Research Statement

Dawn Elizabeth Archey
Department of Mathematics
University of Oregon

My research is in an area of functional analysis known as C^*-algebras. A C^*-algebra is an algebra A over $\mathbb{C}$ having an involution $*\colon A \to A$ and a norm satisfying the following properties. We write a^* for the image of a under $*$. The involution must be conjugate linear and satisfy $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all a and b in A. The norm must satisfy $\|a^*a\| = \|a\|^2$ and the algebra must be complete in this norm. One example is the n by n matrices over $\mathbb{C}$, $M_n(\mathbb{C})$ with the involution given by the conjugate transpose. For a commutative example consider the continuous functions on any compact Hausdorff space X. This algebra is denoted $C(X)$. The operations are pointwise and for the involution we use pointwise complex conjugation.

In particular, for my research I have focused on the properties of crossed product C^*-algebras. Let A be a C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then, as a set, the crossed product $C^*(G, A, \alpha)$ is the group ring $A[G]$. However, the multiplication and involution are skewed by the action α of G on A. If G is not finite but is discrete, we must complete $A[G]$ in a suitable norm. We write α_g instead of $\alpha(g)$. This construction has not only provided new examples of C^*-algebras, but has provided new ways of looking at old and naturally occurring C^*-algebras. For example, consider the irrational rotation algebras A_θ, which were originally described as being generated by elements u and v satisfying the relations $uu^* = 1$, $u^*u = 1$, $vv^* = 1$, $v^*v = 1$ and $uv = e^{2\pi i\theta}vu$. One can also describe A_θ as a crossed product by $\mathbb{Z}$ acting on $C(S^1)$ by rotation by an angle of $2\pi i\theta$.

It is natural to ask which properties of A are shared by the crossed product. In particular we would like to know when $C^*(G, A, \alpha)$ has one or both of the following two properties.

Definition 0.1. A unital C^*-algebra A has **stable rank one** if the invertible elements are dense in A [**?**].

Definition 0.2. A unital C^*-algebra A has **real rank zero** if the invertible self-adjoint elements are dense in the self adjoint elements [**?**].

One reason that these properties are important is because they are satisfied for many C^*-algebras. Additionally, stable rank one, real rank zero, or both are hypotheses of many theorems about C^*-algebras. Finally, it is known that A having stable rank one is not sufficient to guarantee that the stable rank of $C^*(G, A, \alpha)$ is one. There are examples for which the stable rank of the crossed product is two. However, by a theorem of Osaka and Teruya, for any simple unital C^*-algebra and any finite group action, the stable rank of the crossed product is two or less [**?**].

Since real rank zero implies the existence of many projections, we also need a condition guaranteeing the projections are well behaved.

Definition 0.3. For any projections p and q in A, we write $p \sim q$ if there exists an element $v \in A$ such that $v^*v = p$ and $vv^* = q$. In this case we say that p is **(Murray-von Neumann) equivalent** to q. We write $p \precsim q$ if there exists a projection r such that $p \sim r$ and $r \leq q$. In this case we say that p is **(Murray-von Neumann) subequivalent** to q.

Definition 0.4. Let A be a unital C*-algebra. We say that the **order on projections over A is determined by traces** if whenever $n \in \mathbb{N}$ and $p, q \in M_n(A)$ are projections such that $\tau(p) < \tau(q)$ for all tracial states τ on A, then $p \precsim q$.

We will also need a condition on the action.

Definition 0.5. Let A be an infinite dimensional simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the **tracial Rokhlin property** if for every finite set $F \subset A$, every $\epsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \epsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \epsilon$ for all $g \in G$ and all $a \in F$.
(3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
(4) With e as in (3), we have $\|exe\| > 1 - \epsilon$.

I have proven the following theorems which are finite group analogues of known results about actions of $\mathbb{Z}$ [**?**]:

Theorem 0.6. Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the order on projections over $C^*(G, A, \alpha)$ is determined by traces and $C^*(G, A, \alpha)$ has real rank zero.

Theorem 0.7. Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero and stable rank one, and suppose that the order on projections over A is determined by traces. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.

The tracial Rokhlin property has already proven itself useful for proving theorems about crossed products [**?**] and [**?**]. There is a related but strictly stronger notion called the Rokhlin property. For an example of an action with the tracial Rokhlin property, but not the Rokhlin property, let B be any simple C^*-algebra with tracial rank zero. Let $A = B \otimes B$ and let $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to A$ be the action which interchanges the two copies of B. That is, the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ maps to the automorphism $\alpha_2\colon a \otimes b \mapsto b \otimes a$.

There are relatively few actions with the Rokhlin property and many algebras which admit no actions at all with the Rokhlin property. However, there are many examples of actions with the tracial Rokhlin property. Additionally, since the tracial Rokhlin property implies the action is outer, it may be thought of as a strong version of outerness.

It is clear from the definition of the tracial Rokhlin property that it guarantees the existence of at least n projections, where n is the order of the group. In fact, it implies the existence of infinitely many projections. Thus a C^*-algebra with few projections cannot have any action with the tracial Rokhlin property. I have formulated a projectionless generalization of the tracial Rokhlin property. This generalization replaces the projections with positive elements and Murray-von Neumann equivalence with Cuntz equivalence of positive elements.

Definition 0.8. Let x and y be positive elements of a C^*-algebra A. We write $x \precsim y$ if there exist elements r_j in A such that $r_j y r_j^* \to x$ with the convergence in norm. In this case we say

x is **(Cuntz) subequivalent** to y. If $x \precsim y$ and $y \precsim x$, we write $x \equiv y$ and say x is **(Cuntz) equivalent** to y.

It turns out that if p and q are projections and p is Murray-von Neumann subequivalent to q, then p is Cuntz subequivalent to q. Thus the usage of the same symbol for both equivalence relations is justified.

