Freeness of actions of finite groups on C*-algebras: Part 2

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A rough outline

- The Rokhlin property.
- Pointwise outerness.
- The tracial Rokhlin property.
- Applications of the tracial Rokhlin property.
- The Rokhlin property, the tracial Rokhlin property, and freeness.

The Rokhlin property

Reminder of the definition of the Rokhlin property:

Definition

Let A be a separable unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Crossed products by actions with the Rokhlin property

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

Crossed products by actions of finite groups with the Rokhlin property preserve many other structural properties of C*-algebras. See, for example, joint work with Osaka.

The basic idea: Let $e_g \in A$, for $g \in G$, be Rokhlin projections. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Then $w_{g,h} = u_{gh^{-1}} e_h$ defines an approximate system of matrix units in $C^*(G, A, \alpha)$. Let $(v_{g,h})_{g,h \in G}$ be a nearby true system of matrix units. Using the homomorphism $M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A.

Crossed products by actions with the Rokhlin property (continued)

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D. It suffices to consider a finite set of the form $S = F \cup \{u_g \colon g \in G\}$, where F is a finite subset of the unit ball of A and $u_g \in C^*(G, A, \alpha)$ is the canonical unitary implementing the automorphism α_g. So let $F \subset A$ be a finite subset with $\|a\| \le 1$ for all $a \in F$ and let $\varepsilon > 0$.

Set $n = \mathrm{card}(G)$, and set $\varepsilon_0 = \varepsilon/(4n)$.

Crossed products by actions with the Rokhlin property (continued)

Choose $\delta > 0$ such that a system of δ-approximate $n \times n$ matrix units, in which the diagonal approximate matrix units are orthogonal projections summing to 1, can be approximated within ε_0 by a true system of matrix units, with the given diagonal matrix units.

More precisely, choose $\delta > 0$ such that, whenever $(e_{j,k})_{1 \le j,k \le n}$ is a system of matrix units for M_n, whenever B is a unital C*-algebra, and whenever $w_{j,k}$, for $1 \le j, k \le n$, are elements of B such that $\|w_{j,k}^* - w_{k,j}\| < \delta$ for $1 \le j, k \le n$, such that $\|w_{j_1,k_1} w_{j_2,k_2} - \delta_{j_2,k_1} w_{j_1,k_2}\| < \delta$ for $1 \le j_1, j_2, k_1, k_2 \le n$, and such that the $w_{j,j}$ are orthogonal projections with $\sum_{j=1}^n w_{j,j} = 1$, then there exists a unital homomorphism $\varphi \colon M_n \to B$ such that $\varphi(e_{j,j}) = w_{j,j}$ for $1 \le j \le n$ and $\|\varphi(e_{j,k}) - w_{j,k}\| < \varepsilon_0$ for $1 \le j, k \le n$.

Also require $\delta \le \varepsilon/[2n(n+1)]$.

Crossed products by actions with the Rokhlin property (continued)

Apply the Rokhlin property to α with F as given and with δ in place of ε, obtaining projections $e_g \in A$ for $g \in G$.

Define $w_{g,h} = u_{gh^{-1}} e_h$ for $g, h \in G$.

We claim that the $w_{g,h}$ form a δ-approximate system of $n \times n$ matrix units in $C^*(G, A, \alpha)$. We estimate:

$$\|w_{g,h}^* - w_{h,g}\| = \|e_h u_{gh^{-1}}^* - u_{hg^{-1}} e_g\|$$
$$= \|u_{gh^{-1}} e_h u_{gh^{-1}}^* - e_g\| = \|\alpha_{gh^{-1}}(e_h) - e_g\| < \delta.$$

Crossed products by actions with the Rokhlin property (continued)

Recall that $w_{g,h} = u_{gh^{-1}}e_h$ for $g, h \in G$.

We proved $\|w_{g,h}^* - w_{h,g}\| < \delta$.

Next, using $e_g e_h = \delta_{g,h} e_h$ at the second step,

$$
\begin{aligned}
\|w_{g_1,h_1} w_{g_2,h_2} - \delta_{g_2,h_1} w_{g_1,h_2}\| &= \|u_{g_1 h_1^{-1}} e_{h_1} u_{g_2 h_2^{-1}} e_{h_2} - \delta_{g_2,h_1} u_{g_1 h_2^{-1}} e_{h_2}\| \\
&= \|u_{g_1 h_1^{-1}} e_{h_1} u_{g_2 h_2^{-1}} e_{h_2} - u_{g_1 h_1^{-1} g_2 h_2^{-1}} e_{h_2 g_2^{-1} h_1} e_{h_2}\| \\
&= \|u_{g_1 h_1^{-1} g_2 h_2^{-1}} \left(u_{g_2 h_2^{-1}}^* e_{h_1} u_{g_2 h_2^{-1}} - e_{h_2 g_2^{-1} h_1}\right) e_{h_2}\| < \delta.
\end{aligned}
$$

Also, $w_{g,g} = u_1 e_g = e_g$, so the $w_{g,g}$ are orthogonal projections which add up to 1.

