Freeness of actions of finite groups on C*-algebras: Part 3

N. Christopher Phillips

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Outline

- Open problems related to the tracial Rokhlin property. (Not all the problems here are in the survey.)
  - What happens on purely infinite simple C*-algebras?
  - The tracial Rokhlin property and outerness on the von Neumann algebra, if there is more than one tracial state.
  - What if A is not simple?
  - What if A has few projections?
  - Tensor products.
- Some discussion of the tracial Rokhlin property for actions of $\mathbb{Z}$.

The tracial Rokhlin property

Reminder of the definition of the tracial Rokhlin property:

Definition

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

What happens on purely infinite simple C*-algebras?

It is not hard to see that the tracial Rokhlin property implies pointwise outerness. (If α_g is supposed to be $\mathrm{Ad}(u)$, in the definition take the finite set F to be $\{u\}$.)

Problem

Let A be a purely infinite simple separable nuclear unital C*-algebra (a untial Kirchberg algebra), and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a pointwise outer action of a finite group G on A. Does it follow that α has the tracial Rokhlin property?

What happens on purely infinite simple C*-algebras? (continued)

Nakamura proved that pointwise outer action of $\mathbb{Z}$ (that is, an aperiodic automorphism) on a Kirchberg algebra has the Rokhlin property.

There are K-theoretic obstructions to the Rokhlin property for finite group actions even on purely infinite simple unital C*-algebras. For example, no action of $\mathbb{Z}_2 = \{0, 1\}$ group on the Cuntz algebra $\mathcal{O}_3$ can have the Rokhlin property.

To see this: Any action must fix [1], which is a generator of $K_0(\mathcal{O}_3) \cong \mathbb{Z}_2$, and must therefore be trivial on $K_0(\mathcal{O}_3)$. Thus, if e_0 and e_1 are Rokhlin projections, then $[e_1] = [e_0]$, so $[1] = 2[e_0]$, which is not possible.

So the appropriate property to ask for is the tracial Rokhlin property rather than the Rokhlin property.

As far as I know, nobody has looked at this.

The tracial Rokhlin property and outerness on the von Neumann algebra

The following theorem is in my paper with Echterhoff, Lück, and Walters. Its proof is essentially the same as the result for actions of $\mathbb{Z}$, which is joint with Osaka. The methods are originally due to Kishimoto.

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero, and suppose that A has a unique tracial state τ. Let $\pi_\tau \colon A \to L(H_\tau)$ be the Gelfand-Naimark-Segal representation associated with τ. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. Then α has the tracial Rokhlin property if and only if α_g'' is an outer automorphism of $\pi_\tau(A)''$ for every $g \in G \setminus \{1\}$.

The tracial Rokhlin property and outerness on the von Neumann algebra

If A is simple with tracial rank zero and a unique tracial state τ, and G is finite, then $\alpha\colon G \to \mathrm{Aut}(A)$ has the tracial Rokhlin property if $g \mapsto \alpha''_g$ is pointwise outer on $\pi_\tau(A)''$.

The first step of the proof of this direction is to observe that $\pi_\tau(A)''$ is the hyperfinite factor of type II_1 (because A has tracial rank zero). Therefore a pointwise outer action of a finite group on $\pi_\tau(A)''$ has the Rokhlin property in the von Neumann algebra sense. (The errors in the Rokhlin property are controlled in $\| \cdot \|_2$, the norm coming from the trace $[\|a\|_2 = \tau(a^* a)^{1/2}]$, or, roughly equivalently, in the strong operator topology.)

