Affine Lie Superalgebras and Number Theory.

Ian M. Musson

University of Wisconsin-Milwaukee

Fix an integer $d \geq 2$. We consider the number of ways a nonnegative integer m can be written as a sum of d squares. To do this define

$$\square_{d,m} = |\{ (x_1, \ldots, x_d) \in \mathbb{Z}^d \mid m = x_1^2 + \ldots + x_d^2 \}|.$$

It is not hard to see that $\square_{d,m}$ is the coefficient of q^m in $\square(q)^d$ where

$$\square(q) = \sum_{j \in \mathbb{Z}} q^{j^2}. \tag{1}$$

In 1829 C.G.J. Jacobi obtained identities which determine the numbers $\square_{d,m}$ for $d = 2, 4, 6$. For example when $d = 2$ we have

$$\square(q)^2 = 1 - 4 \sum_{j,\ k=1}^{\infty} (-1)^k q^{j(2k-1)}. \tag{2}$$

This says that for $m \geq 2$, $\square_{2,m}$ equals 4 times the difference between the number of divisors of m congruent to 1 mod 4, and the number of divisors congruent to -1 mod 4.

A new approach to this problem was given by V.G. Kac and M. Wakimoto using the representation theory of affine Lie superalgabras. However two key results from their paper were proven only in a special case. These results may be thought of as denominator identities for basic simple (finite dimensional) and affine Lie superalgebras. We give an outline of the proofs in the case of $\mathfrak{gl}(m|n)$.