This proves the claim that we have a system of δ-approximate matrix units.

Crossed products by actions with the Rokhlin property (continued)

Let $(v_{g,h})_{g,h \in G}$ be a system of matrix units for M_n. By the choice of δ, there exists a unital homomorphism $\varphi_0 \colon M_n \to C^*(G, A, \alpha)$ such that $\|\varphi_0(v_{g,h}) - w_{g,h}\| < \varepsilon_0$ for all $g, h \in G$, and $\varphi_0(v_{g,g}) = e_g$ for all $g \in G$. Now define a unital homomorphism $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ by $\varphi(v_{g,h} \otimes d) = \varphi_0(v_{g,1}) d \varphi_0(v_{1,h})$ for $g, h \in G$ and $d \in e_1 A e_1$. Corners of AF algebras are AF, and φ is injective, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$. We complete the proof by showing that every element of S is within ε of an element of D.

Crossed products by actions with the Rokhlin property (continued)

For $g \in G$ we have $\sum_{h \in G} \varphi_0(v_{gh,h}) \in D$ and

$$
\begin{aligned}
\left\| u_g - \sum_{h \in G} \varphi_0(v_{gh,h}) \right\| &\leq \sum_{h \in G} \|u_g e_h - \varphi_0(v_{gh,h})\| \\
&= \sum_{h \in G} \|w_{gh,h} - \varphi_0(v_{gh,h})\| < n\varepsilon_0 \leq \varepsilon.
\end{aligned}
$$

This gives the approximation of the canonical unitaries in the crossed product.

Crossed products by actions with the Rokhlin property (continued)

Now let $a \in F$.

Recall that $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ is defined by $\varphi(v_{g,h} \otimes d) = \varphi_0(v_{g,1}) d \varphi_0(v_{1,h})$ for $g, h \in G$ and $d \in e_1 A e_1$.

The obvious first step in approximating a is to use

$$
\sum_{g \in G} e_g a e_g.
$$

In fact, and this is perhaps the main trick, one needs in the end to (implicitly) use the approximation

$$
\sum_{g \in G} \alpha_g(e_1 \alpha_g^{-1}(a) e_1).
$$

This happens because the definition of φ sends $e_{g,h} \otimes d$, for $d \in e_1 A e_1$, to an element obtained by using (approximately) the action of the group elements g and h.

Set
$$b = \sum_{g \in G} v_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \in M_n \otimes e_1 A e_1.$$

Using $\|e_g a e_h\| \leq \|[e_g, a]\| + \|a e_g e_h\|$, we get
$$\left\| a - \sum_{g \in G} e_g a e_g \right\| \leq \sum_{g \neq h} \|e_g a e_h\| < n(n-1)\delta.$$

We use this, and the inequalities
$$\|\varphi_0(v_{g,1}) e_1 - u_g e_1\| < \varepsilon_0 \quad \text{and} \quad \|e_1 \alpha_g^{-1}(a) e_1 - \alpha_g^{-1}(e_g a e_g)\| < 2\delta,$$

to get
$$\|a - \varphi(b)\| = \left\| a - \sum_{g \in G} \varphi_0(v_{g,1}) e_1 \alpha_g^{-1}(a) e_1 \varphi_0(v_{1,g}) \right\|$$
$$< 2n\varepsilon_0 + \left\| a - \sum_{g \in G} u_g e_1 \alpha_g^{-1}(a) e_1 u_g^* \right\|$$
$$< 2n\varepsilon_0 + 2n\delta + \left\| a - \sum_{g \in G} u_g \alpha_g^{-1}(e_g a e_g) u_g^* \right\|$$
$$< 2n\varepsilon_0 + 2n\delta + n(n-1)\delta \leq \varepsilon.$$

This completes the proof of the theorem.

Freeness and the Rokhlin property

A free action of a finite group on the Cantor set has the Rokhlin property.

Free actions on connected spaces don't, since there are no nontrivial projections.

We will consider mostly simple C*-algebras with many projections. Recall that the irrational rotation algebras UHF algebras have real rank zero. If there are not enough projections, the situation is much less well understood, although there has been some recent progress. In the nonsimple case, rather little is known.

