MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 5

Problem 1 (Problem 12 in Chapter 10 of Rudin's book)**.** For $t \in \mathbb{R}$, use the Residue Theorem to compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x} \right)^2 e^{itx} \, dx.$$

The next problem counts as two ordinary problems.

Problem 2 (Problem 8 in Chapter 10 of Rudin's book)**.** Let P and Q be polynomials such that $\deg(Q) \geq \deg(P) + 2$. Let R be the rational function $R(z) = P(z)/Q(z)$ for $z \in \mathbb{C}$ such that $Q(z) \neq 0$.

(1) Prove that $\int_{-\infty}^{\infty} R(x) \, dx$ is equal to $2\pi i$ times the sum of the residues of R in the upper half plane. (Replace the integral over $[-A, A]$ by the integral over a suitable semicircle, and apply the Residue Theorem.)
(2) What is the analogous statement for the lower half plane?
(3) Use this method to compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx.$$

Problem 3 (Problem 11 in Chapter 10 of Rudin's book)**.** Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \neq 1$. Calculate

$$\int_{0}^{2\pi} \frac{1}{1 - 2\alpha \cos(\theta) + \alpha^2} \, d\theta$$

by integrating $(z - \alpha)^{-1}(z - 1/\alpha)^{-1}$ around the unit circle.

Problem 4 (Problem 13 in Chapter 10 of Rudin's book)**.** Prove that

$$\int_{0}^{\infty} \frac{1}{1 + x^n} \, dx = \frac{\pi/n}{\sin(\pi/n)}$$

for $n \in \mathbb{Z}_{>0}$ with $n \geq 2$.

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