Outerness on the von Neumann algebra (continued)

One now has orthogonal projections in $\pi_\tau(A)''$ which are $\| \cdot \|_2$-approximately permuted by the action and $\| \cdot \|_2$-approximately commute with a finite set. One must convert them into orthogonal projections in A which are $\| \cdot \|$-approximately permuted by the action and $\| \cdot \|$-approximately commute with a finite set. This can be done because there are analogs using $\| \cdot \|_2$ of many results about approximate projections and finite dimensional subalgebras which are proved using semiprojectivity when the approximation is in norm, provided the C*-algebra has real rank zero. Here is a sample result:

Lemma

For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ such that whenever A is a C*-algebra with real rank zero, $T \subset T(A)$, $r \in A$ is a projection, and $p_1, p_2, \ldots, p_n \in rAr$ are projections with $\|p_j p_k\|_{2,\tau} < \delta$ for $j \neq k$ and $\tau \in T$, then there exist mutually orthogonal projections $q_1, q_2, \ldots, q_n \in rAr$ such that $\|q_k - p_k\|_{2,\tau} < \varepsilon$ for all k and all $\tau \in T$.

Outerness on the von Neumann algebra (continued)

Later lemmas are about finite dimensional subalgebras of A. This is why tracial rank zero is needed.

One reason one only gets the tracial Rokhlin property rather than the Rokhlin property is that A has tracial rank zero rather than being AF. But this isn't the main point. One only gets the tracial Rokhlin property even when A is AF. The reason is that the Rokhlin projections $e_g \in A$ that one constructs are $\| \cdot \|_2$-perturbations of projections in $\pi_\tau(A)''$ which add up to 1. Thus, $\sum_{g \in G} e_g \approx 1$ in $\| \cdot \|_2$, which was essentially the definition of the tracial Rokhlin property.

In practice, verifying pointwise outerness on $\pi_\tau(A)''$ is an important method of proving the tracial Rokhlin property, for example for actions on irrational rotation algebras.

What if there is more than one tracial state?

Problem

Is there a related characterization of the tracial Rokhlin property for actions on simple separable unital C*-algebras with tracial rank zero which have more than one tracial state?

Nobody has yet tried to find one, primarily because nobody has yet wanted to use this method to prove the tracial Rokhlin property for any specific action.

One difficulty is that extreme tracial states need not be G-invariant. Even if they are, one must presumably somehow combine Rokhlin towers from $\pi_\tau(A)''$ for different extreme tracial states τ.

Even without direct applications, solving this problem would help understand the meaning of the tracial Rokhlin property.

What if A is not simple?

Recall:

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

What if A is not simple? The difficulty with a naive generalization is that if the nonzero positive element x is in a proper ideal $I \subset A$, than the error projection $1 - e$ is also forced to be in I. This is obviously not right.

Tracial rank zero has been defined for nonsimple C*-algebras. (I don't know if it is yet clear that the definition is the "right" one.)

What if A is not simple? (continued)

Problem

For a suitable version of the tracial Rokhlin property for separable unital C*-algebras which are not necessarily simple, and possibly for a suitable modification of the definition of tracial rank zero, is the theorem above still true?

Besides finding the right definition, one will need a replacement for nonsimple C*-algebras of the result of Jeong and Osaka which was used to show that "tracially small in A" implies "tracially small in $C^*(G, A, \alpha)$".

In general, very little is understood (for actions of any group) about the case in which the algebra A is neither simple nor commutative.

What if A has few projections?

Definition (Archey)

Let A be an infinite dimensional stably finite simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the *projection free tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal positive elements $a_g \in A$ for $g \in G$ with $\|a_g\| = 1$ for all $g \in G$, such that:

1. $\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|a_g c - c a_g\| < \varepsilon$ for all $g \in G$ and all $c \in F$.
3. With $a = \sum_{g \in G} a_g$, we have $\tau(1 - a) < \varepsilon$ for every tracial state τ on A.
4. With $a = \sum_{g \in G} a_g$, the element $1 - a$ is Cuntz subequivalent to an element of the hereditary subalgebra of A generated by x.

(It turns out that Condition (3) is redundant.)

What if A has few projections? (continued)

If A has no nontrivial projections, then obviously no action of a nontrivial group can have the tracial Rokhlin property. Even if there are some nontrivial projections, there may not be "enough".

For example, let Z be the Jiang-Su algebra. Then $Z \otimes Z$ has no nontrivial projections. Archey shows that the tensor flip on $Z \otimes Z$ generates an action of $\mathbb{Z}_2$ with the projection free tracial Rokhlin property.

