Homework 1

1. Section 8

18.2 Reread the proof of Theorem 18.1 with $[a, b]$ replaced by (a, b). Where does it break down? Discuss.

18.4 Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to a number $x_0 \notin S$. Show there exists an unbounded continuous function on S.

18.5 (a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.
(b) Show Example 1 can be viewed as a special case of part (a).

18.6 Prove $x = \cos x$ for some $x \in (0, \frac{\pi}{2})$.

18.8 Suppose f is a real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove there exists x between a and b such that $f(x) = 0$.

18.10 Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist $x, y \in [0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. *Hint*: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

2. Supplement Homework

S1. Provide an example of each or explain why the request is impossible.
- (a) Two functions f and g, neither of which is continuous at 0 but such that $f(x) + g(x)$ and $f(x)g(x)$ are both continuous at 0.
- (b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)+g(x)$ is continuous at 0.
- (c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

S2. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.
- (a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.
- (b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.
- (c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.