Tracial states and the Rokhlin property

Recall (as was mentioned in the first lecture) that if A is a unital C*-algebra, $p, q \in A$ are projections with $\|p - q\| < 1$, and τ is a tracial state on A, then $\tau(p) = \tau(q)$.

Suppose now A has a unique tracial state. (This is true for both UHF algebras and irrational rotation algebras.) Let G be finite, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. In the definition take $\varepsilon = 1$ and $F = \varnothing$. We get projections e_g such that, in particular:

- $\|\alpha_g(e_1) - e_g\| < 1$ for all $g \in G$.
- $\sum_{g \in G} e_g = 1$.

Since τ is unique, we have $\tau \circ \alpha_g = \tau$ for all $g \in G$. So $\tau(e_g) = \tau(e_1)$. It follows that
$$\tau(e_1) = \frac{1}{\mathrm{card}(G)}.$$

The Rokhlin property and A_θ

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and recall that A_θ is generated by unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$. Further recall the action $\alpha \colon \mathbb{Z}_n \to \mathrm{Aut}(A_\theta)$ generated by
$$u \mapsto e^{2\pi i / n} u \quad \text{and} \quad v \mapsto v.$$

This is a noncommutative version of the free action of $\mathbb{Z}_n$ on $S^1 \times S^1$ given by rotation by $e^{2\pi i / n}$ in the first coordinate. Moreover, A_θ has many projections, like $C(X)$ when X is the Cantor set. So one would hope that α has the Rokhlin property.

The Rokhlin property and A_θ (continued)

In fact, *no* action of *any* nontrivial finite group on A_θ has the Rokhlin property! The reason is that there is no projection $e \in A_\theta$ with $\tau(e) = \frac{1}{n}$, for any $n \geq 2$. (Recall that the tracial state τ defines an isomorphism $\tau_*\colon K_0(A_\theta) \to \mathbb{Z} + \theta\mathbb{Z}$.)

For similar reasons, no action of $\mathbb{Z}_2$ on $D = \bigotimes_{n=1}^{\infty} M_3$ has the Rokhlin property. (The tracial state τ defines an isomorphism $\tau_*\colon K_0(D) \to \mathbb{Z}[\frac{1}{3}]$.)

There are more subtle obstructions to the Rokhlin property. For example, $M_2 \otimes \bigotimes_{n=1}^{\infty} M_3$ does have projections with trace $\frac{1}{2}$, but there are still no actions of $\mathbb{Z}_2$ with the Rokhlin property. (One at least needs a copy of M_{2^n} for every n.)

In fact, the Rokhlin property is very rare.

Pointwise outer actions

For simplicity of the crossed product, a much weaker condition suffices.

Definition

An action $\alpha\colon G \to \mathrm{Aut}(A)$ is said to be *pointwise outer* if, for $g \in G \setminus \{1\}$, the automorphism α_g is outer, that is, not of the form $a \mapsto \mathrm{Ad}(u)(a) = uau^*$ for some unitary u in the multiplier algebra $M(A)$ of A.

Theorem (Kishimoto)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a simple C*-algebra A. Suppose that α is pointwise outer. Then $C_r^*(G, A, \alpha)$ is simple.

Digression: When the algebra is not simple

Call an action $\alpha\colon G \to \mathrm{Aut}(A)$ *minimal* if A has no nontrivial α-invariant ideals.

Kishimoto's theorem as stated above is a corollary of a stronger theorem which applies to minimal actions on not necessarily simple C*-algebras. However, the hypothesis in this theorem is stronger than pointwise outerness.

Archbold and Spielberg have done further work in this direction.

If the action is not minimal, the desired conclusion is that every ideal in the crossed product is the crossed product of an invariant ideal in the original algebra. Small examples show that it is necessary to assume "strong pointwise outerness": every group element g is outer on every α_g-invariant subquotient. But we don't know if this is enough.

It seems to be unknown, even for $G = \mathbb{Z}$, whether the reduced crossed product by a pointwise outer minimal action of a countable discrete group is necessarily simple.

Pointwise outerness is not enough

The product type action

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n + 1}.$$

is pointwise outer. However, its crossed product has "too many" tracial states. (A has a unique tracial state, but the crossed product has two extreme tracial states.)

Worse, Elliott has constructed an example of a pointwise outer action α of $\mathbb{Z}_2$ on a simple unital AF algebra A such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero.

Pointwise outerness is thus insufficient for proving classifiability of crossed products.

The tracial Rokhlin property

Definition

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.