On the other hand, if A has tracial rank zero, then the projection free tracial Rokhlin property implies the tracial Rokhlin property.

What if A has few projections? (continued)

The following theorem is due to Archey. Its other hypotheses also hold for the tensor flip on $Z \otimes Z$.

Theorem

Let A be an infinite dimensional stably finite simple unital C*-algebra with stable rank one and with strict comparison of positive elements. Further assume that every 2-quasitrace on A is a trace, and that A has only finitely many extreme tracial states. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the projection free tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.

Problem

Under the hypotheses of the previous theorem, is the order on projections over the crossed product determined by traces? What can one say about its Cuntz semigroup?

How is the projection free tracial Rokhlin property related to outerness on the von Neumann algebra?

Problem

Let A be an infinite dimensional stably finite simple unital C*-algebra, and suppose that A has a unique tracial state τ. Let $\pi_\tau \colon A \to L(H_\tau)$ be the Gelfand-Naimark-Segal representation associated with τ. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. Assume A is a simple unital AH algebra with no dimension growth, or some other suitable structural hypothesis analogous to tracial rank zero. Does it follow that α has the projection free tracial Rokhlin property if and only if α_g'' is an outer automorphism of $\pi_\tau(A)''$ for every $g \in G \setminus \{1\}$?

Another class of C*-algebras that ought to be included is the simple unital direct limits, with no dimension growth, of recursive subhomogeneous algebras. (This includes the Jiang-Su algebra.)

If this is true, what happens when A has more than one tracial state?

Tensor products

It is easy to show that if $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ are actions of a finite group G on unital C*-algebras, and α has the Rokhlin property, then so does the action $\alpha \otimes \beta$ on $A \otimes B$. (It doesn't matter whether one uses the minimal or maximal tensor product.)

Essentially, look at elementary tensors in $A \otimes B$. If $(e_g)_{g \in G}$ is a system of Rokhlin projections for α and the first components of a finite set of elementary tensors in $A \otimes B$, with a suitable tolerance, then $(e_g \otimes 1)_{g \in G}$ will be a system of Rokhlin projections for $\alpha \otimes \beta$ and the given finite set of elementary tensors.

Tensor products (continued)

Problem

Let G be a finite group, let A and B be infinite dimensional simple unital C*-algebras, let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the tracial Rokhlin property, and let $\beta \colon G \to \mathrm{Aut}(B)$ be an arbitrary action. Does it follow that $\alpha \otimes_{\min} \beta \colon G \to \mathrm{Aut}(A \otimes_{\min} B)$ has the tracial Rokhlin property?

(We use the minimal tensor product to ensure simplicity.)

Essentially, one needs to know that "tracially small in A" implies "tracially small in $A \otimes_{\min} B$".

Tensor products (continued)

There is a related result by Lin and Osaka for actions of $\mathbb{Z}$ which have the tracial cyclic Rokhlin property (see below). The assumptions are that A is simple, unital, and has tracial rank zero, that B is simple, unital, and has tracial rank at most one, that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property, and that $\beta \in \mathrm{Aut}(B)$ is arbitrary. The conclusion is that $\alpha \otimes_{\min} \beta \in \mathrm{Aut}(A \otimes_{\min} B)$ has the tracial cyclic Rokhlin property. The same proof gives the following result.

Proposition (Osaka)

Let G be a finite group, let A and B be infinite dimensional simple unital C*-algebras, let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the tracial Rokhlin property, and let $\beta \colon G \to \mathrm{Aut}(B)$ be an arbitrary action. Suppose A has tracial rank zero and B has tracial rank at most one. Then $\alpha \otimes_{\min} \beta \colon G \to \mathrm{Aut}(A \otimes_{\min} B)$ has the tracial Rokhlin property.

The Rokhlin property for actions of $\mathbb{Z}$

For actions of infinite groups, Rokhlin-type properties must use finite subsets of the group. (For one thing, one can't form $\sum_{g \in G} e_g$ when G is infinite and the e_g are nonzero projections.) In fact, they should use Følner sets, so the group should be amenable. We will only talk about $\mathbb{Z}$. (A little is known about $\mathbb{Z}^d$, and essentially nothing about any other amenable group.)

