that, in particular, for all $s_1, s_2 \in [0, 1]$ and $t \in [0,\, 2\pi]$ with $|s_1 - s_2| < \delta$, we have $|\Gamma_{s_1}(t) - \Gamma_{s_2}(t)| < \varepsilon$. Choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \delta$. For all $t \in [0,\, 2\pi]$ and for $j = 1, 2, \ldots, n$, we have

$$|\Gamma_{j/n}(t) - \Gamma_{(j-1)/n}(t)| < \varepsilon \le \mathrm{dist}(K,\, \mathbb{C} \setminus \Omega) \le \mathrm{dist}(K,\, \alpha) \le |\alpha - \Gamma_{(j-1)/n}(t)|.$$

Applying Lemma 6 repeatedly, we get

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_{1/n}}(\alpha) = \cdots = \mathrm{Ind}_{\Gamma_{(n-1)/n}}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha).$$

This completes the proof. □