We expect that if Z is the Jiang-Su algebra, then the action which interchanges the two copies of Z in $Z \otimes Z$ provides an example of an action with the generalized tracial Rokhlin property.

A further assumption seems to be necessary to work around the fact that positive elements do not behave quite as nicely as projections.

Definition 0.9. We say a simple unital C^*-algebra A has **approximate decomposition of positive elements** if for all positive elements a in A, all natural numbers n and all $\epsilon > 0$ there exist orthogonal positive elements $a_0, \ldots, a_n$ in A such that $\|a - \sum_{k=0}^{n} a_k\| < \epsilon$, $a_1 \equiv a_2 \equiv \cdots \equiv a_n$ and $a_0 \precsim a_1$ with equivalence and subequivalence in the sense of the Cuntz.

We suspect that any C^*-algebra which absorbs the Jiang-Su algebra tensorially is an example of an algebra with this property. Let A be a simple unital infinite dimensional C^*-algebra with real rank zero. Then any projection p can be decomposed exactly in the way requested in Definition **??**. Moreover, the elements $a_0, \ldots, a_n$ can be taken to be projections [**?**].

For the remaining portion of my thesis I expect to prove:

Conjecture 0.10. Let A be an infinite dimensional stably finite simple unital C^*-algebra with stable rank one. Assume also that A has approximate decomposition of positive elements. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the generalized tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.

Unlike Theorem **??** and Theorem **??**, the analogue for actions of $\mathbb{Z}$ is not known. See below. We do not ask the analogous question for real rank zero. This is because an algebra with real rank zero has many projections and so we can use the original definition of the tracial Rokhlin property.

A proof of Conjecture **??** would also be of interest as a test of whether the correct generalization of the tracial Rokhlin property has been chosen. It is known that if an action has this generalized tracial Rokhlin property and the algebra is simple with tracial rank zero, then the action has the original tracial Rokhlin property (Lemma 1.8 of [**?**]). Tracial rank zero implies real rank zero and thus the existence of many projections, so this a first indication the generalization has the right definition.

In my future research there are three related problems that I would like to address. The first is to attempt to prove an analogue of Conjecture **??** with the finite group G replaced by $\mathbb{Z}$. I expect the proof to proceed along lines similar to the proof of Conjecture **??**. First, I would hope to identify a large subalgebra of $C^*(G, A, \alpha)$ with matrices over a corner of A and then use the fact that this subalgebra has stable rank one and is large to complete the proof. Unfortunately, replacing G by $\mathbb{Z}$ introduces some complications. The worst of these is that the elements of $\mathbb{Z}$ have infinite order. The effect of this is that the action moves elements out of any previously chosen matrix like part of the algebra. This adds a second layer of approximation and complication. Fortunately, this has already been dealt with in the case where there are many projections [**?**].

The second question is to determine what assumptions on positive elements of A are really necessary in Conjecture **??**.

The third problem is to find a further generalization of the tracial Rokhlin property. With the further generalization, we would hope to prove the crossed product still has stable rank one, even if the hypothesis about decomposition of positive elements is not met. This would be a way of shifting the good behavior required from the algebra to the action. Additionally, under the hypotheses of Conjecture **??**, the further generalization should imply the first generalization of the tracial Rokhlin property.

My research area has some opportunities for undergraduate research projects. Some C^*-algebras have purely algebraic descriptions or important algebraic objects inside them. Such objects could be understood by an undergraduate who knew about algebras and about giving groups by generators and relations. Another possible point of contact is in the area of dynamical systems since many crossed products arise from actions of $\mathbb{Z}$ on a compact space.

The interest of my research lies mainly in its applicability to the classification program. The classification program has been one of the major thrusts in C^*-algebras for the last 15 years. This program is the search for invariants which will distinguish separable, nuclear C^*-algebras up to isomorphism. Most of known theorems deal with simple C^*-algebras. Ideally the invariants used should be relatively computable. One of the most important of these invariants is $K_0(A)$. The group $K_0(A)$ encodes information about projections in $M_n(A)$ up to Murray-von Neumann equivalence. In fact, K_0 is functor which can be considered as a non-commutative homology theory.

Analogously, the Cuntz semigroup encodes information about positive elements up to Cuntz equivalence. Recent work by Brown, Perera, and Toms indicates that the Cuntz semigroup will also be a useful invariant for the purposes of classification [**?**]. Thus far it is known only in a few cases when the crossed product has real rank zero or stable rank one. Theorems with conclusions about the real and stable rank of C^*-algebras are important because of the frequency with which real rank zero and stable rank one appear as hypotheses in classification theorems.

References

[1] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99**(1991), 131–149.

[2] N. P. Brown, F. Perera, and A. S. Toms, *The Cuntz Semigroup, the Elliott Conjecture, and dimension functions on C^*-algebras*, preprint (arXiv: math.OA/0609182)

[3] S. Echterhoff, W. Lueck, N. C. Phillips, and S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$*, preprint (arXiv:math/0609784).

[4] H. Osaka and N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergod. Th. Dynam. Sys. **26**(2006), 1579–1621.

[5] H. Osaka and T. Teruya, *Stable rank for depth 2 inclusions of C^*-algebras* , To appear in Math. Rep. Acad. Sci. Royal. Soc. Canada. (arXiv: math.OA/0602598).

[6] N. C. Phillips, *Every simple higher dimensional noncommutative torus is an AT algebra*, preprint (arXiv:math/0609783)

[7] N. C. Phillips, *Finite cyclic group actions with the tracial Rokhlin property*, preprint (arXiv: math.OA/0609785).

[8] N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, preprint (arXiv math/0609782).

[9] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. (3) **46**(1983), 301–333.