Also, crossed products of unital AF algebras by $\mathbb{Z}$ are never AF (they have nontrival K_1), so one can't get the same result as in the first lecture as for finite groups.

The following form of the Rokhlin property appears in a survey article by Izumi. Earlier versions, often using only one tower, and due to Herman and Ocneanu, and to Kishimoto.

The Rokhlin property for actions of $\mathbb{Z}$ (continued)

Definition

Let A be a simple unital C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the *Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, there are mutually orthogonal projections

$$e_0, e_1, \ldots, e_{n-1}, f_0, f_1, \ldots, f_n \in A$$

such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$ and $\|\alpha(f_j) - f_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.

(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$, and $\|f_j a - a f_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.

(3) $\sum_{j=0}^{n-1} e_j + \sum_{j=0}^{n} f_j = 1$.

Note that $\alpha(e_{n-1} + f_n) \approx e_0 + f_0$.

The Rokhlin property for actions of $\mathbb{Z}$ (continued)

One needs only arbitrarily large values of n, not all of them. However, one must say that there are arbitrarily large n such that for all finite sets F and all $\varepsilon > 0$, Rokhlin projections exist.

One gets no additional generality by allowing more towers, of arbitrary heights at least n, but again, there must be choices of heights such that for all finite sets F and all $\varepsilon > 0$, Rokhlin projections exist.

Unlike for finite groups, there is no known K-theoretic obstruction to the Rokhlin property.

Kishimoto and coauthors have proved structure theorems for crossed products of AF algebras and AT algebras by automorphisms with the Rokhlin property.

The tracial Rokhlin property for actions of $\mathbb{Z}$

The following definition is joint with Osaka:

Definition

Let A be an infinite dimensional stably finite simple unital C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.

(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.

(3) With $e = \sum_{j=0}^{n} e_j$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

If one also requires $\|\alpha(e_n) - e_0\| < \varepsilon$ (and just for arbitrarily large n—above, this makes no difference) we get the *tracial cyclic Rokhlin property* of Lin and Osaka.

The tracial Rokhlin property and the Rokhlin property

Since an error is allowed, it seemed unnecessary to have more than one tower. However, we don't actually know that on a stably finite simple unital C*-algebra, the Rokhlin property implies the tracial Rokhlin property. For example, if there are many tracial states, it might be that for every projection in a tower for the Rokhlin property, there is some tracial state which is large on that particular projection. (Problem: Can this actually happen?)

The Rokhlin property does imply the tracial Rokhlin property if A has tracial rank zero, of if A satisfies the following collection of conditions: A is approximately divisible, every quasitrace on A is a trace, and projections in A distinguish the tracial states of A.

There are no examples of automorphisms known to have the tracial Rokhlin property but known not to have the Rokhlin property.

Crossed products by actions with the tracial Rokhlin property

Here is one new technical difficulty in dealing with crossed products by actions with the tracial Rokhlin property. Namely, if $(e_j)_{0 \leq j \leq n}$ is a Rokhlin tower in the sense of the tracial Rokhlin property, then $\sum_{j=0}^{n} e_j$ is not approximately α-invariant. The point is that $\|\alpha(e_n) - e_0\|$ need not be small.

For proving things like preservation of real rank zero, this turns out not to matter. (This is joint with Osaka.) However, for proving tracial rank zero, there is trouble.

Lin and Osaka have proved that crossed products by automorphisms with the tracial *cyclic* Rokhlin property preserve tracial rank zero. Moreover, if A has tracial rank zero, then sometimes (essentially, if there is enough periodicity in the induced automorphism α_* on $K_0(A)$), then the tracial Rokhlin property implies the tracial *cyclic* Rokhlin property.

However, there are K-theoretic obstructions to the tracial cyclic Rokhlin property.

Crossed products by actions with the tracial Rokhlin property

The following problem remains open.

Problem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let $\alpha \in \mathrm{Aut}(A)$ have the tracial Rokhlin property. Does it follow that $C^*(\mathbb{Z}, A, \alpha)$ has tracial rank zero?