2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

3. With $e = \sum_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

If A is finite, the last condition can be omitted. We will consider only this case.

The tracial Rokhlin property (continued)

The conditions in the definition for the finite case:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.

2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

3. With $e = \sum_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

The first two conditions are the same as for the Rokhlin property.

If the algebra "has enough tracial states", the element x can be omitted, and the third condition replaced by:

- With $e = \sum_{g\in G} e_g$, the projection $1 - e$ satisfies $\tau(1 - e) < \varepsilon$ for all tracial states τ on A.

Comparison: tracial rank zero

The tracial Rokhlin property was motivated by the definition of tracial rank zero (originally called "tracially AF"):

Definition

Let A be a simple unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there exists a projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.

2. $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.

3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.

In both definitions, the strong version (the Rokhlin property, or local approximation by finite dimensional C*-algebras) is supposed to hold only after cutting down by a "large" projection.

What if there are no projections?

We don't know the right definition when A is not simple.

The tracial Rokhlin property clearly requires the presence of a fair number of projections. Recent work of Archey shows how to remove the requirement that A have many projections, at least for simple C*-algebras with well behaved Cuntz semigroup.

For example, the tensor flip on $Z \otimes Z$ (where Z is the Jiang-Su algebra) seems like it deserves to have the tracial Rokhlin property. It does not, because $Z \otimes Z$ has no nontrivial projections. It does have Archey's projection-free version of the tracial Rokhlin property.

We don't know how to define the Rokhlin property in the absence of projections. The discussion later, however, suggests that perhaps one should not expect to find a definition.

Crossed products by actions with the tracial Rokhlin property

The tracial Rokhlin property implies pointwise outerness of the action, but the converse is false.

The tracial Rokhlin property is good for understanding the structure of crossed products.

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha \colon G \to \operatorname{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

This is important because tracial rank zero is a hypothesis in an important classification theorem (due to Lin).

There are examples (such as the one of Elliott mentioned above) which show that this theorem fails if one weakens the condition on the action to pointwise outerness.

Crossed products by actions with the tracial Rokhlin property (continued)

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha \colon G \to \operatorname{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

The idea of the proof is essentially the same as for crossed products of AF algebras by actions with the Rokhlin property. The definitions of both tracial rank zero and the tracial Rokhlin property allow a "small" (in trace) error projection. One must show that the sum of two "small" error projections is again "small".

There is one additional difficulty. The hypotheses give an error which is "small" relative to A. One must prove that it is also "small" relative to $C^*(G, A, \alpha)$. This uses a theorem of Osaka and Jeong.

The tracial Rokhlin property is common

There are many actions which have the tracial Rokhlin property.

- The actions on irrational rotation algebras coming from finite subgroups of $\operatorname{SL}_2(\mathbb{Z})$ have the tracial Rokhlin property.
- The action of $\mathbb{Z}_n$ on an irrational rotation algebra generated by $u \mapsto e^{2\pi i/n} u$ and $v \mapsto v$ has the tracial Rokhlin property.
- The action of $\mathbb{Z}_2$ generated by

$$\bigotimes_{n=1}^{\infty} \operatorname{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_3$$

  has the tracial Rokhlin property.
- The tensor flip on the 2^{∞} UHF algebra has the tracial Rokhlin property.

The first two are not easy to prove.

None of these actions has the Rokhlin property.

An application: Crossed products of rotation algebras by finite groups

Theorem (Joint with Echterhoff, Lück, and Walters; known previously for the order 2 case)

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\alpha \colon G \to \operatorname{Aut}(A_\theta)$ be the action on A_θ of one of the finite subgroups of $\operatorname{SL}_2(\mathbb{Z})$ (of order 2, 3, 4, or 6). Then $C^*(G, A_\theta, \alpha)$ is an AF algebra.

This solved a problem open for some years. The result is initially unexpected, since A_θ itself is not AF. It was suggested by K-theory computations done for rational θ.

The proof uses the tracial Rokhlin property for the action to show that $C^*(G, A_\theta, \alpha)$ has tracial rank zero. One then applies classification theory (specifically, Lin's classification theorem), but one must compute the K-theory of $C^*(G, A_\theta, \alpha)$ and show that it satisfies the Universal Coefficient Theorem. This is done using known cases of the Baum-Connes conjecture.

Crossed products of rotation algebras by finite groups (continued)

$C^*(G, A_\theta, \alpha)$ is AF for finite subgroups $G \subset \mathrm{SL}_2(\mathbb{Z})$. For $G = \mathbb{Z}_2$, much more direct methods are known. For the other cases, our proof, using several different pieces of heavy machinery (the Elliott classification program and the Baum-Connes conjecture), is the only one known.

