Research Statement

Dawn Elizabeth Archey
Department of Mathematics
University of Oregon

My research is in an area of functional analysis known as C^*-algebras. A C^*-algebra is an algebra A over $\mathbb{C}$ having an involution $*\colon A \to A$ and a norm satisfying the following properties. We write a^* for the image of a under $*$. The involution must be conjugate linear and satisfy $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all a and b in A. The norm must satisfy $\|a^*a\| = \|a\|^2$ and the algebra must be complete in this norm. One example is the n by n matrices over $\mathbb{C}$, $M_n(\mathbb{C})$ with the involution given by the conjugate transpose. For a commutative example consider the continuous functions on any compact Hausdorff space X. The operations are pointwise and for the involution we use pointwise complex conjugation.

In particular, for my research I have focused on the properties of crossed product C^*-algebras. Let A be a C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then, as a set, the crossed product $C^*(G, A, \alpha)$ is the group ring $A[G]$. However, the multiplication and involution are skewed by the action α of G on A. If G is not finite but is discrete, we must complete $A[G]$ in a suitable norm. This construction has provided new examples of C^*-algebras, and new ways of looking at old and naturally occurring C^*-algebras. For example, the irrational rotation algebras A_θ were originally given by generators and relations but can also described as crossed products

We would like to know when $C^*(G, A, \alpha)$ has one or both of the following two properties.

Definition 0.1. A unital C^*-algebra A has **stable rank one** if the invertible elements are dense in A [**?**].

Definition 0.2. A unital C^*-algebra A has **real rank zero** if the invertible self-adjoint elements are dense in the self-adjoint elements [**?**].

Thus far it is known only in a few cases when the crossed product has real rank zero or stable rank one. In fact, there are examples for which the stable rank of the crossed product is two. However, by a theorem of Osaka and Teruya, for any simple unital C^*-algebra and any finite group action, the stable rank of the crossed product is two or less [**?**]. Theorems with conclusions about the real and stable rank of C^*-algebras are important because of the frequency with which real rank zero and stable rank one appear as hypotheses in classification theorems.

Thus the interest of my research lies mainly in its applicability to the classification program. The classification program has been one of the major thrusts in C^*-algebras for the last 15 years. This program is the search for invariants which will distinguish separable, nuclear C^*-algebras up to isomorphism. Most of known theorems deal with simple C^*-algebras. Ideally the invariants used should be relatively computable. One of the most important of these invariants is $K_0(A)$. The group $K_0(A)$ encodes information about projections in $M_n(A)$ up to an equivalence relation known as Murray-von Neumann equivalence. In fact, K_0 is functor which can be considered as a non-commutative homology theory.

Since real rank zero implies the existence of many projections, we will need a condition guaranteeing the projections are well behaved with respect to Murray-von Neumann equivalence. This condition is called **order on projections over A is determined by traces**

We will also need a condition on the action. Let α be an action of a finite group G of order n on a C^*-algebra A. Informally speaking, α has the **tracial Rokhlin property** if for every finite subset of A, every $\epsilon > 0$, and every positive element of norm one there exist n mutually orthogonal projections satisfying some nice conditions. Since the tracial Rokhlin property implies the action is outer, it may be thought of as a strong version of outerness. The tracial Rokhlin property has already proven itself useful for proving theorems about crossed products [**?**] and [**?**]. Additionally, there are many examples of actions with the tracial Rokhlin property.

My work has been to generalize known results about actions of $\mathbb{Z}$ (see [**?**]) to the finite group case.

Theorem 0.3. Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the order on projections over $C^*(G, A, \alpha)$ is determined by traces and $C^*(G, A, \alpha)$ has real rank zero.

Theorem 0.4. Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero and stable rank one, and suppose that the order on projections over A is determined by traces. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.

The definition of the tracial Rokhlin property implies the existence of infinitely many projections. Thus a C^*-algebra with few projections cannot have any action with the tracial Rokhlin property. I have formulated a projectionless generalization of the tracial Rokhlin property. This generalization replaces the projections with positive elements and Murray-von Neumann equivalence with a relation defined on all positive elements called Cuntz equivalence. For positive elements a and b which are Cuntz equivalent we write $a \equiv b$.

A further assumption seems to be necessary to work around the fact that positive elements do not behave quite as nicely as projections.

Definition 0.5. We say a simple unital C^*-algebra A has **approximate decomposition of positive elements** if for all positive elements a in A, all natural numbers n and all $\epsilon > 0$ there exist orthogonal positive elements $a_0, \ldots, a_n$ in A such that $\|a - \sum_{k=0}^{n} a_k\| < \epsilon$, $a_1 \equiv a_2 \equiv \cdots \equiv a_n$ and $a_0 \precsim a_1$ with equivalence and subequivalence in the sense of the Cuntz.

We suspect that any C^*-algebra which absorbs the Jiang-Su algebra tensorially is an example of an algebra with this property. Let A be a simple unital infinite dimensional C^*-algebra with real rank zero. Then any projection p can be decomposed exactly in the way requested in Definition **??**. Moreover, the elements $a_0, \ldots, a_n$ can be taken to be projections [**?**].

The next result that I hope to prove is:

Conjecture 0.6. Let A be an infinite dimensional stably finite simple unital C^*-algebra with stable rank one. Assume also that A has approximate decomposition of positive elements. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the generalized tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.

A proof of Conjecture **??** would also be of interest as a test of whether the correct generalization of the tracial Rokhlin property has been chosen. It is known that if an action has this generalized tracial Rokhlin property and the algebra is simple with tracial rank zero, then the action has the original tracial Rokhlin property (Lemma 1.8 of [**?**]). Tracial rank zero implies real rank zero and thus the existence of many projections, so this a first indication the generalization has the right definition.

After proving Conjecture **??**, the first problem I would like to address is to prove an analogue of Conjecture **??** with the finite group G replaced by $\mathbb{Z}$. Unfortunately, replacing G by $\mathbb{Z}$ introduces some complications. The worst of these is that the elements of $\mathbb{Z}$ have infinite order. The effect of this is that the action moves elements out of any previously chosen matrix like part of the algebra. This adds a second layer of approximation and complication. Fortunately, this has already been dealt with in the case where there are many projections [**?**].

My research area has some opportunities for undergraduate research projects. Some C^*-algebras have purely algebraic descriptions or important algebraic objects inside them. Such objects could be understood by an undergraduate who knew about algebras and about giving groups by generators and relations. Another possible point of contact is in the area of dynamical systems since many crossed products arise from actions of $\mathbb{Z}$ on a compact space.

References

[1] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99**(1991), 131–149.

[2] S. Echterhoff, W. Lueck, N. C. Phillips, and S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$*, preprint (arXiv:math/0609784).

[3] H. Osaka and N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergod. Th. Dynam. Sys. **26**(2006), 1579–1621.

[4] H. Osaka and T. Teruya, *Stable rank for depth 2 inclusions of C^*-algebras* , To appear in Math. Rep. Acad. Sci. Royal. Soc. Canada. (arXiv: math.OA/0602598).

[5] N. C. Phillips, *Every simple higher dimensional noncommutative torus is an AT algebra*, preprint (arXiv:math/0609783)

[6] N. C. Phillips, *Finite cyclic group actions with the tracial Rokhlin property*, preprint (arXiv: math.OA/0609785).

[7] N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, preprint (arXiv math/0609782).

[8] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. (3) **46**(1983), 301–333.