MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 6

Problem 1 (Problem 21 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be an open set which contains the closed unit disk. Let f be a holomorphic function on Ω such that $|f(z)| < 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Determine, with proof, the possible numbers of fixed points of f (that is, solutions to the equation $f(z) = z$) in the open unit disk.

Problem 2 (Problem 20 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $f\colon \Omega \to \mathbb{C}$, and let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on Ω. Suppose that $f_n \to f$ uniformly on compact sets in Ω. If the functions f_n are all never zero on Ω, prove that either $f(z) = 0$ for all $z \in \Omega$ or $f(z) \neq 0$ for all $z \in \Omega$. If $U \subset \mathbb{C}$ is open and $f_n(\Omega) \subset U$ for all n, prove that f is constant or $f(\Omega) \subset U$.

Problem 3. Let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence in $C^\infty(S^1)$, the set of C^∞ functions from the circle S^1 to $\mathbb{C}$. Suppose that for every $m \in \mathbb{Z}_{\geq 0}$, the quantity

$$\sup_{n\in\mathbb{Z}_{>0}} \sup_{t\in S^1} |f_n^{(m)}(t)|$$

is finite. Prove that there are $f \in C^\infty(S^1)$ and a subsequence $(f_{k(n)})_{n\in\mathbb{Z}_{>0}}$ of $(f_n)_{n\in\mathbb{Z}_{>0}}$ such that $f_{k(n)}^{(m)} \to f^{(m)}$ uniformly for every $m \in \mathbb{Z}_{\geq 0}$.

Derivatives of functions on S^1 are to be computed by identifying functions on S^1 with 2π-periodic functions on $\mathbb{R}$ in the usual way.

Problem 4 (Problem 13 in Chapter 14 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of injective holomorphic functions on Ω, and suppose that there is a function $f\colon \Omega \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that f is constant or injective. Show by example that both cases can occur.

Problem 5 (Problem 15 in Chapter 14 of Rudin's book)**.** Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disk. Let F be the set of holomorphic functions $f\colon D \to \mathbb{C}$ such that $\mathrm{Re}(f(z)) > 0$ for all $z \in D$ and $f(0) = 1$. Prove that F is a normal family.

Can the condition "$f(0) = 1$" be omitted?

Can the condition "$f(0) = 1$" be replaced with "$|f(0)| \leq 1$"?

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