REFERENCES ON THE RADIUS OF COMPARISON AND RELATED MATERIAL

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This is a collection of references to material related to the radius of comparison, dimension, mean dimension, and some related ideas. It is far from complete. No proofreading has been done.

Dimension theory of topological spaces has a long history, although it is not now a popular topic of research. The reference I use the most is the book of Pears [11]. It discusses much more general spaces than compact metric spaces, and a number of notions of topological dimension. Cohmological dimensions are not there; I have been using the reference [4]. This reference also gives the relation between covering dimension and cohmological dimension, and shows how this is connected to examples with $\dim(X \times Y) < \dim(X) + \dim(Y)$. One warning about [11]: compact spaces are called "bicompact".

The version of the radius of comparison in terms of functionals on $\mathrm{Cu}(A)$ is in [2]. It is called r_A. There is also a version relative to some given $a \in A_{+}$, called $r_{A,a}$, which can be used even if the C*-algebra A is not unital. The number one gets depends strongly on the choice of a. The definition of $\mathrm{rc}(A)$ given in class is more direct, but experience shows it is not suitable for C*-algebras A which are not residually stably finite (*every* quotient A/I is stably finite). When A is residually stably finite, $r_A = \mathrm{rc}(A)$ (Proposition 3.2.3 of of [2]). The direct limit theorem for r_A is Proposition 3.2.4(iii) of of [2]. There is then a direct limit theorem for $\mathrm{rc}(A)$ when every algebra in the direct system is residually stably finite. This isn't in [2], but is Proposition 2.13 of [1]. One warning about [2]: the treatment of the axioms satisfied by $\mathrm{Cu}(A)$ differs from what is common in more recent work.

The proof that the radius of comparison of $C(X)$ is roughly at least half as large as the rational cohomological dimension $\dim_{\mathbb{Q}}(X)$ is in [6]. It relies on the Chern character and Chern classes. In particular, the results proved about failure of comparison are still valid if one adds a trivial projection to both sides. This means that the method can be used in direct limits in which point evaluations are added to ensure simplicity. As far as I know, this has never been done. A logical place for it would have been [5], but the purpose of this paper was different, and for simplicity of exposition it restricted to "solid" spaces instead. The method was used for crossed products in [8].

The proof that $\mathrm{rc}(C(X))$ is roughly at least half as large as $\dim(X)$ is in [12]. It does not rely on Chern classes; in fact, the total Chern class of every vector bundle which implicitly occurs there is trivial. As it stands, this method can't be adapted to direct limits including pint evaluations among the maps, or to crossed products of minimal subshifts.

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Mean dimension was first introduced in [9]. An example of a minimal subshift with nonzero mean dimension (in fact, mean dimension greater than 1) is given there. The first use of this idea to construct counteexamples in C*-algebras is [7]. The most general currently known version of the costruction of minimal actions with nonzero mean dimension is in [3].

The best known results on $\mathrm{rc}(C^*(G, X)) \leq \frac{1}{2}\mathrm{mdim}(G, X)$ are in [10], and the only substantial progress on the reverse inequality is in [8].

References

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