Oregon Invitational Mathematics Tournament

Algebra 2 level exam

May 9, 2009

9am-12pm Solutions

Instructions

There are THREE PARTS to this exam.

- PART I consists of 10 questions with numerical answers.

  Scoring: 6 points for correct answer, 0 points for incorrect answer. No partial credit.

- PART II consists of 7 slightly harder problems with numerical answers.

  Scoring: 9 points for correct answer, 0 points for incorrect answer. No partial credit.

- PART III consists of 4 problems for which you should give full written justifications.

  Scoring: 15 points awarded for each solution with proof, partial credit given.

You should try to spend about one hour on each part of the exam.

Calculators are not allowed.

PART I.

Name:____________________________

School:____________________________

Answer all questions and write your answers in the boxes provided.

1. A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$ and $(2, 0)$ is on the graph of the parabola. What is abc?

   $\boxed{12}$

2. What is the coefficient of x^7 in the polynomial expansion of $(1 + 2x - x^2)^4$?

   $\boxed{-8}$

3. What is the value of $\sqrt{\frac{8^{10}+4^{10}}{8^4+4^{11}}}$?

   $\boxed{16}$

4. We are given that $\sin(x) = 3\cos(x)$. What is the value of $\sin(x)\cos(x)$?

   $\boxed{\frac{3}{10}}$

5. On the globe 17 parallels (lines of latitude) and 24 meridians (lines of longitude) are drawn. Into how many parts do they divide the surface of the globe?

   $\boxed{864}$

6. The average value of all the pennies, nickels, dimes and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

   $\boxed{0}$

7. At a fast-food restaurant, the cost of 3 burgers, 5 drinks and 1 salad is \$23.50, while the cost of 5 burgers, 9 drinks and 1 salad is \$39.50. How much is 2 burgers, 2 drinks and 2 salads?

   $\boxed{15}$

8. If x, y and z are positive integers such that $\dfrac{51}{11} = x - \dfrac{1}{y - \frac{1}{z}}$, find $x + y + z$.

   $\boxed{12}$

9. The price of gas went up 100% and then a month later it went up again, this time by 25%. In another month, after a sharp reduction of x% the price got back to its original level. Find x.

   $\boxed{60}$

10. For how many positive integers n between 1 and 200 (inclusive) is the number n^n a perfect square?

    $\boxed{113}$

PART II. Name:____________________

School:____________________

Answer all questions by writing your answers in the boxes provided.

11. If $f(x)$ is a function such that $2f(1/x) + f(x)/x = x$ for all $x \neq 0$, find $f(2)$.

$\boxed{-1}$

12. Find $m + n$, where m and n are relatively prime positive integers such that

$$\frac{m}{n} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{24^2}\right).$$

$\boxed{73}$

13. Find the number of pairs of positive integers (x, y) such that $x \leq y$ and $1/x + 1/y = 1/6$.

$\boxed{5}$

14. Alice and Bob run on a circular track, starting at the same time at diametrically opposite points of the track. Alice runs clockwise and Bob counterclockwise. Each runner runs at a constant speed. They first meet when Alice has run 100 meters. Next they meet when Bob has run 150 meters from their previous meeting place. Find the length of the track (in meters).

$\boxed{350}$

15. Let M and N be the midpoints of the sides BC and CD of a parallelogram $ABCD$. If the area of the triangle AMN is 15 cm^2, find the area of the parallelogram.

$\boxed{40}$

16. The product of the digits of a four-digit number is 90. How many such numbers are there?

$\boxed{60}$

17. For a semicircle of radius 1, find the side length of a square with two vertices on the diameter and two on the arc of the semicircle.

$\boxed{\frac{2}{\sqrt{5}}}$

PART III. Name:________________________

School:________________________

Solve as many problems as you can by writing your final answers in the boxes provided and giving FULL WRITTEN EXPLANATIONS with a complete justification/proof.

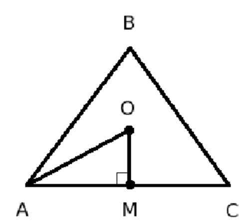
18. Find the length of the altitude (the shortest distance from a vertex to the opposite face) of the regular tetrahedron (triangular pyramid) whose sides are of length 1.

$$\boxed{\sqrt{\frac{2}{3}}}$$

Let h be the length of the altitude and R the distance from the midpoint of a face to a vertex of the same face. By the Pythagorean Theorem, $h^2 + R^2 = 1^2$, so we just need to compute R. Operating on the plane defined by a face of the tetrahedron, it is clear that $|AM| = \frac{1}{2}|AC|$ and that $\angle OAM = 30°$.

Thus $\frac{1}{2} = |AM| = |AO| \cdot cos(30°) = R \cdot \sqrt{\frac{3}{2}}$, and $R = \frac{1}{\sqrt{3}}$.

Finally, $h = \sqrt{1 - R^2} = \sqrt{1 - \dfrac{1}{3}} = \sqrt{\dfrac{2}{3}}$.



19. How many different right triangles are formed by connecting 3 vertices of a regular 10-gon?

$\boxed{40}$

Our 10-gon is inscribed into a circle, so the same is true for any triangle comprised of vertices of the 10-gon. Hence the hypotenuse of each triangle should be a diameter of the circle so that all such triangles are right. Clearly, there are 8 other vertices of the 10-gon not lying on this diameter. Thus, for every diameter we will have 8 triangles. Since there are 5 pairs of diametrically opposite vertices, we get $8 \cdot 5 = 40$.

20. Find a positive integer n such that $1/n < \sqrt{99} - \sqrt{98} < 1/(n-1)$.

$\boxed{20}$

Note that $\sqrt{99} - \sqrt{98} = \dfrac{\left(\sqrt{99} - \sqrt{98}\right)\left(\sqrt{99} + \sqrt{98}\right)}{\sqrt{99} + \sqrt{98}} = \dfrac{1}{\sqrt{99} + \sqrt{98}}$.

Thus

$$\frac{1}{n} < \frac{1}{\sqrt{99} + \sqrt{98}} < \frac{1}{n-1}.$$

This is equivalent to

$$n > \sqrt{99} + \sqrt{98} > n - 1.$$

We claim that $n = 20$:

$\sqrt{99} < \sqrt{100} = 10$ and $\sqrt{98} < \sqrt{100} = 10$, so $\sqrt{99} + \sqrt{98} < 20$.

$$\sqrt{99} + \sqrt{98} > 2\sqrt{98} = 2\sqrt{2 \cdot 49} = 14\sqrt{2}.$$

Now

$$\left(14\sqrt{2}\right)^2 = 196 \cdot 2 = 392 > 361 = 19^2,$$

so $14\sqrt{2} > 19$.

Thus $19 < \sqrt{98} + \sqrt{99} < 20$.

21. The increasing sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all those positive integers which are either powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence.

$\boxed{981}$

Numbers in the sequence are precisely those whose representation in base three has only digits 0 or 1 (i.e. no 2's). In other words, the sequence written in base three looks like 1, 10, 11, 100, 101, 110, 111, ...
i.e. The nth term of our sequence is the number whose representation in base three is the same as the base two representation of n. Since 100_{three} in base two is 1100100_{two}, we see that the 100th number in the sequence is equal to $1100100_{three} = 3^6 + 3^5 + 3^2 = 981$.