Problem
Prove that $C^*(G, A_\theta, \alpha)$ for $G \subset \mathrm{SL}_2(\mathbb{Z})$ of order 3, 4, and 6, by explictly writing down a direct system of finite dimensional C*-algebras and proving directly that its direct limit is $C^*(G, A_\theta, \alpha)$.

An application: Higher dimensional noncommutative tori

Theorem
Every simple higher dimensional noncommutative torus is an AT algebra.

A higher dimensional noncommutative torus is a version of the rotation algebra using more unitaries as generators. An AT algebra is a direct limit of finite direct sums of C*-algebras of the form $C(S^1, M_n)$.

Elliott and Evans proved that A_θ is an AT algebra for θ irrational. A general simple higher dimensional noncommutative torus can be obtained from some A_θ by taking repeated crossed products by $\mathbb{Z}$. If all the intermediate crossed products are simple, the theorem follows from a result of Kishimoto. Using classification theory and the tracial Rokhlin property for actions generalizing the one on A_θ generated by

$$u \mapsto e^{2\pi i/n} u \quad \text{and} \quad v \mapsto v,$$

one can reduce the general case to the case proved by Kishimoto.

Back to the question: What is a free action on a C*-algebra?

Let $\theta \in \mathbb{R}$, and recall that A_θ is the universal C*-algebra generated by unitaries u and v satisfying $vu = e^{2\pi i\theta} uv$. We care here about irrational θ, and about $\theta = 0$. When $\theta = 0$, we get $C(S^1 \times S^1)$.

Recall two actions of $\mathbb{Z}_2$. (We suppress the dependence on θ in the notation.) Let α be the action of $\mathbb{Z}_2$ is generated by

$$u \mapsto -u \quad \text{and} \quad v \mapsto v,$$

and let β be the action of $\mathbb{Z}_2$ is generated by

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*.$$

Both have the tracial Rokhlin property. When $\theta = 0$, we get corresponding actions on $S^1 \times S^1$, given by

$$(\lambda, \zeta) \mapsto (-\lambda, \zeta) \quad \text{and} \quad (\lambda, \zeta) \mapsto (\overline{\lambda}, \overline{\zeta}).$$

Freeness of actions (continued)

The actions of $\mathbb{Z}_2$ on $S^1 \times S^1$:

$$(\lambda, \zeta) \mapsto (-\lambda, \zeta) \quad \text{and} \quad (\lambda, \zeta) \mapsto (\overline{\lambda}, \overline{\zeta}).$$

The first action is free. The second has four fixed points:

$$(1, 1), \quad (1, -1), \quad (-1, 1), \quad \text{and} \quad (-1, -1).$$

Yet both the corresponding noncommutative actions have the tracial Rokhlin property, while neither has the Rokhlin property.

Freeness of actions (continued)

There is a K-theoretic test. Without going into details, here is the statement. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a C*-algebra A,, and let $K_*^G(A)$ be the equivariant K-theory of A. As a group, it is just $K_*(C^*(G, A, \alpha))$. However, it comes with the extra structure of a module over the representation ring $R(G)$ of G.

For a prime ideal $P \subset R(G)$, one can form the "localization" $K_*^G(A)_P$. If A is commutative and unital, then the action is free if and only if these localizations are zero whenever the prime ideal P does not contain a special ideal in $R(G)$, the augmentation ideal $I(G)$.

The action of $\mathbb{Z}_2$ generated by

$$u \mapsto -u \quad \text{and} \quad v \mapsto v$$

has the property that $K_*^G(A_\theta)_P = 0$ for every prime ideal P containing $I(G)$, while the action of $\mathbb{Z}_2$ generated by

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*.$$

does not have this property.

Freeness of actions (continued)

If G is finite, and probably for general compact groups G, the Rokhlin property implies that $I(G)K_*^G(A) = 0$. This is a strictly stronger condition than the one on the previous page involving localizations.

Also, recall that a free action of a finite group on a connected compact metric space does not have the Rokhlin property. In particular,

$$(\lambda, \zeta) \mapsto (-\lambda, \zeta)$$

on $S^1 \times S^1$ generates an action of $\mathbb{Z}_2$ which does not have the Rokhlin property.

It seems that one needs a version of freeness somewhere between the Rokhlin property and the tracial Rokhlin property. I have a possible idea for what this might be, but I don't yet have a serious application beyond the possibility of distinguishing different actions.

The idea seems to potentially apply to actions on the Jiang-Su algebra, where there are no nontrivial projections.