MATH 618 (SPRING 2024, PHILLIPS): HOMEWORK 7

The next problem counts as 1.5 ordinary problems.

Problem 1. Let $w_1, w_2, \ldots \in \mathbb{C} \setminus \{0\}$.

(1) Prove that if $\prod_{n=1}^{\infty} w_n$ converges to a nonzero value, then $\lim_{n \to \infty} w_n = 1$. Show that this can fail if $\prod_{n=1}^{\infty} w_n$ converges to 0.
(2) Prove that $\prod_{n=1}^{\infty} w_n$ converges to a nonzero value if and only if $\sum_{n=1}^{\infty} \log(w_n)$ converges.

The second part counts for about twice as much as the first.

I am sure this can be found in some textbook, but please work out the details yourself.

So as to ensure that $\sum_{n=1}^{\infty} \log(w_n)$ makes sense, take log to be defined on $\mathbb{C} \setminus \{0\}$ by $\log(re^{i\theta}) = \log(r) + i\theta$ when $r > 0$ and $\theta \in (-\pi, \pi]$, with $\log(r)$ using the usual definition of the logarithm as a function $(0, \infty) \to \mathbb{R}$.

There are two annoyances to deal with. First, we haven't formally proved that the definition $\log(re^{i\theta}) = \log(r) + i\theta$ gives a continuous function on $\mathbb{C} \setminus (-\infty, 0]$. (It is certainly not continuous on the domain given above.) You can prove this directly, but there are easier ways to proceed. You can use Problem 6 in Chapter 10 of Rudin's book (which was in a previous homework assignment), but this is overkill. Second, it is not generally true that $\log(ab) = \log(a) + \log(b)$, with any continuous definition on any nonempty neighborhood of 1 in $\mathbb{C}$.

Problem 2 (Problem 19 in Chapter 14 of Rudin's book, plus an additional statement)**.** Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk. If $f \colon D \to D$ is holomorphic and bijective, prove that f extends to a homeomorphism from $\overline{D}$ to $\overline{D}$. (This part is essentially immediate.) Then (this is the main part) exhibit, of course with proof, a bijective homeomorphism $f \colon D \to D$ which does not extend to a continuous function from $\overline{D}$ to $\overline{D}$.

The first part is included mainly for context.

Problem 3 (Problem 22 in Chapter 14 of Rudin's book)**.** Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk. Let Ω be an open square with center at 0 (but with sides not necessarily parallel to the coordinate axes), and let $f \colon D \to \Omega$ is holomorphic, bijective, and satisfy $f(0) = 0$. Prove that $f(iz) = if(z)$ for all $z \in D$. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, prove that $c_n = 0$ unless $n - 1$ is a multiple of 4. Generalize by replacing squares with general rotationally symmetric simply connected regions (of course other than $\mathbb{C}$).

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Problem 4 (Problem 29 in Chapter 14 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a bounded region, fix $a \in \Omega$, and let $f \colon \Omega \to \Omega$ be a holomorphic function such that $f(a) = a$.

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(1) Prove that $|f'(a)| \leq 1$. Hint: set $f_1 = f$ and inductively define $f_{n+1} = f \circ f_n$. (These functions are normally called f^n.) Compute $f_n'(a)$.
(2) If $f'(a) = 1$, prove that $f(z) = z$ for all $z \in \Omega$. Hint: if $f(z) = z + c_m(z - a)^m + c_{M+1}(z-a)^{m+1} + \cdots$, compute the coefficient of $(z-a)^m$ in the expansion of $f_n(z)$.
(3) If $|f'(a) = 1$, prove that f is bijective. Hint: set $\gamma = f'(a)$. Find positive integers $k(1) < k(2) < \cdots$ such that $\lim_{n\to\infty} \gamma^{k(n)} = 1$ and the sequence $(f_{k(n)})_{n \in \mathbb{Z}_{>0}}$ converges uniformly on compacts sets in Ω to some function g. Prove that $g'(a) = 1$. Use Problem 20 in Chapter 10 of Rudin's book (in a previous homework assignment) to prove that $g(\Omega) \subset \Omega$. Use these facts to deduce the desired conclusions for f.

Remark 1. The functions f_n in the hint for part (1) should properly be called f^n. This is consistent with the notation f^{-1} for the inverse function. (So, if $f \colon X \to X$ is invertible then f^{-2} makes sense.) It is standard in work on dynamical systems. For this reason, notation like "$\sin^2(x)$" is bad.

One major exception (another example of there being not enough notation to go around): if X is, say, a compact Hausdorff space, then the set $C(X)$ of continuous functions from X to $\mathbb{C}$ is an algebra, and in particular a ring. With respect to the ring operations, the function $x \mapsto [f(x)]^n$ must be called f^n, and f^{-1} is the function $f^{-1}(x) = 1/f(x)$.

If you are a C*-algebraist and work on crossed products by minimal homeomorphisms, then both meanings may occur in the same sentence.